Existence of Engel structures

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Contents

CHAPTER 1

Introduction

Distributions are subbundles of the tangent bundle of a manifold. It is natural not to consider general distributions but to make geometric assumptions, for example integrability. In this case the distribution is tangent to a foliation. Another possibility is to assume that a distribution is nowhere integrable. Important examples of this type are contact structures on manifolds of odd dimension. Contact structures are hyperplane fields on manifolds of odd dimension which are maximally non–integrable everywhere. On 3–dimensional manifolds properties of contact structures reflect topological features of the underlying manifold in a surprising way.

An Engel structure is a smooth distribution $\mathcal{D}$ of rank 2 on a manifold M of dimension 4 which satisfies the non–integrability conditions

$$\mathrm{rank}[\mathcal{D},\mathcal{D}] = 3 \qquad\qquad \mathrm{rank}[\mathcal{D},[\mathcal{D},\mathcal{D}]] = 4\ ,$$

where $[\mathcal{D},\mathcal{D}]$ consists of those tangent vectors which can be obtained by taking commutators of local sections of $\mathcal{D}$.

If one perturbs a given Engel structure to a distribution which is sufficiently close to $\mathcal{D}$ in the C^2–topology, then the new distribution is again an Engel structure. Moreover all Engel structures are locally isomorphic, i.e. every point has a neighbourhood with local coordinates x, y, z, w such that the Engel structure is the intersection of the kernels of the one–forms

$$\alpha = dz - x dy \qquad\qquad \beta = dx - w dy\ . \tag{1}$$

This normal form was obtained first by F. Engel in [**Eng**].

The stability property described above is called stability in the sense of singularity theory. R. Montgomery has classified the distributions with this stability property.

THEOREM 1.1 (Montgomery, [**Mo1**]). *If a distribution of rank r on a manifold of dimension n is stable in the sense of singularity theory, then $r(n-r) \leq n$. It belongs to one of the following types of distributions.*

n *arbitrary*	$r = 1$	*foliations of rank one*
n *arbitrary*	$r = n-1$	*contact structures if n is odd,* *even contact structures otherwise*
$n = 4$	$r = 2$	*Engel structures*

So Engel structures are special among general distributions and even among the stable distribution types in Theorem 1.1 they seem to be exceptional. On the other hand they appear very naturally. For example a generic plane field on a four–manifold satisfies the Engel conditions almost everywhere. Engel structures can also be constructed from contact structures in a natural way. Certain non–holonomic constraints studied in classical mechanics also lead to Engel structures.

One–dimensional foliations are extensively studied in the theory of dynamical systems. Contact structures have attracted much interest during recent years. On manifolds of dimension 3 the distinction between overtwisted and tight contact structures due to Y. Eliashberg has lead to many interesting results. Using convex integration, one can find even contact structures on all manifolds with vanishing Euler characteristic. Therefore even contact structures seem to be less interesting. In contrast to this, and just like for contact structures, the standard conditions which ensure the validity of an h–principle are not satisfied by Engel structures.

An Engel structure induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} = [\mathcal{D}, \mathcal{D}] \subset TM \tag{2}$$

such that each distribution has corank one in the next one. Here $\mathcal{E}$ is an even contact structure. We say that the foliation $\mathcal{W}$ is associated to the even contact structure. Usually it is called the characteristic foliation of the even contact structure $\mathcal{E}$. The flow of vector fields tangent to the characteristic foliation preserves $\mathcal{E}$.

The existence of the flag (2) implies strong restrictions for the topology of Engel manifolds. The following theorem can be found in [**KMS**]. It was known already to V. Gershkovich. Unfortunately his preprint [**Ger**] was not available to the author.

THEOREM 1.2. *An orientable* 4*–manifold which admits an orientable Engel structure has trivial tangent bundle. Every Engel manifold admits a finite cover which is parallelizable.*

According to [**KMS**] the preprint [**Ger**] suggests an incomplete proof of the converse of Theorem 1.2. The Euler characteristic of an Engel manifold vanishes since there is a non–singular line field on M, or by parallelizability.

In the literature one can find two constructions of Engel structures. The first one is called prolongation. With this method one finds Engel structures on certain S^1–bundles over three–dimensional contact manifolds. The Engel structures obtained in this way are relatively simple, for example their characteristic foliations are given by the fibers of the S^1–bundle. This method is described in [**Mo2**]. The second construction is due to H. J. Geiges, cf. [**Gei**]. It yields Engel structures on parallelizable mapping tori. Its major disadvantage is that one can say nothing about the characteristic foliation or other properties of the Engel structure.

In this thesis we develop three new constructions of Engel manifolds. Our main result is the converse of Theorem 1.2

THEOREM 1.3. *Every parallelizable* 4*–manifold admits an orientable Engel structure.*

Note that Theorem 1.3 can be proved on open manifolds using the h–principle for open, Diff–invariant relations, cf. [**ElM**]. Thus our proof of Theorem 1.3 treats the case of closed manifolds.

1.1. Contact topology

In Chapter 2 we discuss contact structures. Contact structures are maximally non–integrable hyperplane fields on manifolds of odd dimension. In Engel manifolds contact structures appear naturally on hypersurfaces transverse to the characteristic foliation and the theory of contact structures on three–dimensional manifolds will play an important role in our constructions of Engel structures. Therefore we are mostly concerned with the case of manifolds of dimension 3. Much of the material presented here can be found in [**Aeb, EH, Gir1, Ho**].

One of the most important properties of contact structures on closed manifolds is Gray's stability theorem which is valid in all odd dimensions.

THEOREM 1.4 (Gray, [**Gr**]). *Let $\mathcal{C}_t$ be a smooth family of contact structures on a compact manifold. Then all contact structures $\mathcal{C}_t$ are isotopic.*

We will use this theorem frequently. In particular in our first construction of Engel structures we need the construction of the isotopy. We also show that there is a one–to–one correspondence between contact vector fields and differentiable functions on a contact manifold. In Section 2.1.3 we derive the local normal form of contact structures from Darboux's theorem about local normal forms for symplectic manifolds. Like Gray's theorems these results are valid for contact structures on odd dimensional manifolds.

For the remaining part of Chapter 2 we discuss contact structures on 3–manifolds.

In Section 2.2 we discuss Legendrian curves. Legendrian curves are curves which are tangent to the contact structure. We show that every curve is isotopic to a Legendrian one relative to the endpoints. The classical invariants of null–homologous Legendrian curves in a contact manifold are the Thurston–Bennequin number and the rotation number from [**Ben**]. These invariants allow us to distinguish between Legendrian curves up to isotopy through Legendrian curves. Stabilization of Legendrian curves is an efficient method to modify the Legendrian isotopy type of a Legendrian curve. It is explained in Section 2.2.4. One particular property of Legendrian curves is that on a neighbourhood of a Legendrian curve, the contact structure can be brought into a special normal form.

Next we consider convex surfaces in contact manifolds. Convex surfaces are embedded surfaces with Legendrian boundary such that there is a contact vector field transversal to the surface. In Section 2.3 we explain several results about convex surfaces without proofs. Many of the results in this section are due to E. Giroux, cf. [**Gir1**] for closed convex surfaces. Later they were generalized by K. Honda to convex surfaces with Legendrian boundary, cf. [**Ho**].

Most of the results we mention here concern the relation between the contact structure on the neighbourhood of a convex surface and a singular foliation on the surface itself. This singular foliation is defined by those tangent vectors to the surface which are also tangent to the contact structure. It turns out that much information is contained in an associated submanifold – the dividing set – of the surface. For example if the boundary of the surface is connected, then the Thurston–Bennequin invariant and the rotation number of the boundary can be derived from the dividing set using results of Y. Kanda in [**Ka2**].

We also state Eliashberg's classification theorem [**El1**] for overtwisted contact structures on closed manifolds up to isotopy. This theorem will be used at the final stage of the proof of Theorem 1.3.

The results about convex surfaces are used for the construction of bypasses in overtwisted contact manifolds in Section 2.4. Bypasses were introduced by K. Honda in order to relate the dividing sets on two convex surfaces which are isotopic but not isotopic through convex surfaces. In [**Ho**] bypasses are applied for the classification of tight contact structures on lens spaces up to isotopy. Bypasses for convex surfaces can be thought of as analogues of stabilization for Legendrian curves. A difference between these two constructions is the fact that stabilization of a Legendrian curve is always possible independently of the ambient contact structure while bypasses are not always available in tight contact manifolds.

We show that bypasses can be constructed from overtwisted discs in overtwisted contact manifolds. For this one forms the Legendrian connected sum of the boundary of an overtwisted disc and a Legendrian unknot which is constructed from a Legendrian arc on the surface. Contrary to tight contact manifolds, there are no restrictions for the existence of bypasses.

1.2. First results on Engel structures

In Chapter 3, we first define even contact structures and discuss some of their properties. Even contact structure are maximally non–integrable hyperplane fields on even dimensional manifolds. Thus the definition is similar to that of contact structures (just replace even dimensional manifolds by odd dimensional manifolds). Just like contact structures, even contact structures also admit a local normal form. However, there is an important difference between even contact structures and contact structures:

Even contact structures induce a foliation $\mathcal{W}$ of rank one tangent to the even contact structure. Every vector field tangent to the characteristic foliation $\mathcal{W}$ preserves the even contact structure. The presence of the characteristic foliation leads to a significant difference between even contact structures and contact structures. For even contact structures, the analogue of Gray's theorem (Theorem 1.4) is not true. When one modifies the even contact structure, one also modifies the characteristic foliation. But one–dimensional foliations are very sensitive with respect to perturbations. For example closed orbits can break up.

In Section 3.2 we define Engel structures and explain prolongation and the construction of Geiges. We derive Engel's normal form (1). By definition, $[\mathcal{D},\mathcal{D}] = \mathcal{E}$ is an even contact structure if $\mathcal{D}$ is an Engel structure. In this situation the characteristic foliation of $\mathcal{E}$ is tangent to $\mathcal{D}$. Recall that the characteristic foliation is tangent to the even contact structure by definition. As well as in the case of even contact structures Gray's theorem (Theorem 1.4) is not true for Engel structures.

Several theorems which will be used in our constructions concern the behaviour of Engel structures near hypersurfaces transversal to the characteristic foliation. Such a hypersurface N carries the contact structure $\mathcal{E} \cap TN$ and the intersection line field $\mathcal{D} \cap TN$ is Legendrian. The knowledge of this contact structure and of the intersection line field is enough to reconstruct the germ of the Engel structure at N.

There is a geometric interpretation of the condition that $[\mathcal{D},\mathcal{D}]$ is an even contact structure $\mathcal{E}$. As one moves along a leaf of the characteristic foliation, one can compare the Engel structure $\mathcal{D}$ at different points of the same leaf because every flow tangent to the characteristic foliation preserves the even contact structure. The plane field $\mathcal{D}$ rotates around the leaf of the characteristic foliation within the even contact structure. As long as one keeps moving in the same direction, $\mathcal{D}$ rotates without stopping. This is similar to a well known interpretation of the non–integrability condition in the definition of contact structures. It also shows that the even contact structure $\mathcal{E}$ carries a distinguished orientation if it is induced by an Engel structure $\mathcal{D}$, i.e. $\mathcal{E} = [\mathcal{D},\mathcal{D}]$.

In the remaining sections of Chapter 3 we discuss further results about Engel manifolds. In Section 3.3 we prove Theorem 1.2. There is a relation between smooth functions and Engel vector fields in Section 3.5 like for contact vector fields. For Engel structures, the functions which yield Engel vector fields have to satisfy a differential equation which leads to strong restrictions on the functions which really induce Engel vector fields. This differential equation is explained in Section 3.5. Section 3.4 contains proofs of R. Montgomery's results about deformations of certain Engel structures, cf. [**Mo2**]. We finish this chapter with a discussion of the following theorem in Section 3.6

THEOREM 1.5.

(i) *Let $\mathcal{D}_t$ be a smooth family of Engel structures such that the characteristic foliation is independent of t. Then all Engel structures $\mathcal{D}_t$ are isotopic.*
(ii) *Let $\mathcal{E}_t$ be a smooth family of even contact structures such that the characteristic foliation is independent of t. Then all even contact structures $\mathcal{E}_t$ are isotopic.*

While the first part of this theorem was proved by Golubev in [**Gol**], the second part seems not to be discussed in the literature although it is analogous to (i).

1.3. Constructions of Engel manifolds

In this thesis we develop three new methods for the construction of Engel manifolds. We describe them in Chapters 4 to 7. The first and the second construction are similar. They are treated in Chapter 5 and Chapter 6 respectively. The third method is based on Thurston geometries and it is covered in Chapter 7.

In Chapter 4 we explain some of the similarities of the first and the second construction. Here we will usually assume that all Engel structures, the Engel manifolds and the characteristic foliations are oriented. We write ∂_+ for those boundary components where the characteristic foliation points out of the manifold and ∂_- for the remaining boundary components. In this situation, the Engel structures induce oriented contact structures and oriented intersection line fields on all boundary components.

Assume we have an Engel manifold such that the boundary is transversal to the characteristic foliation of the Engel structure. We attach a manifold with boundary to the boundary of the Engel manifold. If we extend the Engel structure to the new manifold it is desirable to achieve that the new boundary is again transversal to the characteristic foliation because then we can repeat the process. This implies that we are not allowed to change the Euler characteristic of M when we attach something along the boundary.

As building blocks we use round handles. A round handle of dimension n and index $k = 0, \ldots, n-1$ is

$$R_k = D^k \times D^{n-k-1} \times S^1 \ .$$

It is attached along the boundary component $\partial_- R_k = S^{k-1} \times D^{n-k-1} \times S^1$. The other boundary component of R_k is $\partial_+ R_k = D^k \times S^{n-k-2} \times S^1$. Round handles of index k and $n-1-k$ are dual to each other, hence $\partial_- R_k \simeq \partial_+ R_{n-1-k}$.

Attaching a round handle to a manifold with boundary does not change the Euler characteristic. Therefore round handles are suitable building blocks for the construction of Engel manifolds. Conversely, every Engel manifold can be decomposed into round handles by the following theorem.

THEOREM 1.6 (Asimov, [**As1**]). *Let M be a manifold of dimension $n \neq 3$. Then M admits a decomposition into round handles if and only if its Euler characteristic is zero. In this case M admits a non–singular Morse Smale vector field.*

In Section 4.1 we sketch a proof of Theorem 1.6. By a result of J. Morgan, the analogous statement is wrong in dimension 3, cf. [**Mor**].

We will frequently use the fact that the diffeomorphism type of the manifold obtained by the attachment of a round handle depends only on the isotopy class of the attaching map. In contrast to ordinary handles, the order in which round handles of the same index are attached is essential.

An important tool in the proof of Theorem 1.6 is the fundamental lemma on round handles (Lemma 4.8). It asserts that if two ordinary handles of consecutive index $k, k+1$ are attached independently to the same connected component of the boundary, then the resulting manifold can also be obtained by attaching one round handle of index k. This lemma allows us to find the Kirby diagram of a round handle body. Conversely one can sometimes find a round handle decomposition of a given manifold from a Kirby diagram.

The model Engel structures on round handles are constructed starting from the prolongation construction. We perturb such Engel structures slightly using a contact vector field on the base manifold. This allows us to determine the characteristic foliation of the perturbed Engel structure. In Section 4.2 we introduce some of the model Engel structures. The model Engel structures on round 1–handles will be used in both constructions.

The particular contact structures we use in the prolongation lead to model Engel structures with different properties. These account for the differences between our first and second constructions.

Let M be an orientable Engel manifold whose boundary is transversal to the characteristic foliation. The conditions under which an Engel structure on M can be extended to $M \cup R_k$ by a fixed model Engel structure on R_k using a fixed attaching map are

(i) the attaching map has to preserve contact structures together with their orientations induced by the Engel structure and
(ii) the attaching map has to preserve the homotopy type of the intersection line field as a Legendrian line field.

The reason why we do not require that the attaching map preserves the intersection line field itself is the existence of a construction which allows us to change the intersection line field on a transversal boundary within its homotopy class. This can be done without changing the contact structure on $\partial_+ M$. This construction is called vertical modification. We explain it in Section 5.2.

1.3.1. The first construction – Connected sums. In Chapter 5 we describe our first construction of Engel structures. In this approach we use model Engel structures on round handles such that the contact structure on the boundary is tight. The model Engel structures depend on a parameter $k \in \mathbb{Z} \setminus \{0\}$.

For the model Engel structures on round handles of index zero and three, there is an obvious identification between $\partial_+ R_0$ and $\partial_- R_3$ which preserves the oriented contact structure and the intersection line fields if one considers the model Engel structures with the same parameter k.

The characteristic foliation of the model Engel structures on round handles of index 1 is spanned by the Liouville vector field W of a symplectic form ω on R_1, i.e. $L_W\omega$ is a positive multiple of ω. The model Engel structures on R_1 are very similar to the model handles used in [**Wei, El2**] for the construction of symplectic handle bodies: The round 1–handles with model Engel structures are also attached along tubular neighbourhoods of Legendrian curves.

The properties of the model Engel structures on round handles of index 2 reflect the duality between round handles of index 1 and 2. Unfortunately, they are not as symmetric as in the case of round handles of index 0 and 3. The characteristic foliation of the model Engel structures on R_2 is again related to a symplectic form ω on R_2. But now that characteristic foliation is spanned by a vector field W with the property that $L_W\omega$ is a negative multiple of ω.

The symmetry between model Engel structures on round handles of index 1 and 2 allows us to construct Engel structures on closed manifolds by an iteration procedure. In order to explain it, we consider first the situation without Engel structures.

Let M_1, M_2 be two manifolds with boundary and let $\psi : \partial M_1 \longrightarrow \partial M_2$ be a diffeomorphism. If we glue a round handle of index one with the attaching map $\varphi_1 : \partial_- R_1 \longrightarrow \partial M_1$, then we can attach a round handle of index 2 to M_2 using the map

$$\varphi_2 = \psi \circ \varphi_1 : \partial_+ R_2 \longrightarrow \partial M_2 .$$

After smoothing corners we obtain new manifolds with boundary

$$\widetilde{M}_1 = M_1 \cup_{\varphi_1} R_1 \qquad \widetilde{M}_2 = M_2 \cup_{\varphi_2} R_2$$

such that the new boundaries can be identified in a natural way by a diffeomorphism $\widetilde{\psi}$. When we identify the boundaries of $\widetilde{M_1}$ and $\widetilde{M_2}$ we obtain a new manifold $\widetilde{M}$. We can also apply the same procedure to $\widetilde{M}_1, \widetilde{M}_2$ and the identification map $\widetilde{\psi}$ of the boundaries.

Now let M_1, M_2 be Engel manifolds with transversal boundaries and oriented characteristic foliation. The diffeomorphism ψ preserves the induced oriented contact structures and oriented intersection line fields on $\partial_+ M_1$ and $\partial_- M_2$.

Assume that R_1 carries a model Engel structure such that the Engel structure on M_1 extends to $\widetilde{M_1}$. In Theorem 5.6 we carry out the construction outlined above. We find an attaching map φ_2 and a model Engel structure on R_2 such that the Engel structure on M_2 extends to $\widetilde{M_2}$. Moreover we construct a map

$$\widetilde{\psi} : \partial_+ \widetilde{M_1} \longrightarrow \partial_- \widetilde{M_2}$$

with properties analogous to the diffeomorphism ψ we started with. From this we obtain a smooth Engel structure on $\widetilde{M}$.

Let us remark that this construction becomes trivial if we consider only the even contact structures induced by the Engel structures. In this situation one can simply reverse the orientation of the characteristic foliation and use $\varphi_2 = \psi \circ \varphi_1$. Then $\widetilde{\psi}$ can be taken to be the obvious identification between the boundaries of new even contact manifolds $\widetilde{M_1}$ and $\widetilde{M_2}$.

The case of Engel structures is more difficult. This is due to fact that an Engel structure with an oriented characteristic foliation induces an orientation of the contact structure on transversal boundaries. For example if one takes a copy of $\widetilde{M_1}$ instead of $\widetilde{M_2}$ and identifies the boundaries by the identity, then the orientations of the contact structures do not fit together. Therefore one does not obtain an Engel structure on the double of $\widetilde{M_1}$ in this way.

For the construction of attaching maps of round 1–handles we use several facts from contact topology. Every embedded circle in a contact manifold is isotopic to a Legendrian curve. In order to change the isotopy classes of Legendrian curves we use stabilizations from Section 2.2.4. It turns out that this method is enough to provide interesting applications of our iteration procedure. It is also sufficient for the proof of Theorem 1.3.

Our first construction can be used to construct Engel structures on manifolds which are not accessible using prolongation or the construction of Geiges. We explain simple examples of this kind in Section 5.5.

If M, M' are two Engel manifolds then their connected sum does not admit an Engel structure since the Euler characteristic of $M\#M'$ is not zero. This can be corrected by adding $S^2 \times S^2$. The main application of our first construction is the following theorem from Section 5.6.

THEOREM 1.7. *Let M, M' be manifolds with Engel structures $\mathcal{D}, \mathcal{D}'$ such that both characteristic foliations admit closed transversals. Then $M\#M'\#(S^2 \times S^2)$ carries an Engel structure which coincides with the old Engel structures on M and M' away from a neighbourhood of the transversals where all connected sums are performed. The characteristic foliation of the new Engel structure again admits a closed transversal.*

If M and M' are parallelizable then the same is true for $M\#M'\#(S^2 \times S^2)$ and there is an Engel structure on $M\#M'\#(S^2 \times S^2)$ by Theorem 1.3. The advantage of Theorem 1.7 is that the given Engel structures are not modified away from a neighbourhood of the closed transversals and Theorem 1.7 does not rely on any specific decompositions of the Engel manifolds into round handles.

The condition in Theorem 1.7 that the characteristic foliations of the Engel structures admit closed transversals can be replaced by an assumption on the number of full twists of the Engel structures $\mathcal{D}, \mathcal{D}'$ in the even contact structures $\mathcal{E}, \mathcal{E}'$ when one moves along leaves of the characteristic foliations. This condition as well as the presence of a hypersurface transversal to the characteristic foliations ensure that we can apply vertical modifications. In the proof of Theorem 1.7, we use vertical modification several times.

Let us mention a special property of our first construction. Assume the Engel structure on M extends to $M \cup R_1$ by a model Engel structure. If the contact structure on $\partial_+ M$ admits a symplectic filling then the same is true for the contact structure on $\partial_+ \widetilde{M}_1$. Thus attaching a round 1–handle with our model Engel structure preserves symplectic fillability of the contact structure on the boundary, cf. [**Wei, El2**]. By a result of Y. Eliashberg and M. Gromov, the contact structures on $\partial_- M_1$ and on $\partial_+ \widetilde{M}_2$ are tight. This is a difference between our first and our second construction of Engel manifolds. In the second construction we systematically produce and use overtwisted contact structures on transversal boundaries.

Another difference concerns dynamical properties of the characteristic foliation. In our first construction the characteristic foliation is described in a very explicit way in the construction of the model Engel structures. In particular all to each round handle in the round handle decomposition corresponds one closed leaf of the characteristic foliation. All closed leaves are hyperbolic.

The constructions of model Engel structures in the second construction do not yield hyperbolic closed leaves and there is no one–to–one correspondence between closed leaves and round handles.

1.3.2. The second construction – Existence theorem. In Chapter 6 we develop our second method for the construction of Engel structures in the proof of the general existence result, Theorem 1.3. One important feature is that in this construction the contact structures on the boundary components $\partial_\pm R_k$ will be overtwisted for many of the model Engel structures. In particular this is the case for all model Engel structures on round handles of index 0 and 3.

In the proof of Theorem 1.3 we need model Engel structures on round handles of index 3 such that the contact structure on the boundary is independent of the model Engel structure and only the homotopy class of the intersection line field varies. With one exception, these model Engel structures can be obtained from the perturbation of a prolonged Engel structure. But the remaining model Engel structure is difficult to find explicitly. Therefore the construction in Section 6.3 is more complicated than the construction of the other model Engel structures.

Another difference is a much larger variety of model Engel structures on round 2–handles. Many of these Engel structures induce an overtwisted contact structure on $\partial_+ R_2$. In particular the induced contact structure on $\partial_+ R_2$ depends on the model Engel structure. Nevertheless, the induced contact structures on $\partial_- R_2$ are essentially the same for all model Engel structures.

The only model Engel structures which are used in both constructions in Chapter 5 and Chapter 6 are the model Engel structures on round 1–handles, as in the first construction. Also the method for the construction of attaching maps of round 1–handles will turn out to be flexible enough in order to prove Theorem 1.3.

Let us briefly explain the proof of Theorem 1.3. We start with a round handle decomposition of a parallelizable oriented manifold M with only one round 3–handle and we fix a trivialization of TM. Suppose we have an oriented Engel structure on a submanifold of M. All distributions in (2) are then oriented. From this we obtain framings which are adapted to the Engel structure. Such trivializations will be called Engel framings.

First we equip the round 0–handle with a model Engel structure such that the Engel framing on R_0 and the given framing are homotopic. This shows that the Engel framing extends from R_0 to a global framing on M. We homotop the given framing such that it coincides with the Engel framing on R_0.

Then we attach the first round 1–handle. As in our first construction of Engel structures we isotope the attaching map and choose a model Engel structure on R_1 such that the Engel structure extends from R_0 to $R_0 \cup R_1$. We can arrange the Engel structure on $R_0 \cup R_1$ such that the given framing on $R_0 \cup R_1$ and the Engel framing on M are homotopic. The analogous statement is true for all subsequent attachments of round 1–handles. This can be carried out such that the contact structure on the boundary remains overtwisted after each attachment of a round 1–handle. We denote the union of the round 0–handle and all round 1–handles in the round handle decomposition of M by M_1.

Let $\varphi_2 : \partial_- R_2 \longrightarrow \partial_+ M_1$ be an attaching map. Recall that all model Engel structures on round 2–handles induce equivalent contact structures on $\partial_- R_2$. In particular the singular foliation on the attaching torus is independent of the model Engel structure. If the contact structure on $\partial_+ M$ is overtwisted, then we can isotope φ_2 such that the resulting embedding preserves the singular foliations. At this point we use the fact that the contact structure on $\partial_+ M$ is overtwisted in an essential way. Using results from contact topology, we can isotope φ_2 further to obtain an attaching map which preserves contact structures.

Once this is achieved, the large variety of model Engel structures on R_2 allows us to pick a model Engel structure such that φ_2 preserves the orientations of the contact structure and the homotopy class of the intersection line fields. This way we obtain an Engel structure on $M_1 \cup R_2$. This construction can be carried out such that the contact structure on the boundary remains overtwisted. In contrast to the attachments of round 1–handles, the Engel framing on $M_1 \cup R_2$ and the given framing on M are not homotopic in general. The same procedure applies for all subsequent attachments of round handles of index 2. Thus we can construct an orientable Engel structure on the union M_2 of round handles with index $0, 1, 2$.

In order to show that we can extend the Engel structure to M, we first show that the Engel framing extends to a framing on M. This is not clear from the construction of the Engel structure on M_2 since we cannot guarantee that the Engel framing and the given framing on M_2 are homotopic. At this point the assumption that there is only one round 3–handle is important.

The fact that we can extend the Engel framing from M_2 to M implies that the contact structure on $\partial_+ M_2$ extends to M as a plane field. But there is a unique homotopy class of plane fields on $S^2 \times S^1$ which extends to $D^3 \times S^1$. According to Eliashberg's classification of overtwisted contact structures, this determines the isotopy class of the contact structure on $\partial_+ M_2$ completely.

This enables us to extend the Engel structure from M_2 to $M = M_2 \cup R_3$ using a model Engel structure on R_3.

1.3.3. The third construction – Thurston geometries. Our last construction is described in Chapter 7. It treats contact structures and Engel structures from a different point of view. In dimension 3 there is the well known list of eight Thurston geometries. We discuss which of these geometries are compatible with contact structures.

We then discuss prolongation in the context of Thurston geometries. This yields Engel structures which are compatible with certain four–dimensional Thurston geometries. The remaining four–dimensional Thurston geometries are treated individually in the last section. We show that the resulting Engel structures are sometimes very similar to Engel structures obtained by the construction of H. J. Geiges. Some examples in this chapter illustrate a problem one encounters when one wants to construct an Engel structure on connected sums $M\#M'\#(S^2 \times S^2)$ without any additional assumptions on the Engel structures as in Theorem 1.7.

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CHAPTER 2

Contact topology

In this chapter we summarize several facts from contact topology. After giving a precise definition we discuss some examples. In particular Example 2.3 of contact structures on the projective bundle associated to a manifold is similar to the construction of Engel structures from contact structures by prolongation in Proposition 3.2.2. In Section 2.1.1 we give a proof of Gray's stability theorem (Theorem 2.4). For us the importance of this theorem is due to the explicit construction of isotopies from families of contact structures. In particular in our first construction of Engel manifolds in Chapter 5 we will use this method frequently.

In Section 2.1.2 we show that there is a correspondence between contact vector fields and differentiable functions. To each function corresponds a contact vector field and vice versa. Locally all contact structures are equivalent and we discuss the normal form for contact structures in Section 2.1.3. This normal form will be used in the theorems about normal forms for even contact structures and Engel structures (Theorem 3.9 and Theorem 3.13).

The results mentioned up to now are valid for contact structures in all odd dimensions. Since an Engel structure induces contact structures on hypersurfaces which are transversal to the characteristic foliation, we will be concerned with contact structures on manifolds of dimension 3.

In the remaining part of this chapter we consider contact structures on 3–manifolds. We discuss Legendrian curves in Section 2.2. This is motivated by the fact that round 1–handles with model Engel structure will be attached along neighbourhoods of Legendrian curves in our constructions of Engel structures. We show that every curve is isotopic to a Legendrian curve (Proposition 2.10).

The two classical invariants of Legendrian knots are the Thurston–Bennequin invariant (Definition 2.15) and the rotation number (Definition 2.17). Using a normal form for contact structures on tubular neighbourhoods of Legendrian curves (Corollary 2.19) explain stabilizations of Legendrian curves. This operation changes the Legendrian isotopy type of an embedded Legendrian curve. We use the Thurston–Bennequin invariant and the rotation number to distinguish Legendrian knots. Stabilization of Legendrian curves is described in Section 2.2.4. Since this operation changes the Thurston–Bennequin invariant and the rotation number, stabilization changes the Legendrian isotopy class. We will use this method for the construction of attaching maps for round 1–handles with model Engel structures (Theorem 5.7 and Theorem 5.8).

Section 2.3 contains some facts about convex surfaces in contact manifolds. An embedded surface is called convex if there is a contact vector field transversal to the surface. Most of the material from this section is contained in [**Gir1, Ho**]. The dividing set of a convex surface consists of those points where the contact structure is tangent to the transversal contact vector field. The results described in this section show that the essential information about the contact structure on a neighbourhood of the convex surface is contained in the dividing set of the surface.

A round 2–handle with a model Engel structure is attached along neighbourhoods of convex tori. The theorems from Section 2.3 will be used to isotop attaching maps of round

2–handles such that they become contact embeddings and for the construction of bypasses in overtwisted contact manifolds (Section 2.4).

We also state Eliashberg's classification theorem for overtwisted contact structures on closed manifolds (Theorem 2.33, [**El1**]). In the construction of model Engel structures on round 3–handles in Section 6.3 and at the final stage of the existence theorem (Theorem 6.1) in Section 6.4 we obtain an overtwisted contact structure on $S^2 \times S^1$ and we can determine the homotopy class of this contact structure viewed as plane field on $S^2 \times S^1$. By Theorem 2.33 this determines the isotopy class of the contact structure.

In Section 2.4 we discuss bypasses in overtwisted contact manifolds. Bypasses were introduced by K. Honda in [**Ho**]. They provide a possibility to isotope convex surfaces through non–convex surfaces. After a bypass is attached to a convex surface it is possible to determine the dividing set on the isotoped surface (Lemma 2.36). In tight contact structures the absence of overtwisted discs and the Bennequin inequality are obstructions to the existence of bypasses. We show that bypasses can be found easily if the surface is disjoint from an overtwisted disc (Proposition 2.37). This enables us to isotope embedded tori in contact manifolds in order to obtain a particular dividing set (Section 6.2). In this way we find attaching maps for round 2–handles with model Engel structures in Section 6.4.

2.1. Basic results on contact structures

DEFINITION 2.1. A contact structure $\mathcal{C}$ on a $2n-1$–dimensional manifold N is a smooth subbundle of TN with corank 1 such that around every point of N there is a 1–form α such that

(i) ker$\alpha = \mathcal{C}$ and
(ii) $d\alpha$ has maximal rank on $\mathcal{C}$.

The second condition is equivalent to $\alpha \wedge (d\alpha)^{n-1} \neq 0$ on the domain of α. Notice that if n is even, the sign of $\alpha \wedge (d\alpha)^{n-1}$ is independent of the choice of α. Then a contact structure induces an orientation of the underlying manifold. In particular every 3–dimensional manifold with contact structure has a preferred orientation. In dimension three, orientability of M is the only obstruction for the existence of a contact structure.

THEOREM 2.2 (Martinet, Lutz, [**Mar**]). *On every closed oriented manifold of dimension* 3*, there exists a contact structure inducing the given orientation. There is a contact structure in every homotopy class of* 2*–plane fields.*

The analogous statement in the case of open manifolds is easily solved using Gromov's h–principle for open, Diff–invariant differential relations as described in [**ElM**]. The following construction of contact structures is very similar to a construction of Engel structures which we will encounter in Proposition 3.15.

EXAMPLE 2.3. Let M be an n–dimensional manifold and consider the projectivization $\mathbb{P}T^*M$ of T^*M. The total space of the bundle pr : $\mathbb{P}T^*M \to M$ has dimension $2n-1$ and carries the distribution

$$\mathcal{C} = \left\{ v \in T_{[\lambda]}\mathbb{P}T^*M \;\middle|\; \mathrm{pr}_*(v) \in \ker(\lambda) \right\} .$$

Notice that ker(λ) is independent of the choice of a representative of $[\lambda]$. In order to show that $\mathcal{C}$ is really a contact structure choose local coordinates $x_1, \ldots, x_n$ on M and the induced local trivialization of T^*M. We write $y_1, \ldots, y_n$ for the coordinates in fiber direction. Then $(x_1, \ldots, x_n, [y_1 : \ldots : y_n])$ are partially homogeneous coordinates on $\mathbb{P}T^*M$. Around $p = (0, \ldots, 0, [1 : 0 : \ldots : 0])$ we obtain local coordinates

$$(x_1, \ldots, x_n, y_2, \ldots, y_n) \longmapsto (x_1, \ldots, x_n, [1 : y_2 : \ldots : y_n]) .$$

In terms of these coordinates

$$\alpha = dx_1 + y_2 dx_2 + \ldots + y_n dx_n \tag{3}$$

is a defining form for $\mathcal{C}$. On can easily check that $\alpha \wedge (d\alpha)^{n-1}$ never vanishes on the domain of our coordinates. We can cover $\mathbb{P}T^*M$ with similar charts. Hence $\mathcal{C}$ is a contact structure.

Every diffeomorphism φ of the base manifold M induces a diffeomorphism $\widetilde{\varphi}$ by

$$[\lambda] \longmapsto [\varphi^{-1*}\lambda]$$

$$\begin{array}{ccc} \mathbb{P}T^*M & \xrightarrow[\widetilde{\varphi}]{} & \mathbb{P}T^*M \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ M & \xrightarrow[\varphi]{} & M\ . \end{array}$$

Let $v \in \mathcal{C}([\lambda])$. Using the commutative diagram above we obtain

$$\left(\varphi^{-1*}\lambda\right)(\mathrm{pr}_*(\widetilde{\varphi}_* v)) = \left(\varphi^{-1*}\lambda\right)(\varphi_*(\mathrm{pr}_* v)) = 0 \ .$$

Therefore $\widetilde{\varphi}$ preserves the contact structures. Not every contact diffeomorphism of $\mathcal{C}$ has to preserve the bundle structure of $\mathbb{P}T^*M$. Hence we do not obtain every contact diffeomorphism this way.

2.1.1. Gray's theorem. The theorem we are going to discuss now is one of the remarkable properties of contact structures. It shows that it may be possible to classify contact structures up to isotopy on compact manifolds. For us, the useful feature of the theorem is the explicit construction of isotopies ψ_s from families $\mathcal{C}_s$ of contact structures such that $\psi_{s*}\mathcal{C}_0 = \mathcal{C}_s$. This construction constitutes the proof.

THEOREM 2.4 (Gray, [**Gr**]). *Let $\mathcal{C}_s, s \in [0,1]$ be a family of contact structures on N which is constant outside of a compact subset of N. Then there is an isotopy ψ_s with the property*

$$\psi_{s*}\mathcal{C}_0 = \mathcal{C}_s \ .$$

PROOF. For the proof we assume that $\mathcal{C}_s$ is defined by a smooth family of one–forms $\alpha(s)$, i.e. we assume that $\mathcal{C}_s$ is transversely orientable. The proof without this assumption is slightly more complicated, it can be found in [**Mar**]. We construct the desired isotopy as the flow of a time–dependent vector field $Z(s)$. This is the unique vector field which is tangent to $\mathcal{C}_s = \ker(\alpha(s))$ and satisfies

$$i_{Z(s)} d\alpha(s) = -\dot{\alpha}(s) \text{ on } \mathcal{C}_s \ . \tag{4}$$

Because $d\alpha(s)$ is a non–degenerate two–form on $\mathcal{C}_s$, such a vector field exists and is uniquely determined. Notice that if $\alpha(s)$ changes while $\mathcal{C}_s$ is constant, the vector field $Z(s)$ is zero since then $\dot{\alpha}(s) = 0$ on $\mathcal{C}_s$. Since $Z(s)$ has compact support, the flow ψ_s is well defined. By construction

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=\sigma} (\psi_s^*\alpha(s)) &= \psi_\sigma^*\left(L_{Z(\sigma)}\alpha(\sigma) + \dot{\alpha}(\sigma)\right) \\ &= 0 \text{ on } \mathcal{C}_s \ . \end{aligned}$$

This shows the existence of a smooth family of functions $f(s)$ such that

$$\left.\frac{d}{ds}\right|_{s=\sigma} (\psi_s^*\alpha(s)) = f(\sigma)\alpha(0) \ .$$

Integrating this expression one can explicitly find a function $F(s)$ with the property that $\psi_s^*\alpha(s) = F(s)\alpha(0)$. Then $\psi_{s*}\mathcal{C}_0 = \mathcal{C}_s$ follows. □

If one can solve (4) without restricting to $\mathcal{C}_s$ for all s, then $f \equiv 0$ and the isotopy ψ_t satisfies $\psi^*\alpha(s) = \alpha(0)$. Under this assumption ψ_s preserves the contact forms and not only contact structures.

2.1.2. Contact vector fields. Let $\mathcal{C}$ be a contact structure on an $2k-1$–dimensional manifold H. We assume that $\mathcal{C}$ is coorientable. In particular it can be defined by a global 1–form α.

DEFINITION 2.5. A vector field X is a *contact vector field* if the local flow of X preserves $\mathcal{C}$.

Associated to a contact form there is a distinguished contact vector field.

LEMMA 2.6. *Let M be an odd-dimensional manifold and α a one-form defining a contact structure. Then there exists a unique vector field R such that $\alpha(R) \equiv 1$ and $i_R d\alpha \equiv 1$.*

PROOF. The rank of TM is odd and $d\alpha$ is a two-form. Since all two-forms have even rank, $d\alpha$ must have a non trivial kernel at every point of M. Furthermore, this kernel is one-dimensional because $d\alpha$ is non degenerate on $\mathcal{C} = \ker \alpha$ and $\mathcal{C}$ has codimension one in TM. Thus the kernel of $d\alpha$ is transversal to $\mathcal{C}$. Since $\mathcal{C}$ is defined by a global form, ker $d\alpha$ is an orientable real line bundle. It is therefore trivial and admits a section X without zeroes and $\alpha(X) \neq 0$ everywhere. Normalizing X we find a vector field R having the desired properties. The construction also shows uniqueness. □

The vector field R from Lemma 2.6 is the *Reeb vector field of* of α.

PROPOSITION 2.7. *The map which assigns to each contact vector field X the function $\alpha(X)$ is a bijection.*

PROOF. We denote the Reeb vector field of α by R. Let $X \in \mathcal{X}(\mathcal{C})$ be a vector field such that $\alpha(X) \equiv 0$. Since X preserves $\mathcal{C}$, there exists a function f such that $L_X\alpha = f\alpha$ and hence

$$i_X d\alpha = f\alpha \; . \tag{5}$$

By assumption X is tangent to $\mathcal{C} = \ker \alpha$. On the other hand, $d\alpha$ is non–degenerate on $\mathcal{C}$. If $X \not\equiv 0$, then there exists a vector field Y tangent to $\mathcal{C}$ such that

$$(i_X d\alpha)(Y) = -\alpha([X,Y]) \not\equiv 0 \; .$$

Since $f\alpha(Y) = 0$, this contradicts (5) and shows injectivity.

Now let f be a smooth function on M. Since $d\alpha\big|_{\mathcal{C}}$ is non–degenerate everywhere there is a unique vector field Y tangent to $\mathcal{C}$ such that

$$(i_Y d\alpha)\big|_{\mathcal{C}} = -df\big|_{\mathcal{C}}.$$

The Reeb vector field spans a complement of $\mathcal{C}$ in TM. Furthermore, $i_Y d\alpha$ vanishes on this complement. Therefore $i_Y d\alpha = df(R)\alpha - df$. This implies that $X = Y + fR$ has the properties

$$L_X\alpha = di_X\alpha + i_X d\alpha = df - df + df(R)\alpha = df(R)\alpha \; .$$

and $\alpha(X) = f$. Beacuse $L_X\alpha$ is a multiple of α the vector field X preserves the contact structure. This proves surjectivity. □

2.1.3. Local normal form for contact structures. All contact structures on $2n+1$–dimensional manifolds are locally diffeomophic. The same is true for even contact structures and Engel structures. Although the Darboux theorem for contact structures is well known we prove it since it will serve as starting point for the analogous theorems for even contact structures and Engel structures. We use the Darboux theorem for symplectic structures.

THEOREM 2.8 (Darboux). *Every symplectic form ω on the $2n$-dimensional manifold M is locally diffeomorphic to the symplectic form*

$$\omega_0 = \sum_{i=1}^{n} dy_i \wedge dx_i$$

on $\mathbb{R}^{2n}$.

The proof of the Darboux theorem for contact structures actually yields more than a standard form for contact structures. As we shall see in the proof, every form α defining the contact structure admits a standard coordinate expression locally. This is due to the following facts.

(i) Every symplectic form has a standard coordinate expression.
(ii) The Reeb vector field of a contact form α preserves the form α and not only the contact structure ker α.

In the case of even contact structures or Engel structures we will only obtain normal forms for distributions and not for defining forms.

THEOREM 2.9. *Let N be a manifold carrying the contact structure $\mathcal{C}$. Around every point $p \in N$ there exists a system of local coordinates $z, x_1, y_1, \ldots, x_n, y_n$ such that $\mathcal{C}$ is defined by*

$$\alpha = dz - \sum_{i=1}^{n} x_i dy_i.$$

PROOF. Let V be a neighbourhood of $p \in N$ such that $\mathcal{C}\big|_V$ is defined by a one–form α. On V we consider the Reeb vector field Z of α. The flow of Z preserves the contact form and not only the contact structure. We fix a contractible hypersurface $H \subset V$ transversal to Z through p.

The restriction of $d\alpha$ to H is a closed two–form. Because the Reeb vector field is transversal to H there is a unique real number λ for each vector $Y \in TH$ such that $Y - \lambda Z \in \mathcal{C}$. Since $i_Y d\alpha = i_{Y-\lambda Z} d\alpha$ and $d\alpha$ is non–degenerate on $\mathcal{C}$ this shows that $(H, d\alpha\big|_H)$ is a symplectic manifold. By Theorem 2.8 we can choose a coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ on a neighbourhood of p in H such that

$$d\alpha\big|_H = -\sum_{i=1}^{n} dx_i \wedge dy_i \ .$$

We assume that H is already small enough. Then $\sigma = \alpha\big|_H + \sum_{i=1}^{n} x_i dy_i$ is a closed form and because we assumed that H is contractible we can choose a function s on H such that $\sigma = ds$. Choose $\varepsilon > 0$ such that the time–t–flow ϕ_t of Z is defined for $t \in (-\varepsilon, \varepsilon)$ on a neighbourhood of p. Let

$$\begin{array}{cccc} \psi : & (-\varepsilon,\varepsilon) \times H & \longrightarrow & N \\ & (z,(x_1,y_1,\ldots,x_n,y_n)) & \longmapsto & \phi_z((x_1,y_1,\ldots,x_n,y_n)) \ . \end{array}$$

Because Z is transversal to H, the image of ψ is a neighbourhood of p. By the implicit function theorem ψ defines a system of local coordinates on some open neighbourhood U

of p in M and by the definition of ψ we have $\partial_z = Z$. Now α is invariant under the flow of Z and $\alpha(Z) \equiv 1$. We write pr for the projection of U to H along the flow lines of ϕ_t. The expression for α in our coordinate system is

$$\alpha = dz - \sum_{i=1}^{n} x_i dy_i + \text{pr}^*(ds) = d(z + s \circ \text{pr}) - \sum_{i=1}^{n} x_i dy_i \ .$$

Since $s \circ \text{pr}$ does not depend on z, the Jacobian of the transformation

$$(z, x_1, y_1, \ldots, x_n, y_n) \longmapsto (z' = z + s \circ \text{pr}, x_1' = x_1, \ldots, y_n' = y_n)$$

at p is represented by the invertible matrix

$$\begin{pmatrix} 1 & \frac{\partial s}{\partial x_1} & \cdots & \frac{\partial s}{\partial y_n} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} .$$

Hence $(z', x_1', \ldots, y_n')$ is a system of local coordinates on a neighbourhood of p such that

$$\alpha = dz' - \sum_{i=1}^{n} x_i' dy_i' \ .$$

□

2.2. Legendrian curves

2.2.1. Existence of Legendrian curves. From now on we restrict ourselves to contact structures on 3–dimensional manifolds. The following statement remains true for higher dimensions and for other non–integrable distributions. Results in this direction can be found in [**Mo3**].

PROPOSITION 2.10. *Let $\gamma : [0,1] \to N$ be a smooth curve in a contact manifold $(N, \mathcal{C})$ of dimension 3. Then γ is isotopic relative to the endpoints to a Legendrian curve $\widetilde{\gamma}$ which can be chosen C^0–close to the original curve γ*

PROOF. By Theorem 2.9 we can cover the image γ with a finite number of open sets U_i of N such that on each U_i there are coordinates x_i, y_i, z_i such that the contact structure is defined by $dz_i - x_i dy_i$. So we treat the case $N = \mathbb{R}^3, \mathcal{C} = \ker(dz - xdy)$ first.

Consider the front–projection of $\mathbb{R}^3$ to the yz–plane. A Legendrian curve can be reconstructed from front–projection as follows. The x–coordinate is determined by the slope of the front–projection since $x = \frac{dz}{dy}$. Conversely, if we want to approximate a given curve γ by a Legendrian curve $\widetilde{\gamma}$, the x–coordinate of γ has to approximate the slope of the front–projection of $\widetilde{\gamma}$.

Fix a piecewise linear curve in the yz–plane which is C^0–close to the front projection of γ. The slope of each linear segment is determined by the x–coordinate of a point on γ whose front–projection is close to the front–projection of the segment. We obtain a piecewise linear curve γ' forming zig–zags close to the front–projection of γ like in Figure 1. Now consider the Legendrian lift $\widetilde{\gamma}'$ of each segment of the zig–zag curve γ'. Each of these segments lifts to a straight Legendrian arc but these arcs do not fit together to form a smooth curve.

In order to connect the endpoints of two consecutive Legendrian segments, we consider the base–projection of $\widetilde{\gamma}'$ to the xy–plane. When the endpoints of two linear segments of γ' meet, the corresponding endpoints of the Legendrian lift have equal y– and z–coordinate. Thus the base–projection of $\widetilde{\gamma}'$ looks like the solid curves in Figure 2. In order to obtain

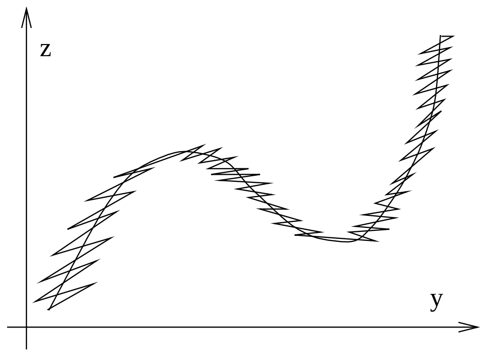


FIGURE 1.

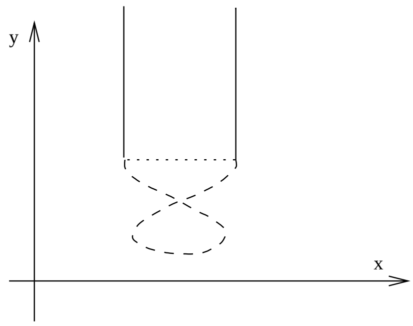


FIGURE 2.

a smooth Legendrian curve close to γ, we have to join the endpoints of two consecutive segments of $\widetilde{\gamma}'$ by short Legendrian curves. Such curves can be easily constructed using the projection to the xy–plane.

There is a unique Legendrian lift of the dashed loop in Figure 2 starting at the endpoint of one Legendrian segment. If the area enclosed by the loop and the straight line between the endpoints of the two Legendrian arcs is zero, the Legendrian lift of the loop connects the endpoints of the two segments.

This proves the theorem for $N = \mathbb{R}^3$ with the standard contact structure. For general $N, \mathcal{C}$ cover the image of γ with Darboux charts and use the construction above for segments p_i, q_i of γ which are contained completely in the domain of one chart. In order to obtain smooth curves one can choose a Darboux chart around $q_{i-1} = p_i$ and replace the Legendrian curve, which is perhaps only piecewise smooth, by a smooth Legendrian segment. □

2.2.2. Contact framings. Let γ be an embedded closed curve in an oriented manifold N of dimension 3. In particular we assume $\dot{\gamma} \neq 0$. Then γ admits a framing, i.e. a trivialization of the normal bundle. We assume that γ is parameterized by $[0, 2\pi]$.

DEFINITION 2.11. When two framings $(S,T), (S',T')$ of a curve γ are homotopic, we write $(S,T) \sim (S',T')$. On the set of framings of γ we define a $\mathbb{Z}$–action by

$$\begin{aligned}\big(m \cdot (S,T)\big)(\gamma(t)) = \big(&\cos(mt)S(\gamma(t)) + \sin(mt)T(\gamma(t) \\ &- \sin(mt)S(\gamma(t) + \cos(mt)T(\gamma(t))\big)\ .\end{aligned} \tag{6}$$

When we reverse the orientation of γ the coorientation of γ changes. Therefore the $\mathbb{Z}$–action on the framings does not depend on the orientation of γ.

LEMMA 2.12. *This $\mathbb{Z}$–action is free and transitive on the homotopy classes of framings of γ which induce the same orientation on the normal bundle of γ.*

From now on we assume that N carries an oriented contact structure which induces the orientation of N. When γ is tangent to the contact structure there is a distinguished class of framings of γ.

DEFINITION 2.13. Curves, line fields or vector fields on a contact manifold are called *Legendrian* if they are tangent to the contact structure.

A framing (S, T) of a closed Legendrian curve γ is an *oriented contact framing* if

(i) S is tangent to the contact structure,
(ii) T is transversal to it,
(iii) $\dot{\gamma}, S$ represents the orientation of the contact structure,
(iv) $\dot{\gamma}, S, T$ represents the orientation of the three–manifold induced by the contact structure.

LEMMA 2.14. *Let γ be an embedded closed Legendrian curve in a manifold N with oriented contact structure $\mathcal{C}$. Then γ has a contact framings and any two of them are homotopic through contact framings.*

PROOF. The contact structure has a nowhere vanishing section along γ, namely $\dot{\gamma}$. Because the contact structure is oriented, we can choose a Legendrian vector field S along γ which is nowhere tangent to γ such that the pair $\dot{\gamma}, S$ induces the orientation of $\mathcal{C}$. The real line bundle $TN/\mathcal{C}$ is trivial since both $\mathcal{C}$ and N are oriented. Therefore we can choose a nowhere vanishing vector field T along γ which is transversal to $\mathcal{C}$ such that $\dot{\gamma}, S, T$ represents the contact orientation of N.

Now suppose that (S, T) and (S', T') are two contact framings of γ. Since T and T' represent the coorientation of $\mathcal{C}$, the family $(S, (1-\tau)T + \tau T'), \tau \in [0, 1]$ is a homotopy between (S, T) and (S, T') through contact framings. Now we have to homotop S to S' within $\mathcal{C}$. Fix an auxiliary Riemannian metric. The angles between $\dot{\gamma}$ and S respectively $\dot{\gamma}$ and S' are contained in the open interval $(0, \pi)$. Thus homotoping S such that it points into the same direction as S' amounts to finding a homotopy between two functions $\gamma \to (0, \pi) \times \mathbb{R}^+$ where the second factor corresponds to the length of a non–zero vector tangent to $\mathcal{C}$. Since $(0, \pi) \times \mathbb{R}^+$ is contractible, there is a homotopy between (S, T) and (S', T') through contact framings. □

We write $\mathrm{fr}_{\mathcal{C}}(\gamma)$ or simply $\mathrm{fr}(\gamma)$ for the homotopy class of framings of γ which contains contact framings.

There are two famous classical invariants for null–homologous Legendrian curves in 3–manifolds with oriented contact structure, namely

- the Thurston–Bennequin invariant and
- the rotation number.

They were introduced in [**Ben**] and allow us to distinguish Legendrian curves up to isotopy through Legendrian curves. We will use slightly modified versions of these classical invariants, but for matters of comparison we recall the definitions from [**Aeb**].

DEFINITION 2.15. Let γ be a Legendrian curve homologous to zero in N. Fix a relative homology class $[\Sigma] \in H_2(N, \gamma; \mathbb{Z})$ which is represented by an oriented surface Σ such that $\partial\Sigma = \gamma$ and γ is oriented as boundary of Σ. A new curve γ' is obtained by pushing γ slightly along a vector field which is transversal to $\mathcal{C}$. The *Thurston–Bennequin invariant* $\mathrm{tb}(\gamma, [\Sigma])$ is the homological intersection number of γ' with Σ.

If $H_2(N;\mathbb{Z}) = 0$ the Thurston–Bennequin invariant can also be defined as linking number of γ' and γ.

REMARK 2.16. A surface Σ bounding γ induces a framing of γ such that $S_\Sigma(t)$ is the inward pointing normal vector of $\partial\Sigma$ and $T_\Sigma(t)$ is transversal to Σ such that $\dot\gamma, S_\Sigma, T_\Sigma$ is positively oriented. Then Σ is oriented by $\dot\gamma, S_\Sigma$. We write $\mathrm{fr}_\Sigma(\gamma)$ for the homotopy class of this framing of γ. The Thurston–Bennequin invariant measures the difference between the framing of γ which is induced by the surface and the contact framing $S_\mathcal{C}, T_\mathcal{C}$

$$\mathrm{tb}(\gamma,[\Sigma])\cdot \mathrm{fr}_\Sigma(\gamma) = \mathrm{fr}_\mathcal{C}(\gamma)\ . \tag{7}$$

If a homotopy class of framings of γ is represented by a framing induced by a surface Σ with $\partial\Sigma = \gamma$, we denote this homotopy class by $\mathrm{fr}_\Sigma(\gamma)$. If Φ is a diffeomorphism of N, the image of a framing is $\Phi_*(S,T) = (\Phi_*S, \Phi_*T)$.

The second classical invariant of a null–homologous oriented Legendrian curve is the rotation number.

DEFINITION 2.17. Let Σ be a connected orientable surface with $\partial\Sigma = \gamma$. Fix an oriented trivialization X, Y of $\mathcal{C}\big|_\Sigma$. Then there are unique functions f_x, f_y such that $\dot\gamma(t) = f_X(t)X + f_Y(t)Y$. The winding number of

$$\begin{aligned} S^1 &\longrightarrow \mathbb{R}^2 \setminus \{(0,0\} \\ t &\longmapsto (f_Y(t), f_Y(t)) \end{aligned}$$

around $(0,0)$ is the *rotation number* $\mathrm{rot}(\gamma,[\Sigma])$.

The rotation number changes sign when we change the orientation of γ while the Thurston–Bennequin invariant does not depend on the orientation of γ.

2.2.3. Tubular neighbourhoods of Legendrian curves. An example of a Legendrian curve in a contact manifold is

$$\begin{aligned} \gamma_0 &= \{(0,0)\} \times S^1 \subset \mathbb{R}^2 \times S^1 = N_0 \\ \alpha_0 &= dy - x dt \end{aligned}$$

with the usual coordinates x, y, t on $\mathbb{R}^2 \times S^1$. The contact structure is $\mathcal{C}_0 = \ker(\alpha_0)$. Now suppose we are given a Legendrian curve γ_1 in a second contact manifold $(N_1, \mathcal{C}_1)$. We want to compare a tubular neighbourhood of γ_1 with $(\gamma_0, N_0, \mathcal{C}_0)$. Let

$$\varphi : N_0 \longrightarrow N_1$$

be an embedding which maps γ_0 to γ_1.

PROPOSITION 2.18. *φ is isotopic relative γ_0 to a contact embedding if and only if φ maps a contact framing of γ_0 to a framing of γ_1 which is homotopic to a contact framing.*

If in addition the contact structures are oriented then under the above condition on framings, φ is isotopic to a contact map preserving oriented contact structures.

PROOF. It is obvious that the condition on the framings is necessary. We now show that it is also sufficient.

If the image of a contact framing of γ_0 is homotopic to a contact framing of γ_1 then the pullback of the contact structure $\varphi_*^{-1}(\mathcal{C}_1)$ is homotopic to $\mathcal{C}_0$ along γ_0. This homotopy induces a fibrewise linear isotopy H_s of $\mathbb{R}^2 \times S^1$ such that $H_0 = \mathrm{id}$ and

$$H_{1*}(\mathcal{C}_0) = \varphi_*^{-1}(\mathcal{C}_1) \qquad \text{along } \gamma_0\ .$$

Hence $\varphi \circ H_1$ is isotopic to φ via $\varphi \circ H_s$ and $\varphi \circ H_1$ preserves the contact structure along γ_0. Moreover $\varphi \circ H_s = \varphi$ along γ_0.

From now on we assume that φ preserves the contact structures along γ_0. Extend the restriction of $\varphi^{-1*}\alpha_0$ to γ_1 to a defining form α' for the contact structure on N_1. For $s \in [0,1]$, let

$$\beta_s = (1-s)\varphi^{-1*}\alpha_0 + s\alpha_1 \ .$$

By our assumption, φ maps contact framings of γ_0 to contact framings of γ_1. In particular φ preserves the orientations which are induced by $\mathcal{C}_0$ and $\mathcal{C}_1$. Since $\varphi^{-1*}\alpha_0$ and α_1 define the same coorientation of $\mathcal{C}_1$ along γ_1, $d(\varphi^{-1*}\alpha_0$ and $d\alpha_1$ define the same orientation of $\mathcal{C}_1$ along γ_1. Hence all four summands in

$$\begin{aligned} \beta_s \wedge d\beta_s = {} & (1-s)^2\varphi^{-1*}\alpha_0 \wedge d\alpha_0 + s^2\alpha_1 \wedge d\alpha_1 \\ & + s(1-s)\alpha_1 \wedge \varphi^{-1*}\alpha_0 + (1-s)s\varphi^{-1*}\alpha_0 \wedge d\alpha_1 \end{aligned}$$

are not negative and one of the first two is positive. There is a tubular neighbourhood U of γ_1 such that β_s defines a contact structure on U for all s.

Now we apply the proof of Theorem 2.4 to β_s on U. The vector field Z_s is the unique time–dependent vector field with

$$\begin{aligned} & \beta_s(Z_s) = 0 \\ & i_{Z_s} d\beta_s = -\dot{\beta}_s \text{ on } \ker(\beta_s) \ . \end{aligned}$$

Let ψ_s be the local flow of Z_s. Along γ_1 the family β_s is constant. This implies $Z_s \equiv 0$ along γ_1. All points on γ_1 are fixed and

$$\psi_{s*}(\ker(\beta_0)) = \ker(\beta_s) \ .$$

Hence $\psi_1 \circ \varphi$ is isotopic to φ and on a neighbourhood of γ_0 we have

$$(\psi_1 \circ \varphi)_*(\ker(\alpha_0)) = \psi_1(\ker(\varphi^{-1*}\alpha_0)) = \ker(\alpha_1) \ .$$

The statement about orientations follows from the fact that the map

$$\begin{aligned} N_0 = \mathbb{R}^2 \times S^1 & \longrightarrow \mathbb{R}^2 \times S^1 = N_0 \\ ((x,y),t) & \longmapsto ((-x,-y),t) \end{aligned}$$

is homotopic to the identity relative γ_0 and it reverses a given orientation of $\mathcal{C}_0$. □

COROLLARY 2.19. *Every closed Legendrian curve γ has a tubular neighbourhood which is diffeomorphic as a contact manifold to $\gamma_0 \subset \mathbb{R}^2 \times S^1$ with the contact structure $dy - xdt$.*

2.2.4. Stabilization of Legendrian curves. We need to manipulate the Legendrian isotopy type of Legendrian curves and Stabilization is a method to do so. Contact framings and rotation numbers can be used to distinguish Legendrian isotopy classes of Legendrian curves.

In order to explain stabilization of Legendrian curves, recall from Corollary 2.19 that a Legendrian curve has a tubular neighbourhood $\mathbb{R}^2 \times S^1$ with coordinates x, y, t, such that the contact structure is defined by $dy - xdt$. The orientation induced by this contact structure is $dx \wedge dy \wedge dt$. The curve $\gamma = \{(0,0)\} \times S^1$ is Legendrian and oriented by ∂_t, We assume that the contact structure is cooriented by ∂_y. This vector field points outwards in Figure 3.

In order to represent Legendrian curves, we project to the tx–space. Let $p, q \in \gamma$. The orientation of the contact structure itself projects to the orientation $dt \wedge dx$ of the tx–space. We modify the arc from q to p of this Legendrian curve as shown by the dashed curve in the upper part of Figure 3. The signed area enclosed by the dashed curve and the projection of $\{(0,0)\} \times S^1$ is zero. This ensures that the Legendrian lift of the dashed curve starting at q really meets the Legendrian curve $\{(0,0)\} \times S^1$. The other stabilization operation

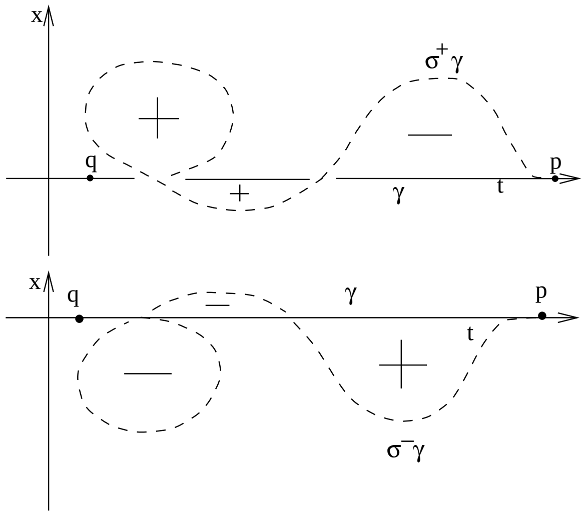


FIGURE 3.

σ^- corresponds to the lower part of Figure 3.The orientation of $\sigma^+\gamma$ is the orientation of γ on the complementary arc p, q of γ. Assume that X is a nowhere vanishing section of the contact structure on N. We can homotop X such that on the part of N where the stabilization of γ is performed $X = \partial_x$. The stabilized Legendrian curve σ^+ has an additional twist compared to γ. With our choices of orientations and a similar argument for $\sigma^-\gamma$ this leads to

$$\begin{aligned} \text{rot}(\sigma^+\gamma, X) &= \text{rot}(\gamma, X) + 1 \\ \text{rot}(\sigma^-\gamma, X) &= \text{rot}(\gamma, X) - 1 \ . \end{aligned} \tag{8}$$

The signs in (8) explain the notation σ^+, σ^-. Now let (S, T) be a contact framing of γ. If we homotop S, T on the arc between q and p suitably, we can assume that $S = \partial_x, T = \partial_y$ along this arc. We can also choose a contact framing (S', T') along $\sigma^+\gamma$ such that $T' = \partial_y$ on the part of $\sigma^+\gamma$ represented in Figure 3. If one performs an ambient isotopy $\psi_s, s \in [0, 1]$ deforming γ to $\sigma^+\gamma$, one obtains

$$\text{fr}_{\mathcal{C}}(\sigma^+\gamma) = 1 \cdot (\psi_{1*}\text{fr}_{\mathcal{C}}(\gamma)) \ . \tag{9}$$

The same statement holds for $\sigma^-\gamma$.

Using (7) we now determine the effect of stabilization on the Thurston–Bennequin invariant in the case when $\gamma = \partial\Sigma$. Let ψ_s be an isotopy of N deforming $\sigma^+\gamma$ to γ. By (7) we have

$$\begin{aligned} \text{fr}_{\mathcal{C}}(\gamma) &= \text{tb}(\gamma, [\Sigma]) \cdot \text{fr}_{\Sigma}(\gamma) \\ \text{fr}_{\mathcal{C}}(\sigma^+\gamma) &= \text{tb}(\sigma^+\gamma, [\psi_1(\Sigma)]) \cdot \text{fr}_{\psi_1(\Sigma)}(\sigma^+\gamma) \end{aligned}$$

Using (9) we obtain

$$\text{tb}(\sigma^+\gamma, [\psi_1(\Sigma)]) = \text{tb}(\gamma, [\Sigma]) - 1 \ .$$

The same expression holds for $\sigma^-\gamma$. When we apply stabilization, the Thurston–Bennequin invariant always decreases. On the other hand the Bennequin inequality, cf. [**Ben**] shows that in some cases the Thurston–Bennequin invariant of all curves in the same (usual) isotopy class has an upper bound. Nevertheless, the following theorem indicates that positive and negative stabilization σ^+ and σ^- provide enough flexibility in many situations.

THEOREM 2.20 (Fuchs, Tabachnikov, [**FT**]). *Let γ_1, γ_2 be Legendrian knots in $\mathbb{R}^3$ with the standard contact structure such that γ_1 and γ_2 represent the same topological knot type. If one applies σ^+ and σ^- to both γ_1 and γ_2 often enough, the resulting curves become isotopic as Legendrian curves.*

Notice that stabilization does not change the parity of the sum of the rotation number and the Thurston–Bennequin invariant

$$\begin{aligned} \text{tb}(\gamma) + \text{rot}(\gamma) &\equiv \text{tb}(\sigma^+\gamma) + \text{rot}(\sigma^+\gamma) \mod 2 \\ &\equiv \text{tb}(\sigma^-\gamma) + \text{rot}(\sigma^-\gamma) \mod 2\,. \end{aligned} \tag{10}$$

For example this sum is always odd for Legendrian knots in $\mathbb{R}^3$ with the standard contact structure.

Finally notice that the effect of stabilization on rotation numbers depends on the orientation of the contact structure. If we orient the contact structure by $-dt \wedge dx$, the effects of σ^+ and σ^- on rotation numbers would be interchanged. However there is always one stabilization σ^+ which increases rotation numbers while σ^- decreases rotation numbers.

2.3. Facts from the theory of convex surfaces

In this section we recall several facts from the theory of contact structures which are used in the proof of Theorem 6.1.

Let $(M, \mathcal{C})$ be a contact manifold. Consider a properly embedded orientable surface Σ. If Σ has a boundary, it is assumed to be Legendrian. On Σ we consider the singular foliation $\mathcal{F} = \mathcal{C} \cap T\Sigma$. Usually this is called the characteristic foliation of Σ. Since in the context of Engel structures there is another characteristic foliation (without singularities), we will refer to $\mathcal{F}$ simply as the *singular foliation* on Σ. The singularities of $\mathcal{F}$ are those points $p \in \Sigma$ where $\mathcal{C}_p = T_p\Sigma$.

If Σ and $\mathcal{C}$ are oriented, the singular foliation is also oriented by the following convention. If p is a non–singular point on Σ, then choose

$$v \in \mathcal{F}_p, v_\Sigma \in T_p\Sigma \setminus \mathcal{F}_p \text{ and } v_{\mathcal{C}} \in \mathcal{C}_p \setminus \mathcal{F}_p$$

such that (v, v_Σ) orients Σ and $(v, v_{\mathcal{C}})$ orients $\mathcal{C}$. Then v represents the orientation of $\mathcal{F}_p$ if $(v, v_{\mathcal{C}}, v_\Sigma)$ is the contact orientation.

Generically, singular points are non–degenerate. We say that a singular point is *elliptic* if its index is $+1$ and *hyperbolic* if the index is -1. When the orientation of $\mathcal{C}$ and the orientation of the surface coincide at a singular point of $\mathcal{F}$, we say that this singularity is *positive*, otherwise it is *negative*. If we orient $\mathcal{F}$ according to our conventions, positive elliptic points are sources and negative elliptic points are sinks.

DEFINITION 2.21. Σ is called *convex* if there is a contact vector field which is transversal to Σ.

Giroux studied convex surfaces in [**Gir1**]. In particular he showed that a closed embedded surface is generically convex (with respect to the C^∞–topology). For surfaces with boundary, the analogous statement is not true in general. For each boundary component $\gamma \subset \partial\Sigma$, we can compare the contact framing with the framing fr_Σ of γ which is induced by the surface. We write $t(\gamma, \text{fr}_\Sigma)$ for the number of counterclockwise full twists of $\mathcal{C}$ with respect to fr_Σ along γ. If γ is a Legendrian knot and Σ is a Seifert surface for γ, then $t(\gamma, \text{fr}_\Sigma)$ is the Thurston–Bennequin invariant.

PROPOSITION 2.22 (Honda, [**Ho**]). *Let Σ be a compact oriented, properly embedded surface with Legendrian boundary, and assume $t(\gamma, \text{fr}_\Sigma) \leq 0$ for all boundary components of Σ. There exists a C^0–small perturbation near the boundary (fixing $\partial\Sigma$) which puts an* *annular neighbourhood A of $\partial\Sigma$ into a standard form, and a subsequent perturbation of the perturbed surface (fixing the annular neighbourhood of $\partial\Sigma$), which makes Σ convex. Moreover, if V is a contact vector field defined on a neighbourhood of A and transverse to $A \subset \Sigma$, then V can be extended to a contact vector field transverse to all of Σ.*

DEFINITION 2.23. Given a convex surface Σ with Legendrian boundary we fix a contact vector field V transversal to Σ. The *dividing set* of Σ is

$$\Gamma_\Sigma = \left\{p \in \Sigma \middle| V(p) \in \mathcal{C}(p)\right\} \ .$$

Giroux showed in [**Gir1**] that Γ_Σ is a submanifold of Σ which is transverse to the singular foliation. Its isotopy class depends only on Σ itself but not on V. From his results it follows immediately that the dividing set of closed convex surfaces Σ is not empty.

DEFINITION 2.24. Let $\mathcal{F}$ be a singular foliation on Σ such that $\partial\Sigma$ is tangent to $\mathcal{F}$. A collection $\Gamma \subset \Sigma$ of closed curves and arcs with end points on $\partial\Sigma$ is said to *divide* $\mathcal{F}$ if on each connected component of the closure of $\Sigma \setminus \Gamma$ there is a smooth volume form ω and a vector field X tangent to $\mathcal{F}$ such that

(i) the divergence of X with respect to ω is positive everywhere and
(ii) X points out of the component wherever $\mathcal{F}$ is transversal to the boundary of the component.

THEOREM 2.25 (Giroux, [**Gir1**]). *If Σ is a convex surface in a contact manifold Γ_Σ divides the singular foliation on Σ.*

If a singular foliation $\mathcal{F}$ on the closed oriented surface Σ is divided by Γ, then there is a positive $\mathbb{R}$–invariant contact structure on $\Sigma \times \mathbb{R}$ such that $\Sigma \times \{0\}$ is convex, the induced singular foliation on $\Sigma \times \{0\}$ is precisely $\mathcal{F}$ and that Γ is the dividing set.

If $\mathcal{C}$ is cooriented by a contact form α and Σ is a closed convex surface, we choose a contact vector field V transversal to Σ such that V followed by the orientation of Σ is the contact orientation. The dividing set Γ separates the region Σ_+ where $\alpha(V)$ is positive from the region Σ_- where $\alpha(V)$ is negative. Let $\chi(\mathcal{C})$ be the Euler class of $\mathcal{C}$ viewed as oriented bundle. Then

$$\langle \chi(\mathcal{C}), [\Sigma] \rangle = \chi(\Sigma_+) - \chi(\Sigma_-) \ .$$

If Σ is the Seifert surface of a Legendrian knot we can derive the classical invariants of $\partial\Sigma$ from Σ_+, Σ_- and Γ as

$$\begin{aligned} \text{tb}(\partial\Sigma) &= -\frac{1}{2}\#(\Gamma \cap \partial\Sigma) \\ \text{rot}(\partial\Sigma) &= \chi(\Sigma_+) - \chi(\Sigma_-) \ . \end{aligned} \tag{11}$$

These formulas are due to Kanda, [**Ka2, Ho**].

The singular foliation is enough to determine the contact structure on a small neighbourhood of a convex surface Σ.

THEOREM 2.26 (Giroux, [**Gir1**]). *Let Σ be a closed orientable convex surface. Two $\mathbb{R}$–invariant contact structures on $\Sigma \times \mathbb{R}$ that induce the same orientation and the same singular foliation on $\Sigma \times \{0\}$ are isotopic. They are conjugate by a diffeomorphism $\varphi \times \mathrm{id}$ and φ is isotopic to the identity through diffeomorphisms of Σ that preserve the singular foliation.*

Next we consider deformations of the singular foliation. Let Σ be a convex surface with Legendrian boundary and fix a transverse contact vector field V. We write $\mathcal{F}_0$ for the singular foliation on Σ.

DEFINITION 2.27. An isotopy Φ_s of a surface Σ is called *admissible* if $\Phi_s(\Sigma)$ is transversal to V for all s.

The following theorem is a generalization of the Giroux flexibility theorem. In Giroux's original statement Σ is assumed to be closed.

THEOREM 2.28 (Giroux, Honda,[**Gir1, Ho**]). *Assume that $\mathcal{F}_1$ is a singular foliation which is divided by Γ_Σ. Then there is an admissible isotopy $\Phi_s, s \in [0,1]$, of Σ such that $\Phi_1(\mathcal{F}_1)$ is the singular foliation on $\Phi_1(\Sigma)$.*

EXAMPLE 2.29. In this example we want to fix some terminology. Consider the $\mathbb{R}$–invariant contact structure

$$\cos(\varphi)dt + \sin(\varphi)dx$$

on $T^2 \times \mathbb{R}$ where x is the coordinate on the $\mathbb{R}$–factor. We say that the singular foliation $\mathcal{F}$ on $T^2 \times \{0\}$ is in *standard form*. The singularities of the singular foliation form two circles $\{\varphi = \pi/2\} \cup \{\varphi = 3\pi/2\}$. Theses are referred to as *Legendrian divides*. The dividing set of $T^2 \times \{0\}$ is

$$\Gamma_{T^2} = \{\varphi = 0\} \cup \{\varphi = \pi\} \ .$$

The curves tangent to ∂_φ are called the *Legendrian ruling*. By Theorem 2.28, the slope of the Legendrian ruling can be changed as long as these Legendrian curves remain transversal to the dividing set. However in our applications we will have an identification of T^2 with $S^1 \times S^1$. We will assume that the Legendrian ruling of a torus in standard form is tangent to the first factor.

Let Σ be a convex surface with Legendrian boundary in a contact manifold. We fix a transversal contact vector field and let Γ_Σ be the corresponding dividing set.

DEFINITION 2.30. A union C of disjoint properly embedded arcs and closed curves on Σ is called *non–isolating* if

(i) C is transverse to Γ_Σ and every arc begins and ends on Γ_Σ.
(ii) every component of $\Sigma \setminus (\Gamma_\Sigma \cup C)$ has a boundary component which intersects Γ_Σ.

The Legendrian realization principle allows us to isotop Σ such that we end up with a collection of Legendrian arcs contained in the singular foliation of the isotoped surface.

THEOREM 2.31 (Kanda, Honda, [**Ka1, Ho**]). *Consider C, a non–isolating collection of properly embedded closed curves and arcs, on a convex surface Σ with Legendrian boundary. Then there exists an admissible isotopy $\Phi_s, s \in [0,1]$ so that*

(i) $\Phi_0 = \mathrm{id}$
(ii) $\Phi_1(\Gamma_\Sigma) = \Gamma_{\Phi_1(\Sigma)}$
(iii) $\Phi_1(C)$ *is Legendrian.*

Let D^2 be an embedded disc with Legendrian boundary. The following dichotomy of contact structures has turned out to be very fruitful.

DEFINITION 2.32. D^2 is called an *overtwisted disc* if all singularities on the boundary have the same sign. A contact structure is called *overtwisted* if it admits an overtwisted disc. A contact structure is *tight* if it is not overtwisted.

Overtwisted discs are often defined by requiring that there are no singularities on the boundary. This is equivalent to our definition by Theorem 2.28. Tight contact structures are more interesting than overtwisted ones in many aspects. More information about tight contact structures can be found in [**Ho**] and the references therein. For our purposes however, the flexibility of overtwisted contact structures will turn out to be very useful.

At the final stage of the construction we will apply the following theorem. A discussion of this theorem as well as of its generalizations can be found in [**Gir2**].

THEOREM 2.33 (Eliashberg, [**El1**]). *If two overtwisted contact structures on a closed manifold are homotopic as plane fields then they are isotopic.*

We will distinguish overtwisted contact structures from tight ones using the following criterion. Sometimes this theorem is referred to as Giroux's criterion.

THEOREM 2.34 (Colin, [**Col**]). *If $\Sigma \neq S^2$ is a convex surface (closed or compact with Legendrian boundary) in a contact manifold $(M, \mathcal{C})$, then Σ has a tight neighbourhood if and only if the dividing set of Σ has no homotopically trivial closed curves. If $\Sigma = S^2$, Σ has a tight neighbourhood if and only if the dividing set has exactly one connected component.*

2.4. Bypasses in overtwisted contact structures

In our construction of Engel manifolds in Chapter 6 we need to manipulate convex tori in overtwisted contact manifolds. This can be done using bypasses. Bypasses where introduced by Honda and they turned out to be useful tools for the understanding of contact structures, cf.[**Ho**].

Recall the following definition of Honda [**Ho**]. We consider a convex surface $\Sigma \subset N$ in a contact manifold $(N, \mathcal{C})$. The surface is either closed or the boundary consists of Legendrian curves. We fix a contact vector field X which is transversal to Σ. Let Γ_Σ be the corresponding dividing set of Σ, i.e.

$$\Gamma_\Sigma = \{p \in \Sigma | X(p) \text{ is tangent to } \mathcal{C}(p)\} .$$

Recall that Γ_Σ is the union of pairwise disjoint embedded curves. Moreover Γ_Σ is transversal to the singular foliation on Σ.

DEFINITION 2.35. A *bypass* for Σ is an embedded half disk D with Legendrian boundary with the following properties:

(i) ∂D is the union of two arcs γ_1, γ_2 which intersect at their endpoints.
(ii) D intersects Σ transversally along γ_1. There are no other intersection points.
(iii) D admits an orientation such that the singular foliation of D along ∂D has the following properties.
  - There are exactly two positive tangencies along γ_1. These are the endpoints of γ_1. They are elliptic.
  - There is exactly one negative tangency on γ_1. It is elliptic.
  - There are only positive tangencies along γ_2. They alternate between elliptic and hyperbolic.
(iv) γ_1 intersects Γ_Σ in exactly three points. The intersections are transversal and correspond to the tangencies of D along γ_1.
(v) The dividing set of D has exactly one connected component.

Requirement (v) in this definition does not appear in [**Ho**]. This is due to the fact the in [**Ho**], all contact structures are tight. In this situation, the dividing set Γ_D of D is determined (up to isotopy) by (i)–(iv). These assumptions imply that the only non–closed component of Γ_D is an arc lying on different connected components of γ_1 when one removes the point of tangency in the interior of γ_1. In overtwisted contact structures however, there could be additional closed components in Γ_D. These are excluded in tight contact manifolds since they would imply the existence of an overtwisted disk in a neighbourhood of D by Theorem 2.34. The bypass attachment lemma (Lemma 2.36) holds only if the dividing set of D has only one connected component.

A bypass allows us to isotope Σ in N such that the resulting surface is again convex and we can determine the dividing set of the new surface up to isotopy.

LEMMA 2.36 (Honda, [**Ho**]). *Assume that D is a bypass for a convex surface Σ. Then there exists a neighbourhood of $\Sigma \cup D \subset N$ which is diffeomorphic to $\Sigma \times [0,1]$ such that*

(i) *$\Sigma \times \{i\}$ is convex for $i = 0, 1$.*
(ii) *The dividing set of $\Sigma \times \{1\}$ can be obtained from the dividing set of $\Sigma \times \{0\}$ as in Figure 4. (In this figure, the bypass is attached to the front. It represents only a neighbourhood of the attaching region of D.)*

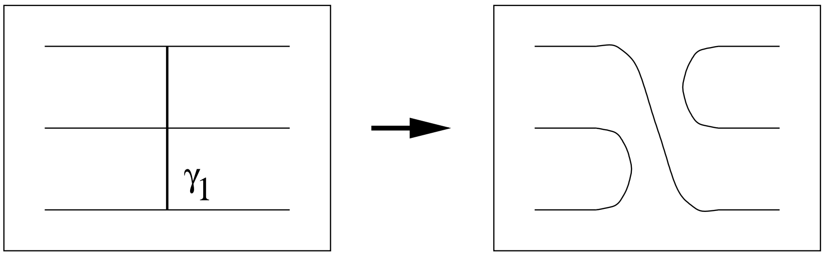


FIGURE 4.

If D_1, D_2 are two bypasses for Σ with $D_1 \cap \Sigma = D_2 \cap \Sigma$ which lie on different sides of Σ such that they fit together smoothly along their intersection then $D_1 \cup D_2$ is an overtwisted disc. In this way, one can think of a bypass as one half of an overtwisted disc. Thus it should be much easier to find bypasses in overtwisted contact manifolds than in tight contact manifolds.

In tight contact structures, the absence of overtwisted discs and the Bennequin inequality lead to obstructions for the existence of bypasses. In overtwisted contact manifolds bypasses are always available.

PROPOSITION 2.37. *Let Σ be a convex surface in a contact manifold, such that there is an overtwisted disc disjoint from Σ. Let $\gamma_1 \subset \Sigma$ be an arc with endpoints on Γ which intersects Γ transversely in three points. Then there is a bypass for Σ which intersects Σ in the Legendrian curve γ_1.*

PROOF. We can assume that γ_1 is already Legendrian. If this is not the case, an application of the Legendrian realization principle (Theorem 2.31) yields an admissible isotopy such that the image of γ_1 in the isotoped surface is Legendrian. The isotopy can be chosen in a small neighbourhood of the original surface Γ and it does not change the dividing set and Σ is still disjoint from D_{ot}.

Consider the image R of γ_1 under the flow φ_t of X for $0 \leq t \leq \varepsilon$. We choose $\varepsilon > 0$ so small that $\gamma_1 = R \cap \Sigma$. The singular foliation on R has the following properties.

(i) The curves $\varphi_t(\gamma_1), 0 \leq t \leq \varepsilon$ are Legendrian.
(ii) Along the segments $\varphi_t(p), 0 \leq t \leq \varepsilon$ of the flow line of $p \in \gamma_1 \cap \Gamma$, the contact structure is tangent to R.

Thus R has Legendrian boundary and it is convex since it admits a dividing set Γ_R. This dividing set is uniquely determined up to isotopy. For example we can choose Γ_R to be the union of the two segments $\varphi_t(q_i), i = 1, 2$ with $0 \leq t \leq \varepsilon$ for two points q_1, q_2 lying in different connected components of $\gamma_1 \setminus (\gamma_1 \cap \Gamma)$.

We orient R such that the tangencies on the boundary of R are positive. By (11), the Thurston–Bennequin invariant and the rotation number of ∂R are

$$\operatorname{tb}(\partial R) = -\frac{1}{2}\#(\Gamma_R \cap \partial R) = -2$$
$$\operatorname{rot}(\partial R) = \chi(R_+) - \chi(R_-) = 1$$

where R_+, R_- are the positive respectively negative parts of $R \setminus \Gamma_R$.

Let D_{ot} be a convex overtwisted disc in N which is disjoint from $R \cup \Sigma$. We orient D_{ot} such that

$$\operatorname{tb}(\partial D_{ot}) = 0$$
$$\operatorname{rot}(\partial D_{ot}) = -1 \ .$$

The idea is to perform a Legendrian connected sum of the knots ∂R and ∂D_{ot}. If one constructs a Seifert surface carefully enough, one obtains a bypass from the Seifert surfaces R and D_{ot}. Let us first explain the Legendrian connected sum of Legendrian knots in a contact manifold. A more general construction for Legendrian knots in two different contact manifolds can be found in [**EH**].

This construction is similar to the one in knot theory. The difference is that in usual knot theory there are two different possibilities to construct the connected sum. The two possibilities arise from the choice of orientations on the knots. For the connected sum of Legendrian knots, there are infinitely many possibilities with different Thurston–Bennequin invariants. One possibility for the Legendrian connected sum of two null–homologous Legendrian knots K_1, K_2 yields a Legendrian knot $K_1\#K_2$ characterized by

(12) $$\operatorname{tb}(K_1\#K_2) = \operatorname{tb}(K_1) + \operatorname{tb}(K_2) + 1$$

(13) $$\operatorname{rot}(K_1\#K_2) = \operatorname{rot}(K_1) + \operatorname{rot}(K_2) \ .$$

We will use only this type of Legendrian connected sums. Let us describe it in a model situation. Consider $\mathbb{R}^3$ with the contact form $dz - x\,dt$ and two Legendrian knots K_1, K_2. We assume that the front projection, i.e. the projection to the z, t–plane, of K_1, K_2 contains two cusp points $p_1 \in K_1$ and $p_2 \in K_2$ lying on the Legendrian curve $\{x = 0, z = 0\}$ as in Figure 5. The x–axis points inwards. We orient the knots as in Figure 5. The Legendrian

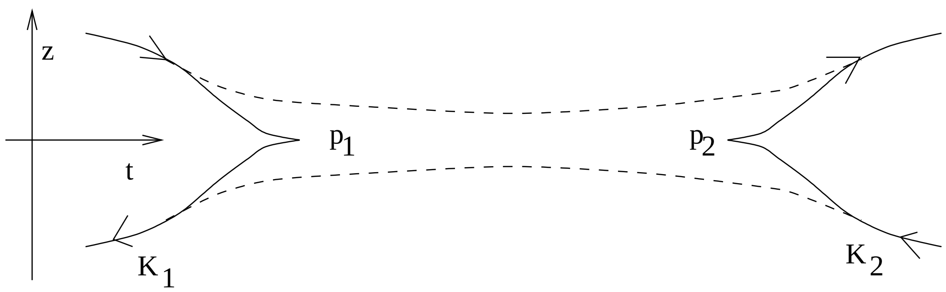


FIGURE 5.

connected sum is then formed using the dashed curves. The base projection, i.e. the projection to the x, t–plane, of this Legendrian connected sum is represented in Figure 6 where the z–axis points inwards. In $\mathbb{R}^3$ with the standard contact structure, the Thurston–Bennequin invariant of a Legendrian knot can be derived from the front projection. According to [**FT**], the Thurston–Bennequin invariant is

(14) $$\operatorname{tb}(K) = \#(\text{positive crossings}) - \#(\text{negative crossings}) - \frac{1}{2}(\text{cusps}) \ .$$

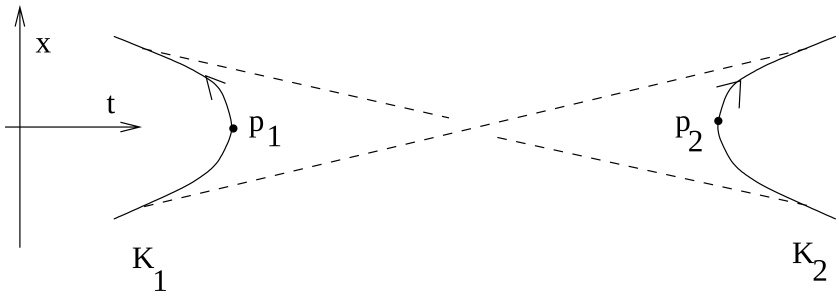


FIGURE 6.

For the definitions we refer to [**FT**]. Since in the Legendrian connected sum we remove two cusps without introducing crossings or cusps, we obtain (12). Equation (13) can be derived directly from Figure 6 or from the front projection using a statement similar to (14) from [**FT**].

Now let us consider Seifert surfaces Σ_1 of K_1 and Σ_2 of K_2. We assume that Σ_1 respectively Σ_2 coincides with translates of K_1 in the negative t–direction respectively of K_2 in the positive t–direction on a neighbourhood of p_1 respectively p_2. We assume that this is the case for the neighbourhood depicted in Figure 5 and Figure 6. If we orient Σ_1 and Σ_2 such that K_1 and K_2 are oriented as boundaries then p_1 is a negative tangency and p_2 is a positive tangency.

We use the ribbon which is bounded by the dashed curves in Figure 5 to form a Seifert surface $\Sigma_1\#\Sigma_2$ for the knot $K_1\#K_2$. There are no tangencies of the ribbon along the dashed curves. The Legendrian connected sum removes the tangencies p_1, p_2 which have different signs. Counting the number of sign changes of the tangencies along $K_1\#K_2$, we recover (12) even if the ambient contact manifold is not $\mathbb{R}^3$ with its standard contact structure.

Hence when we connect a negative tangency of Σ_1 on K_1 with a positive tangency of Σ_2 on K_2 by a Legendrian curve we can form the desired Legendrian connected sum of K_1 and K_2. We apply Corollary 2.19 showing that Legendrian curves have a standard tubular neighbourhood equivalent to the standard contact structure on R^3 we used above. The cusps can be constructed using the base projection in this situation and this can be done through Legendrian isotopies.

For the construction of bypasses we have to be more careful. Up to now all statements concerned only $K_1\#K_2$ but not the interior of the Seifert surface. Condition (v) in Definition 2.35 concerns the interior of the Seifert surface $\Sigma_1\#\Sigma_2$: We have to ensure that the dividing set on the boundary connected sum of $\Sigma_1 = R$ and $\Sigma_2 = D_{ot}$ does not contain any closed component.

The construction of the Legendrian connected sum is performed in a tubular neighbourhood of a Legendrian curve. When we connect the two Seifert surfaces by a ribbon to find a *convex* Seifert surface for $R\#D_{ot}$ we perturb the boundary connected sum $R\#D_{ot}$. We have to ensure that this perturbation can be carried out in a tight region of the contact manifold.

We use the Legendrian Realization principle Theorem 2.31 and the Giroux flexibility theorem Theorem 2.28 to bring the characteristic foliation on D_{ot} in the form indicated in Figure 7. This way we decompose the overtwisted disc into two discs bounded by Legendrian unknots with Thurston Bennequin–invariant -1 and rotation number 0. The two discs are separated by straight Legendrian arcs. The thickened circle in Figure 7 represents the dividing set. The singular foliation near the unknots is in the standard form used in Proposition 2.22. By the last statement in Proposition 2.22, we can now pretend that we

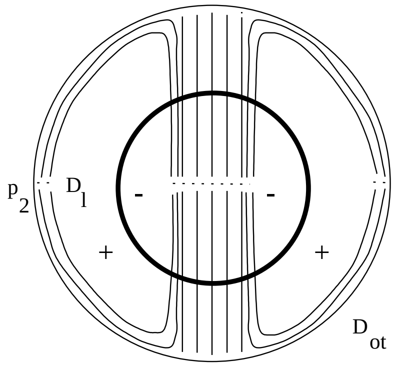


FIGURE 7.

do not form a connected sum of the surfaces R and D_{ot} but a connected sum of R with the left part of D_{ot}. The presence of the Legendrian curves in the middle of D_{ot} prevents an interaction between the left and the right part of D_{ot}.

The union of tubular neighbourhoods of R, the Legendrian arc connecting R with D_{ot} and the left part D_l of D_{ot} can be recovered in tight contact manifolds: D_l can be obtained applying Theorem 2.28 to a bypass. By (12) the Legendrian connected sum of $\partial R \# \partial D_l$ has the Thurston–Bennequin invariant

$$\text{tb}(\partial R \# \partial D_l) = \text{tb}(\partial R) + \text{tb}(\partial D_l) + 1 = -2 \tag{15}$$

This and the fact that $R\#D_l$ has a tight neighbourhood, implies that the dividing set on $R\#D_l$ (after this surface is perturbed to a convex surface) consists of exactly two arcs with endpoints on $\partial R\#\partial D_l$ and no closed components, cf. Theorem 2.34. Note that the notation $R\#D_l$ and $R\#D_{ot}$ is misleading because D_l respectively D_{ot} is *not* a subset of $R\#D_l$ respectively $R\#D_{ot}$ after these surfaces are smoothened and made convex.

If we consider $R\#D_{ot}$ there are only the two possibilities for the isotopy type of the dividing set which are shown in Figure 8. These two possibilities can be distinguished

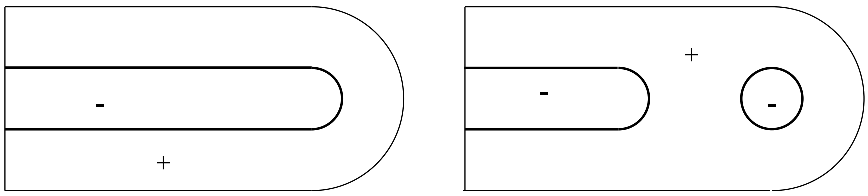


FIGURE 8.

using the rotation number. The boundary of the left part of Figure 8 has rotation number 1, while the right part has rotation number -2. By (13)

$$\text{rot}(\partial R \# \partial D_{ot}) = \text{rot}(R) + \text{rot}(D_{ot}) = 2 - 1 = 1 \ .$$

The remaining conditions (i), (ii) and (iv) in Definition 2.35 are satisfied by construction. The remaining condition (iii) can be achieved using Theorem 2.28. Thus $R\#D_{ot}$ is a bypass. $\square$

CHAPTER 3

First results on Engel structures

In this chapter we start our investigation of Engel structures. An Engel structure $\mathcal{D}$ is a smooth plane field on a 4–dimensional manifold M such that

$$\text{rank}\,[\mathcal{D},\mathcal{D}] = 3 \text{ and rank}\,[\mathcal{D},[\mathcal{D},\mathcal{D}]] = 4\ .$$

This property is sometimes called maximal non–integrability. The distribution $\mathcal{E} = [\mathcal{D},\mathcal{D}]$ is an even contact structure. Even contact structures are defined in a similar way as contact structures on even dimensional manifolds. To each even contact structure one can associate a one–dimensional foliation $\mathcal{W}$. Because of the importance of this foliation we start Chapter 3 starts with a discussion of even contact structures in Section 3.1.

The characteristic foliation $\mathcal{W}$ of an even contact structure $\mathcal{E}$ is tangent to $\mathcal{E}$. All flows which are tangent to $\mathcal{W}$ preserve the even contact structure. This should be compared with contact structures: No non–zero Legendrian vector field preserves the contact structure. If N is a hypersurface transversal to the characteristic foliation $\mathcal{W}$, then $\mathcal{E} \cap TN$ is a contact structure (Lemma 3.5). Using the normal form for contact structures discussed in Theorem 2.9 we proof the analogous theorem for even contact structures (Theorem 3.9).

In Section 3.2 we explain the definition of Engel structures and discuss some examples. Although the characteristic foliation of the even contact structure $\mathcal{E} = [\mathcal{D},\mathcal{D}]$ depends only on $\mathcal{E}$, it is tangent to $\mathcal{D}$. This important observation follows from the defining properties of the characteristic foliation (Lemma 3.11) and the fact that $\mathcal{E} = [\mathcal{D},\mathcal{D}]$. Like contact structures and even contact structures all Engel structures are locally diffeomorphic. The normal form for Engel structures (Theorem 3.13) was obtained first by F. Engel in [**Eng**].

A classical construction of Engel structures is called prolongation. Starting from a contact structure $\mathcal{C}$ one obtains an Engel structure on the space of Legendrian lines $\mathbb{P}\mathcal{C}$ of $\mathcal{C}$ (Proposition 3.15). The characteristic foliation of these Engel structures is given by the leaves of the circle bundle $\mathbb{P}\mathcal{C} \longrightarrow N$. Another construction of Engel structures is due to H. J. Geiges ([**Gei**]). From this method one obtains an Engel structure on the mapping torus of a diffeomorphism of a 3–manifolds if the the mapping torus has trivial tangent bundle (Proposition 3.17).

If one applies prolongation to the contact structure on a hypersurface N transversal to the characteristic foliation, then one obtains a canonical form for the Engel structure on a neighbourhood of N (Theorem 3.19). The germ of the Engel structure along $\mathcal{E}$ depends only on the contact structure $\mathcal{E} \cap TN$ and the intersection line field $\mathcal{D} \cap TN$. Later, we will be concerned with the homotopy class of the intersection line field as Legendrian line field. If $\mathcal{D}$ is oriented one can use rotation numbers to determine the homotopy class of the intersection line field as a Legendrian line field (Section 3.2.4). We can define rotation numbers even for Legendrian curves which are not null-homologous because the intersection line field and the orientation of the contact structure on a transversal boundary provide a global trivialization of the contact structure (cf. Definition 2.17 and Definition 3.23).

In Section 3.2.5 we define the development map. This map can be used to compare the Engel planes at different points of a leaf of the characteristic foliation. Intuitively the development map detects the rotation of $\mathcal{D}$ around the characteristic foliation in the associated even contact structure.

We fix some orientation conventions in Section 3.2.6. On an Engel manifold, the even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ carries a canonical orientation. An orientation of the characteristic foliation induces an orientation of the manifold and vice versa. If the characteristic foliation is oriented, this also induces an orientation of the contact structure on a closed transversal.

In Section 3.3 we discuss the topology of manifolds which admit an Engel structure. Using the presence of the distributions $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ and the relations between their orientations one can easily show that an orientable manifold which admits an orientable Engel structure has trivial tangent bundle (Theorem 3.37).

For Engel structures which are obtained by prolongation R. Montgomery has obtained a complete description of the corresponding deformation germ of these Engel structures in [**Mo2**]. It turns out that the space of possible deformations of prolonged Engel structures has infinite dimension. We explain his results in Theorem 3.41 and Theorem 3.43.

In Section 3.5 we discuss vector fields which preserve a given Engel structure. The results of this section should be compared with Section 2.1.2. We show that Engel vector fields are related to functions which satisfy a condition on their behaviour along the leaves of the characteristic foliation. It turns out that the dimension of the space of Engel vector fields depends on the characteristic foliation. An example where the space of Engel vector fields is 1–dimensional was found by R. Montgomery in [**Mo2**]. We discuss this example in Example 3.49 in a different way using our results about Engel vector fields.

The results about the deformations of prolonged Engel structures imply that Gray's stability theorem (Theorem 2.4) cannot be true for Engel structures without additional assumptions. If one assumes that the characteristic foliation remains constant for a family of Engel structures, then all of these Engel structures are isotopic. This was shown in [**Gol**]. In Section 3.6 we discuss stability theorems for contact structures, even contact structures and Engel structures in a unified setup.

3.1. Even contact structures

DEFINITION 3.1. Let M be a $2n$–dimensional manifold and $\mathcal{E}$ a distribution on M of codimension one. $\mathcal{E}$ is an *even contact structure* if for every local defining 1–form α, the 2–form $d\alpha$ has maximal rank on $\mathcal{E}$.

In other words, $\mathcal{E}$ is an even contact structure if for every local defining form α, the $(2n-1)$–form $\alpha \wedge d\alpha^{n-1}$ has no zeroes. In dimension 4 an equivalent formulation of this condition is $[\mathcal{E}, \mathcal{E}] = TM$. Here $[\mathcal{E}, \mathcal{E}]$ at p consists of all vector which can be obtained as commutators of local sections at p of $\mathcal{E}$.

Since $\mathcal{E}$ has dimension $2n-1$, the rank of $d\alpha\big|_{\mathcal{E}}$ is $2n-2$. Hence $d\alpha\big|_{\mathcal{E}}$ has a kernel $\mathcal{W} \subset \mathcal{E}$ of dimension one. Because

$$d(f\alpha)\big|_{\mathcal{E}} = f\left(d\alpha\big|_{\mathcal{E}}\right) \ ,$$

the line field $\mathcal{W}$ does not depend on the choice of a local defining form α for $\mathcal{E}$.

DEFINITION 3.2. The line field $\mathcal{W}$ is the *characteristic line field* of $\mathcal{E}$. The foliation induced by this line field is called the *characteristic foliation*.

COROLLARY 3.3. *A manifold which admits an even contact structure has vanishing Euler characteristic.*

Very simple examples of even contact structures can be obtained from contact manifolds $(N, \mathcal{C})$ as follows: Let $\pi : M = M \to N$ be a fibre bundle with one–dimensional fibre. Let

$$\mathcal{E} = \left\{ V \in TM \;\middle|\; \pi_*(V) \in \mathcal{C}(\pi(p)) \text{ for } V \in T_pM \right\} \ .$$

This distribution is an even contact structure on M. The tangent space ker(π_*) of the fibers is contained in $\mathcal{E}$ and spans the characteristic line field of $\mathcal{E}$.

Now suppose that W is a vector field tangent to $\mathcal{W}$ and let α be a local defining form of $\mathcal{E}$. By definition of $\mathcal{W}$ we have

$$(L_W\alpha)\big|_{\mathcal{E}} = (i_W d\alpha)\big|_{\mathcal{E}} = 0 \ .$$

Hence $L_W\alpha$ is a multiple of α. This implies that W preserves the even contact structure. Since we have chosen W arbitrary (but tangent to $\mathcal{W}$) we have

LEMMA 3.4. *The characteristic foliation of an even contact structure $\mathcal{E}$ preserves $\mathcal{E}$.*

Another important property of the characteristic line field is the next lemma.

LEMMA 3.5. *Let $\mathcal{E}$ be an even contact structure on M and $\mathcal{W}$ be the characteristic line field of $\mathcal{E}$. If N is a hypersurface transversal to $\mathcal{W}$ then $TN \cap \mathcal{E}$ is a contact structure on H.*

If N' is another transversal such that two interior points $p \in N$ and $q \in N'$ lie on the same leaf $\mathcal{W}_p$ of the characteristic foliation, then the map obtained by following nearby leaves, and thereby identifying a neighbourhood of p in N with a neighbourhood of q in N', preserves the induced contact structures.

PROOF. Let $p \in N$ and α a defining form for $\mathcal{E}$ on a neighbourhood of p. Then $\alpha\big|_N$ is a defining form for the distribution $TN \cap \mathcal{E}$ on N. By the transversality assumption on N, $d\alpha$ is non–degenerate on $TN \cap \mathcal{E}$. Hence $TN \cap \mathcal{E}$ is a contact structure.

The statement about the identification of contact structures follows immediately from Lemma 3.4. □

If n is even, a contact structure on a manifold of dimension $2n-1$ induces an orientation of this manifold. This has consequences for the relation between the orientability the characteristic line field of an even contact structure and the underlying manifold.

PROPOSITION 3.6. *Let $\mathcal{E}$ be an even contact structure on a $4n$–manifold M. Then an orientation of M induces an orientation of the characteristic line field $\mathcal{W}$ and vice versa.*

PROOF. For $p \in M$ choose a local transversal N to $\mathcal{W}$ containing p. By Lemma 3.5, $\mathcal{E}$ induces a contact structure on N. Since N has dimension $4n-1$, the contact structure induces an orientation of N. Hence T_pN has a distinguished orientation. Moreover, again since N is transversal to $\mathcal{W}$, we have $T_pN \oplus \mathcal{W}_p = T_pM$. Thus an orientation of $\mathcal{W}_p$ induces an orientation of T_pM and vice versa.

Since we can identify germs of transversals through p using $\mathcal{W}$, this relation between the orientation of $\mathcal{W}_p$ and T_pM is independent of the choice of the transversal through p by Lemma 3.5. □

Although the definition of even contact structures on even dimensional manifolds is very similar to the definition of contact structures on odd dimensional manifolds, these two structures are of very different nature. One indication for this is the existence of a distinguished line field contained in an even contact structure. More evidence is contained in the following theorem. For the definitions see [**ElM**].

THEOREM 3.7 (McDuff, [**McD**]). *The property of distributions of corank one to be an even contact structure is ample. All forms of the h–principle apply. In particular every even dimensional manifold with vanishing Euler characteristic admits an even contact structure.*

By Corollary 3.3, the condition on the Euler characteristic of the manifold is necessary. The analogous theorem for contact structures or Engel structures is wrong.

Finally we give an example of how even contact structures may arise on exact symplectic manifolds. We will use it in the construction of model Engel structures later.

EXAMPLE 3.8. Let (M,ω) be a symplectic manifold and W a Liouville vector field without zeroes. Hence $\alpha = i_W\omega$ is a nowhere vanishing 1–form and

$$L_W\omega = di_W\omega = \omega$$

by the definition of Liouville vector fields. Since $\mathcal{E} = \ker(\alpha)$ has corank one, $\mathcal{E}$ contains a symplectic subbundle of codimension one in $\mathcal{E}$. So $d\alpha$ has maximal rank on $\ker(\alpha)$ and α defines an even contact structure on M. Since W is a Liouville vector field, $\alpha = i_W d\alpha$ vanishes on $\ker(\alpha)$. So W spans the characteristic line field of $\ker(\alpha)$.

3.1.1. Local normal form for even contact structures. Just like contact structures, even contact structures are locally isomorphic. Still there is a slight difference between the proof of the Darboux theorem for even contact structures and the proof of Theorem 2.9 : Unlike in the case of contact structures, a given defining form does not have a standard expression in general. This is due to the fact that vector fields tangent to $\mathcal{W}$ preserve $\mathcal{E}$ but they do not necessarily preserve α.

A slightly different proof of the Darboux theorem for even contact structures can be found in [**BCG**].

THEOREM 3.9. *Let M be a $2n$-dimensional manifold carrying an even contact structure $\mathcal{E}$ and $p \in M$. Then there is a coordinate system $z, x_1, y_1, \dots, x_{n-1}, y_{n-1}, w$ on a neighbourhood of p such that*

$$dz - \sum_{i=1}^{n-1} x_i dy_i$$

defines $\mathcal{E}$ on this neighbourhood.

PROOF. Consider a foliated chart of the characteristic foliation $\mathcal{W}$ of $\mathcal{E}$ on a neighbourhood U of p

$$\psi : U \longrightarrow \mathbb{R}^{2n-1} \times \mathbb{R}$$

such that $\psi(p) = (0,0)$. Let w denote the coordinate of the second factor in $\mathbb{R}^{2n-1} \times \mathbb{R}$. Then $\psi_*(\mathcal{W}) = \mathrm{span}(\partial_w)$. Let N be the hypersurface corresponding to $\mathbb{R}^{2n-1} \times \{0\}$. It is transversal to the distinguished line field of $\mathcal{E}$. As was shown in Lemma 3.5, the distribution $TN \cap \mathcal{E}$ on N is a contact structure.

By Theorem 2.9, there are coordinates $z, x_1, y_1, \dots, x_{n-1}, y_{n-1}$ on a neighbourhood $V \subset N$ of p in the hypersurface N such that the contact structure $TN \cap \mathcal{E}$ on V is defined by the form

$$\alpha = dz - \sum_{i=1}^{n-1} x_i dy_i \ . \tag{16}$$

Consider the product coordinate system $z, x_1, y_1, \dots, x_{n-1}, y_{n-1}, w$ on a product neighbourhood diffeomorphic to $V \times \mathbb{R}$ of p and let $\mathrm{pr} : V \times \mathbb{R} \to V$ be the projection on the first factor. $\mathcal{E}$ is invariant under the flow of ∂_t by Lemma 3.4. So

$$\mathrm{pr}^*\alpha = dz - \sum_{i=1}^{n-1} x_i dy_i$$

is a defining form for $\mathcal{E}$ on a neighbourhood of p. □

3.2. Engel structures – Definition and first examples

Contact structures are hyperplane fields on manifolds of odd dimension. They usually defined as the kernel of a 1–form without zeros. Therefore contact structures are usually defined using defining forms. It is of course possible to define contact structure using only the distribution it self.

DEFINITION 3.10. An Engel structure is a distribution $\mathcal{D}$ of rank two on a manifold M of dimension four with the following properties.

(i) $\mathcal{E} = [\mathcal{D}, \mathcal{D}] \subset TM$ is a subbundle of rank three.
(ii) $TM = [\mathcal{E}, \mathcal{E}]$.

By $[\mathcal{D}, \mathcal{D}]$ we mean all tangent vectors which are commutators of local sections of $\mathcal{D}$. Obviously $\mathcal{D} \subset [\mathcal{D}, \mathcal{D}]$. In general this is a sheaf of modules over the smooth functions even if $\mathcal{D}$ is a subbundle. Our assumptions assure that $[\mathcal{D}, \mathcal{D}]$ respectively $[\mathcal{E}, \mathcal{E}]$ are really subbundles of TM.

The second condition in the definition of Engel structures implies that $\mathcal{E}$ is an even contact structure. To $\mathcal{E}$ corresponds a line field $\mathcal{W} \subset \mathcal{E}$. The following simple observation will turn out to be very important.

LEMMA 3.11. *If $\mathcal{E}$ is induced by an Engel structure then $\mathcal{W} \subset \mathcal{D}$.*

PROOF. Suppose that $\mathcal{W}_p \not\subset \mathcal{D}_p$. Then choose a local frame X, Y of $\mathcal{D}$ around p and fix a local defining form α for $\mathcal{E}$. Since $d\alpha$ has maximal rank in $\mathcal{E}$

$$d\alpha(X, Y) \neq 0\ .$$

On the other hand we have $[X, Y](p) \in \mathcal{E}_p$ by the definition of $\mathcal{E}$ as $[\mathcal{D}, \mathcal{D}]$. So

$$0 \neq d\alpha(X, Y) = L_X(\alpha(Y)) - L_Y(\alpha(X)) - \alpha([X, Y]) = -\alpha([X, Y])\ .$$

This would imply $[X, Y](p) \notin \mathcal{E}_p$. This is a contradiction to $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$. So $\mathcal{W} \subset \mathcal{E}$. □

DEFINITION 3.12. The foliation induced by $\mathcal{W}$ will be called the *characteristic foliation* of $\mathcal{D}$. A hypersurface in an Engel manifold is *transversal* if it is transversal to $\mathcal{W}$.

By Lemma 3.5 the even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ associated to an Engel structure $\mathcal{D}$ induces a contact structure on a transversal hypersurface.

A distribution of codimension two can by defined locally as the intersection of the kernels of two linearly independent 1–forms. Let α_1, α_2 be 1–forms defining $\mathcal{D}$ locally. The conditions for $\ker\alpha_1 \cap \ker\alpha_2$ to be an Engel structure $\mathcal{D}$ – such that α_1 is a local defining form for $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ – are equivalent to

$$\begin{aligned} \alpha_1 \wedge \alpha_2 \wedge d\alpha_1 = 0 &\Longleftrightarrow [\mathcal{D}, \mathcal{D}] \subset \mathcal{E} \\ \alpha_1 \wedge \alpha_2 \wedge d\alpha_2 \neq 0 &\Longleftrightarrow \mathrm{rank}[\mathcal{D}, \mathcal{D}] = 3 \\ \alpha_1 \wedge d\alpha_1 \neq 0 &\Longleftrightarrow [\mathcal{E}, \mathcal{E}] = TM\ . \end{aligned}$$

Let $\mathcal{D}$ be an Engel structure on M. The result of a perturbation of $\mathcal{D}$ is again an Engel structure if the perturbation is small enough (with respect to the C^2-topology). As we will see, the result of this perturbation is not equivalent to $\mathcal{D}$ in general. Nevertheless, by Theorem 3.13 the germs at $p \in M$ of both Engel structures are equivalent.

3.2.1. Local normal form for Engel structures. Locally, Engel structures have a standard form. According to E. Cartan ([**Car1**]), this normal form was found by Engel for the study of the Monge equation in [**Eng**].

THEOREM 3.13. *Let $\mathcal{D}$ be an Engel structure on M. Every point $p \in M$ has a neighbourhood U with coordinates w, x, y, z such that $\mathcal{D}\big|_U$ is the intersection of the kernels of the* 1*-forms*

$$\alpha_1 = dz - x dy \qquad\qquad \alpha_2 = dx - w dy \; .$$

The even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is defined by α_1.

PROOF. By Theorem 3.9 we can choose local coordinates x, y, z, t on a neighbourhood $U \simeq \mathbb{R}^4$ of p such that the even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ associated to the Engel structure is defined by the form $\alpha_1 = dz - x dy$. The characteristic line field of $\mathcal{E}$ is spanned by ∂_t.

The distribution $\mathcal{D} \cap T(\mathbb{R}^3 \times \{t\})$ is a line field contained in the contact structure $\mathcal{E} \cap T(\mathbb{R}^3 \times \{t\})$ on $\mathbb{R}^3 \times \{t\}$. Hence there are smooth functions a, b defined on U such that $\mathcal{D} \cap T(\mathbb{R}^3 \times \{t\})$ is spanned by

$$X = a\frac{\partial}{\partial x} + b\left(x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) \; .$$

By definition, a and b do not vanish simultaneously. Assume that $b(p) \neq 0$. The Engel structure is spanned by X and ∂_t. Because $\mathcal{D}$ is an Engel structure, the vector field

$$\left[\frac{\partial}{\partial t}, \frac{1}{b}X\right] = \left[\frac{\partial}{\partial t}, \frac{a}{b}\frac{\partial}{\partial x} + \left(x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right)\right]$$

is not contained in $\mathcal{D}$. Therefore

$$\frac{\partial}{\partial t}\left(\frac{a}{b}\right)(p) \neq 0$$

The transformation

$$(x, y, z, t) \longmapsto \left(x, y, z, w = \frac{a}{b}\right)$$

has the Jacobian

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & \frac{\partial}{\partial t}\left(\frac{a}{b}\right) \end{pmatrix} .$$

At p this matrix is invertible. Hence $x, y, z, w = \frac{a}{b}$ is a coordinate system on a neighbourhood of p. In particular the characteristic line field of the associated even contact structures is spanned by ∂_w. This is a non zero multiple of ∂_t. The Engel structure $\mathcal{D}$ is spanned by the vector fields

$$\frac{\partial}{\partial w} \qquad \text{and} \qquad w\frac{\partial}{\partial x} + \left(x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) \; .$$

Thus $\mathcal{D}\big|_U$ is the intersection of the kernels of the one-forms

$$\alpha_1 = dz - x dy \qquad\qquad \alpha_2 = dx - w dy \; .$$

Up to now, we have treated the case $b(p) \neq 0$. In the case $b(p) = 0$ and $a(p) \neq 0$ we would have found the pair

$$\tilde{\alpha}_1 = dz - x dy \qquad\qquad \tilde{\alpha}_2 = dy - w dx$$

of defining forms of $\mathcal{D}$. These forms are equivalent to the one given in the theorem by the coordinate transformation

$$(x, y, z, w) \longmapsto (-y, x, z - yx, -w) \; .$$

□

3.2.2. Examples of Engel structures. Apart from the constructions we present in later chapters, there are two other known construction methods for Engel structures. The first one – called prolongation – is based on contact structures on 3–manifolds. The second construction yields Engel structures on certain mapping tori induced by diffeomorphisms $\psi : N \to N$ of 3–manifolds.

Starting from a contact structure $\mathcal{C}$ on a 3–manifold N one can construct an Engel structure. We consider the equivalence relation

$$v \sim w \text{ for } v, w \in \mathcal{C} \setminus \{0\} \Leftrightarrow v = \lambda w \text{ for some } \lambda \in \mathbb{R}$$

on $\mathcal{C} \setminus \{0\}$. Then the space $\mathbb{P}\mathcal{C} = \mathcal{C} \setminus \{0\}/\sim$ of Legendrian lines is a closed 4–dimensional manifold. By construction, there is a fibration $\mathrm{pr} : \mathbb{P}\mathcal{C} \to N$ sending each Legendrian line to the corresponding base point in N. The fiber is $\mathbb{RP}^1$.

Let $\varepsilon : \mathcal{C} \setminus N \to \mathbb{P}\mathcal{C}$. One can define a distribution of rank two on $\mathbb{P}\mathcal{C}$ by

$$\mathcal{D}_{\mathcal{C}} = \left\{ v \in T_{\varepsilon(l)}\mathbb{P}\mathcal{C} \;\middle|\; \mathrm{pr}_*(v) \in \varepsilon(l) \right\} \ .$$

DEFINITION 3.14. This construction of a distribution on $\mathbb{P}\mathcal{C}$ is called *prolongation*.

Prolongation really yields Engel structures.

PROPOSITION 3.15. *$\mathcal{D}_{\mathcal{C}}$ is an Engel structure on $\mathbb{P}\mathcal{C}$.*

PROOF. Let $p \in N$. The fibers of $\mathbb{P}\mathcal{C}$ are clearly tangent to $\mathcal{D}_{\mathcal{C}}$. Thus $\mathcal{D}_{\mathcal{C}}$ is a subbundle of rank two of $T\mathbb{P}\mathcal{C}$. For $\varepsilon(v) \in \mathbb{P}\mathcal{C}$ choose a local trivialization W, X of $\mathcal{D}_{\mathcal{C}}$ such that W is tangent to the fibers. Let φ_t be the local flow of W. Then by definition

$$\mathrm{pr}_*\big(X\big(\varphi_t(\varepsilon(v))\big)\big) \in \mathcal{C}(\mathrm{pr}(\varepsilon(v)))$$

is a curve transversal to the line $\varepsilon(v)$ in $\mathcal{C}$. Hence

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{pr}_*(X(\varphi_t(\varepsilon(v)))) = \mathrm{pr}_*([W, X]) \notin \varepsilon(v) \ ,$$

so $[W, X](p)$ is not contained in $\mathcal{D}_{\mathcal{C}}$. Thus $[\mathcal{D}_{\mathcal{C}}, \mathcal{D}_{\mathcal{C}}] = \mathrm{pr}^*\mathcal{C}$. This shows that the leaves of the characteristic foliation of $\mathcal{D}_{\mathcal{C}}$ are the fibers of $\mathrm{pr} : \mathbb{P}\mathcal{C} \longrightarrow N$.

We have shown that $\mathrm{pr}_*(X)$ and $\mathrm{pr}_*([W, X])$ span $\mathcal{C}$. Now we restrict pr to a hypersurface through p which is tangent to X. This suffices for the calculation of $[X, [W, X]]$. When we restrict pr to this hypersurface we obtain a local diffeomorphism. Then

$$\mathrm{pr}_*([X, [W, X]]) = [\mathrm{pr}_*(X), \mathrm{pr}_*([W, X])] \notin \mathcal{C}$$

by the definition of contact structures. This shows that $[\mathcal{D}, [\mathcal{D}, \mathcal{D}]]$ has full rank. □

The Engel structures obtained this way are not orientable since the restriction of $\mathcal{D}_{\mathcal{C}}$ to a fiber of $\mathbb{P}\mathcal{C}$ is the the Whitney sum of $T\mathbb{RP}^1$ and the tautological bundle over $\mathbb{RP}^1$. While the first bundle is trivial, the tautological bundle is not orientable. One obtains *orientable* Engel structures when one does the same construction using *oriented* Legendrian lines.

Engel structures constructed by prolongation provide local models for the Engel structure on tubular neighbourhoods of transversal hypersurfaces (cf. Theorem 3.19) and one can obtain automorphisms of these Engel structures from diffeomorphisms a contact structure.

Let N_1 and N_2 be 3–manifolds with contact structures $\mathcal{C}_1, \mathcal{C}_2$ and let $\varphi : N_1 \to N_2$ be a contact diffeomorphism. From φ one can construct a diffeomorphism $\widetilde{\varphi} : \mathbb{P}\mathcal{C}_1 \to \mathbb{P}\mathcal{C}_2$ which preserves the induced Engel structures $\mathcal{D}_1, \mathcal{D}_2$. For $i = 1, 2$ we denote the maps $\mathcal{C}_i \setminus N_i \to \mathbb{P}\mathcal{C}_i$ by κ_i. The following proposition can be found in [**Mo2**], according to this paper it was known before.

PROPOSITION 3.16. *The diffeomorphism*

$$\begin{aligned} \widetilde{\varphi} : \mathbb{P}\mathcal{C}_1 &\longrightarrow \mathbb{P}\mathcal{C}_2 \\ \kappa_1(v) &\longmapsto \kappa_2(\varphi_*(v)) \end{aligned}$$

maps $\mathcal{D}_1$ to $\mathcal{D}_2$. Every diffeomorphism $\mathbb{P}\mathcal{C}_1 \to \mathbb{P}\mathcal{C}_2$ preserving Engel structures is of this form.

PROOF. Consider the map

$$\begin{aligned} \widetilde{\psi} : \mathbb{P}\mathcal{C}_2 &\longrightarrow \mathbb{P}\mathcal{C}_1 \\ \kappa_2(w) &\longmapsto \kappa_1(\varphi_*^{-1} w) \ . \end{aligned}$$

The composition $\widetilde{\psi} \circ \widetilde{\varphi}$ is the identity of $\mathbb{P}\mathcal{C}_1$ since

$$\widetilde{\psi} \circ \widetilde{\varphi}(\kappa_1(v)) = \kappa_1(\varphi_*^{-1}(\varphi_*(v))) = \kappa_1(v)$$

and similarly for $\widetilde{\varphi} \circ \widetilde{\psi}$. Thus $\widetilde{\varphi}$ is a diffeomorphism. Now let Y with base point $\kappa_1(v)$ be tangent to the Engel structure $\mathcal{D}_1$ on $\mathbb{P}\mathcal{C}_1$. The base point of $\widetilde{\varphi}_*(Y)$ is $\kappa_2(\varphi_*(v))$. On the other hand

$$\text{pr}_{2*}\left(\widetilde{\varphi}_*(Y)\right) = \varphi_*(\text{pr}_{1*}(Y)) \tag{17}$$

is contained in $\varphi_*(\kappa_1(v)) = \kappa_2(\varphi_*(v))$ and this is the basepoint of $\widetilde{\varphi}(Y)$. Thus $\widetilde{\varphi}$ preserves Engel structures.

Now let $\Phi : \mathbb{P}\mathcal{C}_1 \to \mathbb{P}\mathcal{C}_2$ be a diffeomorphism preserving Engel structures. Then Φ preserves the characteristic foliations or – equivalently – Φ takes fibers of $\mathbb{P}\mathcal{C}_1$ to fibers of $\mathbb{P}\mathcal{C}_2$, thus the map

$$\begin{aligned} \varphi : N_1 &\longrightarrow N_2 \\ p &\longmapsto \text{pr}_2\left(\Phi\left(\text{pr}_1^{-1}(p)\right)\right) \end{aligned}$$

is well defined. The inverse of φ can be constructed in the same manner so φ is a diffeomorphism. The diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{C}_1 & \xrightarrow{\ \Phi\ } & \mathbb{P}\mathcal{C}_2 \\ \downarrow{\scriptstyle \text{pr}_1} & & \downarrow{\scriptstyle \text{pr}_2} \\ N_1 & \xrightarrow{\ \varphi\ } & N_2 \end{array}$$

commutes. As Φ preserves Engel structures, Φ also preserves the induced even contact structures. The even contact structure $\mathcal{E}_i$ on $\mathbb{P}\mathcal{C}_i$ satisfies $\text{pr}_{i*}\mathcal{E}_i = \mathcal{C}_i$ for $i = 1, 2$. Hence

$$\varphi_*(\mathcal{C}_1) = \varphi_*(\text{pr}_{1*}\mathcal{E}_1) = \text{pr}_{2*}(\Phi_*(\mathcal{E}_1)) = \mathcal{C}_2$$

so φ is a contact diffeomorphism. Let $\widetilde{\varphi} : \mathbb{P}\mathcal{C}_1 \to \mathbb{P}\mathcal{C}_2$ be the induced Engel diffeomorphism. We want to show that $\widetilde{\varphi}^{-1} \circ \Phi$ is the identity map of $\mathbb{P}\mathcal{C}_2$. It is clear that $\widetilde{\varphi}^{-1} \circ \Phi$ preserves each fiber. We want to show that each fiber is preserved pointwise. Let $v \in \mathcal{D}_1(\kappa_1(l))$ be such that $\text{pr}_{1*}(v) \neq 0$. Recall

$$\mathcal{D}_1(\varepsilon_1(l)) = \left\{ w \in T_{\varepsilon_1(l)}\mathbb{P}\mathcal{C}_1 \;\middle|\; \text{pr}_{1*}w \in \varepsilon_1(l) \right\} \ .$$

Now $\widetilde{\varphi}^{-1} \circ \Phi$ preserves $\mathcal{D}_1$. Suppose that $\widetilde{\varphi}^{-1} \circ \Phi(\varepsilon_1(l)) = \kappa_1(l')$. By (17)

$$pr_{1*}(\widetilde{\varphi}_*^{-1}(\Phi_*(v))) = \varphi_*^{-1}(\text{pr}_{2*}(\Phi_*(v))) = \varphi_*^{-1}(\varphi_*(\text{pr}_{1*}(v))) \ .$$

While on the left we have an element of $\kappa_1(l')$, the expression on the right is an element of $\kappa_1(l)$. Thus $\widetilde{\varphi}^{-1} \circ \Phi$ preserves the fibers of $\mathbb{P}\mathcal{C}_1$ pointwise. □

Another construction is due to H.–J. Geiges, [**Gei**]. It shows that parallelizable mapping tori of compact 3–manifolds admit Engel structures without using contact structures. Suppose that $\psi : N \longrightarrow N$ is a diffeomorphism of a compact 3–manifold. Let

$$M = (N \times [0,1])/(x,1) \sim (\psi(x),0) \ .$$

be the mapping torus of ψ. The projection of $N \times [0,1]$ onto the second factor induces a fibration $M \longrightarrow S^1 = [0,1]/0 \sim 1$. We write t for the coordinate on $[0,1]$. The vector field ∂_t on $N \times [0,1]$ induces a vector field X_0 on M.

Now we assume that M is parallelizable. In order to construct a framing of TM such that X_0 is a component, we fix an arbitrary almost quaternionic structure $TM \simeq M \times \mathbb{H}$. Then we obtain a framing

$$X_0, X_1 = iX_0, X_2 = jX_0, X_3 = kX_0 \ .$$

PROPOSITION 3.17 (Geiges, [**Gei**]). *If $n \in \mathbb{N}$ is large enough, the distribution $\mathcal{D}_n$ spanned by X_0 and*

$$Y_n = \frac{1}{n}\left(\cos\left(n^2 t\right) X_1 + \sin\left(n^2 t\right) X_2\right) + X_3$$

is an Engel structure.

PROOF. In order to verify that $\mathcal{D}_n$ is an Engel structure for large n, we calculate the commutators

$$\begin{aligned}
[X_0, Y_n] &= n\left(-\sin(n^2t)X_1 + \cos(n^2t)X_2\right) \\
&\quad + \frac{1}{n}\left(\cos(n^2t)[X_0,X_1] + \sin(n^2t)[X_0,X_2]\right) + [X_0,X_3] \\
[X_0,[X_0,Y_n]] &= n^3\left(-\cos(n^2t)X_1 - \sin(n^2t)X_2\right) + [X_0,[X_0,X_3]] \\
&\quad + 2n\left(-\sin(n^2t)[X_0,X_1] + \cos(n^2t)[X_0,X_2]\right) \\
&\quad + \frac{1}{n}\left(\cos(n^2t)[X_0,[X_0,X_1]] + \sin(n^2t)[X_0,[X_0,X_1]]\right)
\end{aligned}$$

Notice that as n grows to infinity

$$\begin{aligned}
Y_n &\longrightarrow X_3 \\
\frac{1}{n}[X_0,Y_n] &\sim -\sin(n^2t)X_1 + \cos(n^2t)X_2 \\
\frac{1}{n^3}[X_0,[X_0,Y_n]] &\sim -\cos(n^2t)X_1 - \sin(n^2t)X_2 \ .
\end{aligned}$$

Since M is compact, we can choose n so big that

$$X_0, Y_n, [X_0,Y_n], [X_0,[X_0,Y_n]]$$

is a framing of TM. □

Unlike in the case of prolongation it is not possible to determine explicitly the characteristic foliation of Engel structures obtained this way. This is a major disadvantage of this construction.

REMARK 3.18. A mapping torus has vanishing Euler characteristic since there is a vector field without zeroes. One can show that the signature of a four dimensional orientable mapping torus is always zero. However the following example shows that orientable mapping tori do not necessarily admit spin structures.

Let $E \rightarrow T^2$ be a complex line bundle over T^2 with odd first Chern class and let $\underline{\mathbb{C}}$ be the trivial complex line bundle. Consider the $\mathbb{CP}^1$–bundle $M = \mathbb{P}\left(E \oplus \underline{\mathbb{C}}\right)$ obtained from E by fiberwise one–point compactification. Then the normal bundle of the image of the zero section σ of E in M is the pull back of E under σ. Along σ the tangent bundle of M decomposes as a direct sum $TM\big|_\sigma = T\sigma \oplus \sigma^*E$. Hence $TM\big|_\sigma$ has odd first Chern class and therefore TM does not admit a spin structure.

This shows that the condition on orientable mapping tori to be parallelizable is not redundant in dimension 4 and higher.

3.2.3. Tubular neighbourhoods of transversal hypersurfaces. Let M be a manifold with Engel structure $\mathcal{D}$. Suppose that N is a (potentially open) hypersurface which is transversal to the characteristic foliation $\mathcal{W}$ of $\mathcal{D}$. We have seen above that $\mathcal{D}$ induces

- a contact structure $\mathcal{C} = \mathcal{E} \cap TN$ on N and
- a Legendrian line field $\mathcal{L} = \mathcal{D} \cap TN \subset \mathcal{C}$.

If one applies the prolongation construction to $\mathcal{C}$, one obtains the manifold $\mathbb{P}\mathcal{C}$ with its canonical Engel structure. Let

$$\varepsilon : \mathcal{C} \setminus N \longrightarrow \mathbb{P}\mathcal{C}$$

be the projection. We want to compare the Engel structures on a tubular neighbourhood of N in M with the Engel structure $\mathcal{D}_\mathcal{C}$ on $\mathbb{P}\mathcal{C}$ on a neighbourhood of the section

$$\begin{aligned} \sigma : N &\longrightarrow \mathbb{P}\mathcal{C} \\ p &\longmapsto \varepsilon(\mathcal{L}(p)) \ . \end{aligned}$$

The following theorem can be found in [**Mo2**] but according to this article it was known before.

THEOREM 3.19. *Any sufficiently small tubular neighbourhood of N in M is canonically diffeomorphic as an Engel manifold to a tubular neighbourhood of σ.*

PROOF. On N we set $\psi = \sigma$. Since N is transversal to $\mathcal{W}$ we can choose a tubular neighbourhood U of N such that the fibers of U correspond to leaves of the characteristic foliation. Let $\pi : U \to N$ be the bundle projection and $\kappa : \mathcal{C} \setminus N \longrightarrow \mathbb{P}\mathcal{C}$. The leaves of $\mathcal{W}$ are tangent to $\mathcal{D}$. Hence $\pi_*(\mathcal{D}(p))$ is a Legendrian line at the point $\pi(p) \in N$ for $p \in U$. We define

$$\begin{aligned} \psi : U &\longrightarrow \mathbb{P}\mathcal{C} \\ p &\longmapsto \varepsilon(\pi_*(\mathcal{D}(p))) \ . \end{aligned}$$

On N this coincides with our previous definition. Let us first show that ψ is a diffeomorphism onto its image if U is small enough. When restricted to TN, the differential of ψ is injective. By the inverse function theorem it suffices to show that ψ_* maps non–zero vectors which are tangent to the characteristic foliation to non–zero vectors transversal to σ.

Fix a local trivialization W, X of $\mathcal{D}$ around $p \in N$ such that W is tangent to $\mathcal{W}$. Let φ_t be the local flow of W. Then

$$\begin{aligned} \psi_*(W(p)) &= \kappa_* \left(\pi_* \left(\frac{d}{dt}\bigg|_{t=0} X(\varphi_t(p)) \right) \right) \\ &= \kappa_* \left(\pi_*([W, X](p)) \right) \neq 0 \end{aligned}$$

by the definition of Engel structures. (Here the differential κ_* is the differential of κ at $X(p)$.) On the other hand the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \psi\ } & \mathbb{P}\mathcal{C} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathrm{pr}} \\ N & \xrightarrow{\ \mathrm{id}\ } & N \end{array}$$

is commutative. Thus $\mathrm{pr}_*(\psi_*(W)) = 0$. Therefore $\psi_*(W) \neq 0$ is tangent to the fibers of $\mathbb{P}\mathcal{C}$.

In order to show that ψ preserves Engel structures it suffices to prove that $\psi_*(X)$ is tangent to $\mathcal{D}_\mathcal{C}$ since we have already dealt with W. By definition $\psi_*(X) = \varepsilon_*(\pi_*(X(p))$. The Engel structure on $\mathbb{P}\mathcal{C}$ is by definition

$$\mathcal{D}_\mathcal{C}(\kappa(l)) = \left\{ v \in T_{\kappa(l)}\mathbb{P}\mathcal{C} \mid \mathrm{pr}_*(v) \in \kappa(l) \right\}$$

where $\kappa(l)$ is a Legendrian line and $\mathrm{pr} : \mathbb{P}\mathcal{C} \to N$ is the bundle projection.

For $v \in \mathcal{C}_p \setminus \{0\}$ we identify $T_v\mathcal{C}$ with $\mathcal{C}_p \oplus T_pN$. With this identification, the differential of the composed map

$$\mathcal{C} \setminus N \xrightarrow{\ \varepsilon\ } \mathbb{P}\mathcal{C} \xrightarrow{\ \mathrm{pr}\ } N$$

at $v \in \mathcal{C}_p \setminus H$ is just the projection $\mathcal{C}_p \oplus T_pN \longrightarrow T_pN$. Thus

$$\mathrm{pr}_*(\psi_*(X)) = \pi_*(X) \ .$$

This vector is contained in the line $\kappa(\pi_*(X))$. Thus $\psi_*(X)$ is tangent to $\mathcal{D}_\mathcal{C}$. $\square$

3.2.4. Line fields on transversals – Rotation number. Let M be an oriented manifold with an oriented Engel structure $\mathcal{D}$ and let N be a hypersurface transverse to the characteristic foliation $\mathcal{W}$ of $\mathcal{D}$. We fix the canonical orientation of the characteristic foliation. As we have seen, the distribution $TN \cap \mathcal{E}$ is a contact structure on N.

Since $\mathcal{W}$ is contained in $\mathcal{D}$, the intersection $TN \cap \mathcal{D} \subset TN \cap \mathcal{E}$ is a Legendrian line field on N. We orient this line field by the requirement that the orientation of $\mathcal{W}$ followed by the orientation of $TN \cap \mathcal{D}$ is the orientation of $\mathcal{D}$.

DEFINITION 3.20. The oriented Legendrian line field $TN \cap \mathcal{D}$ will be called the *intersection line field* of $\mathcal{D}$ on N.

Of course the intersection line field induces a foliation of rank 1 but this foliation will not play an important role. We will only need the homotopy type of the intersection line field as a Legendrian line field.

First we reduce the problem of distinguishing two Legendrian line fields up to homotopy to the classification of maps $N \longrightarrow S^1$ up to homotopy. For the second step we apply Thom–Pontryagin theory to identify this set with $H^1(N;\mathbb{Z})$.

Let X be a nowhere vanishing Legendrian vector field on the contact manifold $(N, \mathcal{C})$. Choose a section Y of $\mathcal{C}$ such that X, Y is an oriented framing of $\mathcal{C}$. For a Legendrian vector field V there are uniquely determined smooth function f, g such that

$$V = fX + gY \ . \tag{18}$$

We assume that V has no zeroes. Then f and g do not vanish simultaneously. Hence the function

$$\begin{aligned} G(V,X,Y) : N &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ p &\longmapsto (f(p), g(p)) \end{aligned}$$

is well defined. If we start with $X' = hX, Y' = Y$ instead of X, Y with a positive function h, the corresponding map $G(V, X', Y)$ is

$$G(V, X', Y') = \left(\frac{f}{h}, g \right) \ .$$

If we multiply X with a negative function h then we take $Y' = -Y$ instead of Y in order to satisfy the orientation assumption. Then

$$G(V, X', Y') = \left(\frac{f}{h}, -g \right) \ .$$

In both cases the resulting map $G(V,X',Y')$ is homotopic to $G(V,X,Y)$ through maps whose image does not contain 0. For fixed X, the second component Y of the oriented framing is well defined up to multiplication with a positive function and addition of an arbitrary multiple of X. If $Y'=hY+kX$ with $h>0$ then

$$G(V,X,Y')=\left(f-\frac{gk}{h},\frac{g}{h}\right)\ .$$

This is again homotopic to $F(V,X,Y)$. If we start with $V'=hV$ instead of V for a nowhere vanishing function h we have

$$G(V',X,Y)=\frac{G(V,X,Y)}{h}$$

and this is homotopic to $G(V,X,Y)$. Thus the homotopy class of

$$G(V,X,Y):N\longrightarrow\mathbb{R}^2\setminus\{0\}$$

depends only on the Legendrian line fields spanned by V and X and the orientation of $\mathcal{C}$. Hence the homotopy class of

$$F(V,X):N\longrightarrow S^1$$
$$F(V,X)=\frac{G(V,X,Y)}{\|G(V,X,Y)\|}$$

is well defined. In particular the line field spanned by V is homotopic to the line field spanned by X if and only if $F(V,X)$ is homotopic to the constant map.

We denote the set of homotopy classes of maps $N\to S^1$ by $[N;S^1]$. The map

$$\begin{aligned}[N;S^1]&\longrightarrow H^1(N;\mathbb{Z})\\ [F]&\longmapsto\big((\gamma:S^1\to H)\longmapsto\deg(F\circ\gamma)\big)\ .\end{aligned}$$

is bijective. One way to see this is an application of the Thom–Pontryagin construction. A detailed description of this method together with the following theorem can be found in [**Bre**].

THEOREM 3.21 (Thom, Pontryagin). *If N^{n+k} is a compact smooth manifold of dimension $n+k$, then the Thom–Pontryagin construction gives a one–to–one correspondence between the set $\left[N^{n+k};S^n\right]$ and the set of smooth framed cobordism classes of smooth, compact, normally framed k–submanifolds of N^{n+k}.*

In our situation $n=1$ and $k=2$. The k–submanifolds in the theorem are preimages of a regular value of a smooth map $F:N\longrightarrow S^1$ representing a given homotopy class $[F]\in[N;S^1]$. The framed submanifolds are cooriented hypersurfaces in N. These give rise to cohomology classes in $H^1(N;\mathbb{Z})$ as we have explained above. Summarizing we have the following proposition.

PROPOSITION 3.22. *Two orientable Legendrian line fields $\mathcal{F}_1,\mathcal{F}_2$ on N are homotopic through Legendrian line fields if and only if the element in $H^1(N;\mathbb{Z})$ corresponding to $F(\mathcal{F}_1,\mathcal{F}_2)$ is zero.*

It is of course possible to compare $\mathcal{F}_1$ and $\mathcal{F}_2$ with a third framing of the contact structure. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic if and only if we obtain the same class in $H^1(N;\mathbb{Z})$ from the two line fields when we compare $\mathcal{F}_1$ and $\mathcal{F}_2$ with the auxiliary Legendrian line field.

Now let N be a transversal hypersurface in an Engel manifold M. Let $\gamma:S^1\longrightarrow N$ be an oriented Legendrian curve and X a nowhere vanishing section of the contact structure

$\mathcal{C} = \mathcal{E} \cap TN$ spanning the intersection foliation. Since γ is Legendrian, $\dot\gamma$ is a Legendrian vector field along γ.

DEFINITION 3.23. For a Legendrian curve γ in a transversal hypersurface of an Engel manifold, the winding number of $F(\dot\gamma, X)$ around 0 is the *rotation number* of γ.

The rotation number changes sign when we reverse the orientation of the Legendrian curve or when we change the orientation of the contact structure. In particular it changes its sign when we change the orientation of the characteristic foliation of $\mathcal{D}$. It is independent of the orientation of $\mathcal{D}$.

REMARK 3.24. Let us compare Definition 3.23 with the rotation number from contact topology in Definition 2.17. In Definition 2.17 we fix an oriented trivialization of the oriented contact structure on a Seifert surface Σ of the Legendrian knot $\partial\Sigma = \gamma$ and compare $\dot\gamma$ with this trivialization.

If $\mathcal{C}$ is the contact structure on a transversal hypersurface of an Engel manifold with oriented characteristic foliation then $\mathcal{C}$ is oriented. When $\mathcal{D}$ is oriented we can use the intersection line field as the first component of the trivialization of $\mathcal{C}$ over Σ. Thus in this situation the two rotation numbers in Definition 3.23 and Definition 2.17 are equivalent. When the orientation of the contact structure is changed the rotation number changes its sign.

By Proposition 3.22, the homotopy type of a Legendrian line field near a Legendrian curve is classified by the rotation number along this curve.

LEMMA 3.25. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two oriented Legendrian line fields on a closed tubular neighbourhood U of a Legendrian curve γ. Then $\mathcal{F}_1, \mathcal{F}_2$ are homotopic through Legendrian line fields on U if and only if they have the same rotation number along γ.*

The use of the condition on the curve γ to be Legendrian is to single out a distinguished framing of the contact structure along this curve. We then compare the framing of $\mathcal{C}$ along γ defined by X with the framing defined by $\dot\gamma$. If one has a Legendrian line field spanned by V along an arbitrary curve in N one can similarly define a rotation number with respect to this line field using V instead of $\dot\gamma$. Then one can also drop to assumption that γ is Legendrian. This way we define the *rotation number with respect to V*. The analogous statement as Lemma 3.25 is of course true in this more general situation.

3.2.5. Development map. The development map allows us to compare the Engel planes $\mathcal{D}_p$ and $\mathcal{D}_q$ if p and q lie on the same leaf $\mathcal{W}_p$ of the characteristic foliation of a given Engel structure. It was introduced in [**BrH, Mo2**]. The definition of twisting number appears in a slightly modified form in [**Ad**] where it is used to classify Engel structures whose characteristic foliation is given by $N \times I$ or $N \times S^1$ for a 3–manifold N.

Let M be a manifold with Engel structure $\mathcal{D}$. As usual, we have the associated even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ and the characteristic foliation $\mathcal{W} \subset \mathcal{D}$. If U is an open subset of M such that $U/\mathcal{W}$ admits a smooth structure and $\mathrm{pr} : U \longrightarrow U/\mathcal{W}$ is a submersion, then $U/\mathcal{W}$ carries the contact structure $\mathrm{pr}_*(\mathcal{E})$ since $\mathcal{E}$ is invariant along the leaves of $\mathcal{W}$.

DEFINITION 3.26. The *development map* of U is

$$\begin{aligned} \delta_U : U &\longrightarrow \mathbb{P}(\mathrm{pr}_*\mathcal{E}) \\ q &\longmapsto [\mathrm{pr}_*\mathcal{D}(q)] \ . \end{aligned}$$

EXAMPLE 3.27. Let $\mathcal{C}$ be a contact structure on N and $\mathrm{pr} : \mathbb{P}\mathcal{C} \longrightarrow N$ be the bundle projection. The prolonged Engel structure on $\mathbb{P}\mathcal{C}$ is defined by

$$\mathcal{D}(\lambda) = \left\{ v \in T_\lambda \mathbb{P}\mathcal{C} \middle| \mathrm{pr}_* v \in \lambda \right\} \ .$$

Then $\mathbb{P}\mathcal{C} \longrightarrow \mathbb{P}\mathcal{C}/\mathcal{W} = N$ is a submersion. In this case pr is simply the bundle projection. Moreover $\mathrm{pr}_*\mathcal{E} = \mathcal{C}$. Hence the development map of $\mathbb{P}\mathcal{C}$ is a map

$$\delta_{\mathbb{P}\mathcal{C}} : \mathbb{P}\mathcal{C} \longrightarrow \mathbb{P}\mathcal{C} \ .$$

By the definition of the canonical Engel structure on $\mathbb{P}\mathcal{C}$

$$\delta_{\mathbb{P}\mathcal{C}}([l]) = [\mathrm{pr}_*(\mathcal{D}([l]))] = [l] \in \mathbb{P}\mathcal{C} \ ,$$

so $\delta_{\mathbb{P}\mathcal{C}}$ is the identity of $\mathbb{P}\mathcal{C}$.

Let $p \in M$ and $\mathcal{W}_p$ be the leaf of the characteristic foliation containing p. If $\mathcal{W}_p$ is closed we consider the universal covering of a tubular neighbourhood of $\mathcal{W}_p$ with the lifted Engel structure. The universal covering of $\mathcal{W}_p$ is $\widetilde{\mathcal{W}_p}$.

If $p, q \in \widetilde{\mathcal{W}_p}$ we choose a neighbourhood U of the unique segment of $\widetilde{\mathcal{W}_p}$ joining p and q such that $U \longrightarrow U/\mathcal{W}$ is a submersion.

DEFINITION 3.28. The *development map* of $\mathcal{W}_p$ is

$$\begin{aligned} \delta_p : \mathcal{W}_p &\longrightarrow \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p) \simeq \mathbb{R}\mathbb{P}^1 \\ q &\longmapsto \delta_U(q) \ . \end{aligned}$$

$\delta_p(q)$ does not depend on the choice of U. Up to now we used only the fact that $\mathcal{E}$ is invariant along $\mathcal{W}$ and $\mathcal{W} \subset \mathcal{D}$. We did not use the property $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. If $\mathcal{D}$ is an Engel structure we have the following proposition.

PROPOSITION 3.29. *The development map of $\mathcal{W}_p$ is an immersion.*

PROOF. Let $p, q \in \mathcal{W}_p$ and

$$\varphi : \mathbb{P}(\mathcal{E}_q/\mathcal{W}_q) \longrightarrow \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p)$$

the map induced by the leaves of $\mathcal{W}$. Since $\mathcal{E}$ and $\mathcal{W}$ are invariant under flows along $\mathcal{W}$, this is an isomorphism. Moreover $\delta_p = \varphi \circ \delta_q$. So in order to show that δ_p is an immersion it is enough to check this on a neighbourhood of p in $\mathcal{W}_p$.

Now choose a transversal hypersurface H through p and let $\mathcal{C}$ be the induced contact structure on H. Then $\mathcal{C}_p \simeq \mathcal{E}_p/\mathcal{W}_p$. By Theorem 3.19 there is an Engel embedding

$$\psi : U \longrightarrow \mathbb{P}\mathcal{C}$$

of a tubular neighbourhood U of H. We write $\widetilde{\mathcal{W}}$ for the characteristic foliation on $\mathbb{P}\mathcal{C}$. The leaf $\widetilde{\mathcal{W}}(\psi(p))$ is the projectivization of the contact plane $\mathcal{C}(p)$. By Example 3.27

$$\delta_{\psi(p)} : \widetilde{\mathcal{W}}_{\psi(p)} = \mathbb{P}\mathcal{C}(p) \longrightarrow \mathbb{P}\mathcal{C}(p)$$

is the identity map. Then

$$\delta_p = \left(\psi_*\big|_H\right)^{-1} \circ \delta_{\psi(p)} \circ \psi \ .$$

In particular δ_p is an immersion on a neighbourhood of p. □

Fix an orientation of the leaf $\mathcal{W}_p$ of the characteristic foliation through p. p divides $\widetilde{\mathcal{W}_p}$ into two arcs. If $\mathcal{W}_p$ is closed let $\widetilde{\mathcal{W}_p^+}$ be the maximal half–open oriented segment of $\widetilde{\mathcal{W}_p}$ starting at p such that the image of $\widetilde{\mathcal{W}_p^+}$ is mapped injectively to $\mathcal{W}_p$. If $\mathcal{W}_p$ is open, $\widetilde{\mathcal{W}_p^+}$ is the segment of $\mathcal{W}_p$ which starts at p with respect to the given orientation of $\mathcal{W}_p$. Similarly we define $\widetilde{\mathcal{W}_p^-}$.

Let $\mathcal{C}_p$ be the contact plane at p on a local transversal through this point. Consider the development maps

$$\begin{aligned} \delta^+ &: \widetilde{\mathcal{W}_p^+} \longrightarrow \mathcal{C}_p \\ \delta^- &: \widetilde{\mathcal{W}_p^-} \longrightarrow \mathcal{C}_p \end{aligned}$$

DEFINITION 3.30. The *twisting numbers* of p are

$$\begin{aligned}\mathrm{tw}^+(p) &= \#\left\{q \in \widetilde{\mathcal{W}}_p^+ \,\middle|\, \delta^+(q) = \delta^+(p)\right\} \in \mathbb{N} \cup \{\infty\}\\ \mathrm{tw}^-(p) &= \#\left\{q \in \widetilde{\mathcal{W}}_p^- \,\middle|\, \delta^-(q) = \delta^-(p)\right\} \in \mathbb{N} \cup \{\infty\}\ .\end{aligned}$$

A leaf of the characteristic foliation is said to have *finite twisting number* if the twisting number is finite for some (and hence every) point on this leaf. $\mathcal{W}_p$ has *infinite twisting number* if $tw^+(p)$ or $tw^-(p)$ is infinite.

Notice that p is contained in both sets appearing in this definition, so both twisting number are at least 1. By Proposition 3.29, the twisting number is a measure for the number of full twists of the image of $\mathcal{D}_q$ in $\mathcal{C}_p$ when q moves along $\mathcal{W}_p$ away from p as long as it does not reach p again. The last condition is meaningless if $\mathcal{W}_p$ is not closed.

The twisting number has the following application. Consider a local transversal U of the characteristic foliation through p. We orient the contact structure $\mathcal{C}$ on U using the orientation of the even contact structure and the orientation of $\mathcal{W}_p$. Let C_1, C_2 be an oriented framing of $\mathcal{C}$ such that C_1 spans the intersection line field on U.

On $U \times \mathbb{R}$ consider the Engel structure $\mathcal{D}_\infty$ spanned by

$$W = \frac{\partial}{\partial t}, X = \cos(2\pi t)C_1 + \sin(2\pi t)C_2\ .$$

The characteristic foliation of this Engel structure corresponds to the second factor in $U\times\mathbb{R}$, we write δ_p^∞ for the development map of $(p,0)$ in $U \times \mathbb{R}$. For all points $(p,t) \in U \times \mathbb{R}$ the twisting numbers are $\mathrm{tw}^+(p,t) = \mathrm{tw}^-(p,t) = \infty$. There is a unique map

$$\begin{array}{ccccc} \widetilde{\mathcal{W}}_p & \xrightarrow{\varphi} & \{p\} \times \mathbb{R} & & \subset U \times \mathbb{R} \\ \downarrow{\scriptstyle \delta_p} & & \downarrow{\scriptstyle \delta_p^\infty} & & \\ \mathcal{C}_p & = & \mathcal{C}_p & & \end{array}$$

with $\varphi(p) = (p,0)$. By the definition of the twisting number and Proposition 3.29

$$\begin{aligned}\varphi\left(\widetilde{\mathcal{W}}_p^+\right) &\subset \{p\} \times [0, \mathrm{tw}^+(p)]\\ \varphi\left(\widetilde{\mathcal{W}}_p^-\right) &\subset \{p\} \times [-\mathrm{tw}^-(p), 0]\ .\end{aligned}$$

If $\mathcal{W}_p$ is closed, φ extends to an Engel embedding of a tubular neighbourhood of the segment $\widetilde{\mathcal{W}}_p^\pm$. If $\mathcal{W}_p$ is not closed, then for every $q \in \mathcal{W}_p$ the restriction of φ to the segment of $\mathcal{W}_p$ with endpoints p, q extends to an Engel embedding of a tubular neighbourhood of this segment.

Consider a local transversal H of the characteristic foliation of an Engel structure. We write $\mathcal{L}_0$ for its intersection line field. Now consider a homotopy $\mathcal{L}_s$ through Legendrian line fields. We try to find an isotopy H_s of H along the leaves of $\mathcal{W}$ such that, if we identify $H_0 = H$ and H_s using the characteristic foliation, the intersection line field of H_s corresponds to $\mathcal{L}_s$. But if one of the intersection numbers of $\mathcal{W}_p, p \in H$, is finite, such an isotopy does not exist in general. Suppose for example that $tw^-(p) = 1$ and $\mathcal{L}_s(p)$ rotates twice in the sense opposite to the orientation of $\mathcal{C}_p$. Then it is impossible to find the desired isotopy.

The following examples show that all leaves of the characteristic foliation can have finite twisting number even on compact manifolds.

EXAMPLE 3.31. Consider the Engel structure from the normal form for Engel structures $\mathcal{D} = \ker(dz - xdy) \cap \ker(dx - wdy)$ on $\mathbb{R}^4$. For every point $p \in R^4$, we have $\mathrm{tw}^+(p) = \mathrm{tw}^-(p) = 1$.

EXAMPLE 3.32. This example will appear again at the end of Chapter 7. Consider the Lie group Nil^4. The Lie algebra $\mathfrak{nil}^4$ is spanned by W, X, Y, Z with the commutator relations

$$[W, X] = Y \qquad\qquad [X, Y] = Z$$

and all remaining commutators vanish. The left–invariant plane field $\mathcal{D}$ spanned by W, X is an Engel structure. Now Nil^4 is a semidirect product $\mathbb{R}^3 \rtimes \mathbb{R}$. The action of $\mathbb{R}$ on $\mathbb{R}^3$ is given by

$$\exp \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Aut}(\mathbb{R}^3) \ .$$

Thus the characteristic foliation of $\mathcal{D}$ preserves the hypersurfaces $\{t = t_0\}$. The even contact structure $[\mathcal{D}, \mathcal{D}]$ is transversal to these hypersurfaces and $\mathcal{D}$ is never tangent to $\{t = t_0\}$. This shows

$$\mathrm{tw}^+(x, y, z, w) = \mathrm{tw}^-(x, y, z, w) = 1$$

Now Nil^4 contains a discrete subgroup Γ such that Nil^4/Γ is a closed manifold. Thus even on compact manifolds it may happen that every leaf of the characteristic foliation has finite twisting number. Notice that this is also true for the universal coverings of closed leaves of the characteristic foliation, both twisting numbers are 1.

We will encounter the difficulty we just described in Section 5.6. There is a second aspect which makes Engel structures with the property $\mathrm{tw}^+(p) = \mathrm{tw}^-(p) = 1$ for all p particularly interesting.

The following terminology is introduced in [**BrH**] for the study of more general distributions of rank 2. For us, $\mathcal{D}$ is always an Engel structure. A $\mathcal{D}$–curve is a differentiable curve tangent to $\mathcal{D}$. Let $\Omega_{\mathcal{D}}(p, q)$ be the set of $\mathcal{D}$–curves from p to q. We equip $\Omega_{\mathcal{D}}(p, q)$ with the C^1–topology. By Chow's theorem [**Mo3**] we know that $\Omega_{\mathcal{D}}(p, q)$ is not empty.

DEFINITION 3.33. A $\mathcal{D}$–curve $\gamma : [a, b] \longrightarrow M$ is *rigid* if there is a neighbourhood V of γ in $\Omega_{\mathcal{D}}(\gamma(a), \gamma(b))$ such that every $\gamma' \in V$ is a reparameterization of γ.

THEOREM 3.34 (Bryant, Hsu [**BrH**]). *Let $\mathcal{D}$ be an Engel structure on a 4-manifold M and let $\mathcal{W}$ be the characteristic foliation. An immersion $\gamma : [a, b] \longrightarrow M$ which is tangent to $\mathcal{D}$, is rigid if and only if*

(i) *γ is tangent to $\mathcal{W}$ and*
(ii) *the development map*

$$\delta_{\gamma(a)} : \gamma([a, b]) \longrightarrow \mathcal{C}(\gamma(a))$$

is injective except possibly at the endpoints.

Suppose that for every closed leaf $\mathcal{W}_p$ the twisting numbers of $\widetilde{\mathcal{W}}_p$ in the universal covering of a tubular neighbourhood are both one. Assume furthermore that the twisting numbers of the open leaves of $\mathcal{W}$ are also one. Then every immersion of a curve which is tangent to $\mathcal{W}$ is rigid. For example the standard Engel structure on $\mathbb{R}^4$ has this property. We have explained above that such Engel structures exist on compact quotients of Nil^4.

3.2.6. Orientation conventions. By Proposition 3.6, an orientation of the characteristic foliation of an Engel structure induces an orientation of the underlying manifold and vice versa. In addition we have an orientation of $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$.

PROPOSITION 3.35. *If $\mathcal{D}$ is an Engel structure, the even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ has a distinguished orientation.*

PROOF. Let X, Y be local sections of $\mathcal{D}$ around $p \in M$ such that $X(p)$ and $Y(p)$ are linearly independent. Then we orient $\mathcal{E}(p)$ by $X(p), Y(p), [X, Y](p)$. We obtain the same orientation if we interchange X and Y. □

Now let M be an oriented manifold with an oriented Engel structure $\mathcal{D}$. This induces an orientation of the characteristic line field $\mathcal{W}$. Let W be a positive section of $\mathcal{W}$ and let X be a section of $\mathcal{D}$ which is transversal to W such that W, X is an oriented framing of $\mathcal{D}$. By the Engel condition $X, [W, X], [X, [W, X]], W$ spans the tangent bundle of M everywhere. This orientation changes when the orientation of $\mathcal{W}$ is changed but is independent of the choice of the orientation of $\mathcal{D}$.

This leads to the following orientation conventions we will use from now on.

(i) We orient Engel manifolds by $X, [W, X], [X, [W, X]], W$.
(ii) Hypersurfaces which are transversal to the characteristic line field are oriented by the induced contact structure.
(iii) The even contact structure associated to an Engel structure carries its canonical orientation.
(iv) Contact structures on hypersurfaces which are induced by the even contact structure are oriented such that the orientation of the contact structure followed by the orientation of the characteristic line field gives the canonical orientation of the even contact structure.
(v) If in addition the Engel structure $\mathcal{D}$ is oriented, we orient the intersection line field by the convention that the orientation of $\mathcal{W}$ followed by the orientation of the intersection line field is the orientation of $\mathcal{D}$.

If M has a boundary ∂M which is transversal to $\mathcal{W}$ we could orient the boundary such that the orientation of ∂M followed by a normal vector pointing outwards is the orientation of M. On the other hand, the boundary is oriented by the induced contact structure. If the characteristic line field points outward, these two orientations coincide, if $\mathcal{W}$ points into the manifold we obtain opposite orientations.

3.3. Topology of Engel manifolds

An Engel structure $\mathcal{D}$ on M induces a flag of distributions

$$0 \subset \mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM \,. \tag{19}$$

Each of these distributions has corank one in the distribution containing it. This has strong implications for the topology of M. In the following proposition we summarize some relations between the bundles $\mathcal{W}, \mathcal{D}, \mathcal{E}$.

PROPOSITION 3.36. *Let $\mathcal{D}$ be an Engel structure on a 4–manifold M.*

(i) *The is a natural isomorphism between the real line bundles $\Lambda^2\mathcal{D}$ and $\mathcal{E}/\mathcal{D}$.*
(ii) *There is an exact sequence*

$$0 \longrightarrow \mathcal{W} \otimes \frac{\mathcal{E}}{\mathcal{D}} \longrightarrow \mathcal{D} \otimes \frac{\mathcal{E}}{\mathcal{D}} \longrightarrow \frac{TM}{\mathcal{E}} \longrightarrow 0 \,. \tag{20}$$

PROOF. (i) For $X, Y \in \mathcal{D}_p$ we choose local sections $\widehat{X}, \widehat{Y}$ of $\mathcal{D}$ such that $\widehat{X}(p) = X$ and $\widehat{Y}(p) = Y$. Then

$$\begin{aligned} \Lambda^2\mathcal{D}_p &\longrightarrow \mathcal{E}_p/\mathcal{D}_p \\ X \wedge Y &\longmapsto \overline{[\widehat{X}, \widehat{Y}](p)} \end{aligned}$$

is well defined and it is surjective since $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. Thus we have found a bundle isomorphism since $\Lambda^2\mathcal{D}$ and $\mathcal{E}/\mathcal{D}$ have rank 1.

(ii) For $X \in \mathcal{D}_p$ and $V \in \mathcal{E}_p$ choose local sections $\hat{X}$ of $\mathcal{D}$ respectively $\hat{V}$ of $\mathcal{E}_p$ which extend X respectively V. Then we define

$$\begin{aligned} f : \mathcal{D}_p \otimes \frac{\mathcal{E}_p}{\mathcal{D}_p} &\longrightarrow TM_p/\mathcal{E}_p \\ X \otimes \overline{V} &\longmapsto \overline{[\hat{X}, \hat{V}](p)}\ . \end{aligned}$$

This map is independent of the choice of extensions. Because $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$, it is also independent of the choice of a representative $V \in \mathcal{E}_p$ of $\overline{V} \in \mathcal{E}_p/\mathcal{D}_p$.

By the condition $[\mathcal{D}, [\mathcal{D}, \mathcal{D}]] = [\mathcal{D}, \mathcal{E}] = TM$ on Engel structures, f is surjective. Since $\mathcal{E}/\mathcal{D}$ has rank one, every element of $\mathcal{D}_p \otimes \mathcal{E}_p/\mathcal{D}_p$ can be written in the form $X \otimes \overline{V}$. Hence the kernel of f consists of vectors $W \otimes \overline{V}$ such that $[\hat{W}, \hat{V}](p) \in \mathcal{E}_p$. This is exactly the condition that defines the line field $\mathcal{W}$ (cf. Lemma 3.11). Therefore the kernel of k is $\mathcal{W} \otimes \mathcal{E}/\mathcal{D}$. □

If M and $\mathcal{D}$ are both orientable one obtains a stronger result. The following theorem can be found in [**KMS**]. It was already known to V. Gershkovic. Unfortunately his preprint [**Ger**] was not available to the author.

THEOREM 3.37. *Let $\mathcal{D}$ be an oriented Engel structure on an oriented four manifold M. Then the tangent bundle of M trivial.*

PROOF. Consider the flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ of subbundles of TM. The even contact structure $\mathcal{E}$ is oriented without any assumptions on the Engel structure or the underlying manifold. An orientation of M induces an orientation of $\mathcal{W}$ by our conventions. The tangent bundle of M is isomorphic to the sum

$$TM = \mathcal{W} \oplus \frac{\mathcal{D}}{\mathcal{W}} \oplus \frac{\mathcal{E}}{\mathcal{D}} \oplus \frac{TM}{\mathcal{E}} \tag{21}$$

of four real line bundles. Because $\mathcal{D}$ is orientable so are $\mathcal{D}/\mathcal{W}$ and $\mathcal{E}/\mathcal{D}$. So TM is isomorphic to the sum of four trivial line bundles. □

Notice that under the assumption of the theorem, TM is trivial but moreover we can single out a particular trivialization of TM up to homotopy. If we drop the orientability assumptions on M and $\mathcal{D}$ we still have topological obstructions for the existence of an Engel structure on a four–dimensional manifold.

COROLLARY 3.38. *If M admits an Engel structure then there is a covering $\widetilde{M} \longrightarrow M$ with one, two or four sheets such that $\widetilde{M}$ has trivial tangent bundle.*

PROOF. Recall that $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) = H^1(M; \mathbb{Z}_2)$. First consider the 2–sheeted covering $\widetilde{M} \to M$ which corresponds to the subgroup ker$(w_1(M)) \subset \pi_1(M)$. By construction, $\widetilde{M}$ is orientable and we pull back the Engel structure. If the pulled back Engel structure $\widetilde{\mathcal{D}}$ is not yet orientable then consider the 2–sheeted covering $\widetilde{\widetilde{M}}$ of $\widetilde{M}$ corresponding to ker$(w_1(\widetilde{\mathcal{D}}))$. If we pull back $\widetilde{\mathcal{D}}$ to $\widetilde{\widetilde{M}}$ we end up with a orientable manifold carrying an orientable Engel structure. □

Of course (21) follows directly from the existence of the flag (19). From this decomposition of TM into line bundles we can obtain conditions on the Stiefel–Whitney classes $w_i \in H^i(M;\mathbb{Z}_2)$ of an Engel manifold M. By the Whitney formula

$$\begin{aligned} w(TM) &= 1 + w_1(TM) + w_2(TM) + w_3(TM) + w_4(TM) \\ &= (1 + w_1(\mathcal{W})) \cup (1 + w_1(\mathcal{D}) + w_1(\mathcal{W})) \\ &\quad \cup (1 + w_1(\mathcal{E}) + w_1(\mathcal{D})) \cup (1 + w_1(TM) + w_1(\mathcal{E})) \,. \end{aligned}$$

Since $\mathcal{E}$ is canonically oriented $w_1(\mathcal{E}) = 0$. Because transversal hypersurfaces in Engel manifolds are canonically oriented by the induced contact structure we have the relation $w_1(TM) = w_1(\mathcal{W})$. Hence

$$\begin{aligned} w(TM) = 1 &+ w_1(TM) + w_1^2(\mathcal{D}) + w_1(TM) \cup w_1(\mathcal{D}) + w_1^2(TM) \\ &+ w_1^3(TM) + w_1^2(\mathcal{D}) \cup w_1^2(TM) + w_1(\mathcal{D}) \cup w_1^3(TM) \end{aligned}$$

From this we obtain the following proposition

PROPOSITION 3.39. *If M admits an Engel structure then*

$$\begin{aligned} w_3(TM) &= w_1^3(TM) \\ w_4(TM) &= w_1^4(TM) + w_2(TM) \cup w_1^2(TM) \,. \end{aligned}$$

3.4. Deformations of Engel structures

Let $\mathcal{C}$ be a parallelizable contact structure on a 3–manifold N and let V_0, V_1 be Legendrian vector fields such that $\mathcal{C} = \mathbb{R}V_0 \oplus \mathbb{R}V_1$. We view the real projective line as $\mathbb{RP}^1 = S^1/\{\pm 1\}$, the circumference of $\mathbb{RP}^1$ is π. Let $\mathrm{pr} : \mathbb{P}\mathcal{C} \longrightarrow N$ be the projection. Then

$$\begin{aligned} F : N \times \mathbb{RP}^1 &\longrightarrow \mathbb{P}\mathcal{C} \\ (p,\theta) &\longmapsto [\cos(\theta)V_0(p) + \sin(\theta)V_1(p)] \end{aligned} \tag{22}$$

is a well defined diffeomorphism.

DEFINITION 3.40. The image $\Omega(V_0, V_1)$ of $N \times [0, \pi/2]$ under this diffeomorphism is called the *standard domain* associated to the pair of Legendrian vector fields (V_0, V_1). The *standard Engel structure* $\mathcal{D}_0$ in a standard domain is the restriction of the prolonged Engel structure on $\mathbb{P}\mathcal{C}$.

Although the diffeomorphism above depends on the vector fields V_0, V_1, the standard domain depends only on the Legendrian line fields spanned by V_0, V_1. We will use V_0, V_1 to denote the Legendrian vector fields as well as the Legendrian line fields. The projection maps the intersection line fields on the boundary components of $\Omega(V_0, V_1)$ to V_0 respectively V_1.

We equip the set of plane fields of class C^2 on $\Omega(V_0, V_1)$ with the strong C^2–topology. In Theorem 3.41 and Theorem 3.43 we treat with deformations of the standard Engel structure on $\mathbb{P}\mathcal{C}$ and on a standard domain.

It turns out that the space of infinitesimal deformations of $\mathcal{D}_0$ on $\Omega(V_0, V_1)$ up to isotopy has infinite dimension. If $\mathcal{D}_t$ is a deformation of $\mathcal{D}_0$, the characteristic foliation of $\mathcal{D}_t$ is diffeomorphic to the product $N \times I$ if $|t|$ is small enough. In this situation, the characteristic foliation is not responsible for the large number of non–equivalent deformations of $\mathcal{D}_0$. The complexity is due to the presence of two Legendrian line fields on the boundary components of the standard domain. The induced foliations and their relation induced by the characteristic foliation of $\mathcal{D}_t$ account for the fact that the space of infinitesimal deformations of $\mathcal{D}_0$ is infinite dimensional even after we quotient by a suitable equivalence relation.

THEOREM 3.41 (Montgomery, [**Mo2**]).

(i) *Let $\mathcal{D}_t$ be any sufficiently small deformation of the canonical Engel structure $\mathcal{D}_0$ on the standard domain $\Omega(V_0, V_1)$. Then there is a one–parameter family of Legendrian line fields $V_0(t)$ and $V_1(t)$ together with a family of Engel diffeomorphisms*

$$\Phi_t : (\Omega(V_0, V_1), \mathcal{D}_t) \longrightarrow (\Omega(V_0(t), V_1(t)), \mathcal{D}_0) \ .$$

(ii) *For every small variation $(V_0(t), V_1(t))$ of pairs of Legendrian line fields there is an Engel deformation $\mathcal{D}_t$ of the standard Engel structure on the standard domain such that the correspondence constructed in the proof of (i) yields $(V_0(t), V_1(t))$.*

(iii) *Let $\mathcal{D}_t$ be a small deformation of the canonical Engel structure on $\mathbb{P}\mathcal{C} \simeq N \times \mathbb{RP}^1$. If we view N as a section of $\mathbb{P}\mathcal{C}$ then the Poincaré return map of $\mathcal{W}_t$ is a contact diffeomorphism of $(N, \mathcal{C}_t)$ with $\mathcal{C}_t = TN \cap \mathcal{E}_t$.*

(iv) *Any contact isotopy of $(N, \mathcal{C})$ which is close enough to the identity can be realized as the Poincaré return map as in (iii) for some Engel deformation $\mathcal{D}_t$.*

PROOF. (i) We view N as the hypersurface in $\Omega(V_0, V_1)$ corresponding to $N \times \{\pi/4\}$. As the Engel structure varies, the associated even contact structure $\mathcal{E}_t$ and the characteristic line field $\mathcal{W}_t$ also vary. If the variation is small enough, $\Omega(V_0, V_1)$ is foliated trivially by $\mathcal{W}_t$ and N intersects all leaves of $\mathcal{W}_t$ transversely and exactly once. Thus $\mathcal{C}_t = \mathcal{E}_t \cap TN$ is a smooth family of contact structures on N.

As in Theorem 3.19 we construct an Engel embedding of a tubular neighbourhood of $N \subset \Omega(V_0, V_1)$ with the Engel structure $\mathcal{D}_t$ into the Engel manifold $\mathbb{P}\mathcal{C}_t$ associated to the contact structure $\mathcal{C}_t$. The construction of this Engel embedding works for tubular neighbourhoods U of N such that

- U is foliated trivially by $\mathcal{W}_t$
- for all $p \in N$, the segment of $\mathcal{W}_t(p)$ which is contained in U is embedded to $\mathbb{P}\mathcal{C}_t(p)$ by the development map.

Obviously $(\Omega(V_0, V_1), \mathcal{D}_0)$ has these properties, so $(\Omega(V_0, V_1), \mathcal{D}_t)$ has these properties too, provided that $|t|$ is small enough. If the deformation is small enough, $\Omega(V_0, V_1)$ itself has these properties since they are obviously satisfied for $\mathcal{D}_0$. We obtain a family ψ_t of Engel embeddings

$$\psi_t : (\Omega(V_0, V_1), \mathcal{D}_t) \longrightarrow \mathbb{P}\mathcal{C}_t \ ,$$

the Engel structure on $\mathbb{P}\mathcal{C}_t$ is induced by $\mathcal{C}_t$.

Since $\mathcal{C}_t$ is a smooth family of contact structures on N we can apply Gray's theorem 2.4. In order to do so we have to impose an additional condition on the variation $\mathcal{D}_t$: The deformation has to be so small that the time–dependent vector field constructed in the proof of Gray's theorem can be integrated to an isotopy. If N is compact, this condition is automatically satisfied.

Since $\mathcal{C} = \mathcal{C}_0$, there is an isotopy φ_t of N such that $\varphi_{t*}\mathcal{C} = \mathcal{C}_t$. By Proposition 3.16 this induces a smooth family of diffeomorphisms

$$\widetilde{\varphi}_t : \mathbb{P}\mathcal{C} \longrightarrow \mathbb{P}\mathcal{C}_t$$

preserving the canonical Engel structures. The composition

$$\Phi_t = \widetilde{\varphi}_t^{-1} \circ \psi_t : (\Omega(V_0, V_1), \mathcal{D}_t) \longrightarrow \mathbb{P}\mathcal{C}$$

is an Engel embedding. Then pr_* maps the intersection line fields on the boundary components of $\Phi_t(\Omega(V_0, V_1))$ to Legendrian line fields $V_0(t)$ respectively $V_1(t)$ with the property $\Phi_t(\Omega(V_0, V_1)) = \Omega(V_0(t), V_1(t))$.

(ii) Consider a deformation $(V_0(t), V_1(t))$ of (V_0, V_1) through Legendrian line fields. We want to construct a deformation $\mathcal{D}_t$ of the standard Engel structure on the standard domain $\Omega(V_0, V_1)$ from this. Let

$$\widehat{\psi}_t : \Omega(V_0(t), V_1(t)) \longrightarrow \Omega(V_0, V_1)$$

be the diffeomorphism defined by the following conditions

- $\widehat{\psi}_t$ preserves the leaves of the characteristic foliation of $\mathcal{D}$.
- $\widehat{\psi}_t$ preserves the projective structure on the fibers of $N \times \mathbb{RP}^1 \simeq \mathbb{P}\mathcal{C} \longrightarrow N$.
- $\widehat{\psi}_t(p, [V_0(0,p) + V_1(0,p)]) = (p, [V_0(0,p) + V_1(0,p)])$ independently of t.
- $\widehat{\psi}_t(p, [V_i(t,p)]) = (p, [V_i(0,p)])$ for $i = 0, 1$.

The last two conditions determine $\widehat{\psi}_t$ on three disjoint sections of $\mathbb{P}\mathcal{C}$. Since $\widehat{\psi}_t$ is supposed to preserve the projective structure on the leaves of $\mathbb{P}\mathcal{C} \simeq N \times \mathbb{RP}^1$, this determines $\widehat{\psi}_t$ completely. In particular $\widehat{\psi}_0 = \mathrm{id}$. Let $\mathcal{D}_t = \widehat{\psi}_{t*}\mathcal{D}$ on $\Omega(V_0, V_1)$. This is a deformation of the standard Engel structure on $\Omega(V_0, V_1)$.

In the proof of (i) one can use the Engel embedding $\psi_t = \widehat{\psi}_t^{-1}$. Notice that the contact structure on $N \times \{\pi/4\}$ is constant. An application of (i) to the deformation $\mathcal{D}_t$ yields $(V_0(t), V_1(t))$.

(iii) If the variation is small enough N is transversal to $\mathcal{W}_t$ for all t. The claim follows directly from Lemma 3.5 which asserts that the holonomy of $\mathcal{W}_t$ preserves the contact structure on transversals.

(iv) We use the same notation V_0, V_1 for the horizontal lifts of V_0, V_1 to $N \times \mathbb{RP}^1$. The pull back under the diffeomorphism F defined in (22) of the canonical Engel structure on $\mathbb{P}\mathcal{C}$ at (p, θ) is spanned by

$$\begin{aligned} W_0(p,\theta) &= \frac{\partial}{\partial\theta} \\ X(p,\theta) &= \cos(\theta)V_0(p) + \sin(\theta)V_1(p)\ . \end{aligned}$$

Notice that $X(p,\theta) = -X(p,\theta+\pi)$ but (p,θ) and $(p,\theta+\pi)$ represent the same point in $N \times \mathbb{RP}^1$. Since we are only interested in the span of W and X, this ambiguity does not matter. We view N as the hypersurface $N \times \{0\}$. Let $\rho : [0,\pi] \longrightarrow [0,1]$ be a smooth function which is constant near the boundary and $\rho(0) = 0, \rho(\pi) = 1$.

Let $\Phi_t, t \in (-1,1)$ be a contact isotopy of $(N, \mathcal{C})$. For fixed $T \in (-1,1)$ we construct an Engel structure $\mathcal{D}_T$ such that the Poincaré return map of N is Φ_T. For this, we reparameterize the isotopy connecting $\Phi_0 = \mathrm{id}$ and Φ_T using ρ: $\widetilde{\Phi}_t = \Phi_{\rho(t)T}$. Consider the vector field

$$Y_t(p) = \left.\frac{d}{ds}\right|_{s=t} \widetilde{\Phi}_s(p)\ .$$

The flow of Y_t at time T is Φ_T. Let $\widetilde{Y}$ be the horizontal vector field on $N \times \mathbb{RP}^1$ with $\widetilde{Y}(p,\theta) = Y_\theta(p)$. We write $X_\theta, \widetilde{Y}_\theta$ for the vector fields $X(\cdot,\theta), \widetilde{Y}(\cdot,\theta)$ on $N \times \{\theta\}$. Consider the distribution $\mathcal{D}_T$ spanned by

$$\begin{aligned} W(p,\theta) &= \frac{\partial}{\partial\theta} + \widetilde{Y}(p,\theta) \\ X(p,\theta) &= \cos(\theta)V_0(p) + \sin(\theta)V_1(p)\ . \end{aligned}$$

Since $\widetilde{\Phi}_t$ is constant near the endpoints of $[0,\pi]$, this is a smooth distribution of rank two on $\mathbb{P}\mathcal{C}$. The calculation

$$\begin{aligned} [W,X](p,\theta) &= -\sin(\theta)V_0(p) + \cos(\theta)V_1(p) + [\widetilde{Y}, X](p,\theta) \\ [X,[W,X]](p,\theta) &= [V_0, V_1](p,\theta) + [X, [\widetilde{Y}, X]](p,\theta)\ . \end{aligned}$$

shows that $\mathcal{D}_T$ is an Engel structure if Y is small enough, or equivalently, if the isotopy is close enough to the identity in the strong C^2–topology. By construction, the Poincaré return map is the flow of W at time π induces the diffeomorphism Φ_T on $N \times \{0\}$. It remains to show that W spans the characteristic foliation of $\mathcal{D}_T$. The even contact structure $\mathcal{E}_T = [\mathcal{D}_t, \mathcal{D}_T]$ is spanned by $W, X, [W, X]$. Now

$$
\begin{aligned}
[W,[W,X]](p,\theta) = & -\cos(\theta)V_0(p) - \sin(\theta)V_1(p) \\
& -\sin(\theta)[Y_\theta, V_0](p) + \cos(\theta)[Y_\theta, V_1](p) \\
& + [Y_\theta,[Y_\theta,X_\theta]](p)\ ,
\end{aligned}
\tag{23}
$$

and Y_θ is a contact vector field. The sum in the first line is $-X$, the second and the third line are contained in $\mathcal{E}_T$ since Y_θ preserves the contact structure $\mathcal{C}$ on $N \times \{\theta\}$. Hence W is tangent to $\mathcal{E}_T$ and its flow preserves $\mathcal{E}_T$. Thus W spans the characteristic foliation.

Applying the same procedure for all $T \in (-1, 1)$ with the same function ρ, we get a smooth family of Engel structures $\mathcal{D}_T$ on $\mathbb{P}\mathcal{C}$ such that the Poincaré return map of the characteristic line field $\mathcal{W}_T$ is Φ_T. □

We give an example of an Engel deformation similar to those considered in (iii) of Theorem 3.41 in Example 3.49.

By Theorem 3.41 deformations of the prolonged Engel structure $\mathcal{D}_0$ on $\Omega(V_0, V_1)$ respectively on $\mathbb{P}\mathcal{C}$ are equivalent to families of pairs of Legendrian line fields respectively to contact isotopies. Next we define equivalence relations for these objects. In Theorem 3.43 shows that these equivalence relations are compatible.

DEFINITION 3.42. Two Engel deformations $\mathcal{D}_t$ and $\widetilde{\mathcal{D}}_t$ of an Engel structure $\mathcal{D}$ on M represent the same *deformation germ* of $\mathcal{D}$ if there is an isotopy ψ_t of M such that $\psi_{t*}\widetilde{\mathcal{D}}_t = \mathcal{D}_t$ for all t in a neighbourhood of 0.

On a contact manifold $(N, \mathcal{C})$, two deformations $(V_0(t), V_1(t))$ and $(\widetilde{V}_0(t), \widetilde{V}_1(t))$ of (V_0, V_1) through pairs of Legendrian line fields are *equivalent up to contact isotopy* if there is an isotopy φ_t of N which preserves $\mathcal{C}$ and

$$\varphi_{t*}\left(\widetilde{V}_0(t), \widetilde{V}_1(t)\right) = (V_0(t), V_1(t))$$

for all t close enough to 0. Two contact isotopies $\varphi_t, \widetilde{\varphi}_t$ are *equivalent up to t–dependent conjugation* if there is a contact isotopy f_t of $(N, \mathcal{C})$ such that $f_t \circ \varphi_t = f_t \circ \widetilde{\varphi}_t$.

THEOREM 3.43 (Montgomery, [**Mo2**]).

(i) *The space of deformation germs of* $(\Omega(V_0, V_1), \mathcal{D})$ *with its standard Engel structure is canonically isomorphic to the space of deformation germs* $(V_0(t), V_1(t))$ *of* (V_0, V_1) *of pairs of Legendrian line fields on* $(N, \mathcal{C})$ *modulo contact isotopies. This space has infinite dimension.*

(ii) *The space of deformation germs of the standard Engel structure on* $\mathbb{P}\mathcal{C}$ *is equal to the space of deformation germs of the identity through contact isotopies of* $(N, \mathcal{C})$ *modulo t–dependent conjugation:* $\Phi_t \sim g_t \circ \Phi_t \circ g_t^{-1}$.

PROOF. (i) We have constructed deformations of pairs of Legendrian line fields of $(N, \mathcal{C})$ from Engel deformations of $(\Omega(V_0, V_1), \mathcal{D}_0)$ and vice versa in Theorem 3.41. We show next that these constructions are compatible with the equivalence relations in Definition 3.42.

Let $\mathcal{D}_t, \widetilde{\mathcal{D}}_t$ be two equivalent deformation germs of the standard Engel structure on $\Omega(V_0, V_1)$. Then there is an isotopy

$$\psi_t : \Omega(V_0, V_1) \longrightarrow \Omega(V_0, V_1)$$

such that $\psi_{t*}\widetilde{\mathcal{D}}_t = \mathcal{D}_t$. By Theorem 3.41 (i) the deformations $\mathcal{D}_t, \widetilde{\mathcal{D}}_t$ correspond to Legendrian line fields $(V_0(t), V_1(t))$ respectively $(\widetilde{V}_0(t), \widetilde{V}_1(t))$. We want to find a contact isotopy f_t of $(N, \mathcal{C})$ such that $f_{t*}(\widetilde{V}_i(t)) = V_i(t)$ for $i = 0, 1$. Let

$$\varphi_t : (\Omega(V_0, V_1), \mathcal{D}_t) \longrightarrow (\Omega(V_0(t), V_1(t)), \mathcal{D}_0) \subset \mathbb{P}\mathcal{C}$$
$$\widetilde{\varphi}_t : \big(\Omega(V_0, V_1), \widetilde{\mathcal{D}}_t\big) \longrightarrow \big(\Omega(\widetilde{V}_0(t), \widetilde{V}_1(t)), \mathcal{D}_0\big) \subset \mathbb{P}\mathcal{C}$$

be the Engel embeddings used in Theorem 3.41 (i). Then

$$F_t = \varphi_t \circ \psi_t \circ \widetilde{\varphi}_t^{-1} : \Omega(\widetilde{V}_0(t), \widetilde{V}_1(t)) \longrightarrow \Omega(V_0(t), V_1(t))$$

is a diffeomorphism of two standard domains in $\mathbb{P}\mathcal{C}$ preserving $\mathcal{D}_0$. Hence F_t preserves the fibers of $\mathrm{pr} : \mathbb{P}\mathcal{C} \longrightarrow N$ and the map

$$f_t = \mathrm{pr} \circ F_t \circ \mathrm{pr}^{-1} : N \longrightarrow N$$

is a well defined contact map by Proposition 3.16. By the argument in the proof of Proposition 3.16, f_t induces and Engel diffeomorphism $\widetilde{f}_t$ of $\mathbb{P}\mathcal{C}$ which extends F_t. Since F_t maps the boundary of $\Omega(\widetilde{V}_0(t), \widetilde{V}_1(t))$ to the boundary of $\Omega(V_0(t), V_1(t))$ we have

$$f_{t*}(\widetilde{V}_i(t)) = V_i(t) \text{ for } i = 0, 1 \ .$$

Thus the map from equivalent deformation germs of $(\Omega(V_0, V_1), \mathcal{D}_0)$ to the set of deformations germs of pairs of Legendrian line fields modulo contact isotopy is well defined.

Conversely, let $(V_0(t), V_1(t))$ and $(\widetilde{V}_0(t), \widetilde{V}_1(t))$ be deformations of (V_0, V_1) through pairs of Legendrian line fields and let φ_t be a contact isotopy of $(N, \mathcal{C})$ with the property that $\varphi_{t*}(\widetilde{V}_0(t), \widetilde{V}_1(t)) = (V_0(t), V_1(t))$. We write

$$\psi_t : \Omega(V_0(t), V_1(t)) \longrightarrow \Omega(V_0, V_1)$$
$$\widetilde{\psi}_t : \Omega(\widetilde{V}_0(t), \widetilde{V}_1(t)) \longrightarrow \Omega(V_0, V_1)$$

for the maps constructed in Theorem 3.41 (ii). Let $\mathcal{D}_t = \psi_* \mathcal{D}_0$ and $\widetilde{\mathcal{D}}_t = \widetilde{\psi}_* \mathcal{D}_0$ be the corresponding Engel deformations. By Proposition 3.16 the contact isotopy φ_t induces an isotopy $\widetilde{\varphi}_t$ of the standard Engel structure on $\mathbb{P}\mathcal{C}$. Then $\psi_t^{-1} \circ \widetilde{\varphi}_t \circ \widetilde{\psi}_t$ maps $\widetilde{\mathcal{D}}_t$ to $\mathcal{D}_t$. So these deformation germs are equivalent.

It remains to show that the correspondence from Theorem 3.41 (i) and (ii) is independent of choices up to Engel isotopy respectively contact isotopy. We only indicate the argument. The only choice in the proof of Theorem 3.41 (i) was the choice of a section $N \longrightarrow \Omega(V_0, V_1)$, we have chosen the section $N \times \{\pi/4\}$. Any two sections of $\Omega(V_0, V_1)$ are isotopic through a family $\sigma_s, s = [0, 1]$ of sections which are transversal to $\mathcal{W}_t$ for all t close enough to 0. In (i) of Theorem 3.41, the section is identified with N. The isotopy can be used to construct a contact isotopy of $(N, \mathcal{C})$ showing that the family of pairs of Legendrian line fields obtained from $\sigma_0 \simeq N$ and $\sigma_1 \simeq N$ are equivalent up to contact isotopy.

Finally we have to show that the space of deformation germs of pairs of Legendrian line fields modulo contact isotopies has infinite dimension. This is done in two steps. In the first step we relate pairs of Legendrian line fields with ordinary differential equations of second order

$$\frac{d^2y}{dx^2} = G\left(x, y, \frac{dy}{dx}\right) \ . \tag{24}$$

The second step consists of the construction of functional moduli distinguishing equivalence classes of differential equations of type (24). For the second step we refer to [**Arn**] or [**Car2**].

Let V_0, V_1 be a pair of Legendrian line fields on $(N, \mathcal{C})$ such that these line fields span the contact structure and $p \in N$. We consider a flow box $(U, (x, y, z))$ chart for V_0 around p such that $V_0 = \partial_z$ and p has the coordinates $(0, 0, 0)$.

Let H be the plane $\{z = 0\}$ through p and pr $: U \to H$ the projection along the z–direction. Since V_0 and V_1 span a plane field, the projection $\mathrm{pr}_*(V_1(q))$ is a well defined line in TH and we may assume that $\mathrm{pr}_*(V_1(0, 0, 0))$ is tangent to the x–axes in H.

For $(x_0, y_0, z_0) \in U$ let γ be the integral curve of V_1 through this point. If (x_0, y_0, z_0) is close enough to $(0, 0, 0)$ we can view $\mathrm{pr}(\gamma)$ as the graph of a function $y_\gamma(x)$ with $y_\gamma(x_0) = y_0$. Since V_0, V_1 span a contact structure, the slope of the graph varies when the z–coordinate of the base point of γ varies, so

$$\frac{d}{dz}\left(\frac{dy_\gamma}{dx}(x_0)\right) \neq 0 \, .$$

Thus we can replace the z–coordinate by $\widetilde{z} = \frac{dy_\gamma}{dx}(x_0)$ on a small neighbourhood of p. In the new coordinates the contact structure is defined by the 1–form $dy - \widetilde{z}dx$. The second derivative

$$\frac{d^2 y_\gamma}{dx^2}(x_0)$$

is a smooth function F of $(x_0, y_0, \widetilde{z}_0)$ while the slope of $\mathrm{pr}(\gamma)$ at x_0 is just $\widetilde{z}_0$. The projection of an integral curve γ to H satisfies the differential equation

$$\frac{d^2 y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \, . \tag{25}$$

Conversely, a solution g of (25) with initial conditions

$$g(x_0) = y_0 \text{ and } \frac{dg}{dx}(x_0) = \widetilde{z}_0$$

induces the integral curve $(x, g(x), g'(x))$ of V_1 if $(x_0, y_0, \widetilde{z}_0)$ is close enough to $(0, 0, 0)$.

Now let φ_t be a contact isotopy. If we apply the procedure above to the pair of Legendrian line fields $\widehat{V_0} = \varphi_{t*}V_0, \widehat{V_1} = \varphi_{t*}V_1$ we obtain coordinates $\widehat{x}, \widehat{y}, \widehat{z}$ and a function $\widehat{F}$ such that the differential equation

$$\frac{d^2 \widehat{y}}{d\widehat{x}^2} = \widehat{F}\left(\widehat{x}, \widehat{y}, \frac{d\widehat{y}}{d\widehat{x}}\right) \tag{26}$$

corresponds to $\widehat{V_0}, \widehat{V_1}$. By definition $\varphi_t : U \longrightarrow \widehat{U}$ is a contact map which maps the fibration pr $: U \longrightarrow H$ to $\widehat{\mathrm{pr}} : \widehat{U} \longrightarrow \widehat{H}$. As in Example 2.3, this contact map is actually induced by a local diffeomorphism $H \to \widetilde{H}$ which transforms (25) into (26). Hence changing V_0, V_1 by a contact isotopy does not change the equivalence class of the resulting differential equation.

(ii) Choose a section $\sigma : N \longrightarrow \mathbb{P}\mathcal{C} \simeq N \times \mathbb{RP}^1$ and let φ_t be the Poincaré return map for $\mathcal{W}_t$. If f_t is the isotopy from Gray's theorem with the property $f_{t*}\mathcal{C}_0 = \mathcal{C}_t$ then $f_t \circ \varphi_t \circ f_t^{-1}$ is the contact isotopy of $(N, \mathcal{C})$ associated to $\mathcal{D}_t$. Then

$$\psi_t \circ \varphi_t \circ \psi_t^{-1} : \psi_t(N) \longrightarrow \psi_t(N)$$

is the Poincaré return map for $\psi_{t*}\mathcal{W}_t$ of $\psi_t(N)$. Let $h_t : \psi_t(N) \to \psi_0(N)$ be the map induced by the leaves of $\widetilde{\mathcal{W}_t} = \psi_{t*}\mathcal{W}_t$. Then

$$(h_t \circ \psi_t) \circ \varphi_t \circ (h_t \circ \psi_t)^{-1} : \psi_0(N) \to \psi_0(N)$$

is a contact map for the contact structure induced by $\widetilde{\mathcal{E}_t}$ on $\psi_0(N) \simeq N$. Using Gray's theorem again we obtain a contact isotopy of $(N, \mathcal{C})$ which is conjugate to the contact isotopy obtained from $\mathcal{D}_t$.

We omit the converse direction, i.e. conjugate contact isotopies yield equivalent Engel deformations. Finally we show that the correspondence in (ii) does not depend on the choice of a section.

Let σ_0, σ_1 be two sections of $\mathbb{P}\mathcal{C}$. For $|t|$ small enough, σ_0 and σ_1 are both transversal to the characteristic foliation of $\mathcal{D}_t$, we write $\mathcal{C}_t^i$ for the family of contact structures on $\sigma_i, i = 0, 1$. These two sections are identified by the leaves of the characteristic foliation of $\mathcal{D}_t$. This induces a contact map

$$f_t : (\sigma_0(N), \mathcal{C}_t^0) \longrightarrow (\sigma_1(N), \mathcal{C}_t^1)$$

depending smoothly on t. If the deformations is small enough, we can apply Gray's theorem to $\mathcal{C}_0^i, i = 0, 1$ and obtain diffeomorphism Let

$$h_t^0 : (\sigma_0(N), \mathcal{C}_0) \longrightarrow (\sigma_0(N), \mathcal{C}_t^0)$$
$$h_t^1 : (\sigma_1(N), \mathcal{C}_0) \longrightarrow (\sigma_1(N), \mathcal{C}_t^1)$$

preserving the contact structures. Then the contact isotopy from $\mathcal{D}_t$ of $(\sigma_0(N), \mathcal{C}_0) = (N, \mathcal{C})$ and of $(\sigma_1(N), \mathcal{C}_0) = (N, \mathcal{C})$ are conjugate by $g_t = (h_t^1)^{-1} \circ f_t \circ h_t^0$.

Let $\mathcal{D}_t$ and $\widetilde{\mathcal{D}}_t$ be equivalent germs of Engel deformations of the standard Engel structure on $\mathbb{P}\mathcal{C}$. By definition there is an isotopy ψ_t of $\mathbb{P}\mathcal{C}$ such that $\psi_{t*}\widetilde{\mathcal{D}}_t = \mathcal{D}_t$. □

3.5. Engel vector fields

In this section we want to investigate the set of vector fields preserving a given Engel structure on some manifold M. We have already treated the case of contact vector fields in Section 2.1.2. The results we obtain for Engel structures are similar.

DEFINITION 3.44. A vector field preserving the Engel structure is called *Engel vector field*. We denote the Lie algebra of Engel vector fields by $\chi(\mathcal{D})$. A vector field which preserves an even contact structure is an *even contact vector field*.

Of course a vector field which preserves $\mathcal{D}$ also has to preserve the associated even contact structure $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$. Conversely, starting from a vector field preserving $\mathcal{E}$ we can always find an Engel vector field.

LEMMA 3.45. *Let X be a vector field preserving $\mathcal{E}$. Then there is a unique section W of the characteristic line field $\mathcal{W}$ such that $\widetilde{X} = X - W$ preserves $\mathcal{D}$.*

PROOF. Let U be an open subset of M such that $\mathcal{W}$ admits a section W without zeroes on U and such that there is a 1–form β with the property

$$\mathcal{D}\big|_U = \ker(\alpha) \cap \ker(\beta) .$$

We choose a 1–form γ such that γ vanishes on $\mathcal{W}$ such that α, β, γ are linearly independent at each point of U. The characteristic foliation $\mathcal{W}$ of $\mathcal{E}$ is defined by the 3–form $\alpha \wedge d\alpha$. Since X preserves the even contact structure it also preserves the characteristic foliation. The conditions on $\widetilde{X}$ to preserve $\mathcal{D}$ are

(i) $\widetilde{X}$ preserves $\mathcal{E}$, i.e. there is a function g such that $L_{\widetilde{X}}\alpha = g\alpha$, and
(ii) $L_{\widetilde{X}}\beta = g_1\alpha + g_2\beta$ for smooth functions g_1, g_2.

$L_X\beta$ is a linear combination of α, β and γ because it vanishes on W by

$$(L_X\beta)(W) = L_X(\beta(W)) - \beta(L_X W) = 0 .$$

On the other hand $L_W\beta = i_W d\beta$ also vanishes on $\mathcal{W}$. Hence this form can also be written as $a\alpha + b\beta + c\gamma$ with differentiable functions a, b, c on U. We fix a local section Y of

$\mathcal{D}$ which is linearly independent of W. Then the Engel conditions imply $[W,Y] \notin \mathcal{D}$ but $[W,Y] \in \mathcal{E}$. Therefore

$$(L_W\beta)(Y) = -\beta([W,Y])$$

has no zeros. This implies means that h has no zeroes on U. Hence there is a unique function f with the property that

$$L_X\beta - fL_W\beta = L_{X-fW}\beta$$

is a linear combination of α, β. By definition of $\mathcal{W}$, $\widetilde{X} = X - fW$ also preserves $\mathcal{E}$. Hence $\widetilde{X}$ satisfies condition (ii), so $\widetilde{X}$ is an Engel vector field

Now we can cover M by open sets with the properties of U. By the uniqueness of the local construction we obtain a smooth global Engel vector field $\widetilde{X} = X - W$ for a unique section W of the characteristic line field. □

We assume that $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is a coorientable even contact structure with an orientable characteristic foliation. Let α be a defining form of $\mathcal{E}$. As in the case of contact structures treated in Section 2.7 we can associate the function $\alpha(X)$ to each vector field X which preserves $\mathcal{E}$. Unlike in the case of contact structures this function is not arbitrary but it has to satisfy a condition concerning its behaviour along the leaves of $\mathcal{W}$. Let h_W be the function with the property

$$L_W\alpha = h_W\alpha \; . \tag{27}$$

If X preserves $\mathcal{E}$, then $\alpha(X)$ satisfies the identity

$$\begin{aligned} L_W(\alpha(X)) &= i_W d i_X \alpha = i_W L_X\alpha - i_W i_X d\alpha \\ &= i_X i_W d\alpha = h_W \alpha(X) \; . \end{aligned}$$

DEFINITION 3.46. We define the subspace $C^\infty(\alpha)$ of $C^\infty(M)$ by

$$C^\infty(\alpha) = \left\{ f \in C^\infty(M) \;\middle|\; L_W f = h_W f \right\} \; .$$

Note that if we use $W' = gW$ with a nowhere vanishing function g then

$$L_{W'}\alpha = g h_W \alpha \; .$$

If f satisfies $L_W f = h_W f$ then this function also satisfies $L_{W'} f = h_{W'} f$. So $C^\infty(\alpha)$ depends only on the choice of α. The functions in $C^\infty(\alpha)$ play the same role for $\chi(\mathcal{D})$ as $C^\infty(H)$ for the space of contact vector fields.

THEOREM 3.47. *The map which assigns to each Engel vector field X the function $\alpha(X)$ is a bijection onto $C^\infty(\alpha)$.*

PROOF. Suppose that $\alpha(X) \equiv 0$. Then X is tangent to $\mathcal{E}$ and it has the properties which we used to define $\mathcal{W}$. Therefore it is tangent to $\mathcal{W}$. On the other hand the proof of Lemma 3.45 shows that if a vector field is tangent to $\mathcal{W}$ and non–zero, then it does not preserve $\mathcal{D}$. So $X \equiv 0$. This shows injectivity.

In order to prove surjectivity, choose a set T_i of hypersurfaces transversal to $\mathcal{W}$ such that every leaf of $\mathcal{W}$ intersects at least one of these hypersurfaces. Now let $f \in C^\infty(\alpha)$. We apply Proposition 2.7 to $f\big|_{T_i}$ and the contact form $\alpha\big|_{T_i}$ in order to obtain a contact vector field X_i on T_i. Using the flow φ_t of W we can extend X_i to an even contact vector field X_i' on the orbit of T_i.

We now show that $\alpha(X_i') = f$. As a consequence of $L_W\alpha = h_W\alpha$ and $L_W f = h_W f$ we obtain

$$
\begin{aligned}
(\alpha(X_i'))(\varphi_t(p)) &= (\alpha(\varphi_{t*}X_i))\,(\varphi_t(p)) = ((\varphi_t^*\alpha)(X_i))\,(p) \\
&= \exp\left(\int_0^t h_W \circ \varphi_s ds\right)(p) \cdot (\alpha(X_i))(p_i) \\
&= \exp\left(\int_0^t h_W \circ \varphi_s ds\right)(p) \cdot f(p) = f(\varphi_t(p))\,.
\end{aligned}
$$

for $p \in T_i$. Hence X_i' satisfies $\alpha(X_i') = f$. By Lemma 3.45 we can find Engel vector fields $\widetilde{X}_i$ by subtracting appropriate local sections W_i of $\mathcal{W}$ from X_i'.

It remains to show that the vector fields $\widetilde{X}_i$ are restrictions of one global Engel vector field. This follows from injectivity which is already proved. Hence there is a global Engel vector field $\widetilde{X}$ with $\alpha(\widetilde{X}) = f$. □

The set $C^\infty(\alpha)$ depends on the choice of α. A very simple situation occurs when we can choose α such that $L_W(\alpha \wedge d\alpha) = 0$. Since $L_W(\alpha\wedge d\alpha) = i_W\left(d\alpha^2\right)$ this assumption implies $W \in \ker(d\alpha)$. So $L_W\alpha = 0$ and $C^\infty(\alpha)$ consists of smooth functions which are constant along the leaves of $\mathcal{W}$. Whether or not such a choice of α is possible depends only on the characteristic foliation. If $\mathcal{W}$ admits a closed defining form it is said to be *volume–preserving*. Under these assumptions the Engel structure admits an Engel vector field whose properties are similar to those of Reeb vector field, cf.Lemma 2.6.

The following proposition does not require that $\mathcal{E}$ is induced by an Engel structure.

PROPOSITION 3.48. *Let $\mathcal{E}$ be a coorientable even contact structure on a 4–manifold M and let $\mathcal{W}$ be the characteristic foliation. Then the following conditions are equivalent.*

(i) *There is a defining form α for $\mathcal{E}$ and a vector field R such that $\alpha(R) = 1$ and $i_R d\alpha = 0$. The vector field R is well defined only up to addition of a vector field tangent to $\mathcal{W}$.*

If $\mathcal{E} = [\mathcal{D},\mathcal{D}]$ is induced by an Engel structure $\mathcal{D}$ then there is a unique Engel vector field with the same properties as R.

(ii) *$\mathcal{W}$ can be defined by a closed form.*

PROOF. (i) ⇒ (ii) Let α be a defining form for $\mathcal{E}$ and let R be a vector field as in (i). The characteristic foliation is tangent to the kernel of the 3-form $\alpha \wedge (d\alpha)$. Then $d\,(\alpha\wedge d\alpha) = (d\alpha)^2$ is a form of top degree on M. It is zero because $i_R((d\alpha)^2) \equiv 0$. Thus $\mathcal{W}$ can be defined by a closed form.

(ii) ⇒ (i) There is a closed defining form η for $\mathcal{W}$. Let $\widetilde{\alpha}$ be a defining form for $\mathcal{E}$. Then $\widetilde{\alpha}\wedge d\widetilde{\alpha}$ is another defining form for $\mathcal{W}$. Hence there exists a function f without zeroes such that $\eta = f\left(\widetilde{\alpha}\wedge(d\widetilde{\alpha})^{n-1}\right)$. Since both η and $-\eta$ are closed and define $\mathcal{W}$, we may assume $f = e^g > 0$. Then $\alpha = e^{f/2}\widetilde{\alpha}$ is a defining form for $\mathcal{E}$ such that

$$\alpha\wedge d\alpha = f\widetilde{\alpha}\wedge d\widetilde{\alpha} = \eta$$

is closed. Hence $(d\alpha)^2 = 0$ and the kernel of $d\alpha$ is 2–dimensional. Using the non–integrability of $\mathcal{E}$ and the properties of the characteristic foliation one can show that $\mathcal{E}\cap\ker(d\alpha) = \mathcal{W}$.

Choose a complement of $\mathcal{W}$ in ker $d\alpha$. This is also a complement of $\mathcal{E}$ in TM. In particular it is orientable. Thus we can find a nowhere vanishing section R of this complement such that $\alpha(R) = 1$. By construction we have $i_R d\alpha = 0$ so R preserves α and the even contact structure.

If $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is induced by an Engel structure we use Lemma 3.45 to obtain a Reeb vector field for the Engel structure which depends on the choice of the defining form α within the class of one–forms whose exterior derivative has rank 2. □

Let α be a contact form on a 3–manifold N. When we apply the prolongation construction discussed in Section 3.2.2 to the contact structure $\mathcal{C} = \ker(\alpha)$ we obtain an Engel structure $\mathcal{D}$ on the total space of the circle bundle $\mathrm{pr} : \mathbb{P}\mathcal{C} \to N$. Then $\mathrm{pr}^*\alpha$ is a form on $\mathbb{P}\mathcal{C}$ which defines $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$. Obviously $d\mathrm{pr}^*\alpha$ has rank two everywhere. The characteristic foliation of $\mathcal{D}$ is volume preserving since it corresponds to the fibers of a fibers bundle. Among the different lifts of the Reeb vector field R of α to $\mathbb{P}\mathcal{C}$ there is one unique lift $\widetilde{R}$ which preserves $\mathcal{D}$.

The following more interesting example is due to R. Montgomery. In [**Mo2**] it is used to show that the space of infinitesimal automorphisms of an Engel structure can have finite dimension. We use Theorem 3.47 to prove this fact.

EXAMPLE 3.49 ([**Mo2**]). Let Σ be an orientable surface of genus $g(\Sigma) \geq 2$ with a hyperbolic Riemannian metric and let $N = S_1\Sigma \subset T^*\Sigma$ be the circle bundle of 1–forms of unit length. On N there is a 1–form λ defined by

$$\lambda(V) = \alpha(\mathrm{pr}_*(V)) \text{ for } V \in T_\alpha N \ .$$

The contact structure $\ker\lambda$ is trivial because it is coorientable and it is tangent to the orientable circle bundle $S_1T^*\Sigma$.

We fix a trivialization C_1, C_2 of $\mathcal{C}$. Let R be the Reeb vector field of λ. The horizontal lifts of these vector fields to $N \times S^1$ are denoted by the same symbols. We write φ for the coordinate on the second factor of $N \times S^1$. The vector fields

$$\begin{aligned} W_\varepsilon &= \frac{\partial}{\partial\varphi} + \varepsilon R \\ X &= \cos(\varphi)C_1 + \sin(\varphi)C_2 \end{aligned}$$

span an Engel structure $\mathcal{D}_\varepsilon$ if $|\varepsilon|$ is small enough. The characteristic foliation of $\mathcal{D}_\varepsilon$ is spanned by W_ε. A defining form of $\mathcal{E}_\varepsilon = [\mathcal{D}_\varepsilon, \mathcal{D}_\varepsilon]$ is

$$\lambda_\varepsilon = \mathrm{pr}^*\lambda - \varepsilon d\varphi \ .$$

The characteristic foliation $\mathcal{W}_\varepsilon$ is volume preserving because $d\alpha_\varepsilon$ has rank two for all ε. Since

$$\begin{aligned} \alpha_\varepsilon(R) &= \alpha(R) = 1 \\ i_R d\alpha_\varepsilon &= \mathrm{pr}^*\left(i_R d\alpha\right) = 0 \ , \end{aligned}$$

R preserves $\mathcal{E}_\varepsilon$. However R does not preserve $\mathcal{D}_\varepsilon$ in general. By Lemma 3.45 we can find a vector field preserving $\mathcal{D}_\varepsilon$ if we subtract an appropriate multiple of W_ε. Since R is a Reeb vector field it preserves $\mathcal{C}$. With

$$Y = [\partial_\varphi, X] = -\sin(\varphi)C_1 + \cos(\varphi)C_2$$

we can decompose $[R, X] = fX + gY$ as linear combination of X, Y. Then

$$\left[R - \frac{g}{1+g\varepsilon}W_\varepsilon, X\right] = \left(f - \frac{fg}{1+\varepsilon g}\right)X - \left(L_X\left(\frac{g}{1+\varepsilon g}\right)\right)W_\varepsilon$$

is tangent to $\mathcal{D}_\varepsilon$. So the Engel vector field corresponding to $\widetilde{R}$ is

$$\widetilde{R} - \frac{g}{1+g\varepsilon}W_\varepsilon \ .$$

One can easily check that $1 + g\varepsilon$ never vanishes if $\mathcal{D}_\varepsilon$ is an Engel structure.

We can view the characteristic foliation of $\mathcal{D}_\varepsilon$ as the foliation on the mapping torus of the diffeomorphism $\psi_{2\pi\varepsilon}$ where ψ_t is the flow of R on N. The flow of R on N is conjugate to the geodesic flow of Σ on the circle bundle S_1TM. Since geodesic flow of a hyperbolic surface is ergodic, cf. [**Pat**], the only $\psi_{2\pi\varepsilon}$–invariant functions on N are constant. Hence $C^\infty(\alpha_\varepsilon)$ contains exactly the constant functions if $\varepsilon \neq 0$.

By Theorem 3.47 this implies that the space of diffeomorphisms preserving the Engel structure $\mathcal{D}_\varepsilon$ is one–dimensional for $\varepsilon \neq 0$. It has infinite dimension if $\varepsilon = 0$ by Proposition 3.16.

3.6. Analogues of Gray's theorem

We have already discussed Gray's theorem for contact structures in Section 2.1.1. Here we give a proof for similar theorems for even contact structures and Engel structures. These theorems and the proofs can be stated in a very similar way.

We have discussed a deformation of an Engel structure through Engel structures in Example 3.49. In this example, the characteristic foliation of $\mathcal{D}_0$ consists of closed leaves while the characteristic foliations of all other Engel structures in the family have dense leaves. Therefore the assumption on the characteristic foliation in (ii) and (iii) is really necessary.

THEOREM 3.50 (Gray, Golubev, [**Gr, Gol**]). *The following smooth families of distributions on a compact manifold M are parameterized by $t \in [0,1]$.*

(i) *Let $\mathcal{C}_t$ be a family of contact structures on an odd dimensional manifold M. Then there is an isotopy ϕ_t of M such that $\phi_{t*}\mathcal{C}_0 = \mathcal{C}_t$.*
(ii) *Let $\mathcal{E}_t$ be a family of even contact structures on an even dimensional manifold M such that the characteristic line field $\mathcal{W}_t$ is constant. Then there exists an isotopy ϕ_t of M such that $\phi_{t*}\mathcal{E}_0 = \mathcal{E}_t$.*
(iii) *Let $\mathcal{D}_t$ be a family of Engel structures on a four manifold M such that the characteristic line field $\mathcal{W}_t$ is constant. Then there is an isotopy ϕ_t on M such that $\phi_{t*}\mathcal{D}_0 = \mathcal{D}_t$.*

The proof is based on the Moser method. The first case can be found in [**Mar**]. Part (ii) of this theorem seems to be well known to the experts but we did not find a proof in the literature. The third case was treated by A. Golubev in [**Gol**] who uses defining forms. Our proof is an adapted version of the method found in [**Mar**], this has the advantage that we do not restrict ourselves to structures with global defining forms. We first explain some propositions used in all three cases.

We need a description of the tangent bundle of the n-dimensional real projective space in terms of other canonical bundles over $\mathbb{RP}^n$. The tautological bundle τ is defined by

$$\tau = \left\{(v,[x]) \in \mathbb{R}^{n+1} \times \mathbb{RP}^n | v \in [x]\right\} .$$

The other canonical bundle is the universal quotient bundle $Q = \frac{\mathbb{R}^{n+1} \times \mathbb{RP}^n}{\tau}$.

PROPOSITION 3.51. *The tangent bundle of the real projective space is canonically isomorphic to* $\mathrm{Hom}(\tau, Q)$.

PROOF. Let $\kappa : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ be the projection map. The tangent bundle of $\mathbb{R}^{n+1} \setminus \{0\}$ is isomorphic to the trivial vector bundle $\mathbb{R}^{n+1} \setminus \{0\} \times \mathbb{R}^{n+1}$ over $\mathbb{R}^{n+1} \setminus \{0\}$. We claim that

$$\begin{aligned} f : \mathrm{Hom}(\tau, Q) &\longrightarrow T\mathbb{RP}^n \\ \big(\psi : X \longmapsto [Y]\big) &\longmapsto \kappa_*(X,Y) \text{ for } X \neq 0 \end{aligned}$$

is a well defined isomorphism of vector bundles.

Let $\widetilde{Y}$ be another representative of $[Y]$. Then $\widetilde{Y} - Y$ is an element of τ i.e. a multiple of X. Therefore κ_* maps this difference to zero. Now let $\widetilde{X} = \lambda X$ with $\lambda \neq 0$. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow{\kappa} & \mathbb{RP}^n \\ \lambda\cdot \downarrow & & \downarrow \mathrm{id} \\ \mathbb{R}^{n+1} \setminus \{0\} & \xrightarrow[\kappa]{} & \mathbb{RP}^n \end{array}$$

where $\lambda\cdot$ means multiplication by λ. With $\psi(\widetilde{X}) = [\widetilde{Y}]$ we have the relation

$$\kappa_*(\widetilde{X}, \widetilde{Y}) = \kappa_*(\lambda X, \lambda Y) = \kappa_*(X, Y) \ .$$

Therefore f is well defined. It is obviously linear and preserves base points.

Let $\psi \in \ker(f)$. For all $X \in \tau$ in the same fiber as ψ and $\psi(X) = [Y]$, we have $\kappa_*(X, Y) = 0$. This implies $Y \in [X]$ and therefore $\psi(X) = 0 \in Q$. Hence $\psi = 0 \in \mathrm{Hom}(\tau, Q)$. This shows that f is injective.

Finally, both bundles have rank n, so f is an isomorphism. □

Next we introduce some bundles associated to variations of a smooth distribution on M. To this end, it is helpful to interpret a distribution of codimension one on a n–dimensional manifold as a section of the projective bundle $\mathbb{P}T^*M$. A family of distributions corresponds to a family σ_t of sections of the projective bundle. Depending on the case in question, σ_t will be a differentiable family of contact structures, even contact structures or a subdistribution of an even contact structure (this is the Engel case).

Let $\mathrm{pr} : \mathbb{P}T^*M \longrightarrow M$ be the bundle projection. The kernel V of pr_* is a subbundle of $T(\mathbb{P}T^*M)$. Elements of this bundle will be called *vertical*. Pulling back V by σ_t we obtain a family of vector bundles σ_t^*V over M.

PROPOSITION 3.52. *There is a one–to–one correspondence between sections of σ_t^*V and 1–jets of variations of σ_t.*

PROOF. We may assume $t = 0$. Let $[\sigma_s]$ be the 1-jet of a variation of σ_0 represented by σ_s for $s \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$. In order to obtain a section of σ_0^*V, let $p \in M$ and consider the differentiable curve $\sigma_s(p) \in \mathbb{P}T_p^*M$. This curve represents a tangent vector in $T_{\sigma_0(p)}\mathbb{P}T_p^*M$, the tangent vector depends only on the 1–jet of the curve. Since $\mathrm{pr}(\sigma_s(p)) = p$ for all possible s, it is a vertical tangent vector. Thus we get a section of σ_0^*V depending only on the 1-jet of σ_s at $s = 0$. We will denote this section by $\dot{\sigma}_0$.

Now let $X : M \longrightarrow \sigma_0^*V$ be a section. We view X as section of V along σ_0. Extend this to a vertical smooth vector field $\widetilde{X}$ on $\mathbb{P}T^*M$ and let Φ_s be the flow of $\widetilde{X}$. Then $\sigma_s = \Phi_s \circ \sigma_0$ is a variation of σ_0. When we produced a section of $\sigma_0^*V$from this variation as in the beginning of this proof, we obtain X. □

The following notation was already used in Proposition 3.51, nevertheless we hope that no confusion is possible. The tautological bundle τ over $\mathbb{P}T^*M$ is the real line bundle

$$\tau = \{(\alpha, [\psi]) \in \mathrm{pr}^*T^*M | \alpha \in [\psi]\} \ .$$

The universal quotient bundle is $Q = \mathrm{pr}^*T^*M/\tau$. Let $\kappa : T^*M \setminus M \longrightarrow \mathbb{P}T^*M$ be the projection map.

Denote by V_p, τ_p respectively Q_p the bundles V, τ and Q restricted to the fiber $\mathbb{P}T_p^*M$ of $\mathbb{P}T^*M$ over $p \in M$. Thus V_p, τ_p and Q_p are bundles over a real projective space. The tangent bundle of the fiber $\mathbb{P}T_p^*M$ is V_p. By Proposition 3.51, there is a canonical isomorphism between V_p and $\mathrm{Hom}(\tau_p, Q_p)$ for all $p \in M$. We can identify V and $\mathrm{Hom}(\tau, Q)$.

Observe that a fiber of τ over a point $\sigma \in \mathbb{P}T^*M$ consists exactly of the cotangent vectors of M whose kernel (this is a subspace of $T_{\mathrm{pr}(\sigma)}M$) contains the hyperplane represented by σ. The fiber Q_σ can be interpreted as dual vector space of the kernel of σ.

REMARK 3.53. In order to apply the Moser method, we need to define a *Lie derivative for sections in* $\mathbb{P}T^*M$. Let σ be such a section and let X be a smooth vector field on M and $p \in M$. Let ϕ_t be the flow of X. Since every distribution of codimension one has a local defining form, there is a neighbourhood U of p and a one–form ω on U such that $\kappa(\omega) = \sigma\big|_U$ and ω is unique up to multiplication with functions without zeroes on U. The curve $(\phi_t^*\omega)(p)$ represents a tangent vector in $T_{\omega(p)}(T^*M)$ and we define $L_X\sigma(p)$ by

$$(L_X\sigma)(p) := \kappa_*\left(\left.\frac{d(\phi_t^*\omega)(p)}{dt}\right|_{t=0}\right) \in T_{\sigma(p)}\mathbb{P}T^*M\ .$$

This does not depend on the choice of ω since for a smooth function g

$$\left.\frac{d(\phi_t^*(g\omega))(p)}{dt}\right|_{t=0} = \big(L_X g\big)(p)\omega(p) + g(p)\left(\left.\frac{d(\phi_t(\omega))(p)}{dt}\right|_{t=0}\right)$$

and ω lies in the kernel of $\kappa_* : T_{\omega(p)}(T^*M) \longrightarrow T_{\sigma(p)}\mathbb{P}T^*M$. Furthermore, $(L_X\sigma)(p)$ is vertical since

$$\mathrm{pr}\big((\phi_t^*\omega)(p)\big) = p$$

for all t. Thus $L_X\sigma$ is a well defined section of σ^*V.

LEMMA 3.54. *Let σ_t with $t \in [0,1]$ be a differentiable family of smooth sections of $\mathbb{P}T^*M$ and let X_t be a differentiable family of smooth vector fields on M. Let ϕ_t be the flow of X_t. Then the following assertions*

(i) $\phi_t^*\sigma_0 = \sigma_t$ *for all* $t \in [0,1]$

(ii) $L_{X_t}\sigma_t = \dot\sigma_t$ *for all* $t \in [0,1]$

are equivalent.

The notation $\dot\sigma_t$ was defined in the proof of Proposition 3.52.

PROOF. Both conditions are local, thus we can prove the lemma using one–forms representing σ_t on open sets. Let $p \in M$ and $t_0 \in [0,1]$. Choose a neighbourhood U of p such that there exists a differentiable family of one-forms ω_t defined on $\phi_t^{-1}U$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ with $\varepsilon > 0$. We denote $\frac{d}{dt}\omega_t$ by $\dot\omega_t$. Then

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_0}\left(\phi_t^{-1*}\omega_t\right) &= \left.\frac{d}{dt}\right|_{t=t_0}\left(\phi_t^{-1*}\left(\omega_{t_0} + (t-t_0)\dot\omega_{t_0} + o(t-t_0)\right)\right)\\ &= \phi_{t_0}^{-1*}\left(-L_{X_{t_0}}\omega_{t_0} + \dot\omega_{t_0}\right)\end{aligned}$$

on the neighbourhood U of p. This shows that $\phi_t^{-1*}\sigma_t$ is constant if and only if $L_{X_t}\sigma_t = \dot\sigma_t$ for all $t \in [0,1]$, i.e. if and only if there is a family of functions f_t such that $L_{X_t}\omega_t = \dot\omega_t + f_t\omega_t$. □

REMARK 3.55. Let ω be a one-form on M and ϕ a diffeomorphism of M. Then

$$\left(\phi^{-1*}\omega\right)(\phi_*(X_p)) = \omega(\phi_*^{-1}(\phi_*(X_p))) = \omega(X_p).$$

Thus the map

$$\begin{aligned}\ker(\omega) &\longrightarrow \ker(\phi^{-1*}\omega)\\ X &\longmapsto \phi_*X\end{aligned}$$

is a bijection. If σ_0, σ_1 are two sections of $\mathbb{P}T^*M$ such that $\phi^*\sigma_0 = \sigma_1$, then $\ker\,\sigma_0 = \phi_*(\ker\,\sigma_1)$. This relates condition (i) in Lemma 3.54 to the conditions in Theorem 3.50.

Given a family of contact structures respectively even contact structures σ_t on a compact manifold, we consider $\dot{\sigma}_t$ and look for vector fields X_t such that $L_{X_t}\sigma_t = \dot{\sigma}_t$. Then the flow of X_t is an isotopy with the properties stated in Theorem 3.50.

PROOF OF THEOREM 3.50. (i) Let σ_t be a family of contact structures on the $(2n+1)$-dimensional compact manifold M. Consider the map

$$\begin{aligned} \Gamma(\ker(\sigma_t)) = \Gamma(\mathcal{C}_t) &\longrightarrow \sigma_t^* V \\ X &\longmapsto L_X \sigma_t . \end{aligned}$$

This map is linear over smooth functions because $i_X\omega = 0$ for all forms ω representing σ_t on some open set of M. Therefore, its value at $lround0--handlesp$ depends only on X_p and it can therefore be considered as a linear map

$$\begin{aligned} \psi_t \colon \mathcal{C}_t &\longrightarrow \sigma_t^* V \\ X_p &\longmapsto L_X\sigma_t = \kappa_* \left((i_X d\omega)(p) \right), \end{aligned}$$

where ω is a 1-form on a neighbourhood of p representing σ_t.

This map is injective because $d\omega$ is non-degenerate on ker ω by the definition of contact structures. Furthermore, the rank of $\mathcal{C}_t$ is $2n$ and the rank of V is also $2n$. Hence ψ_t is an isomorphism of vector bundles for all t.

Thus for all $t \in [0,1]$, there is a unique section X_t of $\mathcal{C}_t = \ker(\sigma_t)$ such that $L_{X_t}\sigma_t = \dot{\sigma}_t$. Because $\mathcal{C}_t$ is a differentiable family of contact structures, X_t is a differentiable family of smooth vector fields. Since M was assumed to be compact, the flow ϕ_t of X_t is well defined. By Lemma 3.54, ϕ_t has the desired properties. □

PROOF OF THEOREM 3.50. (ii) Let $\mathcal{E}_t = \ker(\sigma_t)$ be a family of even contact structures on the $2n$-dimensional compact manifold M. Recall that we identify Hom(τ, Q) with V and we will interpret the fiber Q_σ over $\sigma \in \mathbb{P}T^*M$ as dual vector space of ker(σ). The bundle σ_t^*V contains all possible first order variations of σ_t, cf. Proposition 3.52, this includes variations of the distinguished line field.

We only consider variations of σ_t such that the characteristic line field of the corresponding even contact structure is constant. The subbundle $V_{\mathcal{W}} \subset V$

$$V_{\mathcal{W}} = \left\{ v \in \text{Hom}(\tau, Q) \middle| v(\omega) \equiv 0 \text{ on } \mathcal{W} \text{ for } \omega \in \tau \right\}$$

takes this restriction into account. It has codimension one in V and the sections of $\sigma_t^*V_{\mathcal{W}}$ correspond exactly to those 1–jets of variations of $\sigma_t$such that $\mathcal{W}$ is contained in all even contact structures of the variation. In particular, $\dot{\sigma}_t$ is a section of $\sigma_t^*V_{\mathcal{W}}$ for all t. Like in the case of contact structures, we consider the linear map

$$\begin{aligned} \psi_t \colon \ker(\sigma_t) = \mathcal{E}_t &\longrightarrow \sigma_t^* V_{\mathcal{W}} \\ X_p &\longmapsto L_X\sigma_t = \kappa_* \left((i_X d\omega)(p) \right). \end{aligned}$$

where ω is representing σ_t on a neighbourhood of $p \in M$. It is well defined since $(i_X d\omega)\big|_{\mathcal{W}} = 0$ by definition of $\mathcal{W}$, recall that $X \in \mathcal{E}_t$. Also by the definition of $\mathcal{W}$, the kernel of ψ_t is precisely $\mathcal{W}$.

The rank of $\mathcal{E}_t$ is $2n-1$, the rank of $V_{\mathcal{W}}$ is therefore $2n-2$. We choose a differentiable family of complements H_t of $\mathcal{W}$ in $\mathcal{E}_t$. This can be done using a constant Riemannian metric on M and taking orthogonal complements. On these complements, ψ_t is injective and both H_t and $\sigma_t^*V_{\mathcal{W}}$ have rank $2n-2$. Hence $\psi_t : \tilde{\mathcal{E}}_t \longrightarrow \sigma_t^*V_{\mathcal{W}}$ is an isomorphism for all t.

Define the vector field X_t as the unique section of H_t satisfying $L_{X_t}\sigma_t = \dot{\sigma}_t$. Because M is supposed to be compact, the flow ϕ_t of X_t is well defined and is an isotopy with

$\phi_t^*\sigma_0 = \sigma_t$ and hence $\phi_{t*}\mathcal{E}_t = \mathcal{E}_0$ by Lemma 3.54 and the remark following this lemma. □

Note that in this case we had to choose complements. If we would have made another choice for these complements we would have obtained a different vector field.

PROOF OF THEOREM 3.50. (iii) Let $\mathcal{D}_t$ be a differentiable family of Engel structures on the compact four manifold M such that the characteristic line field remains constant. By the second case of Theorem 3.50, we can choose an isotopy $\widetilde{\phi}_t$ of M such that $\widetilde{\phi}_t^*\mathcal{E}_t = \mathcal{E}_0$. Thus we may assume that the associated even contact structures of $\mathcal{D}_t$ are constant. We denote this even contact structure by $\mathcal{E}$.

Now the Engel structures $\mathcal{D}_t$ are subbundles of $\mathcal{E}$ of codimension one and they can therefore be interpreted as sections of $\widetilde{\text{pr}} : \mathbb{P}\mathcal{E}^* \longrightarrow M$. For this projective bundle we define again the vertical tangent vectors by $\widetilde{V} = \ker(\widetilde{\text{pr}}_*)$. On $\mathbb{P}\mathcal{E}^*$, there is the tautological bundle $\widetilde{\tau}$ defined by

$$\widetilde{\tau} = \{(\alpha, [\psi]) \in \widetilde{\text{pr}}^*\mathcal{E}^* | \alpha \in [\psi]\}$$

and the universal quotient bundle $\widetilde{Q} = \widetilde{\text{pr}}^*\mathcal{E}/\widetilde{\tau}$.

Again we can identify $\widetilde{V}$ with $\text{Hom}(\widetilde{\tau}, \widetilde{Q})$ by Proposition 3.51. Let $\widetilde{\kappa} : \mathcal{E}^* \setminus M \longrightarrow \mathbb{P}\mathcal{E}^*$ be the projection map. In order to finish the proof, we will need some refinements of tools we have already used. The first concerns the definition of the Lie derivative in Remark 3.53, the second refinement is a special version of Lemma 3.54. Although we will use the notation adapted to our case, the refinements work in general and not only in the case of even contact structures.

REMARK 3.56. Let $\mathcal{E}$ be a distribution of corank one in TM and Y a vector field on M which leaves $\mathcal{E}$ invariant. We consider smooth sections $\widetilde{\sigma}$ of $\mathbb{P}\mathcal{E}^*$. The Lie derivative $L_Y\widetilde{\sigma}$ can be defined as follows. Fix a complement $\mathcal{F}$ of $\mathcal{E}$ in TM. On a small open neighbourhood U of $p \in M$ the section $\widetilde{\sigma}\big|_U$ can be represented be a one–form ω on $\mathcal{E}\big|_U$. We extend ω from $\mathcal{E}\big|_U$ to a one–form $\hat{\omega}$ defined on $TM\big|_U$ by requiring $\widehat{\omega}\big|_{\mathcal{F}} = 0$. We define a Lie derivative by

$$L_Y\tilde{\sigma}(p) = \tilde{\kappa}_*\left((L_X\hat{\omega})\big|_{\mathcal{E}_p}\right) \in T_{\tilde{\sigma}(p)}\mathbb{P}\mathcal{E}^* \ .$$

This does not depend on the choice of $\mathcal{F}$ since for a section X of $\mathcal{E}\big|_U$ we have

$$(L_Y\hat{\omega})(X) = L_Y(\hat{\omega}(X)) - \hat{\omega}(L_YX) = L_Y(\omega(X)) - \omega(L_YX) \ .$$

The last term vanishes since Y preserves $\mathcal{E}$. Hence $L_Y\widetilde{\sigma}\big|_U$ does not depend on the choice of the extension $\widehat{\omega}$. The proof that this definition does not depend on the choice of ω is exactly the same as in Remark 3.53.

LEMMA 3.57. *Let σ_t with $t \in [0,1]$ be a differentiable family of smooth sections of $\mathbb{P}\mathcal{E}^*$ and let X_t be a continuous family of smooth vector fields on M such that X_t preserves $\mathcal{E}$. Let ϕ_t be the flow of X_t. Then the following assertions are equivalent.*

(i) $\phi_t^*\sigma_0 = \sigma_t$ *for all* $t \in [0,1]$.

(ii) $L_{X_t}\sigma_t = \dot{\sigma}_t$ *for all* $t \in [0,1]$.

The definition of $\dot{\sigma}_t$ for sections in $\mathbb{P}\mathcal{E}^*$ is similar to the definition in the case of sections of $\mathbb{P}T^*M$.

PROOF. We first fix a complement of $\mathcal{E}$ in TM and we thereby obtain a smooth family of local one–forms $\hat{\omega}_t$ of locally defined representatives $\omega_t \in \Gamma\mathcal{E}^*$ of $\widetilde{\sigma}_t$. (We extend ω_t by zero on the complement of $\mathcal{E} \subset TM$.) In this situation, we can do the same calculation as in the proof of Lemma 3.54 with $\widehat{\omega}_t$ instead of ω_t. The same arguments as in Lemma 3.54 prove the desired result. □

We only allow variations of the Engel structure such that $\mathcal{W} \subset \mathcal{D}_t$ for all t. So, as in the proof of the second case of Theorem 3.50, we will consider only a subbundle of $\widetilde{V}$, namely the bundle $\widetilde{V}_{\mathcal{W}}$ defined by

$$\widetilde{V}_{\mathcal{W}} = \left\{ \widetilde{v} \in \mathrm{Hom}(\widetilde{\tau}, \widetilde{Q}) \middle| \widetilde{v}(\widetilde{\omega}) = 0 \text{ on } \mathcal{W} \text{ for } \widetilde{\omega} \in \widetilde{\tau} \right\} \ .$$

The rank of $\widetilde{V}_{\mathcal{W}}$ is one. Since $\mathcal{W} \subset \mathcal{D}_t = \ker(\widetilde{\sigma}_t)$ and all sections of $\mathcal{W}$ preserve $\mathcal{E}$, the map

$$\begin{aligned} \tilde{\psi}_t : \mathcal{W} &\longrightarrow \widetilde{\sigma}_t^* \widetilde{V}_{\mathcal{W}} \\ W &\longmapsto L_W \widetilde{\sigma}_t = \widetilde{\kappa}_* \left((i_W d\widetilde{\omega}) \big|_{\mathcal{E}} \right) \end{aligned}$$

is well defined as a map of vector bundles (cf. the proof of the first case of Theorem 3.50). Note that we restrict ourselves to $\mathcal{W}$. We could have taken $\mathcal{D}_t \supset \mathcal{W}$ as domain, but sections $\mathcal{D}_t$ do not preserve $\mathcal{E}$ while sections of $\mathcal{W}$ do. The map $\widetilde{\psi}_t$ is surjective because $\mathcal{D}_t$ is an Engel structure. Both bundles have the same rank.

Thus $\widetilde{\psi}_t$ is an isomorphism of vector bundles. For every $t \in [0, 1]$ we can find a unique section W_t of $\mathcal{W}$ such that $L_{W_t} \widetilde{\sigma}_t = \dot{\widetilde{\sigma}}_t$. The flow of W_t has the desired properties by Lemma 3.57. □

CHAPTER 4

Round handles

A round handle of dimension n and index k is $R_k = D^k \times D^{n-k-1} \times S^1$. Round handle decompositions of manifolds were used by D. Asimov ([**As1, As2**]) for the study of flow manifolds. A flow manifold is a manifold with a non–singular vector field W which is transversal to the boundary. In particular Engel manifolds with transversal boundary are flow manifolds if the characteristic foliation is orientable. We will always write $\partial_+ M$ for those boundary components where W points outwards and $\partial_- M$ for the remaining boundary components.

D. Asimov shows in [**As1**] that every flow manifold can be decomposed into round handles and uses round handle decompositions for the construction of vector fields without zeroes which are structurally stable. One of the most important results in [**As1**] is Theorem 4.6 which says that every manifold of dimension $n \neq 3$ with vanishing Euler characteristic admits a round handle decomposition and a non–singular Morse Smale vector field. J. Morgan showed that the analogous statement is wrong in dimension 3, cf. [**Mor**].

We sketch a proof of Theorem 4.6 using the close relation between ordinary handles and round handles, cf. Lemma 4.8. For the proof of the existence theorem in Chapter 6 we will use round handle decompositions of closed parallelizable manifolds with only one round handle of index 3. Starting from a convex contact structure on a 3–manifold N we construct an Engel structure together with a round handle decomposition of $N \times S^1$. The characteristic foliation of this Engel structure is transversal to the boundary of submanifolds which consist of the round handles. The same method will be used in the construction of model Engel structures on round handles. In Section 4.1.2 we use it in the discussion of a question of J. Adachi, cf. [**Ad**].

In Section 4.2 we describe model Engel structures on round handles. Because of their symmetry we discuss round handles of index 0 and 3 respectively 1 and 2 together. The case of index $0, 3$ in Section 4.2.1 uses a concrete example of a convex contact structure on S^3. In Section 4.2.2 and Section 4.2.3 we construct model Engel structures on round handles of index 1 and 2 which are compatible with a symplectic structure ω, i.e. their characteristic foliation is spanned by a vector field W such that $L_W\omega$ is a constant multiple of ω.

This leads to differences between our first and our second construction. In our first construction there is a one–to–one correspondence between closed leaves of the characteristic foliation and round handles. This is not the case in the second construction. Moreover, in our second construction, the overtwistedness of the contact structures on the boundaries will be important. In the first construction all contact structures on transversal boundaries will be tight.

Some of the properties of the model Engel structures are summarized in Lemma 4.24 and Lemma 4.26. In Section 4.3 we discuss similarities between model Engel structures on round handles of index 1 and 2. In particular we explain how to remove corners when we attach round 1–handles. If we cut of suitable symmetric neighbourhoods of $\partial_+ R_1$ res $\partial_- R_2$, then the smoothened boundaries are again transversal to the characteristic foliation and we can compare the contact structure and the homotopy class of the intersection line field on the new boundaries of R_1 .

These similarities will be used in our first construction of Engel structures in Chapter 5. The model Engel structures on round 1–handles will also be used in our second construction in Chapter 6. In Chapter 6 we will discuss more model Engel structures on round handles of index 2 and 3.

4.1. Generalities

We have shown above that a manifold carrying an Engel structure also admits a line field, namely the characteristic foliation of the Engel structure. Hence the Euler characteristic of M has to vanish. We now look for decompositions of manifolds which reflect this particular property of Engel manifolds.

DEFINITION 4.1. A *flow manifold* is a pair $(M, \partial_- M)$ where M is a smooth connected manifold and $\partial_- M$ is the union of some connected components of the boundary such that there is a vector field V without zeroes on M pointing inward along $\partial_- M$ and outward along $\partial_+ M := \partial M \setminus \partial_- M$. (The cases $\partial M = \emptyset, \partial_- M = \emptyset, \partial_+ M = \emptyset$ are allowed.)

For the proof of the following lemma we refer to [**As1**].

LEMMA 4.2. *$(M, \partial_- M)$ is a flow manifold if and only if $\chi(M) = \chi(\partial_- M) = \chi(\partial_+ M)$.*

Recall that a handle of dimension n and index $k \in \{0, \ldots, n\}$ is defined to be $h_k = D^k \times D^{n-k}$. We write

$$\partial_- h_k = \partial D^k \times \quad D^{n-k}$$
$$\partial_+ h_k = \quad D^k \times \partial D^{n-k} \, .$$

Suppose we have an Engel manifold with transversal boundary. If we attach a handle $h_k = D^k \times D^{4-k}$ of index $k \in \{1, 2, 3, 4\}$ to M, the Euler characteristic changes by $(-1)^k$. Therefore there is no Engel structure on $M \cup H_k$ such that the boundary of $M \cup H_k$ is transversal. In view of the relative simplicity of Engel manifolds on transversal boundaries it is nevertheless desirable to maintain this property.

So instead of attaching handles one should attach building blocks to M without changing the Euler characteristic. Round handles have this property. They were first studied in [**As1, As2**]. In this section we explain the results of [**As1**] we are going to use later.

DEFINITION 4.3. A *round handle* of dimension n and index $k \in \{0, \ldots, n-1\}$ is

$$R_k = D^k \times D^{n-k-1} \times S^1 \, .$$

The boundary ∂R_k contains two subsets

$$\partial_- R_k = \partial D^k \times \quad D^{n-k-1} \times S^1$$
$$\partial_+ R_k = \quad D^k \times \partial D^{n-k-1} \times S^1 \, .$$

We write $x_1, \ldots, x_k$ for the coordinates on D^k, $y_1, \ldots, y_{n-k-1}$ for the coordinates on D^{n-k-1} and t for the usual parameterization of S^1.

Suppose M is a manifold of dimension n and let $\varphi : \partial_- R_k \to \partial M$ be an embedding. Consider the equivalence relation on $M \cup R_k$ generated by $x \sim \varphi(x)$ for $x \in \partial_- R_k$. Just like in the case of ordinary handles, the quotient space

$$\widetilde{M} = M \cup R_k / \sim = M \cup_\varphi R_k$$

is a manifold with corners. The corners correspond to $\partial D^k \times \partial D^{n-k-1} \times S^1 \subset R_k$. There is a canonical procedure to obtain manifolds with boundary from manifolds with corners which is described in [**Dou**]. However this method does not work well when one wants to preserve structures on the boundary. In our situation it will be easier to cut off a part of R_k and we will describe the procedure for the attaching of round one–handles in detail later. The attaching of round handle of index 2, for round 3 handles there are no corners. We say that the resulting manifold $\widetilde{M}$ is obtained from M by attaching a round handle of index k.

Attaching round handles to a flow manifold $(M, \partial_- M)$ one can easily obtain new flow manifolds. Fix a vector field V on M with the properties of the definition above and consider the vector field

$$W_k = -\sum_{l=1}^{k} x_l \frac{\partial}{\partial x_l} + \sum_{l=1}^{n-k-1} y_l \frac{\partial}{\partial y_l} + \frac{\partial}{\partial t}$$

on R_k. Notice that W_k points outward along $\partial_+ R_k$ and inward along $\partial_- R_k$. For a given attaching map $\varphi : \partial_- R_k \to \partial_+ M$ one can refine the attaching procedure such that the vector field W_k extends V to a vector field on $\widetilde{M}$ which shows that $(\widetilde{M}, \partial_- M)$ is a flow manifold. The corners can be smoothened by the standard procedure from [**Dou**] while keeping the vector field smooth and transversal to the boundary. Of course one can also attach R_k by a map $\partial_- R_k \to \partial_- M$ and use $-W_k$ to show that the resulting manifold $(\widetilde{M}, \partial\widetilde{M} \setminus \partial_+ M)$ is again a flow manifold.

If M is even–dimensional every component of the boundary of M has odd dimensionen. Therefore the Euler characteristic of $\partial_- M$ vanishes. Thus $(M, \partial_- M)$ is a flow manifold if and only if $\chi(M) = 0$, independently of the choice of $\partial_- M \subset \partial M$. This shows that attaching a round handle to an even–dimensional flow manifold again yields a flow manifold (with vanishing Euler characteristic). We shall see in Lemma 4.8 that attaching round handles to a compact manifold does never change the Euler characteristic.

REMARK 4.4. Let $M' = M \cup R_k$ be a n–dimensional manifold and $k \leq n-2$. We attach a round handle of index l to $\partial_+ M'$ using an embedding $\varphi : \partial_- R_l \longrightarrow \partial_+ M'$. Then we can isotope φ such that $\varphi\left(\partial D^l \times \{0\} \times S^1\right)$ is transversal to $\{0\} \times \partial D^{n-k-1} \times S^1$. If $l < k$

$$\dim(\partial D^l \times \{0\} \times S^1) + \dim(\{0\} \times \partial D^{n-k-1} \times S^1) = n - 1 - (k - l)\,,$$

hence φ can be isotoped such that $\varphi(\partial D^l \times \{0\} \times S^1)$ is disjoint from $\{0\} \times \partial D^{n-k-1} \times S^1 \subset \partial_+ R_k$. With the flow of a smooth vector field which points radially away from $\{0\} \times \partial D^{n-k-1} \times S^1$, we can isotope φ further to obtain an attaching map $\widetilde{\varphi}$ whose image does not meet $\partial_+ R_k$. Thus $(M \cup R_k) \cup_\varphi R_l$ is diffeomorphic to $(M \cup_{\widetilde{\varphi}} R_l) \cup R_k$. Thus we can rearrange a given round handle decomposition of a manifold such that the round handles are ordered according to their index. Notice that contrary to the case of ordinary handles, two round handles of the same index cannot be interchanged in general.

DEFINITION 4.5. If M is obtained from the disjoint union of finitely many round handles of index 0 by attaching round handles of higher index successively, i.e.

$$M = \left(\dots\left(\bigcup R_0\right) \cup_{\varphi_1} R_{\beta_1} \dots\right) \cup_{\varphi_k} R_{\beta_k}$$

with $\beta_i \in \{1, \dots, n-1\}$ for $i = 1, \dots, k$ we say that we have a *round handle decomposition* of M.

If a closed manifold M admits a round handle decomposition then the Euler characteristic of M has to vanish because we can use the round handle decomposition for the construction of a non–singular vector field. If the dimension of M is 2, one can prove the converse direction using explicit decompositions of the Klein bottle and the torus into round handles. The following theorem treats manifolds whose dimension is at least four.

THEOREM 4.6 (Asimov, [**As1**]). *A closed, connected manifold of dimension* $n \geq 4$ *admits a decomposition into round handles if and only if* $\chi(M) = 0$. *This decomposition can be chosen such that there is only one round* 0*–handle and one round* 3*–handle.*

The statement is trivial in dimension one and it can be checked directly in dimension two, i.e. for T^2 and the Klein bottle. For manifolds of dimension 3, the analogous statement is wrong.

THEOREM 4.7 (Morgan, [**Mor**]). *Let* $P \not\simeq S^2 \times S^1$ *be an orientable prime* 3*–manifold.* M *admits a decomposition into round handles if and only if* P *is the union of non–trivial Seifert spaces attached to one another along components of their boundaries.*

The manifolds formed from Seifert spaces form a special class of 3–manifolds; they were classified by Waldhausen [**Wal**]. The case of non–prime manifolds P is also solved in [**Mor**] when no summand of the decomposition of M is diffeomorphic to $S^2 \times S^1$. Moreover Morgan also shows that $M\#k(S^2 \times S^1)$ admits a round handle decomposition if k is large enough.

One of the ingredients of the proof of Theorem 4.6 is the fact that every smooth manifold admits a decomposition into ordinary handles. This can be shown using Morse theory.

Now let $\varphi_k : \partial_- h_k \to \partial M$ and $\varphi_l : \partial_- h_l \to \partial M$ be attaching maps for ordinary handles. We say that h_k and h_l are attached *independently* if φ_k and φ_l have disjoint images.

The second important ingredient of the proof of Theorem 4.6 is the following lemma.

LEMMA 4.8 (Asimov, [**As1**]). *Let* M *be a manifold and* $k \geq 1$. *Suppose that* $\widetilde{M}$ *is obtained from* M *by attaching a handle of index* k *and a handle of index* $k+1$ *independently to the same connected component of* ∂M. *(If* $k = 1$ *it suffices that only one connected component of* $\partial_- h_1$ *gets mapped to same connected component of* ∂M *as* $\partial_- h_2$*.)*

Then $\widetilde{M}$ *is diffeomorphic to a manifold obtained from* M *by attaching a round handle of index* k.

Conversely a round handle of index k *can be decomposed into a handle of index* k *and a handle of index* $k+1$. *Attaching a round handle to a compact manifold does not change the Euler characteristic.*

This lemma allows us to obtain round handle decompositions of manifolds with a given decomposition into ordinary handles. This will be useful for the construction of explicit examples of Engel manifolds, so we sketch the proof Lemma 4.8.

SKETCH OF PROOF FOR LEMMA 4.8. Fix $p \in \varphi_{k+1}(S^k \times \{0\})$ and an embedded path $c : I \longrightarrow \partial(M \cup_{\varphi_k} h_k)$ with the properties

(i) $c(0) = p$
(ii) $c(1) = \varphi_k((q_1, q_2))$ with $(q_1, q_2) \in S^{k-1} \times S^{n-k-1} \subset \partial_- h_k$
(iii) $c(1/2) = \varphi_k((-q_1, q_2))$
(iv) c does not meet $\varphi_{k+1}(S^k \times \{0\})$ or $\varphi_k(S^{k-1} \times S^{n-k-1})$ at other times.
(v) $\dot{c}$ is orthogonal to $\varphi_{k+1}(S^k \times \{0\})$ and $\varphi_k(S^{k-1} \times S^{n-k-1})$ with respect to some metric.

Such a path exists because h_k and h_{k+1} are attached independently. Now fix a complete vector field C on $\partial(M \cup_{\varphi_k} h_k)$ extending $\dot{c}$. For $\varepsilon > 0$ consider the flow $\psi_{1+\varepsilon}$ of C at time $1+\varepsilon$. Let $\overline{\varphi}_{k+1} = \psi_{1+\varepsilon} \circ \varphi_{k+1}$. Since $\overline{\varphi}_{k+1}$ is isotopic to φ_{k+1} we obtain diffeomorphic manifolds when we attach h_{k+1} using φ_{k+1} or $\overline{\varphi}_{k+1}$. So from now on we use $\overline{\varphi}_{k+1}$. The effect of this operation is that we have dragged the attaching sphere of h_{k+1} over $\partial_+ h_k$. Now consider

$$h_k \cup h_{k+1} \subset (M \cup_{\varphi_k} h_k) \cup_{\overline{\varphi}_{k+1}} h_{k+1} \, .$$

This set can be identified with a round handle of index k. We explain this for $n = 3$ and $k = 1$. The general case is carried out in [**As1**].

We focus on the situation near $c(1)$. The shape of the sets $\varphi_1(\partial_- h_1)$ and $\overline{\varphi}_2(\partial_- h_2) \cap \partial_+ M$ is drawn in Figure 1. The disc is a connected component of $\varphi_1(\partial_- h_1)$ and the

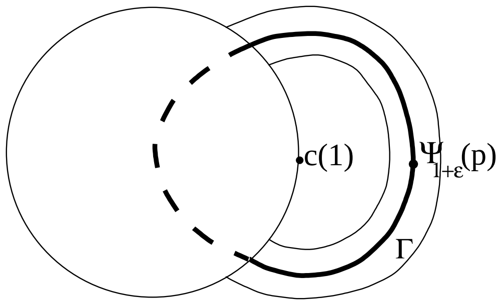


FIGURE 1.

bold arc Γ represents the part of the image of the attaching circle of h_2 under $\overline{\varphi}_2$ which is contained in a neighbourhood of $c(1)$ in ∂M. If one connects the endpoints of Γ in the disc $\varphi_1(\partial_- h_1)$ as indicated by the dashed curve, we can identify the union of the two regions depicted in the figure with $D^1 \times S^1$. This corresponds to one connected component of $\partial_- R_1 = \partial D^1 \times D^1 \times S^1$. This identification extends to an identification of $h_1 \cup h_2$ with R_1.

Now we show how to decompose a round handle of index k into two ordinary handles. Consider $R_k = D^k \times D^{n-k-1} \times S^1$. The last factor S^1 can be decomposed into a one–dimensional 0–handle and a one–dimensional 1–handle, both are diffeomorphic to D^1. Thus R_k can be decomposed into $h_k = D^k \times (D^{n-k-1} \times D^1)$ and $h_{k+1} = (D^k \times D^1) \times D^{n-(k+1)}$. In Figure 2 we give a picture of the case $n = 3, k = 1$. The attaching circle of the 2–handle corresponds to the thick line. The Euler characteristic of a compact manifold

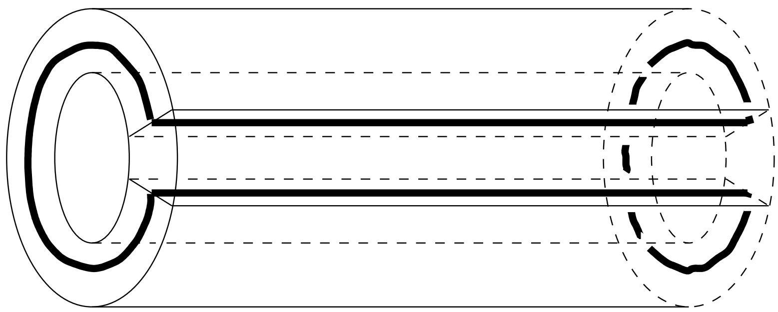


FIGURE 2.

is the difference between the number of handles with even index and the number of handles with odd index in any decomposition of the manifold into ordinary handles. If we attach a round handle of index k this corresponds to the introduction of two handles of consecutive index. Thus the Euler characteristic does not change when one attaches a round handle to a compact manifold. □

For a detailed proof of Theorem 4.6 we refer to [**As1**]. We just sketch the argument in order to show that closed manifolds of dimension at least 4 admit a round handle decomposition with only one round 0–handle and one round 3–handle.

PROOF OF THEOREM 4.6. In order to obtain a round handle decomposition of M with the desired property, we start with a decomposition of M into ordinary handles which contains exactly one handle h_0 of index 0 and exactly one 4–handle. Since the manifold M is not simply connected, there is a handle h_1 of index one. Since M is orientable, the attachment of the first handle of index 1 yields a round 0–handle. If we apply Lemma 4.8 after introducing sufficiently many pairs of cancelling handles of index 2 and 3 respectively 3 and 4, we obtain a decomposition of M into one round 0–handle, several round handles of index 1 together with some ordinary handles of index $2, 3, 4$.

If we introduce a cancelling handle pair of index $2, 3$ we can form a round handle of index 3 from the 4–handle together with the 3–handle we just introduced. This is completely analogous to the formation of a round 0–handle from a pair ordinary 0– and 1–handles.

Now we have obtained a decomposition of M into exactly one round handle of index 0 and 3, several round handles of index $1, 2$ and some ordinary handles of index $2, 3$. These handles are attached to the boundary of the union of all round handles of index $0, 1$. Since the Euler characteristic of M vanishes, there is an equal number of ordinary handles of index $2, 3$. Introducing cancelling pairs of handles of index 2 and 3, one can obtain a round handle decomposition of M without ever introducing an additional 0–handle.

Thus we end up with a round handle decomposition of M with exactly one round handle of index 0 and 3. □

On a manifold with a decomposition into round handles we can construct a non–singular vector field using the vector fields X_k on round handles of index k. Since vector fields similar to the ones occurring this way will appear in the construction of Engel manifolds, we now explain dynamical properties of these vector fields. We first recall some definitions. These can be found e. g. in [**PSm, Sm, Har**].

Let V be a complete vector field on a manifold and let ψ_t be the flow of V.

DEFINITION 4.9. The *non–wandering set* $\Omega(V)$ of V consists of those points p of M with the property that for every neighbourhood U of p and every $T \in \mathbb{R}$ there exists $t > T$ such that $\psi_t(U) \cap U \neq \emptyset$.

For example, every closed orbit of V is contained in $\Omega(V)$. If $X \in TM$ we write $\langle X \rangle$ for the vector space spanned by X.

DEFINITION 4.10. A closed orbit of V of period $T > 0$ is called hyperbolic if the map

$$D\psi_T : \frac{T_pM}{\langle V(p) \rangle} \longrightarrow \frac{T_pM}{\langle V(p) \rangle}$$

has no (complex) eigenvalue with absolute value 1.

For the definition of the stable and unstable manifold of a hyperbolic periodic orbit, as well as for existence and uniqueness results we refer to [**PSm, Sm**].

DEFINITION 4.11. A nowhere vanishing complete vector field V on a manifold M is called *non–singular Morse–Smale vector field* if

(i) $\Omega(V)$ consists of a finite number of closed hyperbolic orbits
(ii) the stable and unstable manifolds of the periodic orbits intersect transversely.

DEFINITION 4.12. A vector field V on a compact manifold M is *structurally stable* if for every vector field V' which is sufficiently close to V in the C^1–topology there exists a homeomorphism of M mapping flow lines of V to flow lines of V'.

Non–singular Morse–Smale vector fields on closed manifolds are structurally stable, cf. [**PSm**]. On a closed manifold with a round handle decomposition we thus can construct a structurally stable vector field without zeroes using X_k on round handles of index k: By definition of X_k, every closed orbit of the vector field constructed this way is hyperbolic. To each round handle corresponds precisely one closed orbit and by compactness we have finitely many closed orbits. The transversality condition is easily achieved by small perturbations of the attaching maps of the round handles.

COROLLARY 4.13 (Asimov, [**As1**]). *Every manifold of dimension $n \geq 4$ with vanishing Euler characteristic admits a structurally stable non–singular vector field.*

Before we give examples of Engel manifolds with a round handle decomposition related to the Engel structure, we want to mention two other applications of round handle decompositions.

THEOREM 4.14 (Asimov, [**As2**]). *Let V be a non–singular vector field on the flow manifold $(M, \partial_- M)$ with $\dim(M) \geq 4$. Then V is homotopic through non–singular vector fields to a non–singular Morse–Smale vector field pointing inward along $\partial_- M$ and outward along $\partial_+ M$.*

Starting from a round handle decomposition Thurston constructed foliations and proved the following celebrated theorem.

THEOREM 4.15 (Thurston, [**Thu1**]). *A closed manifold admits a foliation of codimension one if and only if its Euler characteristic vanishes.*

4.1.1. Engel structures from convex contact structures. In this section we construct first examples of Engel structures which are related to a round handle decompositions of the underlying manifold. Recall the following definition from [**ElG**].

DEFINITION 4.16. A contact structure $\mathcal{C}$ on a manifold M is *convex* if there is a proper Morse function $f : M \to [0, \infty)$ and a complete vector field V such that

(i) V preserves $\mathcal{C}$,
(ii) V is a pseudo–gradient for f, i.e. there is a Riemannian metric and a positive function s on M such that

$$L_V f \geq s \|df\|^2 .$$

Obviously, the zeroes of V are critical points of f. This can be used to show that the zeroes of V are hyperbolic fixed points of the flow of V.

E. Giroux proved in [**Gir1**] that on every oriented manifold of dimension three there is a positive convex contact structure. In order to show this, a suitable handle–decomposition of M is used. Let $\mathcal{C}$ be a contact structure on the 3–manifold M. Suppose that $\mathcal{C}$ is trivial as vector bundle and that there is a vector field V as in Definition 4.16 which was constructed in [**Gir1**]. In particular, let $h = D^k \times D^{3-k}$ be a (standard) handle of index k contained in the decomposition of M. Then V enters h through the boundary component $S^{k-1} \times D^{3-k}$ and leaves h through $D^k \times S^{2-k}$. Each zero of V is in the center of a handle with the same index as the index of the zero of V.

Consider $S^1 \times M$ with the round handle decomposition consisting of products of S^1 with handles contained in the decomposition of M. We fix a trivialization X, Y of $\mathcal{C}$ and we denote the horizontal lifts of V, X, Y to $S^1 \times M$ by the same letters. Using a calculation analogous to (23)from the proof of Theorem 3.41 one can prove the following proposition.

PROPOSITION 4.17. *In this situation, the distribution $\mathcal{D}_k$ on $S^1 \times M$ spanned by*

$$W = \frac{\partial}{\partial t} + V \text{ and } X_k = \cos(kt)X + \sin(kt)Y \tag{28}$$

is an Engel structure on $S^1 \times M$ if we choose V small enough and $k \in \mathbb{Z} \setminus \{0\}$. The characteristic foliation of $\mathcal{D}_k$ is spanned by W. If $k = 0$ we obtain an Engel structure only of $[V, X]$ is linearly independent from X.

Note that we can multiply V with positive real numbers. The characteristic line field of $\mathcal{D}_k$ is spanned by W and we use this to orient the characteristic line field of $\mathcal{D}_k$. Here we use the fact that V preserves the contact structure $\mathcal{C}$. This is a simple instance of the proof of Theorem 3.41 (iv).

Thus we obtain an Engel structure on $S^1 \times M$ together with a decomposition of this manifold into round handles. Each of these round handles contains exactly one closed orbit corresponding to the zero of V in the corresponding handle of the decomposition of M. The characteristic line field is spanned by a vector field whose closed flow lines are hyperbolic. The oriented characteristic line field enters a round handle $S^1 \times D^k \times D^{3-k}$ through $S^1 \times S^{k-1} \times D^{3-k}$ and leaves it through $S^1 \times D^k \times S^{2-k}$.

4.1.2. A question of J. Adachi. At the end of [**Ad**] one can find the following question: *Let $\mathcal{C}_0, \mathcal{C}_1$ be contact structures on a 3–manifold N, which are not isomorphic to each other. Is there an Engel structure $\mathcal{D}$ on $N \times [0,1]$ whose characteristic foliation is transversal to $N \times \{0\}$ and $N \times \{1\}$ and which induces the given contact structures $\mathcal{C}_i$ on $N \times \{i\}$ for $i = 0, 1$?*

For topological reasons the answer to this question is no in general.

LEMMA 4.18. *If $M = N \times I$ is an Engel manifold with transversal boundary such that for an orientation of $\mathcal{W}$, we have $\partial_- M = N \times \{0\}$ and $\partial_+ M = N \times \{1\}$. Then the induced contact structures $\mathcal{C}_i, i = 0, 1$ on $N \times \{i\} \simeq N$ are homotopic as plane fields on N*

PROOF. Since N is an orientable 3–manifold, its tangent bundle is trivial. Fix a framing X, Y, Z of TN. Then X, Y, Z, ∂_t is a framing of $N \times I$. We fix a Riemannian metric such that this framing is orthonormal.

Recall from [**HH**] that the Grassmann manifolds of oriented planes in $\mathbb{R}^3$ respectively $\mathbb{R}^3$ are $\mathrm{Gr}_2(3) \simeq S^2$ respectively $\mathrm{Gr}_2(4) \simeq S^2 \times S^2$. The inclusion $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ induces the diagonal map

$$\Delta : \mathrm{Gr}_2(3) \simeq S^2 \longrightarrow S^2 \times S^2 \simeq \mathrm{Gr}_2(4) .$$

Let $\mathcal{C}_0$ on $N \times \{0\}$ and $\mathcal{C}_1$ on $N \times \{1\}$ be two contact structures and $\mathcal{D}$ and Engel structure on $N \times I$ such that the induces contact structure $\mathcal{C}_i$ on $N \times \{i\}$ for $i = 0, 1$. Without loss of generality we can assume that the characteristic foliation of $\mathcal{D}$ is tangent to ∂_t on neighbourhoods of $\partial(N \times I)$. Let $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$.

When we view $\mathcal{C}_0$ and $\mathcal{C}_1$ as maps from N to $\mathrm{Gr}_2(4)$ the orthogonal complement of $\mathcal{W}$ in $\mathcal{E}$ induces a homotopy

$$H : N \times I \longrightarrow \mathrm{Gr}_2(4)$$

between $\mathcal{C}_0$ and $\mathcal{C}_1$. The composition of H with the projection of pr of $S^2 \times S^2$ to the first factor $S^2 \simeq \mathrm{Gr}_2(3)$ is the desired homotopy between $\mathcal{C}_0$ and $\mathcal{C}_1$ viewed as distributions on N. □

If $\mathcal{C}_0$ and $\mathcal{C}_1$ are homotopic as plane fields, they can still be different as contact structures, for example if $\mathcal{C}_0$ is tight and $\mathcal{C}_1$ is overtwisted. We give an example showing that in this situation, $\mathcal{C}_0, \mathcal{C}_1$ can be cobordant in Adachi's sense.

EXAMPLE 4.19. Let r, φ, z be cylindrical coordinates on $\mathbb{R}^3$. Consider the contact form

$$\alpha = \cos(r^2)dz - \sin(r^2)d\varphi .$$

Let $S^2(r)$ be the sphere of radius r around the origin. The restriction of α to $S^2(r)$ defines a one–dimensional foliation with two elliptic singularities at $z = \pm r$.

If $r^2 < \pi/2$, there are no closed leaves. When $r^2 = \pi/2$, there is one closed leaf and if $r^2 > \pi/2$, there are at least two closed leaves. These bound overtwisted discs in $S^2(r)$. Figure 3 shows the singular foliation for $r^2 = 3\pi/2$. Let $S^2_+ = S^2(\sqrt{3\pi/2})$. Using a theorem form the theory of contact structures (cf. Theorem 2.25) one can show

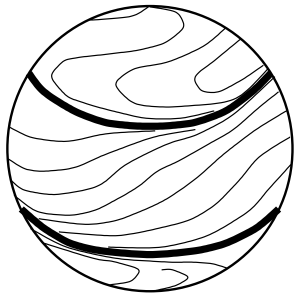


FIGURE 3.

that there is a contact vector field V transversal to these spheres. Without loss of generality we choose V such that it has compact support and positive radial component. Moreover we assume that V is invariant under rotations around the z–axis. Fix a trivialization C_1, C_2 of the contact structure on $\mathbb{R}^3$. If $\varepsilon > 0$ is small enough,

$$W = \frac{\partial}{\partial t} + \varepsilon V \text{ and } X_k = \cos(kt)C_1 + \sin(kt)C_2$$

span an Engel structure on $R^3 \times S^1$. The characteristic foliation is spanned by W. Consider the submanifold

$$M = \left\{1 \leq r \leq \frac{3\pi}{2}\right\} \simeq S^2 \times [0,1] \times S^1$$

of $\mathbb{R}^3 \times S^1$ for an integer $k \neq 0$. It carries an Engel structure and the boundary is transversal to the characteristic foliation. The contact structures on

$$\partial_+ M = S^2_+ \times S^1$$
$$\partial_- M = S^2(1) \times S^1$$

are non–isomorphic: The contact structure on $\partial_- M$ is tight. On the other hand, the contact structure on $\partial_+ M$ is overtwisted since the overtwisted discs contained in $S^2(3\pi/2)$ are still present.

Let pr : $\mathbb{R}^3 \times S^1 \longrightarrow \mathbb{R}^3$ be the projection. The 1–form $\beta_1 = \mathrm{pr}^*\alpha - \mathrm{pr}^*(\alpha(V))dt$ defines an even contact structure on $\mathbb{R}^3$ whose characteristic foliation is transversal to $\partial_+ M$. Since V and α are invariant under rotations around the z–axis, $\alpha(V)$ does not depend on φ. We use spherical coordinates $(\varphi, \vartheta) \in [0, 2\pi) \times [0, \pi]$ on S^2_+. Then

$$\beta_s = -s\cos\left(\frac{3\pi}{2}\sin^2(\vartheta)\right)\sin(\vartheta)d\vartheta - \sin\left(\frac{3\pi}{2}\sin^2(\vartheta)\right)d\varphi - g(\vartheta)dt$$

is a defining form for the contact structure on $\partial_+ M$ for all $s \in [0,1]$. Hence

$$\beta\pm = \sin\left(\frac{3\pi}{2}\sin^2(\vartheta)\right)d\varphi \pm g(\vartheta)dt \tag{29}$$

defines an overtwisted contact structure on $S^2_+ \times S^1$. The contact orientations are different for β_+ and β_-.

4.2. Model Engel structures on round handles

4.2.1. Round handles of index zero and three. The standard contact structure $\mathcal{C}$ on $S^3 \subset \mathbb{R}^4$ is defined by the 1–form

$$\alpha = -y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2 \ ,$$

the corresponding Reeb vector field is

$$R = -y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2} \ .$$

A trivialization of $\mathcal{C}$ is given by

$$\begin{aligned} C_1 &= -y_2 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2} \\ C_2 &= -x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} - y_1 \frac{\partial}{\partial y_2} \ . \end{aligned}$$

Together, R, C_1, C_2 form a framing of S^3 such that $[C_1, C_2] = 2R, [R, C_1] = 2C_2$ and $[R, C_2] = -2C_1$. In order to obtain a model Engel structure on round handles of index 0 and 3 such that the boundary of R_0 respectively R_3 is transversal to the characteristic foliation we consider first $S^3 \times S^1$. The coordinate on the second factor is t. We denote the horizontal lifts of C_1, C_2, R by the same symbols.

On $S^3 \times S^1$ the span of ∂_t and

$$X_k = \cos(kt) C_1 + \sin(kt) C_2$$

is an Engel structure if $k \neq 0$. The characteristic foliation corresponds to the one–dimensional foliation induced by the second factor in $S^3 \times S^1$. We perturb this Engel structure in a similar way as we did in Theorem 3.41 (iii). For $k \neq 0$ consider the distribution $\mathcal{D}_k$ spanned by

$$\begin{aligned} W &= \frac{\partial}{\partial t} + \left(\frac{x_1}{2} R - \frac{x_2}{4} C_1 + \frac{y_2}{4} C_2 \right) \\ X_k &= \cos(kt) C_1 + \sin(kt) C_2 \ . \end{aligned}$$

This perturbation of the initial Engel structure is so small that $\mathcal{D}_k$ is still an Engel structure.

LEMMA 4.20. *For $k \neq 0$, the span $\mathcal{D}_k$ of W, X_k is an Engel structure on $S^3 \times S^1$. The characteristic foliation is spanned by W.*

PROOF. In order to show that $[\mathcal{D}_k, \mathcal{D}_k]$ is a distribution of rank 3 we calculate

$$\begin{aligned} [W, X_k] = &\left(-k \sin(kt) + \frac{1}{4} y_1 \cos(kt) - \frac{3}{4} x_1 \sin(kt) \right) C_1 \\ &+ \left(k \cos(kt) + \frac{3}{4} x_1 \cos(kt) + \frac{1}{4} y_1 \sin(kt) \right) C_2 \ . \end{aligned}$$

This is linearly independent of W and X_k because $[W, X_k]$ has no component in the t–direction and

$$\det \begin{pmatrix} \cos(kt) & -k \sin(kt) + \frac{1}{4} y_1 \cos(kt) - \frac{3}{4} x_1 \sin(kt) \\ \sin(kt) & k \cos(kt) + \frac{3}{4} x_1 \cos(kt) + \frac{1}{4} y_1 \sin(kt) \end{pmatrix} = k + \frac{3}{4} x_1 \neq 0$$

Thus $\mathcal{E} = [\mathcal{D}_k, \mathcal{D}_k]$ is a distribution of rank 3 spanned by C_1, C_2, W. In particular $\mathcal{E}$ is independent of k. Since C_1, C_2 span a contact structure on S^3, $\mathcal{E}$ is an even contact structure.

Let $Z = W - \partial_t$. Then Z can be obtained by applying Proposition 2.7 to the function $x_1/2$. So Z is a contact vector field and $[W, C_1]$ and $[W, C_2]$ are both linear combinations of C_1, C_2. Hence $[W, \mathcal{E}] \subset \mathcal{E}$. This shows that W spans the characteristic foliation of $\mathcal{D}_k$. □

The characteristic foliation of $\mathcal{D}_k$ is transversal to the hypersurface $\{y_1 = 0\} \subset S^3 \times S^1$ since

$$\left\langle \frac{\partial}{\partial y_1}, W \right\rangle = -\frac{1}{2}\left(x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}y_2^2\right) < 0 \ .$$

The only zeroes of $Z = W - \partial_t$ are $(0, \pm 1, 0, 0)$.

Cutting $S^3 \times S^1$ along $\{y_1 = 0\}$ yields two copies of $D^3 \times S^1$. Both carry an Engel structure and the boundary is transversal.

DEFINITION 4.21. The *model Engel structure* $\mathcal{D}_k$ on a round handle $R_0 = D^3 \times S^1$ (respectively $R_3 = D^3 \times S^1$) of index 0 (respectively 3) is the Engel structure $\mathcal{D}_k$ constructed above on $\{y_1 \geq 0\}$ (respectively $\{y_1 \leq 0\}$).

We orient the characteristic foliation of $\mathcal{D}_k$ on R_0 respectively R_3 by W. It points outward along $\partial_+ R_0 = \partial R + 0$ and inward along $\partial_- R_3 = \partial R_3$. The characteristic foliation on R_0 and R_3 has exactly one closed hyperbolic orbit in the center of $D^3 \times S^1$. The model Engel structure itself is oriented by W, X_k.

REMARK 4.22. The model Engel structures $\mathcal{D}_k$ on round handles of index zero respectively three induce equal structures on the boundary. This means that

$$Id : \partial_+ R_0 \simeq \{y_1 = 0\} \times S^1 \longrightarrow \{y_1 = 0\} \times S^1 \simeq \partial_- R_3$$

preserves the induced contact structure and the intersection foliation on the boundaries together with their orientations.

4.2.2. Index one. On a round handle of index one $R_1 = D^1 \times D^2 \times S^1$ we denote the coordinate on D^1 by x, the coordinates on D^2 are y_1, y_2 and the coordinate on S^1 is t. We want to construct different Engel structures on R_1 and discuss some of their properties. Our choices here are motivated by [**Wei, El2**]We start with the construction of an even contact structure.

Consider the symplectic form $\omega = dy_1 \wedge dt + dx \wedge dy_2$. The vector field

$$W_1 = \frac{\partial}{\partial t} + \frac{1}{2}y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} - \frac{1}{2}x\frac{\partial}{\partial x}$$

is a Liouville vector field (up to a factor 2) for ω, i.e.

$$L_{W_1}\omega = \frac{1}{2}\omega \ .$$

Note that W_1 enters R_1 through $\partial_- R_1 = \{\pm 1\} \times D^2 \times S^1$ and points outward along $\partial_+ R_1 = D^1 \times S^1 \times S^1$. By Example 3.8

$$\alpha_1 = i_{W_1}\omega = -dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2$$

defines an even contact structure $\mathcal{E}$ on R_1 whose characteristic line field is W_1. A trivialization of $\mathcal{E}$ is given by W_1 followed by

$$C_1 = y_2\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x} \qquad \text{with } [W_1, C_1] = \frac{1}{2}C_1$$

$$C_2 = \frac{1}{2}x\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \qquad \text{with } [W_1, C_2] = -C_2 \ .$$

Now we look for Engel structures whose associated even contact structure is $\mathcal{E}$. These Engel structures have to be subbundles of $\mathcal{E}$ containing W. For non–zero integers k, let

$$X_k = \cos(kt)C_1 + \sin(kt)C_2 \ .$$

PROPOSITION 4.23. *The span $\mathcal{D}_k$ of W_1, X_k is an Engel structure whose characteristic line field is W_1.*

PROOF. Since $\mathcal{D}_k$ is contained in an even contact structure, it is enough to show that $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$. By definition of the characteristic line field of an even contact structure we have $[W_1, \mathcal{D}_k] \subset [W_1, \mathcal{E}] = \mathcal{E}$. Furthermore

$$[W_1, X_k] = \left(-k\sin(kt) + \frac{1}{2}\cos(kt)\right) C_1 + (k\cos(kt) - \sin(kt))\, C_2 \ .$$

Since $[W_1, X_k]$ has no ∂_t–component, $[W_1, X_k]$ is linearly independent of W_1, X_k if and only if it is linearly independent of X_k or, equivalently, if and only if the determinant

$$\det \begin{pmatrix} \cos(kt) & -k\sin(kt) + \frac{1}{2}\cos(kt) \\ \sin(kt) & k\cos(kt) - \sin(kt) \end{pmatrix} = k - \frac{3}{4}\sin(2kt)$$

never vanishes. But since k is a non–zero integer, this condition is always satisfied. Hence $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$. □

We orient the Engel structure $\mathcal{D}_k$ by W_1, X_k. The canonical orientation of the even contact structure $\mathcal{E} = [\mathcal{D}_k, \mathcal{D}_k]$ is given by W_1, C_1, kC_2. Hence the canonical orientation of $\mathcal{E}$ depends on the sign of k.

Next we summarize some properties of $\mathcal{D}_k$. These properties will be used in later constructions. Since the characteristic line field of $\mathcal{E}$ is transversal to both boundary components of R_1, the even contact structure $\mathcal{E}$ induces a contact structure on $\partial_- R_1$ and $\partial_+ R_1$.

LEMMA 4.24. *The Engel structure $\mathcal{D}_k$ on R_1 has the following properties.*

(i) *On both $\partial_- R_1$ and $\partial_+ R_1$, the orientation of the contact structure is positive with respect to $d\alpha_1$ if $k > 0$ and negative if $k < 0$.*

(ii) *The curves $\gamma_\pm = \{\pm 1\} \times \{0\} \times S^1$ are Legendrian. The rotation number along them is $-|k|$.*

(iii) *The rotation number of the intersection line field with respect to*

$$V = y_2 \frac{\partial}{\partial t} + \frac{1}{2} y_1 \frac{\partial}{\partial x}$$

along $\{0\} \times \{y_1 = 0, y_2 = 1\} \times S^1$ is $-|k|$ and it equals 0 along $\{0\} \times S^1 \times \{0\}$.

PROOF. (i) Let $\widetilde{X}_k, \widetilde{[W_1, X_k]}$ be the projections of $X_k, [W_1, X_k]$ to $\partial_- R_1$ along W_1. The contact structure on $\partial_- R_1$ is spanned and oriented by $\widetilde{X}_k, \widetilde{[W_1, X_k]}$. Now we have to find the sign of $d\alpha_1\left(\widetilde{X}_k, \widetilde{[W_1, X_k]}\right)$. By the definition of the characteristic line field of an even contact structure we find

$$d\alpha_1\left(\widetilde{X_k}, \widetilde{[W_1, X_k]}\right) = \frac{1}{2}\left(k - \frac{3}{4}\sin(2kt)\right) = \begin{cases} > 0 & \text{if } k > 0 \\ < 0 & \text{if } k < 0 \ . \end{cases}$$

This proves the claim on $\partial_- R_1$. The same argument works on $\partial_+ R_1$.

(ii) The contact structure on $\partial_- R_1 = \{\pm 1\} \times D^2 \times S^1$ is defined by the 1–form

$$\alpha\big|_{\partial_- R_1} = -dy_1 + \frac{1}{2} y_1 dt - \frac{1}{2} x dy_2$$

with $x \in \{\pm 1\}$. The tangent space of $\gamma_\pm$ is spanned by ∂_t and y_1 vanishes along $\gamma_\pm$. So these curves are Legendrian. For the calculation of the rotation numbers we first need a framing of the contact structures along $\gamma_\pm$ such that the first vector spans the intersection foliation. The intersection line field is spanned by

$$\begin{aligned}\widetilde{X}_k &= X_k - (-\cos(kt))\left(-\frac{2}{x}W_1\right)\\ &= -\frac{2}{x}\cos(kt)\frac{\partial}{\partial t} - \frac{1}{x}y_1\cos(kt)\frac{\partial}{\partial y_1}\\ &\quad + \left(sin(kt) + \frac{2}{x}y_1\cos(kt)\right)\left(\frac{1}{2}x\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right) .\end{aligned} \tag{30}$$

For later use we have calculated $\widetilde{X}_k$ away from $\gamma_\pm$. Here we only need

$$\widetilde{X}_k = -\frac{2}{x}\cos(kt)\frac{\partial}{\partial t} + \sin(kt)C_2 \qquad \text{along } \gamma_\pm .$$

The second component of an oriented trivialization of the contact structure on $\partial_- R_1$ is the projection $\widetilde{[W_1, X_k]}$ along W to $\{\pm 1\} \times D^2 \times S^1$. For $\widetilde{[W_1, X_k]}$ along $\gamma_\pm$ we obtain

$$\begin{aligned}\widetilde{[W_1, X_k]} &= [W_1, X_k] - \left(k\sin(kt) - \frac{1}{2}\cos(kt)\right)\left(-\frac{2}{x}W_1\right)\\ &= \frac{2}{x}\left(k\sin(kt) - \frac{1}{2}\cos(kt)\right)\frac{\partial}{\partial t} + (k\cos(kt) - \sin(kt))\,C_2 .\end{aligned}$$

The tangent space of $\gamma_\pm$ is spanned and oriented by ∂_t. This vector is the following linear combination of $\widetilde{X}_k$ and $\widetilde{[W_1, X_k]}$

$$\frac{\partial}{\partial t} = \frac{x\left((k\cos(kt) - \sin(kt))\widetilde{X}_k - \sin(kt)\widetilde{[W_1, X_k]}\right)}{-2k + 3\sin(kt)\cos(kt)}$$

Finally, we get the rotation numbers along $\gamma_\pm$ as the winding number around 0 of the map

$$\begin{aligned}\gamma_\pm \simeq S^1 &\longrightarrow \mathbb{R}^2 \setminus \{0\}\\ t &\longmapsto \frac{x\big((k\cos(kt) - \sin(kt)), -\sin(kt)\big)}{-2k + 3\sin(kt)\cos(kt)} .\end{aligned}$$

Thus the winding number is $-|k|$. In particular, the rotation number along γ_+ is the same as the rotation number along γ_-.

(iii)

V is obviously tangent to $\partial_+ R_1$ and since $\alpha_1(V) = 0$, it is a Legendrian vector field. The curve

$$\gamma_1 = \{y_1 = 0, y_2 = 1\} \times \{x = 0\} \times S^1$$

is Legendrian and V equals ∂_t there. In order to find the rotation number of the intersection line field along γ_1 we can use the result for the rotation number along $\gamma_\pm$ form (ii). Notice that all curves

$$\gamma_\pm^c = \{y_1 = 0, y_2 = c\} \times \{x = \pm 1\} \times S^1 \subset \partial_- R_1$$

are isotopic to $\gamma_\pm$ through Legendrian curves. Hence the rotation number along $\gamma_\pm^c$ is independent of c. For $c > 0$ we can transport $\gamma_\pm^c$ together with $\{y_2 > 0\} \subset \partial_- R_1$ to $\partial_+ R_1$ along the leaves of the characteristic foliation to the other boundary component $\partial_+ R_1$ of R_1. The curve $\gamma_\pm^c$ remains Legendrian throughout this process since $y_1 = 0$ along the leaves of $\mathcal{W}_1$ passing through $\gamma_\pm^c$ and the rotation number is always well defined. Hence the rotation number along the resulting curve

$$\{y_1 = 0, y_2 = 1\} \times \{x = f(c)\} \times S^1$$

is also $-|k|$. This curve is isotopic through Legendrian curves to γ_1. So the rotation number along γ_1 is $-|k|$. Notice that if we had started with $c < 0$ we would end up with a curve in $\partial_+ R_1$ having y_2–coordinate -1. Since we obtain the same result if we start with γ_- or γ_+, the argument above also shows that the rotation numbers of the intersection line field along γ_- and γ_+ are equal.

In order to compare the intersection foliation on $\partial_+ R_1$ with the Legendrian line field V along

$$\gamma_2 = \{0\} \times S^1 \times \{0\}$$

we calculate a vector field spanning the intersection line field and then an oriented framing of the contact structure on $\partial_+ R_1$. The first component of this framing is the projection $\widetilde{X}_k$ of X_k along W_1 to $\partial_+ R_1$. We obtain

$$\widetilde{X}_k = -\frac{2y_1 y_2}{2 - y_1^2}\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{2y_2^2}{2 - y_1^2}\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right) .$$

along γ_2. The second component of an oriented framing of the contact structure on $\partial_+ R_1$ is the projection $[\widetilde{W_1, X_k}]$ of $[W_1, X_k]$ along W_1 to $\partial_+ R_1$. Along γ_2 we find

$$[\widetilde{W_1, X_k}] = \frac{2ky_2 - y_1 y_2}{2 - y_1^2}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x} - \frac{ky_1 + y_2^2}{2 - y_1^2}\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right) .$$

For the calculation of the rotation number along $\{0\} \times S^1 \times \{0\}$ with respect to V, we express V as linear combination of $\widetilde{X}_k$ and $[\widetilde{W_1, X_k}]$. We obtain

$$V = \frac{1}{2k}\left((-ky_1 - y_2^2)\widetilde{X}_k + 2y_2^2[\widetilde{W_1, X_k}]\right)$$

The induced map $S^1 \to R^2 \setminus \{0\}$ has winding number zero around 0. Hence the rotation number along $\{0\} \times S^1 \times \{0\}$ is zero. □

4.2.3. Index two. In this section we use the notations $\mathcal{D}_k, \mathcal{E}, X_k, C_1, C_2$ for the definitions of model Engel structures on round handles of index 2. Later, when we deal with the similarities between round handles of index one and two we will add appropriate indices.

In order to construct Engel structures on $R_2 = D^2 \times D^1 \times S^1$, we use the same symplectic form as in the case of index one, so $\omega = dy_1 \wedge dt + dx \wedge dy_2$. The coordinates on D^2 are y_1, y_2, the coordinate on D^1 is x and the coordinate on S^1 is t. We orient R_2 by $\partial_{y_1}, \partial_{y_2}, \partial_x, \partial_t$. Let

$$W_2 = \frac{\partial}{\partial t} - \frac{1}{2}y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2} + \frac{1}{2}x\frac{\partial}{\partial x} .$$

Note that this vector field enters R_2 through $\partial_- R_2 = S^1 \times D^1 \times S^1$ and points outward along $\partial_+ R_2 = D^2 \times \{\pm 1\} \times S^1$. Furthermore, this vector field satisfies

$$L_{W_2}\omega = -\frac{1}{2}\omega .$$

By Example 3.8, the form

$$\alpha_2 = -i_{W_2}\omega = dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2$$

defines an even contact structure $\mathcal{E}$ on R_2 whose characteristic line field is spanned by W_2. As defining form for the even contact structure we use α_2 instead of $i_{W_2}\omega$ because α_2

defines a more convenient coorientation as we shall see in the next section. A trivialization of $\mathcal{E}$ is given by W followed by

$$C_1 = y_2\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x} \qquad \text{with } [W_2, C_1] = -\frac{1}{2}C_1$$
$$C_2 = \frac{1}{2}x\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \qquad \text{with } [W_2, C_2] = C_2\ .$$

For non–zero integers k let

$$X_k = \cos(k(t-4))C_1 + \sin(k(t-4))C_2\ .$$

The shift by 4 in t–direction will be convenient when we compare the model Engel structures on round handles of index one and two, cf. for example Proposition 4.29 (ii).

PROPOSITION 4.25. *The span $\mathcal{D}_k$ of W_2, X_k is an Engel structure on R_2 whose characteristic line field is W_2.*

PROOF. The distribution $\mathcal{D}_k$ is contained in an even contact structure $\mathcal{E}$ and $\mathcal{D}_k$ contains the characteristic line field spanned by W_2 of $\mathcal{E}$. This implies $[\mathcal{D}_k, \mathcal{D}_k] \subset \mathcal{E}$. In order to show $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$ we calculate

$$\begin{aligned}[W_2, X_k] = &\left(-k\sin(k(t-4)) - \frac{1}{2}\cos(k(t-4))\right)C_1 \\ &+ \left(k\cos(k(t-4)) + \sin(k(t-4))\right)C_2\end{aligned}$$

So $[W_2, X_k]$ has no ∂_t–component. It is linearly independent of W, X_k if and only if it is not a multiple of X_k. But the determinant

$$\det\begin{pmatrix} \cos(k(t-4)) & -k\sin(k(t-4) - \frac{1}{2}\cos(k(t-4)) \\ \sin(k(t-4)) & k\cos(k(t-4)) + \sin(k(t-4)) \end{pmatrix} = k + \frac{3}{4}\sin(2k(t-4))$$

never vanishes because $k \in \mathbb{Z}\setminus\{0\}$. Hence $[W_2, X_k]$ and X_k are linearly independent. Hence $\mathcal{D}_k$ is an Engel structure. By construction, $\mathcal{E}$ is the associated even contact structure and therefore the characteristic foliation of $\mathcal{D}_k$ is spanned by W_2. □

As in the case of round handles of index 1, we summarize the characteristic properties of the Engel structures $\mathcal{D}_k$.

LEMMA 4.26. *The Engel structure $\mathcal{D}_k$ on R_2 defined above have the following properties.*

(i) *The orientation of the contact structure on ∂_+R_2 and ∂_-R_2 is positive with respect to $d\alpha_2$ if $k > 0$ and negative if $k < 0$.*
(ii) *The curves $\gamma_\pm = \{\pm 1\}\times\{0\}\times S^1$ are Legendrian. The rotation number along them is $-|k|$.*
(iii) *The rotation number of the intersection line field with respect to*

$$V = y_2\partial_t + \frac{1}{2}y_1\partial_x$$

along $\{0\}\times\{y_1 = 0, y_2 = 1\}\times S^1$ is $-|k|$ and it equals 0 along $\partial D^2\times\{0\}\times\{4\}$.

PROOF. The proof consists of similar calculations as in Lemma 4.24 for the case of index one.

(i) Let $\widetilde{X}_k, \widetilde{[W_2, X_k]}$ be the projections of $X_k, [W_2, X_k]$ to $\partial_+ R_2$ along W_2. As in the case index 1, we calculate $d\alpha_2\left(\widetilde{X}_k, \widetilde{[W_2, X_k]}\right)$. By the defining property of the characteristic foliation

$$d\alpha_2\left(\widetilde{X}_k, \widetilde{[W_2, X_k]}\right) = \frac{1}{2}\left(k + \frac{3}{4}\sin(2k(t-4))\right)$$
$$= \begin{cases} > 0 & \text{if } k > 0 \\ < 0 & \text{if } k < 0\,. \end{cases}$$

The same calculation yields the desired result along $\partial_- R_2$.

(ii) Both curves γ_+ and γ_- are obviously Legendrian. We calculate the projections $\widetilde{X}_k, \widetilde{[W_2, X_k]}$ of $X_k, [W_2, X_k]$ along W_2 to $\partial_+ R_2$. For $\widetilde{X}_k$ we obtain

$$\begin{aligned}\widetilde{X}_k &= X_k - \cos(k(t-4))\left(\frac{2}{x}W_2\right) \\ &= -\frac{2}{x}\cos(k(t-4))\frac{\partial}{\partial t} + \frac{1}{x}y_1\cos(k(t-4))\frac{\partial}{\partial y_1} \\ &\quad + \left(\sin(k(t-4)) + \frac{2}{x}y_2\cos(k(t-4))\right)\left(\frac{x}{2}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right)\,.\end{aligned} \tag{31}$$

This will be needed in the next section. For the moment we need to know only

$$\widetilde{X}_k = -\frac{2}{x}\cos(k(t-4))\frac{\partial}{\partial t} + \sin(k(t-4))C_2 \qquad \text{along } \gamma_\pm\,.$$

It suffices to calculate $\widetilde{[W_2, X_k]}$ only along $\gamma_\pm$. We get

$$\begin{aligned}\widetilde{[W_2, X_k]} &= [W_2, X_k] - \left(-k\sin(k(t-4)) - \frac{1}{2}\cos(k(t-4)\right)\left(\frac{2}{x}W_2\right) \\ &= \frac{2}{x}\left(k\sin(k(t-4)) + \frac{1}{2}\cos(k(t-4))\right)\frac{\partial}{\partial t} \\ &\quad + (k\cos(k(t-4)) + \sin(k(t-4)))C_2\,.\end{aligned}$$

Next we express ∂_t, the tangent vector of $\gamma_\pm$, in terms of the oriented basis $\widetilde{X}_k, \widetilde{[W_2, X_k]}$ of the contact structure on $\partial_+ R_2$

$$\frac{\partial}{\partial t} = \frac{-x\left((k\cos(k(t-4)) + \sin(k(t-4)))\widetilde{X}_k - \sin(k(t-4))\widetilde{[W_2, X_k]}\right)}{2k + 3\sin(k(t-4))\cos(k(t-4))}\,.$$

By definition, the rotation number along $\gamma_\pm$ is the winding number around 0 of

$$\begin{aligned}\gamma_\pm \simeq S^1 &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ t &\longmapsto \frac{-x\big((k\cos(k(t-4)) + \sin(k(t-4))), -\sin(k(t-4))\big)}{2k + 3\sin(k(t-4))\cos(k(t-4))}\,.\end{aligned}$$

Hence the rotation number along $\gamma_\pm$ is $-|k|$.

(iii) V is again a Legendrian vector field on $\partial_- R_2$.The curve

$$\gamma_1 = \{y_1 = 0, y_2 = 1\} \times \{x = 0\} \times S^1$$

is Legendrian and $V = \partial_t$ along this curve. Using the same argument as in Lemma 4.24 (iii) one can show that the rotation number along this curve is $-|k|$.

For the calculation of the rotation number along

$$\gamma_2 = \{0\} \times S^1 \times \{4\}$$

with respect to V we first seek the projections $\widetilde{X}_k, \widetilde{[W_2, X_k]}$ of $X_k, [W_2, X_k]$ along W_2 to $\partial_- R_2$.

Along γ_2 we get

$$\widetilde{X}_k = \frac{2y_1y_2}{2-y_1^2}\frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{2y_2^2}{2-y_1^2}\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right)$$

$$\widetilde{[W_2, X_k]} = \frac{2ky_2 - y_1y_2}{2-y_1^2}\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial}{\partial x} + \frac{y_2^2 + ky_1}{2-y_1^2}\left(y_1\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_1}\right) .$$

The rotation number of the intersection foliation with respect to V along the circle γ_2 is zero (as in the case of index one) since the map $S^1 \to \mathbb{R}^2 \setminus \{0\}$ corresponding to

$$V = \frac{1}{2k}\left((y_1k + y_2^2)\widetilde{X}_k + 2y_2^2\widetilde{[W_2, X_k]}\right)$$

is homotopic to a constant map. □

4.2.4. Derived models. We write $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$ for the model Engel structure with $k \in \mathbb{Z} \setminus \{0\}$ on round handles of index one respectively two. We discuss now the case of index one, the round handles of index two can be treated exactly in the same way.

For all possible k, the Engel structures $\mathcal{D}_k^{(1)}$ induce the same even contact structure. In particular the contact structures on $\partial_- R_1$ and $\partial_+ R_1$ are independent of k. We can obtain different isotopy classes of Engel structures if we apply self–diffeomorphisms of R_1. Let

$$\Theta : R_1 = S^1 \times D^2 \times I \longrightarrow S^1 \times D^2 \times I = R_1$$
$$(t, y, x) \longmapsto (t, \exp(it)y, x) .$$

This generates the isotopy classes of orientation preserving self–diffeomorphisms of R_1. We define

$$\mathcal{D}_{k,m}^{(1)} = \Theta_*^m \mathcal{D}_k^{(1)} . \tag{32}$$

The induced action on homotopy classes of framings of $\gamma_\pm$ is the same as the action defined in (6). By definition, Θ preserves $\partial_+ R_1$ and $\partial_- R_1$ as well as $\gamma_\pm$. As $m \in \mathbb{Z}$ varies, the contact framings of γ_+ induced by $\mathcal{D}_{k,m}^{(1)}$ vary and we obtain all homotopy classes of framings of γ_+ inducing the same orientation. Of course the framings of γ_+ and γ_- do not vary independently. Although the contact structures induced by $\mathcal{D}_{k,m}^{(1)}$ on the boundary are different, they give the same orientation of the boundary.

REMARK 4.27. This is a difference between Weinstein's -1–surgery described in [**Wei**] along one Legendrian knot and our method. In our situation one can realize every oriented framing of γ_+ together with an Engel structure and a symplectic structure on the whole of R_1.

The -1–surgery on Legendrian curves preserves weakly symplectically fillable contact structures. However, the model symplectic structures on ordinary 2–handles $D^2 \times D^2$ from [**El2, Wei**] which induce contact structures on the boundary single out particular framings of the attaching curve.

4.3. Relations between the models on R_1 and R_2

As we have shown in Lemma 4.24 and Lemma 4.26, our model Engel structures on round handles of index 1 and index 2 share many properties. Now we want to look closer at the relations between the induced structures on the boundary components of the round handles. In this section we identify R_1 and R_2 using the obvious map between the two handles. When we still refer to R_1 or R_2 we mean some property of the model Engel structures on R_1 respectively R_2 from the previous sections.

We write $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$ for the model Engel structure with parameter $k \in \mathbb{Z}\setminus\{0\}$ on round handles of index one respectively two. When we use symbols appearing in the constructions of the model Engel structures we add an additional index 1 or 2 depending on the index of the round handle.

PROPOSITION 4.28. *The contact structures induced by the Engel structures $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$ on $\partial_- R_1$ respectively $\partial_+ R_2$ are homotopic through contact structures. The same is true for the pair $\partial_+ R_1, \partial_- R_2$.*

PROOF. The even contact structures on R_1 respectively R_2 induced by $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$ are defined by

$$\begin{aligned} \alpha_1 &= -dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2 && \text{on } R_1 \\ \alpha_2 &= dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2 && \text{on } R_2\ . \end{aligned}$$

Consider the family of vector fields

$$W(s) = (1-2s)\frac{\partial}{\partial t} + \frac{1}{2}y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} - \frac{1}{2}x\frac{\partial}{\partial x}\ .$$

with $s \in [0,1]$. For all s, $W(s)$ is a Liouville vector field of $\omega = dy_1 \wedge dt + dx \wedge dy_2$ up to a factor $1/2$ and $W(s)$ is transversal to the boundary of R_1 for all s. Since $W(s)$ vanishes if and only if $s = 1/2$ and $x = y_1 = y_2 = 0$, the family

$$\alpha(s) = i_{W(s)}\omega = -(1-2s)dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2$$

defines a family of even contact structures on $D^1 \times D^2 \times S^1 \setminus (\{0\} \times \{0\} \times S^1)$ such that the characteristic line field is spanned by $W(s)$. So $\alpha(s)$ induces a family of contact forms on both boundary components of R_1. □

Note that $\alpha(0) = \alpha_1$ and $\alpha(1) = \alpha_2$ while $W(0) = W_1$ but $W(1) = -W_2$.

In the following we want to compare the intersection line fields induced by $\mathcal{D}_k^{(1)}$ and $\mathcal{D}_k^{(2)}$ on both boundary components of R_1 respectively R_2. Since these line fields are Legendrian line fields contained in *different* contact structures, we need to identify the contact structures first. To this end we will apply Gray's theorem (Theorem 2.4) to the family of contact forms used in Proposition 4.28.

Recall that the isotopy in Gray's theorem is obtained as the flow of a time–dependent vector field Z_s associated to a family of contact forms $\alpha(s)$. This vector field is the unique vector field which is tangent to ker$(\alpha(s))$ and satisfies

$$i(Z_s)d\alpha(s) = -\frac{d\alpha(s)}{ds} \text{ on } \ker(\alpha(s))\ . \tag{33}$$

It is an easy consequence of the proof of Theorem 2.4 that if Z_s satisfies the stronger equation

$$i(Z_s)d\alpha(s) = -\frac{d\alpha(s)}{ds} \tag{34}$$

(without restricting to ker$(\alpha(s))$, then the time–τ–flow $\psi(\tau)$ preserves contact forms and not only contact structures since then $f \equiv 0$ in the proof of Theorem 2.4.

4.3.1. The boundary component $\partial_- R_1 = \partial_+ R_2$. First we consider the boundary component $\partial_- R_1$. In order to have flows which are defined for all times, we suppose $(y_1, y_2) \in \mathbb{R}^2$ instead of $(y_1, y_2) \in D^2$ for the moment. The family of contact forms is the restriction of $\alpha(s)$ from Proposition 4.28 to $\partial_- R_1$. We use the same notation for this restriction. In order to find Z_s we have to solve the equations

$$\begin{aligned} i(Z_s)\left(-(1-2s)dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2\right) &= 0 \\ i(Z_s)\frac{1}{2}dy_1 \wedge dt = -2dy_1 \text{ on } \ker(\alpha(s)) &. \end{aligned} \tag{35}$$

The solution Z_s of these equations defined on $\{x = \pm 1\} \times \mathbb{R}^2 \times S^1$ is

$$Z_s = 4\frac{\partial}{\partial t} + \frac{4}{x}y_1\frac{\partial}{\partial y_2}$$

and this vector field even satisfies equation (34). Notice that Z_s does not depend on s. So we write Z_- referring to $\partial_- R_1$ instead of Z_s. The time–τ–flow of this vector field is given by

$$\psi^-(\tau) : (x = \pm 1, y_1, y_2, t) \longmapsto \left(x = \pm 1, y_1, y_2 + \frac{4}{x}y_1\tau, t + 4\tau\right) . \tag{36}$$

On $\{x = \pm 1\} \times \mathbb{R}^2 \times S^1$ this is defined for all τ and ψ^- preserves $\gamma_\pm = \{y_1 = y_2 = 0, x = \pm 1\}$. By construction, the time–τ–flow $\psi^-(\tau)$ of Z_- satisfies

$$\psi^-(\tau)_*(\ker(\alpha(0))) = \ker(\alpha(\tau)) .$$

The following proposition summarizes the relations between the image of the intersection line field induced by $\mathcal{D}_k^{(1)}$ under $\psi^-(1) = \psi^-$ and the intersection line field induced by $\mathcal{D}_k^{(2)}$ on $\partial_- R_1$ respectively $\partial_+ R_2$. For $i = 1, 2$ let $\widetilde{X}_k^{(i)}$ be the projection of the vector field $X_k^{(i)}$ used in the construction of $\mathcal{D}_k^{(i)}$ along W_i to $\partial_- R_1$ respectively $\partial_+ R_2$.

PROPOSITION 4.29. *The time–1–flow ψ^- of Z_-*

(i) *preserves the sets $\{y_1 = 0\}$ and $\gamma_\pm$. Moreover it preserves the orientations of the contact structure induced by $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$.*

(ii) *maps $\widetilde{X}_k^{(1)}$ to a Legendrian vector field which is homotopic to $\widetilde{X}_k^{(2)}$ through Legendrian vector fields. On $\{y_1 = 0\}$ the intersection line fields are preserved (with their orientation given by $\widetilde{X}_k^{(1)}$ respectively $\widetilde{X}_k^{(2)}$). In particular ψ^- preserves the homotopy type of the intersection foliation along $\gamma_\pm$.*

PROOF. (i) That ψ^- preserves $\{y_1 = 0\}$ and $\gamma_\pm$ is obvious from (36). The contact structure along $\{y_1 = 0\}$ induced by $\mathcal{D}_k^{(1)}$ is spanned and oriented by

$$-\frac{1}{x}\frac{\partial}{\partial t}, k\left(\frac{1}{2}x\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)$$

as we have shown in Lemma 4.24 (i). A direct calculation shows

$$\begin{aligned} \left(\psi^-(1)\right)_*\left(-\frac{1}{x}\frac{\partial}{\partial t}\right) &= -\frac{1}{x}\frac{\partial}{\partial t} \\ \left(\psi^-(1)\right)_*\left(k\left(\frac{1}{2}x\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right)\right) &= k\left(\frac{1}{2}x\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right) . \end{aligned}$$

along $\gamma_\pm$. On the other hand we know from Lemma 4.26 (i) that the contact structure induced by $\mathcal{D}_k^{(2)}$ is spanned and oriented by

$$-\frac{1}{x}\frac{\partial}{\partial t}, k\left(\frac{1}{2}x\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right) .$$

This proves (i).

(ii) In the proofs of Lemma 4.24 and Lemma 4.26 we have given expressions for the projections $\widetilde{X}_k^{(1)}$ and $\widetilde{X}_k^{(2)}$. Along $y_1=0$ we get

$$\psi_*^-\left(\widetilde{X}_k^{(1)}\right) = -\frac{2}{x}\cos(k(t-4))\frac{\partial}{\partial t}+\left(\cos(k(t-4))y_2+\frac{1}{2}x\sin(k(t-4))\right)\frac{\partial}{\partial y_1}$$
$$+\left(\sin(k(t-4))+\frac{2}{x}y_2\cos(k(t-4))\right)\frac{\partial}{\partial y_2} .$$

Comparing this expression with (31) one sees that it equals $\widetilde{X}_k^{(2)}$. It is now clear that ψ^- preserves the homotopy type of the intersection foliation along $\gamma_\pm$. □

Away from $\{y_1=0\}$ the statement (ii) of the last lemma is not true. We will use the behaviour of the flow only on a small enough neighbourhood of $\{y_1=0\}$. On this hypersurface the flow of Z_- is complete even on $D^1\times D^2\times S^1\subset D^1\times\mathbb{R}^2\times S^1$.

4.3.2. The boundary component $\partial_+R_1=\partial_-R_2$. Now we carry out the analogous discussion for the other boundary component ∂_+R_1. This is more complicated because of the following reason: When one glues a round handle to a manifold with boundary, one obtains a manifold with corners. In order to get a smooth manifold without corners we cut off a piece of the round handle. So in the case of round handles of index 1, the new boundary component of the manifold with a round handle glued to it is not precisely ∂_+R_1.

As a first approximation we first ignore the effect of smoothing and consider only ∂_+R_1 respectively ∂_-R_2. In order to obtain flows which are defined for all times we assume for the moment that $x\in(-\infty,\infty)$ rather then $x\in[-1,1]$. The Engel structures $\mathcal{D}_k^{(1)}$ and $\mathcal{D}_k^{(2)}$ are defined by the coordinate expressions from the sections above.

We apply the proof of Gray's theorem to the restriction to $\mathbb{R}\times\partial D^2\times S^1$ of the family of 1–forms

$$\alpha(s)=-(1-2s)dy_1+\frac{1}{2}y_1dt-y_2dx-\frac{1}{2}xdy_2 .$$

The restricted family is again denoted by $\alpha(s)$. The kernel of $\alpha(s)$ (restricted to $\mathbb{R}\times S^1\times S^1$) is spanned by

$$y_2\frac{\partial}{\partial t}+\frac{1}{2}y_1\frac{\partial}{\partial x}\quad\text{and}\quad(1-2s)\frac{\partial}{\partial x}+\left(y_1\frac{\partial}{\partial y_2}-y_2\frac{\partial}{\partial y_1}\right)+x\frac{\partial}{\partial t} . \tag{37}$$

The vector field

$$Z_s=\frac{8y_2}{1+y_2^2}V \tag{38}$$

is contained in ker$(\alpha(s))$ and it solves even (34). Again Z_s does not depend on s, we write Z_+ for this vector field. The time–τ–flow of Z_+ is

$$\psi^+(\tau):(x,y_1,y_2,t)\longmapsto\left(x+\frac{4y_1y_2}{1+y_2^2}\tau,y_1,y_2,t+\frac{8y_2^2}{1+y_2^2}\tau\right) .$$

It has the property $\left(\psi^+(\tau)\right)^*\alpha(\tau)=\alpha(0)$. The following lemma describes the behaviour of the time–1–flow ψ^+ of Z_+ with respect to intersection foliations.

PROPOSITION 4.30.

(i) *The line field spanned field* $V = y_2\partial_t + 1/2y_1\partial_x$ *is Legendrian with respect to the contact structure* $\ker(\alpha(s))$ *for all* s *and it is preserved by* $\psi^+(\tau)$.

(ii) ψ^+ *preserves the contact structures induced by* $\mathcal{D}_{1,k}$ *respectively* $\mathcal{D}_{2,k}$ *together with the induced orientations.*

(iii) *The image under* ψ^+ *of the intersection line field of the Engel structure* $\mathcal{D}_k^{(1)}$ *is homotopic to the intersection line field induced by* $\mathcal{D}_k^{(2)}$.

PROOF. (i) V is obviously tangent to ∂_+R_1 and $\alpha(s)(V) = 0$. So V is Legendrian and Z_+ is a multiple of V by (38). Therefore the flow of Z_+ preserves the line field spanned by V.

(ii) Recall that a contact structure on a 3–dimensional manifold induces a canonical orientation of the base manifold. In Lemma 4.24 and Lemma 4.26 we showed that α_1 respectively α_2 defines the right coorientation of the contact structure induced by $\mathcal{D}_k^{(1)}$ respectively $\mathcal{D}_k^{(2)}$ on ∂_+R_1 if $k > 0$ and the wrong coorientation if $k < 0$. Since $\psi^{+*}\alpha_2 = \alpha_1$, the time–1–flow of Z_+ preserves the orientation of the contact structures.

(iii) The flow $\psi^+(\tau)$ preserves the Legendrian curve $\{x = y_1 = 0, y_2 = 1\}$, this curve is Legendrian for all contact structures $\ker(\alpha(s))$. So the rotation number of the image under $\psi^+(\tau)$ of the intersection line field induced by $\mathcal{D}_k^{(1)}$ along this curve is independent of τ. Hence it equals $-|k|$.

In Lemma 4.24 we have shown that along $\{0\} \times S^1 \times \{0\} \subset D^1 \times D^2 \times S^1$, the intersection line field of $\mathcal{D}_k^{(1)}$ is homotopic to the line field spanned by V. Since the flow $\psi^+(\tau)$ preserves V, the same is true for the image under ψ^+ of this intersection line field along the curve $\psi^+(\{0\} \times S^1 \times \{0\})$. Moreover $\psi^+(\{0\} \times S^1 \times \{0\})$ and $\{0\} \times S^1 \times \{0\}$ are isotopic. Together with $\{x = y_1 = 0, y_2 = 1\}$, this curve generates $H_1(\partial_+R_1; \mathbb{Z})$.

By Proposition 3.22 together with (i) and (ii) this proves the claim. □

Finally notice that if we consider $D^1 \times D^2_{\sigma_0} \times S^1$ where the radius of $D^2_{\sigma_0}$ is not 1 but σ_0, then the expression in (38) for the vector field Z_+ obtained by Gray's argument is replaced by

$$Z_+ = \frac{8y_2}{\sigma_0^2 + y_2^2}\left(y_2\frac{\partial}{\partial t} + \frac{1}{2}y_1\frac{\partial}{\partial x}\right) . \tag{39}$$

Of course Proposition 4.30 applies in both situations (38) (where the radius of D^2 is 1) and (39) (where the radius of D^2 is σ_0).

As we have already mentioned, this discussion does only approximate the situation we are in when we glue round 1–handles to manifolds with boundary. In order to obtain manifolds without corners we remove a certain part of the round handle. For the real boundary components, the isotopy relating the two contact structures induced by $\mathcal{D}_k^{(1)}$ and $\mathcal{D}_k^{(2)}$ is more complicated than in the situation above.

We now describe models for

- the gluing of round 1–handles with the Engel structure $\mathcal{D}_k^{(1)}$ to ∂_+M_1 along ∂_-R_1.
- the gluing of round 2–handles with the Engel structure $\mathcal{D}_k^{(2)}$ to ∂_-M_2 along ∂_+R_2.

Then we compare the resulting contact structures and intersection line fields on the boundary of the smooth manifolds $\widetilde{M}_1 = M_1 \cup R_1$ and $\widetilde{M}_2 = M_2 \cup R_2$.

Let M_1 be the subset $|x| \geq 1$ of $\mathbb{R} \times \mathbb{R}^2 \times S^1$. Let M_2 be a copy of M_1. On M_1 we consider the Engel structure defined by the same coordinate expression we used for $\mathcal{D}_k^{(1)}$ while on M_2 we use the expression of the Engel structure $\mathcal{D}_k^{(2)}$.

The round handle of index 1 is the subset $\{|x| \leq 1\} \times D^2 \times S^1$ of $\mathbb{R} \times \mathbb{R}^2 \times S^1$. The Engel structure on M_1 extends obviously to an Engel structure on $M_1 \cup R_1$ and the same statement is true for round handles of index 2 and M_2.

In order to obtain smooth manifolds with boundary, we cut off pieces of R_1 and R_2 as follows. Choose a function $\sigma : D^1 \rightarrow [1/2, 1]$ which is smooth on the interior of D^1 and satisfies

(i) $\sigma(1) = 1$
(ii) $\sigma(-x) = \sigma(x)$
(iii) $\dot{\sigma}(x) \leq 0$ on $x \leq 0$
(iv) $\sigma \equiv \sigma_0$ is constant on $[-1/2, 1/2]$.

Moreover we assume that

$$B = \left\{(x, y_1, y_2, t) \middle| x \in (-1, 1), y_1^2 + y_2^2 = \sigma(x)\right\} \tag{40}$$

together with the part

$$\left\{(x, y_1, y_2, t) \middle| x = \pm 1 \text{ and } y_1^2 + y_2^2 \geq 1\right\}$$

of the boundary of M_1 respectively M_2 is a smooth submanifold of $\mathbb{R} \times \mathbb{R}^2 \times S^1$. It is transversal to $W(s)$ for all s by condition (ii) and (iii).

We remove the points with $y_1^2 + y_2^2 > \sigma(x)$ from R_1 and R_2. The remaining parts will be denoted by $\widetilde{R}_1$ respectively $\widetilde{R}_2$ only for the remaining part of this section. Afterwards we will use R_1 respectively R_2. We obtain smooth manifolds $\widetilde{M}_1 = M_1 \cup \widetilde{R}_1$ and $\widetilde{M}_2 = M_2 \cup \widetilde{R}_2$. Both manifolds now carry smooth Engel structures and the new boundaries are transversal to the characteristic foliation by the conditions (ii) and (iii) on σ.

The following theorem is a refinement of Proposition 4.30 for the situation of the model. We fix some notation first. Let Δ be the curve $\{-1 \leq x \leq 1, t = 0, y_1 = 0, y_2 = \sigma(x)\} \subset R_1$ extended by two straight intervals contained in $\{y_1 = 0\}$ pointing away from R_1 in radial direction, thus only y_2 is increasing along the intervals and Δ is a smooth curve in $\partial_+\widetilde{M}_1$ while $y_1 = 0$.

For the family of contact forms we use the restriction to $\partial_+\widetilde{M}_2$ of

$$\alpha(s) = -(1 - 2s)dy_1 + \frac{1}{2}y_1 dt - y_2 dx - \frac{1}{2}x dy_2$$

with $s \in [0, 1]$. We apply Gray's theorem to this family in order to obtain an isotopy

$$\psi(\tau) : \partial_+\widetilde{M}_1 \longrightarrow \partial_-\widetilde{M}_2$$

such that the image of the contact structure induced by $\mathcal{D}_k^{(1)}$ on $\partial_+\widetilde{M}_1$ is defined by $\alpha(s)$.

THEOREM 4.31. *The isotopy $\psi(\tau)$ constructed above has the following properties.*

(i) $\psi(0)$ *is the identity map $\partial_+\widetilde{M}_1 \rightarrow \partial_-\widetilde{M}_2$ in terms of the coordinates x, y_1, y_2, t.*
(ii) $\psi(1)$ *preserves the contact structures induced by $\mathcal{D}_k^{(1)}$ on $\partial_+\widetilde{M}_1$ respectively by $\mathcal{D}_k^{(2)}$ on $\partial_-\widetilde{M}_2$.*
(iii) $\psi(1)$ *preserves the homotopy type of the intersection line fields.*
(iv) $\psi(\tau)$ *preserves $\{y_1 = 0\}$ and the line field spanned by ∂_t along this hypersurface. This line field is Legendrian with respect to $\alpha(s)$ for all $s \in [0, 1]$.*
(v) $\psi(1)$ *maps the intersection line field of $\mathcal{D}_k^{(1)}$ along Δ to a Legendrian line field which coincides with the intersection line field of $\mathcal{D}_k^{(1)}$ on the boundary points of $\psi(\Delta)$. The two Legendrian line fields are homotopic along $\psi(\Delta)$ relative to the boundary points of this arc.*

PROOF. The statement (i) is a reformulation of the identification between $\widetilde{M}_1$ and $\widetilde{M}_2$. The proofs of (ii) and (iii) are the same as in Proposition 4.30.

(iv) Along $\{y_1 = 0\}$ we have clearly $\alpha(s)(\partial_t) = 0$ for all s. In order to prove the remaining part of (iv), notice that away from the attaching region of $\widetilde{R}_1$ respectively $\widetilde{R}_2$, the claim is true since there the family of contact forms is precisely the one appearing in Proposition 4.29.

For the remaining part B of $\partial_+\widetilde{M}_2$ one can show by a direct calculation which can be found below, that along $\{y_1 = 0\}$ the vector field inducing the Gray isotopy equals

$$\widetilde{Z}_+ = 4\frac{\partial}{\partial t}\ . \tag{41}$$

This vector field obviously preserves $\{y_1 = 0\}$ and ∂_t.

One can expect (41) for the following reason : From Proposition 4.30 we know that the Gray isotopy associated to the restriction of $\alpha(s)$ to ∂_+R_2 preserves $\{y_1 = 0\}$ and the line field spanned by ∂_t. Now $\{y_1 = 0\}$ and the line field spanned by ∂_t are invariant along the characteristic foliation $\mathcal{W}(s)$ of the even contact structure defined by $\alpha(s)$. This foliation is spanned by $W(s)$. By Lemma 3.5, we can identify the contact structure defined by $\alpha(s)$ on ∂_-R_2 with the contact structure defined by $\alpha(s)$ on the smoothened handle $\partial_-\widetilde{R}_2$. We can transfer the vector field which induced the Gray isotopy on ∂_-R_2 to $\partial_-\widetilde{R}_2$. The flow $\widetilde{\psi}(\tau)$ of the vector field on $\partial_+\widetilde{R}_2$ has the property

$$\widetilde{\psi}(\tau)_*\text{ker}(\alpha(0)) = \text{ker}(\alpha(\tau)\ .$$

Unfortunately it is not clear that the vector field we obtained on B is the one we would obtain from Gray's theorem because $\alpha(s)$ is not invariant along $\mathcal{W}(s)$ since $L_{W(s)}\alpha(s) = 1/2\alpha(s)$. Because we want to obtain smooth isotopies on $\partial_-\widetilde{M}_2$ it is better to use one and the same method on $\partial_-\widetilde{M}_2 \setminus B$ and on B to construct the isotopy.

(v) The claim about the intersection line fields at the endpoints of Δ respectively $\psi(\Delta)$ follows from Proposition 4.29 (ii) because the endpoints of Δ lie outside of the attaching region of the round handles and they are contained in $\{y_1 = 0\}$.

We defined $\mathcal{D}_k^{(1)}$ on $\widetilde{M}_1$ to be the span of

$$\begin{aligned}
W_1 &= \frac{\partial}{\partial t} + \frac{1}{2}y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} - \frac{1}{2}x\frac{\partial}{\partial x} \\
&= \frac{\partial}{\partial t} + y_2\frac{\partial}{\partial y_2} - \frac{1}{2}x\frac{\partial}{\partial x} \qquad \text{along } \Delta \\
X_k^{(1)} &= \cos(kt)\left(y_2\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x}\right) + \sin(kt)\left(\frac{1}{2}x\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}\right) \\
&= y_2\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x} \qquad \text{along } \Delta\ .
\end{aligned}$$

A nontrivial linear combination of W_1 and $X_k^{(1)}$ along Δ either has a y_1–component because y_2 is never zero along Δ or the linear combination is in fact a multiple of W_1. In both cases, the linear combination is not colinear with ∂_t.

Similarly, we defined the Engel structure on $\partial_-\widetilde{M}_2$ to be the span of

$$\begin{aligned} W_2 &= \frac{\partial}{\partial t} - \frac{1}{2}y_1\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2} + \frac{1}{2}x\frac{\partial}{\partial x} \\ &= \frac{\partial}{\partial t} - y_2\frac{\partial}{\partial y_2} + \frac{1}{2}x\frac{\partial}{\partial x} \qquad \text{along } \psi(\Delta) \\ X_k^{(2)} &= \cos(k(t-4))\left(y_2\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x}\right) + \sin(k(t-4))\left(\frac{1}{2}x\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\right) \\ &= \left(y_2\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x}\right) \qquad \text{along } \psi(\Delta)\ . \end{aligned}$$

Note that the t–coordinate of $\psi(\Delta)$ is 4. The same argument as above shows that along $\{t = 4, y_1 = 0\}$, the Legendrian vector field ∂_t is never contained in $\mathcal{D}_k^{(2)}$.

So both oriented line fields (the first is the image of the oriented intersection line field on $\partial_+\widetilde{M}_1$ under ψ and the second is the intersection line field induced by $\mathcal{D}_k^{(2)}$ on $\widetilde{M}_2$) are Legendrian for the contact structure induced by $\mathcal{D}_k^{(2)}$ on $\partial_-\widetilde{M}_2$ by construction and they are equal at the end points of $\psi(\Delta)$. Recall that the isotopy ψ preserves ∂_t along $\{y_1 = 0\}$ by (iv). Along this curve, both line fields are never colinear to ∂_t. Since ∂_t is Legendrian along $\psi(\Delta)$, this proves (v). □

PROOF OF (41). Away from the attaching region of R_1 the claim is true since we have shown in Proposition 4.29

$$\begin{aligned} Z_- &= 4\frac{\partial}{\partial t} + \frac{4}{x}y_1\frac{\partial}{\partial y_2} \\ &= 4\frac{\partial}{\partial t} \qquad\qquad \text{along } y_1 = 0 \end{aligned}$$

The remaining part of the boundary is B (for the definition of B see (40)). The subset $\{y_1 = 0\}$ of B has two connected components, we focus on the component with positive y_2. For the other component, the argument is analogous. The tangent space of $B \subset \mathbb{R}\times\mathbb{R}^2\times S^1$ is spanned by

$$\frac{\partial}{\partial t},\ \frac{\partial}{\partial \varphi} = \frac{y_1}{\sigma}\frac{\partial}{\partial y_2} - \frac{y_2}{\sigma}\frac{\partial}{\partial y_1},\ \frac{1}{\sqrt{1+\dot\sigma^2}}\left(\frac{\partial}{\partial x} + \dot\sigma\left(\frac{y_1}{\sigma}\frac{\partial}{\partial y_1} + \frac{y_2}{\sigma}\frac{\partial}{\partial y_2}\right)\right)\ .$$

We will write ∂_r for $\frac{1}{\sigma}(y_1\partial_{y_1} + y_2\partial_{y_2})$. Thus there are functions f, g, h on B such that

$$\widetilde{Z}_+ = f\frac{\partial}{\partial t} + g\frac{\partial}{\partial \varphi} + h\left(\frac{1}{\sqrt{1+\dot\sigma^2}}\left(\frac{\partial}{\partial x} + \dot\sigma\frac{\partial}{\partial r}\right)\right)\ .$$

This vector field has to satisfy the relations

$$\alpha(s)(\widetilde{Z}_+) = 0 \tag{42}$$

$$i(\widetilde{Z}_+)d\alpha(s) = -\dot\alpha(s) \text{ on } \ker(\alpha(s))\ . \tag{43}$$

The first relation (42) yields

$$(1-2s)g - y_2\frac{h}{\sqrt{1+\dot\sigma^2}} - \frac{1}{2}x\frac{\dot\sigma}{\sqrt{1+\dot\sigma^2}}h = 0$$

along $\{y_1 = 0, y_2 > 0\}$ (and hence $y_2 = \sigma(x)$). Solving for h we obtain

$$h = \frac{2(1-2s)\sqrt{1+\dot\sigma^2}}{2y_2 + x\dot\sigma}g\ , \tag{44}$$

note that the denominator is always positive by our assumptions on $\dot\sigma$ and $y_2 > 0$. The relation (43) implies that

$$-gdt - fdy_1 + \frac{h}{\sqrt{1+\dot\sigma^2}}dy_2 - \frac{h\dot\sigma}{\sqrt{1+\dot\sigma^2}}dx = -2\dot\alpha(s) = -4dy_1$$

on $\{y_1 = 0, y_2 > 0\}$. Now along $\{y_1 = 0\}$, the vector field ∂_t is tangent to the kernel of $\alpha(s)$ for all s and it is of course tangent to the boundary of the smoothened handle. Therefore we get $g \equiv 0$. By (44) this implies $h \equiv 0$ on $\{y_1 = 0\}$. All in all we have shown

$$\widetilde{Z}_+ = 4\frac{\partial}{\partial t} \text{ along } \{y_1 = 0\} \; .$$

This proves (41). □

CHAPTER 5

Closed Engel manifolds from round handles

In this chapter we discuss our first construction of Engel manifolds on closed manifolds. The main technical result is Theorem 5.6.

Usually we assume that the characteristic foliation $\mathcal{W}$ of an Engel structure is oriented and that all components of the boundary are transversal to $\mathcal{W}$. Recall that we write ∂_+ for those boundary components where the characteristic foliation points outwards and ∂_- for the remaining boundary components.

In Section 5.1 we explain how one can glue a round 1–handle with a model Engel structure to the transversal boundary of an Engel manifold. The model Engel structure extends an oriented Engel structure from M to an oriented Engel structure on $M \cup_\varphi R_1$ if the attaching map $\varphi : \partial_- R_1 \longrightarrow \partial_+ M$ has the following properties.

(i) φ preserves the oriented contact structures on the boundary.
(ii) φ preserves the oriented intersection line fields.

There is a possibility to modify the Engel structure on M without changing the even contact structure in order to change the intersection line field on $\partial_+ M$ within its homotopy class. This construction, which is described in Section 5.2, is referred to as vertical modification of the boundary. It relies on the fact that $\partial_+ M$ is a closed manifold. If we are allowed to use vertical modifications of the boundary, then we can weaken (ii).

(ii') φ preserves the homotopy class of oriented intersection line fields.

It is not always possible to use vertical modifications if we have to respect a boundary condition when the boundary is not a closed manifold. In this thesis this situation arises only in the proof of Theorem 5.17.

If one attaches a round handle to a manifold, one obtains a manifold with corners. We smoothen corners by the procedure explained in Section 4.3.2.

Assume M_1, M_2 are Engel manifolds with transversal boundary and $\psi : \partial_+ M_2 \longrightarrow \partial_- M$ is a diffeomorphism which preserves oriented contact structures and the intersection line fields. Then there is a smooth Engel structure on $M_1 \cup_\psi M_2$. Let $\varphi_1 : \partial_- R_1 \longrightarrow \partial_+ M_1$ be an attaching map for a round 1–handle such that a model Engel structure on R_1 can be used to extend the Engel structure from M_1 to $\widetilde{M_1} = M_1 \cup \varphi_1 R_1$. In Theorem 5.6 we consider the map $\varphi_2 = \psi \circ \varphi_1 : \partial_+ R_2 \longrightarrow \partial_- M_2$. Recall that round handles of index 1 and 2 are dual to each other.

In a first step we deform the Engel structure on M_2 on a neighbourhood of $\partial_- M_2$ using Gray's theorem (Theorem 2.4). The symmetry between the model Engel structures on round handles of index 1 and 2 discussed in Section 4.3 allows us to find a model Engel structure on R_2 such that the Engel structure on M_2 extends to $\widetilde{M_2} = M_2 \cup_{\varphi_2} R_2$.

In order to remove the corners which appear when the round handles are attached we cut off a suitable piece of R_1 and R_2. This can be done in a symmetric way (we have explained this in Section 4.3.2). Using Gray's theorem again we obtain a diffeomorphism $\partial_+ \widetilde{M_1} \longrightarrow \partial_- \widetilde{M_2}$ which has the same properties as the diffeomorphism ψ we started with.

Using the fact that every curve is isotopic to a Legendrian one and stabilizations, we develop an algorithm which allows us to find attaching maps for round 1–handles for the above procedure. This is summarized in Theorem 5.8. This method turns out to be sufficient for the proof of the existence theorem (Theorem 6.1) in Chapter 6.

It turns out that one can obtain Engel structures on manifolds which are not accessible by prolongation or the construction of H. J. Geiges explained in Section 3.2.2. Such examples are explained in Section 5.5. We use a theorem of J. Hempel who has classified all Abelian groups which appear as subgroup of the fundamental group of a 3–manifold in order to show that the resulting manifolds are not fibrations over S^1 or a 3–manifold.

In Section 5.6 we discuss our main application of Theorem 5.6. Let $\mathcal{D}, \mathcal{D}'$ be Engel structures on the manifolds M, M'. If $\mathcal{D}, \mathcal{D}'$ satisfy an additional condition, then one can use Theorem 5.6 to construct an Engel structure on $M \# M' \# (S^2 \times S^2)$. This is possible if one assumes that the characteristic foliation of $\mathcal{D}$ and $\mathcal{D}'$ admit closed transversals (Theorem 5.14). Another possible assumption on $\mathcal{D}, \mathcal{D}'$ is discussed in Theorem 5.17. In both cases, the additional assumption is used when we apply vertical modifications of the boundary. Using this construction we obtain an Engel structure on $M \# M \# (S^2 \times S^2)$ which coincides with $\mathcal{D}$ respectively $\mathcal{D}'$ away from certain open subsets of M and M'. The Engel structure on $M \# M' \# (S^2 \times S^2)$ satisfies the assumption of Theorem 5.14 respectively Theorem 5.17 again.

In the proof of Theorem 5.14 and Theorem 5.17 the two manifolds are connected using a round 1–handle and a round 2–handle. If one decomposes these round handles into ordinary handles as in Lemma 4.8 one finds the additional summand $S^2 \times S^2$.

5.1. Gluing Engel structures

We first explain how to attach round handles of index one to an Engel manifold with transversal boundary. Then we explain how to glue two Engel manifolds with equivalent transversal boundaries together.

Let M be an Engel manifold with oriented characteristic foliation and transversal boundaries. Assume that a map

$$\varphi : \partial_- R_1 \to \partial_+ M$$

preserves oriented contact structures and intersection line fields where R_1 carries the model Engel structure $\Theta_*^m \mathcal{D}_{1,k}$. Using this model Engel structure we want to extend the Engel structure on M to an Engel structure on $M \cup_\varphi R_1$. Notice that this space is not really a manifold because it has corners. The procedure how to smoothen corners was explained in section 4.3.2.

By Theorem 3.19, the contact structure and the intersection line field on the boundary determine the Engel structure on a collar up to diffeomorphism. We will use Theorem 3.19 to extend the Engel structure on M smoothly to $M \cup_\varphi R_1$.

To this end we extend $R_1 \subset \mathbb{R} \times \mathbb{R}^2 \times S^1$ by the set of points (x, y_1, y_2, t) with

(i) $1 \leq |x| < 1 + \delta$ with $\delta > 0$ (we fix δ later),
(ii) $(y_1, y_2) \in D^2$,
(iii) the leaf of the characteristic foliation of the Engel structure $\mathcal{D}_{m,k}$ on $\mathbb{R} \times \mathbb{R}^2 \times S^1$ through (x, y_1, y_2, t) intersects $\partial_- R_1$.

We write $\widetilde{R}_1$ for the extended round one–handle, cf. Figure 1.

The contact structure on $\partial_- R_1$ respectively $\partial_+ M$ will be denoted by $\mathcal{C}_1$ respectively $\mathcal{C}_M$ and let $\mathcal{L}_1 \subset \mathcal{C}_1$ and $\mathcal{L}_M \subset \mathcal{C}_M$ be the intersection line fields of the Engel structures. By Theorem 3.19 there is a diffeomorphism ψ_1 between a neighbourhood of $\partial_- R_1$ in $\widetilde{R}_1$

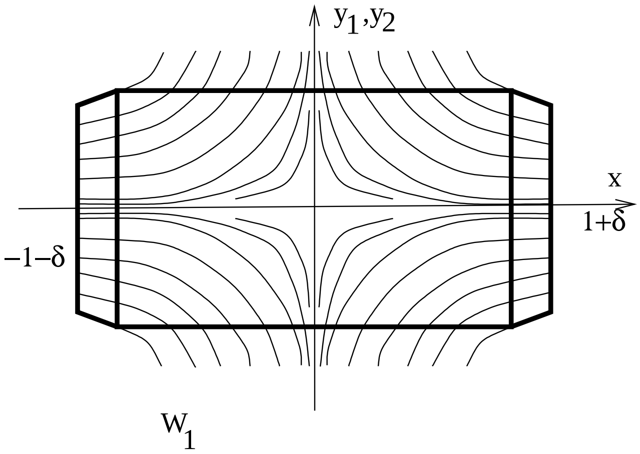


FIGURE 1.

and a neighbourhood of the section

$$\sigma_1 : \partial_- R_1 \longrightarrow \mathbb{P}\mathcal{C}_1$$
$$p \longmapsto [\mathcal{L}_1(p)]$$

such that ψ_1 preserves Engel structures. The analogous statement is true of course for $\partial_- M$ but this time a collar neighbourhood U of $\partial_- M$ gets mapped by ψ_M on one side of the section σ_M corresponding to $\mathcal{L}_M$.

Recall from Proposition 3.16 that a contact map induces an Engel diffeomorphism of the corresponding Engel manifold obtained by prolongation. Hence the embedding φ induces an embedding $\widetilde{\varphi}$ of a neighbourhood of the section $\sigma_1 \subset \mathbb{P}\mathcal{C}_1$ to a subset of $\mathbb{P}\mathcal{C}_M$. By definition of $\widetilde{\varphi}$ we have

$$\widetilde{\varphi} \circ \sigma_1 = \sigma_M \circ \varphi$$

because φ preserves the intersection line fields.

Thus σ_1 gets mapped to σ_M. Up to now we have shown that $\psi_M^{-1} \circ \widetilde{\varphi} \circ \psi_1$ is a diffeomorphism of a small enough neighbourhood of $\partial_- R_1$ in $\widetilde{R_1}$ onto its image and this diffeomorphism preserves Engel structures. We choose δ so that for the corresponding extended round one–handle $\widetilde{R}_1$ the set $\widetilde{R}_1 \setminus R_1$ is contained entirely in this neighbourhood.

The last thing we have to check is that points in $\widetilde{R}_1 \setminus R_1$ get mapped to the collar U of $\partial_+ M$. For this we use that fact that φ preserves the orientation of the contact structure induced by the Engel structures and the characteristic foliations.

Notice that $\psi_M^{-1} \circ \widetilde{\varphi} \circ \psi_1$ preserves Engel structures and in particular characteristic foliations. Recall that we assumed that the characteristic foliations are oriented. Because the orientation of the contact structure on a transversal hypersurface is induced by the Engel structure and the orientation of the characteristic foliation, $\psi_M^{-1} \circ \widetilde{\varphi} \circ \psi_1$ preserves the orientation of the characteristic foliations. By definition, the characteristic foliation on M points outward along $\partial_+ R_1$ and it points inward R_1 along $\partial_- R_1$. This shows that $\widetilde{R}_1 \setminus R_1$ gets mapped on the collar of $\partial_+ M$ by $\psi_M^{-1} \circ \widetilde{\varphi} \circ \psi_1$. Thus we have shown

PROPOSITION 5.1. *Let R_1 carry a model Engel structure and let $\varphi : \partial_- R_1 \to \partial_+ M$ be an embedding which preserves oriented contact structures and oriented intersection line fields. Then we can extend the Engel structure from M to $M \cup_{\widetilde{\varphi}} \widetilde{R}_1$ canonically such that the resulting Engel structure is orientable and smooth away from the corners.*

So under some assumptions on the attaching map $\varphi : \partial_- R_1 \to \partial_+ M$ we can extend the Engel structure on M to an Engel structure on the manifold with corners

$$M \cup_\varphi R_1 \, .$$

Now let M_1 and M_2 be two manifolds with oriented Engel structures $\mathcal{D}_1, \mathcal{D}_2$ such that the boundary of M_i is transversal to the characteristic foliation of $\mathcal{D}_i$ for $i = 1, 2$. Moreover we suppose that the characteristic foliation is also oriented. Let $N_1 \subset \partial_+ M_1$ and $N_2 \subset \partial_- M_2$ be unions of connected components of the boundaries. We denote the induced contact structures on the boundary by $\mathcal{C}_i$ and the intersection line–fields by $\mathcal{F}_i$ for $i = 1, 2$.

THEOREM 5.2. *Let $\varphi : N_1 \longrightarrow N_2$ be a diffeomorphism preserving the oriented contact structures such that $\varphi_*(\mathcal{F}_1) = \mathcal{F}_2$ and the orientations of $\mathcal{F}_1, \mathcal{F}_2$ are preserved. Then one can glue M_1 and M_2 together using φ such that the oriented Engel structures $\mathcal{D}_1$ and $\mathcal{D}_2$ induce an oriented Engel structure on $M = M_1 \cup_\varphi M_2$.*

PROOF. The procedure is similar to Proposition 5.1 but simpler because there are no corners. We extend M_i along N_i vertically by $N_i \times [0, \varepsilon)$ where $\varepsilon > 0$. (If N_i is not compact it may be necessary to allow ε to vary on H_i.) By Theorem 3.19 and Proposition 3.16 applied to $\varphi : N_1 \longrightarrow N_2$, we can identify tubular neighbourhoods of N_1 and N_2.

By the assumption that φ preserves also the orientation of the intersection line field, the Engel structure on $M_1 \cup_\varphi M_2$ is canonically oriented. □

5.2. Vertical modifications of transversal boundaries

Using rotation numbers along Legendrian curves, one can distinguish homotopy classes of oriented Legendrian line fields. Now we want to explain how one can modify the intersection line field within its homotopy class.

Let M be an Engel manifold with transversal boundary. As usual we assume that the characteristic foliation and the Engel structure itself are oriented. This induces an orientation of the contact structure on the boundary. In addition we assume now that the boundary of M is compact. We treat the boundary components $\partial_+ M$ where $\mathcal{W}$ points out of M. The components $\partial_- M = \partial M \setminus \partial_+ M$ can be treated similarly.

Notice that it is not always possible to realize a prescribed change of the intersection line field by an isotopy of the hypersurface in the interior of M. However, when we deal with a transversal boundary we can add an arbitrary number of twists to the leafs passing through it by adding $\partial_+ M \times [0, \infty)$ with a suitable Engel structure.

Because the characteristic foliation $\mathcal{W}$ is transversal to the boundary of M, it is possible to choose a collar $U = \partial_+ M \times (-1, 0]$ of $\partial_+ M$ such that the one–dimensional foliation on U induced by the second factor corresponds to the characteristic foliation of the Engel structure. Since $\partial_+ M$ consists of those boundary components where $\mathcal{W}$ points out of M, the orientation of $\mathcal{W}$ corresponds to the usual orientation of $(-1, 0]$. We write w for the coordinate corresponding to the second factor of U.

Fix a positive section s of the oriented intersection line field on $\partial_+ M$. Furthermore let c be a vector field such that s, c is an oriented trivialization of $\mathcal{C}$. The horizontal lifts of s respectively c to $\partial_+ M \times (-1, 0]$ (or to $\partial_+ M \times (-1, \infty)$) will be denoted by the same letters. We identify $\partial_+ M$ and $\partial_+ M \times \{0\}$. On U the even contact structure $\mathcal{E}$ is spanned by s, c, ∂_w.

There is a unique smooth function $f : U \simeq \partial_+ M \times (-1, 0] \longrightarrow \mathbb{R}$ such that ∂_w and

$$X(p, w) = \cos(f(p, w)) s(p) + \sin(f(p, w)) c(p) \tag{45}$$

span and orient $\mathcal{D}_{(p,w)}$ such that $f(\cdot,0) \equiv 0$. Because $\mathcal{D}$ is an Engel structure, the commutator $[\partial_w, X]$ must be linearly independent of ∂_w, X everywhere. Now s and c are horizontal lifts. Hence

$$\left[\frac{\partial}{\partial w}, X\right](p,w) = \frac{\partial f}{\partial w}(p,w)\big(-\sin(f(p,w))s(p) + \cos(f(p,w))c(p)\big)\ .$$

This vector field has no component in ∂_w–direction. Thus $[\partial_w, X]$ is linearly independent of ∂_w and X if and only if

$$\begin{aligned} 0 \neq \det &\begin{pmatrix} \cos(f(p,w)) & -\frac{\partial f}{\partial w}(p,w)\sin(f(p,w)) \\ \sin(f(p,w)) & \frac{\partial f}{\partial w}(p,w)\cos(f(p,w)) \end{pmatrix} \\ = &\frac{\partial f}{\partial w}(p,w) \end{aligned}$$

holds everywhere. Thus f is either strictly increasing or strictly decreasing along the leaves of $\mathcal{W}$. According to our orientation conventions $\mathcal{E}$ is oriented by $\partial_w, X, [\partial_w, X]$ and this orientation is the orientation given by ∂_w, s, c. Thus the c–component of $[\partial_w, X]$ has to be positive for $t = 0$. This implies

$$\frac{\partial f}{\partial w} > 0\ . \tag{46}$$

Thus we can reparameterize $\partial_+ M \times (-1, 0]$ such that with the new coordinate $\widehat{w} = f(p,w)$ on the second factor of the collar the Engel structure on the collar is defined by

$$X(p,\widehat{w}) = \cos(\widehat{w})s(p) + \sin(\widehat{w})c(p)\ . \tag{47}$$

From now on we use the notation w instead of $\widehat{w}$. We attach $\partial_+ M \times [0,\infty)$ to M along $\partial_+ M$ in the obvious way and extend the Engel structure from M to the new manifold $M \cup \partial_+ M \times [0,\infty)$ by the span of ∂_w, X where X is defined as in (47) on $\partial_+ M \times [0,\infty)$. Note that now s, c are horizontal lifts on $M \times [0,\infty)$. Now we have a smooth Engel structure on $U \cup (\partial_+ M \times [0,\infty)$. The associated even contact structure $\mathcal{E}$ is the span of ∂_w, s, c and the characteristic foliation is spanned by ∂_w.

For a function $g : \partial_+ M \to [0,\infty)$ we define

$$M_g = M \cup_\partial \left\{(p,w) \in \partial_+ M \times [0,\infty) \middle| w \leq g(p)\right\}\ .$$

We will write N_g for $\partial_+ M_g$. By definition of M_g we have

$$N_g = \left\{(p, g(p)) \middle| p \in \partial_+ M\right\}\ .$$

Note that N_g is transversal to the characteristic foliation of $\mathcal{D}$ which, on $\partial_+ M \times [0,\infty)$ is induced by the second factor. By Lemma 3.5 the contact structure $\mathcal{E} \cap TN_g$ on N_g is identified with the contact structure on $\partial_+ M$ by

$$\begin{aligned} \psi_g : \partial_+ M &\longrightarrow N_g \\ p &\longmapsto (p, g(p)))\ . \end{aligned}$$

The manifolds with boundary M and M_g can be identified using a diffeomorphism $M_g \longrightarrow M$ which is a flow along the leaves of the characteristic foliation and such flows preserve the even contact structure. Hence M and M_g are equivalent as manifolds with even contact structure. However they are not equivalent as Engel manifolds because the foliations induced by the intersection line fields on the boundaries are not equivalent in general.

DEFINITION 5.3. The modification of an Engel manifold with boundary described above will be called *vertical modification of the boundary*.

Using this, we can show that every Legendrian line field on $\partial_+ M$ which is homotopic to the original intersection line field can be obtained as intersection line field of an Engel manifold.

THEOREM 5.4. *Let $(M, \mathcal{D})$ be an Engel manifold with transversal boundary and oriented characteristic foliations. If the Legendrian line field $\mathcal{L}$ is homotopic to the intersection line field $\mathcal{L}_\mathcal{D}$ of $\mathcal{D}$, then there is a function*

$$g : \partial_+ M \longrightarrow [0, \infty)$$

such that the intersection foliation on $\partial_+ M_g$ is mapped to $\mathcal{L}$ under the identification $\psi : \partial_+ M \rightarrow \partial_+ M_g$ induced by the characteristic foliation of the Engel structure on M_g.

PROOF. We use parts of the discussion above and the notation introduced there. Let us first assume that the intersection line field is orientable. The non–orientable case can be reduced to this situation. On $\partial_+ M \times [0, \infty)$, the Engel structure is spanned by ∂_w and X where $X(p, w)$ is defined by

$$X(p, w) = \cos(w) s(p) + \sin(w) c(p) \; . \tag{48}$$

By assumption there is a homotopy Z_t of Legendrian vector fields such that Z_0 orients $\mathcal{L}_\mathcal{D}$ and Z_1 orients $\mathcal{L}$. There is a smooth family of functions $\widetilde{g}_s$, $s \in [0, 1]$ such that $\widetilde{g}_0 \equiv 0$ and Z_t is a positive multiple of

$$\cos(\widetilde{g}_t(p)) s(p) + \sin(\widetilde{g}_t(p)) c(p) \; . \tag{49}$$

Because $\partial_+ M$ is compact, there is $m \in \mathbb{N}$ such that $\widetilde{g} \geq -2\pi m$. Now let

$$g = \widetilde{g}(\cdot, 1) + 2\pi m \geq 0 \; .$$

We claim that g has the required properties. By definition of the Engel structure on M_g, the Engel structure is spanned by ∂_w and Z_1 along $\partial_+ M_g$. But by definition Z_1 spans $\mathcal{L}$. So there is $\lambda \in \mathbb{R}$ such that the intersection line field along $\partial_+ M_g$ is spanned and oriented by $Z_1 + \lambda \partial_w$.

The projection of $Z_1 + \lambda \partial_w$ along the leaves of the characteristic foliation to $\partial_+ M$ is therefore $\mathcal{L}$.

This finishes the proof under the assumption that the intersection line field is orientable. If $\mathcal{L}_\mathcal{D}$ is not orientable, we pass to a two–fold covering of $\widetilde{\partial_+ M} \times (-\varepsilon, o]$ of a collar $\partial_+ M$ and pull back the Engel structure and the homotopy H_t connecting the pull back of the intersection line field with the pull back of $\mathcal{L}$. Here H_s is a family of Legendrian line fields. We choose the covering such that the pull back of the intersection line field becomes orientable.

Let f be the non–trivial deck transformation of the covering. We choose the oriented trivialization $\widetilde{s}_1, \widetilde{s}_2$ such that $\widetilde{s}_1$ spans the intersection line field and

$$\widetilde{s}_i(f(p)) = -f_*(\widetilde{s}_i(p)) \text{ and } \widetilde{c}_i(f(p)) = -f_*(\widetilde{c}_i(p)) \; . \tag{50}$$

We also choose a family of Legendrian vector fields Z_t spanning the pull back of the Legendrian line fields such that $Z_t(f(p)) = -f_*(Z_t(p))$. If $Z_t(p)$ is a positive multiple of

$$\cos(\widetilde{g}_t(p)) s(p) + \sin(\widetilde{g}_t(p)) c(p) \; ,$$

then $Z_t(f(p))$ is a positive multiple of

$$\cos(\widetilde{g}_t(p)) s(f(p)) + \sin(\widetilde{g}_t(p)) c(f(p))$$

by (50). Comparing this with (49) we obtain $\widetilde{g}_t(p) = \widetilde{g}_t(f(p))$. Thus the vertical modification of the boundary is actually well defined on $\partial_+ M$ even if the intersection line field is not orientable. $\square$

The assumption on the boundary of M to be compact can be weakened to the assumption on the homotopy to be constant outside of a compact set. We will apply Theorem 5.4 also to embeddings

$$\varphi : N \longrightarrow \partial_+ M$$

where N is a contact manifold with boundary carrying an oriented Legendrian line field $\mathcal{L}$. Assuming that φ preserves oriented contact structures one can compare $\varphi_*\mathcal{L}$ and the intersection line field on $\partial_+ M$. If these Legendrian line fields are homotopic on $\varphi(N)$ one chooses g as above on $\varphi(N) \subset \partial_+ M$ and extends g by a non–negative function to $\partial_+ M$. Using the identification ψ_g of $\partial_+ M_g$ with $\partial_+ M$ induced by the leaves of the characteristic foliation of the Engel structure we can consider

$$\psi_g \circ \varphi : N \longrightarrow \partial_+ M_g$$

This embedding preserves oriented contact structures and intersection line fields.

Notice that if $g(p)$ is a multiple of 2π for $p \in \partial_+ M$, the identification ψ_g preserves the intersection line field at p.

DEFINITION 5.5. If g is a multiple of 2π on some subset U of $\partial_+ M$ we say that the vertical modification does not change the intersection line field on U.

5.3. Doubles

In the first part of this section we explain a major tool for the construction of closed Engel manifolds. Choose a transversal hypersurface N in an Engel manifold M and cut M along this hypersurface. This induces an identification map $\psi : N \longrightarrow N$. Now glue round 1–handles to the domain and the target of ψ such that the Engel structures extend to the round handles. We obtain an Engel manifold which is cut along a hypersurface. Away from a compact set the new hypersurface coincides with N. If the round 1–handles are attached in a symmetric way we can construct an identification map $\widetilde{\psi}$ which coincides with ψ away from a compact subset of the interior of N such that we obtain a new closed Engel manifold. This is done in the proof of Theorem 5.6.

In the second part of this section we discuss the analogue of Theorem 5.6 for round two handles. We show that this construction will only lead to Engel manifolds we could also obtain from the original theorem for round 1–handles.

5.3.1. Adding a round 1–handle. Let M_1 and M_2 be two manifolds with boundary and oriented Engel structures $\mathcal{D}^1$ respectively $\mathcal{D}_2$. We assume that the characteristic foliation of both Engel structures is oriented and transversal to the boundary. Let

$$\psi : \partial_+ M_1 \longrightarrow \partial_- M_2$$

be a diffeomorphism preserving the induced contact structures together with their orientations. In addition to this, we assume that ψ preserves oriented intersection line fields.

Our aim is to attach round handles R_1, R_2 with model Engel structures to both M_1 and M_2 such that the boundaries of the new Engel manifolds $\widetilde{M_1} = M_1 \cup R_1$ and $\widetilde{M_2} = M_2 \cup R_2$ again admit a diffeomorphism

$$\widetilde{\psi} : \partial_+ \widetilde{M_1} \longrightarrow \partial_- \widetilde{M_2}$$

preserving oriented contact structures and the homotopy types of the intersection line fields. A vertical modification of $\widetilde{M_2}$ then leads to a pair of Engel manifolds which can be glued together along their boundary.

Note that R_1 is a round handle of index one and R_2 has index two. We attach R_1 along $\partial_- R_1$ to $\partial_+ M_1$ and R_2 along $\partial_+ R_2 \simeq \partial_- R_1$ to $\partial_+ M_2$. So we will treat R_2 like a round handle of index one.

THEOREM 5.6. *In the situation above, suppose that $\varphi_1 : \partial_- R_1 \to \partial_+ M_1$ is an attaching map which allows us to extend the Engel structure on M_1 to $M_1 \cup_{\varphi_1} R_1$ by the Engel structure $\Theta^m_* \mathcal{D}^{(1)}_k$ on R_1. Then there is an attaching map*

$$\varphi_2 : \partial_+ R_2 \to \partial_- M_2$$

isotopic to $\psi \circ \varphi_1$ and a Engel structure $\mathcal{D}'_2$ on M_2 such that $\mathcal{D}'_2$ extends to R_2 using the model Engel structure $\Theta^m_* \mathcal{D}^{(2)}_k$. $\mathcal{D}'_2$ and $\mathcal{D}_2$ are isotopic. Moreover there is a diffeomorphism

$$\widetilde{\psi} : \partial_+ \widetilde{M}_1 \longrightarrow \partial_- \widetilde{M}_2$$

preserving the oriented contact structures on the boundaries.

The intersection line field on $\partial_+ \widetilde{M}_1$ is mapped by $\widetilde{\psi}$ to a Legendrian line field which is homotopic to the intersection line field on $\partial_- \widetilde{M}_2$.

Let us first sketch the different steps of the proof of Theorem 5.6. We now identify R_1 and R_2. The proof consists of four steps:

(1) Modify the Engel structure on M_2 such that $\varphi_2 = \psi \circ \varphi_1$ is a gluing map for R_2 with the Engel structure $\Theta^m_* \mathcal{D}^{(2)}_k$. To do so, use first Gray's theorem to adapt contact structures and modify the boundary of M_2 vertically in order to achieve that φ_2 preserves the intersection line field on a neighbourhood of $\gamma_\pm$.
(2) Glue R_1 to M_1 and R_2 to M_2 in order to obtain $\widetilde{M}_1$ and $\widetilde{M}_2$.
(3) Apply Gray's theorem again in order to isotope the obvious map between $\widetilde{M}_1$ and $\widetilde{M}_2$ to a map which preserves oriented contact structures.
(4) Show that the resulting map preserves the homotopy type of the intersection line fields. This requires some analysis of the isotopy obtained in the third step.

PROOF OF THEOREM 5.6. On R_1 and R_2 we use the model Engel structure corresponding to the same parameters m, k. Our aim is of course to compare the present situation with the model discussed in Theorem 4.31. For this, it is convenient to use the coordinates

$$x = \pm 1, \Theta^{-m*} y_1, \Theta^{-m*} y_2, t$$

on R_1 and R_2. During this proof we use the notation x, y_1, y_2, t for the *new* coordinates. Then the Engel structure $\Theta^m_* \mathcal{D}^{(1)}_k$ is defined by the usual expressions for W_1 and $X^{(1)}_k$ and the analogous statement is true on R_2. By assumption $\varphi_1 : \partial_- R_1 \to \partial_+ M_1$ preserves oriented contact structures and oriented intersection line fields. Using φ_1 we identify $\partial_- R_1$ with its image $U \subset \partial_+ M_1$. In particular we obtain coordinates on U which we denote again by $x = \pm 1, y_1, y_2, t$. The contact structure on U is defined by the 1–form

$$\beta_0 = -dy_1 + \frac{1}{2} y_1 dt - \frac{1}{2} x dy_2 \tag{51}$$

with $x = \pm 1$. Moreover, the intersection line field on U is the same as in the model, it is spanned by $\widetilde{X}^{(1)}_k$.

So on U we have exactly the same situation as in the model for gluing round 1–handles. Now on $\psi(U)$ we have the coordinates

$$x' = x = \pm 1,\ y'_1 = \psi^{-1*} y_1,\ y'_2 = \psi^{-1*} y_2,\ t' = \psi^{-1*} t\ .$$

But on $\psi(U)$ the contact structure induced by the Engel structure on M_2 does *not* have the expression we used in the model for the gluing of round 2–handles but it is defined by β_0. In order to obtain the situation of the model on a subset of $\psi(U)$, we modify the Engel manifold $(M_2, \mathcal{D}_2)$ in two steps.

For the first step choose a smooth function $\rho : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with the properties

(i)
$$\rho(r) = \begin{cases} 1 & \text{if } r \in \left[0, \frac{1}{10}\right] \\ 0 & \text{if } r \in \left[\frac{9}{10}, \infty\right) \end{cases}$$

(ii)
$$\left|\frac{d\rho}{dr}(r)\right| < \frac{1}{2r} \; .$$

A function ρ with the desired properties exists because
$$\int_{1/10}^{9/10} \frac{1}{2r} dr > 1 \; .$$

Using ρ we will define a family of 1–forms β_s. The kernel of β_0 defines the original contact structure. The conditions (i),(ii) ensure that the deformed distributions $\ker(\beta_s)$ are also contact structures. Let $r = \sqrt{{y_1'}^2 + {y_2'}^2}$. For $s \in [0,1]$ consider the family of 1–forms

$$\beta_s = -\left(1 - 2s\rho(r)\right) dy_1' + \frac{1}{2} y_1' dt' - \frac{1}{2} x' dy_2' \tag{52}$$

By construction, β_s is constant on a neighbourhood of the boundary of $\psi(U)$. We extend β_s to the whole of $\partial_- M_2$ using a fixed defining form for the contact structure outside of $\psi(U) \subset \partial_- M_2$ coinciding with β_s near the boundary of $\psi(U)$. For all $s \in [0,1]$, the 1–form β_s defines a contact structure on $\partial_- M_2$ since

$$\begin{aligned} \beta_s \wedge d\beta_s &= \left(-s \frac{y_1' y_2'}{r^2} r \frac{d\rho}{dr}(r) + \frac{1}{4} x'\right) dy_1' \wedge dy_2' \wedge dt' \\ &= \begin{cases} > \left(-\frac{r}{2}\frac{1}{2r} + \frac{1}{4}\right) dy_1' \wedge dy_2' \wedge dt' = 0 & \text{if } x' = 1 \\ < \left(\frac{r}{2}\frac{1}{2r} - \frac{1}{4}\right) dy_1' \wedge dy_2' \wedge dt' = 0 & \text{if } x' = -1 \; . \end{cases} \end{aligned}$$

On $\{r < 1/10\} \subset \psi(U)$, β_1 has the same coordinate expression as the contact structure which appeared in the model for the gluing of round 2–handles. The homotopy is constant away from the compact subset $\psi(U) \subset \partial_- M_2$. So we can apply Gray's theorem to the family β_s and we obtain an isotopy Φ_s of $\partial_- M_2$ with the property
$$\Phi_{s*}\left(\ker\beta_0\right) = \ker\beta_s \; .$$

Using Φ we modify the Engel structure on M_2. Choose a collar $\partial_- M_2 \times [0,1]$ of $\partial_- M_2 = \partial_- M_2 \times \{0\}$ such that the characteristic foliation of the Engel structure corresponds to the foliation given by the second factor of the collar. Fix a smooth function $g : [0,1] \to [0,1]$ which is constant near the boundary of the interval with $g(0) = 1$ and $g(1) = 0$. Let
$$\begin{aligned} \Phi' : \partial_- M_2 \times [0,1] &\longrightarrow \partial_- M_2 \times [0,1] \\ (p,s) &\longmapsto \left(\Phi_{g(s)}(p), s\right) \end{aligned}$$

and extend this diffeomorphism by the identity to the whole of M_2. Instead of $\mathcal{D}_2$ we consider now the Engel structure $\mathcal{D}_2' = \Phi'_* \mathcal{D}_2$ on M_2 but we do not change the coordinates. Thus the contact structure induced and oriented by $\mathcal{D}_2'$ on $\partial_- M_2$ is defined by β_1. This 1–form defines the coorientation induced by $\mathcal{D}_2'$ if $k > 0$. If $k < 0$, β_1 and $\mathcal{D}_2'$ define opposite coorientations of the contact structure. On $\{r < 1/10\} \subset \psi(U)$ the Engel structure $\mathcal{D}_2'$ induces a contact structure which is defined by a 1–form having the same expression as the contact structure in the model.

Unfortunately, the intersection line field on $\partial_- M_2$ with the modified Engel structure $\mathcal{D}_2'$ does not coincide with the intersection line field in the model for gluing round 2–handles even on $\{r < 1/10\}$ where we have the right contact structure. However, by Proposition 4.29 (ii), the intersection line field of $\mathcal{D}_2'$ is already the one appearing in the model on the subset $\{y_1 = 0\}$ of $\{r < 1/10\}$.

In the second step of the modification of the initial Engel manifold M_2, we use a vertical modification of the boundary $\partial_- M_2$ to achieve that the intersection line field on $\{r < 1/10\}$ coincides with the intersection line field in the model for gluing round 2–handles on $\{r < 1/10\}$. By Theorem 5.4, this is possible since the rotation number of the intersection line field of $\mathcal{D}_2'$ along the Legendrian curve $\{x = \pm 1, y_1 = y_2 = 0\}$ is $-|k|$. On $\{y_1 = 0\} \cap \{r < 1/10\}$, the intersection line field already was the one of the model situation. So we may assume that along $\{y_1 = 0, r < 1/10\}$ the intersection line field remains unchanged even on $\{y_1 = 0\} \subset \psi(U)$. We also assume that the intersection line field remains unchanged outside of $\psi(U)$.

From now on we use the notations $\mathcal{D}_2'$ and M_2 for the Engel structure on the manifold obtained by vertical modification. By construction of the modified Engel manifold M_2

$$\varphi_2 = \psi \circ \varphi_1 : \partial_+ R_2 \cap \{r \leq 1/20\} \longrightarrow \partial_- M_2$$

is a gluing map for a round 2–handle with the model Engel structure $\Theta_*^m \mathcal{D}_k^{(2)}$. Let

$$\widetilde{M}_1 = M_1 \cup_{\varphi_1} \left(\left\{ r \leq \frac{1}{20} \right\} \cap R_1 \right)$$
$$\widetilde{M}_2 = M_2 \cup_{\varphi_2} \left(\left\{ r \leq \frac{1}{20} \right\} \cap R_2 \right)$$

be the manifolds obtained from $M_1 \cup_{\varphi_1} R_1$ and $M_2 \cup_{\varphi_2} R_2$ after smoothing corners as in Section 4.3.1. We write $\widetilde{\mathcal{D}}_1$ respectively $\widetilde{\mathcal{D}}_2'$ for the Engel structure obtained on $\widetilde{M}_1$ respectively $\widetilde{M}_2$. We extend the coordinates $x = \pm 1, y_1, y_2, t$ respectively x', y_1', y_2', t to a system of coordinates on R_1 respectively R_2 in the obvious way. In particular x varies now. Let V be the complement of $U = \varphi_1(\partial_- R_1)$ in $\partial_+ M_1$. There is a diffeomorphism

$$\psi' : \partial_+ \widetilde{M}_1 \longrightarrow \partial_- \widetilde{M}_2$$

defined as follows: Away from U let $\psi' = \psi$. On $\widetilde{U} = \widetilde{M}_1 \setminus V$ let ψ' be the identity map in terms of the coordinates x, y_1, y_2, t. These two definitions fit to a smooth diffeomorphism since we obtained the coordinates on $\partial_- M_2$ by ψ. On V, ψ' preserves oriented contact structures but not on $\widetilde{U}$.

The push–forward by ψ' of the contact structure on $\partial_+ \widetilde{M}_1$ and the contact structure on $\partial_- \widetilde{M}_2$ induced by $\widetilde{\mathcal{D}}_2'$ are homotopic, the homotopy is given by the family of 1–forms

$$\widetilde{\beta}_s = -(1 - 2s\rho(r))dy_1' + \frac{1}{2} y_1' dt' - \frac{1}{2} x' dy_2' - y_2' dx' \ . \tag{53}$$

As usual, $\widetilde{\beta}_s$ is constant on $\psi(V)$. Notice that $\beta_s = \widetilde{\beta}_s$ on $\{1/20 \leq r \leq 1\}$ since we have $x' = \pm 1$ and so $dx' = 0$ there. The push forward of the contact structure on $\partial_+ \widetilde{M}_1$ is defined by $\widetilde{\beta}_0$ while the actual contact structure on $\partial_- \widetilde{M}_2$ is defined by $\widetilde{\beta}_1$.

Applying Gray's theorem to this family of contact forms we obtain an isotopy

$$\widetilde{\Phi}_s : \partial_- \widetilde{M}_2 \longrightarrow \partial_- \widetilde{M}_2 \ .$$

On $\{r \leq 1/10\}$, the family $\widetilde{\beta}_s$ inducing this isotopy coincides with the family of 1–forms in the proof of Theorem 4.31 apart from the fact that there we had round 2–handles $D^2 \times D^1 \times S^1$ where the radius of the D^2–factor is one while here it is $1/20$. Let

$$\widetilde{\psi} = \widetilde{\Phi}_1 \circ \psi' : \partial_+ \widetilde{M}_1 \longrightarrow \partial_- \widetilde{M}_2 \ .$$

This map preserves the contact structures induced by the Engel structures $\widetilde{\mathcal{D}}_1$ respectively $\widetilde{\mathcal{D}}_2'$. Moreover $\widetilde{\psi}$ preserves the orientation of the contact structures since on V, we have $\widetilde{\psi} = \psi$ and ψ has this property by assumption.

It remains to show that $\widetilde{\psi}$ preserves the homotopy type of the intersection line fields. On V we have by definition $\widetilde{\psi} = \psi$ so $\widetilde{\psi}$ has the desired property on V. By Proposition 3.22 it now suffices to show that ψ preserves the homotopy type of the intersection line field only along some curves which meet $\partial_+ R_1$. These curves have to be chosen such that together with the curves contained in V, they generate $H_1(\partial_+ \widetilde{M}_1; \mathbb{Z})$.

Let σ be the function appearing in the smoothing procedure as explained bore formulating Theorem 4.31 and let $\sigma(0) = c$. Let $\gamma = \{x = y_1 = 0, y^2 = c\} \times S^1 \subset R_1$. The rotation number along this curve is preserved by $\widetilde{\psi}$ by Theorem 4.31 (iii). The same is true for $\gamma = \{x = 0\} \times \partial D^2 \times S^1$.

Now let γ represent any homology class in $H_1(\partial_+ \widetilde{M}_1; \mathbb{Z})$. Let $\Delta = \{y_1 = 0, t = 0\} \subset \partial_+ \widetilde{M}_1$ with endpoints $\{x = \pm 1, y_1 = 0, y_2 = 1/15, t = 0\}$, cf.Theorem 4.31. So the endpoints of Δ lie in the region where the isotopy of the model situation in Theorem 4.31 and our isotopy induced by $\widetilde{\beta}_s$ coincide because $\rho(r) = 1$ for $r < 1/10$. Since it is enough to treat a complete set of generators of $H_1(\partial_+ \widetilde{M}_1; \mathbb{Z})$ we can assume that

$$\gamma \cap \left\{ r \leq \frac{1}{15} \right\} \subset \Delta$$

$$\gamma \cap \widetilde{U} \subset \{y_1 = 0\} .$$

By Theorem 4.31 (v), the diffeomorphism $\widetilde{\psi}$ maps the intersection line field at the endpoints of Δ to the intersection line of $\mathcal{D}_2'$ at the endpoints of $\widetilde{\psi}(\Delta)$. Moreover $\widetilde{\psi}$ preserves the homotopy type of the intersection line fields relative to the boundary points of Δ.

Now along $\{y_1 = 0, r \geq 1/15\}$ the isotopies induced by β_s and $\widetilde{\beta}_s$ coincide and both preserve $\{y_1 = 0\}$. This can be checked by a calculation similar to the construction of the flow ψ^- in Proposition 4.29. Now on the one hand, we did not change the intersection line field along $y_1 = 0$ when we modified M_2 vertically. On the other hand, the intersection line field induced by $\mathcal{D}_2'$ on $\{y_1 = 0, r \geq 1/15\}$ is by definition the image of the intersection line field induced by $\mathcal{D}_1$ on $\partial_+ M_1$ under $\Phi_1 \circ \psi$ where Φ_s is the isotopy obtained from β_s.

Hence $\widetilde{\psi}$ preserves the intersection line field along $\gamma \setminus \Delta$. This shows that $\widetilde{\psi}$ preserves the homotopy type of the intersection line fields. □

5.3.2. Adding a round 2–handle. Whether Theorem 5.6 is also true for round 2–handles is not clear at least to the author. It seems to be difficult to find a deformation of the contact structure which is constant away from a neighbourhood of the attaching region of R_2 like in (52) or (53).

Assume that the construction of the maps φ_2 and $\widetilde{\psi}$ in the proof of Theorem 5.6 also works for round 2–handles. We want to show that using this hypothetical construction we obtain no new Engel manifolds.

Let M_1, M_2 be oriented Engel manifolds with transversal boundary and $\psi : \partial_+ M_1 \longrightarrow \partial_- M_2$ as in Theorem 5.6. The attaching map $\varphi_1 : \partial_- R_2 \longrightarrow \partial_+ M_1$ is supposed to preserve oriented contact structures and intersection line fields. We attach a round 2–handle with some model Engel structure in order to obtain the Engel manifold $M_1 \cup_{\varphi_1} R_2$. Let

$$\varphi_2 : \partial_+ R_1 \longrightarrow \partial_- M_2$$

$$\widetilde{\psi} : \partial_+ \left(M_1 \cup_{\varphi_1} R_2\right) \longrightarrow \partial_- \left(M_2 \cup_{\varphi_2} R_1\right)$$

be the maps constructed as in the proof of Theorem 5.6. The double $\widetilde{M}$ of $M \cup_\varphi R_2$ is

$$\widetilde{M} = (M_1 \cup_{\varphi_1} R_2) \cup_{\widetilde{\psi}} (R_1 \cup_{\varphi_2} M_2)$$

We apply the argument from Remark 4.4 to $\widetilde{\psi}$. The radial vector field

$$y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}$$

on ∂_+R_2 preserves the contact structure and we can extend this vector field to a global contact vector field on ∂_+M by Proposition 2.7. Using the flow of this vector field we deform $\widetilde{\psi}$ to $\widetilde{\Psi}$. Then obtain the Engel manifold

$$(M_1\cup_{\varphi_1}R_2)\cup_{\widetilde{\Psi}}(R_1\cup_{\varphi_2}M_2)\ .$$

But now we can interchange R_1 and R_2 and we end up with a double which is decomposed into $M_1\cup R_1$ and $M_2\cup R_2$. The attaching map of R_1 is the restriction of $\widetilde{\Psi}$ to $\partial_-R_1\subset\partial_-(M_2\cup_{\varphi_2}R_1)$ and similarly for the attaching map of R_2. The gluing map

$$\partial_+(M_1\cup R_1)\longrightarrow\partial_-(M_2\cup R_2)$$

can be defined piecewise. Away from ∂_+R_1 it is $\widetilde{\Psi}$ while on ∂_+R_1 the gluing map is φ_2. This is isotopic through contact diffeomorphisms to the result of the construction given in the proof of Theorem 5.6 applied to the initial data

$$\widetilde{\Psi}:\partial_-R_1\longrightarrow\partial_+M_1$$
$$\psi:\partial_+M_1\longrightarrow\partial_-M_2\ .$$

This is an Engel manifold we can obtain from Theorem 5.6 for round 1–handles. Thus even if Theorem 5.6 were true for round 2–handles it would only lead to Engel manifolds which can be obtained using Theorem 5.6.

5.4. Modifications of rotation numbers and framings

Let M be a manifold with boundary and an Engel structure $\mathcal{D}$. We suppose that $\mathcal{D}$ as well as the characteristic line field $\mathcal{W}$ is oriented. The other distributions associated to an Engel structure are then oriented by our conventions. We suppose that the boundary of M is transversal to $\mathcal{W}$. Starting from an embedding

$$\varphi:\partial_-R_1\longrightarrow\partial_+M$$

we want to determine whether φ can be isotoped to a map $\widetilde{\varphi}$ which preserves oriented contact structures and intersection line fields of a model Engel structure $\mathcal{D}^{(1)}_{k,m}$. Then we can attach R_1 using $\widetilde{\varphi}$ instead of φ and extend the Engel structure from M to $M\cup_{\widetilde{\varphi}}R_1$. This manifold is diffeomorphic to $M\cup_\varphi R_1$ since φ and $\widetilde{\varphi}$ are isotopic.

A necessary condition is that φ preserves the orientations on ∂_-R_1 respectively ∂_+M induced by the contact structures. Recall that all contact structures obtained from $\mathcal{D}^{(1)}_{k,m}$ induce the same orientation on ∂_-R_1.

Let $\gamma_\pm=S^1\times\{0\}\times\{\pm1\}\subset\partial_-R_1$. Using Proposition 2.10 we can isotope φ to an embedding φ' such that $\varphi'(\gamma_\pm)$ are two Legendrian curves. The next step would be a choice of model Engel structure. Whether or not φ' can be isotoped to a contact embedding with respect to the contact structure induced by the model Engel structure on ∂_-R_1 of course depends on the choice of the model Engel structure. Here we want to determine under which conditions it is possible to choose a model Engel structure on R_1 such that we can isotope φ' to an embedding allowing us to extend the Engel structure using the model. We assume that the isotopy is constant along $\gamma_\pm$.

The answer will be of course in terms of contact framings and rotation numbers of $\varphi'(\gamma_\pm)$. Although we have fixed particular Legendrian curves in the isotopy class of $\varphi(\gamma_\pm)$, it will turn out that the condition we will find will not depend on this choice. It is a condition depending only on the isotopy class of φ. From now on we assume that φ already maps $\gamma_\pm$ to Legendrian curves.

Recall that $\partial_- R_1$ has two connected components. We will write φ_+ respectively φ_- for the restriction of φ to $\{x = 1\} \times D^1 \times S^1$ respectively $\{x = -1\} \times D^2 \times S^1$. A contact framing of $\gamma_\pm$ induced by the model Engel structure $\mathcal{D}^{(1)}_{k,m}$ on R_1 will be denoted by $\mathrm{fr}(\gamma_\pm, m)$. If σ is a Legendrian curve in $\partial_+ M$ we write $\mathrm{fr}(\sigma)$ for a contact framing of σ. When two framings (S,T) and (S',T') of a fixed curve are homotopic we write $(S,T) \sim (S',T')$.

By Lemma 2.12 there exist $n_+, n_- \in \mathbb{Z}$ such that

$$\varphi_{+*}\big(\mathrm{fr}(\gamma_+,0)\big) \sim n_+ \cdot \mathrm{fr}(\varphi_+(\gamma_+))$$
$$\varphi_{-*}\big(\mathrm{fr}(\gamma_-,0)\big) \sim n_- \cdot \mathrm{fr}(\varphi_-(\gamma_-))\ .$$

The following theorem gives a criterion whether one can stabilize φ_+ and φ_- in order to meet the conditions on framings and rotation numbers.

THEOREM 5.7. *We can choose $k \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}$ and stabilize the attaching map $\varphi_\pm$ such that the modified maps have the following properties with respect to the Engel structure $\mathcal{D}^{(1)}_{k,m}$ on R_1*

(i) *the stabilized attaching map sends a contact framing of $\gamma_\pm$ to a framing $\varphi_\pm(\gamma_\pm)$ which is homotopic to a contact framing,*

(ii) *the rotation numbers of $\mathcal{D}$ along the stabilized Legendrian curves obtained from $\varphi_+(\gamma_+)$ and $\varphi_-(\gamma_-)$ are both equal to k*

if and only if the condition

$$n_+ + \mathrm{rot}(\varphi_+(\gamma_+)) \equiv n_- + \mathrm{rot}(\varphi_-(\gamma_-)) \mod 2 \tag{54}$$

is satisfied

PROOF. Throughout this proof k will denote a nonzero integer which will be fixed at the end.

Recall that $n_+, n_- \in \mathbb{Z}$ satisfy

$$\varphi_{+*}\big(\mathrm{fr}(\gamma_+,0)\big) \sim n_+ \cdot \mathrm{fr}(\varphi_+(\gamma_+))$$
$$\varphi_{-*}\big(\mathrm{fr}(\gamma_-,0)\big) \sim n_- \cdot \mathrm{fr}(\varphi_-(\gamma_-))$$

Because φ_+ is orientation preserving

$$m \cdot \big(\varphi_{+*}(S,T)\big) \sim \varphi_{+*}\big(m \cdot (S,T)\big)$$

holds for every framing (S,T) of γ_+. The analogous statement with φ_-, γ_- is also true. If we use the Engel structure $\mathcal{D}^{(1)}_{k,m}$ instead of $\mathcal{D}^{(1)}_{k,0}$ on R_1 we obtain

$$\varphi_{+*}\big(\mathrm{fr}(\gamma_+,m)\big) \sim (m+n_+) \cdot \mathrm{fr}(\varphi_+(\gamma_+))$$
$$\varphi_{+*}\big(\mathrm{fr}(\gamma_-,m)\big) \sim (m+n_-) \cdot \mathrm{fr}(\varphi_-(\gamma_-))\ .$$

From the discussion in Section 2.2.4 it follows that both positive and negative twists have the following effect on contact framings

$$\left(\sigma^\pm\varphi_+\right)_* (\mathrm{fr}(\gamma_+,m)) \sim (n_+ + m - 1) \cdot \mathrm{fr}\left(\sigma^\pm\left(\varphi_+(\gamma_+)\right)\right)$$
$$\left(\sigma^\pm\varphi_-\right)_* (\mathrm{fr}(\gamma_-,m)) \sim (n_- + m - 1) \cdot \mathrm{fr}\left(\sigma^\pm\left(\varphi_-(\gamma_-)\right)\right)\ .$$

Since we want the stabilized embedding $\widetilde{\varphi}_\pm$ to map contact framings of $\gamma_\pm$ to a framing of $\widetilde{\varphi}_\pm(\gamma_\pm)$ which is homotopic to a contact framing, we have to apply positive or negative stabilization $(n_+ + m)$–times respectively $(n_- + m)$–times to φ_+ respectively φ_-. Since there is (in general) no inverse procedure to stabilization we have to achieve that $n_+ + m$ and $n_- + m$ are both non–negative.

Depending on how often we apply σ^+ and σ^- respectively, we get different results for the rotation numbers since by Section 2.2.4

$$\text{rot}\left(\left(\sigma^+(\varphi_+)\right)(\gamma_+)\right) = \text{rot}\left(\varphi_+(\gamma_+)\right) + 1$$
$$\text{rot}\left(\left(\sigma^-(\varphi_+)\right)(\gamma_+)\right) = \text{rot}\left(\varphi_+(\gamma_+)\right) - 1\ ,$$

and similarly for γ_-. If $n_+^+, n_+^-, n_-^+, n_-^- \in \mathbb{N}_0$ satisfy

$$\begin{aligned} n_+ + m &= n_+^+ + n_+^- \geq 0 \\ n_- + m &= n_-^+ + n_-^- \geq 0\ , \end{aligned} \tag{55}$$

we get the following effect on rotation numbers

$$\text{rot}\left(\left((\sigma_+)^{n_+^+}(\sigma_-)^{n_+^-}\varphi_+\right)(\gamma_+)\right) = \text{rot}(\varphi_+(\gamma_+)) + n_+^+ - n_+^-$$
$$\text{rot}\left(\left((\sigma_+)^{n_-^+}(\sigma_-)^{n_-^-}\varphi_-\right)(\gamma_-)\right) = \text{rot}(\varphi_-(\gamma_-)) + n_-^+ - n_-^-\ .$$

We want equal and non–zero rotation numbers after stabilization. This can be achieved if and only if we can solve (55) and

$$\begin{aligned} n_-^+ - n_-^- - n_+^+ + n_+^- &= \text{rot}(\varphi_+(\gamma_+)) - \text{rot}(\varphi_-(\gamma_-)) \\ \text{rot}(\gamma_+) + n_+^+ - n_+^- &\neq 0 \end{aligned} \tag{56}$$

with nonnegative integers $n_+^+, n_+^-, n_-^+, n_-^-$ and $m \in \mathbb{Z}$. Then we can take

$$\begin{aligned} k &= \text{rot}(\varphi_+(\gamma_+)) + n_+^+ - n_+^- \\ &= \text{rot}(\varphi_-(\gamma_-)) + n_-^+ - n_-^-\ . \end{aligned}$$

Considering the equations (55) $\mod 2$ and comparing this with

$$n_-^+ - n_-^- - n_+^+ + n_+^- = \text{rot}(\varphi_+(\gamma_+)) - \text{rot}(\varphi_-(\gamma_-)) \mod 2$$

we see that (54) is a necessary condition for the solvability of (56) and (55). If (54) is satisfied, this system of equations admits solutions in $\mathbb{Z}$. If we choose m large enough, we can achieve $n_+^+, n_+^-, n_-^+, n_-^- \in \mathbb{N}_0$. □

We want to explain the meaning of (54) in more topological terms. For this we consider an orientation preserving attaching map $\varphi_\pm : \partial_- R_1 \to \partial_+ M$. The Engel structure on M determines a trivialization of TM which is well defined up to homotopy. We can pull back a trivialization of the boundary $\partial_+ M$. In order to obtain a trivialization of the tangent bundle of R_1 on $\partial_- R_1$ we add an inward pointing vector field. If we want to extend an Engel structure on M over R_1 we have to be able to extend the trivialization on M to $M \cup_\varphi R_1$. This is possible if and only if the pull back trivialization of TR_1 on $\{x = -1\}$ is homotopic to the pullback trivialization on $\{x = 1\}$. The homotopy between these two trivializations then provides an obvious extension of the trivialization on $\partial_- R_1$ to R_1.

Whether or not it is possible to extend the trivialization on M to $\widetilde{M} = M \cup_\varphi R_1$ depends only on the isotopy class of $\varphi_\pm$ and the trivialization on M.

Now assume that for an even contact structure $\Theta_*^m(\ker(\alpha_1))$ on R_1 we have isotoped $\varphi_\pm$ to a map (again denoted by $\varphi_\pm$) that preserves contact structures together with their orientations. This is always possible (for suitable m) by the arguments used in the proof of Theorem 5.7. The present situation corresponds to $n_+ = n_- = 0$ in the above notation. The even contact structure $\ker(\alpha_1)$ has a trivialization over the whole of R_1. We compare the pull back trivialization with a given trivialization of TR_1 in order to see whether it is possible to extend the pull back trivialization. Since $(R_1, \partial_- R_1)$ retracts onto $(\{y_1 = y_2 =$

$0\}, \gamma_+ \cup \gamma_-)$ it suffices to consider the extension problem on this cylinder. Comparing the pull back framing with the given framing on γ_+ respectively γ_-, we obtain maps

$$f_+ : S^1 = \gamma_+ \longrightarrow GL(4)$$
$$f_- : S^1 = \gamma_- \longrightarrow GL(4)$$

and the extension problem can be solved if and only if f_- and f_+ represent the same element in $\pi_1(GL(4)) = \pi_1(SO(4)) = \mathbb{Z}_2$.

Now that $\varphi_\pm$ preserves contact structures and orientations, the homotopy class of the pull back trivialization is fixed by the homotopy class of the trivialization of the contact structure on the two components of $\partial_- R_1$. The homotopy class of the pull back trivialization can be determined by the rotation number with respect to the given framing of TR_1. So f_- and f_+ are homotopic if and only if

$$\text{rot}(\varphi_+(\gamma_+)) \equiv \text{rot}(\varphi_-(\gamma_-)) \mod 2 .$$

Since we have achieved $n_+ = n_- = 0$ this corresponds to (54).

Thus if we start with an attaching map $\varphi_\pm$ and end up with a map which violates (54) then there is *no* map isotopic to $\varphi_\pm$ which could be used to glue a round 1–handle to M and extend Engel structures on M.

Since M has trivial tangent bundle, $w_2(TM) = 0$. When we attach a round handle of index 1 to M we add the cylinder $\{y_1 = y_2 = 0\}$ to the 2-skeleton of (a triangulation or CW–decomposition of) M. Condition (54) ensures that the given trivialization of the tangent bundle extends over the cylinder. In particular, the tangent bundle of $\widetilde{M}$ is trivial over the 2–skeleton of $\widetilde{M}$. Thus (54) makes sure that the second Stiefel–Whitney class remains zero after we glued the round handle to M.

Now if $\varphi : \partial_- R_1 \longrightarrow \partial_+ M$ is an embedding such that $\varphi(\gamma_\pm)$ are Legendrian curves and φ preserves contact framings and rotation numbers along $\gamma_\pm$ then by Proposition 2.18 we can isotope φ relative to $\gamma_\pm$ such that the resulting map preserves the contact structure on a tubular neighbourhood of $\gamma_\pm$. For $0 < s \leq 1$

$$\partial_- R_1 \longrightarrow \partial_- R_1$$
$$(x = \pm 1, y_1, y_2, t) \longmapsto (x, sy_1, sy_2, t)$$

is a contact isotopy. This shows

THEOREM 5.8. *Assume that $\varphi : \partial_- R_1 \longrightarrow \partial_+ M$ is an embedding, the trivialization of TM induced by the Engel structure can be extended to $M \cup_\varphi R_1$.*

Then there is a model Engel structure on R_1 such that φ is isotopic to an embedding $\widetilde{\varphi}$ which preserves contact structures.

5.5. New Engel manifolds – Doubles

As a first application, we give examples of Engel manifolds whose fundamental group contains relatively big Abelian subgroups. This topological property can be used to show that the manifolds we construct are not total spaces of fibrations over the circle or a 3–manifold. In particular, these Engel manifolds are not covered by the Geiges construction or prolongation.

LEMMA 5.9. *Let M be a manifold and $H \subset \partial M$ a connected component of the boundary. Consider $\widetilde{M} = M \cup_{\text{id}_H} M$. Then $i : M \hookrightarrow \widetilde{M}$ induces an inclusion*

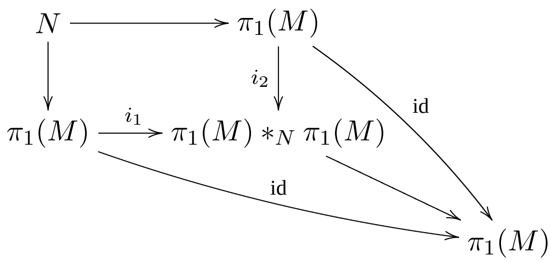
$$i_\# : \pi_1(M) \longrightarrow \pi_1(\widetilde{M}) .$$

If all elements of $\pi_1(M)$ have representatives which are contained in H, i.e. the inclusion $H \longrightarrow M$ induces an epimorphism of fundamental groups, then $i_\#$ is bijective.

PROOF. For all fundamental groups we use a fixed base point in H. Let $N = \pi_1(H)$. By the theorem of Seifert–van Kampen, the inclusions of M respectively H into $\widetilde{M}$ induce an isomorphism between the fundamental group of $\widetilde{M}$ and

$$\pi_1(M) *_N \pi_1(M) .$$

Applying the universal property of the amalgamated product we can find a unique homomorphism $\pi_1(M) *_N \pi_1(M) \longrightarrow \pi_1(M)$ such that the diagram



commutes. i_1 respectively i_2 maps $\pi_1(M)$ to the first respectively second factor of the amalgamated product. In particular $\pi_1(M) \to \pi_1(M) *_N \pi_1(M) \simeq \pi_1(\widetilde{M})$ is induced by the inclusion $M \to \widetilde{M}$ and injective.

The amalgamated product $\pi_1(M) *_N \pi_1(M)$ can be defined as the free product of $\pi_1(M)$ with itself divided by the normal subgroup generated by

$$\left\{ i_1(a)\,(i_2(a))^{-1} \middle| a \in N = \pi_1(H) \right\} .$$

If $\pi_1(H) = N \to \pi_1(M)$ is surjective, we can replace in every word representing an element of $\pi_1(M) *_N \pi_1(M)$ all letters coming from the second factor in the free product by elements coming from the first factor. Then $i_\#$ is also surjective. $\square$

THEOREM 5.10. *Let G be a group which admits a presentation*

$$G = \langle g_0, g_1, \ldots, g_k | \, r_1, \ldots, r_k \rangle$$

such that for all $i \in \{1, \ldots, k\}$ the relation r_i involves only the generators $g_0, \ldots, g_{i-1}$. Then one can obtain a closed Engel manifold whose fundamental group is isomorphic to G using our first construction of Engel structures.

PROOF. We apply Theorem 5.6 inductively to construct a pair of Engel handle bodies using only round handles of index 1 and 0. Starting point for the construction is the Engel structure on $S^3 \times S^1$ described in Section 4.2.1. The fundamental group $\pi_1(S^3 \times S^1) \simeq \mathbb{Z}$ satisfies the assumptions in the theorem and proves it for $k = 0$. Notice that $\{y_1 = 0\} \simeq S^2 \times S^1$ is transversal to the characteristic foliation. Thus $S^3 \times S^1$ can be obtained from one round handle of index 0 and one round handle of index 3 by an identification of the boundaries of the handles.

Now we come to the inductive step. Suppose that we have an Engel manifold M with fundamental group

$$G_j = \langle g_0, \ldots, g_j | r_1, \ldots, r_j \rangle .$$

We assume that M can be cut along a connected transversal hypersurface H into two pieces M_1 and M_2 which are diffeomorphic, and we assume that the characteristic foliation points out of ∂M_1 and into M_2 along ∂M_2. We denote the identification of the boundaries by ψ. This map preserves oriented contact structures and intersection line fields. In order to apply Lemma 5.9 we suppose furthermore that if we identify M_1 with M_2 then with respect to this identification ψ is isotopic to the identity of the boundary. We assume also that the generators $g_0, \ldots, g_j \in \pi_1(M)$ have representatives which are contained in $\partial_+ M_1$ (we choose the base point in $\partial_+ M_1$). Notice that all these assumptions are satisfied in the case of $S^3 \times S^1$.

In order to apply Theorem 5.6 we need an attaching map $\varphi_1 : \partial_- R_1 \to \partial_+ M_1$ for a round handle of index one. Because the generators $g_0, \dots, g_j$ have representatives contained in $\partial_+ M_1$, the same is true for r_{j+1}. By Proposition 2.10 we can choose a Legendrian representative $\widehat{\gamma}_+$ of $r_{j+1} \in G_j$ and a homotopically trivial Legendrian curve $\widehat{\gamma}_-$. We push away $\widehat{\gamma}_\pm$ from the basepoint by a very short distance. For dimension reasons we can assume that the curves $\widehat{\gamma}_\pm$ are now disjoint from a fixed set of curves representing $g_0, \dots, g_j$.

Fix a model Engel structure on R_1 and an orientation preserving embedding

$$\varphi'_1 : \partial_- R_1 \longrightarrow \partial_+ M_1$$

mapping $\gamma_\pm$ to $\widehat{\gamma}_\pm$. We may assume that φ'_1 satisfies (54) in Theorem 5.7. If not, we change the framing of φ_1 along γ_-. Applying Theorem 5.7, Theorem 5.8 and a suitable vertical modification of $\partial_+ M_1$ and $\partial_- M_2$, we find a model Engel structure on R_1 and an attaching map φ_1 such that the Engel structure extends from M to $\widetilde{M}_1 = M \cup_{\varphi_1} R_1$.

By Theorem 5.6 we can attach a round 2–handle to $\partial_- M_2$ such that we can extend the Engel structure on M_2 to $\widetilde{M}_2 = M_2 \cup R_2$. Moreover we obtain a diffeomorphism

$$\widetilde{\psi} : \partial_+ \widetilde{M}_1 \longrightarrow \partial_- \widetilde{M}_2$$

which allows us to glue $\widetilde{M}_1$ and $\widetilde{M}_2$ together along the boundary by Theorem 5.2. We obtain a closed Engel manifold $\widetilde{M}$. By construction, $\widetilde{M}_1$ and $\widetilde{M}_2$ are diffeomorphic as manifolds and $\widetilde{\psi}$ is isotopic to the identity with respect to this identification.

We now show that $\widetilde{M}$ has fundamental group G_{j+1}. Notice that the round handle induces the relation r_{j+1} by sliding the curve $\widehat{\gamma}_+$ from $x = 1$ to $x = -1$. This way, $\widehat{\gamma}_+$ becomes homotopically trivial. Choosing representatives of r_{j+1} which lie on the boundary of the attaching region, we can perform this homotopy completely in the boundary of $\widetilde{M}_1$. Moreover the fundamental group of $\widetilde{M}_1$ has one additional generator g_{j+1} which is represented by a curve joining the two ends of R_1 in the round handle together with a curve joining the two components of the attaching region in the remaining part of ∂M_1. In particular, the new generator of the fundamental group of $\widetilde{M}_1$ can be represented by a curve which lies completely in $\partial \widetilde{M}_1$. Thus $\widetilde{M}_1$ has fundamental group $\pi_1(\widetilde{M}_1) = G_{j+1}$. Since $\widetilde{M}$ is isotopic to the double of $\widetilde{M}_1$, the fundamental group of $\widetilde{M}$ is G_{j+1} by Lemma 5.9.

Finally note that we have shown that $\widetilde{M}$ satisfies the same hypothesis as M did in the inductive step if we cut along $\widetilde{H} = \partial_- \widetilde{M}_1 \subset \widetilde{M}$. □

We want to show that many of the Engel manifolds obtained from Theorem 5.10 do not fiber over S^1 or a 3–manifold. The next proposition shows that such fibrations have special topological properties. It is based on the following theorem about the fundamental group of 3–manifolds.

THEOREM 5.11 (Hempel, [**Hem**] p. 84). *Let G be a finitely generated Abelian group. If G is a subgroup of $\pi_1(M)$ for some three–manifold M, then G is isomorphic to one of*

$$\mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},\ \mathbb{Z} \oplus \mathbb{Z}_2 \text{ or } \mathbb{Z}_n$$

for some integer n. In particular $\mathrm{rank}(G) \leq 3$.

Using this theorem one could find several criteria for deciding whether a given four–manifold fibers over the circle or over a three–manifold. In the following proposition we explain one possibility.

PROPOSITION 5.12. *Let M be a connected 4–manifold. If M is a fibration over a three–manifold or a circle then the rank of every Abelian subgroup of $\pi_1(M)$ is at most 4.*

PROOF. Suppose M fibers over the circle S^1 with fiber N. Let $i : N \longrightarrow M$ be the inclusion of a fiber and pr the bundle projection. Without loss of generality we assume that N is connected. The long exact sequence of homotopy groups yields

$$\pi_2(S^1) = \{0\} \longrightarrow \pi_1(N) \xrightarrow{i_\#} \pi_1(M) \xrightarrow{\mathrm{pr}_\#} \pi_1(S^1) \simeq \mathbb{Z} \longrightarrow \{0\}\ . \tag{57}$$

Let G be an Abelian subgroup of $\pi_1(M)$. Either $\mathrm{pr}_\#$ is zero on G or the image $\mathrm{pr}_\#(G)$ is isomorphic to $\mathbb{Z}$. In the first case $i_\#^{-1}(G)$ is isomorphic to G. Since $i_\#^{-1}(G)$ is a subgroup of $\pi_1(N)$, we have $\mathrm{rank}(G) \leq 3$. In the second case choose a generator h of $\mathrm{pr}_\#^{-1}(1)$. The map $\mathbb{Z} \longrightarrow G$ which maps n to $n \cdot h$ induces a splitting of the short exact sequence of Abelian groups obtained from (57)

$$0 \longrightarrow i_\#^{-1}(G) \xrightarrow{i_\#} G \xrightarrow{\mathrm{pr}_\#} Z \simeq \mathrm{pr}_\#(G) \longrightarrow 0\ .$$

This induces an isomorphism $G \simeq i_\#^{-1}(G) \times \mathbb{Z}$. By Theorem 5.11 the rank of G is smaller or equal than 4.

Now suppose that M fibers over a three–manifold N with fiber S^1. We use the same notation for the inclusion of a fiber and the bundle projection as above. Applying the long exact sequence of homotopy groups again we obtain

$$\pi_2(N) \longrightarrow \pi_1(S^1) \simeq \mathbb{Z} \xrightarrow{i_\#} \pi_1(M) \xrightarrow{\mathrm{pr}_\#} \pi_1(N) \longrightarrow \{1\}\ . \tag{58}$$

The image of $\pi_2(N)$ is a subgroup of $\mathbb{Z}$, therefore it is either $\{0\}$ or isomorphic to $\mathbb{Z}$. In the first case, we have a short exact sequence

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(N) \longrightarrow \{0\}\ , \tag{59}$$

in the second case there is an integer n such that $\mathrm{im}(\pi_2(M)) = n\mathbb{Z}$ and we get

$$\{0\} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(N) \longrightarrow \{0\} \tag{60}$$

from (58). Now let G be an Abelian subgroup of $\pi_1(M)$. The image $\mathrm{pr}_\#(G)$ is an Abelian subgroup of $\pi_1(N)$. We have $\mathrm{rank}(i_\#^{-1}(G)) \leq 1$. Since (59) and (60) are exact

$$\mathrm{rank}(i_\#^{-1}(G)) - \mathrm{rank}(G) + \mathrm{rank}(\mathrm{pr}_\#(G)) = 0$$

and hence by Theorem 5.11

$$\mathrm{rank}(G) \leq 1 + \mathrm{rank}(\mathrm{pr}_\#(G)) \leq 4\ .$$

□

EXAMPLE 5.13. It is of course easy to find a presentation of a group satisfying the assumption of Theorem 5.10 and containing an Abelian subgroup of rank 5. One of the simplest is

$$\left\langle g_0, \ldots, g_{11} \middle| r_2 = g_0 g_1 {g_0}^{-1} {g_1}^{-1}, r_3 = g_0 g_2 {g_0}^{-1} {g_2}^{-1}, \ldots, r_{11} = g_3 g_4 {g_3}^{-1} {g_4}^{-1} \right\rangle\ .$$

We have 10 relations. Here $g_0, \ldots, g_4$ generate $\mathbb{Z}^5$.

5.6. Connected sums

Let M, M' be two Engel manifolds with Engel structures $\mathcal{D}, \mathcal{D}'$. The connected sum $M \# M'$ does not admit an Engel structure because the Euler characteristic of this connected sum is -2. Introducing an additional summand $S^2 \times S^2$, one can sometimes circumvent this problem if some condition on the Engel structure is satisfied.

THEOREM 5.14. *Let M, M' be manifolds with Engel structures $\mathcal{D}, \mathcal{D}'$ such that both characteristic foliations admit closed transversals. Then $M\#M'\#(S^2 \times S^2)$ carries an Engel structure which coincides with the old Engel structures on M and M' away from a neighbourhood of the transversals where all connected sums are performed. The characteristic foliation of the new Engel structure again admits a closed transversal.*

PROOF. Let us assume for the moment that M, M' are oriented. Fix the induced orientation of the characteristic foliation $\mathcal{W}$ of $\mathcal{D}$ respectively $\mathcal{W}'$ of $\mathcal{D}'$. Choose closed transversals N respectively N' of $\mathcal{W}$ respectively $\mathcal{W}'$. We cut the manifolds along these hypersurfaces and obtain new manifolds with boundary. These will be denoted again by M respectively M'. The boundary of each manifold M and M' has two connected components

$$\begin{aligned} \partial_+ M &\simeq N \simeq \partial_- M \\ \partial_+ M' &\simeq N' \simeq \partial_- M' \, . \end{aligned}$$

There is a natural identification

$$\psi : \partial_+ M \cup \partial_+ M' \longrightarrow \partial_- M \cup \partial_- M'$$

which satisfies the assumptions of Theorem 5.6. We choose contractible Darboux charts $(x, y, z), U \subset \partial_+ M$ and $(x', y', z'), U' \subset \partial_+ M'$ for the contact structures. We fix an orientation of the intersection line fields on U and U'. It is not necessary to orient the intersection line field on the entire hypersurfaces N, N' for vertical modifications (cf. Theorem 5.4).

In each chart choose a Legendrian unknot K respectively K' with rotation number -1 and Thurston–Bennequin invariant -2. According to [**El3**] this determines $K \subset U$ and $K' \subset U'$ uniquely up to Legendrian isotopy within U, U'. One can obtain K, K' by negative stabilization of the Legendrian unknot with Thurston–Bennequin invariant -1.

We equip R_1 with model Engel structure $\mathcal{D}_{0,1}^{(1)}$. Recall

$$\gamma_\pm = \{\pm 1\} \times \{(0,0)\} \times S^1 \subset \partial D^1 \times D^2 \times S^1 = \partial_- R_1 \, .$$

The contact framing of $\gamma_\pm$ is S^1–invariant. Choose an attaching map φ_0 for R_1 which preserves oriented contact framings and maps γ_+ to K and γ_- to K'. The rotation number along $\gamma_\pm$ is also -1.

Thus φ_0 preserves oriented contact framings and the homotopy class of the intersection line fields. As a consequence we can isotope φ_0 such that the resulting attaching map φ_1 preserves oriented contact structures. Throughout the isotopy $\gamma_\pm$ is mapped to $\varphi_0(\gamma_\pm)$.

With a vertical modification of $\partial_+ M \cup \partial_+ M'$ we can achieve that φ_1 also preserves oriented intersection line fields and not only their homotopy types. After this vertical modification, ψ no longer preserves the intersection line field. We apply a vertical modification to $\partial_- M \cup \partial_- M'$ to restore this property.

Using Theorem 5.6 we obtain an attaching map for a round 2–handle with a model Engel structure

$$\varphi_2 : \partial_+ R_2 \longrightarrow \partial_- M \cup \partial_- M' \, .$$

For $i = 1, 2$ we attach R_i using φ_i. The modified boundary components are denoted by $\partial_\pm \widetilde{M}$. Theorem 5.6 also yields a diffeomorphism

$$\widetilde{\psi} : \partial_+ \widetilde{M} \longrightarrow \partial_- \widetilde{M}$$

preserving oriented contact structures and intersection line fields up to homotopy. Using a vertical modification for the last time in this proof, we finally obtain a closed connected Engel manifold $\widetilde{M}$ when we identify the two boundary components $\partial_\pm \widetilde{M}$.

The hypersurfaces N, N' are still contained in $\widetilde{M}$ and they are transversal to the characteristic foliation of the Engel structure we have constructed. It remains to show that $\widetilde{M}$ is diffeomorphic to $M\#M'\#(S^2\times S^2)$. In order to show this, we construct $M\#M'\#(S^2\times S^2)$ using ordinary handles and we apply Lemma 4.8 to identify $M\#M'\#(S^2\times S^2)$ with the manifold obtained from the construction above.

Using an orientation preserving attaching map $\hat{\varphi}_1$, we attach a one–handle connecting ∂_+M and ∂_+M'. We do the same with ∂_-M and ∂_-M' using the attaching map $\hat{\varphi}_2 = \psi\circ\hat{\varphi}_1$. If we identify the new boundaries now in the natural way, we obtain $M\#M'$.

Choose a ball in ∂_+M which is disjoint from the attaching region of the one–handle. Attach a 2–handle along an unknot contained in this ball with framing -4 to ∂_+M. The

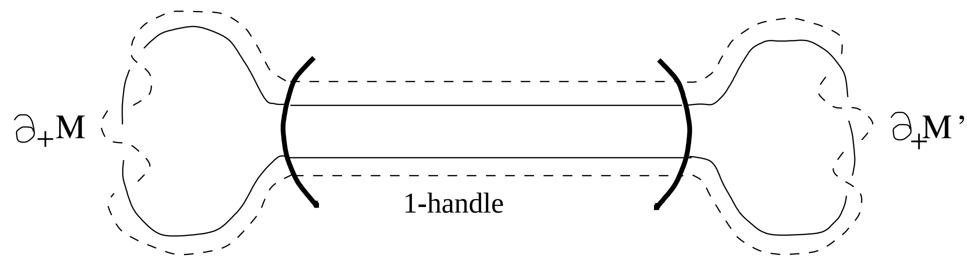


FIGURE 2.

handles of index 1 and 2 are attached independently and we can use Lemma 4.8: As in the proof of that lemma, we first slide the 2–handle over the 1–handle. Figure 2 shows the attaching curve of the 2–handle after the slide. The framing is indicated by the dashed curve and the two arcs represent the boundary of the 1–handle. After we identify the two ordinary handles with a round handle of index one, we may assume that the attaching map of the round two handle has framing -2 at both ends. Then the attaching map of the round one–handle is isotopic to the attaching map φ_0 we started with at the beginning of this proof.

Thus if we attach a one–handle and a two–handle as above to both $\partial_+M\cup\partial_+M'$ and $\partial_-M\cup\partial_-M'$ in a symmetric way and identify the new boundaries, then we obtain a manifold diffeomorphic to $\widetilde{M}$.

On the other hand, the one–handles account for the direct sum $M\#M'$. When we want to show $\widetilde{M}\simeq M\#M'\#(S^2\times S^2)$, we have to understand the two–handles. If one attaches a two–handle to D^4 along an unknot with framing -4, the second two–handle coming from the double is attached along a zero–framed meridian of the unknot. Two consecutive handle slides show that one can use the zero–framing on both unknots without changing the diffeomorphism type of the manifold, cf. [**GoS**] p. 144. We obtain the usual Kirby diagram of $S^2\times S^2$. This proves the claim under the assumption that M and M' are oriented.

We assume for simplicity that M' is orientable. This assumption can be dropped in the same way as for M. If M is not orientable, there are two possibilities. Either N is coorientable or not. If N is coorientable, we orient the characteristic foliation on a tubular neighbourhood of N. This suffices to carry out the proof above. If N is not coorientable, the situation is slightly more complicated. If we cut M along N, the boundary of the resulting manifold is a connected two–fold covering of N. The non–trivial deck transformation ψ interchanges points, which correspond to the same point in N. The restriction of $\mathcal{W}$ to ∂M can be oriented by an outward pointing section. This orientation of $\mathcal{W}$ near ∂M induces an orientation of the contact structure on ∂M.

Now we choose Darboux charts U and $\psi(U)$ and perform the same construction as above. On U we orient $\mathcal{W}$ such that it points out of M and on $f(U)$ such that it points inwards. These orientations are not compatible with an orientation of $\mathcal{W}$ on ∂M but this does not matter. Whenever we apply vertical modification to arrange the intersection line fields on U, the intersection line field on $\psi(U)$ does not really change since $g \equiv 2k\pi, k \in \mathbb{N}$ there (g is the function appearing in the vertical modification). The same statement is true for vertical modifications of $\psi(U)$. Thus we can pretend that we can apply vertical modifications on U and $\psi(U)$ independently. As before we do not need an oriented intersection line field but only on orientable contact structure on ∂M. Then the proof carries over to this situation. □

In order to apply Theorem 5.14, one has to find Engel structures whose characteristic foliation admits a closed transversal. This is true for the Engel structures we shall construct in the proof of Theorem 6.1. The following example shows that closed transversals do not always exist.

EXAMPLE 5.15. Let N be an orientable 3–manifold such that TN has an orientable subbundle with non–trivial Euler class $e \in H^2(N;\mathbb{R})$. By Theorem 2.2, there is a contact structure $\mathcal{C}$ on N which is homotopic to the original subbundle.

Now the prolongation construction yields an Engel structure on $\mathbb{P}\mathcal{C}$. The leaves of the characteristic foliation are the fibers of the S^1–bundle pr : $\mathbb{P}\mathcal{C} \longrightarrow N$, the Euler class of this S^1–bundle is $e \neq 0$. In particular, the characteristic foliation of the Engel structure on $\mathbb{P}\mathcal{C}$ does not admit a closed transversal.

Engel structures obtained this way are so simple that they can be easily deformed to Engel structures which satisfy the assumption of Theorem 5.14. For this, choose a contractible Darboux chart $((x,y,z),U) \simeq \mathbb{R}^3$ in M. Choose a contact vector field V with compact support in U such that V has a non–degenerate sink at the origin. Fix a trivialization $\mathrm{pr}^{-1}(U) \simeq U \times S^1$ and write t for the coordinate on the S^1–factor.

On $\mathrm{pr}^{-1}(U)$, the Engel structure $\mathcal{D}$ is spanned by $W = \partial_t$ and a second vector field X. For $\varepsilon > 0$ small enough, the distribution $\mathcal{D}_\varepsilon$ spanned by $W_\varepsilon = \partial_t + \varepsilon V$ and X is still an Engel structure. Since V is a contact vector field, the characteristic foliation on $\mathrm{pr}^{-1}(U)$ is spanned by W_ε. If S^2 is a small sphere around the origin in U which is transversal to V, then $\mathrm{pr}^{-1}(S^2) \simeq S^2 \times S^1$ is a closed transversal of $\mathcal{W}_\varepsilon$.

Hence we can apply Theorem 5.14 to Engel structures obtained by prolongation after we perturb them slightly.

COROLLARY 5.16. *If $(N_1, \mathcal{C}_1)$ and $(N_2, \mathcal{C}_2)$ are manifolds with orientable contact structure, then $\mathbb{P}\mathcal{C}_1 \# \mathbb{P}\mathcal{C}_2 \# (S^2 \times S^2)$ admits an Engel structure.*

Starting from contact structures on $S^3, T^3, S^2 \times S^1$ which are trivial as vector bundles, we find Engel structures on manifolds like

$$N = T^4 \# (S^2 \times T^2) \# (S^2 \times S^2)$$
$$M_k = k(S^3 \times S^1) \# (k-1)(S^2 \times S^2)$$

using Corollary 5.16. One can show that it is impossible to construct an Engel structure on M_k using prolongation or the method of Geiges, although M_k is the total space of a circle bundle over a 3–manifold.

We return to the proof of Theorem 5.14 and discuss the meaning of the assumption that both Engel structures have characteristic foliations which admit a closed transversal. We do not make explicit use of the fact that N and N' are *closed* transversals. But implicitly, this assumption is used when we apply vertical modification.

Let us recall the construction of vertical modifications of transversal boundaries from Section 5.2. The aim is to change the intersection line field on a transversal boundary within its homotopy class of Legendrian line fields. Assume for simplicity that the intersection line field is orientable. When we modify $\partial_+ M$ vertically, we first attach $\partial_+ M \times [0,\infty)$ to $\partial_+ M$. On $\partial_+ M \times [0,\infty)$, the Engel structure is spanned by

$$W = \frac{\partial}{\partial t}, X = \cos(t)s + \sin(t)c \tag{61}$$

where s spans the intersection foliation on $\partial_+ M$ and s, c is an oriented trivialization of the contact structure. The modified Engel manifold is then defined using a positive function $g : \partial_+ M \longrightarrow R^+$ as follows

$$M_g = M \cup \left\{(p,t) \in \partial_+ M \times [0,\infty) \middle| t \leq g(p)\right\} \ .$$

Suppose that $U \subset M$ and $U' \subset M'$ are compact hypersurfaces with boundary transversal to the characteristic foliations. Then we can try to cut along U and U' and perform the construction of Theorem 5.14. When we cut along U, U' we do not obtain manifolds with boundary, the problematic points are the boundary points of U, U', but if we carry out all constructions in the interior of U and U' without changing anything on a neighbourhood of ∂U and $\partial U'$, this does not cause problems. We orient $\mathcal{W}$ and $\mathcal{W}'$ near U and U'. We use the notation similar to the notation used in the proof of Theorem 5.14, i.e. $\partial_+ M \simeq U$, etc.

Assume that $\mathcal{L}$ is the intersection line field on U and $\mathcal{L}'$ is another Legendrian line field $\mathcal{L}$ such that the homotopy $H_s, s \in [0,1]$ connecting them is constant away from a compact subset in U. Now consider U as a hypersurface in the Engel manifold $U \times \mathbb{R}$ (not $U \times [0,\infty)$) with the Engel structure defined as in (61). From H_s one can construct a function g with the following properties.

(i) g has compact support in U.
(ii) If one identifies $U \times \{0\}$ and $U_g = \{(p, g(p)) \in U \times \mathbb{R}\}$ using the characteristic foliation of the Engel structure, the intersection line field on U_g is mapped to $\mathcal{L}'$.

Then the intersection line field on the boundary of $M_g \subset U \times \mathbb{R}$ has the desired form. Unfortunately it is not possible to perform this construction in $M \cup U \times [0,\infty)$ in general. If $g(p)$ is negative, the corresponding point of U_g would lie in the interior of the manifold M with the original Engel structure. But it is not true in general that, as one moves along $\mathcal{W}_p \subset M$, the Engel structure rotates around $\mathcal{W}$ in $\mathcal{E}$ often enough.

If for all $p \in U$ the twisting number defined in Definition 3.30 satisfies the condition

$$\text{tw}^-(p) > |g(p)| + 1 \ , \tag{62}$$

then it is possible to embed the relevant piece of $U \times \mathbb{R}$, namely

$$\left\{(p,t) \middle| g(p) \leq t\right\} \subset U \times \mathbb{R}$$

into $M \cup U \times [0,\infty)$ such that the Engel structures are preserved.

Using this observation, one can replace the assumption in Theorem 5.14 that the characteristic foliations of the Engel structures admit closed transversals by a condition on the twisting numbers of leaves of $\mathcal{W}$ respectively $\mathcal{W}'$ passing trough a compact transversal hypersurface U respectively U'.

THEOREM 5.17. *Let M, M' carry Engel structures $\mathcal{D}, \mathcal{D}'$ such that there are non-closed leaves $\mathcal{W}_0$ through $p_0 \in M$ and $\mathcal{W}_0'$ through $q_0 \in M'$ of the characteristic foliations such that*

$$\text{tw}^{\pm}(p_0) \geq C \textit{ and } \text{tw}^{\pm}(q_0) \geq C \tag{63}$$

for some constant C which is independent of the Engel structures.

Then there is an Engel structure on $M\#M'\#(S^2 \times S^2)$ *which coincides with the Engel structure on* M, M' *outside of neighbourhoods of* p_0, q_0 *where all connected sums are formed. There is a point in* $M\#M'\#(S^2 \times S^2)$ *which satisfies condition* (63)*.*

If $\mathcal{W}_0$ *or* $\mathcal{W}_0'$ *are closed, the same conclusion holds if one replaces* C *by* $2(C+1)$ *in* (63) *in the condition on the closed leaf.*

PROOF. We perform the construction in a model situation. The relevant part of this model situation can be recovered in all Engel manifolds satisfying the assumptions of the theorem. The constant C will appear right after the discussion of the model construction. We start with the description of the model situation and how it arises in Engel manifolds. First we assume the case that both $\mathcal{W}_0$ and $\mathcal{W}_0'$ are open.

Choose a chart $V \subset M$ around p_0 and coordinates w, x, y, z such that the Engel structure is defined as the intersection of the kernels of

(64) $$\alpha = dz - xdy \text{ and } \beta = dx - wdy\ .$$

We may assume that the coordinates of p_0 are $(0,0,0,0)$. Let U be a closed 3–ball with constant w–coordinate through p_0. U is transversal to the characteristic foliation. We orient the normal bundle $\mathcal{W}$ of U by ∂_w. In an analogous way we choose a chart V' and a 3–ball U' in M' such that q_0 has the coordinates $(0,0,0,0)$.

By definition of the development map (cf. Definition 3.26) and of the twisting numbers (cf. Definition 3.30), there is a neighbourhood $\widetilde{U}$ of $\mathcal{W}_p$ such that $\widetilde{U}/\mathcal{W}$ is a well defined smooth manifold and $\widetilde{U} \longrightarrow \widetilde{U}/\mathcal{W}$ is a smooth submersion. We can identify a neighbourhood of $p_0 \in \widetilde{U}/\mathcal{W}$ with a neighbourhood of $p_0 \in U$. we assume that this neighbourhood is actually U itself. According to the definition of tw$^{\pm}$ and by continuity we can assume that for all points $p \in U$, the twisting numbers tw$^{\pm}(p) \geq C - 1$. On M' we proceed in the same manner.

Rescaling the coordinates appropriately, we can achieve that U contains $[-1,1]^3$. We carry out all constructions within this domain. Equip V and V' with a Riemannian metric such that $\partial_w, \partial_x, \partial_y, \partial_z$ is an orthonormal frame. Let s, s' be sections of the intersection line field on U, U' with unit length and let c, c' be two sections of the contact structure on U, U' such that s, c respectively s', c' form an oriented orthonormal frame of the contact structure on U and U'.

Now consider the manifolds $U \times \mathbb{R}$ respectively $U' \times \mathbb{R}'$ with the Engel structures

$$\widetilde{\mathcal{D}} = \text{span}\left\{\frac{\partial}{\partial w}, \cos(w)s(p) + \sin(w)c(p)\right\}$$
$$\widetilde{\mathcal{D}}' = \text{span}\left\{\frac{\partial}{\partial w'}, \cos(w')s(p') + \sin(w')c(p')\right\}\ .$$

We apply the procedure the proof of Theorem 5.14 to the Engel manifolds $U \times \mathbb{R}$ and $U \times \mathbb{R}'$. The only difference is the restriction to transversal modifications which do not change anything on open neighbourhoods of the boundaries of U and U'. The function g which characterizes the vertical modification has compact support in U and similarly for U'.

There is yet another small complication when we want to apply vertical modifications. This appears after we attach the round handles. To explain this we focus on the round 1–handle. With the exception of the unstable manifold of the periodic orbit in the center of R_1 all leaves of the characteristic foliation contain a segment $\{p\} \times (-\infty, a(p)] \subset U \times \mathbb{R}$. For all points p on these leaves $tw^-(p) = \infty$ follows. On the other hand all points p on leaves of $\mathcal{W}$ which are contained in the unstable manifold also have have the property tw$^-(p) = \infty$. Hence we can apply vertical modification also after we attached the round 1–handle.

Throughout this construction, vertical modification is applied several times. Let $\widetilde{C}_+$ and $\widetilde{C}_-$ be the maximal and the minimal value of all the functions which occur when vertical modifications of the boundary are applied.

We have performed the construction in a model situation. If

$$C \geq \max\{\widetilde{C}_+, \widetilde{C}_-\} + 1 ,$$

this procedure can be carried out with $U \subset M$ and not only with $U = U \times \{0\} \subset U \times \mathbb{R}$ since then we recover the relevant piece of the Engel manifolds $U \times \mathbb{R}$ respectively $U' \times \mathbb{R}$ in M respectively M'. The constant C does not depend on the Engel manifolds $(M, \mathcal{D})$ and $(M', \mathcal{D}')$.

If $\mathcal{W}_0$ is not closed, we have to ensure that the vertical modifications on the boundary $\partial_+ M$ (we use the term *boundary* although we do not really have a manifold with boundary) and the vertical modifications of $\partial_- M$ never interfere. This is ensured when we replace C by $2(C+1)$ in (63). □

We do not try to determine the constant C in this theorem. The theorem can be applied if $\text{tw}^{\pm}(p_0) = \text{tw}^{\pm}(q_0) = \infty$ or when one can enlarge the twisting numbers by a perturbation or an explicit construction like in the following example.

EXAMPLE 5.18. Let $N, \mathcal{C}$ be a contact manifold and let C_1, C_2 be a trivialization of $\mathcal{C}$. Then on $N \times S^1$ we have the usual Engel structure spanned by

$$\cos(kt)C_1 + \sin(kt)C_2$$

and the tangent space of the fibers of the projection $N \times S^1 \longrightarrow N$. If we choose k big enough we can apply Theorem 5.17.

Let us finally point out that the conditions (63) are not always fulfilled, e.g. the Engel manifolds obtained from Nil^4 in Example 3.32 or the standard Engel structure on $\mathbb{R}^4$ do not satisfy (63).

CHAPTER 6

The existence theorem

In this chapter we discuss our second construction of Engel structures. We prove the converse of Theorem 3.37.

THEOREM 6.1. *Every parallelizable closed manifold of dimension* 4 *admits an oriented Engel structure.*

Note that on open 4–manifolds with trivial tangent bundle, an Engel structure can be constructed using the h–principle for open, Diff–invariant relations, cf. [**ElM**]. The proof of Theorem 6.1 covers this chapter. First we give an overview.

Let M be a closed 4–manifold with trivial tangent bundle. Fix a round handle decomposition of M with exactly one round 3–handle and a trivialization of TM. The round handle decomposition can be chosen such that round handles are attached according to their index. We write M_1 for the manifold with boundary containing only the round handles of index zero and one. M_2 will contain all round handles of index zero, one and two.

The strategy of the proof is to perform the attachments of round handles one after the other and to show that each time the Engel structure we have already constructed can be extended by a model Engel structure on the round handle.

We will show that until the last attachment of a round handle of index 1, we can homotop the original trivialization such that it coincides with a distinguished *Engel trivialization* on the round handle body. In particular after we have attached the last round 1–handle the Engel trivialization extends to the entire manifold M.

Then we attach the round 2–handles. At this stage we will make use of the flexibility of singular foliations of tori in overtwisted contact manifolds. Together with the fact that the Engel trivialization on M_1 extends to M this will allow us to show that when we attach a round 2–handle R_2 to M' we can isotope the attaching map and find a suitable model Engel structure extending the given Engel structure to $M' \cup R_2$.

In general, the Engel trivialization on $M' \cup R_2$ and the given trivialization are not homotopic relative to M'. After the attachment of the last round 2–handle with a model Engel structure it is therefore not clear if the Engel trivialization on M_2 extends over the whole of M. This is a necessary condition for the possibility to extend the Engel structure on M_2 to the whole of M.

At this point we use the fact that we did not start with an arbitrary round handle decomposition but one with only one round 3–handle. So we are left with exactly one round 3–handle over which we have to extend the Engel structure as well as the Engel trivialization. On the other hand the Engel trivialization on M_2 is not arbitrary: The component corresponding to the characteristic foliation of the Engel structure is transversal to ∂M_2. Together these two facts will allow us to show that the Engel trivialization can be extended to M.

This in turn will be used to pick a model Engel structure on the round 3–handle such that the Engel structure on M_2 can be extended to the whole of M. This finishes the proof.

Let us compare our proof and the following characterization of parallelizable manifolds.

THEOREM 6.2 (Hirzebruch, Hopf, [**HH**]). *An orientable 4–manifold has trivial tangent bundle if and only if*

(i) *the Euler characteristic vanishes,*
(ii) *the second Stiefel–Whitney class is zero, i.e.* $w_2(M) = 0$ *and*
(iii) *the signature* $\sigma(M)$ *of* M *is zero.*

Since we start with a round handle decomposition, the condition on the Euler characteristic is used throughout the proof, cf. Theorem 4.6. The second Stiefel–Whitney class $w_2(M)$ of an orientable 4–manifold M is zero if and only if TM is trivial on the 2–skeleton of M. When one decomposes a round handle of index 1 respectively 2 as in Lemma 4.8 one obtains an ordinary 2–handle and another handle of index 1 respectively 3. Thus we use condition (ii) at two stages of the proof: First when we attach round 1–handles (Theorem 5.8) and later when we attach round 2–handles (Claim (1) and (2) of the proof of Theorem 6.1 in Section 6.4). Finally we use the vanishing of the signature at the final stage of the proof when we show that the Engel trivialization extends from M_2 to M.

We rely on several facts from the theory of contact structures. We have summarized them in Chapter 2. On the round 1–handles we use the same model Engel structures as in our first construction in Chapter 5. In Section 5.4 we have shown that when ever the Engel trivialization extends from M to $M \cup_\varphi R_1$, then we can isotope the attaching map such that the Engel structure can be extended to $M \cup_\varphi R_1$ by a model Engel structure on R_1.

In Section 6.1 and Section 6.3 we define model Engel structures on round handles of index two and three. In particular for round handles of index 2 we obtain a large variety of model Engel structures. Still the contact structure on $\partial_- R_2$ is equivalent for all model Engel structures. We do not describe the characteristic foliation in the interior of R_2 but we ensure only that it is transversal to both boundary components. At this point we use the fact that every contact vector field on a submanifold can be extended to a global contact vector field by Proposition 2.7.

In order to isotope attaching maps for round 2–handles to contact embeddings we use bypasses in overtwisted contact structures (Section 2.4) in Section 6.2. The proof of Theorem 6.1 is given in Section 6.4.

6.1. Model Engel structures on round handles of index 2

In this section, we construct Engel structures on round handles of index 2. Recall that such a handle is defined to be

$$R_2 = D^2 \times I \times S^1 .$$

We have already constructed model Engel structures on round 2–handles in Section 4.2.3. Now we want to get model Engel structures with properties as in the next proposition.

PROPOSITION 6.3. *Given integers* $n \in \mathbb{Z}$ *and* $k \in \mathbb{Z} \setminus \{0\}$, *there is a model Engel structure on* R_2 *with the following properties.*

(i) *The characteristic foliation of* $\mathcal{D}$ *can be oriented such that it points*
$$\text{outwards along } \partial_+ R_2 = D^2 \times \partial I \times S^1$$
$$\text{inwards along } \partial_- R_2 = \partial D^2 \times I \times S^1 .$$
(ii) *The singular foliation on* $T_0^2 = \partial D^2 \times \{0\} \times S^1$ *is divided by two homotopically non–trivial curves. It is in standard form. The Legendrian ruling corresponds to the first factor of* $T_0 = \partial D^2 \times \{0\} \times S^1$. *The dividing curves are tangent to the last factor. In particular* T_0^2 *is convex.*
(iii) *The rotation number of the intersection line field along* $\gamma = \partial D^2 \times \{0\} \times \{0\}$ *(with its orientation as boundary* ∂D^2*) is* $2n$.

(iv) *The rotation number of the intersection line field along the Legendrian divides (with the canonical orientation of the last factor of $\partial D^2 \times \{0\} \times S^1$) is $k \neq 0$.*
(v) *The orientation of the contact structure on $\partial_+ R_2$ can be chosen freely.*

All model Engel structures induce the same contact structure on a neighbourhood of $T_0^2 \subset \partial_- R_2$.

REMARK 6.4. The conditions (iii),(iv) and the orientation of the contact structure on $\partial_- R_2$ determine the homotopy class of the intersection line field as Legendrian line field. This is explained in Proposition 3.22.

PROOF. The proof is by an explicit construction. We will choose the even contact structure first. The rotation number along $\partial D^2 \times \{0\} \times \{1\}$ is (up to sign) already determined by this choice. The starting point is a singular foliation $\mathcal{F}$ on a disc D^2. On D^2 we use polar coordinates (r, φ). Choose $\mathcal{F}$ such that

(i) on the collar $A = \{r > 1/2\} = \partial D^2 \times (1/2, 1]$, $\mathcal{F}$ is defined by $\cos(\varphi)dr$.
(ii) $\mathcal{F}$ admits a dividing set Γ containing the straight arc γ_0 from $(r = 1, \varphi = 0)$ to $(r = 1, \varphi = \pi)$.
(iii) except for γ_0, all components of Γ are closed and bound a disc containing no other components of Γ. All closed components lie in the same part of $D^2 \setminus \gamma_0$.

Figure 1 shows a possible $\mathcal{F}$ such that the dividing set has two connected components in the lower half disc. The thickened curves divide $\mathcal{F}$. Similar singular foliations can be found for one or more such components. By Theorem 2.25 we can choose an $\mathbb{R}$–invariant positive

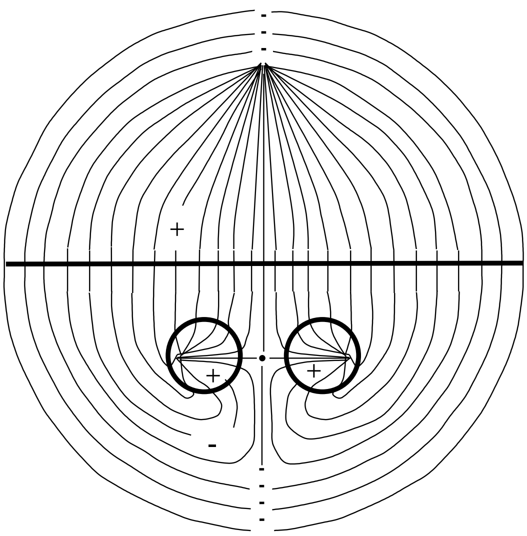


FIGURE 1.

contact form α on $D^2 \times \mathbb{R}$ such that the induced singular foliation on $D^2 \times \{0\}$ is $\mathcal{F}$. Let $\mathcal{C} = \ker(\alpha)$. The coordinate corresponding to the $\mathbb{R}$–factor is x. We may assume that on $A \times \mathbb{R}$ we have

$$\alpha = \cos(\varphi)dr + \sin(\varphi)dx \ .$$

This choice fixes an orientation of the contact structure. In order to find a contact vector field V and a 2–handle $h_2 \subset D^2 \times \mathbb{R}$ such that V is transversal to ∂h_2, we need to take some care since we know nothing about the region $r < 1/2$, except that ∂_x is a contact vector field everywhere. We focus first on $A \times \mathbb{R}$. Let g_1, g_2 be functions depending only on x. The contact vector field V associated to the function

$$h = g_1(x)\cos(\varphi) + g_2(x)\sin(\varphi)$$

can be determined using the proof of Proposition 2.7. We obtain

$$V = g_1(x)\frac{\partial}{\partial r} - \left(g_1'(x)\cos^2(\varphi) + g_2'(x)\sin(\varphi)\cos(\varphi)\right)\frac{\partial}{\partial\varphi} + g_2(x)\frac{\partial}{\partial x}$$

We choose the functions g_1, g_2 such that

$$g_1(x) = \begin{cases} 0 & \text{for } |x| \geq 1 \\ -1 & \text{for } |x| \leq \frac{3}{4} \end{cases}$$

$$g_2(x) = \begin{cases} a & \text{for } x \geq \frac{3}{4} \\ -a & \text{for } x \leq -\frac{3}{4} \\ 0 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}\,. \end{cases} \tag{65}$$

for a positive constant a. For this choice of g_1, g_2, the contact vector field V on $A \times \mathbb{R}$ can be extended by $a \cdot \operatorname{sgn}(x)\partial_x$ on $|x| \geq 1$ to a smooth contact vector field which we still denote by V. Finally we extend V to a contact vector field on the whole of $D^2 \times \mathbb{R}$. For this it is enough to extend the function $\alpha(V)$ to a smooth function and then to apply Proposition 2.7, the extension will have zeroes in general. It is transversal to $\partial D^2 \times [-3/4, 3/4]$ and points inwards. Now consider the pair of hypersurfaces defined by the equation

$$|x| = \frac{5}{4} - \frac{r^2}{2}\,.$$

Since $r \leq 1$, both are contained in the region $|x| \geq 3/4$. Thus $g_2 = \pm a$ depending on the sign of x.

$$L_V\left(x - \frac{5}{4} + \frac{r^2}{2}\right) = rg_1(x)a \qquad \text{if } x > 0$$

$$L_V\left(x + \frac{5}{4} - \frac{r^2}{2}\right) = -rg_1(x) - a \qquad \text{if } x < 0\,.$$

Thus if we fix a big enough, V is transversal to the hypersurfaces $\{|x| = 5/4 - r^2/2\}$ and it points outwards. We define

$$h_2 = \left\{(r,\varphi,x)\,\middle|\, |x| \leq 5/4 - r^2/2\right\}\,.$$

h_2 is an ordinary handle of index 2 such that V is transversal to both boundary components. By our construction, V has the desired orientations along $\partial_\pm h_2$. Figure 2 shows h_2 and V along the boundary of h_2.

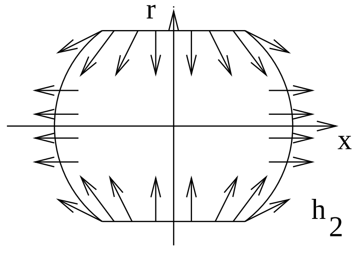


FIGURE 2.

The rotation number of $\partial D^2 \subset h_2$ with respect to the contact structure $\ker(\alpha) = \mathcal{C}$ can be determined using the formula in (11). If Γ_{D^2} contains n closed components lying in the open half disc $\{\varphi \in (0,\pi)\}$ then

$$\mathrm{rot}_{\mathcal{C}}(\partial D^2) = (1-n) - (1+n) = -2n \ .$$

If all closed components of Γ are contained in the open half disc $\{\varphi \in (\pi, 2\pi)\}$ then we obtain $\mathrm{rot}_{\mathcal{C}}(\partial D^2) = 2n$.

Now fix an oriented trivialization C_1, C_2 of $\mathcal{C}$. We assume that

$$C_1 = \frac{\partial}{\partial\varphi}, C_2 = -\sin(\varphi)\frac{\partial}{\partial r} + \cos(\varphi)\frac{\partial}{\partial x} \tag{66}$$

near the point $\{\varphi = \pi/2, r = 1, x = 0\} \subset \partial_- h_2$. We consider the horizontal lifts of C_1, C_2, V on

$$R_2 = h_2 \times S^1 \ .$$

Let $\pi : R_2 \longrightarrow h_2$ be the projection. The coordinate on S^1 will be denoted by t. For $k \in \mathbb{Z} \setminus \{0\}$ the distribution $\mathcal{D}_k$ spanned by

$$\begin{aligned} W &= \frac{\partial}{\partial t} + \varepsilon V \\ X_k &= \cos(kt)C_1 + \sin(kt)C_2 \end{aligned}$$

is an Engel structure if $\varepsilon > 0$ is small enough, cf. Proposition 4.17. Since V is a contact vector field,

$$\begin{aligned} [W, X_k] = &-k\sin(kt)C_1 + k\cos(kt)C_2 \\ &+ \varepsilon\left(\cos(kt)[V, C_1] + \sin(kt)[V, C_2]\right) \end{aligned} \tag{67}$$

is tangent to $\pi_*^{-1}(\mathcal{C})$. The characteristic foliation of this Engel structure is spanned by W. This vector field is transversal to $\partial_\pm R_2$ and it points in the desired directions. The even contact structure $\mathcal{E} = [\mathcal{D}_k, \mathcal{D}_k]$ on R_2 is defined by

$$\beta = \pi^*\alpha - \varepsilon\pi^*(\alpha(V))dt \ .$$

Let $\widetilde{A} = A \times \{-1/2 \leq x \leq 1/2\} \times S^1$. Using the expressions for V, α, h and our choices of g_1, g_2 we obtain

$$\begin{aligned} \beta &= (\cos(\varphi)dr + \sin(\varphi)dx) - \varepsilon(g_1(x)\cos(\varphi) + g_2(x)\sin(\varphi))dt \\ &= \cos(\varphi)dr + \sin(\varphi)dx + \varepsilon\cos(\varphi)dt \ . \end{aligned}$$

on $\widetilde{A}$. The contact structure on $\partial_- R_2$ is defined by

$$\beta\big|_{\partial_- R_2} = \sin(\varphi)dx + \varepsilon\cos(\varphi)dt - \varepsilon g_2(x)\sin(\varphi)dt \ . \tag{68}$$

Restricting β to $T_0^2 = \partial D^2 \times \{0\} \times S^1$ we obtain

$$\beta\big|_{T_0^2} = \varepsilon\cos(\varphi)dt \ .$$

Thus the characteristic foliation on T_0^2 is in standard form. The curves $\varphi = \pi/2$ and $\varphi = 3\pi/2$ are the Legendrian divides. The Legendrian ruling is tangent to the foliation given by the first factor in $T_0^2 = \partial D^2 \times \{0\} \times S^1$.

For $k > 0$, the orientation of the even contact structure is W, C_1, C_2. If $k < 0$ we obtain the converse orientation $W, C_1, -C_2$.

The rotation number of the intersection line field along the Legendrian curve $\partial D^2 \times \{0\} \times \{1\}$ compares the framing ∂_φ of $\mathcal{E}/\mathcal{W}$ with the image of $\mathcal{D}_k$ in $\mathcal{E}/\mathcal{W}$. Notice that ∂_φ is nowhere tangent to $\mathcal{W}$ and that t is constant on $D^2 \times \{0\} \times \{1\}$. Hence the rotation number along the boundary of this disc is independent of k. By Remark 3.24, we can determine the rotation number from the singular foliation $\mathcal{F}$ we started with.

If $k > 0$, the orientation of $\mathcal{E}/\mathcal{W}$ defined by C_1, C_2 (used in particular for the calculation (11)) and the orientation of $\mathcal{E}/\mathcal{W}$ induced from the orientation of $\mathcal{E}$ and $\mathcal{W}$ coincide. Thus if $k > 0$, the rotation number along $\partial D^2 \times \{0\} \times \{1\}$ is the same as the rotation number $\mathrm{rot}_{\mathcal{C}}(\partial D_2)$ we have obtained from (11).

If $k < 0$, the rotation number of the intersection line field along $\partial D^2 \times \{0\} \times \{1\}$ has the opposite sign since now the orientation of $\mathcal{E}/\mathcal{W}$ induced by the Engel structure and the orientation defined by C_1, C_2 are opposite.

Let us now calculate the rotation number along the Legendrian divide $\{\varphi = \pi/2\} \times \{0\} \times S^1 \subset T_0^2$. Here we use the particular choice of the framing C_1, C_2 near $\{\varphi = \pi/2, r = 1, x = 0\} \in \partial_- h_2$. Since $V = -\partial_r$, the terms in the second line of (67) vanish, i.e.

$$[W, X_k] = -k\sin(kt)C_1 + k\cos(kt)C_2 \,.$$

We write $\widetilde{X}_k, \widetilde{[W, X_k]}$ for the projection of $X_k, [W, X_k]$ to $\partial_- R_2$ along W. By our assumption (66) on the framing C_1, C_2 near $\{\varphi = \pi/2, r = 1, x = 1\} \in \partial_- h_2$, the contact structure on $\partial_- R_2$ is spanned and oriented by

$$\widetilde{X}_k = X_k - \frac{\sin(kt)}{\varepsilon}W = \cos(kt)\frac{\partial}{\partial\varphi} - \frac{\sin(kt)}{\varepsilon}\frac{\partial}{\partial t}$$
$$\widetilde{[W, X_k]} = [W, X_k] - \frac{k\cos(kt)}{\varepsilon}W = -k\sin(kt)\frac{\partial}{\partial\varphi} - k\frac{\cos(kt)}{\varepsilon}\frac{\partial}{\partial t}$$

along the Legendrian divide in T_0^2 with $\varphi = \pi/2$. Along this Legendrian divide we obtain the following expression for ∂_t:

$$\frac{\partial}{\partial t} = \varepsilon\left(-\sin(kt)\widetilde{X}_k - \frac{1}{k}\cos(kt)\widetilde{[W, X_k]}\right) \,.$$

Hence the rotation number along the Legendrian divide $\{\varphi = \pi/2\} \subset T_0^2$ is $-|k|$. One obtains the same result for $\{\varphi = 3\pi/2\}$. Together with the rotation number along the Legendrian rulings $\partial D^2 \times \{0\} \times \{t\}$ and the orientation of the contact structure on $\partial_- R_2$ this determines the homotopy class of the intersection line field as Legendrian line field on $\partial_- R_2$ completely.

Let us summarize the properties of the model Engel structures $\mathcal{D}_k$ we have obtained up to now. Recall that $\mathcal{D}_k$ depends not only on k but also on the choice of the dividing set at the beginning of the construction. We can choose $n \in \mathbb{Z}$ freely, $|n|$ is the number of closed components of Γ. Since we have fixed the contact form on $A \times \mathbb{R}$, the contact structure on $\partial_- R_2$ depends only on the choice of V near the boundary.

	Orientation of $\mathcal{E}/\mathcal{W}$	Rotation number $\partial D^2 \times \{0\} \times \{1\}$	Rotation number Legendrian divides
$k > 0$	C_1, C_2	$2n$	$-\lvert k\rvert$
$k < 0$	$C_1, -C_2$	$-2n$	$-\lvert k\rvert$

The model Engel structures with positive rotation numbers along the Legendrian divides can be obtained by applying the involution

$$\iota : R_2 \longrightarrow R_2$$
$$(r, \varphi, x, t) \longmapsto (r, \varphi, -x, -t) \,.$$

This diffeomorphism preserves the contact structure on $\{-1/2 \leq x \leq 1/2\} \subset \partial_- R_2$, cf. (68), but it reverses the orientation of the Legendrian divides. In particular we can compare the orientations of the contact structure and the homotopy class of the intersection line fields with the corresponding properties of $\mathcal{D}_k$. The model Engel structures $\iota_* \mathcal{D}_k$ cover the cases which are missing in the table above. □

Let M be an Engel manifold with transversal boundary and fix a model Engel structure on R_2. We want to determine under which conditions an attaching map

$$\psi_0 : \partial_- R_2 \longrightarrow \partial_+ M$$

can be isotoped so that the resulting map ψ_1 preserves contact structures. The following proposition is a first step in this direction. We assume that $\psi_0(T_0^2)$ is a convex surface. This can be achieved by a C^∞–small perturbation of ψ_0.

PROPOSITION 6.5. *If ψ_0 respects the orientations induced by the contact structures and the restriction of ψ_0 to T_0^2 preserves the isotopy class of the dividing set, then ψ_0 can be isotoped to a contact embedding.*

PROOF. Let $T_M^2 = \psi_0(T_0^2)$ and Γ_M be the dividing set of T_M^2. We first isotope ψ_0 to a map $\widetilde{\psi}$ which maps the dividing set of T_0^2 to the dividing set of T_M^2. We do this in such a way that throughout the isotopy, T_0^2 is mapped to T_M^2. Since T_M^2 is convex, it has a tubular neighbourhood $U \simeq T_M^2 \times \mathbb{R}$ such that the contact structure on U is mapped to an $\mathbb{R}$–invariant contact structure on $T_M^2 \times \mathbb{R}$.

We isotope $\widetilde{\psi}$ in order to achieve that the image of the isotoped map is contained in U. This isotopy can be chosen to be constant along T_0^2. The map obtained from this isotopy will still be denoted by $\widetilde{\psi}$. Now the image of the singular foliation on T_0^2 under $\widetilde{\psi}$ and the singular foliation on T_M^2 have the same dividing set.

By Theorem 2.28, there is an isotopy of $\widetilde{\psi} : \partial_- R_2 \longrightarrow T^2 \times \mathbb{R}$ to an embedding $\widehat{\psi}$ such that this map preserves the singular foliation on T_0^2. Since the isotopy is admissible, the surface $\widehat{\psi}$ is transversal to the second factor of $T_M^2 \times \mathbb{R}$.

We identify T_0^2 and $\widehat{\psi}(T_0^2)$. From this identification we get coordinates φ, z, t on U such that T_0^2 corresponds to $z = 0$, ∂_z is the canonical vector field on $U \simeq T_M^2 \times \mathbb{R}$ which is tangent to the second factor. By Giroux's Theorem 2.26, we may assume that the contact structure on U is defined by the z–invariant contact form

$$\beta_0 = \varepsilon \cos(\varphi) dt + \sin(\varphi) dz \; .$$

Consider the embedding

$$\begin{aligned} \psi' : \partial_- R_2 &\longrightarrow T^2 \times \mathbb{R} \\ (\varphi, t, x) &\longmapsto (\varphi, t, z = x) \; . \end{aligned}$$

Since ψ_0' preserves the orientation induced by the contact structures, the restrictions of $\widehat{\psi}_*$ and ψ'_* to T_0^2 are homotopic. By the uniqueness theorem for tubular neighbourhoods, these maps are isotopic. The claim would follow immediately if the contact structure on $\partial_- R_2$ were invariant under ∂_x. Unfortunately we cannot make this assumption, but we can modify ψ' using Gray's theorem.

For this, we use several constants and some notation from Proposition 6.3. We apply Gray's theorem for the following family of contact structures. Let

$$\beta_s = \varepsilon \cos(\varphi) dt + \sin(\varphi) dz - s\varepsilon g_2(z) \sin(\varphi) dt \; ,$$

where g_2 is a smooth extension of the function we used in (65) such that g_2 has compact support and depends only on z. When one compares β_1 with the expression (68) for the contact structure on $\partial_- R_2$ one has to remember that we assumed that $g_1 \equiv 0$ in (68) on $\partial_- R_2$ respectively $\partial_- h_2$. Recall also that $dr = 0$ on these boundary components. Because

$$\beta_s \wedge d\beta_s = \varepsilon d\varphi \wedge dz \wedge dt$$

is independent of s, the family β_s is a family of contact forms. Consider the induced isotopy F_s of $T^2 \times \mathbb{R}$. Since $g_2(z) = 0$ for $-1/2 \leq z \leq 1/2$, F_s is the identity near $T_0 \times \{0\}$

and F_s has compact support. Moreover $F_s^*\beta_s$ is a multiple of β_0. Hence $\psi_1 = F_1^{-1} \circ \psi'$ preserves contact structures and is isotopic to ψ_0 relative to T_0^2. □

Suppose we are given an attaching map $\psi : \partial_- R_2 \longrightarrow \partial_+ M$. In order to find a model Engel structure on R^2 and an attaching map $\widetilde{\psi}$ isotopic to ψ such that the Engel structure extends to $M \cup_{\widetilde{\psi}} R_2$ it is enough to understand how one can manipulate the isotopy class of the dividing set of an embedded torus in a contact manifold. In an overtwisted contact manifold this can be done efficiently using the bypasses we obtained in Proposition 2.37. We will discuss this in Section 6.2.

Notice that in our list of model Engel structures on round 2–handles the case that the rotation number along the Legendrian divides is zero is not contained. It will turn out that it is always possible to arrange the attaching map of R_2 such that model Engel structures of this type are not needed.

6.2. Tori in overtwisted contact manifolds

Our model Engel structures on round 2–handles share one property, namely the singular foliation on $T_0^2 = \partial D^2 \times \{0\} S^1 \subset \partial_- R_2$ is the same for all our models. Now suppose that M is an Engel manifold with transversal boundary and $\varphi_2 : \partial_- R_2 \longrightarrow \partial_+ M$ is an attaching map which preserves the contact orientations of $\partial_- R_2$ and $\partial_+ M$.

If we want to attach R_2 to M and extend the Engel structure from M to $M \cup R_2$, then we have to ensure that the attaching map preserves contact structures. By Proposition 6.5 it suffices to modify φ_2 such that after the deformation the image of T_0^2 is convex and the attaching map preserves the isotopy class of the dividing sets. Recall that the dividing set of T_0^2 consists of two homotopically non–trivial circles.

Let $(N, \mathcal{C})$ be an overtwisted contact manifold. Using Lemma 2.36 and Proposition 2.37 we can perform the desired modification. In the following proposition we focus on the image T^2 of the attaching map and isotope only this torus. It is clear how to obtain the desired isotopy from this.

The following example shows that Theorem 6.7 is wrong when one drops the assumption that $\mathcal{C}$ is overtwisted.

EXAMPLE 6.6. On $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ consider the contact structure defined by $\alpha_n = \cos(2n\pi z)dx + \sin(2n\pi z)dy$ for $n \in \mathbb{N}$. Using the results in [**Ka1**], one can show that for $n \geq 2$ the torus $T^2 = \{y = 0\} \subset T^3$ is not isotopic to a convex surface whose dividing set consists of two components.

THEOREM 6.7. *Let T^2 be an embedded torus in an overtwisted contact manifold $(N, \mathcal{C})$. Assume that $\mathcal{C}$ is orientable and that the Euler class of the restriction of $\mathcal{C}$ to T^2 is zero. Then we can isotope T^2 such that after the isotopy the singular foliation on the torus is in standard form. Moreover we can prescribe the slope of the dividing curves.*

After the isotopy, the complement of a tubular neighbourhood of T^2 contains an overtwisted disc.

PROOF. It suffices to find a convex torus which is isotopic to the original one such that the dividing set consists of two homotopically non–trivial components which have the desired slope. Using the Giroux flexibility theorem (Theorem 2.28) one can arrange the singular foliation on T^2 such that T^2 is in standard form. We will frequently use Proposition 2.37. The following figures represent the dividing set on a torus before and after the bypass attachment. The thickened curve represents the attaching curve γ_1 of the bypass.

1^{st} Step: Assume that $N \setminus T^2$ contains no overtwisted disc. Let D_{ot} be an overtwisted disc. We perturb the embedding of T^2 such that it becomes transversal to D_{ot}. Using an extension of a radial vector field on D_{ot} we can isotope T^2 such that after the isotopy $T^2 \cap D_{ot} = \emptyset$.

Without loss of generality, we assume that D_{ot} is convex. In particular, there is a neighbourhood diffeomorphic to $D_{ot} \times (-1, 1)$ which is foliated by overtwisted discs. In the following we will always ensure that after each modification of the embedding of T^2 there is an overtwisted disc which is disjoint from the deformed torus: If D is a bypass for T^2, we choose the neighbourhood of $T^2 \cup D$ such that its complement still contains overtwisted discs.

In the following steps we attach bypasses to T^2 in order to obtain the desired configuration of dividing curves. Notice that the dividing set of a convex closed surface is never empty. In all figures in this proof the rectangle represents the torus (i.e. opposite edges are identified in the usual way). The thickened arc represents the segment γ_1 of the boundary of a bypass.

2^{nd} Step: In this step we remove all homotopically non–trivial components of the dividing set. If there are no such components we continue with step 3.

If the dividing set contains more than two homotopically non–trivial components, we reduce the number of its components of the dividing set using the bypass attachments in Figure 3 often enough. We end up with a dividing set which contains two homotopically

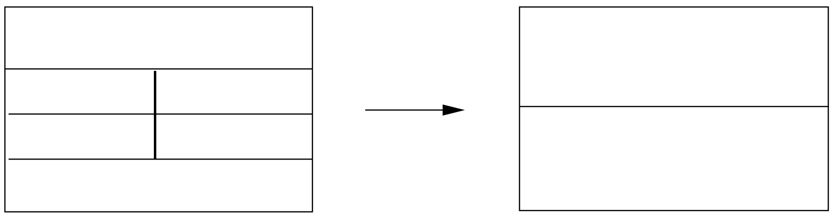


FIGURE 3.

non–trivial curves. We remove these components with the bypass attachment in Figure 4

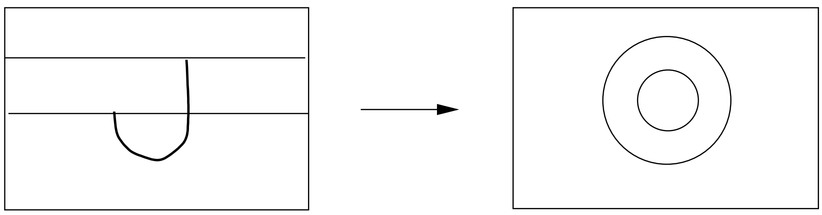


FIGURE 4.

3^{rd} Step: Using the the bypass attachment in Figure 5, we obtain two new components of the dividing set. Their slope depends on the bypass. When we fix an identification $T^2 \simeq S^1 \times S^1$, we can achieve that the new components of the dividing set are isotopic to $\{p\} \times S^1$ for $p \in S^1$. The dashed curve represents this circle.

4^{th} Step: We are left with a convex torus whose dividing set contains exactly two homotopically non–trivial dividing curves σ_1, σ_2 with the desired slope. If this is the entire dividing set we are done. Otherwise we consider the two annuli $T^2 \setminus (\sigma_1 \cup \sigma_2)$.

If only one of these annuli contains other components of the dividing set Γ, we claim that there is at least one component of Γ which bounds a disc $\widetilde{D}$ which contains another component of Γ. Assume that this is not true. Then $T^2 \setminus \Gamma$ contains $r > 0$ discs, one

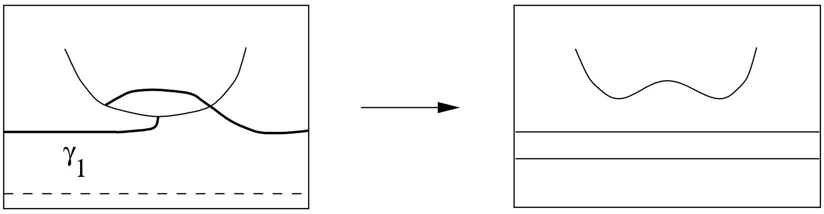


FIGURE 5.

annulus and one annulus with r holes. The annulus and the discs have the same sign when one chooses a contact form and a contact vector field which is transversal to Γ. In this situation, the Euler number of the restriction of $\mathcal{C}$ to T^2 is

$$\langle \chi(\mathcal{C}), [T^2] \rangle = \chi(T^2_+) - \chi(T^2_-) = \pm 2r \neq 0\ . \tag{69}$$

The sign depends on the orientations of T^2 and of the contact structure. But (69) contradicts our assumption on the Euler class of $\mathcal{C}$. In order to reduce the number of connected components of Γ we perform a bypass attachment as the one indicated in Figure 6. Notice

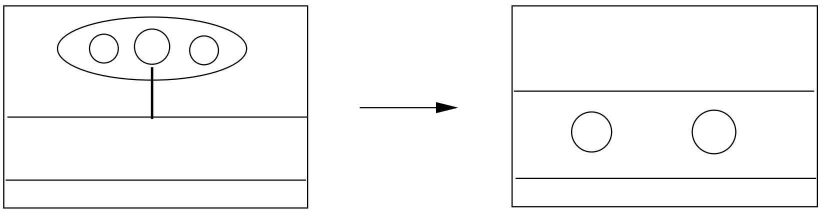


FIGURE 6.

that this does not affect the homotopically non–trivial dividing curves.

If both annuli $T^2 \setminus (\sigma_1 \cup \sigma_2)$ contain connected components of Γ we reduce the number of components using the bypass attachment in Figure 7. Again this does not change the

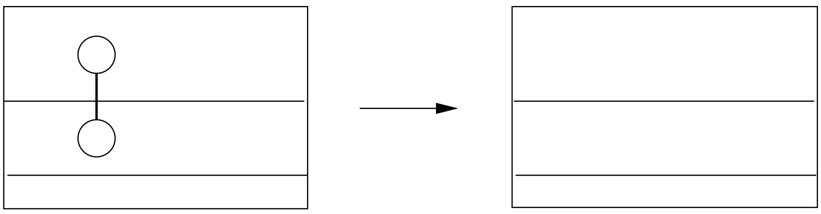


FIGURE 7.

number and the slope of homotopically non–trivial dividing curves.

If we apply the last step often enough we end up with the desired configuration of dividing curves on T^2. $\square$

REMARK 6.8. P. Ghiggini has suggested a different approach to Theorem 6.7. For this, assume that T^2 is convex and consider a tubular neighbourhood $T^2 \times [-1, 1]$ such that the contact structure is $\mathbb{R}$–invariant. We want to replace the given contact structure by a contact structure which is homotopic to the given contact structure relative to $T^2 \times \{\pm 1\}$ such that the singular foliation on $T^2 \times \{0\}$ has the desired shape.

Now the homotopy between the two contact structure induces an isotopy between the two contact structures by Theorem 2.33. Actually this isotopy can be chosen to be constant outside a larger tubular neighbourhood of T^2 as long as this neighbourhood contains an overtwisted disc. This isotopy shows that the torus with the desired singular foliation is also contained in the original contact manifold and it is isotopic to $T^2 \times \{0\}$.

It remains to construct the desired contact structure on $T^2 \times [-1,1]$. For this We start with the layer $T^2 \times [-1,-1/2]$ and attach bypasses as in the proof of Theorem 6.7. Here we attach abstract bypasses, i.e. bypasses which are not contained in $(N,\mathcal{C})$. Using the bypass attachment lemma (Lemma 2.36) we obtain contact structures on layers $T^2 \times (t, t+\varepsilon)$ such that on one components of the boundary of the layer we have the dividing set before the bypass attachment and on the other boundary component we have the dividing set after the bypass attachment. After a finite number of bypass attachments, we have the desired singular foliation. Then we perform more bypass attachments in order to get back the old singular foliation on $T^2 \times \{1\}$. This block can be used to replace the original contact structure on $T^2 \times [-1,1]$. The new contact structure is homotopic to the old contact structure.

Thus we can use bypasses effectively to modify singular foliations of tori in overtwisted contact manifolds $(N,\mathcal{C})$ which are trivial as bundles. This will always be the case in our applications. Now we show that a bypass attachment also affects framings. For our purpose, it will be enough to show that a particular bypass attachment has an effect on framings. Honda described this effect in more detail, cf. Proposition 4.7 in [**Ho**].

Let X be a nowhere vanishing section of $\mathcal{C}$. If N is the transversal boundary of an Engel manifold with orientable Engel structure, then we can take the intersection line field for X.

Assume that $T^2 = S^1 \times S^1$ is an embedded surface in N such that the singular foliation is in standard form. We fix an identification $T^2 = S^1 \times S^1$ such that the Legendrian divides are tangent to curves $\{p\} \times S^1$.

We write v for the coordinate on the first factor and t for the second. A small tubular neighbourhood of T^2 is diffeomorphic as a contact manifold to $T^2 \times \mathbb{R}$ with the $\mathbb{R}$–invariant contact structure defined by

$$\alpha_0 = \cos(v)dt - \sin(v)dx \tag{70}$$

where x is the coordinate on $\mathbb{R}$. The curves $\{v = 0\}$ and $\{v = \pi\}$ are the Legendrian divides. Let

$$\begin{aligned} C_1 &= \sin(v)\frac{\partial}{\partial t} + \cos(v)\frac{\partial}{\partial x} \\ C_2 &= \frac{\partial}{\partial v}\ . \end{aligned}$$

This is a framing of the contact structure on $\mathcal{C}$ such that C_1 is tangent to the Legendrian ruling and C_2 is tangent to the Legendrian divides. We orient $\mathcal{C}$ by C_1, C_2.

The rotation numbers of X along $S^1 \times \{0\}$ and $\{0\} \times S^1$ compare the framing induced by X with the framing C_1, C_2. In the following lemma we assume that the complement of a tubular neighbourhood of T^2 contains an overtwisted disc. In our application this will always be the case since T^2 is obtained from an application of Theorem 6.7.

LEMMA 6.9. *Assume that the rotation number of X along the Legendrian divides is zero and that it is even along $S^1 \times \{0\}$. We attach a bypass as in Figure 8 to T^2 and bring the characteristic foliation in standard form such that the Legendrian ruling is still tangent to the foliation from the first factor in $S^1 \times S^1$.*

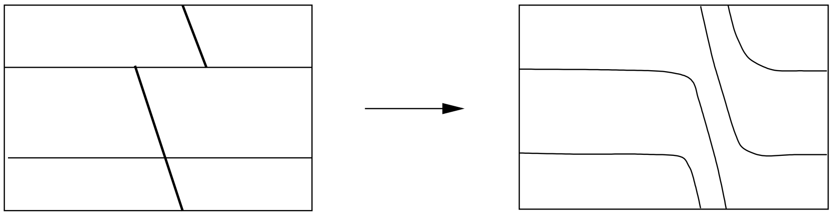


FIGURE 8.

Then the rotation number along the Legendrian divides in the isotoped torus is odd (and therefore non–zero) while the rotation number along $S^1 \times \{0\}$ remains even. After the isotopy, the complement of a small tubular neighbourhood contains an overtwisted disc.

PROOF. We use some notation from the bypass attachment lemma and we extend the coordinate system x, υ, t to $T^2 \times I$. The bypass attachment in Figure 8 changes the dividing set by a right handed Dehn twist. By Proposition 2.37 such a bypass attachment is possible. On a neighbourhood of $T^2 \times \{0\}$, the contact structure is defined by the form α_0 from (70).

By Theorem 2.26, the contact structure on a neighbourhood of $T^2 \times \{1\}$ is isotopic to the contact structure defined by

$$\alpha_1 = \cos(\upsilon + t)dt - \sin(\upsilon + t)dx$$

and we may therefore assume that it is really defined by this form. The two vector fields

$$C_1' = \sin(\upsilon + t)\frac{\partial}{\partial t} + \cos(\upsilon + t)\frac{\partial}{\partial x}$$
$$C_2' = \frac{\partial}{\partial \upsilon} \; .$$

define a framing of the contact structure on this neighbourhood. We want to compare the framing C_1', C_2' of $\mathcal{C}$ with the vector field X. Let

$$R = \cos(\upsilon)\frac{\partial}{\partial t} - \sin(\upsilon)\frac{\partial}{\partial x} \qquad \text{near } T^2 \times \{0\}$$
$$R' = \cos(\upsilon + t)\frac{\partial}{\partial t} - \sin(\upsilon + t)\frac{\partial}{\partial x} \qquad \text{near } T^2 \times \{1\} \; .$$

Both C_1, C_2, R and C_1', C_2', R' represent the contact orientation. Now we compare the framings C_1, C_2, R respectively C_1', C_2', R' with $\partial_\upsilon, \partial_t, \partial_x$, this last framing is also compatible with the contact orientation on $T^2 \times I$.

Consider first the annulus $A = \{0\} \times S^1 \times I \subset T^2 \times I$. We compare the framings C_1, C_2, R respectively C_1', C_2', R' with $\partial_\upsilon, \partial_t, \partial_x$ along the two circles ∂A (t is varying while υ is constant). The resulting maps

$$\partial A \supset \{0\} \times S^1 \times \{0\} \longrightarrow \mathrm{SO}(3)$$
$$\partial A \supset \{0\} \times S^1 \times \{1\} \longrightarrow \mathrm{SO}(3)$$

are not homotopic. The first map is actually constant while the second map represents the non–zero element in $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.

Hence the framings C_1, C_2, R and C_1', C_2', R' do not extend from ∂_A to A. The same is true if we take the framing $C_1', -C_2', -R'$ instead of C_1', C_2', R'. Now for one of these two framings, the third component, i.e. R' or $-R'$, can be extended from $T^2 \times \{1\}$ to a vector field on $T^2 \times I$ which is transversal to the contact structure over $T^2 \times I$ and coincides with R on $T^2 \times \{0\}$. We will pretend that this is true for R'; for the converse situation, we can argue similarly. The extension will be denoted by R.

There is another framing of TM along A formed by X, Y, R, where Y is a section of the contact structure such that X, Y, R represents the contact orientation. Now if the rotation number of X with respect to the framing C_1, C_2, R along $\{0\} \times S^1 \times \{0\}$ respectively the framing C_1', C_2', R' along $\{0\} \times S^1 \times \{1\}$ were both even, then this would imply that the framings C_1, C_2, R and C_1', C_2', R' of TN along ∂A actually extend over A. This is a contradiction to the above. Thus the rotation number X along $\{0\} \times S^1 \times \{1\}$ is odd. We write r_1 for this number.

Now consider the annulus $\widetilde{A} = S^1 \times \{0\} \times I$. It is easy to show that the framings C_1, C_2, R respectively C_1', C_2', R' of TN extend from $\partial\widetilde{A}$ to $\widetilde{A}$. This implies that the parity of the rotation number of X along the Legendrian curves $S^1 \times \{0\} \subset T^2 \times \{i\}$ is the same parity for $i = 0, 1$. By assumption it is even. Actually the rotation numbers along both boundary components of $\widetilde{A}$ are equal (we orient both boundary components using the orientation of the S^1–factor in $\widetilde{A}$) since the bypass attachment in Figure 8 can be chosen disjoint from $\widetilde{A}$.

The Legendrian divides in $T^2 \times \{1\}$ are the curves

$$\{(x, \pi/2 - x) \subset S^1 \times S^1 = T^2\}$$
$$\{(x, 3\pi/2 - x) \subset S^1 \times S^1 = T^2\}$$

The rotation number along the Legendrian divide compares X with the framing C_1', C_2' of the contact structure over $T^2 \times \{1\}$. It equals the difference of r_1 and the rotation number of X along $S^1 \times \{0\} \times \{1\}$. It is therefore odd (and non–zero). □

Using this lemma, we will be able to arrange an embedding of a torus in an overtwisted contact manifold such that the rotation number along Legendrian divides is non–zero at the expense of changing the slope of the Legendrian divides. This makes it unnecessary to close the gap in our list of model Engel structures on R_2. The construction of the corresponding Engel structure on round handles of index 3 indicates that this would be complicated.

6.3. Model Engel structures on R_3

In this section we want to construct model Engel structures on round 3–handles

$$R_3 = D^3 \times S^1$$

such that the characteristic foliation is orientable and transversal and inward pointing to $\partial_- R_3 = S^2 \times S^1$. We want the induced contact structure on $\partial_- R_3$ to be overtwisted. In each homotopy class of plane fields on $S^2 \times S^1$ there is a unique (up to isotopy) overtwisted positive contact structure by Theorem 2.33.

We show in Lemma 6.10 that there is a unique homotopy class of plane fields which extends to $D^3 \times S^1$. This will be the homotopy class of plane fields on $\partial_- R_3$ which will arise in our models as contact structure on the boundary. Unlike in the case of round 2–handles, we have to cover *all* possible homotopy classes of intersection line fields.

It is possible to realize many homotopy classes of intersection line fields using by the method used in Example 4.19. This way we obtain all but one homotopy class of intersection line fields (the missing homotopy class corresponds to $k = 0$ in Example 4.19) . Of course one can try to guess an Engel structure on $D^3 \times S^1$ whose intersection line field represents the missing homotopy class. Unfortunately, it turns out to be difficult to do this directly.

The idea in the following construction is to use a decomposition of D^3 into pieces Z and h_2. While h_2 is an ordinary handle of index 2 and dimension 3, Z is a solid torus.

Both pieces carry contact structures such that the boundaries are convex. We will apply the Giroux flexibility theorem to find a gluing map

$$\varphi : \partial_+ h_2 \longrightarrow \partial_- Z$$

inducing a contact structure on $Z \cup_\varphi h_2$. If we think of Z as an ordinary 0–handle D^3 with a 1–handle $I \times D^2$ attached to it, the 1–handle and the 2–handle form a cancelling handle pair. So after the gluing we end up with D^3. However when one takes the contact structures on the pieces into account, we will obtain an overtwisted contact structure on D^3 while initially, the contact structures on Z and h_2 are tight.

On $Z \times S^1$ and $R_2 = h_2 \times S^1$, the model Engel structures with transversal boundary representing all homotopy classes of intersection fields can be found easily from additional structures we will define on h_2 and Z. From φ we obtain an attaching map

$$\psi : \partial_+ R_2 \longrightarrow \partial_- Z \times S^1$$

which preserves contact structures. From this construction one obtains model Engel structures representing *all* homotopy classes of intersection line fields on the boundary of

$$(Z \times S^1) \cup_\psi R_2 = (Z \cup_\varphi h_2) \times S^1 \simeq D^3 \times S^1 = R_3 .$$

Let us first define some contact structures and vector fields on Z and h_2.

6.3.1. Structures on Z. On $\mathbb{R}^2 \times S^1$ we use the coordinates x, y, s. Let $\alpha_Z = dx + y\,ds$. This is a positive contact form. The induced contact structure $\mathcal{C} = ker(\alpha_Z)$ is invariant under

$$V(Z) = \frac{\partial}{\partial s} - \varepsilon \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right)$$
$$L_{V(Z)}\alpha_Z = \varepsilon(d(-x) - y\,ds) = -\varepsilon\alpha_Z$$

for all $\varepsilon > 0$. The contact vector field V_Z is transversal and inward pointing along the boundary $\partial_- Z$ of the solid torus $Z = D^2 \times S^1$. The singular foliation on $\partial_- Z$ is in standard form. It is represented in Figure 10, we write θ for the angular coordinate in $\mathbb{R}^2$. The dividing set corresponds to the two solid curves, the dashed curve and the two thickened segments will be needed later. Consider the trivialization

$$C_1(Z) = \frac{\partial}{\partial y} \text{ and } C_2(Z) = \frac{\partial}{\partial s} - y\frac{\partial}{\partial x} .$$

These vector fields satisfy the commutator relations

$$[V(Z), C_1(Z)] = \varepsilon\frac{\partial}{\partial y} = \varepsilon C_1(Z)$$
$$[V(Z), C_2(Z)] = 0 .$$

For $\kappa \in \mathbb{Z}$ we consider

$$X_\kappa(Z) = \cos(\kappa s)(C_1(Z) + C_2(Z)) + \sin(\kappa s)C_1(Z) .$$

This vector field satisfies the commutator relation

$$[V(Z), X_\kappa(Z)] = -\kappa\sin(\kappa s)(C_1(Z) + C_2(Z)) + \kappa\cos(\kappa s)C_1(Z)$$
$$+ \varepsilon(\cos(\kappa s) + \sin(\kappa s))C_1(Z) .$$

If we fix $\varepsilon = 1/3$, this is linearly independent of $X_\kappa(Z)$ for all $\kappa \in \mathbb{Z}$ since

$$\det\begin{pmatrix} \cos(\kappa s)+\sin(\kappa s) & \begin{matrix} -\kappa\sin(\kappa s)+\kappa\cos(\kappa s) \\ +\varepsilon(\cos(\kappa s)+\sin(\kappa s)) \end{matrix} \\ & \\ \cos(\kappa s) & -\kappa\sin(\kappa s) \end{pmatrix} =$$
$$= -\kappa - \varepsilon(\cos^2(\kappa s)+\sin(\kappa s)\cos(\kappa s)) \ .$$

If $\kappa \neq 0$ the last expression is never zero since the absolute value of the last term is bounded by $1/2$. On the other hand, if $\kappa = 0$ then the last expression equals $-\varepsilon$. So for $\varepsilon = 1/3$ and for all $\kappa \in \mathbb{Z}$ we have shown that

$$X_\kappa(Z) \text{ and } [V(Z), X_\kappa(Z)]$$

are linearly independent sections of the contact structure $\mathcal{C}_Z$. From now on we fix

$$V(Z) = \frac{\partial}{\partial s} - \frac{1}{3}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) \text{ on } \mathbb{R}^2 \times S^1 \ .$$

6.3.2. Structures on h_2. The contact structure we use here is taken from [**Gir1**]. Let $h_2 = D^2 \times I \subset \mathbb{R}^3$ and equip h_2 with the positive contact form $\alpha_h = dz + y\,dx + 2x\,dy$. Note that h_2 not exactly the same as in the construction of model Engel structures on round handles of index 2. The contact structure $\mathcal{C}_h = \ker(\alpha)$ is invariant under the vector field

$$V(h) = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$
$$L_{V(h)}\alpha_h = dz + 2x\,dy + y\,dx = \alpha \ .$$

This vector field is transversal to both boundary components of

$$h_2 = \left\{(x,y,z) \in \mathbb{R}^3 \middle| x^2+z^2 \leq 1 \text{ and } |y| \leq 1\right\} \ .$$

It points inwards along $\partial_- h_2 = D^2 \times \{\pm 1\}$ and outward along $\partial_+ h_2 = \partial D^2 \times I$. Later we are going to attach h_2 along $\partial_+ h_2$ to $\partial_- Z$. This is different from the usual conventions because in the end, we want an inward pointing contact vector field transversal to the boundary of $Z \cup h_2$. The framing $C_1(h) = y\partial_z - \partial_x, C_2(h) = 2x\partial_z - \partial_y$ satisfies the commutator relations

$$[V, C_1(h)] = -y\frac{\partial}{\partial z} - \left(y\frac{\partial}{\partial z} - 2\frac{\partial}{\partial x}\right) = -2C_1(h)$$
$$[V, C_2(h)] = 4x\frac{\partial}{\partial z} - \left(2x\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right) = C_2(h) \ .$$

We fix the following vector fields and note the commutator relations

$$X_+(h) = C_1(h) + C_2(h) \qquad [V(h), X_+(h)] = -2C_1(h) + C_2(h)$$
$$X_-(h) = C_1(h) - C_2(h) \qquad [V(h), X_-(h)] = -2C_1(h) - C_2(h) \ .$$

The orientation $X_+(h), [V(h), X_+(h)]$ of the contact structure is the same as the orientation $C_1(h), C_2(h)$ while $X_-(h), [V(h), X_-(h)]$ represents the opposite orientation. Figure 9 shows the singular foliation on $\partial_+ h_2$. The dashed line is $\{y = 0\}$, the two thickened segments correspond to the dividing set. There are two hyperbolic singular points and θ is the angular coordinate in the x,z–plane. It is defined by the 1–form $(\cos(\theta) - y\sin(\theta))d\theta + 2\cos(\theta)dy$.

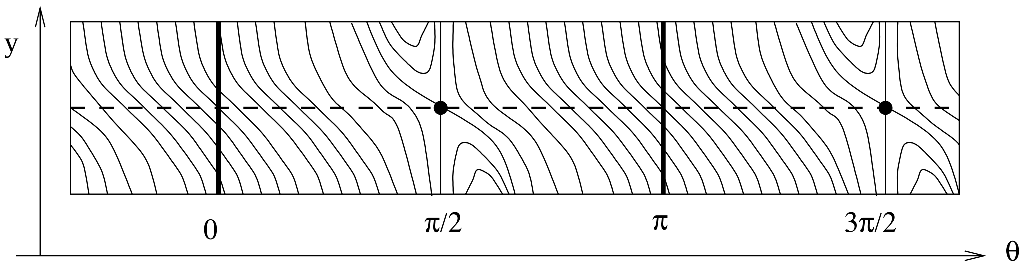


FIGURE 9.

6.3.3. Combining Z and h_2. The contact structure on h_2 and the contact vector field $V(h)$ appear in the construction of convex contact structures in [**Gir1**]. The construction of an attaching map for h_2 to a manifold with contact structure and convex boundary (like Z) is carried out in detail in [**Gir1**]. We therefore give only an outline.

Following [**Gir1**] we will construct an embedding of a neighbourhood $U \subset h_2$ of $\partial_+ h_2$

$$\varphi : (U, \partial_+ h_2) \longrightarrow (Z', \partial_- Z')$$

which preserves contact structures and maps $V(h)$ to $V(Z)$. As we have already mentioned this is different form our usual convention that boundary where a certain vector field points inwards is attached to a boundary component where the vector fields points inwards. We write Z' instead of Z because the solid torus will be deformed while the contact vector field $V(Z)$ will remain unchanged.

Let $A = \partial_+ h_2$. We orient A and $\partial_- Z$ so that the orientation of A respectively $\partial_- Z$ followed by the contact vector fields $V(h)$ respectively $V(Z)$ is the contact orientation. Choose an orientation preserving embedding

$$\widetilde{\varphi} : A \longrightarrow \partial_- Z$$

such that $\partial D_2 \times \{0\} \subset \partial_+ h_2$ gets mapped to a curve σ with the following properties.

(i) σ intersects one component of the dividing set Γ_Z of ∂Z transversely in two points. We denote this component by γ.
(ii) σ does not meet the other component of Γ_Z.
(iii) σ is isotopic to γ.

The dashed curve in Figure 10 has these properties. Moreover we assume that the image of $\widetilde{\varphi}$ is a tubular neighbourhood U of σ whose intersection with Γ_z consists of two segments γ_1, γ_2. Let $\mathcal{F}_h$ be the singular foliation on $\partial_- h_2$. We construct a singular foliation $\mathcal{F}$ on $\partial_+ Z$ such that

(i) $\mathcal{F}$ is divided by Γ_Z.
(ii) $\mathcal{F}$ coincides with $\varphi_*(\mathcal{F}_h)$.

Figure 11 shows such a singular foliation on one of the annuli $\partial Z \setminus \Gamma_Z$. Each annulus contains an arc of σ.

On the other annulus we can choose a foliation in an analogous way such that the two singular foliations form a smooth singular foliation on $\partial_- Z$. If all singularities have the same sign, the new singular foliation is again divided be Γ_Z. The foliation in Figure 11 is an instance of a more general construction on p. 660 of [**Gir1**]. The dashed curve represents a segment of σ. It passes through a hyperbolic singularity of $\mathcal{F}$. A neighbourhood of the dashed curve carries a foliation which is equivalent to the singular foliation on a part of $\partial_+ h_2$.

Now we apply Giroux flexibility theorem to ∂Z. By Theorem 2.28, there is an admissible isotopy $f_\tau, \tau \in [0,1]$, of ∂Z such that $f_1 \circ \widetilde{\varphi}$ preserves characteristic foliations.

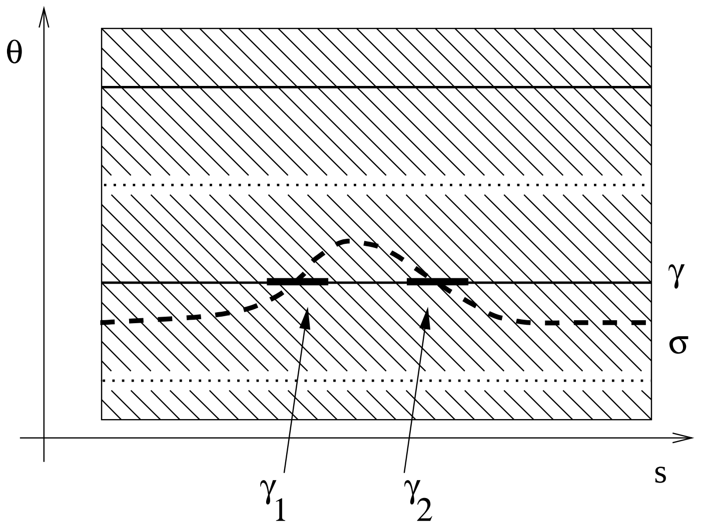


FIGURE 10.

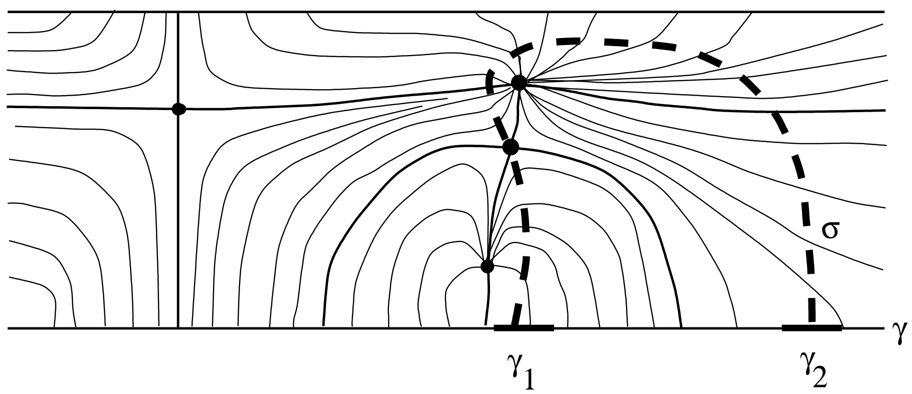


FIGURE 11.

Moreover, $f_1 \circ \widetilde{\varphi}$ extends to a small neighbourhood of A in h_2. The extension φ' is uniquely determined by the requirement that $\varphi'_*(V(h)) = V(Z)$. The surface $f_1(\partial_- Z)$ is the boundary of a solid torus Z' whose boundary is also convex and transversal to $V(Z)$.

In order to find a map φ which preserves contact structures and satisfies $\varphi'_*(V(h)) = V(Z)$ one applies Theorem 2.26. Now extend the $V(Z)$–invariant contact structure from a neighbourhood of $\partial_- Z$ to an $\mathbb{R}$–invariant contact structure on $\partial_- Z' \times \mathbb{R}$ such that $V(Z)$ corresponds to the vector field induced by the second factor on $\partial_- Z' \times \mathbb{R}$.

On the image of φ', the contact structure $\varphi'_*(\mathcal{C}_h)$ is also $\mathbb{R}$–invariant. This contact structure can be extended to an $\mathbb{R}$–invariant contact structure on $\partial_- Z' \times \mathbb{R}$ such that the singular foliation on $\partial_- Z'$ induced by this contact structure coincides with $\mathcal{F}$. The procedure how to find this extension is described in [**Gir1**] ("Sous–Lemma 3.3" of chapter 3). The application of Theorem 2.26 then yields the desired attaching map φ. It maps the dividing set of $\partial_+ h_2$ to γ_1, γ_2.

We glue h_2 to Z' using φ. On the resulting space we get a contact structure and a contact vector field V which coincides with $V(Z)$ on Z' and with $V(h)$ on h_2. After we cut of a suitable piece of h_2 in order to smoothen corners we obtain a manifold which can be identified with D^3. This can be done in such a way that V is transversal to ∂D^3 and inward pointing.

In particular the boundary of $Z' \cup h_2$ is convex and we can deduce the dividing set of $\partial_- D^3$. From the component γ of $\Gamma_{Z'}$ we remove the two segments γ_1, γ_2 when we attach h_2. The contact vector field on h_2 is tangent the contact structure along the band $\{z = 0\}$. Hence on D^3, the endpoints of $\gamma \setminus (\gamma_1 \cup \gamma_2)$ are connected such that we obtain two components of the dividing set of $\partial_- D^3$. The other component from the dividing set of Z' is not affected by the gluing procedure. Thus the dividing set of $\partial_- D^3$ has three connected components. By Theorem 2.34, this implies that the contact structure on D^3 is overtwisted.

6.3.4. Model Engel structures. From now on we write Z for the deformed solid torus Z'. The original torus will play no role anymore. Let us consider h_2 and Z separately again. We have an embedding of a neighbourhood U of $\partial_+ h_2$

$$\varphi : (U, \partial_+ h_2) \longrightarrow (\mathbb{R}^2 \times S^1, \partial_- Z)$$

which preserves contact structures and maps $V(h)$ to $V(Z)$. Now consider the vector field X_+ on h_2. Its image under φ is homotopic (as a section of $\mathcal{C}_Z$) to

$$X_\kappa(Z) = \cos(\kappa s)(C_1(Z) + C_2(Z)) + \sin(\kappa s) C_1(Z)$$

for exactly one $\kappa \in \mathbb{Z}$. We fix this κ. Let $X(Z) = X_\kappa(Z)$. In Section 6.3.1 we showed that $[V(Z), X(Z)]$ is linearly independent of $X(Z)$ everywhere. This defines an orientation of $\mathcal{C}_Z$. We choose $X(h) = X_+(h)$ or $X(h) = X_-(h)$ such that φ preserves the orientation of the contact structures for the orientation $X(h), [V(h), X(h)]$ of $\mathcal{C}_h$. Let

$$Y(h) = [V(h), X(h)]$$
$$Y(Z) = [V(Z), X(Z)] \, .$$

In the following we denote by $X(h), Y(h), X(Z), Y(Z)$ also the horizontal lift of the respective vector field to $h \times S^1$ respectively $Z \times S^1$. The coordinate on the second factor will be denoted by t. For $k \in \mathbb{Z}$ and $\eta > 0$ consider the distributions

$$\mathcal{D}_k(h) \text{ spanned by } W(h) = \frac{\partial}{\partial t} + \eta V(h) \text{ and}$$
$$\widetilde{X}_k(h) = \cos(kt) X(h) + \sin(kt) Y(h)$$
$$\mathcal{D}_k(Z) \text{ spanned by } W(Z) = \frac{\partial}{\partial t} + \eta V(Z) \text{ and}$$
$$\widetilde{X}_k(Z) = \cos(kt) X(Z) + \sin(kt) Y(Z)$$

on $h \times S^1$ respectively $Z \times S^1$. These distributions are Engel structures for all $k \in \mathbb{Z}$ if $\eta > 0$ is small enough. In particular the case $k = 0$ is allowed. For example

$$\begin{aligned} [W(h), \widetilde{X}_k(h)] &= -k \sin(kt) X(h) + k \cos(kt) Y(h) \\ &\quad + \eta \left(\cos(kt) Y(h) + \sin(kt) [V(h), Y(h)] \right) \, . \end{aligned} \tag{71}$$

This shows that $\eta > 0$ can be chosen independently from k. The commutator vector field $[W(h), \widetilde{X}_k(h)]$ is linearly independent of $W(h), \widetilde{X}_k(h)$ for $k = 0$ since

$$[W(h), \widetilde{X}_0(h)] = \eta Y(h) \, .$$

This is linearly independent of $\widetilde{X}_0(h)$ by construction and it has no ∂_t–component. For $k \neq 0$ it is obvious from (71) that $\widetilde{X}_k(h)$ and $[W(h), \widetilde{X}_k(h)]$ are linearly independent. In the same way one sees that $\mathcal{D}_k(Z)$ is an Engel structure for all $k \in \mathbb{Z}$.

Let $\mathcal{E}(Z) = [\mathcal{D}_k(Z), \mathcal{D}_k(Z)]$. This even contact structure is independent of k. If we intersect $\mathcal{E}(Z)$ with the tangent bundle of the first factor of $Z \times S^1$ we obtain a distribution $\widetilde{\mathcal{C}}_Z$. This is the horizontal lift of the contact structures on Z to $Z \times S^1$. We use the analogous statements and notations for h instead of Z. Consider the embedding

$$\widetilde{\varphi} = \varphi \times \mathrm{Id} : (U, \partial_- h_2) \times S^1 \longrightarrow (\mathbb{R}^2 \times S^1) \times S^1 \ .$$

It is clear from the construction of φ and from the choice of structures on Z respectively h_2 that this embedding has the following properties.

(i) It maps the even contact structure $\mathcal{E}(h)$ to $\mathcal{E}(Z)$.
(ii) $\widetilde{\varphi}_*(W(h)) = W(Z)$.
(iii) $\widetilde{\varphi}$ maps $\widetilde{X}_k(h)$ to a section of $\widetilde{\mathcal{C}}(Z)$ which is homotopic to $\widetilde{X}_k(Z)$ among nowhere vanishing sections of $\widetilde{\mathcal{C}}(Z)$.
(iv) It preserves the orientations of $\mathcal{E}(h)$ respectively $\mathcal{E}(Z)$ which are induced from $\mathcal{D}_k(h)$ respectively $\mathcal{D}_k(Z)$.

Hence we can apply vertical modification from Theorem 5.4 in order to obtain Engel structures $\mathcal{D}_k$ on

$$R_3 = D^3 \times S^1 = (Z' \cup_\varphi h_2) \times S^1 = (Z' \times S^1) \cup_{\widetilde{\varphi}} (h_2 \times S^1) \ .$$

We write W for the vector field obtained from $W(Z)$ and $W(h)$. The even contact structure which is spanned by the horizontal lifts of $\mathcal{C}_Z$ respectively $\mathcal{C}_h$ and W will be denoted by $\mathcal{E}$. This is the even contact structure $[\mathcal{D}_k, \mathcal{D}_k]$.

The vector field W is transversal to $\partial_- R_3$ and points into R_3. Let $\mathcal{C}_\partial$ be the contact structure on the boundary. By construction of R_3, the surface $\partial_- D^3 \times \{p\}$ is convex for $p \in S^1$. Its dividing set has three connected components. By the Giroux criterion (Theorem 2.34) the contact structure on $\partial_- R_3 = S^2 \times S^1$ is overtwisted.

Let us summarize what we have. The induced orientation of the even contact structure $\mathcal{E}(h)$ coincides with the orientation $W(h), X(h), Y(h)$ respectively $W(Z), X(Z), Y(Z)$ for $k \geq 0$. If $k < 0$ we obtain the opposite orientations. As oriented bundle we can identify the contact structure on the boundary with $\mathcal{E}/W$. For each homotopy class of Legendrian fields we have obtained an Engel structure whose intersection line field on $\partial_- R_3$ is this homotopy class and such that the contact structure carries an orientation induced by the Engel structures. It remains to construct model Engel structures which induces the opposite orientations.

This can be done in a similar way as in the case of round 2–handles at the end of Proposition 6.3. We use a self diffeomorphism of R_3 which preserves the contact structure on the boundary but reverses its orientation.

LEMMA 6.10. *There is a unique homotopy class of orientable plane fields on* $S^2 \times S^1 = \partial D^3 \times S^1$ *which extends to* $D^3 \times S^1$.

PROOF. Recall from [**HH**] that the Grassmann manifolds of oriented planes in $\mathbb{R}^3$ respectively $\mathbb{R}^4$ are $\mathrm{Gr}_2(3) \simeq S^2$ respectively $\mathrm{Gr}_2(4) \simeq S^2 \times S^2$. The inclusion $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ induces the diagonal map

$$\Delta : \mathrm{Gr}_2(3) \simeq S^2 \longrightarrow S^2 \times S^2 \simeq \mathrm{Gr}_2(4)$$

Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two plane fields on $S^2 \times S^1$ who extend to the interior of $D^3 \times S^1$. We view $\mathcal{C}_0, \mathcal{C}_1$ as maps from $S^2 \times S^1$ to $\mathrm{Gr}_2(3)$ and their extensions as maps from $D^3 \times S^1$ to $\mathrm{Gr}_2(4)$. Because $\{0\} \times S^1$ is a strong deformation retract of $D^3 \times S^1$ and $\mathrm{Gr}_2(4)$ is simply connected, the extensions of $\mathcal{C}_0$ and $\mathcal{C}_1$ are homotopic. This way we obtain a homotopy of $\mathcal{C}_0$ and $\mathcal{C}_1$ in $T(D^3 \times S^1)\big|_{S^2 \times S^1}$. Using the projection of $\mathrm{Gr}_2(4) \simeq S^2 \times S^2$ onto the first factor, we obtain a homotopy between $\mathcal{C}_0$ and $\mathcal{C}_1$. □

Since the contact structure on $\partial_- R_3$ is overtwisted and represents the unique homotopy class of plane fields which extends to $D^3 \times S^1$ we can apply Theorem 2.33. It implies that the contact structure $\mathcal{C}$ on $\partial_- R_3 = S^2 \times S^1$ is isotopic to the contact structure defined by

$$\beta_\pm = \sin\left(\frac{3\pi}{2}\sin^2(\vartheta)\right) d\alpha \pm g(\vartheta)dt \,.$$

Here we use spherical coordinates $\alpha, \vartheta \in [0, 2\pi) \times [0, \pi]$ on S^2 and the 1–forms from (29). Whether one has to take β_+ or β_- depends on the relation between the contact orientation of $\partial_- R_3$ and the identification of R_3 with $D^3 \times S^1$.

Now consider the involution

$$\begin{aligned} \iota : S^2 \times S^1 &\longrightarrow S^2 \times S^1 \\ (\vartheta, \alpha, t) &\longmapsto (\vartheta, -\alpha, -t) \end{aligned}$$

It extends to $D^3 \times S^1$, the points $\vartheta = 0, \pi$ are fixed and it has the property

$$\iota^* \beta_\pm = -\beta_\pm \,.$$

We denote the extension to $D^3 \times S^1$ also by ι. Let $p \in S^2$ such thatι maps $\gamma = \{p\} \times S^1$ to itself. Let $C_1(\gamma), C_2(\gamma)$ be a framing of $\mathcal{C}$ along γ such that $C_1(\gamma)$ is invariant under ι while $\iota_*(C_2(\gamma)) = -C_2(\gamma)$.

Now the intersection line field of $\mathcal{D}_k$ along γ is homotopic to one of the following ι–invariant sections of $\mathcal{C}$

$$\cos(lt)C_1 + \sin(lt)C_2$$

with $l \in \mathbb{Z}$. Thus the intersection line field of the Engel structure $\overline{\mathcal{D}}_k = \iota_* \mathcal{D}_k$ is homotopic to the intersection line field of $\mathcal{D}_k$. But $\mathcal{D}_k$ and $\overline{\mathcal{D}}_k$ induce different orientations on $\mathcal{C}$. We have shown the following proposition.

PROPOSITION 6.11. *Fix an orientation of the contact structure $\mathcal{C}$ on $\partial_- R_3$ and an orientable Legendrian line field $\mathcal{L}$. There is exactly one Engel structure among the model Engel structures $\mathcal{D}_k, \overline{\mathcal{D}}_k, k \in \mathbb{Z}$ such that the intersection line field is homotopic to $\mathcal{L}$ and the induced orientation of $\mathcal{C}$ is the preassigned orientation.*

6.4. Proof of Theorem 6.1

Before we put the ingredients together in order to prove Theorem 6.1, let us remark that statements analogous to Proposition 5.1 are true for round handles of index 2 and 3: Assume a round handle carries a model Engel structure and let M be an oriented Engel manifold with transversal boundary and oriented characteristic foliation. Whenever an attaching map

$$\varphi : \partial_- R \longrightarrow \partial_+ M$$

preserves contact structures, their orientation induced by the Engel structure and the oriented intersection line field, we can attach the round handle such that we obtain an oriented Engel structure on $M \cup_\varphi R$. The characteristic foliation is again transversal to the boundary.

Recall from Theorem 3.37 that an oriented Engel structure on an oriented manifold induces a decomposition

$$TM \simeq \mathcal{W} \oplus \frac{\mathcal{D}}{\mathcal{W}} \oplus \frac{\mathcal{E}}{\mathcal{D}} \oplus \frac{TM}{\mathcal{E}} \tag{72}$$

of oriented real line bundles. We fix a Riemannian metric. Then (72) induces a trivialization of TM. Assume that we have an Engel structure on $N \subset M$. Then an *Engel trivialization* on N is a trivialization which coincides with the trivialization on N we just described.

PROOF OF THEOREM 6.1. Let M be a closed parallelizable manifold of dimension 4 and fix a trivialization $TM \simeq M \times \mathbb{R}^4$ of the tangent bundle of M. We consider a round handle decomposition of M such that there is exactly one round 3–handle and one round 0–handle. Such a decomposition of M exists by Theorem 4.6. Thus M is decomposed

$$M = \Big(\ldots\Big(\Big(\ldots\Big(R_0 \cup_{\varphi_1^1} R_1^1\Big)\ldots\cup_{\varphi_1^{r_1}} R_1^{r_1}\Big)\cup_{\varphi_2^1} R_2^1\Big)\ldots\cup_{\varphi_2^{r_2}} R_2^{r_2}\Big)\cup_{\varphi_3} R_3\ .$$

The attaching maps φ_1^i, φ_2^i are indexed by i (and not powers of maps φ_1, φ_2). We will frequently isotope the attaching maps but this will not be reflected in the notation.

Start with the round handles of index 0. As model Engel structure on a round 0–handle, we take the model Engel structure $\mathcal{D}_0$ from Section 6.3 which corresponds to $k = 0$ and reverse the orientation of the characteristic foliation such that it points out of $R_3 \simeq R_0$ along the boundary. Assume that the orientation induced by $\mathcal{D}_0$ on R_0 does not coincide with the orientation of M given by $TM \simeq M \times \mathbb{R}^4$. In this case consider an automorphism ι_0 of R_0 which reverses the orientation of R_0. Then we equip R_0 with the Engel structure $\iota_{0*}\mathcal{D}_0$ instead. This way, we ensure that the Engel orientation and the orientation of M coincide on R_0.

We compare the trivialization $M \times \mathbb{R}^4$ and the Engel trivialization on R_0 along the curve $\{0\} \times S^1 \subset D^3 \times S^1 \subset R_0$. This defines a map

$$g_1 : S^1 = \{0\} \times S^1 \longrightarrow \mathrm{SO}(4)\ .$$

Since $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$, this map is either homotopic to zero or it represents the non–zero element of $\pi_1(\mathrm{SO}(4))$. In the latter case we apply again an automorphism of $R_0 = D^3 \times S^1$. We use the usual coordinates (x, y, z) on D^3 and t on S^1. Let

$$\begin{aligned} F_0 : R_0 = D^3 \times S^1 &\longrightarrow D^3 \times S^1 = R_0 \\ \big((x,y,z),t\big) &\longmapsto \big((\cos(t)x + \sin(t)y, -\sin(t)x + \cos(t)y, z), t\big)\ . \end{aligned}$$

We push–forward the Engel structure on R_0 by F_0. The trivialization induced by the new Engel structure and the given trivialization $TM \simeq M \times \mathbb{R}^4$ are now homotopic along $\{0\} \times S^1 \subset D^3 \times S^1 \subset R_0$. Since this curve is a strong deformation retract of R_0 we can homotop the given trivialization $TM \simeq M \times \mathbb{R}^4$ such that it coincides with the Engel trivialization on R_0. The contact structure on $\partial_+ R_0$ is overtwisted by construction.

In the following we will assume that the attaching maps of the round handles preserve the orientation induced by the contact structure on the boundary when we equip the round handle with a model Engel structure. Since an orientation of the characteristic foliation induces an orientation of an Engel manifold and vice versa, this condition ensures that the Engel trivialization on the round handle and the trivialization of TM define the same orientation. If an attaching map φ does not preserve the contact orientation, then we replace φ by $\varphi \circ \iota$ where ι is the orientation reversing involution on round handles induced by the diffeomorphism $\iota(t) = -t$ of the S^1–factor.

Let M_1^{i-1} be the round handle body obtained from R_0 and $R_1^1, \ldots, R_1^{i-1}$. Assume that we have attached all round handles R_1^j with $j \leq i-1$ and that we have extended the Engel structure over all these round handles of index 1 such that the contact structure on $\partial_+ M_1^{i-1}$ is overtwisted. Assume moreover that throughout this process we have homotoped the trivialization of M such that it coincides with the Engel trivialization on the round handle body we have treated so far.

Hence the Engel trivialization on M_1^{i-1} can be extended to $M_1^{i-1} \cup_{\varphi_1^i} R_1^i$. By Theorem 5.8, we can isotope φ_1^i to an attaching map $\widetilde{\varphi}_1^i$ such that the Engel structure on M_1^{i-1} extends to an Engel structure on $M_1^i = M_1^i \cup_{\varphi_1^i} R_1^i$ using a model Engel structure on R_1 from Section 4.2.2.

In order to ensure that the contact structure on $\partial_+ M_1^i$ is again overtwisted, we isotope φ_1^i before the application of Theorem 5.8 such that its image is disjoint from an overtwisted disc in $\partial_+ M_1^{i-1}$. For this, assume that $\varphi_1^i(\gamma_\pm)$ is transversal to an overtwisted disc D_{ot} and let p be a point on D_{ot} which does not lie on $\varphi_1^i(\gamma_\pm)$. Then use the flow of a radial vector field centered at p to isotope $\varphi_1^i(\gamma_\pm)$ such that the image of $\gamma_\pm$ becomes disjoint from D_{ot}. The remaining steps, like making the attaching curves Legendrian and stabilization, can be carried out in a small tubular neighbourhood which is also disjoint from D_{ot}.

Unfortunately, the Engel trivialization and the original trivialization of M need not to be homotopic on M_1^i relative to M_1^{i-1}. We can arrange this by applying a suitable self–diffeomorphism of R_1^i. Let $\gamma_\pm$ be the attaching curves $\{\pm 1\} \times \{0\} \times S^1 \subset \partial_- R_1^i$ with their orientation from the S^1–factor and consider

$$I \times \{0\} \times S^1 \subset R_1^i = I \times D^2 \times S^1 \, .$$

This cylinder can be decomposed into a 1–cell $e_1 = I \times \{0\} \times \{1\}$ and a 2–cell e_2. The 1–cell is attached to M^{i-1} using the restriction of φ_1^i. The 2–cell e_2 is attached along γ. This path is formed from the consecutive paths e_1 from $-1 \in I$ to $1 \in I$, $\varphi_1^i(\gamma_+)$ with the positive orientation, $-e_1$ and finally $\varphi_1^i(\gamma_-)$ with the orientation inverse to the given one.

We first modify the Engel structure on R_1^i such that the new Engel trivialization is homotopic to the given orientation along e_1 relative to the endpoints of e_1. Let $\rho : I = [-1,1] \longrightarrow [0, 2\pi]$ be a smooth function which is constant near the boundary, $\rho(-1) = 0, \rho(1) = 2\pi$. Then consider the diffeomorphism

$$\begin{aligned} F_1 : R_1^i = I \times D^2 \times S^1 &\longrightarrow I \times D^2 \times S^1 = R_1^i \\ (x, y_1, y_2, t) &\longmapsto \begin{array}{l} (x, \cos(\rho(x))y_1 + \sin(\rho(x))y_2, \\ -\sin(\rho(x))y_1 + \cos(\rho(x))y_2, t) \end{array} \, . \end{aligned}$$

As in the case of round zero handles we now use the fact $\pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$. If the Engel trivialization and the given trivialization of M are not yet homotopic along e_1 relative to the boundary points, then we push forward the model Engel structure on R_1^i using F_1. The properties of ρ ensure that we obtain again a smooth Engel structure on M_1^i but the trivialization induced by the new Engel structure is homotopic to the given trivialization along e_1 relative to the boundary.

Next consider the 2–cell e_2. Both the Engel trivialization and the given trivialization of M extend from $\gamma = \partial e_2$ to e_2. Since $\pi_2(\mathrm{SO}(4))$ is trivial, this extension is unique up to homotopy relative to γ.

Now $M_1^{i-1} \cup e_1 \cup e_2$ is a strong deformation retract of M_1^i relative to M_1^{i-1}. Thus we can extend the Engel structure from M_1^{i-1} to M_1^i such that the Engel trivialization and the given trivialization of M are homotopic relative to M_1^{i-1}. The attaching region of the round 1–handle can be chosen so small that in its complement there is an overtwisted disc. Thus the contact structure on $\partial_+ M_1^i$ is still overtwisted.

In the next step we attach round 2–handles. We are no longer able to ensure that the Engel trivialization and the given trivialization on M are homotopic after we attach round 2–handles. Assume that we have already attached the first $i-1$ round 2–handles such that on the resulting handle body M_2^{i-1} we have an Engel structure extending the Engel structure on M_1. The contact structure on the boundary is assumed to be overtwisted. Consider the attaching map

$$\varphi_2^i : \partial_- R_2^i \longrightarrow \partial_+ M_2^{i-1} \, .$$

The contact structure on $\partial_+ M_2^{i-1}$ is orientable and it has an oriented section, namely the intersection line field. Thus the Euler class of the contact structure, viewed as bundle, vanishes. By assumption, the contact structure is overtwisted. According to Theorem 6.7 we can isotope φ_2^i such that the singular foliation on the image of φ_2^i,

$$T_0^2 \simeq \partial D^2 \times \{0\} \times S^1 \subset \partial_- R_2^i ,$$

is in standard form. Moreover, we can assume that the Legendrian divides γ_1, γ_2 are tangent to $\varphi_2^i(\{p\} \times \{0\} \times S^1)$ with $p \in \partial_D$. The Legendrian ruling can be chosen to be tangent to the foliation induced by the first factor of $\partial D^2 \times \{0\} \times S^1$. Finally, Theorem 6.7 ensures that the attaching region of R_2^i is contained in a neighbourhood U_i of $\varphi_2^i(T_0^2)$ which is disjoint from some overtwisted disc.

In order to find a model Engel structure on R_2^i which extends the Engel structure on M_2^{i-1} to an Engel structure on $M_2^i = M_2^{i-1} \cup_{\varphi_2^i} R_2^i$ we are left with several difficulties. These concern the homotopy class of the intersection line field as a Legendrian line field.

(1) We have to show that the rotation number along the Legendrian rulings is even.
(2) We have to ensure that the rotation number along the Legendrian divides is not zero.

If we can ensure these two additional conditions we can apply Proposition 6.3 and Proposition 6.5 to find a model Engel structure on R_2^i and an isotopy of φ_2^i such that the new attaching map

- has an image which is contained in a tubular neighbourhood $U_i \simeq T^2 \times \mathbb{R}$.
- preserves the orientation of $\partial_- R_2^i$ and $\partial_+ M_2^{i-1}$ which is induced by the respective contact structure.
- preserves contact structures together with the orientations which are induced by the Engel structures.
- maps the intersection line field on $\partial_- R_2^i$ to a Legendrian line field on $\partial_+ M_2^i$ which is homotopic to the intersection line field of the Engel structure on M_2^{i-1}, cf. Remark 6.4.

After a suitable vertical modification of $\partial_+ M_2^{i-1}$, we can attach R_2^i such that the model Engel structure on R_2^i extends the Engel structure on M_2^{i-1} smoothly. By Theorem 6.7 and Lemma 6.9 the complement of a small tubular neighbourhood of the image of T_0^2 under the isotoped attaching map contains an overtwisted disc. If we choose the attaching region small enough, the contact structure on $\partial_+ M_2^i$ is still overtwisted. At this stage we use that there is a trivialization of TM over all ordinary handles of index ≤ 2, this corresponds to the condition that the second Stiefel–Whitney class of M vanishes, cf. Theorem 6.2.

We now show that we can always achieve the two conditions above with the following assertions. Let

$$\gamma = \partial D^2 \times \{0\} \times \{1\} .$$

Claim (1) : The Engel trivialization on M_2^{i-1} extends to a trivialization of TM over $D^2 \times \{0\} \times S^1 \subset R_2^i$

PROOF OF CLAIM (1). Let $S^1 = I_0 \cup I_1$ be the union of two closed intervals which have only boundary points in common. We assume that $1 \in S^1$ is contained in the interior of I_0. We decompose the round 2–handle R_2^i into one ordinary handle of index 2 and one ordinary handle of index 3

$$\begin{aligned} R_2^i = D^2 \times I \times S^1 &= (D^2 \times (I \times I_0)) \cup ((D^2 \times I_1) \times I) \\ &\simeq (D^2 \times D^2) \cup (D^3 \times I) = h_2^i \cup h_3^i . \end{aligned}$$

With this identification, the attaching curve of h_2^i is γ. The attaching map $\overline{\varphi}_2^i$ of h_2^i is the restriction of φ_2^i to $\partial_- h_2^i \subset \partial_- R_2^i$. The attaching map of h_3^i can also be described using φ_2^i and an identification of $\partial_+ h_2^i$ with parts of $\partial_- h_3^i$ which are obvious from the decomposition. But we will not need the attaching map of h_3^i explicitly.

The claim only involves the 2–handle h_2^i but not the 3–handle h_3^i. Thus we can remove h_3^i.

Recall that $\partial_+ R_2^{i-1} = D^2 \times \{\pm 1\} \times S^1$ has dimension three. Thus if we perturb $\overline{\varphi}_2^i$ slightly, the attaching curve γ of the 2–handle becomes disjoint from the circles $\{0\} \times \{\pm 1\} \times S^1$ in $\partial_+ R_2^{i-1}$. Thus using the flow of a vector field which points away from these circles, we can isotope $\overline{\varphi}_2^i$ such that its image does not intersect $\partial_+ R_2^{i-1}$.

We remove R_2^{i-1} from our round handle body. Now we can apply the same procedure with R_2^{i-2}. Iterating this procedure, we can isotope $\overline{\varphi}_2^i$ such that in the end its image is contained in $\partial_+ M_1$.

As we have shown above, the Engel trivialization extends from M_1 to the whole of M. In particular it can be extended over h_2^i when this handle is attached to $\partial_+ M_1$. But in order to achieve this, we have only isotoped the attaching map.

This shows that the Engel trivialization on M_2^{i-1} extends over h_2^i also with the original attaching map. □

Claim (2) : The rotation number along $\varphi_2^i(\gamma)$ is even.

PROOF OF CLAIM (2). For this let us fix a model Engel structure on R_2^i and isotope φ_2^i so that it preserves the contact structure on a neighbourhood of the image of $\gamma \subset \partial_- R_2^i$. We homotop the Engel trivialization on M_2^{i-1} such that the only component of the framing which is not tangent to $\partial_+ M_2^{i-1}$ is W. Then we pull back the Engel trivialization on $\partial_+ M_2^{i-1}$ to a framing on $\partial_- R_2^i$.

Strictly speaking, this does not make sense because φ_2^i is a map to $\partial_+ M_2^{i-1}$ but the Engel framing has one component which is transversal to this boundary. This is the vector field W which orients the characteristic foliation. But since on R_2 the characteristic foliation is also oriented by a vector field W_R which is transversal to $\partial_- R_2^i$, we can take W_R as pull–back of W.

Since we have assumed that the attaching map preserves contact structures, the pull back of the component of the Engel framing which is orthogonal to the even contact structure is transversal to the even contact structure on R_2. Without loss of generality, we can choose these components of the Engel framings such that they are preserved by φ_2^i. Thus by definition the pullback framing and the Engel framing on R_2^i have two common components. When we want to compare the pull back framing with the Engel trivialization along γ it is therefore enough to consider the rotation numbers along γ.

By the definition of the model Engel structures, the rotation number of the Engel trivialization on R_2^i is even. If the rotation number of the pull back framing along γ is odd, then the pull back framing and the Engel framing are not homotopic along γ. But this implies that the pull back framing can not be extended over the disc $D^2 \times \{0\} \subset h_2$. This is a contradiction to Claim (1). □

Claim (3) : We can isotope φ_2^i such that $\varphi_2^i(T_0^2)$ is in standard form and the rotation number along the Legendrian divides is not zero.

PROOF OF CLAIM (3). Assume $\varphi_2^i(T_0^2)$ is in standard form and the rotation number along the Legendrian divides is zero. By Claim (2), we can apply Lemma 6.9. Thus we achieve that the rotation number along the Legendrian divides is not zero at the expense of changing the slope of the dividing curves by a right handed Dehn twist. □

This shows that we can extend the Engel structure from M_2^{i-1} to M_2^i by a model Engel structure from Section 6.1. If we really have applied Lemma 6.9 in Claim (3) then we have to modify our model Engel structures slightly by a push forward with the diffeomorphism

$$\delta : R_2^i = D^2 \times I \times S^1 \longrightarrow D^2 \times I \times S^1$$
$$((y_1, y_2), x, t) \longmapsto ((\cos(t)y_1 - \sin(t)y_2, \sin(t)y_1 + \cos(t)y_2), x, t)\ .$$

When we restrict this diffeomorphism to the torus $\partial D^1 \times \{0\} \times S^1$ this performs a right handed Dehn twist.

Assume that we have an extension of the Engel trivialization on M_2^{i-1}. This is the case for $i = 1$. Then unfortunately the given trivialization and the Engel trivialization on M_2^i are not homotopic relative to M_2^{i-1} in general. This is due to the fact that $\pi_3(\mathrm{SO}(4)) \simeq \mathbb{Z} \times \mathbb{Z}$. Hence if we decompose R_2^i into ordinary handles h_2^i, h_3^i of index 2 and 3, the extension of the Engel trivialization on M_2^{i-1} is unique up to homotopy over h_2^i but there are many non–homotopic possible extensions over h_3^i.

After the last attachment of a round 2–handle, we have extended the Engel structure to M_2. When we want to extend the Engel structure from M_2 to M, the Engel trivialization has to extend, too. Once we have shown that this is really the case, we can choose a model Engel structure on R_3 such that the Engel structure extends to M.

Claim (4) : The Engel trivialization extends from M_2 to M.

PROOF OF CLAIM (4). First we reduce the problem to bundles of rank 3. The first component W of the Engel trivialization is transversal to ∂M_2 by construction. Thus W extends to a vector field without zeroes on M. We equip M with an almost quaternionic structure such that the Engel framing and W, IW, JW, KW coincide on $M_1 \cup h_2^1 \ldots \cup h_2^{r_2}$. Then we can choose a trivialization of the orthogonal complement $\mathcal{W}^\perp$ of W in M. (This trick is from Geiges [**Gei**]). For the remaining part of the proof of Claim (4), we consider $\mathcal{W}^\perp$.

We decompose all round 2–handles into ordinary handles h_2^j, h_3^j for $1 \leq j \leq r_2$ of index 2 and 3 and we rearrange the handles such that the 2–handles are attached to M_1. We have already shown in Claim (1) that the Engel trivialization of $\mathcal{W}^\perp$ on M_1 extends to $M_1 \cup h_2^1 \cup \ldots \cup h_2^{r_2}$ and because $\pi_2(\mathrm{SO}(3))$ is trivial, the extension of the trivialization over these handles is unique up to homotopy. Therefore, the Engel trivialization on $M_1 \cup h_2^1 \cup \ldots h_2^{r_2}$ also extends to M.

Finally we decompose the round 3–handle into an ordinary 3–handle $\widehat{h}_3$ and one ordinary 4–handle $\widehat{h}_4$. We have shown that on the 2–skeleton the SO(3)–bundle $\mathcal{W}^\perp$ is trivial. Therefore we can lift it to a S^3–bundle. (Recall that $\mathrm{Spin}(3) = \mathrm{SU}(2) = S^3$.) Since $\pi_2(S^3)$ is trivial, the trivialization of $\mathcal{W}^\perp$ extends from M_2 to $\widehat{h}_3$. We fix such an extension.

The obstruction for the extension of the trivialization of $\mathcal{W}^\perp$ from the union of all ordinary handles of index ≤ 3 to M is a cochain x in the cellular cochain group $C^4(M, \mathbb{Z})$ which depends on the choice of extensions of the trivialization over the 3–handles and on the handle decomposition itself. The cochain x represents a class $[x] \in H^4(M, \pi_3(S^3)) = H^4(M, \mathbb{Z})$ which does not depend on the choice of trivializations on the 3–handles or the handle decomposition. According to [**GoS**] p. 31, $[x]$ is the second Chern class of the $SU(2)$–bundle. As we have showed $\mathcal{W}^\perp$ is trivial, hence $[x] = 0$.

Recall that $c_2(\mathcal{W}^\perp) = p_1(\mathcal{W}^\perp) = p_1(TM) = 3\sigma(M)$ by the signature theorem of Hirzebruch, so in this step of the proof we use the fact that the signature of a parallelizable 4–manifold is zero, cf. Theorem 6.2.

The handle decomposition of M contains exactly one 4–handle and M is an oriented closed manifold $C^4(M, \mathbb{Z}) = H^4(M, \mathbb{Z})$. Thus $[x] = 0$ implies $x = 0$. Therefore the Engel trivialization of $\mathcal{W}^\perp$ extends from M_2 to M although the Engel trivialization on M_2 may not be homotopic to the trivialization of M we fixed at the beginning of the proof. □

The contact structure $\mathcal{C}$ on $\partial M_2 = S^2 \times S^1$ is overtwisted. By Claim (4) the Engel trivialization extends from M_2 to M. By Lemma 6.10 $\mathcal{C}$ is contained in the unique homotopy class of orientable plane fields which extends from $S^2 \times S^1$ to $D^3 \times S^1$. According to Theorem 2.33, φ_3 can be isotoped such that it preserves the contact structure on R_3 when we equip R_3 with a model Engel structure. (Recall that this contact structure is the same for all models.)

Now we chose the model Engel structure on R_3 such that the orientation of the contact structure as well as the homotopy type of the intersection line fields is preserved. This is possible by Proposition 6.11. This proves the theorem. □

CHAPTER 7

Geometric examples

In this chapter we discuss Engel structures from a different point of view. If a manifold X admits an Engel structure which is invariant under the action of a discrete group Γ such that X/Γ is again a smooth manifold, then we obtain an Engel structure on X/Γ. A rich source of group actions are Thurston geometries. Let us summarize some facts about Thurston geometries ([**Thu2**]).

DEFINITION 7.1. Let X be a simply connected, complete Riemannian manifold and G the group of isometries of X. The pair (X, G) is called *Thurston geometry* if

(i) G acts transitively on X
(ii) the stabilizer of a point $x \in X$
$$\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$$
is compact
(iii) G contains a discrete subgroup Γ such that X/Γ is a compact manifold.

One Thurston geometry (X_1, G_1) is said to be *equivalent* to another Thurston geometry (X_2, G_2) if there is a diffeomorphism $\psi : X_1 \to X_2$ such that $\psi \circ G_1 \circ \psi^{-1}$ is a subgroup in G_2. Note that this is *not* an equivalence relation. If Γ is a lattice in G then X/Γ is said to have X–geometry.

If (X, G) is a Thurston geometry and $H \subset G$ is a subgroup such that (X, H) is also a Thurston geometry then (X, G) and (X, H) are equivalent. Therefore one usually only considers Thurston geometries (X, G) where G is the maximal group with the properties in Definition 7.1. Notice that G is not required to be connected. We write G_0 for the identity component of G and $\mathbb{H}^n$ for the hyperbolic space of dimension n.

For $\dim(X) = 3$, Thurston classified all possible Thurston geometries up to equivalence in [**Thu2**] as follows

Isomorphism type of $\mathrm{Stab}_0(x)$	Isometry class of X
SO(3)	$S^3, \mathbb{H}^3, \mathbb{R}^3$
SO(2)	$S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$
	Nil^3, $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$
$\{1\}$	Sol^3

We will describe the Riemannian metrics and isometry groups later. The source we use for this is [**Thu2**]

In dimension 4, Filipkiewicz obtained the following classification of Thurston geometries up to equivalence in [**Fil**]. The following list can be found in [**Wa1**].

Isomorphism type of $\mathrm{Stab}_0(x)$	Isometry class of X
SO(4)	$S^4, \mathbb{H}^4, \mathbb{R}^4$
U(2)	$\mathbb{CP}^2, \mathbb{H}^2(\mathbb{C})$
SO$(2) \times$ SO(2)	$S^2 \times S^2, S^2 \times \mathbb{R}^2, S^2 \times \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^2, \mathbb{H}^2 \times \mathbb{H}^2$
SO(3)	$S^3 \times \mathbb{R}, \mathbb{H}^3 \times \mathbb{R}$
SO(2)	Nil$^3 \times \mathbb{R}$, $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$, Sol$^4_0$
$\{1\}$	Nil^4, Sol$^4(m,n)$, Sol$^4_1$

The geometries $\mathrm{Sol}^4(m,n)$ are indexed by positive integers m,n. We will give more details later in the discussion of Engel structures on geometric manifolds. The product geometry $\mathrm{Sol}^3\times\mathbb{R}$ is included in $\mathrm{Sol}^4(m,n)$. The descriptions of the isometry groups are essentially from [**Wa1, Wa2**].

We say that a contact structure respectively an Engel structure on X is geometric if it is invariant under a subgroup H of the isometry group G such that (X,H) is a Thurston geometry.

In Section 7.1 we discuss which 3–dimensional Thurston geometries admit a geometric contact structure. The contact structures on $X = S^3, \mathrm{Nil}^3, \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ will appear later in Section 7.2 in the discussion of geometric Engel structures. For these geometries, the contact plane at a point p is invariant under a 1–dimensional subgroup of the isometry group. The only other Thurston geometry which is compatible with a contact structure is Sol^3.

Section 7.2 and Section 7.3 contain a discussion of Engel structures compatible with Thurston geometries. Many of the 4–dimensional Thurston geometries do not admit an Engel structure for topological reasons. The remaining geometries can be treated in two different ways.

The geometries $S^3\times\mathbb{R}, \mathrm{Nil}^3\times\mathbb{R}$ respectively $\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}$ can be treated starting from contact structures on S^3, Nil^3 respectively $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Here we use a construction similar to prolongation (cf. Proposition 3.15).

The other 4–dimensional Thurston geometries are treated individually in Section 7.3. It turns out that these geometries admit geometric Engel structures which are similar to those obtained by the construction of H. J. Geiges (cf. Proposition 3.17).

Let us emphasize that we treat only the existence of geometric contact structures respectively Engel structures but we do not classify them up to isomorphism.

7.1. Geometric contact manifolds

In the following we seek geometric contact structures in dimension 3. We show only their existence but we do not classify them.

DEFINITION 7.2. A *geometric contact structure* is a triple $(X,\mathcal{C},G)$ where $(X,\mathcal{C})$ is a contact manifold and G is a group of diffeomorphisms of X which preserve $\mathcal{C}$. Moreover, (X,G) is assumed to be a Thurston geometry. A geometric contact structure is called *maximal* if its isometry group consists of all orientation preserving isometries in G.

When the identity component $G_0\subset G$ acts freely, one cannot expect that geometric contact structures are unique. Any small G_0–equivariant perturbation of the contact structure will yield again a geometric contact structure which is invariant under the action of the identity component. The perturbed contact structure is no longer invariant under all connected components of G. We will show this in some cases but we do not classify geometric contact structures up to equivalence. The following table summarizes the existence results.

Thurston geometry	geometric contact structure	maximal
S^3	yes	no
$\mathbb{R}^3, \mathbb{H}^3$	no	no
$S^2\times\mathbb{R}, \mathbb{H}^2\times\mathbb{R}$	no	no
$\mathrm{Nil}^3, \widetilde{\mathrm{Sl}}(2,\mathbb{R})$	yes	yes
Sol^3	yes	yes

The cases are grouped according to the corresponding maximal Thurston geometry.

7.1.1. $X = S^3$**.** The full isometry group is $G = \mathrm{O}(4)$ acting on $S^3 \subset \mathbb{R}^3$. The metric on S^3 is a multiple of the standard metric with constant curvature.

We identify S^3 with $SU(2)$. Choosing a plane $\mathcal{C}_e$ in $T_eSU(2) = \mathfrak{su}(2)$ we obtain a left–invariant distribution $\mathcal{C}$ of rank 2 on S^3. If $X, Y \in \mathfrak{su}(2)$ span $\mathcal{C}_e$ then $[X, Y]$ is linearly independent of X, Y. Hence the commutator of two linearly independent left–invariant sections of $\mathcal{C}$ is nowhere tangent to $\mathcal{C}$. This shows that $\mathcal{C}$ is a contact structure. By definition it is invariant under $\mathrm{SU}(2) \subset \mathrm{O}(4)$. We want to determine the maximal subgroup of isometries of S^3 which preserve $\mathcal{C}$.

Using a suitable element of $h \in \mathrm{Stab}(e)$ we can achieve that $h_*\mathcal{C}_e$ is the complex subspace of T_eS^3. Then $h_*\mathcal{C}$ is the standard contact structure on S^3 which is defined by

$$\alpha = x_1dy_1 - y_1dx_2 + x_2dy_2 - y_2dx_2 \ .$$

The orthogonal complement of the standard contact structure on S^3 is tangent to the Hopf fibration. The fiber of the Hopf fibration through p is the intersection of the orthogonal complement $\mathcal{C}^{bot}$ of $\mathcal{C}(p) \subset T_p\mathbb{C}^2 = \mathbb{C}^2$ with S^3 and $\mathcal{C}^\perp$ is again a complex subspace of $\mathbb{C}^2$. Moreover the map

$$\begin{aligned} \mathrm{conj} : S^3 &\longrightarrow S^3 \\ (x_1, y_1, x_2, y_2) &\longmapsto (x_1, -y_1, x_2, -y_2) \end{aligned}$$

preserves ker(α) but it reverses the coorientation of $\mathcal{C}$.

Hence $g \in \mathrm{O}(4)$ preserves $h_*\mathcal{C}$ if and only if g preserves the complex subspaces of $\mathbb{C}^2$. Because g is an isometry it also preserves the action of i up to multiplication with ± 1. So g preserves $\mathcal{C}$ if and only if

$$g \in \mathrm{Gl}(2, \mathbb{C}) \cap \mathrm{O}(4) \ .$$

If g anticommutes with i then $g \circ \mathrm{conj} \in U(2)$. Thus we have shown that the subgroups of isometries preserving $\mathcal{C}$ is to

$$H = \mathrm{U}(2) \cup (\mathrm{U}(2) \circ \mathrm{conj}) \subset \mathrm{O}(4) \ .$$

So very lattice $\Gamma \subset H$ gives rise to a contact structure on S^3/Γ. The manifolds obtained this way include all lens spaces. In [**Wo**] one can find more spherical space forms corresponding to subgroups $\Gamma \subset H$.

7.1.2. $X = \mathbb{R}^3$**.** The metric is the flat metric and the maximal group of isometries is $\mathbb{R}^3 \rtimes \mathrm{O}(3)$ acting in the obvious way on $\mathbb{R}^3$.

Suppose that $G \subset \mathbb{R}^3 \rtimes \mathrm{O}(3)$ acts transitively on $\mathbb{R}^3$. Then G must contain $\mathbb{R}^3$ since all elements of $\mathrm{O}(3)$ fix the origin of $\mathbb{R}^3$. So a contact structure which is invariant under the action of G is invariant under the action of $\mathbb{R}^3$ on itself. But every translation invariant plane field on $\mathbb{R}^3$ is integrable. Therefore there is no geometric contact structure which is equivalent to the Thurston geometry $(\mathbb{R}^3, \mathbb{R}^3 \rtimes \mathrm{O}(3))$.

7.1.3. $X = \mathbb{H}^3$**.** The metric on $\mathbb{H}^3$ is the usual hyperbolic metric and its isometry group is $G \simeq \mathrm{PSl}(2, \mathbb{C}) \rtimes \mathbb{Z}_2$ where the non–zero element in $\mathbb{Z}_2$ acts on $\mathrm{PSl}(2, \mathbb{C})$ by composition with a reflection along a fixed hyperbolic plane.

The maximal isometry group has two connected components and taking $H = G_0$ yields a non–maximal Thurston geometry $(\mathbb{H}^3, H)$.

In order to show that there are no other non–maximal geometries equivalent to $\mathbb{H}^3$, we prove that there is no subgroup H of G which has codimension at least one and acts transitively on $\mathbb{H}^3$ such that there is a lattice $\Gamma \subset H$. Assume that H has the desired properties.

Since G_0 has finite index in G we can assume that $H \subset G_0$. We fix a basepoint $x_0 \in \mathbb{H}^3$. Then G_0 is foliated by $\{g \in G_0 | gx_0 = x\}$ for $x \in \mathbb{H}^3$. A subgroup H which acts transitively on $\mathbb{H}^3$ has to be transversal to this foliation and in particular to the stabilizer $K = \mathrm{Stab}(x_0) = \mathrm{SU}(2)/\{\pm 1\} \simeq \mathrm{SO}(3)$ of x_0. This is a maximal compact subgroup of G. If H is not connected then by transversality each connected component of H meets K. Since K is compact and H is supposed to be a closed subgroup, H has only finitely many connected components.

Hence $\Gamma \cap H_0$ has finite index in Γ and the same is true for $H_0 \subset H$ and we can restrict ourselves to connected groups H. The assumption that H contains a lattice implies that H cannot be an algebraic subgroup of G_0 by the Borel density theorem:

THEOREM 7.3 (Borel,[**VGS**]). *Let G be a real algebraic group and Γ a lattice in G. Then the closure $\overline{\Gamma}^a$ with respect to the Zariski topology on G contains a normal cocompact subgroup G' of G.*

It remains to show that every connected Lie subgroup H of G_0 which is transversal to K is contained in an algebraic Lie subgroup of the same dimension. The question which subalgebras of the Lie algebra of an algebraic group correspond to algebraic subgroups is studied for example in [**Bor**]. Such subalgebras will be called *algebraic*. We use the following results from chapter II.7 of [**Bor**].

THEOREM 7.4 (Chevalley, [**Bor**]). *Let $\mathfrak{g}$ be the Lie algebra of an algebraic Lie group G_0 and $\mathfrak{h}$ a subalgebra which corresponds to a Lie subgroup of G_0.*

(i) *$\mathfrak{h}$ is algebraic if it is spanned by algebraic Lie subalgebras.*
(ii) *$[\mathfrak{h}, \mathfrak{h}]$ is algebraic.*

Since H acts transitively on $\mathbb{H}^3$, its dimension is at least three. We denote the Lie algebra of H by $\mathfrak{h}$ and $\mathfrak{h}^{(1)} = [\mathfrak{h}, \mathfrak{h}]$. Obviously $\mathfrak{h}^{(1)}$ is a subalgebra of $\mathfrak{h}$.

If $\dim(H) = 3$ and $\mathfrak{h}^{(1)} = \mathfrak{h}$, then H is algebraic. If $\mathfrak{h}^{(1)} \neq \mathfrak{h}$ then $\mathfrak{h}^{(1)}$ would be two dimensional and solvable or Abelian. Hence H and Γ would be solvable. This leads to a compact hyperbolic manifold with solvable fundamental group and to a contradiction to Preissmann's theorem. Hence $\mathfrak{h}^{(1)} = \mathfrak{h}$ and $\mathfrak{h}$ is algebraic.

If $\dim(H) = 5$ then by transversality $H \cap K$ is a subgroup of K of dimension 2. But $K = \mathrm{SO}(3)$ has no such subgroups.

We are left with the case $\dim(H) = 4$. If the dimension of $\mathfrak{h}^{(1)}$ or $[\mathfrak{h}^{(1)}, \mathfrak{h}^{(1)}]$ is less than three, then we obtain a hyperbolic manifold with solvable fundamental group and a contradiction to Preissmann's theorem (as above). The remaining case is $\dim(\mathfrak{h}^{(1)}) = 3$ and $[\mathfrak{h}^{(1)}, \mathfrak{h}^{(1)}] = \mathfrak{h}^{(1)}$. In particular $\mathfrak{h}^{(1)}$ is algebraic. In view of (i) of Theorem 7.4 it suffices to find an algebraic complement of $\mathfrak{h}^{(1)}$ in $\mathfrak{h}$.

Since $\mathfrak{h}$ is transversal to $\mathfrak{k}$ there is a vector V spanning $\mathfrak{h} \cap \mathfrak{k}$. If we conjugate H with arbitrary elements g of K we obtain subgroups of G which correspond to non–maximal geometries. Without loss of generality we can assume

$$V = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}) \; .$$

It is to see that the subgroups of $\mathrm{PSl}(2, \mathbb{C})$ corresponding to V and iV are algebraic. Moreover iV is not contained in the Lie–algebra $\mathfrak{k}$ of K. We now show that one of the two vectors V, iV together with $\mathfrak{h}^{(1)}$ generates $\mathfrak{h}$.

Since $\mathfrak{h}$ is transversal to $\mathfrak{k}$, there is an element of the form $iV + W$ with $W \in \mathfrak{k}$ in $\mathfrak{h}$. Consider

$$[V, iV + W] = [V, W] \in \mathfrak{k} \; .$$

If V and W were linearly independent then by the commutator relations (73) of $\mathfrak{k} = \mathfrak{su}(2)$ this would imply that $[V, W]$ is linearly independent of V. But then $\dim(\mathfrak{k} \cap \mathfrak{h}) \geq 2$ and this is a contradiction to our assumptions on H. Thus V, W are not linearly independent and we may assume $W = 0$. Let V, C_1, C_2 be a basis of $\mathfrak{k}$ such that

$$[V, C_1] = 2C_2, \quad [C_1, C_2] = 2V, \quad [C_2, V] = 2C_1\ . \tag{73}$$

Then $\mathfrak{h}$ is generated as a real vector space by

$$V, iV, \widehat{V}_1 = iC_1 + (a_1C_1 + a_2C_2), \widehat{V}_2 = iC_2 + (b_1C_1 + b_2C_2)$$

with real numbers a_1, a_2, b_1, b_2. It is clear from the commutator relations (73) that we can obtain at most one of the vectors V, iV by forming commutators of the basis vectors of $\mathfrak{h}$ described above. Because V, iV and $\mathfrak{h}^{(1)}$ are algebraic Lie algebras the same is true for $\mathfrak{h}$ by Theorem 7.4.

Thus we have shown that, apart from $H = G_0$, there is no non–maximal geometry equivalent to $(\mathbb{H}^3, G)$. Since there is no plane field which is invariant under the action of G_0, there is no geometric contact structure equivalent to $(\mathbb{H}^3, G)$.

7.1.4. $X = S^2 \times \mathbb{R}$**.** This is the obvious product geometry. The full isometry group is the product of the isometry group of S^2 and $\mathbb{R}$. It has four connected components.

Suppose that $\mathcal{C}$ is a geometric contact structure on X. Since G acts transitively, $\mathcal{C}$ is either everywhere tangent to the foliation corresponding to the first factor of $S^2 \times \mathbb{R}$ or $\mathcal{C}$ is everywhere transversal to it. The first case is impossible since contact structures have no integral surfaces. The second case is impossible since it would imply the existence of a nowhere vanishing line field $\mathcal{C} \cap TS^2$ on S^2.

7.1.5. $X = \mathbb{H}^2 \times \mathbb{R}$**.** This is the second product geometry. The isometry group G is the product of the isometry groups of the factors. As in the case of $S^2 \times S^1$ it has four connected components.

The subgroups H of G for which $(\mathbb{H}^2 \times \mathbb{R}, H)$ is a Thurston–geometry have dimension 3 or 4. In the four–dimensional case, H is the union of several connected components of G. Now we want to show that there is no three-dimensional group H which acts transitively on $\mathbb{H}^2 \times \mathbb{R}$ and contains a cocompact lattice. Suppose that H is such a group. Then H has to be transversal to the stabilizer of a fixed point $x_0 \in \mathbb{H}^2$. Let $\mathfrak{h}$ be the Lie algebra of H.

Since H acts transitively, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{sl}(2, \mathbb{R})$ has dimension at least two. Let T span the Lie algebra of $\mathrm{Isom}(\mathbb{R})$. Since H acts transitively along the real line $\mathbb{R}$ through $x_0 \in \mathbb{H}^2$, we can consider a smooth path in H such that the image of x_0 under the action of the group elements on this path is contained in the real line. Hence $\mathfrak{h}$ contains a vector of the form $T + w$ where T corresponds to the Lie algebra of $\mathbb{R}$ while $w \in \mathfrak{so}(2) \subset \mathfrak{sl}(2, \mathbb{R})$ is tangent to the stabilizer of x_0.

Because T lies in the center of $\mathfrak{g}$ and $\mathfrak{h}'$ is transversal to w, the Lie algebra generated by $T + w$ and $\mathfrak{h}'$ actually contains $\mathfrak{sl}(2, \mathbb{R})$ for $w \neq 0$. This a contradiction to our initial assumptions. Thus $w = 0$ and the identity component of H is the product of $\mathbb{R}$ with a two–dimensional subgroup of $\mathrm{PSl}(2, \mathbb{R})$. In particular H and Γ are solvable. Since H has to be transversal to the foliation of $\mathrm{PSl}(2, \mathbb{R})$ whose leaves are given by $\{g \in \mathrm{PSl}(2, \mathbb{R}) | gx_0 = x\}$ for $x \in \mathbb{H}^2$, the connected component of the identity of H has finite index in H. Hence we can assume that H itself is connected. We apply the following theorem to $R = \mathbb{R}$.

THEOREM 7.5 (Wang, [**Rag**]). *Let H be a connected Lie group and R its radical. Assume that H/R has no compact factors. Let Γ be a lattice in H and $\pi : H \to H/R$ the natural map. Then $\pi(\Gamma)$ is discrete in H/R.*

Hence $\pi(\Gamma)$ is a discrete group. As $\mathbb{H}^2$ is connected and the stabilizer of $x \in \mathbb{H}^2$ under the action of $\pi(\Gamma)$ varies continuously with x, the stabilizer of x under the action of $\pi(\Gamma)$ is independent of x. We choose an element $g \in \Gamma$ of this stabilizer. Then g preserves distinct points of the hyperbolic plan which means that g acts by the identity on $\mathbb{H}^2$. Thus $\pi(\Gamma)$ acts freely on $\mathbb{H}^2$ with compact quotient. On the other hand $\pi(\Gamma)$ is solvable. This leads to a contradiction to Preissmann's theorem.

The only plane field which is invariant under the action of the identity component of G is tangent to the foliation induced by first factor of $\mathbb{H}^2 \times \mathbb{R}$. So there is no geometric contact structure equivalent to $(\mathbb{H}^2 \times \mathbb{R}, G)$.

7.1.6. $X = \mathrm{Nil}^3, \widetilde{\mathrm{Sl}}(2,\mathbb{R})$**.** Here there are natural geometric contact structures. Remember that the stabilizer of a point is one–dimensional for these two geometries. In both cases, the contact plane at $x \in X$ is the $\mathrm{Stab}(x)$–invariant subspace of T_xX.

The nilpotent 3–dimensional Lie group Nil^3 has the description

$$\mathrm{Nil}^3 = \left\{ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \, x, y, z \in \mathbb{R} \right\}$$

with matrix multiplication. Let X, Y, Z be the left invariant vector fields with $X(e) = \partial_x, Y(e) = \partial_y, Z(e) = \partial_z$. The contact structure on Nil^3 is the left invariant plane field $\mathcal{C}$ spanned by X, Y. Since $[X, Y] = Z$, $\mathcal{C}$ is really a contact structure. There is a fibration

$$\begin{array}{c} \mathrm{pr} : \mathrm{Nil}^3 \longrightarrow \mathbb{R}^2 \\ \left(\begin{array}{ccc} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \longmapsto (x, y) \end{array}$$

which is transversal to the contact structure. The assumption on X, Y, Z to be an orthonormal basis defines a metric on Nil^3. Then pr_* is a Riemannian submersion. The isometry group G_{Nil} of Nil^3 consists of lifts of those isometries of $\mathbb{R}^2$ which lift to contact automorphisms of Nil^3. It has two connected components.

DEFINITION 7.6. Let Γ be a lattice in G_{Nil} such that the quotient X/Γ is a smooth manifold. Then X/Γ is called an *Infranil–manifold*. If $\Gamma \subset \mathrm{Nil}^3$ then X/Γ is a *Nil–manifold*.

By definition of Nil^3–geometry, every Infranil–manifold inherits a contact structure from Nil^3.

EXAMPLE 7.7. All diffeomorphism types of Nil–manifolds can be obtained by using the lattice Γ_k generated by

$$a = \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), b = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right), c = \left(\begin{array}{ccc} 1 & 0 & 1/k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

for $k \in \mathbb{Z} \setminus \{0\}$. The quotient X/Γ_k is a S^1–bundle over T^2 with Euler number k.

Now we turn to $X = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Recall that $\mathrm{Sl}(2,\mathbb{R})$ acts freely and transitively on the unit–tangent bundle $S_1T\mathbb{H}^2$ of the hyperbolic plane H^2. The connection 1–form α of the hyperbolic metric is a defining form for a distribution transversal to the fibers of $\mathrm{pr} : S_1T\mathbb{H}^2 \to \mathbb{H}^2$. Because the curvature $d\alpha$ is the lift of a non–zero multiple of the volume form on $\mathbb{H}^2$, $d\alpha$ is non-degenerate on $\ker(\alpha)$. Hence $\mathcal{C} = \ker(\alpha)$ is a contact structure.

The metric on $S_1T\mathbb{H}_2$ is defined to be left–invariant under the action of $\mathrm{Sl}(2,\mathbb{R})$ such that $\mathcal{C}$ is everywhere orthogonal to the fibers of $S_1T\mathbb{H}^2$. Now we lift the contact structure and the metric to the universal cover $\widetilde{\mathrm{S_1T\mathbb{H}^2}} = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$.

The isometries of this Thurston geometry are lifts of isometries of the hyperbolic plane. Again this group has two components.

EXAMPLE 7.8. All bundles of unit tangent vectors of closed hyperbolic surfaces are examples of compact quotients of the $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$–geometry.

7.1.7. $X = \mathrm{Sol}^3$**.** The group Sol^3 can be described as semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$. We write x, y for the coordinates on $\mathbb{R}^2$ and t for the coordinate on $\mathbb{R}$. The action $\psi : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is given by

$$\psi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} .$$

For the metric on X we can take any left invariant metric. In order to have a simple description of the isometry group we choose the metric on the Lie algebra $\mathfrak{sol}^3$ such that the plane corresponding to $\mathbb{R}^2$ and the line corresponding to $\mathbb{R}$ in the semidirect product $\mathbb{R}^2 \rtimes \mathbb{R}$ are orthogonal. Then we obtain the following additional isometries of X

$$\begin{aligned} r_1 &: (x,y,t) \longmapsto (-x,y,t) \\ r_2 &: (x,y,t) \longmapsto (x,-y,t) \\ \rho &: (x,y,t) \longmapsto (y,x,-t) . \end{aligned}$$

The maximal isometry group of Sol^3 has eight connected components. Four of them contain orientation preserving isometries.

If an isometry of Sol^3 is preserves a contact structure, then it must be orientation preserving. Let X, Y, T be the left–invariant vector fields induced by $\partial_x, \partial_y, \partial_t$. Then $[X,Y] = 0, [T,X] = X, [T,Y] = -Y$. A distribution which is invariant under Sol^3, $r_1 \circ r_2$ and ρ is

$$\mathcal{C} = \mathrm{span}(T, X+Y) .$$

This defines a contact structure since $[T, X+Y] = X - Y$. It is invariant under the action of four of the eight connected components of the isometry group of Sol^3.

7.2. Geometric Engel manifolds – Prolongation

DEFINITION 7.9. A *geometric Engel structure* is a triple $(X, \mathcal{D}, G)$ where $(X, \mathcal{D})$ is an Engel manifold and G is a group of diffeomorphisms of X which preserve $\mathcal{D}$. Moreover, (X, G) is supposed to be a Thurston geometry.

Generally we will always seek a connected group which is maximal among the isometries preserving $\mathcal{D}$. In order to find more connected components we use the following remark. As in the case of contact structures we treat only the existence of geometric Engel structures nut we do not classify them.

REMARK 7.10. Every isometry preserving an Engel structure $\mathcal{D}$ has to preserve the induced flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TX .$$

Hence the identity component of the stabilizer of a point $x \in X$ in the group of isometries $G_{\mathcal{D}}$ preserving an Engel structure $\mathcal{D}$ acts trivially on T_xX. In particular $G_{\mathcal{D}}$ has dimension four.

An element g of the isometry group $G_{\mathcal{D}}$ which fixes a point $x \in X$ preserves the subspaces $\mathcal{W}, \mathcal{D}, \mathcal{E}$ of T_xX. Suppose that W, V, Y, Z is an orthonormal basis of T_xX such that

$$\begin{aligned} \mathcal{W} &= \mathbb{R}W & \mathcal{D} &= \mathcal{W} \oplus \mathbb{R}V \\ \mathcal{E} &= \mathcal{D} \oplus \mathbb{R}Y & T_xX &= \mathcal{E} \oplus \mathbb{R}Z . \end{aligned}$$

Then the action of G has to preserve the basis W, V, Y, Z of T_xX. Now recall that $\mathcal{E}$ is canonically oriented and that an orientation of $\mathcal{W}$ induces an orientation of T_xX, cf. Section 3.2.6. If we reverse the orientation of $\mathcal{W}$ we also have to reverse the orientation of X and every Engel diffeomorphism has to preserve the orientation of $\mathcal{E}$. Thus if $g \in \mathrm{Stab}(x)$ acts non–trivially on T_xX then $g_*$is one of the following maps

$$\begin{aligned} \varphi_1 : T_xX &\longrightarrow T_xX \\ W &\longmapsto -W \\ V &\longmapsto -V \\ Y &\longmapsto Y \\ Z &\longmapsto -Z \end{aligned} \qquad \begin{aligned} \varphi_2 : T_xX &\longrightarrow T_xX \\ W &\longmapsto -W \\ V &\longmapsto V \\ Y &\longmapsto -Y \\ Z &\longmapsto -Z \end{aligned} \qquad \begin{aligned} \varphi_{12} : T_xX &\longrightarrow T_xX \\ W &\longmapsto W \\ V &\longmapsto -V \\ Y &\longmapsto -Y \\ Z &\longmapsto Z \end{aligned}$$

Notice that $\varphi_1 \circ \varphi_2 = \varphi_{12}$. The stabilizer of x has either one, two or four elements. In the last case $\mathrm{Stab}(x) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Let us summarize the result. The proof of the following theorem covers the remaining part of this chapter. Note that Theorem 7.11 concerns only the existence of geometric Engel structures. It does not contain a complete classification.

THEOREM 7.11. (i) *There is no geometric Engel structure* $(X, \mathcal{D}, G)$ *such that* (X, G) *is equivalent to one of the geometries* S^4, $\mathbb{H}^4$, $\mathbb{CP}^2$, $\mathbb{H}^2(\mathbb{C})$, $S^2 \times S^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $S^2 \times \mathbb{H}^2$, $S^2 \times \mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{H}^2 \times \mathbb{R}^2$, $\mathbb{H}^3 \times \mathbb{R}$.

(ii) *For each of the following geometries, there exists a geometric Engel structure* $(X, \mathcal{D}, G)$ *such that* (X, G) *is equivalent to it:*

$$S^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2, \mathbb{R}) \times \mathbb{R}, \mathrm{Sol}^4(m, n), \mathrm{Sol}^4_0, \mathrm{Sol}^4_1, \mathrm{Nil}^4 .$$

The maximal group of isometries preserving the Engel structure constructed in the proof has four components for all these geometries except $\mathrm{Sol}^4(m, n)$ *and* Sol^4_0.

(iii) *The only maximal Thurston–geometry which is compatible with a geometric Engel structure is* Nil^4.

In a first step we will obtain geometric Engel structures for non–maximal geometries equivalent to $S^3 \times \mathbb{R}$, $\mathrm{Nil}^3 \times \mathbb{R}$ and $\widetilde{Sl}(2, \mathbb{R}) \times \mathbb{R}$ using a construction similar to prolongation. The remaining cases will be treated in Section 7.3.

For the remaining part of this section, X will be one of the three–dimensional geometric contact structures S^3, Nil^3, $\widetilde{\mathrm{Sl}}(2, \mathbb{R})$. The contact structures described in the last section all appear at the same stage of the proof of the classification of 3–dimensional Thurston geometries in [**Thu2**] on p. 184. In these cases, prolongation can be modified such that it gives rise to geometric Engel structures.

A modification is necessary since if we apply prolongation naively on the geometric contact structure on X, then we obtain an Engel structure on the universal cover $X \times \mathbb{R}$ of $S_1\mathcal{C}$ but the natural group action of $X \times \mathbb{R}$ on itself by left–multiplication does *not* preserve the Engel structure since the Engel structure is not invariant under translations in the $\mathbb{R}$–direction. Recall that the $\mathbb{R}$–factor corresponds to the characteristic foliation of a prolonged Engel structure.

Recall that in the cases $X = \mathrm{Nil}^3$ and $X = \widetilde{\mathrm{Sl}}(2, \mathbb{R})$, the maximal isometry group preserves a contact structure and that the stabilizer of a point in Nil^3 and $\widetilde{\mathrm{Sl}}(2, \mathbb{R})$ acts by isometries on the contact plane through this point. For the geometric contact structure on S^3, the maximal group of isometries preserving the contact structure is $U(2) \cup (U(2) \circ \mathrm{conj})$.

Now

$$U(2)\cap\mathrm{Stab}(1,0)=\left\{\left(\begin{array}{cc}1&0\\0&e^{it}\end{array}\right)\middle|\,t\in\mathbb{R}\right\} \tag{74}$$

acts by rotations on the contact plane through $(1,0)\in S^3\subset\mathbb{C}^2$. Complex conjugation induces a reflection of this contact plane.

Now consider the universal cover $\widetilde{G}_{\mathcal{C}}$ of the identity component $G_{\mathcal{C},0}$ of $G_{\mathcal{C}}$ and its Lie algebra $\mathfrak{g}$. The action of $G_{\mathcal{C},0}$ is free and transitive on the unit sphere bundle $S^1\mathcal{C}$. So the universal cover $\widetilde{G}_{\mathcal{C}}$ acts freely and transitively on $\widetilde{S^1\mathcal{C}}\simeq X\times\mathbb{R}$. The choice of a basepoint $(p,0)\in X\times\mathbb{R}$ yields an identification $\widetilde{G}_{\mathcal{C}}\simeq X\times\mathbb{R}$ as spaces. Notice that with the obvious group structure on $X\times\mathbb{R}$, this identification is not a homomorphism of groups. Let pr $:X\times\mathbb{R}\to X$ be the projection onto the first factor. Then the following diagram commutes

$$\begin{array}{ccc}\widetilde{G}_{\mathcal{C}}\times(X\times\mathbb{R}) & \longrightarrow & X\times\mathbb{R}\\ \Big\downarrow{\scriptstyle(\Lambda,\mathrm{pr})} & & \Big\downarrow{\scriptstyle\mathrm{pr}}\\ G_{\mathcal{C},0}\times X & \longrightarrow & X\end{array}\;.$$

Let $0\neq W\in\mathfrak{g}$ be tangent to $\mathrm{Stab}(p)\subset\widetilde{G}_{\mathcal{C}}$ and let $0\neq V\in\mathfrak{g}$ be such that the vector field $\widetilde{V}$ on $X\times\mathbb{R}$ associated to V is projected to $\mathcal{C}$ by pr. At this point where we do something very similar to prolongation (cf. Proposition 3.15). Let $\widetilde{W}$ be the vector field on $X\times\mathbb{R}$ which corresponds to W. Since the stabilizer of $p\in X$ under the action of $\widetilde{G}_{\mathcal{C}}$ acts by rotation on $\mathcal{C}(p)$

$$\widetilde{W},\widetilde{V},[\widetilde{W},\widetilde{V}]$$

are linearly independent vector fields. Because the action of $\widetilde{G}_{\mathcal{C}}$ preserves the contact structure on $\mathcal{C}$, the projection of $\mathrm{pr}_*([\widetilde{W},\widetilde{V}])$ is contained in the contact structure on X. Again since $\mathrm{Stab}(p)$ acts by rotations on the contact plane through p the commutator $[\widetilde{W},[\widetilde{W},\widetilde{V}]]$ also projects to the contact structure on X. On the other hand, because $\mathcal{C}$ is a contact structure, $[\widetilde{V},[\widetilde{W},\widetilde{V}]]$ is linearly independent from $\widetilde{W},\widetilde{V},[\widetilde{W},\widetilde{V}]$.

PROPOSITION 7.12. *The left–invariant plane field spanned by $\widetilde{W},\widetilde{V}$ on $X\times\mathbb{R}$ is a geometric Engel structure which is invariant under the action of $\widetilde{G}_{\mathcal{C}}$. The characteristic foliation is tangent to the second factor of $X\times\mathbb{R}$.*

PROOF. Since the action of $\widetilde{G}_{\mathcal{C}}$ on $X\times\mathbb{R}$ is free and transitive it remains only to show that $\widetilde{G}_{\mathcal{C}}$ contains a cocompact lattice. For $X=S^3$ we obtain a lattice from the deck transformations of the universal covering $\widetilde{G}_{\mathcal{C}}\to G_{\mathcal{C}}$. For $X=\mathrm{Nil}^3,\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ we can obtain a lattice as follows. In $G_{\mathcal{C}}$ consider a lattice Γ which exists by assumption. Then the preimage $\widetilde{\Gamma}$ of Γ under the universal covering map is a lattice in $\widetilde{G}_{\mathcal{C}}$. □

Before we continue with the remaining Thurston geometries let us explain how to identify $\widetilde{G}_{\mathcal{C}}$ with subgroup of the maximal isometry group of $X\times\mathbb{R}$. We will identify $X\times\mathbb{R}$ and $\widetilde{G}_{\mathcal{C}}$ several times. Moreover we obtain all connected components of the group of Engel structure preserving isometries.

7.2.1. $X=S^3$**.** The identity component of $G_{\mathcal{C}}$ of the geometric contact structure on S^3 is $U(2)$. In order to show that the universal covering $\widetilde{U(2)}$ occurs as a subgroup of the isometry group of $S^3\times S^1$, consider the subgroup

$$\widetilde{U(2)}=\left\{(A,t)\in U(2)\times\mathbb{R}\,\middle|\,\det(A)=e^{it}\right\}\subset U(2)\times\mathbb{R}\,.$$

The map $\widetilde{U(2)} \to U(2)$ sending (A,t) to A is a universal covering of $U(2)$. It acts in the obvious way on $S^3 \times \mathbb{R}$ and it acts on S^3 if one drops the $\mathbb{R}$–factor.

The stabilizer of $(1,0) \in S^3 \subset \mathbb{C}^2$ was already described in (74). The group isomorphism

$$\begin{aligned} \mathrm{SU}(2) \times \mathbb{R} &\longrightarrow \widetilde{U(2)} \\ (A,t) &\longmapsto \left(e^{it/2}A, t\right) \end{aligned}$$

shows that $(S^3 \times \mathbb{R}, \widetilde{U(2)})$ and $(S^3 \times \mathbb{R}, S^3 \times \mathbb{R})$ are equivalent to each other.

Now we determine all components of the group of Engel structure preserving isometries of $S^3 \times \mathbb{R}$. Consider the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$. We can describe this Lie algebra using generators and relations

$$\begin{array}{lll} A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & C = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ {[A,B] = 2C} & {[B,C] = 2A} & {[C,A] = 2B}\ . \end{array}$$

In order to obtain the Lie algebra of $U(2)$ we add a tangent vector W of the stabilizer of $(1,0)$. Hence

$$W = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} .$$

The new commutator relations are

$$[W,A] = -B \qquad [W,B] = A \qquad [W,C] = 0$$

The span of W, A is an Engel structure on $\widetilde{U(2)}$ and A is tangent to the standard contact structure on $SU(2) = S^3$. The two isomorphisms of $U(2)$

$$A \longmapsto (A^T)^{-1} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$$

lift to isomorphisms of $\widetilde{U(2)}$ such that the first (second) map realizes φ_2 (φ_{12}) from Remark 7.10. Thus the group of Engel preserving isometries consists of four components. The identity component is $\widetilde{U(2)}$.

7.2.2. $X = \mathrm{Nil}^3$**.** We identify Nil^3 with the upper triangular matrices

$$\mathrm{Nil}^3 = \left\{ [x,y,z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ with } x,y,z \in \mathbb{R} \right\} .$$

Let G be the group of isometries of Nil^3. Remember that every isometry of Nil^3 preserves the contact structure on Nil^3. There is a fibration

$$\begin{aligned} \pi : \mathrm{Nil}^3 &\longrightarrow \mathbb{R}^2 \\ (x,y,z) &\longmapsto (x,y) \end{aligned}$$

which is a Riemannian submersion for the flat metric on $\mathbb{R}^2$ and a G–invariant metric on Nil^3.

We give an explicit description of the isometry group of Nil^3–geometry. Every isometry of Nil^3 projects to an isometry of $\mathbb{R}^2$. Conversely, we can lift every isometry φ of $\mathbb{R}^2$ to an isometry $\widetilde{\varphi}$ of Nil^3 as follows. Fix a point $q \in \pi^{-1}(\varphi(0))$. For $p \in \mathrm{Nil}^3$ choose a Legendrian curve γ starting at $(0,0,0) \in \mathrm{Nil}^3$ and consider the image $\varphi(\pi(\gamma))$ of its projection. Since the contact structure is transversal to the fibers of π, there is a unique lift of $\varphi(\pi(\gamma))$ to a Legendrian curve starting at q. We define $\widetilde{\varphi}(p)$ to be the endpoint of the lifted curve.

This definition is independent of the choice of γ: Let γ, γ' be two Legendrian curves from $(0,0,0)$ to p. Then the signed area enclosed by the closed curve formed of $\pi(\gamma)$ and $\pi(\gamma')$ is zero. Since φ is an isometry, the same is true for the image of these two curves. This ensures that the Legendrian lifts of these curves starting at q have the same endpoint in Nil^3. Note that the lift of isometries of $\mathbb{R}^2$ to isometries of Nil^3 is unique up to shifts in z–direction.

If we lift the standard representation of $\mathrm{O}(2)$ on $\mathbb{R}^2$ to an action of $\mathrm{O}(2)$ on Nil^3 we get an explicit description of the isometry group of Nil^3–geometry as $G = \mathrm{Nil}^3 \rtimes \mathrm{O}(2)$ with

$$\begin{aligned} e^{it} \cdot [x+iy, z] &= \left[(\cos(t) + i\sin(t))(x+iy), z - \sin^2(t)xy - \sin(2t)\frac{(x^2-y^2)}{4}\right] \\ \alpha \cdot [x+iy, z] &= [-x+iy, -z] \end{aligned}$$

where α denotes the reflection of $\mathbb{R}^2$ along the y–axis.

In order to show that the above geometric Engel structure induces a Thurston geometry equivalent to Nil$^3 \times \mathbb{R}$–geometry, consider the embedding

$$\begin{aligned} \widetilde{G} = \mathrm{Nil}^3 \rtimes \mathbb{R} &\longrightarrow (\mathrm{Nil}^3 \rtimes \mathrm{O}(2)) \times \mathbb{R} \subset \mathrm{Isom}(\mathrm{Nil}^3 \times \mathbb{R}) \\ (g,t) &\longmapsto \left(\left(g, e^{it}\right), t\right) \ . \end{aligned}$$

The Lie algebra of $\widetilde{G}$ is generated by X, Y, Z, W where $X, Y, Z \in \mathfrak{nil}^3$ and W is tangent to the stabilizer of the unit element in Nil^3 under the action of $\widetilde{G}$. Then the Lie algebra of $\widetilde{G}$ satisfies the commutator relations

$$\begin{array}{lll} [X,Y] = Z & [Y,Z] = 0 & [X,Z] = 0 \\ [W,X] = Y & [W,Y] = -X & [W,Z] = 0 \ . \end{array}$$

In particular, this Lie algebra is solvable but not nilpotent. Hence this geometry is *not* equivalent to $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$. In **[Wa1]** this Lie algebra is mentioned as a non–maximal Thurston geometry (denoted by H'_X) but in **[Wa2]**, Wall claims that this is actually not a geometry because it does not admit a lattice.

We now show that the group of isometries of Nil$^3 \times \mathbb{R}$ which preserve the Engel structure has four components. The identity component is Nil$^3 \rtimes \mathbb{R}$. The maps

$$\begin{aligned} \mathrm{Nil}^3 \rtimes \mathbb{R} &\longrightarrow \mathrm{Nil}^3 \rtimes \mathbb{R} \\ ((x,y,z),t) &\longmapsto ((-x,y,-z),-t) \\ ((x,y,z),t) &\longmapsto ((x,-y,-z),-t) \\ ((x,y,z),t) &\longmapsto ((x,-y,-z),t) \end{aligned}$$

are group isomorphisms realizing all non–trivial possibilities in Remark 7.10.

Finally we consider the other non–maximal Thurston geometry equivalent to Nil$^3 \times \mathbb{R}$–geometry, namely $(\mathrm{Nil}^3 \times \mathbb{R}, \mathrm{Nil}^3 \times \mathbb{R})$. Let A_1, A_2 be left invariant vector fields spanning the distribution $\mathcal{D}$. Thus $\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is spanned by A_1, A_2 and

$$[T + A_1, A_2] = \lambda Z$$

for some $\lambda \in \mathbb{R}$. But $\mathbb{Z}$ lies in the center of $\mathfrak{nil}^3$. Therefore $\mathcal{D}^3 = \mathcal{D}^2$ and $\mathcal{D}$ is not an Engel structure. Thus there is no geometric Engel structure $(\mathrm{Nil}^3 \times \mathbb{R}, \mathcal{D}, \mathrm{Nil}^3 \times \mathbb{R})$.

7.2.3. $X = \widetilde{\mathrm{Sl}}(2,\mathbb{R})$**.** Let us first describe the isometry group of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Recall that the entire isometry group of this Thurston geometry preserves the contact structure on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. If we fix a point $p \in \mathbb{H}^2$ then we obtain a fibration $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \to \mathbb{H}^2$ such that the fiber over $q \in \mathbb{H}^2$ is

$$\{g \in \widetilde{\mathrm{Sl}}(2,\mathbb{R}) | g \cdot p = q\}$$

The typical fiber is $\mathbb{R}$ and the projection map is a Riemannian submersion.

The isometry group G of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is a semidirect product generated by three types of isometries.

- The elements of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ acting by left–translations on $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ are of course isometries.
- If we lift a rotation of $\mathbb{H}^2$ around p to a contact preserving isometry of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ which preserves the unit element, we obtain isometries of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. This group is SO$(2)$.
- The lift of a reflection of $\mathbb{H}^2$ along a geodesic through p also yields an isometry. These lifts also have to reverse the orientation of the fibers.

However we do not work out the lifts explicitly. We will only treat the connected component of the identity of the isometry group. The isometries of the second type form the stabilizer of the unit element of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ under the action of $G = \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \rtimes \mathrm{SO}(2)$.

Again we want to find a concrete embedding of $\widetilde{G}$ into the isometry group of $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$ such that the action of the stabilizer of a point $q \in \widetilde{\mathrm{Sl}}(2,\mathbb{R})$ under the action of $\widetilde{G}$ is a translation of the real line lying over q.

As generators for $\mathfrak{sl}(2,\mathbb{R})$ we use

$$A = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad C = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We write W for the standard generator of the Lie algebra of the stabilizer of $p \in \mathbb{H}^2$ under the action of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$. Now B corresponds to rotations of $\mathbb{H}^2$ around p. Thus the Lie algebra of the isometry group of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ satisfies the commutator relations

$$\begin{aligned} &[C,A] = B && [C,B] = A && [A,B] = -C \\ &[W,A] = C && [W,B] = 0 && [W,C] = -A\,. \end{aligned}$$

The embedding

$$\begin{aligned} \widetilde{G} = \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \rtimes \mathbb{R} &\longrightarrow \left(\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \rtimes \mathrm{SO}(2)\right) \times \mathbb{R} = \mathrm{Isom}\left(\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}\right) \\ (g,t) &\longmapsto \left(\left(g, e^{it}\right), t\right) \end{aligned}$$

shows that we end up with a geometry which is equivalent to $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$–geometry. The map defined by

$$\begin{aligned} &A \longmapsto A && B \longmapsto B && C \longmapsto C \\ &W \longmapsto T + B \end{aligned}$$

is an isomorphism of the Lie algebras of $\widetilde{G}$ and $\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}$ where we write T for a generator of the Lie algebra of the factor $\mathbb{R}$. This shows that we actually obtained a geometric Engel structure such that the induced Thurston geometry is isomorphic to the non–maximal Thurston geometry $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R})$.

Thus the left invariant vector fields $T+B, A$ on $(\widetilde{\mathrm{Sl}}(2,\mathbb{R}) \times \mathbb{R}, \widetilde{\mathrm{Sl}}(2,\mathbb{R})) \times \mathbb{R}$ is an Engel structure whose characteristic foliation is tangent to $T+B$. In the notation of Remark 7.10,

X corresponds to A and $[T+B,A]=C$ corresponds to Y. The maps

$$\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}\longrightarrow\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}$$
$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix},t\right)\longmapsto\left(\begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22}\end{pmatrix},-t\right)$$
$$(A,t)\longmapsto\big((A^T)^{-1},t\big)$$

are isomorphisms of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}$ which realize φ_1 and φ_{12} from Remark 7.10. Thus the maximal group of Engel preserving isometries has four components.

7.3. Geometric Engel manifolds – Remaining geometries

For several Thurston–geometries (X,G) in dimension 4 there is no invariant orientable Engel structure since X has non–trivial tangent bundle. For the cases

$$S^4,\mathbb{CP}^2,S^2\times S^2$$

this is obvious.

If $X=S^2\times Y$ for a two–dimensional geometry Y, then it is easy to show that there is no geometric Engel structure $(X,\mathcal{D},G)$: Assume that $\mathcal{D}$ were such an Engel structure. Then $[\mathcal{D},\mathcal{D}]=\mathcal{E}$ is a G–invariant even contact structure. Since G is supposed to act transitively on X, $\mathcal{E}$ is either everywhere tangent to the first factor in $S^2\times Y$ or it is transversal to it. Since every surface tangent to an even contact structure must be tangent to the characteristic foliation, we would obtain a line field on the sphere, which is of course impossible. Thus $\mathcal{E}$ is everywhere transversal to the spheres and hence $\mathcal{E}$ induces a foliation on each sphere $S^2\times\{y\}$ for $y\in Y$. Again this is a contradiction.

There are other geometries for which topological arguments show the non–existence of Engel structures.

- Hyperbolic four–manifolds have positive Euler characteristic.
- According to [**Wa2**], manifolds with an $\mathbb{H}^2\times\mathbb{H}^2$–structure have positive Euler characteristic.
- Manifolds with an $\mathbb{H}^2(\mathbb{C})$–structure have positive signature and Euler characteristic, cf. [**Wa2**].

Hence the geometries $\mathbb{H}^4,\mathbb{H}^2\times\mathbb{H}^2,\mathbb{H}^2(\mathbb{C})$ do not admit any Engel structure.

We have already covered the geometries $S^3\times\mathbb{R},\mathrm{Nil}^3\times\mathbb{R},\widetilde{\mathrm{Sl}}(2,\mathbb{R})\times\mathbb{R}$ in the preceding section. The remaining geometries are

$$\mathbb{R}^4,\mathbb{H}^3\times\mathbb{R},\mathbb{H}^2\times\mathbb{R}^2,\mathrm{Sol}^4(n,m),\mathrm{Sol}^4_0,\mathrm{Sol}^4_1,\mathrm{Nil}^4\ .$$

We will treat these geometries individually.

7.3.1. $X=\mathbb{R}^4$**.** All subgroups of the isometry group of $\mathbb{R}^4$ which act transitively on $\mathbb{R}^4$ must contain the translations of $\mathbb{R}^4$. The only translation invariant plane fields on $\mathbb{R}^4$ are foliations. Thus there is no geometric Engel structure for this geometry.

7.3.2. $X=\mathbb{H}^3\times\mathbb{R}$**.** The maximal isometry group the product of the isometry groups of each factor. It has four connected components and it has dimension 7. The subgroups H of G which consist of connected components of G obviously yield non–maximal Thurston geometries $(\mathbb{H}^3\times\mathbb{R},H)$.

In order to show that there is no subgroup $H\subset G$ of codimension at least one such that $(\mathbb{H}^3\times\mathbb{R},H)$ is a Thurston geometry we can argue like in the case of contact structures on the three–dimensional geometry $\mathbb{H}^2\times\mathbb{R}$. Again we can assume that H is connected and apply Theorem 7.5 with $R=\mathrm{Isom}_0(\mathbb{R})\subset G_0$. Thus $(\mathbb{H}^3,H))$ is a Thurston geometry since, like in the case $\mathbb{H}^2\times\mathbb{R}$ the image $\pi(\Gamma)$ of a cocompact lattice $\Gamma\subset G_0$ is again a discrete group which acts freely with compact quotient. Since $\mathbb{H}^3$ has no nontrivial subgeometries this implies that $\pi(H)$ is Isom$_0(\mathbb{H}^3)$. Hence H has at least dimension 6. If the dimension of H is seven then $H = G_0$.

We are left with the case that H has dimension 6. Since H acts transitively on $\mathbb{H}^3 \times \mathbb{R}$, the intersection $H \cap$ Isom$_0(\mathbb{H}^3)$ has dimension 5. In particular the intersection of H with the maximal compact subgroup $K \simeq$ SO(3) of Isom$(\mathbb{H}^3)$ has dimension two. But this is impossible. The assumption that H has dimension 6 leads to a contradiction.

There is no plane field which is invariant under the action of G_0. Hence there is no geometric Engel structure for the geometry $\mathbb{H}^3 \times \mathbb{R}$.

7.3.3. $X = \mathbb{H}^2 \times \mathbb{R}^2$**.** The maximal isometry group G is the product of the isometry groups of the factors, it has dimension 6 and four connected components. If a subgroup $H \subset G$ is the union of connected components of G we have a Thurston geometry $(\mathbb{H}^2 \times \mathbb{R}^2, H)$. The plane fields which are invariant under G_0 are tangent to one of the factors of $\mathbb{H}^2 \times \mathbb{R}^2$. Thus there are no geometric Engel structures for these Thurston geometries.

We now look for Thurston geometries $(\mathbb{H}^2 \times \mathbb{R}^2, H)$ such that H has dimension less than 6. Let H be such a subgroup of G. Since the stabilizer of a point in $\mathbb{H}^2 \times \mathbb{R}^2$ is compact and H has to be transversal to the stabilizers of points in $\mathbb{H}^2 \times \mathbb{R}^2$, we can assume that H is connected (cf. the case $\mathbb{H}^2 \times \mathbb{R}$).

Let $\mathfrak{h}$ be the Lie algebra of H. Since $\mathfrak{h}$ has dimension 4 or 5 there is a non–zero vector $w \in \mathfrak{h}$ which is tangent to $\mathfrak{sl}(2, \mathbb{R})$. On the other hand, by Theorem 7.5 and Preissmann's theorem, the projection of H to Isom$(\mathbb{H}^2)$ has dimension 3. Thus there are elements $T_1 + v_1, T_2 + v_2$ such that T_1, T_2 are tangent to Isom$(\mathbb{R}^2)$ and v_1, v_2, w span $\mathfrak{sl}(2, \mathbb{R})$. Hence $w, [w, T_1 + v_1] = [w, V_1], [w, T_2 + v_2] = [w, v_2]$ span $\mathfrak{sl}(2, \mathbb{R})$. Since $\mathfrak{h}$ is a subalgebra, $\mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{h}$.

On the other hand if we apply Theorem 7.5 again, we see that $\Gamma \cap \mathbb{R}^2$ is a lattice. Thus $H \cap$ Isom$(\mathbb{R}^2)$ is a Thurston geometry which is equivalent to $\mathbb{R}^2$. Thus $H \cap$ Isom$(\mathbb{R}^2)$ contains $\mathbb{R}^2$. Hence H is either G_0 or the product Isom$(\mathbb{H}^2) \times \mathbb{R}^2$.

By Remark 7.10 there is no geometric Engel structure in this case.

7.3.4. $X =$ Sol$^4(m, n)$**.** Let m, n be positive integers such that the zeroes of

$$P(m, n) = -\lambda^3 + m\lambda^2 - n\lambda + 1 \tag{75}$$

are real numbers which are pairwise different. Other possible configurations of the zeroes of $P(m, n)$ will be discussed below. Let $e^\alpha, e^\beta, e^\gamma$ be the zeroes of $P(m, n)$ with $\alpha + \beta + \gamma = 0$ and $\alpha > \beta > \gamma$. The solvable Lie group Sol$^4(m, n)$ is defined as $\mathbb{R}^3 \rtimes \mathbb{R}$ with the action

$$t \longmapsto \psi(t) = \exp \begin{pmatrix} \alpha t & 0 & 0 \\ 0 & \beta t & 0 \\ 0 & 0 & \gamma t \end{pmatrix}$$

of $\mathbb{R}$ on $\mathbb{R}^3$. The characteristic polynomial of

$$A(m, n) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -n & m \end{pmatrix} \tag{76}$$

is $P(m, n)$. Hence $A(m, n)$ and $\psi(1)$ are conjugate. In particular, there is a matrix $\mathfrak{A}(m, n) \in \mathfrak{sl}(3, \mathbb{R})$ such that $\exp(\mathfrak{A}(m, n)) = A(m, n)$. The groups Sol$^4(m, n)$ and

$$\mathbb{R}^3 \rtimes \mathbb{R} \text{ with } t \in \mathbb{R} \text{ acting by } \exp(t\mathfrak{A}(m, n))$$

are isomorphic Lie groups. The second group contains the lattice $\mathbb{Z}^3 \rtimes \mathbb{Z}$.

For $(m, n), (m', n')$ as above we obtain isomorphic Lie groups if and only if the corresponding triples (α, β, γ) and $(\alpha', \beta', \gamma')$ are proportional.

In the case $m = n \geq 4$ we have $\beta = 0$ and $\alpha = -\gamma \in \mathbb{R}$. We obtain Sol$^3 \times \mathbb{R}$, the eigenvector of $\psi(1)$ for the eigenvalue e^β corresponds to the second factor of Sol$^3 \times \mathbb{R}$. In general, two Lie groups Sol$^4(m,n)$, Sol$^4(m',n')$ are isomorphic if and only if (α,β,γ) and (α',β',γ') are proportional.

The Lie algebra $\mathfrak{sol}^4(m,n)$ is generated by X_1, X_2, X_3, T and the commutator relations

$$[T,X_1] = \alpha X_1 \qquad [T,X_2] = \beta X_2 \qquad [T,X_3] = \gamma X_3$$

and the remaining commutators vanish. The left–invariant plane field $\mathcal{D} = \mathrm{span}(T, X_1 + X_2 + X_3)$ satisfies

$$\begin{aligned} \mathcal{D}^2 &= \mathcal{D} + [\mathcal{D},\mathcal{D}] = \mathcal{D} \oplus \mathbb{R}(\alpha X_1 + \beta X_2 + \gamma X_3) \\ \mathcal{D}^3 &= \mathcal{D}^2 + [\mathcal{D},\mathcal{D}^2] = \mathcal{D}^2 \oplus \mathbb{R}(\alpha^2 X_1 + \beta^2 X_2 + \gamma^2 X_3) \ . \end{aligned}$$

Since α, β, γ are pairwise different, this implies that $\mathcal{D}$ is an Engel structure. The characteristic line field is spanned by $X_1 + X_2 + X_3$.

The action of $\varphi \in \mathrm{Stab}(e)$ on $\mathfrak{sol}^4(m,n)$ is given by

$$\begin{aligned} &X_1 \longmapsto \pm X_1 \qquad X_2 \longmapsto \pm X_2 \qquad X_3 \longmapsto \pm X_3 \\ &T \longmapsto T \ . \end{aligned} \tag{77}$$

Thus the isometry group of Sol$^4(m,n)$ has eight components. Two of these preserve the Engel structure described above. In the notation of Remark 7.10, only φ_2 can be realized by an isometry of Sol$^4(m,n)$.

7.3.5. $X =$ Sol$_0^4$**.** We now treat the case when m, n are such that (75) has two different complex solutions $\lambda, \overline{\lambda}$ and a real solution $|\lambda|^{-2}$ different from $0, 1$. The Lie group Sol$^4(\lambda)$ associated to these parameters is $\mathbb{R}^3 \rtimes \mathbb{R} = (\mathbb{C} \oplus \mathbb{R}) \rtimes \mathbb{R}$ with the action of $\mathbb{R}$ defined by

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathrm{Gl}(\mathbb{C} \oplus \mathbb{R}) \\ t &\longmapsto \Big((u,x) \longmapsto \Big(e^{t\lambda}u, e^{-2\Re(\lambda)}x\Big)\Big) \end{aligned} \tag{78}$$

acts by isometries on a Riemannian manifold which is independent of the concrete values of m, n. We thus get only one new maximal Thurston geometry which we denote by Sol$_0^4$.

The Lie group Sol$_0^4$ is the semidirect product $(\mathbb{C} \oplus \mathbb{R}) \rtimes \mathbb{R}$ with the action of $\mathbb{R}$ on $\mathbb{C} \oplus \mathbb{R} = \mathbb{R}^2 \oplus \mathbb{R}$ given by

$$t \longmapsto \exp \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & -2t \end{pmatrix} .$$

This Lie group does not admit a lattice, cf. **[Hil]** p. 137, but still we can obtain a Thurston geometry from this Lie group. The metric $e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2$ is left–invariant and, compared with the geometries Sol$^4(m,n)$ from the previous section, it admits additional isometries of the complex plane. The identity component of the full isometry group of Sol$_0^4$ is the semidirect product

$$\mathrm{Isom}_0(\mathrm{Sol}_0^4) = \mathrm{Sol}_0^4 \rtimes \mathrm{SO}(2) \simeq (\mathbb{C} \oplus \mathbb{R}) \rtimes (\mathbb{R} \times S^1) \ .$$

We can embed the group Sol$^4(\lambda)$ defined in (78) into Isom$_0$(Sol$_0^4$) by

$$\begin{aligned} (\mathbb{C} \oplus \mathbb{R}) \rtimes \mathbb{R} &\longrightarrow \mathrm{Isom}(\mathrm{Sol}_0^4) \\ ((u,x),t) &\longmapsto ((u,x),(\Re(\lambda)t, \exp(i\Im(\lambda)t))) \ . \end{aligned}$$

In this way we obtain discrete subgroups of Isom(Sol$_0^4$) which act on Sol$_0^4$ such that the quotient is a compact manifold. Hence (Sol$_0^4$, Isom(Sol$_0^4$)) is really a Thurston geometry.

The different parameter values m, n such that (75) has two complex solutions $\lambda, \overline{\lambda}$ with $|\lambda| \neq 1$ give rise to different non–maximal Thurston geometries $(\mathrm{Sol}_0^4, \mathrm{Sol}^4(\lambda))$ which depend on λ.

The Lie algebra of $\mathrm{Sol}^4(\lambda)$ is generated by U_1, U_2, V, T with the commutator relations

$$[T, U_1] = \Re(\lambda)U_1 + \Im(\lambda)U_2 \qquad [T, U_2] = -\Im(\lambda)U_1 + \Re(\lambda)U_2$$
$$[T, V] = -2\Re(\lambda)V$$

and all other commutators vanish. Now consider the plane field $\mathcal{D}(\lambda)$ on Sol_0^4 which is left–invariant under the action of $\mathrm{Sol}^4(\lambda)$ and which corresponds to $T, U_1 + V$. By the commutator relations given above

$$\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}] = \mathcal{D} \oplus \mathbb{R}\left(\Re(\lambda)U_1 + \Im(\lambda)U_2 - 2\Re(\lambda)V\right)$$
$$\mathcal{D}^3 = \mathcal{D}^2 + [\mathcal{D}, \mathcal{D}^2] = \mathcal{D}^2 \oplus \mathbb{R}\left(\left(\Re^2(\lambda) - \Im^2(\lambda)\right)U_1 + 2\Re(\lambda)\cdot\Im(\lambda)U_2 + 4\Re^2(\lambda)V\right) .$$

Since $\Re(\lambda) \neq 0$ and $\Im(\lambda) \neq 0$ we have a $\mathrm{Sol}^4(\lambda)$–invariant Engel structure $\mathcal{D}(\lambda)$ on Sol_0^4 for all possible values of λ. The characteristic foliation is spanned by $U_1 + V$.

In order to determine the maximal subgroup of $\mathrm{Isom}(\mathrm{Sol}_0^4)$ which preserves the Engel structure $\mathcal{D}(\lambda)$ it suffices to apply Remark 7.10. As in the case $\mathrm{Sol}^4(m, n)$ we find only the isometry

$$\mathrm{Sol}_0^4 \longrightarrow \mathrm{Sol}_0^4$$
$$((u, x), t) \longmapsto ((-u, -x), t)$$

In the notation of Remark 7.10 this corresponds to φ_2.

7.3.6. $X = \mathrm{Sol}_1^4$**.** While in the last two sections we considered semidirect products of $\mathbb{R}$ with $\mathbb{R}^3$ we now consider the semidirect product $\mathrm{Sol}_1^4 = \mathrm{Nil}^3 \rtimes \mathbb{R}$ where the action of $t \in \mathbb{R}$ on Nil^3 is defined by

$$t \cdot [x, y, z] \longmapsto \left[e^{-t}x, e^t y, z\right] .$$

In this geometry points have discrete stabilizers. We write $T \in \mathfrak{sol}_1^4$ for the generator of the Lie algebra of $\mathbb{R}$. For the generators of $\mathfrak{nil}^3$ we write X, Y, Z. These generators satisfy the commutator relations

$$[T, X] = -X \qquad [T, Y] = Y \qquad [X, Y] = Z .$$

and the remaining commutators vanish. The orthogonal complement of center $\mathbb{R}Z$ is an even contact structure It is spanned by T, X, Y and the characteristic foliation is spanned by T. If T, X, Y span the even contact structure which is induced by a left–invariant Engel structure $\mathcal{D}$, this Engel structure has to contain T. We choose $\mathcal{D} = \mathrm{span}(T, X + Y)$. Then

$$\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}] = \mathrm{span}(T, X + Y, -X + Y) .$$

Hence $\mathcal{D}$ is a left–invariant Engel structure whose characteristic foliation is spanned by T. Again we try to determine which connected components of the isometry group of Sol_1^4–geometry preserve $\mathcal{D}$. According to **[Wa2]**, the action of the stabilizer of e on $\mathfrak{sol}_1^4$ is given by

$$X \longmapsto aX \qquad Y \longmapsto bY \qquad Z \longmapsto abZ$$
$$T \longmapsto T$$

with $a, b = \pm 1$ or

$$X \longmapsto Y \qquad Y \longmapsto X \qquad Z \longmapsto -Z$$
$$T \longmapsto -T .$$

Thus the isometry group of Sol_1^4 has eight connected components. The Engel structure is preserved by the second kind of isometries preserving e and by the first kind for $a = b = \pm 1$. The isometries preserving $\mathcal{D}$ are contained in four of the eight connected components.

REMARK 7.13. Before going on to the missing geometry Nil^4, we want to explain the Engel structures obtained from the solvable geometries $X = \mathrm{Sol}^4(m,n), \mathrm{Sol}_0^4$ and Sol_1^4. We focus on manifolds X/Γ where Γ is constructed as explained in the section about the geometry $\mathrm{Sol}^4(m,n)$. Now X viewed as a manifold is a product $\mathbb{R}^3 \times \mathbb{R} = \mathrm{Nil}^3 \times \mathbb{R}$. Let $\Gamma' = \Gamma \cap \mathbb{R}^3$ respectively $\Gamma' = \Gamma \cap \mathrm{Nil}^3$. This group acts on the manifold $\mathbb{R}^3 = \mathrm{Nil}^3$ such that the quotient is a compact manifold. The projection $X \longrightarrow \mathbb{R}$ induces a fibration

$$\pi : X/\Gamma \longrightarrow S^1$$

with fiber N. Thus X/Γ is the mapping torus of a diffeomorphism of $N = \mathbb{R}^3/\Gamma'$ which preserves a given decomposition of TN into a sum of line fields. We write T for the suspension vector field. We call a section normal if it has unit length with respect to an invariant metric.

Recall that if $X_0 = \partial_t, X_1, X_2, X_3$ is a framing of a parallelizable mapping torus, the span of

$$X_0 \quad \text{and} \quad Y_k = \frac{1}{k}\left(\cos(k^2 t)X_1 + \sin(k^2 t)X_2\right) + X_3 \tag{79}$$

is an Engel structure if $k \in \mathbb{N}$ is big enough by Proposition 3.17. Instead of (79) we now use a simpler version of Geiges's construction, namely we consider the span of

$$X_0 \quad \text{and} \quad Y_k = \cos(kt)X_1 + \sin(kt) + X_3 \; . \tag{80}$$

for $k \in \mathbb{N}$ and a fixed framing $X_0 = \partial_t, X_1, X_2, X_3$.

In the case of $X = \mathrm{Sol}^4(m,n)$, the construction of Geiges as in (80) applied to the framing tangent to the $X_0 = T, X_1 + X_2, X_1 - X_2, X_3$ works already for $k = 0$ and it yields the Engel structure we obtained above.

The case Sol_1^4 is also simple. Here N is a Nil^3–manifold with its canonical contact structure and the suspension map ψ preserves this contact structure. Moreover, the contact structure can be decomposed in the sum of two line bundles $\mathcal{C} = \mathcal{C}^s \oplus \mathcal{C}^u$ which is preserved by ψ. The restriction of ψ_* to the contact planes behaves like the differential of an Anosov diffeomorphism. If one applies the Geiges construction to a normal framing tangent to $T, X+Y, X-Y, Z$ one obtains an Engel structure already for $k = 0$.

The case $X = \mathrm{Sol}_0^4$ is slightly more complicated. Let $\Gamma \subset \mathrm{Sol}^4(\lambda)$ be a lattice constructed as described in the section about $\mathrm{Sol}^4(m,n)$. If one considers the span of normal sections a_1, a_2, v of the line fields U_1, U_2, V, the span of T and

$$\cos(kt)a_1 + \sin(kt)a_2 + v$$

is a contact structure for $k = \Im(\lambda)$.

7.3.7. $X = \mathrm{Nil}^4$**.** The Lie algebra $\mathfrak{nil}^4$ is generated by W, V, Y, Z with the commutator relations

$$[V, W] = Y \qquad\qquad [V, Y] = Z \; ,$$

the remaining commutators vanish. One can choose a left–invariant metric on Nil^4 such that W, V, Y, Z is an orthonormal basis. The left–invariant distribution $\mathcal{D}$ spanned by W, V is an Engel structure, the characteristic line field is spanned by W. The even contact structure of $\mathcal{D}$ is spanned by W, V, Y, i.e. it is orthogonal to the center $\mathbb{R}Z$ of $\mathfrak{nil}^4$. Moreover $\mathcal{D}$ is orthogonal to

$$[\mathfrak{nil}^4, \mathfrak{nil}^4] = \mathrm{span}(Y, Z) \; .$$

The distribution spanned by Y, Z is integrable. The isometry group of Nil^4 has four connected components ([**Wa2**]). The isometries which preserve $e \in$ Nil^4 and which are not contained in Nil^4 act on $\mathfrak{nil}^4$ by

$$W \longmapsto aW \qquad V \longmapsto bV$$
$$Y \longmapsto abY \qquad Z \longmapsto aZ$$

with $a, b = \pm 1$. Thus the entire isometry group of Nil^4 preserves the Engel structure $\mathcal{D}$.

DEFINITION 7.14. Let Γ be a subgroup of the isometry group of Nil^4–geometry which acts freely on Nil^4 such that the quotient Nil$^4/\Gamma$ is a compact manifold. Then Nil$^4/\Gamma$ is called *infranil–manifold*. If $\Gamma \subset$ Nil^4 then Nil$^4/\Gamma$ is a Nil^4*–manifold*.

REMARK 7.15. We have shown that every infranil–manifold carries an Engel structure. In order to relate Engel structures obtained this way with other known constructions, we focus on Nil^4–manifolds, i.e. we consider manifolds Nil$^4/\Gamma$ with $\Gamma \subset$ Nil^4. Such manifolds are parallelizable.

With the action of $\mathbb{R}$ on $\mathbb{R}^3$ given by

$$\varphi(t) = \exp \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$$

Nil^4 is isomorphic to $\mathbb{R}^3 \rtimes \mathbb{R}$. In this presentation, the generators of $\mathfrak{nil}^4$ are the left–invariant vector fields which, if we view them as elements of T_eNil$^4 \simeq T_e(\mathbb{R}^3 \times \mathbb{R})$, are

$$W(e) = \frac{\partial}{\partial a_3} \qquad V(e) = \frac{\partial}{\partial t}$$
$$Y(e) = \frac{\partial}{\partial a_2} \qquad Z(e) = \frac{\partial}{\partial a_1}$$

where a_1, a_2, a_3 are coordinates on $\mathbb{R}^3$ and t is the coordinate on the second factor of $\mathbb{R}^3 \rtimes \mathbb{R}$. In particular we have the fibration

$$\begin{aligned} \text{pr} : \text{Nil}^4 = \mathbb{R}^3 \rtimes \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ (a_1, a_2, a_3, t) &\longmapsto (a_3, t) \end{aligned}$$

which descends to a fibration pr_Γ : Nil$^4/\Gamma \longrightarrow T^2$ if the image of $\Gamma \subset$ Nil^4 under pr is a lattice in $\mathbb{R}^2$.

According to [**Dek**], every discrete subgroup of Nil^4 has a presentation

$$\Gamma = \Big\langle a, b, c, d \Big| [b, a] = c^\alpha d^\beta, [c, a] = d^\gamma, [c, b] = 1, [a, d] = [b, d] = [c, d] = 1 \Big\rangle \ .$$

with $\alpha > 0$ and $\gamma > 0$. A group Γ with this presentation is generated by

$$a = ((0, 0, 0), 1) \qquad b = ((0, \alpha\gamma/2 - \beta, \alpha\gamma), 0)$$
$$c = ((0, -\gamma, 0), 0) \qquad d = ((1, 0, 0), 0)$$

The image of Γ under pr is $\alpha\gamma\mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{R}^2$. Thus the map Nil$^4 \to \mathbb{R}^2$ induces a fibration

$$\text{Nil}^4/\Gamma \longrightarrow T^2 \ .$$

Since the diffeomorphism type of a Nil^4–manifold is classified by the fundamental group Γ, this shows that every Nil^4–manifold fibers over T^2. In particular, Nil$^4/\Gamma$ fibers over S^1 and it is parallelizable. This relates the Engel structure on Nil^4–manifolds to the construction of Geiges.

Using the lattice Γ given above we obtain Engel structures on T^2–bundles over T^2 which are transversal to the fibers. Two Nil^4–manifolds Nil$^4/\Gamma_1$, Nil$^4/\Gamma_2$ are diffeomorphic via conjugation with an element of the affine transformations $\mathrm{Aff}(\mathrm{Nil}^4)$ of Nil^4, cf. [**Dek**]. Since $\mathrm{Aff}(\mathrm{Nil}^4)$ is the semidirect product between Nil^4 and the group of automorphisms of Nil^4, this does not imply that Nil$^4/\Gamma_1$ and Nil$^4/\Gamma_2$ are diffeomorphic as Engel manifolds.

EXAMPLE 7.16. We want to show that among the infranil–manifolds there are *non–orientable* manifolds with an Engel structure. Thus we obtain new Engel manifolds this way which of course are finitely covered by manifolds which carry an Engel structure by the construction of Geiges. We rely on the description of Ue for Nil^4–manifolds, cf. [**Ue**].

All Nil^4–manifolds admit Seifert fibrations with fiber T^2 over T^2, the Klein bottle K , the annulus or the Möbius band. Now all T^2–bundles over $S = T^2, K$ can be obtained as follows.

(i) Using a representation

$$\rho : \pi_1(S) \longrightarrow \mathrm{Diff}(T^2)$$

in order to construct a flat T^2–bundle $p : M' \to S$. The isomorphism type of the fibration depends only on the conjugacy class of the representation.

(ii) Choose a disc $D \subset S$ and remove $p^{-1}(D)$ form M'. The T^2–bundle $p^{-1}(D)$ over D is trivial, hence $\partial p^{-1}(D) \simeq \partial D \times T^2$. We view T^2 as $\mathbb{R}^2/\mathbb{Z}^2$ and $S^1 = \mathbb{R}/\mathbb{Z}$. For integers a, b we glue $p^{-1}(D)$ to $M' \setminus \mathrm{int}\left(p^{-1}(D)\right)$ using the map

$$\begin{aligned} S^1 &\longrightarrow \mathrm{Diff}(T^2) \\ t &\longmapsto ([x,y] \longmapsto [x+at, y+bt]) \ . \end{aligned}$$

The Nil^4–manifold among the T^2–bundles over T^2 are obtained for the representations defined by

$$\begin{aligned} \rho : \pi_1(T^2) \simeq \mathbb{Z}^2 &\longrightarrow \mathrm{Diff}(T^2) \\ (1,0) &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ (0,1) &\longmapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \end{aligned}$$

with $\lambda \in \mathbb{Z}$ and $\lambda, b \neq 0$. The T^2–bundles over the Klein bottle K which admit a Nil^4–structure correspond to

$$\begin{aligned} \rho : \pi_1(K) \simeq \mathbb{Z}_2 \times \mathbb{Z} &\longrightarrow \mathrm{Diff}(T^2) \\ (1,0) &\longmapsto \begin{pmatrix} 1 & c \\ 0 & -1 \end{pmatrix} \\ (0,1) &\longmapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \end{aligned}$$

with $c \in \{0,1\}, \lambda \in \mathbb{Z}$ and $\lambda, b \neq 0$.

Now the monodromy of TK along a curve γ representing the torsion element in $\pi_1(K)$ is orientation preserving. On the other hand, the monodromy of the T^2–bundle over γ is orientation reversing since $(1,0)$ is mapped to an orientation reversing diffeomorphism of T^2. Hence the total space of the T^2 bundles over K which admit a Nil^4–structure is not orientable. So, although these spaces fiber over the circle, one cannot apply the construction of Geiges to these manifolds.

REMARK 7.17. The examples of Engel structures obtained in this chapter are volume preserving. In all these cases, the characteristic foliation is spanned by a left–invariant vector field and the volume form of an invariant Riemannian metric provides a volume form which is preserved by the vector field spanning $\mathcal{W}$. So $\mathcal{W}$ is really defined by a closed form.

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