Noncommutative gauge theory and κ-deformed spacetime

Lutz Möller

Dissertation an der Fakultät für Physik
der Ludwig-Maximilians-Universität München
März 2004

Noncommutative gauge theory and κ-deformed spacetime

Dissertation
an der Fakultät für Physik
der Ludwig-Maximilians-Universität München

vorgelegt von Lutz Möller
aus München

München, den 19. März 2004

Datum der Abgabe der Dissertation: 19. März 2004

Datum der mündlichen Prüfung: 19. Juli 2004

Erstgutachter: Prof. Dr. Julius Wess

Zweitgutachter: Prof. Dr. Ivo Sachs

Zusammenfassung

Feldtheorien auf nichtkommutativen (NC) Räumen werden untersucht als realistische Erweiterungen des Standardmodells der Elementarteilchenphysik. Vor allem werden zwei Modelle mit nicht vertauschenden operatorwertigen Koordinaten betrachtet: Kanonisch NC Räume mit $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ und der κ-deformierte Raum mit $[\hat{x}^\mu, \hat{x}^\nu] = ia^\mu \hat{x}^\nu - ia^\nu \hat{x}^\mu$. Diese NC Räume werden auf gewöhnlichen Funktionen durch Sternprodukte dargestellt.

Die deformierte Multiplikation erzwingt, dass in einer Eichtheorie auf einem NC Raum das Eichpotential nicht Werte in einer Lie Algebra annimmt, sondern in deren Einhüllenden Algebra. Diese NC Eichtheorie kann jedoch so formuliert werden, dass die Freiheitsgrade mit denen der kommutativen Eichtheorie übereinstimmen. Somit kann die Eichtheorie auf der Basis jeder Lie Algebra definiert werden, sie wird rein algebraisch aus einem Konsistenzprinzip konstruiert und hier aufgefächert in der Einhüllenden Algebra zur zweiten Ordnung in θ berechnet. Der Zusammenhang mit der Seiberg-Witten-Abbildung der Stringtheorie wird ausführlich diskutiert, ebenso Auswirkungen der Freiheiten dieser Konstruktion für physikalische Theorien. Dieser Ansatz der Auffächerung in θ versteht sich als effektive Theorie. Daher wird die Quantenfeldtheorie des Standardmodells zwar nicht im Ultravioletten abgeschirmt, das in der NC Feldtheorie notorische UV-IR Problem wird aber a priori umgangen.

Der κ-deformierte Raum ist ein NC Raum mit einer deformierten Symmetriestruktur. Diese Symmetrie wird durch eine Hopfalgebra beschrieben und deren Eigenschaften werden hier aus der Konsistenz mit den NC Vertauschungsbeziehungen hergeleitet. Ableitungsoperatoren werden ausschöpfend diskutiert, ebenso algebraische Vektorfelder und zwei verschiedene Definitionen von Differentialformen. Neu ist die Einführung eines NC Differentialkalküls mit genau n Einsformen in n Dimensionen. Alle abstrakt definierten Größen werden auf gewöhnlichen Funktionen durch ableitungswertige Operatoren dargestellt. Es werden Fortschritte erzielt bei der Definition eines eichinvarianten Integrals über dem κ-deformierten Raum, das zugleich invariant unter der deformierten Symmetrie ist.

Abschließend wird die Eichtheorie für den κ-deformierten Raum konstruiert, aufgefächert im Deformationsparameter bis zur zweiten Ordnung. Lagrangefunktionen und Wirkungen werden berechnet. Eichfelder sind für Räume mit deformierter Symmetrie ableitungswertig und koppeln nicht-trivial mit anderen Feldern. Diese Modelle sagen keine neuen Teilchen vorher, sondern Wechselwirkungs-Vertices und für den κ-deformierten Fall auch neue Propagatoren. Die explizite Berechnung dieser Theorie für das Standardmodell kann zu messbaren Korrekturen führen, z.B. zu im Standardmodell verbotenen Zerfällen.

Danksagungen

An vielen der Forschungsprojekten, die zusammen die vorliegende Doktorarbeit ergeben, habe ich gemeinsam mit Kollegen des Lehrstuhl für mathematische Physik an der Ludwig-Maximilians-Universität München gearbeitet. Viele der vorgestellten Ergebnisse und Ideen sind auf diese sehr fruchtbare Zusammenarbeit zurückzuführen. Für die vielfältige Unterstützung, Anregungen und konstruktive Kritik bin ich daher sehr zu Dank verpflichtet insbesondere gegenüber Marija Dimitrijević, Branislav Jurčo, Frank Meyer, Frosso Tsouchnika und Julius Wess (in alphabetischer Reihenfolge). Ehemalige Mitglieder dieses Instituts, mit denen ich ebenfalls eng zusammenarbeitete und die daher mit zu den in dieser Arbeit vorgestellten Ergebnissen beitrugen, sind Larisa Jonke, Stefan Schraml, Peter Schupp und Michael Wohlgenannt. Ich bin diesen Kollegen und auch denjenigen, die nicht namentlich aufgeführt sind, dankbar für die kollegiale und produktive Zusammenarbeit und für viele anregende und lehrreiche Diskussionen.

Besonders möchte ich Julius Wess danken für die intensive Zusammenarbeit, von der ich überaus viel lernen konnte. Einen Großteil meiner Begeisterung für die mathematische Physik verdanke ich ihm. Seine beeindruckende physikalische Intuition, sein außerordentliches Talent für überzeugende Erläuterungen und sein wissenschaftlicher Arbeitsstil waren während der vier Jahre von Diplomarbeit und Promotion stete Motivation und Inspiration.

Ich danke dem Max-Planck-Institut für Physik, München, für die Anstellung als wissenschaftlicher Mitarbeiter von November 2000 bis Oktober 2003. Diese Position ermöglichte es mir, die in dieser Doktorarbeit zusammengefasste Forschung auszuführen. Ebenso möchte ich dem Max-Planck-Institut für die effiziente und schnelle Verwaltung danken. Ich danke der LMU München für das sehr gute Arbeitsumfeld und die Anstellung als wissenschaftlicher Mitarbeiter im Wintersemester 2003/04.

Ebenfalls danke ich meiner Verlobten Sabine Fischer und meinen Eltern Luise and Klaus Möller für ihre ununterbrochene, intensive Unterstützung während meines gesamten Studiums und der Promotion.

Contents

Chapter 1

Introduction

Shortly after quantum mechanics was discovered and the phase space of classical mechanics was found to be deformed to an algebra of noncommuting operators of position and momentum, Heisenberg [1] tried to apply such a deformation also to coordinate spacetime itself. This generalisation was not guided by experimental input as in the case of quantum mechanics, which was a mathematical framework for the empirical description of atoms. In contrast, Heisenberg hoped that noncommutative (NC) coordinates $[\hat{x}^\mu, \hat{x}^\nu] \sim \theta^{\mu\nu}$ formulated in terms of operators leading to uncertainty relations $\Delta\hat{x}^\mu\Delta\hat{x}^\nu \gtrsim |\theta^{\mu\nu}|$, could eliminate the infamously divergent electron self-energy. In the earliest stages of the development of quantum electrodynamics it was recognised that these divergences persist and fundamentally haunt any quantum field theory (QFT). At that time Snyder [2] for the first time discussed NC coordinates (of the Lie algebra type) in a published scientific article. However, the mathematical tools to treat such theories appropriately were not sufficiently developed then.

Soon, renormalisation was found to be an elegant and successful way to accommodate the QFT divergences in terms of running coupling constants, e.g. [3]. Nonetheless, renormalisation presupposes some higher scale where new physics sets in. This may be the grand-unified scale or the quantum gravity scale. But again beyond a truly fundamental scale physics should start to be prevalent, which is different from ordinary QFT. It is an open question, which framework describes such a setting best, M-theory or spin foam models or a yet entirely unknown theory. At any rate the usual concept of spacetime should dissolve beyond this scale and become fuzzy, leading to spacetime uncertainty relations. It has been argued [4], that an operational definition of localisation at a Planck scale leads to the creation of a black hole, such that spacetime becomes inaccessible and at least operationally fuzzy beyond the Planck scale. This intuitive expectation of Planck-scale fuzziness coincides with Heisenberg's and Snyder's intuition about quantising spacetime in terms of NC coordinates.

Several developments in mathematics paved the way such that during the late eighties and throughout the nineties of the 20^{th} century, NC spaces have become attractive models for physics again. A key mathematical insight is contained in the Gel'fand-Naimark theorem [5]. It states that locally compact Hausdorff spaces X and the commutative C^*-algebra $C_0(X)$ ($C(X)$ for a compact Hausdorff space) can be mapped into each other as categories by a contravariant functor. This theorem provides the essential insight that a manifold can equally well be described by (an appropriately restricted) class of functions on the manifold. The space underneath may be ignored completely, all the important information is contained as well in the (appropriately restricted) algebra of functions. This point of view is actually quite close to modern physics, diffeomorphism invariance in general relativity removes the background on which field theory is defined.

The Gel'fand-Naimark theorem allows several generalisations. Especially important is the observation that the C^*-algebra of functions need not necessarily be commutative. A commutative C^*-algebra may be deformed continuously with a small parameter and since the Gel'fand-Naimark theorem still holds in this case, the manifold disappears, to which the commutative C^*-algebra is isomorphic. The deformed C^*-algebra is not the algebra of functions on some manifold-type space anymore, the space beneath is a NC space, i.e. a NC algebra of coordinates. This is the setting which is described by Alain Connes' NC Geometry [6].

Another important mathematical development is the concept of Hopf algebras as generalisations of Lie groups and their Lie algebras [7]. The important aspect of Hopf algebras is that they can be deformed continuously [8], [9]. Therefore they can take over the role of groups and, acting on the algebra of functions on a deformed space, describe a deformed symmetry. Non-trivially deformed Hopf algebras with particularly suitable properties (quasitriangularity) for physical applications are called quantum groups [10], [11].

Deformed Hopf algebras and quantum groups have a well-defined and rich representation theory, but typically these are lattice-like structures [12], with deformed eigenfunctions such as q-hypergeometric functions. The representation theory of deformed Hopf algebras in the usual approach provides a non-perturbative description of a NC space. Similarly, Connes' NC geometry tries to understand generic and non-perturbative NC features.

The ansatz in this thesis is quite different from these approaches, although it would be very interesting to understand possible overlaps. As interesting as the non-perturbative descriptions are by themselves, the results are only partially useful from a physical perspective. If a NC space is supposed to be a model of the real world, then its low energy description must be a commutative manifold, since measurements in particle physics and astrophysics at high energies are in perfect accordance with such a description. What we can expect to measure at best in future experiments are corrections to the commutative description, at first or second order in a deformation parameter.

The framework of deformation quantisation, also a rather new branch of mathematics [13], [14], [15], provides a natural setting for connecting the deformed and the undeformed spacetime. Deformation quantisation allows to describe all properties, which essentially make up a NC space, in a perturbative way order by order in a small, formal parameter. To zeroth order in the parameter, the commutative setting is recaptured. The key tool of deformation quantisation is to rewrite the deformed multiplication of the NC coordinate algebra in terms of a $\star$-product for the ordinary coordinate algebra, i.e. a perturbative formal power series of derivative operators acting on ordinary functions.

The main momentum for the research work in NC field theory in recent years came from a setting, where such a $\star$-product is used. It was discovered that in the low energy limit the correlation functions of the endpoints of open strings in a background NS-NS B-field can be described by a $\star$-product NC field theory [16], [17], [18]. This result has created an own area of research with a focus on NC field theories with a constant tensor of the noncommutativity $\theta^{\mu\nu}$ [19], [20]. In this NC space ordinary Lorentz symmetry a priori is broken [21] because of the background field and there is no deformed symmetry either.

Mostly these theories have been treated in a summed-up version, i.e. the effect of the noncommutativity is summed into a factor multiplying the vertices in the momentum-space NC QFT. This additional factor creates new non-planar Feynman diagrams, which do not improve the ultraviolet renormalisation behaviour, but create additional infrared divergences [22], [23]. Especially important is that the UV and the IR divergences are intrinsically connected, the non-perturbative and the perturbative regime cannot be separated. This property of these string inspired NC models is actually quite attractive considering it as a theory of the quantum gravity regime. The reason is that the UV-IR mixing reflects the fact that in general relativity two energy scales are related to a given object, the Compton wavelength $\sim \frac{\hbar}{E}$ and the Schwarzschild radius $\sim G_N E$ [24].

Still, this UV-IR mixing is disappointing from the point of view of field theory extensions of the Standard Model. It shows that NC field theories are not automatically better behaved than ordinary field theories. Of course, the reason may be that not the most suitable NC model has been chosen. The breaking of Lorentz invariance by the constant background field could be restored, if $\theta^{\mu\nu}$ transforms tensor-like. Since this results in a general x-dependent $\star$-product, it would be more reasonable to use an x-dependent $\star$-product from the outset.

Among the general x-dependent $\star$-products, there are some special cases, for which the corresponding NC spaces are stable under the action of a deformed symmetry structure. These special cases might have a different high-energy behaviour than the case with constant $\theta^{\mu\nu}$. Indeed it has been suggested that not the noncommutativity of coordinates will improve renormalisability, but instead the deformation and braiding of the symmetry structure [25]. Whether this suggestion can be put into practice is an open question, but it motivates further to focus not on the string-inspired models, but on NC models with a deformed symmetry. One of the main topics of this thesis is therefore the in-depth discussion of the κ-deformed space [26], [27], [28], a NC model with a deformed symmetry structure.

The general philosophy followed in this thesis is that we accept the lessons learnt from non-perturbative NC models (representations of quantum groups, NC geometry, summed-up NC field theory). We would like to regard NC field theory as a potential extension of ordinary QFT. The scale at which this extension sets in is not fixed by the model itself, this requires experimental input. It is conceivable that this scale is not too far away from currently accessible accelerator energies. But since the renormalisation behaviour of NC field theory is not improved, we have to regard it as an effective field theory. We expand the NC field theory at a certain order in the small deformation parameter, before it is quantised. Because of the finite number of expansion terms, the theory is infrared regulated a priori. It provides (power-counting) non-renormalisable higher order operators which are exactly pinned down by the NC structure. The benefit of this θ-expanded theory is therefore not its improved high-energy behaviour, but that it can deliver precise experimental predictions for new higher-dimensional operators and therefore new physical effects.

This approach also avoids several complications in summed-up NC field theories. For example, these models allow only $U(n)$ gauge theory, although work-arounds have be defined [29], [30]. There are no restrictions in our approach [31], [32], [33] concerning the choice of gauge group in NC gauge field theory. However, the possibility to work with arbitrary gauge groups requires to generalise the concept of Lie algebra gauge theories to gauge enveloping algebras. In addition, the Standard Model gauge groups can be lifted as a tensor product into the NC regime and there is no restriction concerning the admissible charges of the individual representations of the NC gauge theory. In addition the theory seems to be anomaly free [34], [35].

This reworking of the Standard Model [36] as the zeroth order of a NC gauge theory reflects a general philosophy of our approach, which we also use in the framework of the κ-deformed space [37]. Since we do not necessarily regard the theory as a model for Planck-scale physics, we may choose a basis in the algebras describing the NC space, based on the requirement that the resulting theory coincides in as many respects as possible with the commutative regime. For example, for κ-space we choose the algebraic commutation relations of the symmetry generators such that they coincide with those of the undeformed symmetry algebra. This means that the theory can have the identical particle content like QFT on ordinary spacetime, with ordinary spinors and vectors etc. In addition we choose the gauge theory structures such that they coincide with the Standard Model gauge groups [38]. Such a deformation of the symmetry Lie algebra which leaves the algebraic sector of the symmetry invariant is always possible for semi-simple Lie algebras. This follows from the Gerstenhaber-Whitehead theorem [39].

Therefore the NC models can be designed in such a way that they predict no new particles, but “only” new dynamics, i.e. interaction vertices, and new kinematics, i.e. propagators. These predictions can be fixed almost uniquely. Therefore they provide corrections to Standard Model predictions or even predict new effects which are forbidden in the Standard Model [40]. These new effects appear because of the NC structure which is realised on ordinary spacetime in terms of formal power series in derivative operators [37]. In addition to the non-local $\star$-product with its arbitrary number of derivatives on products of functions, the deformed symmetry generators are realised by highly non-local derivative operators[1]. Even more, forms, vector fields and other geometric quantities become in a concrete way derivative-valued. Such a model has been dubbed “cogravity” by S. Majid [41], i.e. gravity in momentum space.

[1] In the expanded approach, only a finite number of derivatives appear at every order.

This thesis is structured as follows: In the second chapter we present the $\star$-product formalism as a perturbative realisation of NC coordinate algebras rather than as a deformation of a given Poisson structure (top-down rather than bottom-up). A focus is put on equivalent $\star$-products realising different ordering prescriptions of the NC algebra.

The $\star$-product is the only ingredient necessary to formulate NC gauge theory for the canonical NC space with a constant tensor $\theta^{\mu\nu}$. This is the content of the third chapter. We construct the enveloping algebra-valued gauge theory from a consistency condition such that its degrees of freedom coincide with those of ordinary Lie algebra gauge theory. This construction has been the content of an article [33] published jointly with B. Jurčo, S. Schraml, P. Schupp and J. Wess. The connection with the Seiberg-Witten map appearing in string theory is explored, but the main focus is on expanding the fields of the NC gauge theory for obtaining an effective Lagrangian description. The results obtained up to second order in $\theta^{\mu\nu}$ are presented here for the first time. Another important aspect is to understand and interpret the freedom present in this constructive approach and its effect on physical theories.

The non-perturbative properties of the κ-deformed space are derived in chapter four from consistency with the commutation relations of the NC coordinates. Several derivative operators are discussed in depth, as well as algebraic vector fields and two different definitions of differential forms. A NC differential calculus with n one-forms on an n-dimensional space is a conceptual novelty. All quantities defined in the abstract setting are represented in terms of derivative operators on commutative functions, such that a perturbative realisation becomes viable. Parts of this and parts of the sixth chapter are a condensed and revised version of a series of articles [37], [38], [42] published together with M. Dimitrijević, L. Jonke, F. Meyer, E. Tsouchnika, J. Wess and M. Wohlgenannt.

In the fifth chapter we discuss the definition of a gauge invariant integral on κ-deformed space. We perform a detailed analysis of the action of the deformed symmetry on this integral, aiming at a fully satisfactory definition of an action functional for κ-deformed gauge theory. The integral is formulated in the $\star$-product language, therefore we do not take into account non-perturbative concepts of integrals on NC spaces, such as sums over lattice points.

In the last chapter we determine the gauge fields and Lagrangians on κ-deformed space expanded in the deformation parameter up to second order. The most important observation is that gauge potentials become derivative-valued for spaces with deformed symmetries, they acquire non-trivial couplings to other fields.

Chapter 2

NC spaces and $\star$-products

2.1 NC spaces as abstract coordinate algebras

Noncommutative (NC) spaces $\mathcal{A}_{\hat{x}}$ are factor spaces, i.e. they are associative algebras freely generated by n abstract coordinates $\hat{x}^\mu$ ($\mu = 1, \ldots, n$) and divided by the ideal generated by the commutation relations:

$$\hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu = [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}(\hat{x}), \tag{2.1}$$

where $\theta^{\mu\nu}(\hat{x})$ a priori may be an arbitrary polynomial of the coordinates [43]. In other words, $\mathcal{A}_{\hat{x}}$ is the enveloping algebra of the coordinates, i.e. arbitrary coordinate combinations modulo the commutation relations (2.1)

$$\mathcal{A}_{\hat{x}} = \frac{\mathbb{C}[[\hat{x}^{\mu_1}, \hat{x}^{\mu_2}, \ldots, \hat{x}^{\mu_n}]]}{[\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}(\hat{x})}. \tag{2.2}$$

Among the polynomials which a priori might appear on the right hand side of (2.1), there are only three possibilities which fulfil the Poincaré-Birkhoff-Witt (PBW) property. The NC space $\mathcal{A}_{\hat{x}}$ possesses the PBW property, if considered as a graded algebra, the subspace of monomials of a certain degree in the NC coordinates (with a given order) has the same dimension as the corresponding subspace of monomials in commuting coordinates. The grading can be thought of as counting the number of coordinates in an ordered monomial. Therefore the monomials in PBW NC coordinates represent a basis for the polynomial algebra, they can be mapped one-to-one to monomials of commuting coordinates.

There are three types of NC spaces with the PBW property:

$$\begin{aligned} \text{Canonical NC spaces} \quad & [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \\ \text{Lie algebra spaces} \quad & [\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda, \\ \text{Quantum spaces} \quad & \hat{x}^\mu \hat{x}^\nu = qR^{\mu\nu}_{\kappa\lambda} \hat{x}^\kappa \hat{x}^\lambda. \end{aligned} \tag{2.3}$$

In the canonical case, $\theta^{\mu\nu}$ is a second rank tensor with mass[1] dimension (-2), $C^{\mu\nu}_\lambda$ are the Lie algebra structure constants with mass dimension (-1) and $R^{\mu\nu}_{\kappa\lambda}$ is the dimensionless

[1] As usual, we set $c = \hbar = 1$. We will introduce *another*, dimensionless $\hbar$ in a second.

braid- or simply R-matrix of the quantum space with q its deformation parameter. The canonical NC space will be discussed in sections 2 and 3, while the Lie algebra-type κ-deformed space will be the topic of sections 4 and 5. Quantum or q-deformed spaces are closely related to the κ-deformed space, since its symmetry algebra is a contraction of the symmetry algebra of a q-deformed space.

Symplectic structures have non-degenerate tensors $\theta^{\mu\nu}(\hat{x})$, i.e. considered as a bilinear form, non-degenerate $\theta^{\mu\nu}(\hat{x})$ means that $\theta^{\mu\nu}(\hat{x})\hat{x}_1^\mu\hat{x}_2^\nu = 0$ for all $\hat{x}_1^\mu$ implies $\hat{x}_2^\mu = 0$ and vice versa. $\theta^{\mu\nu}(\hat{x})$ for the Lie algebra and quantum spaces are of course degenerate at the origin.

The PBW theorem establishing the PBW property of $\mathcal{A}_{\hat{x}}$ requires to fix an ordering prescription on the coordinates. There are several possible orderings for a given abstract algebra of n coordinates, the most useful are normal ordering (NO)

$$(\hat{x}^1)^{i_1}(\hat{x}^2)^{i_2}(\hat{x}^1)^{i_3}(\hat{x}^n)^{i_4}\dots(\hat{x}^1)^{i_5}\cdots \stackrel{\mathrm{NO}}{\longrightarrow} (\hat{x}^1)^{i_1+i_3+i_5+\cdots}(\hat{x}^2)^{i_2+\cdots}\dots(\hat{x}^n)^{i_4+\cdots}+\dots\ , \tag{2.4}$$

and symmetric ordering (SO), e.g.

$$\hat{x}^\mu\hat{x}^\nu \stackrel{\mathrm{SO}}{\longrightarrow} \frac{1}{2}(\hat{x}^\mu\hat{x}^\nu+\hat{x}^\nu\hat{x}^\mu) = \,:\hat{x}^\mu\hat{x}^\nu:_{\mathrm{SO}}\ . \tag{2.5}$$

The ordering is a secondary structure imposed on the NC space, the commutator (2.1) is automatically factored out of the enveloping algebra (2.2). However, for any calculation an ordering is indispensable. It is necessary to impose an ordering to write a polynomial element of $\mathcal{A}_{\hat{x}}$ in a unique way

$$\hat{f}(\hat{x}) = \sum_{i=0}^{N} f_{\mu_1\dots\mu_i} : \hat{x}^{\mu_1}\dots\hat{x}^{\mu_i} : . \tag{2.6}$$

With an ordering imposed, a function $\hat{f}(\hat{x})$ can be fully characterised by its expansion coefficients $f_{\mu_1\dots\mu_i}$. Another way to understand the necessity of an ordering is to consider the multiplication of two functions:

$$\hat{f}(\hat{x})\cdot\hat{g}(\hat{x}) \longrightarrow \widehat{f\cdot g}(\hat{x}), \tag{2.7}$$

where the new polynomial $\widehat{f\cdot g}$ a priori is an arbitrary element of the enveloping algebra of the coordinates. In order to be an element of the NC space $\mathcal{A}_{\hat{x}}$, the commutation relations (2.1) have to be factored out again. This requires to impose an ordering.

Having defined the algebra of NC polynomials, we may generalise to all functions which can be expanded in terms of a formal power series:

$$\hat{f}(\hat{x}) = \sum_{i=0}^{\infty} f_{\mu_1\dots\mu_i} : \hat{x}^{\mu_1}\dots\hat{x}^{\mu_i} : . \tag{2.8}$$

We assume convergence of this formal power series, being aware that serious problems might be hidden under the carpet. For example, the definition of the $\star$-product on all smooth functions is not possible. Even worse, the domain of convergence might be zero or a set of measure zero or the set of convergent functions might be empty. The NC structure itself is even not sufficient to determine the subalgebra of admissible functions. However, the theory of deformation quantisation tells us [46] that it is a viable strategy to work with formal power series expansions.

2.2 ⋆-products

The framework of deformation quantisation [13] allows to map the associative algebra of functions on a NC space to an algebra of functions on a commutative space by means of ⋆-products.

The enveloping algebra of commuting coordinates is called $\mathcal{A}_x = \mathbb{C}[[x^1, x^2, \dots, x^n]]$. Functions on commutative space are elements of $\mathcal{A}_x$, the usual point-wise multiplication sends $\cdot : \mathcal{A}_x \times \mathcal{A}_x \to \mathcal{A}_x$. The two algebras $\mathcal{A}_x$ and $\mathcal{A}_{\hat{x}}$ are obviously different because of (2.1), but the point-wise multiplication can be deformed $\cdot \to \star : \mathcal{A}_x \to \mathcal{A}_x[[\hbar]]$ in terms of a formal parameter $\hbar$, such that $\mathcal{A}_x[[\hbar]]$ is isomorphic to $\mathcal{A}_{\hat{x}}$. This deformation of the point-wise product is the ⋆-product, it has to reproduce (2.1)

$$x^\mu \star x^\nu - x^\nu \star x^\mu = [x^\mu \overset{\star}{,} x^\nu] = i\theta^{\mu\nu}(x). \tag{2.9}$$

A ⋆-product is a deformation of a Poisson structure on a manifold. It can be expanded as a formal power series in the parameter $\hbar$

$$\star = \sum_0^\infty \hbar^r M_r, \quad \text{with } \mathbb{C} - \text{bilinear maps } M_r : \mathcal{A}_x \times \mathcal{A}_x \to \mathcal{A}_x, \tag{2.10}$$

such that ⋆: $\mathcal{A}_x[[\hbar]] \times \mathcal{A}_x[[\hbar]] \to \mathcal{A}_x[[\hbar]]$ is characterised by the following properties:

- ⋆ is a deformation of the point-wise product: $M_0(f(x), g(x)) = f(x) \cdot g(x)$.
- ⋆ is a Poisson-structure $M_1(f(x), g(x)) - M_1(g(x), f(x)) = \{f(x), g(x)\}$ to first order, i.e. it is antisymmetric, it fulfils the Jacobi-identity and the Leibniz rule.
- ⋆ is associative: $f(x) \star (g(x) \star h(x)) = (f(x) \star g(x)) \star h(x)$.
- There exists an identity, which is stable under ⋆: $f(x) = \mathbf{1} \star f(x) = f(x) \star \mathbf{1}$.

Writing the Poisson structure as $\{f, g\} = i\theta^{\mu\nu}(x)\partial_\mu f(x)\partial_\nu g(x)$, the antisymmetric tensor $\theta^{\mu\nu}(x)$ has to fulfil the Jacobi identity:

$$\theta^{\kappa\lambda}(x)\partial_\lambda\theta^{\mu\nu}(x) + \theta^{\mu\lambda}(x)\partial_\lambda\theta^{\nu\kappa}(x) + \theta^{\nu\lambda}(x)\partial_\lambda\theta^{\kappa\mu}(x) = 0. \tag{2.11}$$

The existence of ⋆-products for general Poisson manifolds was only shown recently in [15]. Deformation quantisation has been developed historically in order to deform classical Hamiltonian mechanics to quantum mechanics, therefore the deformation parameter is called $\hbar$. In our case, coordinate space itself is deformed. Poisson structures on coordinate space usually arise because of background fields. It has been shown already by Peierls [47], beautifully accounted in [48], that in a strong magnetic field the coordinates of a massless particle restricted to the lowest Landau level have non-vanishing Poisson brackets, i.e. they do not commute.

Without the presence of a specific background field, *any* deformation of a commutative manifold is as good as any other, since the first order Poisson structure is not fixed[2]. We assume that it is only necessary to conserve the point-wise product as the zeroth order in the $\star$-product. The first order properties of the $\star$-product then follow from requiring that the $\star$-product reproduces (2.9). We will not adopt the point of view that the $\star$-product under consideration is a quantisation of a Poisson structure (given e.g. by a background field), rather that it is a realisation of an a priori interesting NC structure.

There are several ways to construct a $\star$-product. A particularly efficient way of computing $\star$-products is Weyl quantisation [50], [31]. The Fourier-transform $\tilde{f}(k)$ of a function of n commuting variables $f(x^1, \ldots, x^n)$ can be associated in a unique way to an operator $W(f)$ of NC variables $\hat{x}^\mu$:

$$W(f)(\hat{x}) = (2\pi)^{-\frac{n}{2}} \int \mathrm{d}^n k \;\; : e^{i\hat{x}^\mu k_\mu} : \tilde{f}(k), \quad \text{with} \quad \tilde{f}(k) = (2\pi)^{-\frac{n}{2}} \int \mathrm{d}^n x \;\; e^{-ix^\mu k_\mu} f(x). \tag{2.12}$$

The exponential $: e^{i\hat{x}^\mu k_\mu} :$ is ordered according to the ordering prescription chosen for the abstract algebra, e.g. symmetric ordering or normal ordering. Weyl quantisation is a scheme independent of a specific ordering.

If the product of two Weyl quantised operators closes, it can be associated with a deformed product of the original functions $(f \star g)(x)$:

$$W(f)(\hat{x}) \cdot W(g)(\hat{x}) = (2\pi)^{-n} \int\int \mathrm{d}^n k \;\, \mathrm{d}^n p \;\; : e^{i\hat{x}^\mu k_\mu} :: e^{i\hat{x}^\nu p_\nu} : \tilde{f}(k)\tilde{g}(p) = W(f \star g)(\hat{x}). \tag{2.13}$$

The exponentials $: e^{i\hat{x}^\mu k_\mu} :: e^{i\hat{x}^\nu k_{nu}} :$ have to be rearranged into one exponential according to the ordering prescription.

$$: e^{i\hat{x}^\mu k_\mu} :: e^{i\hat{x}^\nu p_\nu} :=: e^{i\hat{x}^\mu \chi_\mu(k,p)} :=: e^{i\hat{x}^\mu (k_\mu + p_\mu) + i\hat{x}^\mu \tilde{\chi}_\mu(k,p)} : \;. \tag{2.14}$$

The functions $\tilde{\chi}_\mu(k,p)$ contain the information on the NC structure and inverse Fourier transformation allows to convert the momenta k and p into derivatives on the functions $f(x)$ and $g(x)$:

$$\begin{aligned} (f \star g)(x) &= (2\pi)^{-n} \int\int \mathrm{d}^n k \;\, \mathrm{d}^n p \;\; : e^{i\hat{x}^\mu k_\mu} :: e^{i\hat{x}^\nu p_\nu} : \tilde{f}(k)\tilde{g}(p) \\ &= \lim_{\substack{y \to z \\ z \to x}} e^{ix^\mu \tilde{\chi}_\mu(\partial_y, \partial_z)} f(y) g(z) \\ &= \mathsf{m}\big(\lim_{\substack{y \to z \\ z \to x}} e^{ix^\mu \tilde{\chi}_\mu(\partial \otimes \partial)} f(y) \otimes g(z) \big). \end{aligned} \tag{2.15}$$

[2]NC space like the κ-deformed space preserving a deformed symmetry structure are considered preferable though.

Note that on the right hand side of (2.15) there is no ordering symbol, since inverse Fourier transformation gives an expression in terms of commutative quantities.

In the last line, we have written the $\star$-product in tensor notation. m denotes the multiplication map, $\mathsf{m} : \mathcal{A}_x[[\hbar]] \otimes \mathcal{A}_x[[\hbar]] \rightarrow \mathcal{A}_x[[\hbar]]$. The $\star$-product of two functions $f(x)$ and $g(x)$ is a mapping to another function $(f \star g)(x)$, which is again in the same algebra of functions $\mathcal{A}_x[[\hbar]]$. In most of the discussions in this thesis, we will be interested in properties of NC spaces arising because of a deformed symmetry action with a deformed Leibniz rule. The mathematically correct formulation of this action (involving an operation called coproduct, explained in section 4.2) can be notationally cumbersome. Therefore we will also write $\star$-products in a mathematically incorrect way, suppressing the multiplication map m and the essential property that $(f \star g)(x)$ again is an element of $\mathcal{A}_x[[\hbar]]$. Thus, we will frequently write $f(x) \star g(x)$ instead of $(f \star g)(x)$.

The Baker-Campbell-Hausdorff (BCH)-formula provides a scheme to perform the rearrangement as in (2.14) for symmetric ordering[3]. For canonically NC spaces Weyl quantisation using the BCH-formula gives the Moyal-Weyl product, with $x^\mu \tilde{\chi}_\mu(k,p) = \frac{i\hbar}{2}\theta^{\mu\nu}k_\mu p_\nu$:

$$(f \star g)(x) = \lim_{\substack{y\to z\\ z\to x}} e^{\frac{i\hbar}{2}\theta^{\mu\nu}\partial_{y^\mu}\partial_{z^\nu}} f(y)g(z) \equiv \mathsf{m}\big(\lim_{\substack{y\to z\\ z\to x}} e^{\frac{i\hbar}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu} f(y)\otimes g(z)\big). \tag{2.16}$$

We use the following abbreviations for tensor products of derivatives

$$\partial_\mu \otimes 1 = \partial_{y^\mu} = \frac{\partial}{\partial y^\mu}, \qquad 1\otimes\partial_\mu = \partial_{z^\mu} = \frac{\partial}{\partial z^\mu}, \qquad \partial_\mu\otimes\partial_\nu = \frac{\partial}{\partial y^\mu} + \frac{\partial}{\partial z^\nu}. \tag{2.17}$$

Differentiation w.r.t. y acts on the first and differentiation w.r.t. z acts on the second term in the tensor product of two functions.

Note that for x-dependent $\star$-products, the limes is performed after expanding the exponential, therefore coordinates x explicitly appearing in the exponent of the $\star$-product are not differentiated.

An example of an x-dependent $\star$-product is the BCH $\star$-product for Lie algebra NC spaces. It can be expanded in terms of the Lie algebra structure constants $C_\lambda^{\mu\nu}$:

$$\begin{aligned}(f\star g)(x) \;=\; & \mathsf{m}\Big(\exp\Big(\frac{i\hbar}{2}x^\lambda C_\lambda^{\mu\nu}\partial_\mu\otimes\partial_\nu + \frac{\hbar^2}{12}x^\lambda C_\lambda^{\rho\sigma}C_\rho^{\mu\nu}(\partial_\sigma\otimes 1 - 1\otimes\partial_\sigma)\partial_\mu\otimes\partial_\nu + \\ & + \frac{i\hbar^3}{24}x^\lambda C_\lambda^{\alpha\beta}C_\alpha^{\rho\sigma}C_\rho^{\mu\nu}\partial_\beta\partial_\mu\otimes\partial_\sigma\partial_\nu + \dots\Big)\; f(x)\otimes g(x)\Big).\end{aligned} \tag{2.18}$$

2.3 The κ-deformed space and its $\star$-products

The κ-deformed space is a NC space of the Lie algebra type[4]:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC_\lambda^{\mu\nu}\hat{x}^\lambda, \tag{2.19}$$

[3]The BCH product has been related to the Kontsevich $\star$-product, cp. [52] and [53].

[4]The properties of the κ-deformed space will be discussed in a more elaborate way in chapter 4.

with the structure constants $C^{\mu\nu}_{\lambda}$ fulfilling the Jacobi identity (2.11):

$$C^{\kappa\lambda}_{\rho}C^{\mu\nu}_{\lambda} + C^{\mu\lambda}_{\rho}C^{\nu\kappa}_{\lambda} + C^{\nu\lambda}_{\rho}C^{\kappa\mu}_{\lambda} = 0. \tag{2.20}$$

The κ-deformed space has the structure constant $C^{\mu\nu}_{\lambda} = a^{\mu}\delta^{\nu}_{\lambda} - a^{\nu}\delta^{\mu}_{\lambda}$, therefore

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = ia^{\mu}\hat{x}^{\nu} - ia^{\nu}\hat{x}^{\mu}. \tag{2.21}$$

The n-dimensional Euclidean κ-deformed space is characterised by a vector a^{μ} with mass-dimension (-1). We may choose the coordinate system in such a way that the coordinate $\hat{x}^n$ is parallel in direction to the vector a^{μ}, $a^{\mu} = a\delta^{\mu}_{n}$. The $n-1$ orthogonal coordinates $\hat{x}^1, \ldots, \hat{x}^{n-1}$ commute among each other, but not with $\hat{x}^n$:

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j, \qquad [\hat{x}^i, \hat{x}^j] = 0, \quad \forall i, j \in \{1, \ldots, n-1\}. \tag{2.22}$$

The structure constants for κ-deformed space with $a^{\mu} = a\delta^{\mu}_{n}$ are $C^{\mu\nu}_{\lambda} = a(\delta^{\mu}_{n}\delta^{\nu}_{\lambda} - \delta^{\nu}_{n}\delta^{\mu}_{\lambda})$. They allow to considerably simplify the expression for the symmetrically ordered BCH $\star$-product (2.18):

$$C^{\mu_1\nu_1}_{\lambda}C^{\mu_2\nu_2}_{\mu_1}C^{\mu_3\nu_3}_{\mu_2}\ldots C^{\mu_k\nu_k}_{\mu_{k-1}1} = (-1)^{k-1}a^{k-1}\delta^{\nu_1}_{n}\delta^{\nu_2}_{n}\ldots\delta^{\nu_{k-1}}_{n}C^{\mu_k\nu_k}_{\lambda}. \tag{2.23}$$

Expanding the BCH $\star$-product up to second order in the deformation parameter $\hbar$ gives therefore (suppressing the multiplication map m)

$$\begin{aligned} f(x) \star_{SO} g(x) &= f(x)g(x) + \frac{i\hbar a}{2}x^j(\partial_n f(x)\partial_j g(x) - \partial_j f(x)\partial_n g(x)) \\ &\quad - \frac{\hbar^2 a^2}{8}x^j x^k(\partial_n^2 f(x)\partial_j\partial_k g(x) - 2\partial_j\partial_n f(x)\partial_k\partial_n g(x) + \partial_j\partial_k f(x)\partial_n^2 g(x)) \\ &\quad - \frac{\hbar^2 a^2}{12}x^j(\partial_n^2 f(x)\partial_j g(x) - \partial_n\partial_j f(x)\partial_n g(x) - \partial_n f(x)\partial_n\partial_j g(x) + \partial_j f(x)\partial_n^2 g(x)) + \ldots\ . \end{aligned} \tag{2.24}$$

Normal ordering is the second natural ordering imposed on the κ-deformed space. The only non-commuting coordinate $\hat{x}^n$ (in the sense of (2.22)) can be ordered either to the furthest left or the furthest right in any monomial. Normal ordering with all $\hat{x}^n$ to the left is reproduced by the $\star$-product $\star_L$ and the opposite ordering by the $\star$-product $\star_R$. Both can be obtained by Weyl quantisation:

$$\begin{aligned} f(x) \star_L g(x) &= \lim_{\substack{y\to x\\ z\to x}} \exp\left(x^j\partial_{y^j}(e^{-i\hbar a\partial_{z^n}} - 1)\right) f(y)g(z), \\ f(x) \star_R g(x) &= \lim_{\substack{y\to x\\ z\to x}} \exp\left(x^j\partial_{z^j}(e^{i\hbar a\partial_{y^n}} - 1)\right) f(y)g(z). \end{aligned} \tag{2.25}$$

2.4 Conjugation and equivalent $\star$-products

NC spaces have been defined as abstract algebras over the complex numbers. There is an additional operation defined on the complex number field, complex conjugation

$\overline{x+iy} = x - iy$. Generalising this operation to the algebra $\mathcal{A}_{\hat{x}}$, the conjugation $^\dagger : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$ acts as an involution

$$(\hat{f}(\hat{x}) \cdot \hat{g}(\hat{x}))^\dagger = \hat{g}^\dagger(\hat{x}) \cdot \hat{f}^\dagger(\hat{x}), \tag{2.26}$$

on $\mathbb{C}$-numbers it acts as complex conjugation. Coordinates are defined to be hermitian algebra elements $(\hat{x}^\mu)^\dagger = \hat{x}^\mu$, this is required by the commutative limit. Conjugation will be discussed more extensively in section 5.2.

The commutation relations (2.3) are invariant under conjugation. The imaginary phase i in (2.3) guarantees the hermiticity of the coordinates for the canonical and the Lie algebra NC space. The quantum space commutator is invariant under conjugation as well, since the braid matrix is real.

However, an arbitrary function $\hat{f}(\hat{x})$ of the coordinates (2.8) with an arbitrary ordering in general is not invariant under conjugation, even if the expansion coefficients $f_{\mu_1 \ldots \mu_i}$ are real. Because of the involution property, the order of coordinates is reversed and commuting the coordinates back into the original fixed order generates additional terms.

The symmetric ordering uniquely fixes a conjugation-invariant ordering. In the symmetrically ordered case, the conjugation properties of a function $\hat{f}(\hat{x})$ depend on the expansion coefficients alone, cp. (2.8). Thus, the symmetric ordering and therefore the symmetric $\star$-product are a preferable basis to work with. The symmetric $\star$-product is invariant under conjugation up to the involution. We denote conjugation on commutative quantities by a bar:

$$\overline{f(x) \star_{SO} g(x)} = \overline{g(x)} \star_{SO} \overline{f(x)} = \overline{g}(x) \star_{SO} \overline{f}(x). \tag{2.27}$$

Equation (2.27) can be checked using the BCH formula, e.g. for Lie algebras (2.18) and explicitly in (2.24). We call the symmetric $\star$-product $\star_{SO}$ a *hermitian* $\star$-product, because of its importance we drop the subscript (SO), $\star_{SO} \equiv \star$.

Although we will work only with the symmetric $\star$-product in physical applications, the normal ordered $\star$-products are interesting as well. Many constructions in κ-deformed space in the literature are worked out using the normal ordered $\star$-products [54], [55]. In this thesis we need them to derive a closed formula for the symmetric $\star$-product (2.24). For this derivation, closed expressions for a coordinate $\star$-multiplied from the left or from the right to an arbitrary function $f(x)$ are needed:

$$\begin{aligned}
x^j \star_L f(x) &= x^j e^{-ia\partial_n} f(x), & f(x) \star_L x^j &= x^j f(x), \\
x^n \star_L f(x) &= x^n f(x), & f(x) \star_L x^n &= \left(x^n - iax^k \partial_k\right) f(x), \\
\\
x^j \star_R f(x) &= x^j f(x), & f(x) \star_R x^j &= x^j e^{ia\partial_n} f(x), \\
x^n \star_R f(x) &= \left(x^n + iax^k \partial_k\right) f(x), & f(x) \star_R x^n &= x^n f(x).
\end{aligned} \tag{2.28}$$

These relations immediately follow from (2.25). There are similar relations for the symmetric ⋆-product:

$$\begin{aligned}
x^j \star f(x) &= x^j \frac{ia\partial_n}{e^{ia\partial_n}-1} f(x), \\
f(x) \star x^j &= x^j \frac{-ia\partial_n}{e^{-ia\partial_n}-1} f(x), \\
x^n \star f(x) &= \left(x^n - \frac{x^k \partial_k}{\partial_n} \left(\frac{ia\partial_n}{e^{ia\partial_n}-1} - 1 \right) \right) f(x), \\
f(x) \star x^n &= \left(x^n - \frac{x^k \partial_k}{\partial_n} \left(\frac{-ia\partial_n}{e^{-ia\partial_n}-1} - 1 \right) \right) f(x).
\end{aligned} \tag{2.29}$$

The relations (2.29) follow from (2.23) and a property of the BCH formula [52]:

$$\exp(\hat{x}_1)\cdot\exp(\hat{x}_2) = \exp\left(\hat{x}_1+\hat{x}_2+\frac{1}{2}[\hat{x}_1,\hat{x}_2]+\sum_{k=2}^{\infty}\frac{B_k}{k!}[\hat{x}_1,[\hat{x}_1,\ldots[\hat{x}_1,\hat{x}_2]\ldots]]\right)+\mathcal{O}(\hat{x}_2^2), \tag{2.30}$$

where B_k are the Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and all odd $B_{2n+1} = 0$:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_n x^n}{n!}. \tag{2.31}$$

Recall that we have argued that the ordering in the abstract coordinate algebra is secondary, provided that there is *any* ordering. Therefore the ⋆-products corresponding to different ordering prescriptions should be equivalent. There should be a $\mathbb{C}$-linear, invertible operator T implementing this equivalence of ⋆-products T: $\mathcal{A}_x[[\hbar]] \to \mathcal{A}_x[[\hbar]]$. This operator can be expanded as a formal power series in the deformation parameter $\hbar$

$$T = 1 + \sum_{j=0}^{\infty} \hbar^j T_j, \quad \text{with} \quad T_j : \mathcal{A}_x \to \mathcal{A}_x, \tag{2.32}$$

such that

$$\begin{aligned}
T(f(x) \star' g(x)) &= T(f(x)) \star'' T(g(x)), \\
\text{or} \qquad f(x) \star' g(x) &= T^{-1}\big(T(f(x)) \star'' T(g(x))\big).
\end{aligned} \tag{2.33}$$

This definition indeed specifies an equivalence relation between ⋆-products (i.e. reflexiveness, symmetry, transitivity).

The equivalence classes of ⋆-products which are quantisations of symplectic manifolds (i.e. their Poisson tensor is non-degenerate) are isomorphic to the second deRham-cohomology class of the manifold, considered as a formal power series $\hbar$ [51]. Therefore all ⋆-products for deformations of flat $\mathbb{R}^n$ with constant, non-degenerate tensor of the noncommutativity $\theta^{\mu\nu}$ are equivalent. In this case all ⋆-products can be obtained by choosing different ordering schemes. This simple observation is not true for arbitrary $\theta^{\mu\nu}(x)$ [49]. The tensor of the noncommutativity of κ-deformed space however is degenerate at the origin; but all $\star$-products discussed in this thesis correspond to different orderings and are equivalent.

We implement the observation of equivalence of $\star$-products to derive a closed symbolic form for the symmetric $\star$-product. To this end, equivalence operators T relating the $\star$-products $\star$, $\star_L$ and $\star_R$ (2.25) are constructed explicitly. The closed form of $\star$ has been found on similar arguments before [57], [58].

For definiteness, we first relate $\star$ and $\star_L$, $T(f \star g) = T(f) \star_L T(g)$. Comparing the expansion in $\hbar$ of different $\star$-products (2.24) and (2.25), up to first order in $\hbar$, T is given by

$$T = 1 + ci\hbar a x^j \partial_j \partial_n + \dots, \tag{2.34}$$

with a real constant c to be determined. The equivalence operator T depends on x^j, but not on x^n, since T and ∂_n commute. Choosing $f(x) = x^\mu$ in (2.33) we obtain $T(x^\mu) = x^\mu$ and by means of (2.29)

$$T(x^j \star_L g(x)) = T(x^j) \star T(g(x)) \;\Rightarrow\; T(x^j e^{-i\hbar a\partial_n} f(x)) = x^j \frac{i\hbar a\partial_n}{e^{i\hbar a\partial_n} - 1} T(f(x)). \tag{2.35}$$

We may multiply (2.35) with $e^{i\hbar a\partial_n}$ from the left on both sides and rewrite it as

$$T x^l f(x) = x^l \frac{-i\hbar a\partial_n}{e^{-i\hbar a\partial_n} - 1} T f(x), \quad \text{or} \quad [T, x^l] = \frac{\partial T}{\partial \partial_l} = x^l \left(\frac{-i\hbar a\partial_n}{e^{-i\hbar a\partial_n} - 1} - 1 \right) T. \tag{2.36}$$

Similarly, we obtain

$$[T, x^n] = \frac{\partial T}{\partial \partial_n} = -x^l \frac{\partial_l}{\partial_n} \left(\frac{-i\hbar a\partial_n}{e^{-i\hbar a\partial_n} - 1} - 1 \right) T. \tag{2.37}$$

On the right hand side ∂_l acts on T as well. These two simple differential equations have the solution

$$T = \lim_{z\to x} \exp\left(x^i \partial_{z^i} \left(\frac{-i\hbar a\partial_{z^n}}{e^{-i\hbar a\partial_{z^n}} - 1} - 1 \right) \right). \tag{2.38}$$

Next we determine T^{-1}, $T \cdot T^{-1} = T^{-1} \cdot T = 1$. We expect $T^{-1} = 1 + ci\hbar a x^i \partial_i + \dots$ and note

$$\begin{aligned} &\lim_{y\to x} \exp\left(x^i \partial_{y^i} f_1(\partial_{y^n}) \right) \cdot \lim_{z\to y} \exp\left(y^i \partial_{z^i} f_2(\partial_{z^n}) \right) = \\ &\qquad \lim_{z\to x} \exp\left(x^i \partial_{z^i} \left(f_1(\partial_{z^n}) + f_2(\partial_{z^n}) + f_1(\partial_{z^n}) f_2(\partial_{z^n}) \right) \right), \end{aligned} \tag{2.39}$$

as can be checked by a power series expansion. The result is

$$T^{-1} = \lim_{z\to x} \exp\left(x^i \partial_{z^i} \left(\frac{e^{-i\hbar a\partial_{z^n}} - 1}{-i\hbar a\partial_{z^n}} - 1 \right) \right). \tag{2.40}$$

Similarly the operator $T'(f \star g) = T'(f) \star_R T'(g)$ relating the symmetric and the right-normal ordered $\star$-product can be calculated with the result

$$\begin{aligned} T' &= \lim_{z\to x} \exp\left(x^i \partial_{z^i} \left(\frac{i\hbar a\partial_{z^n}}{e^{i\hbar a\partial_{z^n}} - 1} - 1 \right) \right), \\ T'^{-1} &= \lim_{z\to x} \exp\left(x^i \partial_{z^i} \left(\frac{e^{i\hbar a\partial_{z^n}} - 1}{i\hbar a\partial_{z^n}} - 1 \right) \right). \end{aligned} \tag{2.41}$$

With the solution (2.38) the symmetric ⋆-product can be constructed:

$$
\begin{aligned}
f(x) \star g(x) &= \lim_{\substack{y \to x \\ z \to x}} T\Big(T^{-1}(f(y) \star_L T^{-1}(g(z))\Big) \\
&= \lim_{w \to x} \exp\Big(x^j \partial_{w^j} (\frac{-i\hbar a \partial_{w^n}}{e^{-i\hbar a \partial_{w^n}} - 1} - 1)\Big) \lim_{\substack{y \to w \\ z \to w}} \exp\Big(w^j \partial_{y^j} (e^{-i\hbar a \partial_{z^n}} - 1)\Big) \cdot \qquad (2.42) \\
& \lim_{\substack{u \to y \\ t \to z}} \Big(\exp\Big(y^j \partial_{u^j} (\frac{e^{-i\hbar a \partial_{u^n}} - 1}{-i\hbar a \partial_{u^n}} - 1)\Big) f(u)\Big) \Big(\exp\Big(z^j \partial_{t^j} (\frac{e^{-i\hbar a \partial_{t^n}} - 1}{-i\hbar a \partial_{t^n}} - 1)\Big) g(t)\Big).
\end{aligned}
$$

Contracting all limits, this result is written in a compact way ($\partial_n = \partial_{y^n} + \partial_{z^n}$):

$$
\begin{aligned}
f(x) \star g(x) &= \lim_{\substack{y \to x \\ z \to x}} \exp\left(x^j \partial_{y^j} \left(e^{-i\hbar a \partial_{z^n}} \frac{-i\hbar a \partial_n}{e^{-i\hbar a \partial_n} - 1} \frac{e^{-i\hbar a \partial_{y^n}} - 1}{-i\hbar a \partial_{y^n}} - 1 \right) \right. \\
& \qquad \left. + x^j \partial_{z^j} \left(\frac{-i\hbar a \partial_n}{e^{-i\hbar a \partial_n} - 1} \frac{e^{-i\hbar a \partial_{z^n}} - 1}{-i\hbar a \partial_{z^n}} - 1 \right) \right) f(y) g(z). \quad (2.43)
\end{aligned}
$$

Chapter 3

Gauge theories on canonical NC spaces

In this chapter we discuss the construction of gauge theories on NC spaces. We restrict our attention to NC spaces with canonical noncommutativity. This setting already reveals many generic features of NC gauge theory.

In section 3.1 we review the construction of NC gauge theories in the framework of covariant coordinates. The terminology of this section is inspired by the appearance of NC gauge theory in string theory and it focuses on NC gauge theory of inner derivations. In section 3.2 enveloping algebra-valued gauge theories are constructed from scratch for exterior derivatives. We present a general scheme to construct gauge theories in a θ-expanded, i.e. perturbative way. This scheme gives results identical to the well-known Seiberg-Witten map. In section 3.3, the constructions of 3.2 are performed explicitly up to second order in $\theta^{\mu\nu}$. The action of NC gauge theory and the Standard Model is the content of sections 3.5 and 3.8. In section 3.6, the freedom in the construction of enveloping algebra-valued gauge theory is discussed in depth. The emphasis is on understanding how the construction of actions is affected by this freedom. In section 3.7, the existence of the enveloping algebra gauge theory is proved to all orders. Sections 3.1, 3.7 and 3.8 mostly review results of other authors.

3.1 Covariant coordinates

NC field theory has turned into a very active field of research[1] since the discovery [16] and [17] that the correlation functions on the boundary of a disc (i.e. the string world sheet) of an open string σ-model in a constant, closed and non-degenerate background B-field

$$\mathcal{S}_B = \int_D \mathrm{d}^2\sigma B_{\mu\nu}\partial_a x^\mu \epsilon^{ab}\partial_b x^\nu, \tag{3.1}$$

[1] Another reason for the interest in NC field theories from string theory in recent years is due to [59].

can be described by a NC field theory:

$$\langle f_1(x(t_1)) \cdot \dots \cdot f_n(x(t_n)) \rangle = \int_{\partial D} \mathrm{d}t f_1 \star \dots \star f_n. \tag{3.2}$$

The correlation functions are time-ordered $t_1 < \dots < t_n$ and $\star$ is the Moyal-Weyl $\star$-product with constant Poisson tensor $\theta^{\mu\nu}$. Since the boundaries of the string world sheet are the end points of the open string in target space, i.e. they make up or live on a D-brane, the world-volume of the D-brane is a NC space. Note that this description is valid only at low energies, in the decoupling limit with zero slope

$$\theta^{\mu\nu} = 2\pi\alpha' \Big(\frac{1}{g + 2\pi\alpha' B}\Big)^{\mu\nu} \stackrel{\alpha' \to 0}{\longrightarrow} (\frac{1}{B})^{\mu\nu}. \tag{3.3}$$

Formula (3.2) can be derived from correlation functions of exponential operators

$$\langle \prod_n e^{ip_\mu^n x^\mu(t_n)} \rangle = e^{-\frac{i}{2}\sum_{n>m} p_\mu^n \theta^{\mu\nu} p_\nu^m \epsilon(\tau_n - \tau_m)} \delta(\sum p^n). \tag{3.4}$$

This formula represents the Weyl-quantisation of n ordered exponentials (cp. section 2.2).

Seiberg and Witten [18] studied the effect of a slight perturbation of the B-field by a $U(n)$ gauge field strength $B' = B + \mathrm{d}a$ with a a gauge potential ($B'_{\mu\nu} = B_{\mu\nu} + \partial_\mu a_\nu - \partial_\nu a_\mu$). This perturbation leads to an additional term in the action which is integrated over the boundary of the disk:

$$\mathcal{S}_a = \int_{\partial D} \mathrm{d}t \;\; a_\mu(x(t)) \partial_t x^\mu(t), \tag{3.5}$$

Gauge invariance of the action under $\delta_\alpha a_\mu = \partial_\mu \alpha$ is in fact not automatically safeguarded in the quantum theory. In this sense gauge invariance is respected only in the Pauli-Villars scheme:

$$\delta_\alpha \mathcal{S}_a = \int_{\partial D} \mathrm{d}t \;\; \partial_\mu \alpha \partial_t x^\mu(t) = \int \mathrm{d}t \;\; \partial_t \alpha. \tag{3.6}$$

On the contrary, point-splitting regularisation leads to an invariant integral provided that the gauge transformation has the form

$$\delta_{\hat\alpha} \hat a_\mu = \partial_\mu \hat\alpha - i \hat a_\mu \star \hat\alpha + i \hat\alpha \star \hat a_\mu, \tag{3.7}$$

with $\star$ the Moyal-Weyl $\star$-product. Since physical results have to be independent of the use of a particular regularisation scheme, these two settings have to be related. This is the statement of the Seiberg-Witten map: Physical descriptions obtained via different regularisation schemes have to be equivalent, therefore there has to be a map relating the commutative and the NC $U(n)$ gauge theory such that:

$$\begin{aligned} \hat a_\mu(a_\nu) + \delta_{\hat\alpha} \hat a_\mu(a_\nu) &= \hat a_\mu(a_\nu + \delta_\alpha a_\nu), \\ \hat\alpha(a_\nu) + \delta_{\hat\beta} \hat\alpha(a_\nu) &= \hat\alpha(a_\nu + \delta_\beta a_\nu). \end{aligned} \tag{3.8}$$

The Seiberg-Witten map states that there is a NC gauge theory which can equivalently be described by commutative gauge theory via the identification (3.8).

Seiberg and Witten solved the equation (3.8) by expanding the NC gauge transformation in terms of powers of θ derived from the $\star$-product, arriving at

$$\begin{aligned}
\hat{a}_\mu(a_\nu) &= a_\mu - \frac{1}{4}\theta^{\kappa\lambda}\{a_\kappa, \partial_\lambda a_\mu + f_{\lambda\mu}\} + \mathcal{O}(\theta^2), \\
\hat{\alpha}(\alpha, a_\nu) &= \alpha - \frac{1}{4}\theta^{\kappa\lambda}\{a_\kappa, \partial_\lambda \alpha\} + \mathcal{O}(\theta^2), \\
\hat{f}_{\mu\nu}(a_\nu) &= f_{\mu\nu} - \frac{1}{4}\theta^{\kappa\lambda}\big(\{a_\kappa, (\partial_\lambda + \mathcal{D}_\lambda) f_{\mu\nu}\} - 2\{f_{\kappa\mu}, f_{\lambda\nu}\}\big) + \mathcal{O}(\theta^2).
\end{aligned} \tag{3.9}$$

and then, summing up (3.9) to all orders into a differential equation for varying θ, i.e. for different $\star$-products arising from similar Poisson structures $\theta' = \theta + \delta\theta$:

$$\begin{aligned}
\delta\hat{a}_\mu(\theta) &= \delta\theta^{\kappa\lambda}\frac{\partial}{\partial\theta^{\kappa\lambda}}\hat{a}_\mu(\theta) &&= -\frac{1}{4}\theta^{\kappa\lambda}\{\hat{a}_\kappa \overset{\star}{,} (\partial_\lambda \hat{a}_\mu + \hat{f}_{\lambda\mu})\}, \\
\delta\hat{\alpha}_\mu(\theta) &= \delta\theta^{\kappa\lambda}\frac{\partial}{\partial\theta^{\kappa\lambda}}\hat{\alpha}(\theta) &&= -\frac{1}{4}\theta^{\kappa\lambda}\{\hat{a}_\kappa \overset{\star}{,} \partial_\lambda \hat{\alpha}\}, \\
\delta\hat{f}_{\mu\nu}(\theta) &= \delta\theta^{\kappa\lambda}\frac{\partial}{\partial\theta^{\kappa\lambda}}\hat{f}_{\mu\nu}(\theta) &&= -\frac{1}{4}\theta^{\kappa\lambda}\big(\{\hat{a}_\kappa \overset{\star}{,} (\partial_\lambda + \mathcal{D}_\lambda)\hat{a}_\mu\} - 2\{\hat{f}_{\kappa\mu}, \hat{f}_{\lambda\nu}\}\big).
\end{aligned} \tag{3.10}$$

These are the all-orders solutions of the Seiberg-Witten map, non-trivial field redefinitions of the gauge fields, written in terms of a varying tensor of the noncommutativity. Recall that varying θ is equivalent to adding a field strength to the field strength $B' = B + \mathrm{d}a$. The fluctuations of the D-brane are described by NC Yang-Mills theory.

There is a suitable description of this string theoretical setting in terms of the gauge theory of NC inner derivations [60], [61], [62]. We will show now that the extra term in the action obtained from a variation of the B-field can be described as a change of coordinates:

$$x^\mu \to x'^\mu = x^\mu + \theta^{\mu\nu}\hat{a}_\nu. \tag{3.11}$$

The result of this analysis will be that a change of the background field $B \to B + \mathrm{d}a$, which serves as the Poisson tensor for the NC description of spacetime, generates translations of the coordinates.

Suppose a Poisson structure is fixed and we consider a NC gauge transformation of a field which transforms under a gauge transformation, e.g. from the left

$$\psi \to e_\star^{i\hat{\alpha}} \star \psi \quad \text{or infinitesimally} \quad \delta_{\hat{\alpha}}\psi = i\hat{\alpha} \star \psi. \tag{3.12}$$

We have introduced the $\star$-exponential function $e_\star^{i\hat{\alpha}}$. This has to be interpreted as a formal power series, where ordinary multiplication in every summand is replaced by $\star$-multiplication: $e_\star^{i\hat{\alpha}} = 1 + i\hat{\alpha} - \frac{1}{2}\hat{\alpha} \star \hat{\alpha} - \frac{i}{6}\hat{\alpha} \star \hat{\alpha} \star \hat{\alpha} + \ldots$. Therefore $e_\star^{i\hat{\alpha}} \star e_\star^{-i\hat{\alpha}} = 1$. These $\star$-exponentials replace finite gauge transformations in the NC regime (for the subtleties, cp. [63]).

The field ψ in (3.12) cannot simply be multiplied from the left with a function f with $\delta_{\hat{\alpha}} f = 0$ without spoiling the gauge covariance (infinitesimally)

$$\delta_\alpha(f \star \psi) = if \star \hat{\alpha} \star \psi \neq i\hat{\alpha} \star f \star \psi. \tag{3.13}$$

Multiplying a field with a function can be reconstituted as a gauge covariant operation introducing covariant functions

$$\mathfrak{D}f = f + \mathfrak{A}_f, \tag{3.14}$$

which transform as

$$\mathfrak{D}f \to e_\star^{i\hat{\alpha}} \star \mathfrak{D}f \star e_\star^{-i\hat{\alpha}}, \quad \text{or infinitesimally} \quad \delta_{\hat{\alpha}}\mathfrak{D}f = i[\hat{\alpha} \stackrel{\star}{,} \mathfrak{D}f]. \tag{3.15}$$

The gauge potential $\mathfrak{A}_f$ transforms as follows:

$$\mathfrak{A}_f \to e_\star^{i\hat{\alpha}} \star [f \stackrel{\star}{,} e_\star^{-i\hat{\alpha}}] + e_\star^{i\hat{\alpha}} \star [\mathfrak{A}_f \stackrel{\star}{,} e_\star^{-i\hat{\alpha}}], \tag{3.16}$$

or infinitesimally

$$\delta_{\hat{\alpha}}\mathfrak{A}_f = -i[f \stackrel{\star}{,} \hat{\alpha}] - i[\mathfrak{A}_f \stackrel{\star}{,} \hat{\alpha}]. \tag{3.17}$$

There is a gauge field strength corresponding to $\mathfrak{A}_f$ as well, it is defined just like curvature in differential geometry

$$\mathfrak{F}_{(f,g)} = [\mathfrak{D}f \stackrel{\star}{,} \mathfrak{D}g] - \mathfrak{D}[f \stackrel{\star}{,} g]. \tag{3.18}$$

For the case of a constant Poisson structure $\theta^{\mu\nu}$ and $f = x^\mu$, we obtain the covariant coordinates:

$$x^\mu \to X^\mu = x^\mu + \hat{A}^\mu, \quad \text{with} \quad \delta_{\hat{\alpha}}\hat{A}^\mu = -i[x^\mu \stackrel{\star}{,} \hat{\alpha}] - i[\hat{A}^\mu \stackrel{\star}{,} \hat{\alpha}]. \tag{3.19}$$

Since

$$-i[x^\mu \stackrel{\star}{,} \hat{\alpha}] = \theta^{\mu\nu}\partial_\nu\hat{\alpha}, \quad \Rightarrow \quad \delta_{\hat{\alpha}}\hat{A}^\mu = \theta^{\mu\nu}\partial_\nu\hat{\alpha} - i[\hat{A}^\mu \stackrel{\star}{,} \hat{\alpha}], \tag{3.20}$$

the covariant coordinate is exactly the expression we interpreted as a translation in (3.11) $x'^\mu = x^\mu + \theta^{\mu\nu}\hat{a}_\nu$. Here $\hat{a}_\nu$ is the Yang-Mills type gauge potential as in (3.11), which transforms as (cp. (3.7))

$$\hat{a}_\nu \to ie_\star^{i\hat{\alpha}} \star \partial_\nu e_\star^{-i\hat{\alpha}} + e_\star^{i\hat{\alpha}} \star \hat{a}_\nu \star e_\star^{-i\hat{\alpha}}, \quad \text{or inf.} \quad \delta_{\hat{\alpha}}\hat{a}_\nu = \partial_\nu\hat{\alpha} - i[\hat{a}_\nu \stackrel{\star}{,} \hat{\alpha}]. \tag{3.21}$$

The field strength for the gauge potential $\hat{a}_\nu$ is:

$$\hat{f}_{\mu\nu} \to e_\star^{i\hat{\alpha}} \star \hat{f}_{\mu\nu} \star e_\star^{-i\hat{\alpha}}, \quad \text{with} \quad \hat{f}_{\mu\nu} = \partial_\mu\hat{a}_\nu - \partial_\nu\hat{a}_\mu - i[\hat{a}_\mu \stackrel{\star}{,} \hat{a}_\nu]. \tag{3.22}$$

This NC Yang-Mills type field strength is related to $\mathfrak{F}_{(f,g)}$ in the following way:

$$i\hat{f}_{\mu\nu}\theta^{\rho\mu}\theta^{\sigma\nu} = \mathfrak{F}_{(x^\rho,x^\sigma)} = [x^\rho \stackrel{\star}{,} \mathfrak{A}_{x^\sigma}] - [x^\sigma \stackrel{\star}{,} \mathfrak{A}_{x^\rho}] + [\mathfrak{A}_{x^\rho} \stackrel{\star}{,} \mathfrak{A}_{x^\sigma}]. \tag{3.23}$$

The first interesting aspect of covariant coordinates is exactly the relation between (3.21) and (3.11): Gauge potentials can be introduced without having to fix a differential calculus on the NC space first, inner derivations are sufficient. This allows to construct a large variety of concepts on the NC space without much additional algebraic or geometric structure. The derivative in (3.20) is an ordinary derivative originating from the $\star$-product.

For inner derivations the antisymmetrised Hochschild cohomology or Chevalley cohomology $\mathcal{C}^p = \mathrm{Hom}(\mathcal{A}_k^{\wedge p}, \mathcal{A}_x)$ of antisymmetrised p-th tensor powers of the algebra of coordinates (multiplied with the $\star$-product) takes over the role of the deRham cohomology of an exterior differential calculus. There is an analogon of the deRham differential $\mathbf{d}_\star$ on a p-"form" $\mathcal{C} \in \mathcal{C}^p$ formulated with the Gerstenhaber bracket

$$(\mathbf{d}_\star\mathcal{C})(f_1,\ldots,f_{p+1}) = f_1\star\mathcal{C}(f_2,\ldots,f_{p+1})-\mathcal{C}(f_1\star f_2,\ldots,f_{p+1})+\ldots(-1)^{p+1}\mathcal{C}(f_1,\ldots,f_p)\star f_{p+1}. \tag{3.24}$$

In this cohomology based on inner derivations, gauge theory can be discussed in full analogy to the usual treatment of gauge theories in the deRham cohomology. A gauge potential is a Chevalley one form (3.16), a field strength is a Chevalley two form, for which the Bianchi identity following from $\mathbf{d}_\star^2 = 0$ is valid. Equation (3.18) can be written as

$$\mathfrak{F}_{(f,g)} = (\mathbf{d}_\star\mathfrak{A})_{(f,g)} + \mathfrak{A}_f \wedge \mathfrak{A}_g, \quad \text{with} \quad \mathbf{d}_\star\mathfrak{F} + \mathfrak{A}\wedge\mathfrak{F} - \mathfrak{F}\wedge\mathfrak{A} = 0. \tag{3.25}$$

The second interesting aspect of covariant coordinates is the relation between (3.19) and (3.11):

The covariantiser $\mathfrak{D} = 1 + \mathfrak{A}$ (3.14) makes the $\star$-multiplication with a covariantised function a gauge-invariant operation. Therefore it generates an equivalence of two situations: whether a field is multiplied with a covariantised function or not makes no difference from the point of view of the gauge transformation. On the other hand, gauge transformations also generate translations of the NC space (3.11). Therefore gauge transformations generate an equivalence between two different NC structures, corresponding to two different quantised Poisson structures. But both NC structures have their respective gauge structure associated. Therefore the translation has to be an equivalence between two different NC gauge structures, cp. [91]. This equivalence is implemented by an operator, which due to the correspondence between gauge transformations and translations is exactly the covariantiser $\mathfrak{D}$.

A cautionary remark concerning equivalent $\star$-products is in order: The solution of the Seiberg-Witten map is a flow in the space of equivalent $\star$-products (3.10). But the setting of section 2.4, where explicit equivalence operators between different $\star$-products have been constructed, is not sufficient. There the topic of gauge theory was completely ignored, the operators T (2.38) do not know about the gauge degrees of freedom. But if a NC space is endowed with an additional gauge theory structure, than changing from one $\star$-product to another, the gauge degrees of freedom have to dragged along in a compatible way.

The construction of $\mathfrak{D}$ as an equivalence operator between different $\star$-structures with their respective gauge structure associated will be sketched now. The presentation is based on the approach of [61], constructing the Seiberg-Witten map from Poisson manifold quantities, which are lifted into a quantum version by means of Kontsevich's formality map.

One of the key observations concerning NC gauge theory is that even an Abelian gauge theory in the commutative regime turns into a non-Abelian gauge theory in the NC regime due to the $\star$-product. If this property is to be reconstructed from Poisson manifold quantities, a prescription has to be be found that turns an Abelian Poisson manifold quantity into a non-Abelian one in a suitable way.

The Poisson tensor on a Poisson manifold is a bivector which defines an analogon of $\mathbf{d}_\star$, i.e. a differential on functions $\mathbf{d}_\theta f = -\theta^{\mu\nu}\partial_\mu f \partial_\nu$. $\mathbf{d}_\theta f$ is the Hamiltonian vector field corresponding to f. A Poisson manifold vector field (this is an Abelian gauge potential, it is not the analogon to the non-Abelian gauge potential $\mathfrak{A}$ yet) is $\mathbf{a}_\theta = a_\mu \mathbf{d}_\theta x^\mu$. Similarly there is an Abelian field strength $\mathbf{f}_\theta = \mathbf{d}_\theta \mathbf{a}_\theta$, which is a bivector field $\mathbf{f}_\theta = -\frac{1}{2}\theta^{\mu\kappa} f_{\kappa\lambda}\theta^{\lambda\nu}\partial_\mu \wedge \partial_\nu$.

A change of coordinates for the Poisson manifold is described by a parametric deformation θ_t with $\theta_0 = \theta$ and $\partial_t\theta_t = \mathbf{f}_{\theta_t}$, leading to the solution $\theta_t = \theta\frac{1}{1+tf\theta}$. Compare this result to the change in the Poisson structure adding a fluctuation to the B-field: $\left(\frac{1}{B}\right)'^{\mu\nu} = \left(\frac{1}{B+f}\right)^{\mu\nu}$.

There is a generalisation of the Lie bracket, the Schouten-bracket $[\cdot,\cdot]_S$ (see [61]), which allows to write $\partial_t\theta_t = -[\mathbf{a}_{\theta_t}, \theta_t]_S$. This Schouten-bracket can be integrated to a flow on the Poisson manifold

$$\rho_a^* = e^{\mathbf{a}_\theta + \partial_t} e^{-\partial_t}|_{t=0}. \tag{3.26}$$

relating two Poisson structures θ $(t = 0)$ and θ' $(t = 1)$. The result (3.26) follows from $e^{-\partial_t} f(t) = f(t-1)$, inserting $(\mathbf{a}_\theta + \partial_t) f(t) = 0$, therefore $e^{\mathbf{a}_\theta+\partial_t} f(t) = f(t)$. The flow ρ_a^* depends on the components a_μ of the Abelian vector field $\mathbf{a}_\theta$.

This exponentiated Abelian vector field (3.26) has exactly the right properties to regard it as the non-Abelian Poisson manifold analogon A_a of $\mathfrak{A}$: $A_a = \rho_a^* - 1$.

Quantising the two equivalent Poisson structures, equivalent $\star$-products are obtained. Formality maps [15] associate to Poisson polyvectors (e.g. a vector or a bivector) a poly-differential operator, which is a realisation of an operator on the NC space. The formality quantisation of $\mathbf{a}_\theta$ is

$$\mathbf{a}_\star = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} U_{n+1}(\mathbf{a}_\theta, \theta, \ldots, \theta), \tag{3.27}$$

with $\mathbf{a}_\theta$ a vector and θ the Poisson bivector, resulting in a differential operator $a_\star$ (the "Abelian" NC gauge potential). Similarly the "Abelian" NC field strength

$$\mathbf{f}_\star = \sum_{n=0}^{\infty} \frac{(i\hbar)^{n+1}}{n!} U_{n+1}(\mathbf{f}_\theta, \theta, \ldots, \theta), \tag{3.28}$$

with $\mathbf{f}_\star = \mathbf{d}_\star \mathbf{a}_\star$ is a quantised version of $\mathbf{f}_\theta = \mathbf{d}_\theta \mathbf{a}_\theta$ ($\mathbf{d}_\star$ as in (3.27)). The Kontsevich formality maps allow to carry over the construction of the flow in the space of equivalent Poisson structures (for details [61]) and to generate the non-Abelian gauge potential $\mathfrak{A}$ (cp. [64]) as

$$\mathfrak{A} = e^{\mathbf{a}_\star + \partial_t} e^{-\partial_t}|_{t=0} - 1. \tag{3.29}$$

Through $\mathbf{a}_\star$ and $\mathbf{a}_\theta$, the non-Abelian NC gauge potential again is a function of the Abelian Poisson manifold gauge potential component a_μ.

Thus, we have shown the construction of the equivalence operator $\mathfrak{D} = 1 + \mathfrak{A}$, which on the basis of the arguments given above is identical to the covariantiser of a function.

The Seiberg-Witten map comes into the game, if we consider an infinitesimal gauge transformation on a_μ, $a_\mu \to a_\mu + \partial_\mu \alpha$, that does not change the Poisson structure and the $\star$-products. But it induces a change

$$\mathbf{a}_\theta \to \mathbf{a}_\theta + \mathbf{d}_\theta \alpha, \quad \to \quad \mathbf{a}_\star \to \mathbf{a}_\star + \mathbf{d}_\star \tilde{\alpha}, \tag{3.30}$$

where $\tilde{\alpha}$ is a formality quantised version of α. Because of (3.29) it also induces a change in $\mathfrak{A}$: it generates a NC gauge transformation on $\mathfrak{A}$ such that

$$\mathfrak{A}_{a_\mu + \partial_\mu \alpha} = \mathfrak{A}_{a_\mu} + \hat{\delta}_{\hat{\alpha}} \mathfrak{A}_{a_\mu}. \tag{3.31}$$

This is exactly the Seiberg-Witten equation. The tricky bits and the complicated mathematics is of course hidden in Kontsevich's formalism, which we have not even touched upon. The power of the formality map [15], [65] which can be defined for arbitrary $\theta^{\mu\nu}(x)$, allows to transfer all structures described in the Poisson manifold language into a fully quantised setting. This discussion will lead us too far, we refer to the literature, [61], [62].

To summarise this section, recall that the fluctuations of the background field B can either be interpreted as a change of the noncommutativity underlying a fixed spacetime (i.e. a gauge transformation), or as a change of the brane configuration itself in a fixed background, in other words a translation. More arbitrary transformations are possible for non-constant $\theta^{\mu\nu}(x)$, these describe a curving of the brane [66]. A remarkable feature is that also such transformations can be implemented via NC gauge transformations, without the specification of generators of translation. Inner derivations are entirely sufficient. The covariant coordinates then do not have the simple form (3.11) anymore. Inner derivations are particularly well suited to discuss settings with varying $\theta^{\mu\nu}(x)$ [67], [68], [69].

In the rest of this thesis we will not follow the train of thought of this section. We will take into account an exterior calculus explicitly. The exterior differential calculus of the commutative space can be transferred to the canonically NC regime, since derivatives commute with the x-independent $\star$-product. For spaces with deformed symmetries the exterior differential calculus is one of the most interesting features.

3.2 Enveloping algebra-valued gauge theories

It has been shown in [33] that the results of Seiberg and Witten's seminal paper [18], connecting commutative and NC gauge theory, can also be obtained in a setting entirely independent of string theory. The ansatz of [33] uses only algebraic properties of the canonically NC space, i.e. properties of the $\star$-product. In particular it does not use the properties of a deformation quantisation of a Poisson structure, e.g. from a background field.

The most important result of [33] is that no restrictions exist on the admissible gauge groups. However, the disadvantage of this constructive setting is that no statements about uniqueness or existence can be made a priori. However, the approach can be formalised and the existence of this construction can be shown by complete induction (cp. section 3.7). Uniqueness is discussed in sections 3.4 and 3.6.

We start considering this constructive approach towards NC gauge theory by fixing the notations for infinitesimal gauge transformations on a commutative space [70]:

$$\delta_\alpha \psi_i^0(x) = i\alpha_a(x)(T^a)_{ij}\psi_j^0(x). \tag{3.32}$$

The field $\psi^0(x)$ is in a representation of an arbitrary Abelian or non-Abelian gauge group. Our discussion uses the more general non-Abelian setting, Abelian simplifications are not spelt out. The index i refers to the components of the representation, $\psi^0(x)$ is in addition a function of the commutative coordinates x. A local[2] gauge transformation maps this field $\psi^0(x)$ by left multiplication with a x-dependent matrix to the transformed field $\left(\delta_\alpha\psi^0\right)_i(x)$. The matrices T^a are the generators of the Lie algebra of the gauge group:

$$[T^a, T^b] = if_c^{ab}T^c. \tag{3.33}$$

We can considerably simplify our notation by absorbing the index of the components of the representation and by keeping the generators of the gauge Lie algebra and the x-dependence of ψ^0 and α implicit, $\alpha \equiv \alpha_a(x)T^a$ and $\psi^0 \equiv \psi^0(x)$:

$$\delta_\alpha\psi^0 = i\alpha\psi^0. \tag{3.34}$$

Since the generators T^a form a Lie algebra, the commutator of two infinitesimal gauge transformations closes:

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi^0 = \alpha\beta\psi^0 - \beta\alpha\psi^0 = \delta_{-i[\alpha,\beta]}\psi^0 \equiv \delta_{\alpha\times\beta}\psi^0. \tag{3.35}$$

A gauge transformation acts in the following way on the conjugate transpose of a field $(\psi^0)^\dagger$ (for later use we concentrate on Dirac fields $\overline{\psi^0} = (\psi^0)^\dagger\gamma^0$, with γ^0 the matrix implementing conjugation as usual):

$$\delta_\alpha\overline{\psi^0} = -i\overline{\psi^0}\alpha, \quad \text{such that} \quad \delta_\alpha(\overline{\psi^0}\psi^0) = 0. \tag{3.36}$$

Kinetic terms $\partial_\mu\psi^0$ are not gauge invariant anymore, but the derivatives can be gauged (i.e. can be made gauge covariant) by adding a gauge potential:

$$\begin{aligned} \delta_\alpha(\mathcal{D}_\mu^0\psi^0) &= \delta_\alpha\big((\partial_\mu - iA_\mu^0)\psi^0\big) \stackrel{!}{=} i\alpha(\mathcal{D}_\mu^0\psi^0), \\ \Rightarrow \quad \delta_\alpha A_\mu^0 &= \partial_\mu\alpha - i[A_\mu^0, \alpha]. \end{aligned} \tag{3.37}$$

The gauge potential transforms in the adjoint representation, like the field strength $F_{\mu\nu}^0$, which is constructed from the commutator of two covariant derivatives

$$\begin{aligned} F_{\mu\nu}^0 &= i[\mathcal{D}_\mu^0, \mathcal{D}_\nu^0] = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0 - i[A_\mu^0, A_\nu^0], \\ \Rightarrow \quad \delta_\alpha F_{\mu\nu}^0 &= i[\alpha, F_{\mu\nu}^0], \\ [\mathcal{D}_\lambda^0, F_{\mu\nu}^0] + [\mathcal{D}_\mu^0, F_{\nu\lambda}^0] + [\mathcal{D}_\nu^0, F_{\lambda\mu}^0] &= 0, \quad \text{Bianchi identity.} \end{aligned} \tag{3.38}$$

[2]For $\alpha_a = const$, a constant global transformation, the NC transformation is identical to the commutative one.

As in the commutative setting, we start to consider gauge theory on NC space based on an infinitesimal gauge transformation. The NC space is represented on the algebra of functions of commutative variables by a $\star$-product (cp. section 2.2). Therefore (3.32) is replaced by

$$\delta_{\hat{\Lambda}}\hat{\psi}(\hat{x}) = i\hat{\Lambda}(\hat{x})\cdot\hat{\psi}(\hat{x}), \quad \Rightarrow \quad \delta_\Lambda\psi(x) = i\Lambda(x)\star\psi(x). \tag{3.39}$$

As before, the gauge transformation of the field $\psi(x)$ is implemented by local left $\star$-multiplication of $\psi(x)$ with a NC function $\Lambda(x)$. The term *local $\star$-multiplication* has to be taken with a grain of salt, since the $\star$-product involves an arbitrary number of derivatives and is therefore highly non-local. Representations for which a field $\psi(x)$ is multiplied from the right are possible as well. Covariant derivatives, gauge potentials and field strengths are constructed from (3.39) just as in the commutative case.

Yet there is a problem. Suppose the gauge parameter $\Lambda(x)$ implementing the infinitesimal gauge transformation is Lie algebra-valued, i.e. it can be written in terms of the generators of a Lie algebra $\Lambda_a(x)T^a$. Then the x-dependent function of the NC gauge parameter is multiplied with the $\star$-product, therefore two gauge transformations do not commute and in general do not close anymore as in (3.35)

$$\begin{aligned}(\delta_{\Lambda_1}\delta_{\Lambda_2}-\delta_{\Lambda_2}\delta_{\Lambda_1})\psi(x) &= \Lambda_1(x)\star\Lambda_2(x)\star\psi(x)-\Lambda_2(x)\star\Lambda_1(x)\star\psi(x)\\ &= \frac{1}{2}[T^a,T^b]\{\Lambda_{1,a}(x)\stackrel{\star}{,}\Lambda_{2,b}(x)\}\star\psi(x)\\ &\quad +\frac{1}{2}\{T^a,T^b\}[\Lambda_{1,a}(x)\stackrel{\star}{,}\Lambda_{2,b}(x)]\star\psi(x)\stackrel{!}{=}\delta_{\Lambda_1\times\Lambda_2}\psi(x).\end{aligned} \tag{3.40}$$

The anti-commutator $\{T^a,T^b\}$ in the gauge transformation $\delta_{\Lambda_1\times\Lambda_2}\psi(x)$ imposes restrictions on the admissible gauge groups. Only a $U(n)$ Lie algebra gauge theory allows to express the anti-commutator $\{T^a,T^b\}$ again in terms of the generators [18].

The only alternative is that the concept of Lie algebra gauge theories has to be generalised. This is the approach followed here. Taking the commutation relations (3.33) as a starting point, the Lie algebra and its enveloping algebra are the only mathematical objects fulfilling these relations, independently of a specific representation. The enveloping algebra $\mathcal{A}_T$ of the Lie algebra is an infinite-dimensional algebra freely generated by T and divided by the ideal generated by the commutation relations (3.33). Note that the product of two generators is not the matrix product in a particular representation, but the tensor product. An ordering prescription has to be specified for concrete calculations. As in the discussion of NC coordinates in section 2.1, symmetric ordering is suitable because of the invariance under conjugation:

$$\mathcal{A}_T = \{T, :T^aT^b := \frac{1}{2}\{T^a,T^b\},\ldots, :T^{a_1}\cdots T^{a_l} := \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\cdots T^{\sigma(a_l)}),\ldots\}. \tag{3.41}$$

Obviously, the worrisome anti-commutator of (3.40) is in the enveloping algebra as well, since $\mathcal{A}_T$ is "large enough". Therefore we have resolved the predicament (3.40), the commutator of two enveloping algebra-valued gauge transformations remains enveloping algebra-valued. However, $\mathcal{A}_T$ is also "too large". To see the problem, we expand an enveloping algebra-valued gauge parameter in terms of the basis of $\mathcal{A}_T$:

$$\Lambda = \Lambda^0_a T^a + \Lambda^1_{ab} : T^a T^b : + \cdots + \Lambda^n_{a_1 a_2 \ldots a_{n+1}} : T^{a_1} T^{a_2} \ldots T^{a_{n+1}} : + \ldots \quad . \tag{3.42}$$

Just as Λ depends on an infinite number of components $\Lambda^n_{a_1 a_2 \ldots a_{n+1}}$, all other quantities of this enveloping algebra-valued gauge theory have an infinite number of components. To show this, we introduce the other quantities of NC gauge theory explicitly.

Since the Moyal-Weyl $\star$-product is x-independent, it commutes with derivatives and derivatives have the ordinary Leibniz rule: $\partial_\mu(f(x) \star g(x)) = (\partial_\mu f(x)) \star g(x) + f(x) \star (\partial_\mu g(x))$. Therefore a covariant derivative on the canonically NC space along with a NC gauge potential and the NC field strength are introduced as generalisations of the Lie algebra covariant derivative, gauge potential and field strength (3.37) and (3.38):

$$\begin{aligned} \delta_\Lambda(\mathcal{D}_\mu\psi) &= \delta_\Lambda\big(\partial_\mu\psi - iA_\mu \star \psi\big) \stackrel{!}{=} i\Lambda \star \partial_\mu\psi + \Lambda \star A_\mu \star \psi = i\Lambda \star \mathcal{D}_\mu\psi, \\ \Rightarrow \quad \delta_\Lambda A_\mu &= \partial_\mu\Lambda - i[A_\mu \stackrel{\star}{,} \Lambda] \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} F_{\mu\nu} &= i[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu], \\ \Rightarrow \quad \delta_\Lambda F_{\mu\nu} &= i[\Lambda \stackrel{\star}{,} F_{\mu\nu}], \\ [\mathcal{D}_\lambda \stackrel{\star}{,} F_{\mu\nu}] + [\mathcal{D}_\mu \stackrel{\star}{,} F_{\nu\lambda}] + [\mathcal{D}_\nu \stackrel{\star}{,} F_{\lambda\mu}] &= 0, \quad \text{Bianchi identity.} \end{aligned} \tag{3.44}$$

The behaviour of A_μ under gauge transformations (3.43) shows that an enveloping algebra-valued gauge parameter directly implies an enveloping algebra-valued gauge potential and field strength. This situation is physically untenable, since it implies an infinite number of degrees of freedom of physical fields.

The only possibility to reduce this infinite number is to demand that all higher-order degrees of freedom, e.g. of $\Lambda^n_{a_1 a_2 \ldots a_{n+1}}$, depend on the degrees of freedom up to some fixed order. The most plausible solution is to demand that all $\Lambda^n_{a_1 a_2 \ldots a_{n+1}}$ for $n > 0$ depend on the degrees of freedom present of zeroth order, the Lie algebra gauge parameter $\Lambda^0_a T^a$. If such a reduction of degrees of freedom is possible, it means that the gauge theory on NC spaces can be related to and is determined entirely by the gauge theory on commutative space. Especially the number of degrees of freedom in this case would be identical to the commutative case.

If this reduction is possible, a *tower* in the enveloping algebra is defined. Taking into account the "size" of the enveloping algebra, the reduction from the full enveloping algebra to the tower is severe. In contrast to the string theory setting, there is no principle to ensure that the reduction to the commutative degrees of freedom is indeed possible. We have to perform actual calculations.

To this end we perform an explicit construction order by order in the parameter of the noncommutativity. The gauge parameter $\Lambda = \Lambda_\alpha$ of course has to depend on a given Lie algebra gauge parameter α as in (3.32), while A_μ depends on A^0_μ etc.

We use as a consistency condition that two consecutive gauge transformations have to close into another one. From this consistency condition we perform the construction of the enveloping algebra gauge theory. If the NC quantities depend on the Lie algebra quantities only, the gauge variation δ_{Λ_α} can be reduced to δ_α, since the gauge variation can be applied to each of the Lie algebra factors in the expansion of the NC quantities separately:

$$(\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1})\psi = \delta_{\Lambda_1\times\Lambda_2}\psi \quad \Rightarrow \quad (\delta_{\alpha_1}\delta_{\alpha_2} - \delta_{\alpha_2}\delta_{\alpha_1})\psi = \delta_{\alpha_1\times\alpha_2}\psi, \tag{3.45}$$

if α_i are the zeroth, Lie algebra components of the Λ_i.

Applying the consistency condition, we find that the commutative gauge potential A^0_μ appears in the expansion of all quantities of NC gauge theory. Therefore

$$\Lambda_\alpha := \Lambda[\alpha, A^0_\mu], \qquad A_\mu := A_\mu[A^0_\mu], \qquad \text{and} \quad F_{\mu\nu} := F_{\mu\nu}[F^0_{\mu\nu}, A^0_\mu], \tag{3.46}$$

where the square brackets denote functional dependence. In particular, these quantities depend on the Lie algebra quantities and an arbitrary number of derivatives on them. Still, the functionals are supposed to be local in the sense that at any finite order in the expansion, only a finite number of derivatives appears. To avoid notational clutter, we will keep this functional dependence implicit.

Since Λ_α depends on A^0_μ explicitly, a gauge transformation of Λ_α is not identical zero $\delta_\alpha\Lambda_\beta \neq 0$, while $\delta_\alpha\beta = 0$ is still valid. We have to include terms taking this into account in (3.45). Therefore the consistency relation for enveloping algebra-valued gauge theory is:

$$\begin{aligned} (\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = \delta_{-i[\alpha,\beta]}\psi &= \delta_{\alpha\times\beta}\psi, \\ \Leftrightarrow \quad i(\delta_\alpha\Lambda_\beta)\star\psi - i(\delta_\beta\Lambda_\alpha)\star\psi + \Lambda_\alpha\star\Lambda_\beta\star\psi - \Lambda_\beta\star\Lambda_\alpha\star\psi &= i\Lambda_{\alpha\times\beta}\star\psi, \end{aligned} \tag{3.47}$$

and since this equation must be true for all fields ψ:

$$i\delta_\alpha\Lambda_\beta - i\delta_\beta\Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{\alpha\times\beta}. \tag{3.48}$$

This consistency condition has the virtue of being an equation of the gauge parameter Λ_α alone. Once we find solutions for (3.48), it is possible to solve

$$\delta_\alpha\psi = i\Lambda_\alpha\star\psi, \tag{3.49}$$

$$\delta_\alpha A_\mu = \partial_\mu\Lambda_\alpha - i[A^\mu \stackrel{\star}{,} \Lambda_\alpha], \quad \text{etc.} \tag{3.50}$$

We assume that there is a "tower" in the representation of the enveloping algebra, such that ψ is a functional of the fields ψ^0 with the transformation property (3.34) and the gauge potential A^0_μ. These fields ψ may be in an arbitrary representation of the enveloping algebra gauge theory, as long as the infinitesimal gauge transformation is implemented by left $\star$-multiplication with the gauge parameter Λ_α (e.g. the fundamental representation).

The consistency condition (3.48) will be the starting point for our construction of a NC gauge theory. Since the gauge theory is driven into the enveloping algebra because of the NC $\star$-product, instead of (3.42) we expand Λ_α in terms of $\hbar$:

$$\Lambda_\alpha = \alpha + \hbar\Lambda^1_\alpha + \hbar^2\Lambda^2_\alpha + \dots . \tag{3.51}$$

The expansions (3.42) and (3.51) do not coincide. There are other suitable expansions, e.g. in terms of the number of factors of gauge potentials A^0_μ. This expansion allows some interesting all-orders summations [33], [73]. We will expand the $\star$-product order by order in the parameter of the noncommutativity, solve the resulting equation, reinsert the solution for calculating the next order etc.

Correspondingly, ψ, A_μ and $F_{\mu\nu}$ are expanded in terms of $\hbar$

$$\psi = \psi^0 + \hbar\psi^1 + \hbar^2\psi^2 + \dots , \tag{3.52}$$
$$A_\mu = A^0_\mu + \hbar A^1_\mu + \hbar^2 A^2_\mu + \dots , \quad \text{etc.} \tag{3.53}$$

3.3 θ-expanded solutions up to second order

The expansions (3.51), (3.52) and (3.53) are defined in such a way that (3.49) coincides with (3.34) and (3.50) with (3.37) in zeroth order. In addition, $\delta_\alpha\beta = 0$ and the $\star$-product to zeroth order is the point-wise product, therefore (3.48) is (3.35) to zeroth order.

We expand (3.48) to first order in $\hbar$:

$$i(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + [\alpha \stackrel{\star}{,} \beta]|_{\mathcal{O}(\hbar)} = i\Lambda^1_{\alpha\times\beta}. \tag{3.54}$$

Inserting the explicit form of the Moyal-Weyl $\star$-product, (3.54) is

$$i(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] - i\Lambda^1_{\alpha\times\beta} = -\frac{i\hbar}{2}\theta^{\mu\nu}\{\partial_\mu\alpha, \partial_\nu\beta\} . \tag{3.55}$$

To first order in $\hbar$ the NC structure contributes a term from the expansion of the $\star$-product, which prohibits Λ^1_α equal zero. Equation (3.55) is an inhomogeneous linear equation in Λ^1_α with the solution [33]:

$$\Lambda^1_\alpha = -\frac{1}{4}\theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\} + c\,\theta^{\mu\nu}[A^0_\mu, \partial_\nu\alpha], \tag{3.56}$$

That (3.56) solves (3.55) can be seen using $\delta_\alpha\beta = 0$ and $\delta_\alpha A^0_\mu = \partial_\mu\alpha - i[A^0_\mu, \alpha]$.

For hermitian fields A^0_μ and hermitian gauge parameters α this solution is hermitian only for purely imaginary c. If Λ^1_α is not required to be hermitian, c may reach arbitrary complex values. The part of (3.56) parametrised by c is the solution of the homogeneous equation

$$i(\delta_\alpha\Lambda^1_\beta - \delta_\beta\Lambda^1_\alpha) + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] - i\Lambda^1_{\alpha\times\beta} = 0, \tag{3.57}$$

and it has to be added to any special solution of the inhomogeneous equation (3.55). Homogeneous equations such as (3.57) appear at every order

$$\Delta\Lambda^k := i(\delta_\alpha\Lambda^k_\beta - \delta_\beta\Lambda^k_\alpha) + [\alpha, \Lambda^k_\beta] + [\Lambda^k_\alpha, \beta] - i\Lambda^k_{\alpha\times\beta} = 0. \tag{3.58}$$

We have introduced the shorthand $\Delta\Lambda^k$ (cp. [92]) for the terms which appear on the left hand side in the inhomogeneous equation (3.55) and in the homogeneous equation (3.57). The structure of these terms on the left hand side is identical at every order k in $\hbar$.

The solution (3.56) marks the first appearance of a freedom or ambiguity in the solutions of (3.48). We will defer the discussion of such ambiguities until section 3.6. For the rest of this section we choose the special solution $c = 0$. In a sense that will be discussed in section 3.4, there is a class of solutions, which may be called natural solutions; $c = 0$ is the natural solution in first order.

To second order in $\hbar$ (3.48) reads:

$$\Delta\Lambda^2 = \frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\mu\partial_\kappa\alpha, \partial_\nu\partial_\lambda\beta] - [\Lambda^1_\alpha, \Lambda^1_\beta] - \frac{i}{2}\theta^{\mu\nu}\Big(\{\partial_\mu\Lambda^1_\alpha, \partial_\nu\beta\} - \{\partial_\mu\Lambda^1_\beta, \partial_\nu\alpha\}\Big). \quad (3.59)$$

Using (3.56) with $c = 0$, we have found the following solution for (3.59) in [33]

$$\begin{aligned}\Lambda^2_\alpha \; = \; & \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(- 4\{\partial_\mu\alpha, \{A^0_\kappa, \partial_\lambda A^0_\nu\}\} - i\{\partial_\mu\alpha, \{A^0_\kappa, [A^0_\nu, A^0_\lambda]\}\} \\ & -i\{A^0_\nu, \{A^0_\lambda, [\partial_\mu\alpha, A^0_\kappa]\}\} + 2i[\partial_\mu\partial_\kappa\alpha, \partial_\nu A^0_\lambda] \\ & -2[\partial_\nu A^0_\lambda, [\partial_\mu\alpha, A^0_\kappa]] + 2i[[A^0_\nu, A^0_\lambda], [\partial_\mu\alpha, A^0_\kappa]]\Big).\end{aligned} \quad (3.60)$$

Similar solutions have been found independently in [71] and [72].

Adding a particular solution of the homogeneous equation in second order

$$\begin{aligned}\Lambda^2_{\alpha\ \mathrm{hom}} \; = \; & \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{\{A^0_\mu, \partial_\lambda A^0_\nu\}, \partial_\kappa\alpha\} + i\{A^0_\nu, \{A^0_\lambda, [\partial_\mu\alpha, A^0_\kappa]\}\} \\ & +\{A^0_\kappa, \{\partial_\mu\alpha, \partial_\lambda A^0_\nu\}\} + \{A^0_\mu, \{A^0_\kappa, \partial_\nu\partial_\lambda\alpha\}\} - 2[[A^0_\mu, \partial_\lambda\alpha], F^0_{\kappa\nu}]\Big),\end{aligned} \quad (3.61)$$

the solution (3.60) can be brought to the "natural" form:

$$\begin{aligned}\Lambda^{2'}_\alpha = \Lambda^2_\alpha + \Lambda^2_{\alpha\ \mathrm{hom}} \; = \; & \frac{1}{32}x^\rho x^\sigma C^{\mu\nu}_\rho C^{\kappa\lambda}_\sigma\Big(\{A^0_\mu, \{\partial_\nu A^0_\kappa, \partial_\lambda\alpha\}\} + \{A^0_\mu, \{A^0_\kappa, \partial_\nu\partial_\lambda\alpha\}\} \\ & +\{\{A^0_\mu, \partial_\nu A^0_\kappa\}, \partial_\lambda\alpha\}\} - \{\{F^0_{\mu\kappa}, A^0_\nu\}, \partial_\lambda\alpha\} - 2i[\partial_\mu A^0_\kappa, \partial_\nu\partial_\lambda\alpha]\Big).\end{aligned} \quad (3.62)$$

With these solutions for Λ_α up to second order, we determine the field ψ (3.52) from (3.49) up to second order. To first order in $\hbar$ we obtain:

$$\Delta_\alpha\psi^1 := \delta_\alpha\psi^1 - i\alpha\psi^1 = i\Lambda^1_\alpha\psi^0 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi^0. \quad (3.63)$$

Note the definition of the operator Δ_α, it is defined for any ψ^k: $\Delta_\alpha\psi^k = \delta_\alpha\psi^k - i\alpha\psi^k$. Using solution (3.56) for Λ^1_α, we find

$$\psi^1 = -\frac{1}{2}\theta^{\mu\nu}A^0_\mu\partial_\nu\psi^0 + \frac{i}{4}\theta^{\mu\nu}A^0_\mu A^0_\nu\psi^0. \quad (3.64)$$

Similarly the next order,

$$\Delta_\alpha\psi^2 = i\Lambda^2_\alpha\psi^0 + i\Lambda^1_\alpha\psi^1 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\Lambda^1_\alpha\partial_\nu\psi^0 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi^1 - \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa\alpha\partial_\nu\partial_\lambda\psi^0, \quad (3.65)$$

is analysed. In [33] we have found a solution, using (3.56) for Λ^1_α and (3.60) for Λ^2_α:

$$\begin{aligned}\psi^2 &= \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(-4i\partial_\mu A^0_\kappa\partial_\nu\partial_\lambda\psi^0 + 4A^0_\mu A^0_\kappa\partial_\nu\partial_\lambda\psi^0 + 8A^0_\mu\partial_\nu A^0_\kappa\partial_\lambda\psi^0 \\ &\quad -4A^0_\mu\partial_\kappa A^0_\nu\partial_\lambda\psi^0 - 4iA^0_\mu A^0_\nu A^0_\kappa\partial_\lambda\psi^0 + 4iA^0_\kappa A^0_\nu A^0_\mu\partial_\lambda\psi^0 - 4iA^0_\nu A^0_\kappa A^0_\mu\partial_\lambda\psi^0 \\ &\quad +4\partial_\nu A^0_\kappa A^0_\mu\partial_\lambda\psi^0 - 2\partial_\mu A^0_\kappa\partial_\nu A^0_\lambda\psi^0 + 4iA^0_\mu A^0_\lambda\partial_\kappa A^0_\nu\psi^0 + 4iA^0_\mu\partial_\kappa A^0_\nu A^0_\lambda\psi^0 \\ &\quad -4iA^0_\mu\partial_\nu A^0_\kappa A^0_\lambda\psi^0 + 3A^0_\mu A^0_\nu A^0_\lambda A^0_\kappa\psi^0 + 4A^0_\mu A^0_\kappa A^0_\nu A^0_\lambda\psi^0 + 2A^0_\mu A^0_\lambda A^0_\kappa A^0_\nu\psi^0\Big).\end{aligned} \tag{3.66}$$

Again we can add a solution of a homogeneous equation $\Delta_\alpha\psi^2 = 0$ to arrive at the following solution, which can be obtained directly using (3.56) with $c = 0$ for Λ^1_α and (3.62) for Λ^2_α:

$$\begin{aligned}\psi^{2'} &= -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\partial_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 + iA^0_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 - i\partial_\kappa A^0_\mu A^0_\nu\partial_\lambda\psi^0 + iF^0_{\kappa\mu}A^0_\nu\partial_\lambda\psi^0 \\ &\quad -iA^0_\nu\partial_\kappa A^0_\mu\partial_\lambda\psi^0 + 2iA^0_\nu F^0_{\kappa\mu}\partial_\lambda\psi^0 + 2A^0_\mu A^0_\kappa A^0_\nu\partial_\lambda\psi^0 - A^0_\mu A^0_\nu A^0_\kappa\partial_\lambda\psi^0\Big) \\ &\quad -\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(2\partial_\kappa A^0_\mu\partial_\lambda A^0_\nu\psi^0 - 2i\partial_\kappa A^0_\mu A^0_\lambda A^0_\nu\psi^0 + 2iA^0_\nu A^0_\lambda\partial_\kappa A^0_\mu\psi^0 \\ &\quad +i[[\partial_\kappa A^0_\mu, A^0_\nu], A^0_\lambda]\psi^0 + 4iA^0_\nu F^0_{\kappa\mu}A^0_\lambda\psi^0 - A^0_\kappa A^0_\lambda A^0_\mu A^0_\nu\psi^0 + 2A^0_\kappa A^0_\mu A^0_\nu A^0_\lambda\psi^0\Big).\end{aligned} \tag{3.67}$$

The conjugate field $\overline{\psi} = \psi^\dagger\gamma^0$ is obtained by conjugation of ψ, assuming that A^0_μ is hermitian. For example, $\overline{\psi}^0 = \overline{\psi^0}$ and $\overline{\psi}^1 = \overline{\psi^1}$:

$$\overline{\psi}^1 = \frac{1}{2}\theta^{\mu\nu}\partial_\mu\overline{\psi}^0 A^0_\nu + \frac{i}{4}\theta^{\mu\nu}\overline{\psi}^0 A^0_\mu A^0_\nu. \tag{3.68}$$

The enveloping algebra gauge potential (3.53) is determined by expanding (3.50):

$$\begin{aligned}\Delta_\alpha A^1_\sigma &= \delta_\alpha A^1_\sigma - i[\alpha, A^1_\sigma] = \partial_\sigma\Lambda^1_\alpha - i[A^0_\sigma, \Lambda^1_\alpha] + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^0_\sigma, \partial_\nu\alpha\}, \\ \Delta_\alpha A^2_\sigma &= \partial_\sigma\Lambda^2_\alpha - i[A^0_\sigma, \Lambda^2_\alpha] - i[A^1_\sigma, \Lambda^1_\alpha] + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^1_\sigma, \partial_\nu\alpha\} \\ &\quad +\frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^0_\sigma, \partial_\nu\Lambda^1_\alpha\} + \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\kappa\partial_\mu A^0_\sigma, \partial_\lambda\partial_\nu\alpha].\end{aligned} \tag{3.69}$$

Note the definition $\Delta_\alpha A^k_\sigma = \delta_\alpha A^k_\sigma - i[\alpha, A^k_\sigma]$. A hermitian solution for A^1_σ with Λ^1_α as in (3.56) is:

$$A^1_\sigma = -\frac{1}{4}\theta^{\mu\nu}\Big(\{A^0_\mu, \partial_\nu A^0_\sigma\} - \{F^0_{\mu\sigma}, A^0_\nu\}\Big), \tag{3.70}$$

with $F^0_{\mu\nu}$ the Lie algebra field strength (3.38). In [33] we have constructed A^2_σ with Λ^1_α as in (3.56) and Λ^2_α as in (3.60)

$$\begin{aligned}A^2_\sigma &= \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(4i[\partial_\kappa\partial_\mu A^0_\sigma, \partial_\lambda A^0_\nu] - 2i[\partial_\kappa\partial_\sigma A^0_\mu, \partial_\lambda A^0_\nu] + 4\{A^0_\kappa, \{A^0_\mu, \partial_\nu F^0_{\lambda\sigma}\}\} \\ &\quad +2[[\partial_\kappa A^0_\mu, A^0_\sigma], \partial_\lambda A^0_\nu] - 4\{\partial_\lambda A^0_\sigma, \{\partial_\mu A^0_\kappa, A^0_\nu\}\} + 4\{A^0_\kappa, \{F^0_{\lambda\mu}, F^0_{\nu\sigma}\}\} \\ &\quad -i\{\partial_\sigma A^0_\nu, \{A^0_\lambda, [A^0_\mu, A^0_\kappa]\}\} - i\{A^0_\mu, \{A^0_\kappa, [\partial_\sigma A^0_\nu, A^0_\lambda]\}\} \\ &\quad +4i[[A^0_\mu, A^0_\lambda], [A^0_\kappa, \partial_\nu A^0_\sigma]] - 2i[[A^0_\mu, A^0_\lambda], [A^0_\kappa, \partial_\sigma A^0_\nu]] \\ &\quad +\{A^0_\kappa, \{[A^0_\lambda, A^0_\mu], [A^0_\nu, A^0_\sigma]\}\} + [[A^0_\mu, A^0_\lambda], [A^0_\kappa, [A^0_\nu, A^0_\sigma]]] \\ &\quad -\{A^0_\mu, \{A^0_\kappa, [A^0_\lambda, [A^0_\nu, A^0_\sigma]]\}\}\Big).\end{aligned} \tag{3.71}$$

Adding solutions of the homogeneous equation $\Delta_\alpha A^2_\sigma = 0$, we obtain a solution which can be obtained directly using (3.56) and (3.62):

$$\begin{aligned} A^{2'}_\sigma \;=\; & \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{\{A^0_\kappa,\partial_\lambda A^0_\mu\},\partial_\nu A^0_\sigma\} - \{\{F^0_{\kappa\mu},A^0_\lambda\},\partial_\nu A^0_\sigma\} - 2i[\partial_\kappa A^0_\mu,\partial_\lambda\partial_\nu A^0_\sigma] \\ & -\{A^0_\mu,\{\partial_\nu F^0_{\kappa\sigma},A^0_\lambda\}\} - \{A^0_\mu,\{F^0_{\kappa\sigma},\partial_\nu A^0_\lambda\}\} + \{A^0_\mu,\{\partial_\nu A^0_\kappa,\partial_\lambda A^0_\sigma\}\} \\ & +\{A^0_\mu,\{A^0_\kappa,\partial_\nu\partial_\lambda A^0_\sigma\}\} + \{\{A^0_\kappa,\partial_\lambda F^0_{\mu\sigma}\},A^0_\nu\} - \{\{\mathcal{D}^0_\kappa F^0_{\mu\sigma},A^0_\lambda\},A^0_\nu\} \\ & -2\{\{F^0_{\mu\kappa},F^0_{\sigma\lambda}\},A^0_\nu\} + 2i[\partial_\kappa F^0_{\mu\sigma},\partial_\lambda A^0_\nu] - \{F^0_{\mu\sigma},\{A^0_\kappa,\partial_\lambda A^0_\nu\}\} + \{F^0_{\mu\sigma},\{F^0_{\kappa\nu},A^0_\lambda\}\}\Big). \end{aligned} \tag{3.72}$$

The Lie algebra covariant derivative $\mathcal{D}^0_\sigma$ used in this solution has been introduced in (3.37). The gauge potential A_σ allows the definition of an enveloping algebra-valued covariant derivative according to (3.43)

$$\mathcal{D}_\sigma\psi = \partial_\sigma\psi - iA_\sigma \star \psi. \tag{3.73}$$

Since the derivative ∂_σ is undeformed, we only have to add the expanded solution for A_σ.

In order to express the enveloping algebra-valued field strength

$$F_{\rho\sigma} = i[\mathcal{D}_\rho,\mathcal{D}_\sigma] = \partial_\rho A_\sigma - \partial_\sigma A_\rho - i[A_\rho \stackrel{\star}{,} A_\sigma], \tag{3.74}$$

in terms of Lie algebra-valued quantities, we insert (3.70) and (3.72) into (3.74). To first order in $\hbar$ we obtain:

$$F^1_{\rho\sigma} = -\frac{1}{4}\theta^{\kappa\lambda}\Big(\{A^0_\kappa,\partial_\lambda F^0_{\rho\sigma}\} - \{\mathcal{D}^0_\kappa F^0_{\rho\sigma},A^0_\lambda\} - 2\{F^0_{\rho\kappa},F^0_{\sigma\lambda}\}\Big). \tag{3.75}$$

We could have used the covariant transformation behaviour $\delta_\alpha F_{\rho\sigma} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\rho\sigma}]$ to construct the field strength. This reproduces only the first two terms in (3.75). The third, fully covariant term is a specific solution for $F^1_{\rho\sigma}$ as in (3.74).

In second order in $\hbar$, the field strength $F^2_{\rho\sigma}$ is calculated from A^2_σ as in (3.72):

$$\begin{aligned} F^2_{\rho\sigma} \;=\; & \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(2\{\{A^0_\mu,\partial_\nu A^0_\kappa\},\partial_\lambda F^0_{\rho\sigma}\} - 2\{\{F^0_{\mu\kappa},A^0_\nu\},\partial_\lambda F^0_{\rho\sigma}\} + 2\{A^0_\kappa,\partial_\lambda\{A^0_\mu,\partial_\nu F^0_{\rho\sigma}\}\} \\ & -2\{A^0_\kappa,\partial_\lambda\{\mathcal{D}^0_\mu F^0_{\rho\sigma},A^0_\nu\}\} - 4\{A^0_\kappa,\partial_\lambda\{F^0_{\rho\mu},F^0_{\sigma\nu}\}\} - 4i[\partial_\mu A^0_\kappa,\partial_\nu\partial_\lambda F^0_{\rho\sigma}] \\ & +i\{[\{A^0_\mu,\partial_\nu A^0_\kappa\},F^0_{\rho\sigma}],A^0_\lambda\} - i\{[\{F^0_{\mu\kappa},A^0_\nu\},F^0_{\rho\sigma}],A^0_\lambda\} \\ & +i\{[A^0_\kappa,\{A^0_\mu,\partial_\nu F^0_{\rho\sigma}\}],A^0_\lambda\} - i\{[A^0_\kappa,\{\mathcal{D}^0_\mu F^0_{\rho\sigma},A^0_\nu\}],A^0_\lambda\} \\ & -2i\{[A^0_\kappa,\{F^0_{\rho\mu},F^0_{\sigma\nu}\}],A^0_\lambda\} + i\{[A^0_\kappa,F^0_{\rho\sigma}],\{A^0_\mu,\partial_\nu A^0_\lambda\}\} \\ & -i\{[A^0_\kappa,F^0_{\rho\sigma}],\{F^0_{\mu\lambda},A^0_\nu\}\} + 2\{\{\partial_\mu A^0_\kappa,\partial_\nu F^0_{\rho\sigma}\},A^0_\lambda\} \\ & +2[\partial_\mu[A^0_\kappa,F^0_{\rho\sigma}],\partial_\nu A^0_\lambda] + 2\{\{\mathcal{D}^0_\mu F^0_{\rho\kappa},A^0_\nu\},F^0_{\sigma\lambda}\} - 2\{\{A^0_\mu,\partial_\nu F^0_{\rho\kappa}\},F^0_{\sigma\lambda}\} \\ & +4\{\{F^0_{\mu\rho},F^0_{\nu\kappa},\},F^0_{\sigma\lambda}\} + 2\{F^0_{\rho\kappa},\{\mathcal{D}^0_\mu F^0_{\sigma\lambda},A^0_\nu\}\} - 2\{F^0_{\rho\kappa},\{A^0_\mu,\partial_\nu F^0_{\sigma\lambda}\}\} \\ & +4\{F^0_{\rho\kappa},\{F^0_{\mu\sigma},F^0_{\nu\lambda}\}\} + 4i[\partial_\mu F^0_{\rho\kappa},\partial_\nu F^0_{\sigma\lambda}]\Big). \end{aligned} \tag{3.76}$$

While all previous results have been worked out by hand from scratch, (3.76) has been guessed in the framework presented in the subsequent section, and checked afterwards.

In section 3.1 we have discussed covariant coordinates and covariant functions. To round off the presentation, we also quote briefly the results for θ-expanded covariant functions. As stated in section 3.1, the covariant coordinate $X^\mu = x^\mu + \mathcal{A}^\mu$ is determined from

$$\delta_\alpha X^\mu \star \psi \stackrel{!}{=} i\Lambda_\alpha \star X^\mu \star \psi, \quad \Rightarrow \quad \delta_\alpha \mathcal{A}^\mu = -i[x^\mu \stackrel{\star}{,} \Lambda_\alpha] - i[\mathcal{A}^\mu \stackrel{\star}{,} \Lambda_\alpha]. \tag{3.77}$$

Since $[x^\mu \stackrel{\star}{,} \Lambda_\alpha] = -\theta^{\mu\nu}\partial_\nu\Lambda_\alpha$, obviously $\mathcal{A}^\mu = -\theta^{\mu\nu}A_\nu$ with A_ν the all orders solution for the gauge potential derived in this and the previous section.

Multiplying a function $f(x)$ with $\delta_\alpha f(x) = 0$ from the left to a module of the gauge group spoils covariance. But the covariant function $\mathcal{D}(f)$ of f, constructed as the function plus a covariantising function f_A, reconstitutes gauge covariance

$$\delta_\alpha(\mathcal{D}(f) \star \psi) = i\Lambda_\alpha \star \mathcal{D}(f) \star \psi. \tag{3.78}$$

This implies the transformation law for f_A:

$$\delta_\alpha f_A = -i[f \stackrel{\star}{,} \Lambda_\alpha] - i[f_A \stackrel{\star}{,} \Lambda_\alpha]. \tag{3.79}$$

The solution for the covariant function is therefore up to second order:

$$f \to \mathcal{D}(f) = f + f_A = f - \hbar\theta^{\mu\nu}A^0_\mu\partial_\nu f - \hbar^2\theta^{\mu\nu}A^1_\mu\partial_\nu f + \frac{\hbar^2}{4}\theta^{\mu\nu}\theta^{\rho\sigma}\{A^0_\mu, A^0_\rho\}\partial_\nu\partial_\sigma f + \dots \ . \tag{3.80}$$

where A^1_μ is as in (3.92). At third order we obtain for f_A

$$\begin{aligned} f^3_A \ = \ & -\theta^{\mu\nu}A^2_\mu\partial_\nu f + \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}\{A^1_\mu, A^0_\rho\}\partial_\nu\partial_\sigma f + \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}\{A^0_\mu, A^1_\rho\}\partial_\nu\partial_\sigma f \\ & + \frac{1}{4}\theta^{\mu\nu}\theta^{\rho\sigma}\{A^0_\mu \stackrel{\star^1}{,} A^0_\rho\}\partial_\nu\partial_\sigma f - \frac{1}{6}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}A^0_\mu A^0_\rho A^0_\kappa\partial_\nu\partial_\sigma\partial_\lambda f \\ & + \frac{i}{12}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}[\partial_\mu A^0_\kappa, A^0_\rho]\partial_\nu\partial_\sigma\partial_\lambda f + \frac{1}{24}\theta^{\mu\nu}\theta^{\rho\sigma}\theta^{\kappa\lambda}\partial_\mu\partial_\rho A^0_\kappa\partial_\nu\partial_\sigma\partial_\lambda f + \dots \ . \end{aligned} \tag{3.81}$$

Ignoring the terms in the last line, this third order solution seems to indicate an all-orders solution

$$\mathcal{D}(f) \sim e_\star^{-\hbar\theta^{\mu\nu}A_\mu\partial_\nu} f.$$

The $\star$-exponential has been defined in (3.12) and the derivatives appearing in the exponential only act on the function f. The last term in (3.81) arises due to the third order term in the expansion of the $\star$-product. This term can be accommodated easily using the product $\star_2$:

$$\mathcal{D}(f) \sim e_\star^{-\hbar\theta^{\mu\nu}A_\mu\partial_\nu} \star_2 f, \quad \text{with} \quad (f \star_2 g)(x) = \mathsf{m}\Big(\frac{\sin(\frac{i\hbar}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu)}{\frac{i\hbar}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} f(x) \otimes g(x)\Big).$$

This product appears in many places in NC gauge theory, e.g. [64], [74], [73], and its appearance can be understood because only odd powers of the $\star$-product contribute to the covariant function. $\star_2$ is not a $\star$-product itself. However, even with this addition, the first term in the last row of (3.81) does not fit into the condensed notation, the all-orders solution for the covariant function will be even more complicated:

$$\mathcal{D}(f) = e_\star^{-\hbar\theta^{\mu\nu}A_\mu\partial_\nu} \star_2 f + \mathcal{O}(\hbar^3). \tag{3.82}$$

3.4 Structure of θ-expanded gauge theory

The solutions (3.56), (3.60) for the gauge parameter Λ_α and (3.70), (3.71) for the gauge potential A_σ have been worked out in tedious manual calculations. Thus, it has been shown that indeed up to second order, the construction of enveloping algebra gauge theory is possible. However, the structure of these solutions is quite in-transparent. We have quoted the alternative solutions (3.62) and (3.72) related to the first set by solutions of the homogeneous equations. We will now show that this second set of solutions has a rigid structure. The following derivation has been sketched in [75], but only for Λ_α (a different discussion of the order by order solution is contained in [76]). This structure is derived from the Seiberg-Witten flow equations (stated in section 3.1):

$$\frac{\partial \Lambda_\alpha}{\partial \hbar} = -\frac{1}{4}\theta^{\mu\nu}\{A_\mu \overset{\star}{,} \partial_\nu \Lambda_\alpha\}. \tag{3.83}$$

This differential equation (3.83) describes the change in the gauge parameter along a trajectory in θ-space, i.e. for varying noncommutativity $\theta^{\mu\nu}(\hbar)$ (in the language of equivalent $\star$-products). We use this flow equation for a Taylor expansion of Λ_α around $\alpha = \Lambda^0_\alpha$:

$$\Lambda_\alpha = \alpha + \hbar\Lambda^1_\alpha + \hbar^2\Lambda^2_\alpha + \cdots = \alpha + \hbar\frac{\partial \Lambda_\alpha}{\partial \hbar}|_{\hbar=0} + \frac{1}{2!}\hbar^2\frac{\partial^2 \Lambda_\alpha}{\partial \hbar^2}|_{\hbar=0} + \ldots \ . \tag{3.84}$$

Therefore we obtain

$$\Lambda^1_\alpha = \frac{\partial \Lambda_\alpha}{\partial \hbar}|_{\hbar=0} = -\frac{1}{4}\theta^{\mu\nu}\{A_\mu \overset{\star}{,} \partial_\nu\Lambda_\alpha\}|_{\hbar=0} = -\frac{1}{4}\theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\}, \tag{3.85}$$

$$\begin{aligned}\Lambda^2_\alpha &= \frac{1}{2}\Big(\frac{\partial^2\Lambda_\alpha}{\partial\hbar^2}\Big)_{\hbar=0} = \frac{1}{2}\Big(\frac{\partial}{\partial\hbar}\frac{\partial\Lambda_\alpha}{\partial\hbar}\Big)_{\hbar=0} = -\frac{1}{8}\theta^{\mu\nu}\Big(\frac{\partial}{\partial\hbar}\{A_\mu \overset{\star}{,} \partial_\nu\Lambda_\alpha\}\Big)_{\hbar=0} = \\ &= -\frac{1}{8}\theta^{\mu\nu}\Big(\{\frac{\partial A_\mu}{\partial\hbar} \overset{\star}{,} \Lambda_\alpha\} + \{A_\mu(\frac{\partial}{\partial\hbar}\overset{\star}{,})\Lambda_\alpha\} + \{A_\mu \overset{\star}{,} \frac{\partial\Lambda_\alpha}{\partial\hbar}\}\Big)_{\hbar=0} \\ &= -\frac{1}{8}\theta^{\mu\nu}\Big(\{A^1_\mu, \alpha\} + \{A^0_\mu \overset{\star^1}{,} \alpha\} + \{A^0_\mu, \Lambda^1_\alpha\}\Big),\end{aligned} \tag{3.86}$$

and similarly

$$\begin{aligned}\Lambda^3_\alpha &= \frac{1}{3!}\Big(\frac{\partial^3\Lambda_\alpha}{\partial\hbar^3}\Big)_{\hbar=0} = -\frac{1}{24}\theta^{\mu\nu}\Big(\{\frac{\partial^2 A_\mu}{\partial\hbar^2} \overset{\star}{,} \Lambda_\alpha\} + \{A_\mu(\frac{\partial^2}{\partial\hbar^2}\overset{\star}{,})\Lambda_\alpha\} + \{A_\mu \overset{\star}{,} \frac{\partial^2\Lambda_\alpha}{\partial\hbar^2}\} + \\ &+2\{\frac{\partial A_\mu}{\partial\hbar}(\frac{\partial}{\partial\hbar}\overset{\star}{,})\Lambda_\alpha\} + 2\{\frac{\partial A_\mu}{\partial\hbar} \overset{\star}{,} \frac{\partial\Lambda_\alpha}{\partial\hbar}\} + 2\{A_\mu(\frac{\partial}{\partial\hbar}\overset{\star}{,})\frac{\partial\Lambda_\alpha}{\partial\hbar}\}\Big)_{\hbar=0}.\end{aligned} \tag{3.87}$$

We have introduced the convenient notation:

$$f(x) \star^1 g(x) = f(x)\big(\frac{\partial}{\partial\hbar}\star\big)g(x)_{\hbar=0} = \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x). \tag{3.88}$$

And similarly for all higher order terms

$$f(x)\star g(x) = f(x)\cdot g(x) + \hbar f(x)\star^1 g(x) + \hbar^2 f(x)\star^2 g(x) + \cdots \ . \tag{3.89}$$

Equations (3.86) and (3.87) depend on derivatives of A_σ w.r.t. $\hbar$, therefore these equations are not solvable by themselves, but only in combination with a similar flow equation for the NC gauge potential A_σ:

$$\frac{\partial A_\sigma}{\partial \hbar} = -\frac{1}{4}\theta^{\mu\nu}(\{A_\mu \stackrel{\star}{,} \partial_\nu A_\sigma\} - \{F_{\mu\sigma} \stackrel{\star}{,} A_\nu\}). \tag{3.90}$$

Taylor expanding the gauge potential in $\hbar$, we want to identify the Taylor expansion coefficients with the solutions found in section 3.3:

$$A_\sigma = A_\sigma^0 + \hbar A_\sigma^1 + \hbar^2 A_\sigma^2 + \cdots = A_\sigma^0 + \hbar \frac{\partial A_\sigma}{\partial \hbar}|_{\hbar=0} + \frac{1}{2!}\hbar^2 \frac{\partial^2 A_\sigma}{\partial \hbar^2}|_{\hbar=0} + \dots\ . \tag{3.91}$$

Therefore we obtain

$$\begin{aligned} A_\sigma^1 &= \frac{\partial A_\sigma}{\partial \hbar}|_{\hbar=0} = -\frac{1}{4}\theta^{\mu\nu}\Big(\{A_\mu \stackrel{\star}{,} \partial_\nu A_\sigma\} - \{F_{\mu\sigma} \stackrel{\star}{,} A_\nu\}\Big)_{\hbar=0} \\ &= -\frac{1}{4}\theta^{\mu\nu}\Big(\{A_\mu^0, \partial_\nu \alpha\} - \{F_{\mu\sigma}^0, A_\nu^0\}\Big), \end{aligned} \tag{3.92}$$

$$\begin{aligned} A_\sigma^2 &= \frac{1}{2}\Big(\frac{\partial^2 A_\sigma}{\partial \hbar^2}\Big)_{\hbar=0} = -\frac{1}{8}\theta^{\mu\nu}\Big(\frac{\partial}{\partial \hbar}(\{A_\mu \stackrel{\star}{,} \partial_\nu A_\sigma\} - \{F_{\mu\sigma} \stackrel{\star}{,} A_\nu\})\Big)_{\hbar=0} \\ &= -\frac{1}{8}\theta^{\mu\nu}\Big(\{A_\mu^1, \partial_\nu A_\sigma^0\} + \{A_\mu^0 \stackrel{\star^1}{,} \partial_\nu A_\sigma^0\} + \{A_\mu^0, \partial_\nu A_\sigma^1\} \\ &\qquad -\{F_{\mu\sigma}^1, A_\nu^0\} - \{F_{\mu\sigma}^0 \stackrel{\star^1}{,} A_\nu^0\} - \{F_{\mu\sigma}^0, A_\nu^1\}\Big). \end{aligned} \tag{3.93}$$

In summary, the result obtained from the coupled flow equations is related to the solutions found in 3.3, identifying

$$\begin{aligned} \frac{\partial \Lambda_\alpha}{\partial \hbar}|_{\hbar=0} &= \Lambda_\alpha^1, & \frac{\partial^2 \Lambda_\alpha}{\partial \hbar^2}|_{\hbar=0} &= 2\Lambda_\alpha^2, & &\dots, \\ \frac{\partial A_\sigma}{\partial \hbar}|_{\hbar=0} &= A_\sigma^1, & \frac{\partial^2 A_\sigma}{\partial \hbar^2}|_{\hbar=0} &= 2A_\sigma^2, & &\dots, \\ \frac{\partial \star}{\partial \hbar}|_{\hbar=0} &= \star^1, & \frac{\partial^2 \star}{\partial \hbar^2}|_{\hbar=0} &= 2\star^2, & &\dots\ . \end{aligned} \tag{3.94}$$

The result (3.56) (for $c = 0$) is obviously identical to the result derived in this section for Λ_α^1. The result (3.62) can be assembled into

$$\begin{aligned} \Lambda_\alpha^2 &= \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{A_\mu^0, \{\partial_\nu A_\kappa^0, \partial_\lambda \alpha\}\} + \{A_\mu^0, \{A_\kappa^0, \partial_\nu \partial_\lambda \alpha\}\} + \{\{A_\mu^0, \partial_\nu A_\kappa^0\}, \partial_\lambda \alpha\}\} \\ &\qquad -\{\{F_{\mu\kappa}^0, A_\nu^0\}, \partial_\lambda \alpha\} - 2i[\partial_\mu A_\kappa^0, \partial_\nu \partial_\lambda \alpha]\Big) \\ &= -\frac{1}{8}\,\theta^{\mu\nu}\Big(\{A_\mu^0, \partial_\nu \Lambda_\alpha^1\} + \{A_\mu^1, \partial_\nu \alpha\} + \{A_\mu^0 \stackrel{\star^1}{,} \partial_\nu \alpha\}\Big) = -\frac{1}{8}\,\theta^{\mu\nu} \sum_{\substack{i,j,k=0 \\ i+j+k=1}}^{1} \{A_\mu^i \stackrel{\star^j}{,} \partial_\nu \Lambda_\alpha^k\}. \end{aligned} \tag{3.95}$$

The last expression is a symbolic short hand which should be read in the following way: recursively insert the particular solution (3.56) for Λ_α^1, for the $\star$-product and (3.60) for A_μ^1 and expand. Apply partial derivatives on products according to the Leibniz rule.

Similarly we find for Λ^3_α:

$$\begin{aligned}\Lambda^3_\alpha &= -\frac{1}{12}\,\theta^{\mu\nu}\Big(\{A^2_\mu,\partial_\nu\alpha\} + A^0_\mu \overset{\star^2}{,}\, \partial_\nu\alpha\} + \{A^0_\mu,\partial_\nu\Lambda^2_\alpha\} \\ &\qquad +\{A^0_\mu \overset{\star^1}{,}\, \partial_\nu\Lambda^1_\alpha\} + \{A^1_\mu \overset{\star^1}{,}\, \partial_\nu\alpha\} + \{A^1_\mu,\partial_\nu\Lambda^1_\alpha\}\Big) \\ &= -\frac{1}{12}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=2}}^{2}\{A^i_\mu \overset{\star^j}{,}\, \partial_\nu\Lambda^k_\alpha\}.\end{aligned} \tag{3.96}$$

We have checked explicitly that this choice of Λ^3_α fulfils the consistency condition. Therefore the solution of the enveloping algebra gauge potential can be written in a condensed way, to arbitrary order:

$$\Lambda^n_\alpha = -\frac{1}{4n}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1}\{A^i_\mu \overset{\star^j}{,}\, \partial_\nu\Lambda^k_\alpha\}. \tag{3.97}$$

We emphasise that this result has not been proven in the constructive enveloping algebra approach. But we have checked that the results up to third order coincide up to solutions of homogeneous equations.

Similarly, the result (3.70) is identical to A^1_σ derived in this section and (3.72) can be rewritten in terms of

$$\begin{aligned}A^2_\sigma &= -\frac{1}{8}\,\theta^{\mu\nu}\Big(\{A^1_\mu,\partial_\nu A^0_\sigma\} + \{A^0_\mu \overset{\star^1}{,}\, \partial_\nu A^0_\sigma\} + \{A^0_\mu,\partial_\nu A^1_\sigma\} \\ &\qquad -\{F^1_{\mu\sigma},A^0_\nu\} - \{F^0_{\mu\sigma} \overset{\star^1}{,}\, A^0_\nu\} - \{F^0_{\mu\sigma}A^1_\nu\}\Big) \\ &= -\frac{1}{8}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=1}}^{1}\Big(\{A^i_\mu \overset{\star^j}{,}\, \partial_\nu A^k_\sigma\} - \{F^i_{\mu\sigma} \overset{\star^j}{,}\, A^k_\nu\}\Big).\end{aligned} \tag{3.98}$$

This formula for A^2_σ generalises for higher order A^n_σ as well, we obtain

$$A^n_\sigma = -\frac{1}{4n}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1}\Big(\{A^i_\mu \overset{\star^j}{,}\, \partial_\nu A^k_\sigma\} - \{F^i_{\mu\sigma} \overset{\star^j}{,}\, A^k_\nu\}\Big). \tag{3.99}$$

The two expressions (3.97) and (3.99) for Λ^n_α and A^n_σ are the compactest formulation of the n-th order in the θ-expanded gauge parameter and the gauge potential, these are what we have called the "natural" solution in the previous section. They allow a recursive calculation to arbitrary order. However, in contrast to the approach in the previous section, both quantities have to be solved in parallel. The ansatz using the consistency condition (3.48) allows a solution in terms of the Λ_α alone.

The enveloping algebra-valued field strength

$$F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho - i[A_\rho \overset{\star}{,}\, A_\sigma], \tag{3.100}$$

with the transformation property

$$\delta_\alpha F_{\rho\sigma} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\rho\sigma}], \tag{3.101}$$

is needed as well as a recursive expression, since $F^i_{\rho\sigma}$ appears explicitly in the formal expression (3.99). This recursive construction has not been performed in the literature so far. To expand the field strength, we insert the result (3.99) into (3.100). To obtain a recursion formula we need the full solution (3.99). In first order we obtain:

$$F^1_{\rho\sigma} = -\frac{1}{4}\theta^{\mu\nu}\Big(\{A^0_\mu, \partial_\nu F^0_{\rho\sigma}\} - \{\mathcal{D}^0_\mu F^0_{\rho\sigma}, A^0_\nu\} - 2\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}\Big). \tag{3.102}$$

The solution for the field strength $F^2_{\rho\sigma}$ to second order indicates the general solution to arbitrary order:

$$F^2_{\rho\sigma} = -\frac{1}{8}\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=1}}^{1}\Big(\{A^i_\mu \stackrel{\star^j}{,} \partial_\nu F^k_{\rho\sigma}\} - 2\{F^i_{\rho\mu} \stackrel{\star^j}{,} F^k_{\sigma\nu}\}\Big) + \frac{1}{8}\theta^{\mu\nu}\sum_{\substack{i,j,k,l=0\\ i+j+k+l=1}}^{1}\{\mathcal{D}^i_\mu F^j_{\rho\sigma} \stackrel{\star^k}{,} A^l_\nu\}. \tag{3.103}$$

We have used the $\star$-Bianchi identity for enveloping algebra field strengths:

$$\mathcal{D}_\nu F_{\rho\sigma} + \mathcal{D}_\rho F_{\sigma\nu} + \mathcal{D}_\sigma F_{\nu\rho} = 0. \tag{3.104}$$

Different powers of $\hbar$ define a grading, therefore there are Bianchi identities for every order in $\hbar$. For example, the first order Bianchi identity reads

$$\partial_\nu F^1_{\rho\sigma} - i[A^0_\nu, F^1_{\rho\sigma}] - i[A^1_\nu, F^0_{\rho\sigma}] - i[A^0_\nu \stackrel{\star^1}{,} F^0_{\rho\sigma}] + (\text{cycl.perm.}) = 0. \tag{3.105}$$

To obtain a closed formula to all orders, A_ρ as in (3.99) is inserted into (3.100):

$$\begin{aligned}
F^n_{\rho\sigma} &= \partial_\rho A^n_\sigma - \partial_\sigma A^n_\rho - i\sum_{\substack{i,j,k=0\\ i+j+k=n}}^{n}[A^i_\rho \stackrel{\star^j}{,} A^k_\sigma] \\
&= -\frac{1}{4n}\,\theta^{\mu\nu}\sum_{i+j+k=0}^{n-1}\Big(\{\partial_\rho A^i_\mu \stackrel{\star^j}{,} \partial_\nu A^k_\sigma\} + \{A^i_\mu \stackrel{\star^j}{,} \partial_\nu\partial_\rho A^k_\sigma\} - \{\partial_\rho F^i_{\mu\sigma} \stackrel{\star^j}{,} A^k_\nu\} \\
&\qquad -\{F^i_{\mu\sigma} \stackrel{\star^j}{,} \partial_\rho A^k_\nu\} - \{\partial_\sigma A^i_\mu \stackrel{\star^j}{,} \partial_\nu A^k_\rho\} - \{A^i_\mu \stackrel{\star^j}{,} \partial_\nu\partial_\sigma A^k_\rho\} \\
&\qquad +\{\partial_\sigma F^i_{\mu\rho} \stackrel{\star^j}{,} A^k_\nu\} + \{F^i_{\mu\rho} \stackrel{\star^j}{,} \partial_\sigma A^k_\nu\}\Big) \\
&\quad -\frac{1}{4n}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n}}^{n}\ \sum_{\substack{r,s,t=0\\ r+s+t=i-1}}^{i-1}\Big([\{A^r_\mu \stackrel{\star^s}{,} \partial_\nu A^t_\rho\} \stackrel{\star^j}{,} A^k_\sigma] - [\{F^r_{\mu\rho} \stackrel{\star^s}{,} A^t_\nu\} \stackrel{\star^j}{,} A^k_\sigma] \\
&\qquad +[A^j_\rho \stackrel{\star^k}{,} \{A^r_\mu \stackrel{\star^s}{,} \partial_\nu A^t_\sigma\}] - [A^j_\rho \stackrel{\star^k}{,} \{F^i_{\mu\sigma} \stackrel{\star^j}{,} A^k_\nu\}]\Big) \\
&\quad +\frac{1}{2n}\,\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1}\{\partial_\mu A^i_\rho \stackrel{\star^j}{,} \partial_\nu A^k_\sigma\} \\
&= -\frac{1}{4n}\,\theta^{\mu\nu}\sum_{\substack{i,j,k,(l)=0\\ i+j+k(+l)=n-1}}^{n-1}\Big(\{A^i_\mu \stackrel{\star^j}{,} \partial_\nu F^k_{\rho\sigma}\} - \{\mathcal{D}^i_\mu F^j_{\rho\sigma} \stackrel{\star^k}{,} A^l_\nu\} - 2\{F^i_{\rho\mu} \stackrel{\star^j}{,} F^k_{\sigma\nu}\}\Big).
\end{aligned} \tag{3.106}$$

In the first identity we have used (3.99) and the fact that the n-th order of the $\star$-product can be resolved into the $(n-1)$-th order. In the second identity we have merged the two sums into one. We have used the n-th order Bianchi identity. The idea for this derivation came from studying Stora's proof [93] of the existence of the enveloping algebra gauge theory to all orders (see section 3.7).

It would be nice to have an equation similar to (3.83) and (3.90) also for a field ψ, which transforms by left multiplication with the gauge parameter. But the obvious first guess based on formula (3.64) is *not* correct:

$$\frac{\partial\psi}{\partial\hbar} = -\frac{1}{4}\theta^{\mu\nu}(2A_\mu \star \partial_\nu\psi - iA_\mu \star A_\nu \star \psi).$$

The reason is that there is a certain asymmetry, because by definition Λ_α is multiplied to ψ only from one side. However, we have found a setting which allows a closed flow equation also for ψ. In this setting A_ρ and Λ_α are not hermitian, and therefore also less symmetric quantities. It is possible to formulate flow equations also for non-hermitian gauge parameters and gauge potentials (for arbitrary gauge groups):

$$\frac{\partial\Lambda_\alpha}{\partial\hbar} = -\frac{1}{2}\theta^{\mu\nu}A_\mu \star \partial_\nu\Lambda_\alpha, \tag{3.107}$$

and

$$\frac{\partial A_\sigma}{\partial\hbar} = -\frac{1}{2}\theta^{\mu\nu}(A_\mu \star \partial_\nu A_\sigma - F_{\mu\sigma} \star A_\nu). \tag{3.108}$$

These flow equations are not motivated from string theory, we have worked them out perturbatively and have checked that they can be solved consistently up to second order.

These flow equations have solutions, which also solve the consistency equation (3.48):

$$\begin{aligned}
\Lambda_\alpha^1 &= -\frac{1}{2}\theta^{\mu\nu}A_\mu^0\partial_\nu\alpha, \qquad \Lambda_\alpha^2 = -\frac{1}{4}\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=1}}^{1} A_\mu^i \star^j \partial_\nu\Lambda_\alpha^k,\\
\Lambda_\alpha^n &= -\frac{1}{2n}\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1} A_\mu^i \star^j \partial_\nu\Lambda_\alpha^k.
\end{aligned} \tag{3.109}$$

Similarly the following non-hermitian gauge potentials are solutions of equation (3.50):

$$\begin{aligned}
A_\sigma^1 &= -\frac{1}{2}\theta^{\mu\nu}(A_\mu^0\partial_\nu A_\sigma^0 - F_{\mu\sigma}^0 A_\nu^0), \qquad A_\sigma^2 = -\frac{1}{4}\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=1}}^{1}(A_\mu^i \star^j \partial_\nu A_\sigma^k - F_{\mu\sigma}^i \star^j A_\nu^k),\\
A_\sigma^n &= -\frac{1}{2n}\theta^{\mu\nu}\sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1}(A_\mu^i \star^j \partial_\nu A_\sigma^k - F_{\mu\sigma}^i \star^j A_\nu^k).
\end{aligned} \tag{3.110}$$

Note that in order for these recursive solutions to be true, only the non-hermitian quantities are recursively inserted into (3.109) and (3.110), not the hermitian ones. We have to distinguish carefully between the two recursive expressions (3.97)/(3.99) and (3.109)/(3.110).

With this non-hermitian solutions for Λ_α and A_σ, the following all-orders solution for ψ can be constructed (checked up to second order, with strong hints that higher order terms are correct as well):

$$
\begin{aligned}
\psi^1 &= -\frac{1}{2}\theta^{\mu\nu} A^0_\mu \partial_\nu \psi^0, \\
\psi^2 &= -\frac{1}{4}\theta^{\mu\nu} \sum_{\substack{i,j,k=0\\ i+j+k=1}}^{1} A^i_\mu \star^j \partial_\nu \psi^k \\
&= -\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(i\partial_\kappa A^0_\mu \partial_\lambda \partial_\nu \psi^0 - 2A^0_\mu(\partial_\nu A^0_\kappa)\partial_\lambda \psi^0 - A^0_\mu A^0_\kappa \partial_\nu \partial_\lambda \psi^0 + F^0_{\mu\kappa} A^0_\nu \partial_\lambda \psi^0\big), \\
\psi^n &= -\frac{1}{2n}\theta^{\mu\nu} \sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1} A^i_\mu \star^j \partial_\nu \psi^k,
\end{aligned}
\tag{3.111}
$$

The adjoint field $\overline{\psi}$ is obtained by conjugating the result for ψ, keeping in mind that Λ_α and A_σ are not hermitian in the ansatz (3.107) and (3.108).

$$
\overline{\psi}^n = \overline{\psi^n} = \frac{1}{2n}\theta^{\mu\nu} \sum_{\substack{i,j,k=0\\ i+j+k=n-1}}^{n-1} \partial_\mu \psi^i \star^j A^k_\nu,
$$

The non-hermitian formulae (3.111) and (3.112) have been derived here for the first time, to obtain a better understanding of the structure of the enveloping algebra-valued gauge theory. We do not use these terms for the formulation of a physical theory, since they do not provide hermitian actions.

The solutions (3.97) and (3.109) can be related to each other by adding non-hermitian solutions of the homogeneous equation $\Delta\Lambda^n = 0$, just as (3.99) can be related to (3.110) by solutions of $\Delta_\alpha A^n_\sigma = 0$.

3.5 Constructing actions

There are typically two ways of constructing quantised field theory in terms of a Fock space for particles with a finite spin or finite helicity for massless particles. The first option is to perform quantisation in terms of a Lagrangian which is quantised either canonically or by functional integration. The second is Wigner's classification of particles in terms of positive energy representations of the (universal covering of the) Poincaré group [77].

In the context of NC spaces, where the underlying symmetry of spacetime is at least unclear, the second approach is very cumbersome and the notion of a local field might has to be changed [78], although it can be done in some cases, e.g. for the q-Minkowski spacetime it has been performed successfully [79]. In the absence of a fully understood notion of the underlying symmetry, it is easier to choose the first route. This applies especially to canonical NC spacetime, in which ordinary Lorentz invariance is broken.

In this thesis we will therefore analyse possibilities to construct Lagrangians and based on them action functionals for a NC field theory. The quantisation of these models has been discussed [80], [81]. What has not yet been achieved is the construction of a full-fledged quantum field theory.

Most of all we are interested in Lagrangians which fit into the framework of NC gauge theory. We choose among the possible Lagrangians for NC gauge theory those which are direct generalisations of the Lagrangians of the Standard Model of particle physics. We will omit the discussion of other interesting Lagrangians such as Born-Infeld (but the approach would be analogous). With this restriction to Standard Model-type Lagrangians we could of course miss out on models which might be more suited to NC spaces, cp. [100]. Also we omit pure scalar field theory, the Einstein-Hilbert Lagrangian and supersymmetric Lagrangians. All these models by now have been discussed in the literature, e.g. [82], [83].

We focus on constructing NC generalisations of the following Lagrangians:

Yang-Mills	$\mathcal{L}^0_{\text{YM}} = \text{Tr}(F^0_{\mu\nu}F^{0\mu\nu})$	(3.112)
Minimally coupled fermions	$\mathcal{L}^0_{\text{MCF}} = i\overline{\psi^0}\gamma^\mu\mathcal{D}^0_\mu\psi^0$	(3.113)
Minimally coupled scalars	$\mathcal{L}^0_{\text{MCS}} = (\mathcal{D}^{0\mu}\phi^0)^\dagger(\mathcal{D}^0_\mu\phi^0)$	(3.114)
Mass term for scalars	$\mathcal{L}^0_{\text{MS}} = m(\phi^0)^\dagger\phi^0$	(3.115)
Potential for scalars	$\mathcal{L}^0_{\text{PotS}} = k((\phi^0)^\dagger\phi^0)((\phi^0)^\dagger\phi^0)$	(3.116)
Yukawa terms	$\mathcal{L}^0_{\text{Y}} = g\overline{\psi^0}\phi^0\psi^0$	(3.117)

Tr indicates a trace over the matrix indices of the generators of the underlying non-Abelian gauge group.

Requiring that the NC generalisations of (3.112) to (3.117) are the Standard Model Lagrangians in zeroth order in the deformation parameter $\hbar$, the most natural approach is to replace every commutative field by its NC analogue. It is then possible to reconnect the NC Lagrangians to the commutative ones via the $\star$-product and the expansions of the enveloping algebra-valued fields:

$$\hat{\mathcal{L}}_{\text{YM}} = \tilde{c}\text{Tr}(\hat{F}_{\rho\sigma}\hat{F}^{\rho\sigma}) \longrightarrow \mathcal{L}_{\text{YM}} = \tilde{c}\text{Tr}(F_{\rho\sigma}\star F^{\rho\sigma}) \tag{3.118}$$

$$\hat{\mathcal{L}}_{\text{MCF}} = i\overline{\hat{\psi}}\gamma^\mu\hat{\mathcal{D}}_\mu\hat{\psi} \longrightarrow \mathcal{L}_{\text{MCF}} = i\overline{\psi}\star\gamma^\mu\mathcal{D}_\mu\psi \tag{3.119}$$

$$\hat{\mathcal{L}}_{\text{MCS}} = (\hat{\mathcal{D}}^\mu\hat{\phi})^\dagger(\hat{\mathcal{D}}_\mu\hat{\phi}) \longrightarrow \mathcal{L}_{\text{MCS}} = (\mathcal{D}^\mu\phi)^\dagger\star(\mathcal{D}_\mu\phi) \tag{3.120}$$

$$\hat{\mathcal{L}}_{\text{MS}} = m\hat{\phi}^\dagger\hat{\phi} \longrightarrow \mathcal{L}_{\text{MS}} = m\phi^\dagger\star\phi \tag{3.121}$$

$$\hat{\mathcal{L}}_{\text{PotS}} = k(\hat{\phi}^\dagger\hat{\phi})(\hat{\phi}^\dagger\hat{\phi}) \longrightarrow \mathcal{L}_{\text{PotS}} = k(\phi^\dagger\star\phi)\star(\phi^\dagger\star\phi) \tag{3.122}$$

$$\hat{\mathcal{L}}_{\text{Y}} = \overline{\hat{\psi}}\hat{\phi}\hat{\psi} \longrightarrow \mathcal{L}_{\text{Y}} = g\psi^\dagger\star\phi\star\psi \tag{3.123}$$

In the previous sections we have determined the expansions up to the second order in $\hbar$ of all the fields mentioned. Therefore we only need to insert them into (3.118) up to (3.123). The only topic not treated so far is the Higgs field ϕ. We postpone the discussion of Yukawa terms and the kinetic, mass and interaction term of the Higgs field until section 3.8. Instead we discuss here the non-Standard Model Lagrangian for massive fermions

$$\hat{\mathcal{L}}_{\text{MF}} = m\overline{\hat{\psi}}\hat{\psi} \longrightarrow \mathcal{L}_{\text{MF}} = m\overline{\psi}\star\psi. \tag{3.124}$$

First we consider the NC Yang-Mills Lagrangian. The zeroth order is by definition identical to the commutative counterpart. For higher orders note that we only need the θ-expansion for $F^j_{\rho\sigma}$, the dual field strength $F^{j\rho\sigma}$ is obtained raising the indices with a formal metric:

$$F_{\rho\sigma} \star F^{\rho\sigma} = g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} \star F_{\rho\sigma}, \tag{3.125}$$

therefore the order $F_{\rho\sigma} \star F^{\rho\sigma}$ vs. $F^{\rho\sigma} \star F_{\rho\sigma}$ is unimportant. We obtain in first order

$$\begin{aligned} F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^1)} &= F^0_{\rho\sigma} \star^1 F^{0\rho\sigma} + F^1_{\rho\sigma} F^{0\rho\sigma} + F^0_{\rho\sigma} F^{1\rho\sigma} = \\ &= \frac{i}{2}\theta^{\mu\nu} \mathcal{D}^0_\mu F^0_{\rho\sigma} \mathcal{D}^0_\nu F^{0\rho\sigma} + \frac{1}{2}\theta^{\mu\nu}\{\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}, F^{0\rho\sigma}\} \\ &\quad -\frac{1}{2}\theta^{\mu\nu}\{A^0_\mu, \partial_\nu(F^0_{\rho\sigma}F^{0\rho\sigma})\} + \frac{i}{4}\theta^{\mu\nu}\{A^0_\mu, [A^0_\nu, (F^0_{\rho\sigma}F^{0\rho\sigma})]\}. \end{aligned} \tag{3.126}$$

As well, we obtain the second order NC Yang-Mills Lagrangian simply by inserting the results of the previous section:

$$\begin{aligned} & F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^2)} = \\ & -\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\mu\mathcal{D}^0_\kappa F^0_{\rho\sigma}\mathcal{D}^0_\nu\mathcal{D}^0_\lambda F^{0\rho\sigma} + \frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}[\mathcal{D}^0_\kappa\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}, \mathcal{D}^0_\lambda F^{0\rho\sigma}] \\ & +\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{[\mathcal{D}^0_\kappa F^0_{\rho\mu}, \mathcal{D}^0_\lambda F^0_{\sigma\nu}], F^{0\rho\sigma}\} + \frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{\{F^0_{\mu\kappa}, F^0_{\nu\rho}\}, F^0_{\lambda\sigma}\}, F^{0\rho\sigma}\} \\ & -\frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, \partial_\lambda\{\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}, F^{0\rho\sigma}\}\} + \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [A^0_\lambda, \{\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}, F^{0\rho\sigma}\}]\} \\ & -\frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, \partial_\lambda\left(\mathcal{D}^0_\mu F^0_{\rho\sigma}\mathcal{D}^0_\nu F^{0\rho\sigma}\right)\} - \frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [A^0_\lambda, \left(\mathcal{D}^0_\mu F^0_{\rho\sigma}\mathcal{D}^0_\nu F^{0\rho\sigma}\right)]\} \\ & +\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, \partial_\lambda\{A^0_\mu, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}\} - \frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [A^0_\lambda, \{A^0_\mu, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}]\} \\ & -\frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, \partial_\lambda\{A^0_\mu, [A^0_\nu, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\}\} - \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [A^0_\lambda, \{A^0_\mu, [A^0_\nu, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\}]\} \\ & +\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, \partial_\lambda A^0_\mu\}, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} - \frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, \partial_\mu A^0_\lambda\}, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} \\ & -\frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, \partial_\lambda A^0_\mu\}, [A^0_\nu, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} + \frac{i}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, \partial_\mu A^0_\lambda\}, [A^0_\nu, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} \\ & -\frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [\{A^0_\mu, \partial_\nu A^0_\lambda\}, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} + \frac{i}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [\{A^0_\mu, \partial_\lambda A^0_\nu\}, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} \\ & -\frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, [\{A^0_\mu, [A^0_\nu, A^0_\lambda]\}, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} - \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, [A^0_\lambda, A^0_\mu]\}, [A^0_\nu, \left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} \\ & -\frac{i}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{A^0_\kappa, [A^0_\lambda, A^0_\mu]\}, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} + \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\{\mathcal{D}^0_\kappa F^0_{\rho\sigma}, F^0_{\mu\lambda}\}, \mathcal{D}^0_\nu F^{0\rho\sigma}] \\ & -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\kappa A^0_\mu, \partial_\lambda\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)] - \frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\kappa A^0_\mu, \partial_\lambda[A^0_\nu, \left(F^0_{\rho\sigma}, F^{0\rho\sigma}\right)]] \\ & -\frac{1}{16}\theta^{\mu\nu}\theta^{\kappa\lambda}\{A^0_\kappa, \{\partial_\mu A^0_\lambda, \partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}\} + \frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\{F^0_{\mu\rho}, F^0_{\nu\sigma}\}, \{F^{0\ \rho}_{\kappa}, F^{0\ \sigma}_{\lambda}\}\}. \end{aligned} \tag{3.127}$$

Obviously (3.126) and (3.127) are not explicitly gauge covariant. This lack of covariance is cured in the action. Many of the non-covariant terms in (3.127) are of the form $\theta^{\kappa\lambda}\{A^0_\kappa, \partial_\lambda X\} - \frac{i}{2}\theta^{\kappa\lambda}\{A^0_\kappa, [A^0_\lambda, X]\}$. Under an integral these non-covariant terms are rearranged into a field strength:

$$\begin{aligned}\mathrm{Tr}\int \mathrm{d}x\,\theta^{\kappa\lambda}\Big(\{A^0_\kappa,\partial_\lambda X\}-\frac{i}{2}\{A^0_\kappa,[A^0_\lambda,X]\}\Big) &= \mathrm{Tr}\int \mathrm{d}x\,\theta^{\kappa\lambda}\{(-\partial_\lambda A^0_\kappa+\frac{i}{2}[A^0_\lambda,A^0_\kappa]),X\}\\ &= \frac{1}{2}\mathrm{Tr}\int \mathrm{d}x\,\theta^{\kappa\lambda}\{F^0_{\kappa\lambda},X\}.\end{aligned} \tag{3.128}$$

We quote two relations which have been used frequently while deriving (3.127):

$$\begin{aligned}&\theta^{\mu\nu}\Big(\{\{A^0_\mu,\partial_\nu X\},Y\}+\{X,\{A^0_\mu,\partial_\nu Y\}\}\Big)=\\ &\qquad=\theta^{\mu\nu}\Big(\{A^0_\mu,\partial_\nu\{X,Y\}\}+\{[A^0_\mu,X],\partial_\nu Y\}+\{\partial_\mu X,[A^0_\nu Y]\}\Big),\\ &\theta^{\mu\nu}\Big(\{\{A^0_\mu,[A^0_\nu,X]\},Y\}+\{X,\{A^0_\mu,[A^0_\nu,Y]\}\}\Big)=\\ &\qquad=\theta^{\mu\nu}\Big(\{A^0_\mu,[A^0_\nu,\{X,Y\}]\}+2\{[A^0_\mu,X],[A^0_\nu,Y]\}\Big).\end{aligned} \tag{3.129}$$

The large number of apparently unrelated terms in (3.127) raises the question whether there is a way to obtain higher order terms in the Lagrangian systematically. However, it seems that no recursive relations as in 3.4 are possible for composite structures such as Lagrangians and that there is no other way to construct these Lagrangians than by such tedious calculations.

Next we discuss mass terms and gauge coupling terms for fermion fields ψ transforming from the left. Again the zeroth order coincides with the commutative Lagrangian by definition. For higher order terms the θ-expanded solutions for ψ, $\overline{\psi}$ and A_μ are inserted. The fermion mass term reads:

$$m\overline{\psi}\star\psi|_{\mathcal{O}(\hbar^1)} = m\left(\overline{\psi^0}\star^1\psi^0+\overline{\psi^1}\psi^0+\overline{\psi^0}\psi^1\right)=m\frac{i}{2}\theta^{\mu\nu}\mathcal{D}^0_\mu\overline{\psi}^0\mathcal{D}^0_\nu\psi^0 \tag{3.130}$$

$$\begin{aligned}m\overline{\psi}\star\psi|_{\mathcal{O}(\hbar^2)} &= m\left(\overline{\psi^0}\star^2\psi^0+\overline{\psi^1}\star^1\psi^0+\overline{\psi^0}\star^1\psi^1+\overline{\psi^1}\psi^1+\overline{\psi^2}\psi^0+\overline{\psi^0}\psi^2\right)=\\ &= m\left(-\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\mu\mathcal{D}^0_\kappa\overline{\psi^0}\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0-\frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\mu\overline{\psi^0}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0\right).\end{aligned} \tag{3.131}$$

In these expressions the covariant derivative is evaluated on a conjugate field as $\mathcal{D}^0_\mu\overline{\psi^0}=\partial_\mu\overline{\psi^0}+i\overline{\psi^0}A^0_\mu$. The fermionic mass Lagrangians have an interestingly clean structure, cp. the complicated expressions for ψ^1 and ψ^2. In these θ-expanded Lagrangians terms appear in which $\star$-product partial derivatives get replaced by covariant derivatives and there are higher order terms where there is a "braiding" between factors of $\theta^{\mu\nu}$, implemented by field strengths: $\theta^{\mu\nu}\theta^{\kappa\lambda}F^0_{\mu\kappa}$.

Similar features appear in the θ-expansion of the Lagrangian describing the minimal coupling of matter fields to gauge potentials:

$$i\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^1)} = -\frac{1}{2}\theta^{\mu\nu}\mathcal{D}^0_\mu\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0+\frac{i}{2}\theta^{\mu\nu}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}\mathcal{D}^0_\nu\psi^0 \tag{3.132}$$

$$\begin{aligned}i\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^2)} = & -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\mathcal{D}^0_\mu\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0+\frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\overline{\psi^0}F^0_{\lambda\mu}\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0\\ & -\frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda\left(F^0_{\mu\rho}\mathcal{D}^0_\nu\psi^0\right)-\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho\left(\mathcal{D}^0_\kappa F^0_{\mu\rho}\right)\mathcal{D}^0_\lambda\mathcal{D}^0_\nu\psi^0\\ & -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0-\frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0.\end{aligned} \tag{3.133}$$

In this coupling Lagrangian, a field strength which is braided with a tensor θ appears already in first order. In second order these terms become quite complicated. Similar structures are present also in the Yang-Mills Lagrangian, although they are hardly recognisable in the complicated expression.

The construction of an action requires the definition of an integral. Integration is in general difficult to implement for NC spaces. It is misleading to expect that the integral is related to summing the field values over points, since NC spaces are "pointless". Therefore most of the usual intuitions concerning integration have to be dropped. A detailed discussion of integration is crucial for κ-deformed spacetime, in the canonical case the integral can be defined consistently with some minimal requirements. In particular, the integral should have the trace property, to be able to form a gauge invariant action from a gauge-covariant Lagrangian and to have a variational principle. Therefore we demand:

$$\int \mathrm{d}x\ \ f\star g=\int \mathrm{d}x\ \ g\star f. \tag{3.134}$$

If the integral allows the application of Stokes' theorem, we may partially integrate the derivatives of the $\star$-product for $\theta^{\mu\nu} = const$ and obtain, because of the antisymmetry of $\theta^{\mu\nu}$,

$$\int \mathrm{d}x\ \ f\star g=\int \mathrm{d}x\ \ fg=\int \mathrm{d}x\ \ gf=\int \mathrm{d}x\ \ g\star f. \tag{3.135}$$

This definition of integral is sufficient for the canonical NC space. We defer the reader to chapter 5 for further details about NC integration.

With the trace property we see immediately that an action constructed by an integral over the fermionic Lagrangians is gauge invariant:

$$\begin{aligned}\mathcal{S}_{\mathrm{F}} &= \int \mathrm{d}^n x\ \ \overline{\psi}\star(i\gamma^\rho\mathcal{D}_\rho-m)\psi \quad\longrightarrow \\ \delta_\alpha\mathcal{S}_{\mathrm{F}} &= \int \mathrm{d}^n x\ \ \Big(-i\overline{\psi}\star\Lambda_\alpha\star(i\gamma^\rho\mathcal{D}_\rho-m)\psi+i\overline{\psi}\star(i\gamma^\rho\mathcal{D}_\rho-m)\Lambda_\alpha\star\psi\Big)=0,\end{aligned} \tag{3.136}$$

by definition of the covariant derivative.

The commutative Yang-Mills action is multiplied with a numerical factor $-\frac{1}{4}$, $\frac{1}{2}$ from the trace over normalised generators of the gauge group, and another $\frac{i^2}{2}$ because field strengths are two-forms. In the NC enveloping algebra gauge theory, the trace cannot be fixed in such a straightforward way, the choice of trace is restricted only by [84]

$$\frac{1}{g^2} = \sum_{\rho} c_\rho \mathrm{Tr}(\rho(T^a)\rho(T^a)). \tag{3.137}$$

Here g is the coupling of the gauge group, ρ denotes a representation of the generators of the Lie algebra, the parameters c_ρ may be chosen freely as long as they fulfil (3.137). Since the sum over generators is not performed, the numerical factor of $\mathcal{S}_{\text{YM}}$ in our approach is an as far arbitrary real constant $\tilde{c}$:

$$\mathcal{S}_{\text{YM}} = \tilde{c}\,\mathrm{Tr}\int \mathrm{d}^n x\; F_{\rho\sigma} \star F^{\rho\sigma}. \tag{3.138}$$

Introducing $\tilde{c}$ also avoids confusing this numerical factor with other factors from the expansion of the enveloping algebra. The trace property ensures gauge invariance of the Yang-Mills action:

$$\delta_\alpha \mathcal{S}_{\text{YM}} = \tilde{c}\,\mathrm{Tr}\int \mathrm{d}^n x\; \Big(i[\Lambda_\alpha \stackrel{\star}{,} F_{\rho\sigma}] \star F^{\rho\sigma} + iF_{\rho\sigma} \star [\Lambda_\alpha \stackrel{\star}{,} F^{\rho\sigma}]\Big) = 0,$$

Partial integration and the trace property lead to the following result for the $\theta^{\mu\nu}$-expanded Yang-Mills Lagrangian:

$$\tilde{c}\,\mathrm{Tr}\int \mathrm{d}x\; F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^0)} = \tilde{c}\,\mathrm{Tr}\int \mathrm{d}x\; F^0_{\rho\sigma} F^{0\rho\sigma}, \tag{3.139}$$

$$\tilde{c}\,\mathrm{Tr}\int \mathrm{d}x\; F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^1)} = \tilde{c}\theta^{\mu\nu}\mathrm{Tr}\int \mathrm{d}x\; \big(2F^0_{\rho\mu}F^0_{\sigma\nu}F^{0\rho\sigma} - \frac{1}{2}F^0_{\mu\nu}F^0_{\rho\sigma}F^{0\rho\sigma}\big). \tag{3.140}$$

Unfortunately, we have not been able yet to formulate the second order Yang-Mills action in an explicitly covariant way, because of the technical complexity of the calculations. The θ-expanded fermionic mass term reads:

$$m\int \mathrm{d}x\; \overline{\psi} \star \psi|_{\mathcal{O}(\hbar^0)} = m\int \mathrm{d}x\; \overline{\psi^0}\psi^0, \tag{3.141}$$

$$m\int \mathrm{d}x\; \overline{\psi} \star \psi|_{\mathcal{O}(\hbar^1)} = -\frac{m}{4}\theta^{\mu\nu}\int \mathrm{d}x\; \overline{\psi^0}F^0_{\mu\nu}\psi^0, \tag{3.142}$$

$$\begin{aligned} m\int \mathrm{d}x\; \overline{\psi} \star \psi|_{\mathcal{O}(\hbar^2)} = {} & m\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\; \big(\frac{i}{8}\overline{\psi^0}(\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu\psi^0 - \frac{1}{8}\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}\psi^0 \\ & + \frac{1}{32}\overline{\psi^0}F^0_{\kappa\lambda}F^0_{\mu\nu}\psi^0\big), \end{aligned} \tag{3.143}$$

and the minimally gauge coupled fermionic action is:

$$
\begin{aligned}
i\int \mathrm{d}x\,\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^0)} &= i\int \mathrm{d}x\,\overline{\psi}^0\gamma^\rho\mathcal{D}^0_\rho\psi^0, &(3.144)\\
i\int \mathrm{d}x\,\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^1)} &= \theta^{\mu\nu}\int \mathrm{d}x\,\Big(-\frac{i}{4}\overline{\psi}^0\gamma^\rho F^0_{\mu\nu}\mathcal{D}^0_\rho\psi^0-\frac{i}{2}\overline{\psi^0}\gamma^\rho F^0_{\rho\mu}\mathcal{D}^0_\nu\psi^0\Big), &(3.145)\\
i\int \mathrm{d}x\,\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^2)} &= \theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\,\Big(-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0-\frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0\\
&\quad-\frac{1}{8}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda\mathcal{D}^0_\rho\psi^0-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\kappa\mu}F^0_{\lambda\nu}\mathcal{D}^0_\rho\psi^0\\
&\quad-\frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0\\
&\quad+\frac{i}{32}\overline{\psi^0}\gamma^\rho F^0_{\kappa\lambda}F^0_{\mu\nu}\mathcal{D}^0_\rho\psi^0\Big). &(3.146)
\end{aligned}
$$

3.6 Ambiguities of enveloping algebra gauge theory

Starting from the consistency condition (3.48), there is an ambiguity in the construction of the enveloping algebra-valued gauge parameter Λ_α; this has been remarked already in section 3.3. Even worse, there are not only ambiguities in constructing Λ_α order by order in $\hbar$, but also additional ambiguities in constructing fields and the gauge potential. In this section we investigate these ambiguities thoroughly to first order and also discuss the most interesting second order ambiguities. We will focus on understanding how these ambiguities affect the definition of the θ-expanded actions.

The freedom (3.56) has been discussed shortly in [33], along with a discussion of field redefinition ambiguities. Freedom in the Seiberg-Witten map was discussed at length in [71], [86] in the string-theory context. More discussion is included in [88] and [75]. However, these approaches do not discuss the meaning of the freedom for the construction of the action (the Yang-Mills action was discussed under this aspect in [71]). All these texts ignore the discussion of fermionic matter.

All terms which parametrise an ambiguity or freedom in constructing the enveloping algebra gauge theory have to be solutions of the homogeneous equations:

$$
\begin{aligned}
\Delta\Lambda^n &= 0,\\
\Delta_\alpha\psi^n &= 0, &(3.147)\\
\Delta_\alpha A^n_\sigma &= 0.
\end{aligned}
$$

In the construction of the NC gauge parameter there are essentially two types of ambiguities, which we call covariant and non-covariant ambiguities. The first order ambiguity presented in (3.56), i.e. a solution of $\Delta\Lambda^1=0$:

$$\Lambda^{1,c_1}_\alpha = c_1\theta^{\mu\nu}[A^0_\mu,\partial_\nu\alpha], \qquad (3.148)$$

is of the *non-covariant* type, since a derivative of α is multiplied with the non-covariant gauge potential A^0_μ. Calling this type of terms non-covariant might be misleading, but this terminology is only needed in this section. Most frequently, this freedom is called a gauge-ambiguity. A *covariant* term in contrast is a term, in which $\partial_\nu \alpha$ is multiplied with a covariant quantity $\tilde{F}_\mu$, i.e. $\delta_\alpha \tilde{F}_\mu = i[\alpha, \tilde{F}_\mu]$:

$$\Lambda_\alpha^{1,c_\Lambda} = c_\Lambda \theta^{\mu\nu} \{\tilde{F}_\mu, \partial_\nu \alpha\}_\pm. \tag{3.149}$$

In contrast to the homogeneous equations for the gauge parameter Λ_α, the homogeneous equations for the fields and the gauge potential allow only covariant ambiguities, with arbitrary covariant terms $\tilde{F}_{\mu\nu}$ and $\tilde{F}_{\mu\rho\nu}$:

$$\psi^{1,c_\psi} = c_\psi \theta^{\mu\nu} \tilde{F}_{\mu\nu} \psi^0, \tag{3.150}$$

$$A_\rho^{1,c_A} = c_A \theta^{\mu\nu} \tilde{F}_{\rho\mu\nu}. \tag{3.151}$$

The notation $\{\ ,\ \}_\pm$ in (3.149) refers to a commutator or to an anti-commutator, depending on the sign. So far we have not demanded that the ambiguities should fulfil additional requirements such as hermiticity. To discuss the ambiguities without requiring hermiticity is possible, but increases the number of possible terms enormously (cp. the terms in [71] for a field in the adjoint representation). We will therefore demand that Λ_α^n and A_σ^n are hermitian. Fields ψ^n have to be constructed from hermitian quantities. In addition, we do not discuss terms in which indices of $\theta^{\mu\nu}$ are lowered with a metric as in [80], terms with θ with lowered indices are entirely different ambiguities.

The only first-order hermitian freedom for Λ_α^1 of the non-covariant type is:

$$\Lambda_\alpha^{1,c_1} = ic_1 \theta^{\mu\nu} [A_\mu^0, \partial_\nu \alpha], \quad c_1 \in \mathbb{R}. \tag{3.152}$$

To see the effect of this freedom, we plug this term into equations such as:

$$\delta_\alpha \psi^{1,c_1} = i\Lambda_\alpha^{1,c_1} \psi^0, \tag{3.153}$$

leading to an additional term for the fermion field

$$\psi^{1,c_1} = -c_1 \theta^{\mu\nu} A_\mu^0 A_\nu^0 \psi^0, \qquad \text{and} \qquad \overline{\psi^{1,c_1}} = c_1 \theta^{\mu\nu} \overline{\psi}^0 A_\mu^0 A_\nu^0. \tag{3.154}$$

Similarly this freedom generates via

$$\delta_\alpha A_\rho^{1,c_1} = \partial_\rho \Lambda_\alpha^{1,c_1} - i[A_\rho^0, \Lambda_\alpha^{1,c_1}], \tag{3.155}$$

an additional term for the gauge potential:

$$A_\rho^{1,c_1} = ic_1 \theta^{\mu\nu} \big([\partial_\rho A_\mu^0, A_\nu^0] - i[[A_\rho^0, A_\mu^0], A_\nu^0]\big) = ic_1 \theta^{\mu\nu} \big(\partial_\rho (A_\mu^0 A_\nu^0) - i[A_\rho^0, A_\mu^0 A_\nu^0]\big). \tag{3.156}$$

Adding these terms up, we see that the fermionic mass Lagrangian $\mathcal{L}_{\mathrm{MF}}^{1,c_1}$ and the minimally gauge coupled Lagrangian $\mathcal{L}_{\mathrm{MCF}}^{1,c_1}$ derived from the freedom parametrised by c_1 are identically zero.

The field strength corresponding to (3.156) is

$$F_{\rho\sigma}^{1,c_1} = c_1 \theta^{\mu\nu} [[F_{\rho\sigma}^0, A_\mu^0], A_\nu^0], \tag{3.157}$$

and the trace of the corresponding Yang-Mills Lagrangian is therefore

$$\mathcal{L}^{1,c_1}_{\rm YM} = -c_1\theta^{\mu\nu}{\rm Tr}(A^0_\mu A^0_\nu F^0_{\rho\sigma}F^{0\rho\sigma} - F^0_{\rho\sigma}F^{0\rho\sigma}A^0_\mu A^0_\nu) = 0. \tag{3.158}$$

The first order non-covariant hermitian freedom does not contribute to the action.

Next we consider the covariant freedom in first order. The only covariant quantity with one index in zeroth order is the covariant derivative $\tilde{F}^0_\mu = \mathcal{D}^0_\mu$. Before accepting a term with a covariant derivative acting freely to the right as an admissible covariant ambiguity, we have to define precisely what such a term means. First of all, the resulting ambiguity has to be hermitian. The hermiticity of a freely acting derivative can be checked via partial integration

$$\overline{\chi(x)\partial_\mu} = \overleftarrow{\partial}_\mu\overline{\chi}(x) \overset{\text{part. int.}}{\longrightarrow} -(\partial_\mu\overline{\chi}(x)) - \overline{\chi}(x)\partial_\mu \tag{3.159}$$

Therefore allowing such a term would lead to the following hermitian solution

$$\Lambda^{1,c_2}_\alpha = ic_2\theta^{\mu\nu}\{\mathcal{D}^0_\mu, \partial_\nu\alpha\} = -2ic_2\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu + c_2\theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\}, \tag{3.160}$$

because of antisymmetry of $\theta^{\mu\nu}$.

The problem with such a term is not its hermiticity or that we could not calculate fields from it or gauge potentials, at least formally. The problem is that if we allow such a term with a derivative acting freely to the right, it has to be included also into the calculation of the consistency condition (3.48) to higher orders in $\hbar$. The second order consistency condition involving a term like (3.160) cannot be solved. Similarly, a hermitian solution of the homogeneous equation for the gauge potential such as

$$A^{1c_3}_\rho = ic_3\theta^{\mu\nu}\mathcal{D}^0_\nu F^0_{\rho\mu} + 2ic_3\theta^{\mu\nu}F^0_{\rho\mu}\mathcal{D}^0_\nu. \tag{3.161}$$

cannot be used, since its contribution to the equation for A^2_σ cannot be solved. Therefore we prohibit such derivative-valued solutions for all quantities in the canonical NC gauge theory. In contrast, such derivative-valued quantities appear in a natural, solvable way in κ-deformed gauge theory because of the deformed symmetry[3].

In contrast, a term such as

$$\Lambda^{1,c_4}_\alpha = c_4\theta^{\mu\nu}\{\theta^{\kappa\lambda}\mathcal{D}^0_\kappa F^0_{\mu\lambda}, \partial_\nu\alpha\}, \tag{3.162}$$

is hermitian and allowed and formally at first order in $\hbar$. But in fact it is of order $(\theta^{\mu\nu})^2$ and only because of the splitting of $\theta^{\mu\nu}$ into $\hbar\theta^{\mu\nu}$ this term is first order in $\hbar$. There are circumstances in which it is sensible to take this splitting serious (e.g. in the discussion of the renormalisation behaviour, [90]), but here we regard (3.162) as a second order term.

The only remaining hermitian covariant ambiguities are therefore solutions of the homogeneous equations (3.147) for ψ,

$$\psi^{1c_5} = c_5\theta^{\mu\nu}F^0_{\mu\nu}\psi^0, \tag{3.163}$$

[3]This was the motivation to investigate whether derivative-valued terms could appear as part of the freedom in canonically NC gauge theory.

and the gauge potential A_ρ,

$$A_\rho^{1c_6} = c_6 \theta^{\mu\nu} \mathcal{D}^0_\rho F^0_{\mu\nu}. \tag{3.164}$$

That this freedom for the gauge potential is unique can be seen using the Bianchi identity. The field strength corresponding to $A_\rho^{1c_6}$ is

$$F^{1c_6}_{\rho\sigma} = i[\mathcal{D}^{1,c_6}_\rho, \mathcal{D}^0_\sigma] + i[\mathcal{D}^0_\rho, \mathcal{D}^{1,c_6}_\sigma] = -ic_6\theta^{\mu\nu}[F^0_{\rho\sigma}, F^0_{\mu\nu}]. \tag{3.165}$$

Since this field strength is a commutator, it does not contribute to the Yang-Mills Lagrangian. There are no other ambiguities than (3.165) for the field strength, since it is strictly calculated from the gauge potential and not constructed as a solution of the transformation law, compare the discussion in the previous sections. Therefore the Yang-Mills action is not affected by the covariant ambiguities at first order, but the two ambiguities (3.163) and (3.164) introduce an ambiguity in the fermionic action:

$$\mathcal{S}^{1,c_5,c_6}_{\text{MCF, MF}} = \int \mathrm{d}^n x \left(2c_5\theta^{\mu\nu}\overline{\psi^0}F^0_{\mu\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + (c_5+c_6)\theta^{\mu\nu}\overline{\psi^0}i\gamma^\rho(\mathcal{D}^0_\rho F^0_{\mu\nu})\psi^0\right). \tag{3.166}$$

The significance of this result can be seen comparing (3.166) with the fermionic action derived in the previous section from particular solutions of the inhomogeneous equations:

$$\begin{aligned}\mathcal{S}^1_{\text{MCF, MF}} + \mathcal{S}^{1,c_5,c_6}_{\text{MCF, MF}} \;=\; & \int \mathrm{d}^n x \Big(\underbrace{(2c_5 - \frac{1}{4})}_{d_1}\theta^{\mu\nu}\overline{\psi^0}F^0_{\mu\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 \qquad (3.167)\\ & -\frac{i}{2}\theta^{\mu\nu}\overline{\psi^0}\gamma^\rho F^0_{\rho\mu}\mathcal{D}^0_\nu\psi^0 + \underbrace{(c_5+c_6)}_{d_2}\theta^{\mu\nu}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\nu F^0_{\rho\mu})\psi^0\Big).\end{aligned}$$

Choosing $c_5 = \frac{1}{8}$ and $c_6 = -\frac{1}{8}$ two terms can be set to zero. We can equally well choose any other value for d_1 and d_2; this will be as consistent with the structure of the enveloping algebra gauge theory.

But physics must not depend on a choice of a gauge, therefore any prediction based on a particular value of d_i is unphysical. The choice of the value of c_6 in the tower of the enveloping algebra choice is in fact a gauge choice of the commutative Lie algebra gauge theory. Typically, [33], [94], [75], this type of covariant freedom parametrised by c_6 is called a field redefinition of A_σ. In the field redefinition perspective two constructions of the enveloping algebra gauge theory, one before taking a field redefinition $A^0_\mu \to A^0_\mu + \tilde{F}_\mu$ and one after, may look different. But they have to be equivalent, since they do only depend on a freedom in the Lie algebra gauge theory. This freedom leads to

$$\Lambda^1_\alpha \sim \theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\} \quad \longrightarrow \quad \Lambda^{1'}_\alpha \sim \theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\} + \theta^{\mu\nu}\{\tilde{F}_\mu, \partial_\nu\alpha\}. \tag{3.168}$$

The covariant ambiguities are exactly such field redefinitions. We have chosen the term "covariant ambiguity", since ψ^0 is not redefined in terms of fermionic degrees of freedom, but by multiplication with $F^0_{\mu\nu}$. From the point of view of constructing an enveloping algebra gauge theory from the transformation behaviour, the two terms parametrised by

c_5 and c_6 are ignored by this construction, they have to be included as an ambiguity. This type of ambiguity is also called "covariant" in [71].

We may conclude that the only new term in the NC fermionic action at first order in θ is

$$\mathcal{S}^{1,\text{invariant}}_{\text{MCF, MF}} = -\frac{i}{2}\theta^{\mu\nu}\int \mathrm{d}^n x \overline{\psi^0}\gamma^\rho F^0_{\rho\mu}\mathcal{D}^0_\nu\psi^0, \quad (3.169)$$

since the two other terms are proportional to a covariant ambiguity. In particular, no mass term appears at first order in θ.

Next let us turn to the ambiguities appearing at second order. We take over the restrictions on the ambiguities from first order: these terms have to be hermitian, not derivative-valued and solutions of the homogeneous equations.

The non-covariant freedom in the gauge parameter Λ^{1,c_1}_α has the following effect on the consistency condition in second order:

$$\Delta\Lambda^{2,c_1}_\alpha = -[\Lambda^{1,c_1}_\alpha \stackrel{\star^1}{;} \beta] - [\alpha \stackrel{\star^1}{;} \Lambda^{1,c_1}_\beta] - [\Lambda^{1,c_1}_\alpha, \Lambda^1_\beta] - [\Lambda^1_\alpha, \Lambda^{1,c_1}_\beta] - [\Lambda^{1,c_1}_\alpha, \Lambda^{1,c_1}_\beta]. \quad (3.170)$$

This equation is solved by the following terms

$$\begin{aligned}\Lambda^{2,c_1}_\alpha &= c_1\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(-\frac{i}{2}\{[\partial_\mu A^0_\kappa, A^0_\lambda], \partial_\nu\alpha\} - \frac{1}{4}\{[[A^0_\mu, \partial_\kappa\alpha], A^0_\nu], A^0_\lambda\} \\ &\quad + \frac{1}{4}\{[[A^0_\mu, A^0_\kappa], A^0_\nu], \partial_\lambda\alpha\}\Big) - \frac{c_1^2 i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}[[A^0_\mu, A^0_\nu], [A^0_\kappa, \partial_\lambda\alpha]]. \quad (3.171)\end{aligned}$$

Again this freedom does not contribute at all to the Lagrangian, since Λ^{2,c_1}_α leads to the additional terms in ψ^{2,c_1}:

$$\begin{aligned}\psi^{2,c_1} &= c_1\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(-\frac{i}{2}[\partial_\mu A^0_\kappa, A^0_\lambda]\partial_\nu\psi^0 + \frac{1}{2}A^0_\mu A^0_\nu A^0_\kappa\partial_\lambda\psi^0 - \frac{i}{4}A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0\Big) \\ &\quad + \frac{c_1^2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0. \quad (3.172)\end{aligned}$$

and the gauge potential

$$\begin{aligned}A^{2,c_1}_\rho &= c_1\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(-\frac{i}{2}\{[\partial_\mu A^0_\kappa, A^0_\lambda]\partial_\nu A^0_\rho\} + \frac{1}{4}\{[[A^0_\mu, A^0_\kappa], A^0_\nu], (F^0_{\lambda\rho} + \partial_\lambda A^0_\rho)\} \\ &\quad - \frac{1}{4}\{[[A^0_\mu, (F^0_{\kappa\rho} + \partial_\kappa A^0_\rho)], A^0_\nu], A^0_\lambda\}\Big) \quad (3.173) \\ &\quad - \frac{c_1^2}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(i[[A^0_\mu, A^0_\nu], [\partial_\rho A^0_\kappa, A^0_\lambda]] + [[A^0_\mu, A^0_\nu], [[A^0_\rho, A^0_\kappa], A^0_\lambda]]\Big).\end{aligned}$$

Plugging these terms into the second order fermionic mass and minimally coupled Lagrangian, all contributions (in c_1^2 and in c_1) drop out. Similarly the Λ_α freedom in c_1 does not contribute to the Yang-Mills Lagrangian. We conclude that the non-covariant freedom $\Lambda^{1c_1}_\alpha$ is irrelevant concerning Lagrangians.

Next we investigate in which sense the first order covariant ambiguities reappear as additional ambiguities at second order. For example the ambiguity parametrised by c_5:

$\psi^{1,c_5} = c_5\theta^{\mu\nu}F^0_{\mu\nu}\psi^0$ enters into the determining equation for ψ^2:

$$\begin{aligned} \Delta_\alpha\psi^{2,c_5} &= i\Lambda^1_\alpha\psi^{1,c_5} + i\alpha\star^1\psi^{1,c_5}, \\ \Rightarrow \quad \psi^{2,c_5} &= -\frac{c_5}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}A^0_\kappa\partial_\lambda(F^0_{\mu\nu}\psi^0) + \frac{ic_5}{4}A^0_\kappa A^0_\lambda F^0_{\mu\nu}\psi^0. \end{aligned} \tag{3.174}$$

The inhomogeneous equation is identical to the one for ψ^1, since ψ^{1,c_5} is effectively a field redefinition. Therefore the c_5-parametrised Lagrangian and action to second order are:

$$\begin{aligned} \mathcal{L}^{2,c_5}_{\text{MF}} &= \frac{ic_1}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathcal{D}^0_\kappa(\overline{\psi^0}F^0_{\mu\nu})\mathcal{D}^0_\lambda\psi^0 + \mathcal{D}^0_\kappa\overline{\psi^0}\mathcal{D}^0_\lambda(F^0_{\mu\nu}\psi^0)\big) + c_5^2\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}F^0_{\mu\nu}F^0_{\kappa\lambda}\psi^0, \\ \mathcal{S}^{2,c_5}_{\text{MF}} &= (c_5^2 - \frac{c_5}{2})\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}^n x\ \overline{\psi^0}F^0_{\mu\nu}F^0_{\kappa\lambda}\psi^0, \\ \mathcal{L}^{2,c_5}_{\text{MCF}} &= -\frac{c_1}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\mathcal{D}^0_\kappa(\overline{\psi^0}F^0_{\mu\nu})\gamma^\rho\mathcal{D}^0_\lambda\mathcal{D}^0_\rho\psi^0 + \mathcal{D}^0_\kappa\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda(\mathcal{D}^0_\rho(F^0_{\mu\nu}\psi^0)) \\ &\quad -i\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0 - i\overline{\psi^0}\gamma^\rho F^0_{\kappa\rho}\mathcal{D}^0_\lambda(F^0_{\mu\nu}\psi^0) \\ &\quad +ic_5^2\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}F^0_{\mu\nu}\gamma^\rho\mathcal{D}^0_\rho(F^0_{\kappa\lambda})\psi^0)\Big), \\ \mathcal{S}^{2,c_5}_{\text{MCF}} &= i\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}^n x\Big((c_5^2 - \frac{c_5}{4})\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}\mathcal{D}^0_\rho(F^0_{\kappa\lambda}\psi^0) - \frac{c_5}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\lambda}\mathcal{D}^0_\rho\psi^0 \\ &\quad +\frac{c_5}{2}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0 + \frac{c_5}{2}\overline{\psi^0}\gamma^\rho F^0_{\kappa\rho}\mathcal{D}^0_\lambda(F^0_{\mu\nu}\psi^0)\Big). \end{aligned} \tag{3.175}$$

The gauge potential ambiguity proportional to c_6 leads in second order to

$$A^{2,c_6}_\rho = c_6\frac{1}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{\partial_\mu(\mathcal{D}^0_\rho F^0_{\kappa\lambda}), A^0_\nu\} - \frac{i}{2}\{[A^0_\mu, (\mathcal{D}^0_\rho F^0_{\kappa\lambda})], A^0_\nu\}\Big). \tag{3.176}$$

For the minimally coupled fermion action this means in second order:

$$\begin{aligned} \mathcal{L}^{2,c_6}_{\text{MF}} &= \frac{ic_6}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\mathcal{D}^0_\kappa\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda((\mathcal{D}^0_\rho F^0_{\mu\nu})\psi^0) + \overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\kappa\mathcal{D}^0_\rho F^0_{\mu\nu})\mathcal{D}^0_\lambda\psi^0\big), \\ \mathcal{S}^{2,c_6}_{\text{MF}} &= c_6\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}^n x\big(-\frac{1}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}(\mathcal{D}^0_\rho F^0_{\kappa\lambda})\psi^0 + \frac{i}{2}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\kappa\mathcal{D}^0_\rho F^0_{\mu\nu})\mathcal{D}^0_\lambda\psi^0\big). \end{aligned} \tag{3.177}$$

We will analyse the effect of these ambiguities below.

In addition to the first order effects to second order quantities, there are of course also intrinsic second order ambiguities. It has not been possible to find an exhaustive list of all second order non-covariant and covariant ambiguities. For example, the terms in (3.61) are together a non-covariant solution of the homogeneous equation. We have analysed the three non-covariant ambiguities

$$\begin{aligned} \Lambda^{2,c_7}_\alpha &= c_7\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\{\partial_\mu A^0_\kappa, \partial_\nu\partial_\lambda\alpha\} + i\{\partial_\mu A^0_\kappa, [\partial_\nu\alpha, A^0_\lambda]\}\Big), \\ \Lambda^{2,c_8}_\alpha &= c_8\theta^{\mu\nu}\theta^{\kappa\lambda}\{\partial_\mu A^0_\nu, [A^0_\kappa, \partial_\lambda\alpha]\}, \\ \Lambda^{2,c_9}_\alpha &= c_9\theta^{\mu\nu}\theta^{\kappa\lambda}\{[A^0_\mu, A^0_\nu], [A^0_\kappa, \partial_\lambda\alpha]\}. \end{aligned} \tag{3.178}$$

These lead to the following terms for fields ψ:

$$\begin{aligned} \psi^{2,c_7} &= ic_7\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A^0_\kappa\partial_\nu A^0_\lambda\psi^0, \\ \psi^{2,c_8} &= c_8\theta^{\mu\nu}\theta^{\kappa\lambda}\big(i\{\partial_\mu A^0_\nu, A^0_\kappa A^0_\lambda\}\psi^0 + 2A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0\big), \\ \psi^{2,c_9} &= c_9\theta^{\mu\nu}\theta^{\kappa\lambda}2A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0, \end{aligned} \tag{3.179}$$

and for the gauge potential

$$\begin{aligned} A_\rho^{2,c_7} &= ic_7\theta^{\mu\nu}\theta^{\kappa\lambda}\big(\{\partial_\rho\partial_\mu A^0_\kappa, \partial_\nu A^0_\lambda\} - i\{[A^0_\rho, \partial_\mu A^0_\kappa], \partial_\nu A^0_\lambda\}\big), \\ A^{2,c_8} &= c_8\theta^{\mu\nu}\theta^{\kappa\lambda}\big(i\{\partial_\mu A^0_\nu, A^0_\kappa A^0_\lambda\}\psi^0 + 2A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0\big), \\ A^{2,c_9} &= c_9\theta^{\mu\nu}\theta^{\kappa\lambda}2A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\psi^0. \end{aligned} \tag{3.180}$$

A rather fast calculation shows that the Lagrangians built from these fields are identically zero. This leads us to believe that the non-covariant ambiguities are altogether irrelevant from the point of view of Lagrangians and actions. We may concentrate on the covariant ambiguities.

We do not want to collect encyclopedically all covariant ambiguities either, but have a look at the Lagrangians in section 3.5. We would like to know, whether these Lagrangians can be obtained as well by covariant ambiguities. The θ-expanded fermionic action (3.144) reads:

$$\begin{aligned} &\int \mathrm{d}x\, \overline{\psi} \star (i\gamma^\rho\mathcal{D}_\rho - m)\psi|_{\mathcal{O}(\hbar^2)} = \\ &\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\, \Big(\frac{1}{32}\overline{\psi^0}F^0_{\kappa\lambda}F^0_{\mu\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \frac{1}{8}\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 \\ &\qquad + \frac{i}{8}\overline{\psi^0}(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0 \\ &\qquad - \frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0 - \frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0 - \frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0\Big). \end{aligned} \tag{3.181}$$

The following covariant ambiguities are possible at second order:

$$\begin{aligned} \psi^{2,c_{10}} &= c_{10}\theta^{\mu\nu}\theta^{\kappa\lambda}F^0_{\mu\nu}F^0_{\kappa\lambda}\psi^0, \\ \psi^{2,c_{11}} &= c_{11}\theta^{\mu\nu}\theta^{\kappa\lambda}F^0_{\kappa\mu}F^0_{\lambda\nu}\psi^0, \\ \psi^{2,c_{12}} &= ic_{12}\theta^{\mu\nu}\theta^{\kappa\lambda}(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda\psi^0. \end{aligned} \tag{3.182}$$

They lead to the following terms in the action:

$$\begin{aligned} \mathcal{S}^{2,c_{10}}_{\text{MCF, MF}} &= c_{10}\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\, \big(2\overline{\psi^0}F^0_{\mu\nu}F^0_{\kappa\lambda}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + i\overline{\psi^0}\gamma^\rho\{\mathcal{D}^0_\rho F^0_{\mu\nu}, F^0_{\kappa\lambda}\}\psi^0\big), \\ \mathcal{S}^{2,c_{11}}_{\text{MCF, MF}} &= c_{11}\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\, \big(2\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + i\overline{\psi^0}\gamma^\rho\{\mathcal{D}^0_\rho F^0_{\kappa\mu}, F^0_{\lambda\nu}\}\psi^0\big), \\ \mathcal{S}^{2,c_{12}}_{\text{MCF, MF}} &= c_{12}\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\, \big(2i\overline{\psi^0}(\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 \\ &\qquad + \frac{1}{2}\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\rho\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu\psi^0\big). \end{aligned} \tag{3.183}$$

The covariant freedom (3.182) shows that the first three terms in (3.181), especially all fermionic mass terms, are of the type of a field redefinition and therefore should vanish. In contrast the last four terms in (3.181) are not affected by this freedom and can also not arise because of a field redefinition of A_μ, since we have argued above that derivative-valued field redefinitions of A_μ have to be excluded, since they violate the enveloping algebra structure. However, the last term in (3.181) is identical to one of the terms which carry over from first order covariant ambiguities (3.175). Therefore the only physically relevant terms in the fermionic action to second order are

$$\int \mathrm{d}x\, \overline{\psi} \star (i\gamma^\rho \mathcal{D}_\rho - m)\psi|_{\mathcal{O}(\hbar^2)} = \theta^{\mu\nu}\theta^{\kappa\lambda} \int \mathrm{d}x \Big(-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0 \\ -\frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0 - \frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0\Big). \quad (3.184)$$

From the point of view of the theory of consistent deformations (cp. [85]), the enveloping algebra is a nontrivial deformation of type 1. This means that although the gauge transformation is trivially deformed ($\delta_{\Lambda_\alpha} \equiv \delta_\alpha$), the deformation is non-trivial, it cannot only be obtained via field redefinitions (both the Yang-Mills action and the fermionic interaction term (3.169) are nontrivial at first and second order).

3.7 Cohomology of enveloping algebra gauge theory

In the previous section we discussed at length the freedom in constructing the enveloping algebra-valued gauge theory and how much of it is dangerous and worrisome for physics. Again, we have been thrown back to tedious calculations. The quite complicated terms which appear already at second order in $\theta^{\mu\nu}$ motivate the goal goal to formulate the enveloping algebra-valued gauge theory in terms of a more rigid mathematical language. Indeed, shortly after the article [33] had appeared, several groups [92], [93] and [94] started to investigate the cohomological structure of the enveloping algebra gauge theory. Here we state the main results [75] and will see what this means for the results derived above. We emphasise that for actual calculations, this approach is not easier or better, but it enlightens the underlying structure.

In all the approaches stated above, the gauge parameter Λ_α is reinterpreted as a ghost field (see also [89]), with the exception of [94], [88], where the antifield formalism is used. We denote the ghost field Λ to keep track of its role as a gauge parameter. The ghost field of course is Grassmannian, in addition it is (non-Abelian) enveloping algebra-valued. The introduction of the ghost field allows to rewrite the complicated consistency condition (3.48) by means of a BRST operator

$$s\Lambda = i\Lambda \star \Lambda. \quad (3.185)$$

To summarise the notation, the BRST operator is the same as in the commutative Lie algebra gauge theory, it is nilpotent, commutes with derivatives and has a graded Leibniz rule:

$$\begin{aligned} s\lambda &= i\lambda\lambda, & s\Lambda &= i\Lambda\star\Lambda, \\ sa_\mu &= \partial_\mu\lambda - i[A^0_\mu, \lambda], & sA_\mu &= \partial_\mu\Lambda - i[A_\mu \stackrel{\star}{,} \Lambda], \\ s(f\cdot g) &= (sf)\cdot g + (-1)^{(\mathrm{deg} f)} f\cdot(sg), & s(f\star g) &= (sf)\star g + (-1)^{(\mathrm{deg} f)} f\star(sg), \\ s^2 &= 0, & [s, \partial_\mu] &= 0. \end{aligned} \quad (3.186)$$

As in the previous section we assume that there is an expansion of Λ in terms of $\hbar$, $\Lambda = \lambda + \sum_n \hbar^n \Lambda^n$. and that the zeroth order term in $\hbar$ is λ. The higher orders should be expressible in terms of A^0_μ and λ and their derivatives. Altogether Λ should have ghost number one at every order Λ^n. The gauge potential A_μ, a functional of A^0_μ, has ghost number zero at every order, $A_\mu = A^0_\mu + \sum_n \hbar^n A^n_\mu$.

Representations of a Lie algebra can in a natural way be lifted to representations of their enveloping algebra. The operator Δ introduced in (3.58) takes the following form:

$$\Delta\Lambda^n = s\Lambda^n - i\{\lambda, \Lambda^n\} = \Xi^n, \qquad \Delta A^n_\mu = sA^n_\mu - i[\lambda, A^n_\mu] = \Upsilon^n_\mu, \tag{3.187}$$

where Ξ^n and Υ^n_μ parametrise the inhomogeneous terms at order n. Δ removes the covariant part of the transformation behaviour under s. The action of Δ on Λ^n is the generic action on an odd quantity, the action on A^n_μ the generic action on an even quantity. On the Lie algebra quantities we observe $\Delta\lambda = -i\lambda\cdot\lambda$ and $\Delta a_\mu = \partial_\mu\lambda$; Δ is an anti-derivation with ghost-number one and it is nilpotent. Therefore $\Delta\Upsilon^n_\mu = \Delta\Xi^n = 0$.

Especially we see that $\Lambda'^n = \Lambda^n + \Delta\xi^n$ is again a solution of $\Delta\Lambda^n = \Xi^n$. The ghost number zero quantity ξ^n parametrises the freedom of the non-covariant or gauge type in constructing the enveloping algebra gauge theory presented in the last chapter.

For example, the non-covariant ambiguity $\Lambda^{c_1}_\alpha = ic_1\theta^{\mu\nu}[A^0_\mu, \partial_\nu\alpha]$ is of the form

$$\Delta\big(ic_1\theta^{\mu\nu}A^0_\mu A^0_\nu\big) = \Lambda^{c_1}_\alpha. \tag{3.188}$$

This transfers into the ambiguity in $A^1_\mu = D^0_\mu\Lambda^{c_1}_\alpha + S_\mu$, where S_μ is a covariant ambiguity or a field redefinition. This is exactly the result of the previous section. The three second order non-covariant ambiguities that we investigated in section 3.6 are derived from

$$\begin{aligned}
\Delta\big(c_7\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A^0_\kappa \partial_\nu A^0_\lambda\big) &= \Lambda^{c_7}_\alpha,\\
\Delta\big(c_8\theta^{\mu\nu}\theta^{\kappa\lambda}\{\partial_\mu A^0_\nu, A^0_\kappa A^0_\lambda\}\big) &= \Lambda^{c_8}_\alpha,\\
\Delta\big(c_9\theta^{\mu\nu}\theta^{\kappa\lambda}A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda\big) &= \Lambda^{c_9}_\alpha.
\end{aligned} \tag{3.189}$$

In [75] it was argued that the ambiguities of the non-covariant type (gauge ambiguities) are an infinitesimal version of the so-called Stora invariance of the consistency condition

$$\begin{aligned}
\Lambda &\longrightarrow \Lambda' = G^{-1}\star\Lambda\star G + iG^{-1}\star sG,\\
A_\mu &\longrightarrow A'_\mu = G^{-1}\star A_\mu\star G + iG^{-1}\star\partial_\mu G,
\end{aligned} \tag{3.190}$$

where G is an arbitrary $\star$-invertible local functional of ghost number zero, since

$$\begin{aligned}
s\Lambda' &= -G^{-1}\star sG\star G^{-1}\star\Lambda\star G + iG^{-1}\star\Lambda\star\Lambda\star G - G^{-1}\star\Lambda\star sG - iG^{-1}\star sG\star G^{-1}\star sG\\
&= i\big(G^{-1}\star\Lambda\star G + iG^{-1}\star sG\big)\star\big(G^{-1}\star\Lambda\star G + iG^{-1}\star sG\big) = i\Lambda'\star\Lambda'.
\end{aligned} \tag{3.191}$$

Note that the minus sign of the third term in the fist line is due to ghost number one of Λ. To get from the first to the second line, insert $1 = G\star G^{-1}$ in the second and third term. A_μ is treated analogously.

If Λ^n exists $\forall n$, i.e. if $\Delta\Lambda^n = \Xi^n$ is true $\forall n$, then the enveloping algebra-valued gauge theory can be constructed to all orders. In cohomological language, Λ^n exists $\forall n$, if the coboundary operator Δ maps cohomology classes $\Lambda^n + \Delta\xi^n$ to a Δ-exact Ξ^n.

The most straightforward proof, by complete induction, for the existence of the enveloping algebra gauge theory of non-Abelian gauge groups is due to Stora [93], we shortly sketch the idea. At zeroth order $s\lambda = i\lambda\lambda$ and assume that for $i = 1, \ldots, n-1$ it has been shown that $s\Lambda^i = i\sum_{r+s+t=i} \Lambda^r \star^s \Lambda^t$ (the notation is identical to that of section 3.4). Therefore at n-th order:

$$s\Lambda^n = i \sum_{r+s+t=n} \Lambda^r \star^s \Lambda^t = i\{\lambda, \Lambda^n\} + i \sum_{\substack{r+s+t=n \\ r\neq n, t\neq n}} \Lambda^r \star^s \Lambda^t = i\{\lambda, \Lambda^n\} + \Xi^n. \tag{3.192}$$

Since $\Delta\cdot = s \cdot - i\{\lambda, \cdot\}$, we have to calculate $\Delta\Xi^n$. If $\Delta\Xi^n = 0$, the enveloping algebra gauge theory exists to all orders. Since Ξ^n has ghost number one less than Λ^n, Δ is defined differently:

$$\begin{aligned}
\Delta\Xi^n &= s\Xi^n - i[\lambda, \Xi^n] = \sum_{\substack{r+s+t=n \\ r\neq n, t\neq n}} \big(i(s\Lambda^r) \star^s \Lambda^t - i\Lambda^r \star^s (s\Lambda^t) + \lambda\Lambda^r \star^s \Lambda^t - \Lambda^r \star^s \Lambda^t\lambda\big) \\
&= -\sum_{\substack{p+q+r+s+t=n \\ p+q+r\neq n, t\neq n}} (\Lambda^p \star^q \Lambda^r) \star^s \Lambda^t + \sum_{\substack{p+q+r+s+t=n \\ p\neq n, r+s+t\neq n}} \Lambda^p \star^q (\Lambda^r \star^s \Lambda^t) \\
&\quad + \sum_{\substack{r+s+t=n \\ r\neq n, t\neq n}} \big(\lambda\Lambda^r \star^s \Lambda^t - \Lambda^r \star^s \Lambda^t\lambda\big).
\end{aligned} \tag{3.193}$$

Now performing some re-orderings of the summations, e.g.

$$-\sum_{\substack{p+q+r+s+t=n \\ p+q+r\neq n, t\neq n}} (\Lambda^p \star^q \Lambda^r) \star^s \Lambda^t = -\sum_{p+q+r+s+t=n} (\Lambda^p \star^q \Lambda^r) \star^s \Lambda^t + \sum_{p+q+r=n} (\Lambda^p \star^q \Lambda^r)\lambda + \lambda\lambda\Lambda^n, \tag{3.194}$$

and using associativity of the $\star$-product, one can quickly see that indeed $\Delta\Xi^n = 0$, if $\Delta\Xi^i = 0, \forall i = 0, \ldots, n-1$. To finish the proof, the construction of a homotopy operator is needed, which shows that the cohomology of Δ is empty for ghost number two:

$$K\Delta + \Delta K = 1. \tag{3.195}$$

The homotopy operator is the "inverse" of Δ and has ghost number (-1). Of course Δ is not invertible because of $\Delta^2 = 0$, but if $\Delta\Lambda^n = \Xi^n$, then $(K\Delta + \Delta K)\Xi^n = \Xi^n$, with $\Lambda^n = K\Xi^n$ if $\Delta\Xi^n = 0$.

The hands-on construction that we have performed in sections 3.2 and 3.3 was quick and effective, because of experience with these calculations. The homotopy operator K is a more systematic way of performing this construction [92], [87]. First observe that λ appears in Ξ^n always only as $\partial_\mu\lambda = b_\mu$.

$$\Delta A^0_\mu = b_\mu, \qquad \Delta b_\mu = 0. \tag{3.196}$$

A linear operator $\tilde{K}$ is introduced which inverts this action

$$\tilde{K}b_\mu = A^0_\mu, \qquad \tilde{K}A^0_\mu = 0. \tag{3.197}$$

and therefore $(\tilde{K}\Delta + \Delta\tilde{K}) = 1$ on b_μ and A^0_μ. $\tilde{K}$ has a graded Leibniz rule, commutes with covariant derivatives, anti-commutes with s, is nilpotent on A^0_μ and b_μ and has ghost number (-1). K is obtained from $\tilde{K}$ by the following definition: On monomials $f(A^0_\mu, b_\mu)$ of degree N in A^0_μ and b_μ (counting all occurrences), $Kf(A^0_\mu, b_\mu)$ acts as $\frac{1}{N}\tilde{K}f(A^0_\mu, b_\mu)$.

In the approach of [87], a scheme was given to determine $\Delta\Xi^n = 0$ in the algebra of A^0_μ and b_μ. To this end an additional condition has to be imposed. Changing the abstract algebra, on which the operators Δ and K are defined from A^0_μ and λ to A^0_μ and b_μ, it is not automatically given anymore that $\Delta F_{\mu\nu} = 0$. The algebra generated by A^0_μ and b_μ is not free. The reason is that the definition $\partial_\mu\lambda = b_\mu$ forgets that derivatives commute and $\partial_\nu b_\mu = \partial_\mu b_\nu$ is not safeguarded anymore in the algebra generated by A^0_μ and b_μ. Therefore $\Delta F_{\mu\nu} = 0$ and $[F_{\mu\nu}, \cdot] - i[[\mathcal{D}_\mu, \mathcal{D}_\nu], \cdot] = 0$ have to be imposed a constraints. In general $\Delta\Upsilon^k_\mu = 0$ will therefore not be true, $\Delta\Upsilon^k_\mu$ will be proportional to the constraints, e.g. at first order

$$\Delta\Upsilon^1_\mu = \frac{1}{2}\theta^{\kappa\lambda}[\Delta F^0_{\mu\mu}, B^0_\lambda]. \tag{3.198}$$

The solution is to enlarge the algebra by an element $f_{\mu\nu}$, for which it is demanded $\Delta f_{\mu\nu} = 0$ and $f_{\mu\nu} = F_{\mu\mu}$. This allows to implement the constraint. In this scheme the proof can be carried out as well that the enveloping algebra gauge theory exists to all orders, though more cumbersome than the above construction.

3.8 NC Standard Model and phenomenology

We have deferred the discussion of Lagrangians involving the Higgs field in section 3.5. The new aspect of the analysis of the Higgs field is that it is sandwiched between two fermionic fields in different representations in the Yukawa Lagrangian. From the left it is multiplied with the left-handed component of the fermions and from the right it is multiplied with the right-handed component. In the Standard Model, the left- and right-handed components of the fermions transform under different representations, therefore the Higgs field also has to transform in different ways from the left and from the right. We have already treated fields transforming from the left and from the right (the complex conjugate field), therefore we only have to combine the two transformations. This is also called the hybrid Seiberg-Witten map [84]. Note that two different gauge transformations mean two different gauge potentials A_σ and A'_σ, a left transformation with Λ_α and A_μ and a right transformation with Λ'_α and A'_μ:

$$\delta_\alpha\phi = i\Lambda_\alpha \star \phi - i\phi \star \Lambda'_\alpha, \tag{3.199}$$

$$\phi = \phi^0 + \hbar\phi^1 + \hbar^2\phi^2 + \ldots, \tag{3.200}$$

$$\phi^1 = -\frac{1}{2}\theta^{\mu\nu}A^0_\mu\partial_\nu\phi^0 + \frac{i}{4}\theta^{\mu\nu}A^0_\mu A^0_\nu\phi^0 + \frac{1}{2}\theta^{\mu\nu}\partial_\mu\phi^0 A'^0_\nu + \frac{i}{4}\theta^{\mu\nu}\phi^0 A'^0_\mu A'^0_\nu. \tag{3.201}$$

Similarly the higher orders are obtained. The Yukawa Lagrangian

$$\mathcal{L}_{\mathrm{Y}} = \overline{\psi} \star \phi \star \psi, \tag{3.202}$$

can be expanded as

$$\mathcal{L}^1_{\mathrm{Y}}\mathcal{O}(\hbar) = \frac{i}{2}\theta^{\mu\nu}\big(\mathcal{D}^0_\mu\overline{\psi^0}\mathcal{D}^0_\nu(\phi^0\ \psi^0) + \overline{\psi^0}\ \mathcal{D}'^0_\mu\phi^0\ \mathcal{D}'^0_\nu\psi^0\big). \tag{3.203}$$

Note that in the first term only the derivative of the covariant derivative $\mathcal{D}^0_\mu\phi = \partial_\mu\phi^0 - iA^0_\mu\phi^0$ acts on ψ^0. In contrast $\mathcal{D}'^0_\mu\phi^0 = \partial_\mu\phi^0 + i\phi^0 A'^0_\mu$.

The covariant derivative $\mathcal{D}_\mu$ of ϕ has to be covariant w.r.t. the transformation of both sides:

$$\mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu \star \phi + i\phi \star A'_\mu. \tag{3.204}$$

Expanding the covariant derivative in first order we obtain:

$$\begin{aligned}\mathcal{D}_\mu\phi|_{\mathcal{O}(\hbar^1)} &= \frac{1}{2}\theta^{\kappa\lambda}\big(F^0_{\kappa\mu}\mathcal{D}^0_\lambda\phi^0 - A^0_\kappa\mathcal{D}^0_\lambda\mathcal{D}^0_\mu\phi^0 - \frac{i}{2}A^0_\kappa A^0_\lambda\partial_\mu\phi^0 \\ &\quad -\mathcal{D}'^0_\kappa\phi^0 F'^0_{\lambda\mu} + \mathcal{D}'^0_\kappa\mathcal{D}'^0_\mu\phi^0 A'^0_\lambda - \frac{i}{2}\partial_\mu\phi^0 A'^0_\kappa A'^0_\lambda\big).\end{aligned} \tag{3.205}$$

The kinetic term of the action of the Higgs field

$$\mathcal{S}_{\mathrm{MCS}} = \int \mathrm{d}^n x\ \ (\mathcal{D}^\mu\phi)^\dagger \star (\mathcal{D}_\mu\phi), \tag{3.206}$$

can be evaluated in an obvious way with these expansions, since the integral is cyclic. Similarly we would treat the mass term

$$\mathcal{L}_{\mathrm{MS}} = m\phi^\dagger \star \phi, \tag{3.207}$$

and the quartic potential of the Higgs field

$$\mathcal{L}_{\mathrm{Pot}} = k(\phi^\dagger \star \phi) \star (\phi^\dagger \star \phi). \tag{3.208}$$

We do not present the explicit form of the actions, it is obvious how to proceed, cp. [36]. The NC Higgs mechanism has been presented first in [97], in the context of the string inspired models.

Most of the problems that appear in NC versions of the Standard Model (cp. [98]) can be solved by the enveloping algebra gauge theory approach. We quote the results of [36], since this work shows the power of the enveloping algebra-valued approach developed in this thesis.

In the NC gauge theory derived from D-brane physics, it has been found that only charges (+1), (0) and (−1) are allowed for matter fields [95], [96]. The reason is simple: the rigid NC gauge theory allows only the left (fundamental), trivial or right (anti-fundamental) representation, corresponding to these three charges. In contrast, we have seen that the enveloping algebra can be constructed in such a way that it is entirely determined by the Lie algebra level through the consistency condition. Therefore we may also construct several copies of NC gauge potentials $a_\mu^{(n)}$, corresponding to several copies of NC photons, all of which are determined by one physical photon which is a Lie algebra quantity.

The generator of $U(1)$, which was so far kept implicit in $A_\mu = eQa_\mu$ is then written explicitly (e is the electric charge). $\star$-Multiplying A_μ to a fermionic field with charge n we obtain

$$A_\mu \star \psi^{(n)} = ea_\mu \star (Q\psi^{(n)}) = eq^{(n)} a_\mu^{(n)} \star \psi^{(n)}. \tag{3.209}$$

Therefore, for every charge present in the Standard Model, a separate gauge potential has to be defined in the enveloping algebra, which is subsumed in the gauge field $A_\mu = eQa_\mu$. The quantities of the NC gauge theory become dependent on this specific charge. Thus, for every charge n there is a field strength, covariant derivative etc.

$$\begin{aligned} f_{\mu\nu}^{(n)} &= \partial_\mu a_\nu^{(n)} - \partial_\nu a_\mu^{(n)} + ieq^{(n)} [a_\mu^{(n)} \stackrel{\star}{,} a_\nu^{(n)}], \\ \mathcal{D}_\mu^{(n)} \psi^{(n)} &= \partial_\mu \psi^{(n)} - ieq^{(n)} a_\mu^{(n)} \star \psi^{(n)}, \\ \delta_\alpha a_\mu^{(n)} &= \partial_\mu \Lambda_\alpha^{(n)} + ieq^{(n)} [\Lambda_\alpha^{(n)} \stackrel{\star}{,} a_\nu^{(n)}], \\ \delta_\alpha \psi^{(n)} &= ieq^{(n)} \Lambda_\alpha^{(n)} \star \psi^{(n)}. \end{aligned} \tag{3.210}$$

It is an open question whether there have to be separate kinetic terms for every single NC photon $a_\mu^{(n)}$ [36].

The gauge group of the Standard Model is the tensor product $U(1) \otimes SU(2) \otimes SU(3)$. Treated in the context of $U(n)$-model building, the tensorisation of the NC analogs is very non-trivial [99]. But the Standard Model tensor gauge group can be lifted immediately into the NC regime, working with enveloping algebra gauge potentials V_μ. This gauge potential V_μ is a sum of gauge potentials corresponding to the 1+3+8 gauge bosons of the Standard Model

$$\mathsf{V}_\mu^0 = g' A_\mu^0 Y + g \sum_{a=1}^{3} B_{\mu a}^0 T_{SU(2)}^a + g_S \sum_{a=1}^{8} G_{\mu a}^0 T_{SU(3)}^a, \tag{3.211}$$

where Y is the generator of hyper-charge and T^a are the generators of the two non-Abelian gauge groups of the Standard Model. Gauge transformations on this gauge potential are implemented by a Standard Model gauge parameter:

$$\alpha = g' \alpha^Y Y + g \sum_{a=1}^{3} \alpha_a^{SU(2)} T_{SU(2)}^a + g_S \sum_{a=1}^{8} \alpha_a^{SU(3)} T_{SU(3)}^a. \tag{3.212}$$

The full gauge potential and gauge parameter are used to construct the enveloping algebra gauge theory,

$$\Lambda_\alpha = \alpha - \frac{1}{4} \theta^{\mu\nu} \{\mathsf{V}_\mu^0, \partial_\nu \alpha\} + \dots \ , \tag{3.213}$$

and

$$\mathsf{V}_\kappa = \mathsf{V}^0_\kappa - \frac{1}{4}\theta^{\mu\nu}\left(\{\mathsf{V}^0_\mu, \partial_\nu \mathsf{V}^0_\kappa\} - \{\mathsf{F}^0_{\mu\kappa}, \mathsf{V}^0_\nu\} + \dots \quad . \tag{3.214}$$

For charged fermionic fields, a representation ρ has to be fixed,

$$\Psi^{(n)} = \Psi^{0,(n)} - \frac{1}{2}\theta^{\mu\nu}\left(\rho_{(n)}(\mathsf{V}^0_\mu)\partial_\nu\psi^{0,(n)} - \frac{i}{2}\rho_{(n)}(\mathsf{V}^0_\mu)\rho_{(n)}(\mathsf{V}^0_\nu)\psi^{0,(n)}\right) + \dots \quad . \tag{3.215}$$

The representations are those of the ordinary Standard Model, left handed doublets, right handed singlets and a two-component Higgs field.

$$\Psi^{(n)}_L = \begin{pmatrix} L^{(n)}_L \\ Q^{(n)}_L \end{pmatrix}, \qquad \Psi^{(n)}_R = \begin{pmatrix} e^{(n)}_R \\ u^{(n)}_R \\ d^{(n)}_R \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{3.216}$$

Similarly all other NC quantities are constructed by the usual prescription presented in section 3.3. The action of the NC Standard Model is obtained by sandwiching together all the quantities described, according to the scheme presented in the previous sections.

The crucial observation is that new nonlinear couplings appear between the gauge bosons, corresponding to different sectors of the Standard Model. Also neutral particles may couple to the photon through NC effects. For all further properties of the NC Standard Model, we refer to [36]. We finish the discussion of θ-expanded canonical NC gauge theory with an overview of the current status of experimental predictions.

This model predicts operators of higher dimension in the action which are clearly power-counting non-renormalisable[4]. This is acceptable, if the theory is regarded as an effective physical theory, presupposing other physics at some higher scale. In contrast the truncation of the expansion at some fixed order in θ provides an IR cut-off. Conceptually [103] this means that at large distances spacetime is a commutative manifold ($|in\rangle$ and $|out\rangle$-states are defined in the ordinary sense at a large, commutative distance). At short distances, the manifold picture of spacetime is changed, NC effects appear. The crossover might be describable in terms of a phase transition. Note that if this phase-transition picture is taken serious, broken Lorentz invariance of canonical NC models becomes a less severe problem, since an inertial system is preferred from the outset. Attempts have been made [67] to accommodate a step-function in a description of varying noncommutativity $\theta(x)$. It has been shown [34], [35] that these models are anomaly-free.

Several experimental predictions have been calculated for NC versions of the Standard Model. Most of these phenomenological studies, e.g. [106], treat $\theta^{\mu\nu}$ as an ether-like field pervading spacetime in the neighbourhood of the earth. Such a "global" background field with an explicit breaking of Lorentz invariance seems quite untenable from the perspective of the UV-IR breaking NC QFT and indeed the calculated experimental bounds are very high (see below). In contrast we will now focus on predictions of IR-regulated, θ-expanded models.

[4] Compare this ansatz to the UV-IR-plagued but renormalisable matrix-model theory of [100], [101].

The focus in [40] and [104] is on the gauge sector, triple gauge boson vertices are calculated which are forbidden in the Standard Model. These vertices originate from combining the different gauge potentials of the Standard Model in the tensor gauge potential V_μ. At first order in θ, V_μ acquires a self coupling of all Standard Model gauge potentials. This leads, among many other new interactions, to the potentially measurable decay of the Z boson into two γ or two gluons. This decay is forbidden in the Standard Model because of bosonic statistics and angular momentum conservation.

However, the pure gauge sector of NC gauge theory is not very well suited to derive bounds on the NC scale, because of the dependence on the choice of the trace. We multiplied the Yang-Mills action with a free parameter $\tilde{c}$ in (3.137) to indicate the indeterminacy in defining the trace. Although ongoing analysis manages to constrain further the trace in the electroweak sector [104], this freedom limits the predictiveness of the pure gauge sector.

Studies of θ-expanded NC effects in the fermionic sector are scarce, [21], [107]. NC corrections to masses of off-shell quarks in θ-expanded QCD at one loop are discussed in [107]. They compare the differences in the hyperfine splitting variations of Cs and Hg atomic clocks over the year, since these two nuclei should respond differently to the effective background field of the NC structure. They obtain a bound on the NC scale of $\gtrsim 10^{17} GeV$. This result certainly rules out a low-energy ether-like NC pervading space and time at large distances. However, this bound is not necessarily restrictive in the IR-regulated theory and it was shown [105] that in the θ-expanded theory this bound is invalidated altogether, imposing the equations of motion on the matrix element describing the interaction.

In [102] a lower bound on the noncommutativity has been derived based on the decay of plasmons into neutrino - anti-neutrino pairs, which would generate an energy loss in globular clusters. Since the NC Standard Model contains a coupling of neutral particles to photons, a new decay channel into neutrinos becomes available. The argument for a bound of $\gtrsim 100 GeV$ is that this new channel should not overly contribute to the known channels, otherwise a too large neutrino flux would wash out cluster structures in a measurable way. In [103], another bound is found considering corrections to the neutrino dipole moment by the NC coupling of the neutrino to photons. The bound of $\gtrsim 161 TeV$ is based on the requirement that this new contribution does not dominate the usual Penguin diagram contributions for massive neutrinos.

Chapter 4

The κ-deformed Euclidean space

The κ-deformed space is introduced as a factor space, i.e. as an abstract associative algebra $\mathcal{A}_{\hat{x}}$ over the complex numbers, generated freely by n coordinates $\hat{x}^\mu$, with an ideal generated by the Lie algebra commutation relations with the structure constant $C_\lambda^{\mu\nu} = a^\mu \delta_\lambda^\nu - a^\nu \delta_\lambda^\mu$. This characterisation of an abstract space is due to [43]. The definitions introduced for a general NC space in chapter 2 apply.

The vector a^μ which characterises the κ-deformed space can be rotated into the n-th direction $a^\mu = a\delta_n^\mu$ without loss of generality, then $C_\lambda^{\mu\nu} = a(\delta_n^\mu \delta_\lambda^\nu - \delta_n^\nu \delta_\lambda^\mu)$. In the following, we take this rotation of a^μ into the n-th direction as given. The most important formulae for the generic case are stated in the appendix. It is reasonable to perform this rotation and to distinguish between the coordinates parallel and orthogonal to a^μ, since the calculations simplify a lot.

The κ-deformed space for $a^\mu = a\delta_n^\mu$ has a singled-out coordinate $\hat{x}^n$ which does not commute with the coordinates orthogonal to it

$$[\hat{x}^n, \hat{x}^j] = ia\hat{x}^j, \qquad [\hat{x}^i, \hat{x}^j] = 0, \qquad i, j = 1, 2, \ldots, n-1. \tag{4.1}$$

In most discussions of κ-deformed spacetime, $\hat{x}^n$ is taken to be the time coordinate of a four-dimensional Minkowski spacetime. This has probably historic reasons, κ-spacetime has been introduced first as the translational sector of the κ-Poincaré group [26], [27], [57]. The κ-Poincaré group has been constructed [26] as a contraction limit of a q-deformed quantum group and it consists of finite deformed symmetry transformations in four dimensions. There is a dual [109] symmetry structure, the infinitesimal symmetry transformations in four dimensions. Many conventions of this historic derivation still persist.

The restriction to four dimensions had already been lifted in [110]. The identification of $\hat{x}^n$ with the direction of time is an additional and arbitrary choice from the point of view of the abstract algebra (4.1). NC spacetime is not a metric space or even a manifold. The metric in our setting is a prescription of how to perform summations in the algebra $\mathcal{A}_{\hat{x}}$. Therefore the signature of a metric a priori specifies only the signs in the contraction of two quantities, one with upper and one with lower index. The structure of this formal metric may be quite arbitrary, interesting analyses treat $\hat{x}^n$ as lightlike [111] or work with non-diagonal metrics [112].

In our approach $\hat{x}^n$ is an arbitrary direction of an n-dimensional Euclidean space. The Euclidean setting is chosen for transparency of the calculus[1], it generalises immediately to the Minkowski setting [38]. Therefore indices can be lowered or raised with the formal metric $g_{\mu\nu} = g^{\mu}{}_{\nu} = g^{\mu\nu} = \delta^{\mu\nu}$ at will, doubly appearing indices are summed (Einstein convention). Greek letters run from $1, \ldots, n$ and Roman letters from $1, \ldots, n-1$.

Our definition of the κ-deformed space (4.1) uses a length scale a in place of the more common $\kappa = \frac{1}{a}$. Which convention is more convenient depends on whether calculations are performed mostly in coordinate space (as is done here, then a is more natural) or in momentum space (as is done in many other approaches, then an inverse length or mass scale κ is more natural).

The length scale a is the small parameter of this model. Thinking of a as a fixed and possibly experimentally testable length scale, the limiting process to commutative quantities should better be performed with a dimensionless parameter $\hbar$, $a \to \hbar a$ (cp. section 2.3). This parameter $\hbar$ can be taken to zero without harm. In this chapter, we will not isolate $\hbar$ to avoid clutter.

As in the previous section, we consider field theory on NC space in a perturbation expansion around the commutative field theory. In this section we construct additional geometrical quantities for the κ-deformed space such as derivatives, symmetry generators etc. All these quantities are treated as formal power series. For example, inverting a quantity such as $\hat{x}^\mu$ is a priori not possible. We only use expansions as formal power series. The scale a is used also for expansions in the abstract algebra, i.e. before taking a representation on commutative quantities. This use of the same deformation parameter of two very different power series expansion, one in the abstract algebra, one for the representation on commutative space, allows to fix a commutative limit in a unique way. We require that for all constructions the commutative quantities are recovered in the limit $a \to 0$. Therefore all NC quantities have to have the same (mass) dimensionality as their commutative counterparts.

4.1 Linear derivatives on the κ-deformed space

Derivatives $\widehat{\partial}_\mu$ are introduced as maps on the coordinate algebra, $\widehat{\partial}_\mu$: $\mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$ [44], [45]. We demand that these derivatives

- respect the factor space, i.e. they have to be consistent with (4.1);
- are a deformation of ordinary derivatives, i.e. $[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \mathcal{O}(a)$;
- commute among themselves $[\widehat{\partial}_\mu, \widehat{\partial}_\nu] = 0$.

These restrictions on derivatives $\widehat{\partial}_\mu$ are weak, there exists a large number of possible solutions

$$[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \sum_j a^j (\widehat{\partial}_{(\mu,\nu)})^j . \tag{4.2}$$

[1]Otherwise signs would have to be changed according to $\hat{x}^n$ time-like or space-like.

The symbolic notation refers to all terms of a power series expansion in the derivatives $\widehat{\partial}_\mu$, which are consistent with the index structure.

Under the additional condition that the commutator $[\widehat{\partial}_\mu, \hat{x}^\nu]$ is at most linear in the derivatives, there are exactly three one-parameter families of solutions $\hat{\partial}^{c_1}_\mu$, $\hat{\partial}^{c_2}_\mu$ and $\hat{\partial}^{c_3}_\mu$:

$$\begin{array}{llll}
[\hat{\partial}^{c_1}_n, \hat{x}^n] = 1 + c_1 ia\hat{\partial}^{c_1}_n, & [\hat{\partial}^{c_2}_n, \hat{x}^n] = 1 + iac_2\hat{\partial}^{c_2}_n, & [\hat{\partial}^{c_3}_n, \hat{x}^n] = 1 + 2ia\hat{\partial}^{c_3}_n, & \\
[\hat{\partial}^{c_1}_n, \hat{x}^j] = 0, & [\hat{\partial}^{c_2}_n, \hat{x}^j] = ia(1+c_2)\hat{\partial}^{c_2}_j, & [\hat{\partial}^{c_3}_n, \hat{x}^j] = ia\hat{\partial}^{c_3}_j, & \\
[\hat{\partial}^{c_1}_i, \hat{x}^n] = ia\hat{\partial}^{c_1}_i, & [\hat{\partial}^{c_2}_i, \hat{x}^n] = 0, & [\hat{\partial}^{c_3}_i, \hat{x}^n] = iac_3\hat{\partial}^{c_3}_i, & (4.3) \\
[\hat{\partial}^{c_1}_i, \hat{x}^j] = \delta^j_i, & [\hat{\partial}^{c_2}_i, \hat{x}^j] = \delta^j_i(1 + iac_2\hat{\partial}^{c_2}_n), & [\hat{\partial}^{c_3}_i, \hat{x}^j] = \delta^j_i. &
\end{array}$$

The real parameters c_i are not fixed by consistency with (4.1). We prefer to work with one particular choice in the following, $\hat{\partial}^{c_1=0}_\mu$. For brevity, $\hat{\partial}^{c_1=0}_\mu$ is denoted as $\hat{\partial}_\mu$.

There is always more than one set of linear derivatives (consistent with the coordinate algebra) on NC spaces of the Lie algebra type

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\lambda \hat{x}^\lambda. \tag{4.4}$$

If we denote the commutator of coordinates and derivatives linear in $\widehat{\partial}_\mu$ as

$$[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + i\rho^{\nu\lambda}_\mu \widehat{\partial}_\lambda, \tag{4.5}$$

we obtain two conditions on $\rho^{\nu\lambda}_\mu$ from consistency with (4.4):

$$\rho^{\mu\nu}_\lambda - \rho^{\nu\mu}_\lambda = C^{\mu\nu}_\lambda, \qquad \rho^{\mu\nu}_\lambda \rho^{\kappa\sigma}_\nu - \rho^{\kappa\nu}_\lambda \rho^{\mu\sigma}_\nu = C^{\mu\kappa}_\nu \rho^{\nu\sigma}_\lambda. \tag{4.6}$$

All three one-parameter sets of derivatives $\hat{\partial}^{c_i}_\mu$ (4.3) fulfil the conditions (4.6). With the freedom indicated by the parametrisation in (4.3), we have exhausted all linear derivatives.

Commutation relations with coordinates can be generalised to Leibniz rules. Commuting for example the derivative $\hat{\partial}_\mu$ with an ordered monomial $f(\hat{x})$ using (4.3) this yields for $\hat{\partial}_n$ the Leibniz rule of an ordinary derivative on commutative space:

$$\hat{\partial}_n(\hat{f} \cdot \hat{g}) = (\hat{\partial}_n \hat{f}) \cdot \hat{g} + \hat{f} \cdot (\hat{\partial}_n \hat{g}). \tag{4.7}$$

Because of this undeformed Leibniz rule the derivatives $\hat{\partial}_\mu$ are a particularly suitable set of linear derivatives. The derivative $\hat{\partial}_j$ shifts every factor of $\hat{x}^n$ by ia. This shift can be implemented by the operator $e^{ia\hat{\partial}_n}$:

$$e^{\pm ia\hat{\partial}_n}(\hat{f} \cdot \hat{g}) = (e^{\pm ia\hat{\partial}_n}\hat{f}) \cdot (e^{\pm ia\hat{\partial}_n}, \hat{g}) \qquad (e^{\pm ia\hat{\partial}_n} f(\hat{x}^j, \hat{x}^n)) = f(\hat{x}^j, (\hat{x}^n \pm ia)).$$

Therefore the Leibniz rule for $\hat{\partial}_j$ is

$$\hat{\partial}_j(\hat{f} \cdot \hat{g}) = (\hat{\partial}_j \hat{f}) \cdot \hat{g} + (e^{ia\hat{\partial}_n}\hat{f}) \cdot (\hat{\partial}_j \hat{g}). \tag{4.8}$$

Similarly we determine the Leibniz rules for $\hat{\partial}^{c_1}_\mu$

$$\begin{aligned}
\hat{\partial}^{c_1}_j(\hat{f} \cdot \hat{g}) &= (\hat{\partial}^{c_1}_j \hat{f}) \cdot \hat{g} + ((1 + iac_1\hat{\partial}^{c_1}_n)^{\frac{1}{c_1}}\hat{f}) \cdot (\hat{\partial}^{c_1}_j \hat{g}), \\
\hat{\partial}^{c_1}_n(\hat{f} \cdot \hat{g}) &= (\hat{\partial}^{c_1}_n \hat{f}) \cdot \hat{g} + ((1 + iac_1\hat{\partial}^{c_1}_n)\hat{f}) \cdot (\hat{\partial}^{c_1}_n \hat{g}),
\end{aligned} \tag{4.9}$$

for $\hat{\partial}^{c_2}_\mu$

$$\begin{aligned}\hat{\partial}^{c_2}_j(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_2}_j\hat{f})\cdot((1+iac_2\hat{\partial}^{c_2}_n)\hat{g}) + \big((1+iac_2\hat{\partial}^{c_2}_n)^{\frac{c_2+1}{c_2}}\hat{f}\big)\cdot(\hat{\partial}^{c_2}_j\hat{g}),\\ \hat{\partial}^{c_2}_n(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_2}_n\hat{f})\cdot\hat{g} + \big((1+iac_2\hat{\partial}^{c_2}_n)\hat{f}\big)\cdot(\hat{\partial}^{c_2}_n\hat{g}),\end{aligned} \tag{4.10}$$

and for $\hat{\partial}^{c_3}_\mu$

$$\begin{aligned}\hat{\partial}^{c_3}_j(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_3}_j\hat{f})\cdot\hat{g} + \big((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_l\hat{\partial}^{c_3}_l)^{\frac{1}{2}}\hat{f}\big)\cdot(\hat{\partial}^{c_3}_j\hat{g}),\\ \hat{\partial}^{c_3}_n(\hat{f}\cdot\hat{g}) &= (\hat{\partial}^{c_3}_n\hat{f})\cdot\hat{g} + \big((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_l\hat{\partial}^{c_3}_l)\hat{f}\big)\cdot(\hat{\partial}^{c_3}_n\hat{g})\\ &\quad +iac_3\big((1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_l\hat{\partial}^{c_3}_l)^{\frac{1}{2}}\hat{\partial}^{c_3}_k\hat{f}\big)\cdot(\hat{\partial}^{c_3}_k\hat{g}).\end{aligned} \tag{4.11}$$

That there is such a variety of derivatives with linear commutation relations with the coordinates is disturbing at first sight. But all three families $\hat{\partial}^{c_i}_\mu$ can be mapped into each other. The derivatives $\hat{\partial}_\mu$ ($c_1 = 0$) are mapped to the derivatives $\hat{\partial}^{c_1}_\mu$ for arbitrary c_1 in the following way:

$$\hat{\partial}^{c_1}_j = \hat{\partial}_j, \qquad \hat{\partial}^{c_1}_n = \frac{e^{iac_1\hat{\partial}_n}-1}{iac_1}. \tag{4.12}$$

The role of the shift operators is played by the following operators, in terms of $\hat{\partial}^{c_1}_\mu$

$$\Big(\frac{1}{1+ic_1a\hat{\partial}^{c_1}_n}\Big)^{\frac{1}{c_1}} = e^{-ia\hat{\partial}_n} \quad\text{and}\quad (1+ic_1a\hat{\partial}^{c_1}_n)^{\frac{1}{c_1}} = e^{ia\hat{\partial}_n}. \tag{4.13}$$

The Leibniz rule (4.9) is identical to (4.8) using (4.12) and afterwards setting $c_1 = 0$.

The derivatives $\hat{\partial}^{c_2}_\mu$ can be expressed in terms of $\hat{\partial}_\mu$ as well:

$$\hat{\partial}^{c_2}_n = \frac{e^{iac_2\hat{\partial}_n}-1}{iac_2}, \qquad \hat{\partial}^{c_2}_j = \hat{\partial}_j e^{iac_2\hat{\partial}_n}, \quad e^{ia\hat{\partial}_n} = (1+ic_2a\hat{\partial}^{c_2}_n)^{\frac{1}{c_2}}. \tag{4.14}$$

The map from $\hat{\partial}_\mu$ to derivatives $\hat{\partial}^{c_3}_\mu$ reads

$$\hat{\partial}^{c_3}_n = \frac{e^{2ia\hat{\partial}_n}-1}{2ia} + \frac{iac_3}{2}\hat{\partial}_k\hat{\partial}_k, \qquad \hat{\partial}^{c_3}_j = \hat{\partial}_j, \qquad e^{ia\hat{\partial}_n} = (1+2ia\hat{\partial}^{c_3}_n + a^2c_3\hat{\partial}^{c_3}_l\hat{\partial}^{c_3}_l)^{\frac{1}{2}}. \tag{4.15}$$

Because of their simple Leibniz rule, the derivatives $\hat{\partial}_\mu$ will be the preferable basis in the algebra of derivatives to develop our formalism. The maps (4.12), (4.14) and (4.15) allow to reformulate the entire formalism, which we will develop in the following based on $\hat{\partial}_\mu$, in terms of the three one-parameter families of linear derivatives.

There is even more freedom in defining derivatives, if the condition is lifted that the commutator (4.2) has to be linear in the derivatives. We will introduce several such derivatives, for which we lift the condition of linearity, but impose other conditions.

We emphasise in comparison with section 3.1 that the derivatives defined in this section are exterior derivatives. There are no elements in the algebra of functions of NC coordinates (i.e. formal power series) that have the property: $[\hat{f}_\mu(\hat{x}), \hat{x}^\nu] = \delta^\nu_\mu + \ldots$. Non-polynomial inverses of coordinates $\frac{1}{\hat{x}^\mu}$ are not elements of the algebra of functions.

Nevertheless the commutator of a coordinate with a function can be written in such a way that they appear identical to the Leibniz rules for $\hat{\partial}_\mu$. This will be discussed shortly. The identities (2.28) and (2.29) imply the following relations, which hold independently of a specific ordering:

$$x^j \star f(x) = \big(e^{ia\partial_n} f(x)\big) \star x^j, \quad \text{and} \quad x^n \star f(x) = f(x) \star x^n - iax^k\partial_k f(x). \tag{4.16}$$

Because of the properties of the $\star$-products involving a coordinate, this is identical to

$$\begin{aligned} (-iax^j\partial_n)\big(f \star g\big) &= \big((-iax^j\partial_n)f\big) \star g + \big(e^{ia\partial_n} f\big) \star \big((-iax^j\partial_n)g\big), \\ (-iax^k\partial_k)\big(f \star g\big) &= \big((-iax^k\partial_k)f\big) \star g + f \star \big((-iax^k\partial_k)g\big). \end{aligned} \tag{4.17}$$

Having introduced derivatives $\hat{\partial}_\mu$, we can lift these relations into the abstract algebra. We will show later that the NC quantities $\hat{\partial}_n$ and $\hat{x}^k\hat{\partial}_k$ and the commutative quantities ∂_n and $x^k\partial_k$ can be identified:

$$\hat{x}^j \hat{f}(\hat{x}) = \big(e^{ia\hat{\partial}_n} \hat{f}(\hat{x})\big)\hat{x}^j, \quad \text{and} \quad \hat{x}^n \hat{f}(\hat{x}) = \hat{f}(\hat{x})\hat{x}^n - ia\hat{x}^k\hat{\partial}_k \hat{f}(\hat{x}). \tag{4.18}$$

The inner derivations $[\hat{x}^\mu, \hat{f}(\hat{x})]$ therefore have the Leibniz rules

$$\begin{aligned} [\hat{x}^j, (\hat{f} \cdot \hat{g})] &= [\hat{x}^j, \hat{f}] \cdot \hat{g} + \big(e^{ia\hat{\partial}_n} \hat{f}\big) \cdot [\hat{x}^j, \hat{g}], \\ [\hat{x}^n, (\hat{f} \cdot \hat{g})] &= [\hat{x}^n, \hat{f}] \cdot \hat{g} + \hat{f} \cdot [\hat{x}^n, \hat{g}], \end{aligned} \tag{4.19}$$

which are identical to

$$\begin{aligned} \hat{x}^j(e^{ia\hat{\partial}_n} - 1)\big(\hat{f} \cdot \hat{g}\big) &= \big(\hat{x}^j(e^{ia\hat{\partial}_n} - 1)\hat{f}\big) \cdot \hat{g} + \big(e^{ia\hat{\partial}_n} \hat{f}\big) \cdot \big(\hat{x}^j(e^{ia\hat{\partial}_n} - 1)\hat{g}\big), \\ (-ia\hat{x}^k\hat{\partial}_k)\big(\hat{f} \cdot \hat{g}\big) &= \big((-ia\hat{x}^k\hat{\partial}_k)\hat{f}\big) \cdot \hat{g} + \hat{f} \cdot \big((-ia\hat{x}^k\hat{\partial}_k)\hat{g}\big). \end{aligned} \tag{4.20}$$

Equation (4.20) therefore allows the definition of a gauge theory of inner derivations as in section 3.1.

Note that the inner derivations do not commute (cp. $[\partial_\mu, \partial_\nu] = 0$):

$$[\hat{x}^n, [\hat{x}^j, \hat{f}(\hat{x})]] - [\hat{x}^j, [\hat{x}^n, \hat{f}(\hat{x})]] \neq 0, \tag{4.21}$$

in general, since $[\hat{x}^n, [\hat{x}^j, \hat{x}^n]] - [\hat{x}^j, [\hat{x}^n, \hat{x}^n]] = a^2\hat{x}^j$.

This observation finishes the discussion of inner derivations in this thesis.

4.2 $SO_a(n)$ as symmetry Hopf algebra

In contrast to the canonical NC space discussed in chapter 3, the κ-deformed space $\mathcal{A}_{\hat{x}}$ allows the definition of a symmetry action. Since κ-deformed space is n-dimensional Euclidean, the symmetry structure is a deformation of the n-dimensional group of rotations, called $SO_a(n)$

$$SO_a(n) : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}. \tag{4.22}$$

A symmetry is defined in our approach by generators $\hat{M}^{\mu\nu}$ of an abstract algebra $SO_a(n)$ over $\mathbb{C}$ and the maps $\hat{M}^{\mu\nu} : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$. These maps are implemented by specifying commutation relations with the generators $\hat{x}^\mu$ of the coordinate algebra $\mathcal{A}_{\hat{x}}$. These commutation relations have to respect the factor space (4.1). In addition, to zeroth order in a they have to coincide with those of the generators of the symmetry Lie group acting on an n-dimensional, Euclidean commutative manifold. The commutation relations therefore have to be

$$[\hat{M}^{\mu\nu}, \hat{x}^\lambda] = \delta^{\mu\lambda}\hat{x}^\nu - \delta^{\nu\lambda}\hat{x}^\mu + \mathcal{O}(a). \tag{4.23}$$

It is straightforward to see that there have to be $\mathcal{O}(a)$ terms in $[\hat{M}^{\mu\nu}, \hat{x}^\lambda]$, otherwise the relations (4.1) would not be respected[2]:

$$[\hat{M}^{\mu\nu}, \Big([\hat{x}^n, \hat{x}^j] - ia\hat{x}^j\Big)] = -ia\delta^{\mu j}\hat{x}^n + ia\delta^{\mu n}\hat{x}^j \neq 0.$$

We construct the commutation relations (4.23) order by order in a. The generators of rotations $\hat{M}^{rs}$ and $\hat{N}^l = \hat{M}^{nl}$ should appear at most linearly on the right hand side of the commutators (4.23) . The only terms admissible in $\mathcal{O}(a)$ therefore involve the generators of rotations ($\hat{M}^{rs}$ and $\hat{N}^l$) exactly once. Higher order terms in a have to be accompanied by derivatives because of dimensional reasons, $\hat{M}^{\mu\nu}$ have to have mass dimension zero (cp. the representation of commutative orbital rotations $M^{\mu\nu}_{\text{orb}} = x^\nu\partial_\mu - x^\mu\partial_\nu$). Also the indices have to match on both sides of (4.23). All these conditions imply that terms of higher order than linear in a vanish. The unique solution which is consistent with (4.1) and which forms a bialgebra is therefore [37]:

$$\begin{aligned}
[\hat{M}^{rs}, \hat{x}^n] &= 0, \\
[\hat{M}^{rs}, \hat{x}^j] &= \delta^{rj}\hat{x}^s - \delta^{sj}\hat{x}^r, \\
[\hat{N}^i, \hat{x}^n] &= \hat{x}^i + ia\hat{N}^i, \\
[\hat{N}^i, \hat{x}^j] &= -\delta^{ij}\hat{x}^n - ia\hat{M}^{ij}.
\end{aligned} \tag{4.24}$$

These commutators respect the coordinate algebra relations $[\hat{M}^{\mu\nu}, ([\hat{x}^n, \hat{x}^j] - ia\hat{x}^j)] = 0$. The deformed generators $\hat{M}^{\mu\nu}$ have the commutation relations of the Lie algebra of $SO(n)$ among themselves

$$\begin{aligned}
[\hat{M}^{rs}, \hat{M}^{tu}] &= \delta^{rt}\hat{M}^{su} + \delta^{su}\hat{M}^{rt} - \delta^{ru}\hat{M}^{st} - \delta^{st}\hat{M}^{ru}, \\
[\hat{M}^{rs}, \hat{N}^i] &= \delta^{ri}\hat{N}^s - \delta^{si}\hat{N}^r, \\
[\hat{N}^i, \hat{N}^j] &= \hat{M}^{ij}.
\end{aligned} \tag{4.25}$$

We have not shown that commutation relations among the generators of rotations other than (4.25) are not consistent with (4.1) and (4.24). But because of dimensional reasons and because of the index structure this possibility seems quite unlikely.

[2]Starting from an orbital representation $\hat{M}^{\mu\nu} = \hat{x}^\nu\hat{\partial}_\mu - \hat{x}^\mu\hat{\partial}_\nu$, these commutation relations close, but this orbital $\hat{M}^{\mu\nu}$ does not lead to a bialgebra.

Although the algebra of rotations is undeformed (4.25), the action on coordinates is deformed (4.24). Therefore we call the algebra generated by $\hat{M}^{\mu\nu}$ the algebra of $SO_a(n)$ rotations. The Gerstenhaber-Whitehead theorem [39] states that (the enveloping algebras of) semi-simple Lie algebras are rigid with respect to deformations of their algebraic structure. A nontrivial deformation can take place however w.r.t. the Hopf structure, to which the commutation relations with the coordinates conceptually belong [113]. A remark is in order concerning our notation: the generators of $SO_a(n)$ $\hat{M}^{\mu\nu}$ are infinitesimal deformed rotations. Therefore the Lie algebra $\mathfrak{so}(n)$ is deformed. However, only the enveloping algebra of the Lie algebras can be deformed consistently, we should use $\mathcal{U}_a(\mathfrak{so}(n))$. The notation $SO_a(n)$ should be considered as an abbreviation.

Consistency with the coordinate algebra $\mathcal{A}_{\hat{x}}$ leads directly to the so called bicrossproduct basis of the κ-deformed Euclidean algebra, first defined in [28]. The bicrossproduct basis is singled out by (4.25) in contrast to the so called classical basis which has been obtained contracting the q-anti-de Sitter Hopf algebra $SO_q(3,2)$ [26]. The classical and the bicrossproduct basis are related by a nonlinear change of variables. For all further constructions, consistency with (4.1) and (4.25) is the crucial touchstone. Note that we have chosen a particular point of view w.r.t. the κ-deformed case, ignoring the fact that the coordinate algebra can be treated as the translational sector of the κ-deformed Euclidean *group* (this is the Hopf algebra dual to the κ-deformed Euclidean algebra). Instead, only the infinitesimal generators of the dual κ-deformed algebra $SO_a(n)$ are taken as generators of a deformed symmetry.

In (4.8) we have generalised commutation relations $[\hat{\partial}_\mu, \hat{x}^\nu]$ to Leibniz rules. Similarly we want to construct the action of the generators of rotation on products of functions. With a slight abuse of terminology we dub their action Leibniz rule as well. We find that their action cannot be written in terms of $\hat{M}^{\mu\nu}$ alone, their action involves the derivatives $\hat{\partial}_\mu$:

$$\begin{aligned} \hat{M}^{rs}\,(\hat{f}\cdot\hat{g}) &= (\hat{M}^{rs}\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{M}^{rs}\hat{g}), \\ \hat{N}^{i}\,(\hat{f}\cdot\hat{g}) &= (\hat{N}^{i}\hat{f})\cdot\hat{g} + (e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{N}^{i}\hat{g}) - ia(\hat{\partial}_j\hat{f})\cdot(\hat{M}^{ij}\hat{g})\,. \end{aligned} \tag{4.26}$$

The equations (4.8) and (4.26) are the coproducts of the derivatives and generators of rotations:

$$\begin{aligned} \Delta\hat{\partial}_n &= \hat{\partial}_n\otimes 1 + 1\otimes\hat{\partial}_n, \\ \Delta\hat{\partial}_i &= \hat{\partial}_i\otimes 1 + e^{ia\hat{\partial}_n}\otimes\hat{\partial}_i, \\ \Delta\hat{M}^{rs} &= \hat{M}^{rs}\otimes 1 + 1\otimes\hat{M}^{rs}, \\ \Delta\hat{N}^{i} &= \hat{N}^{i}\otimes 1 + e^{ia\hat{\partial}_n}\otimes\hat{N}^{i} - ia\hat{\partial}_j\otimes\hat{M}^{ij}\,. \end{aligned} \tag{4.27}$$

The notion of coproduct can sensibly be used in this context since the generators of the κ-deformed symmetry are elements of a Hopf algebra $SO_a(n)$. A Hopf algebra [114] is an algebra, at the same time a coalgebra and equipped with an additional operation called the antipode, such that all operations are compatible. A Hopf algebra is characterised by the specification of five operations on elements of a vector space (in this case over $\mathbb{C}$).

The operations constituting an *algebra* are familiar, but we rephrase them in an unfamiliar, Hopf algebra language. The multiplication m of vector space elements or the product is a homomorphism $\mathsf{m} : SO_a(n) \otimes SO_a(n) \to SO_a(n)$ with $\mathsf{m}(\mathcal{W} \otimes \mathcal{V}) = \mathcal{W} \cdot \mathcal{V} \in SO_a(n)$ if $\mathcal{W}, \mathcal{V} \in SO_a(n)$. This simply means that the product of two algebra elements has to close in the algebra. The unit η is a homomorphism $\eta : \mathbb{C} \to SO_a(n)$ defining a unit element in the algebra which behaves like a complex number ($\mathsf{m}(\mathcal{W} \otimes \eta) = \mathsf{m}(\eta \otimes \mathcal{W}) = \mathcal{W}$).

The concept of *coalgebra* is in an abstract sense dual to the concept of an algebra. For a coalgebra two operations on vector space elements have to be specified: the coproduct $\Delta(\mathcal{W})$ and the counit $\epsilon(\mathcal{W})$. The coproduct $\Delta : SO_a(n) \to SO_a(n) \otimes SO_a(n)$ is a homomorphism of complex algebras and coassociative:

$$(\Delta \otimes 1)\Delta(\mathcal{W}) = (1 \otimes \Delta)\Delta(\mathcal{W}), \qquad \forall \mathcal{W} \in SO_a(n). \tag{4.28}$$

As can be seen from (4.27), the coproduct of $SO_a(n)$ is obviously not cocommutative.

In the language of representations, the coproduct specifies how a coalgebra element $\mathcal{W} \in SO_a(n)$ acts on products of representations (if V and W are modules of $SO_a(n)$, then $V \otimes W$ is a $SO_a(n) \otimes SO_a(n)$-module).

The counit is a homomorphism of complex algebras $\epsilon : SO_a(n) \to \mathbb{C}$ and fulfils

$$(\epsilon \otimes 1)\Delta(\mathcal{W}) = \mathbb{C} \otimes \mathcal{W} \simeq \mathcal{W} \simeq \mathcal{W} \otimes \mathbb{C} = (1 \otimes \epsilon)\Delta(\mathcal{W}), \qquad \forall \mathcal{W} \in SO_a(n), \tag{4.29}$$

where $\simeq$ denotes the fact that the tensor product with a complex number is the same as multiplication with that number. The counit describes the action of the coalgebra on the zero-dimensional representation.

For a *bialgebra*, the algebra aspects and the coalgebra aspects have to be compatible, e.g. $\Delta(\mathcal{W} \cdot \mathcal{V}) = \Delta(\mathcal{W}) \cdot \Delta(\mathcal{V})$. This compatibility condition involves an (obvious) twist operation, which has not been denoted here. The condition has been checked explicitly for (4.27). In addition it is required that $\Delta\eta = \eta \otimes \eta$, $\eta(\epsilon) = 1$ and $\epsilon(\mathcal{W}) \otimes \epsilon(\mathcal{V}) = \epsilon(\mathcal{W} \cdot \mathcal{V})$.

For a *Hopf algebra*, the antipode $S(\mathcal{W})$ which is an anti-homomorphism has to be defined such that it is compatible with all other operations:

$$\mathsf{m}\big((S \otimes 1)\Delta(\mathcal{W})\big) = \eta(\epsilon(\mathcal{W})). \tag{4.30}$$

The antipode is the analog of the inverse element of groups; in the language of representations, it states the action on the dual representation.

For $SO_a(n)$ the counit and the antipode are:

$$\begin{aligned}
\epsilon(\hat{\partial}_n) &= 0, & S(\hat{\partial}_n) &= -\hat{\partial}_n, \\
\epsilon(\hat{\partial}_i) &= 0, & S(\hat{\partial}_i) &= -\hat{\partial}_i e^{-ia\hat{\partial}_n}, \\
\epsilon(\hat{M}^{rs}) &= 0, & S(\hat{M}^{rs}) &= -\hat{M}^{rs}, \\
\epsilon(\hat{N}^i) &= 0, & S(\hat{N}^i) &= -\hat{N}^i e^{-ia\hat{\partial}_n} - ia\hat{M}^{ik}\hat{\partial}_k e^{-ia\hat{\partial}_n} - ia(n-1)\hat{\partial}_i e^{-ia\hat{\partial}_n}.
\end{aligned} \tag{4.31}$$

For groups the inverse of the inverse is the identity and the dual of the dual representation is again the original one. Applying the antipode twice, this is obviously not necessarily the case for a deformed Hopf algebra such as $SO_a(n)$. We obtain $S^2(\mathcal{W}) = \mathcal{W}$ for $\mathcal{W} \neq \hat{N}^i$ and for $\hat{N}^i$

$$S^2(\hat{N}^i) = \hat{N}^i + ia(n-1)\hat{\partial}_i \neq \hat{N}^i. \tag{4.32}$$

We have introduced $\hat{M}^{\mu\nu}$ in (4.24) as the generators of $SO_a(n)$ rotations. Since the coproduct (4.27) involves derivatives, we can consistently deform - as a Hopf algebra - only the Lie algebra of the inhomogeneous $SO(n)$, the Euclidean group. The Hopf algebra $SO_a(n)$ is the symmetry structure obtained from deforming the Lie algebra of the Euclidean group.

We prefer to work in a basis, in which the commutation relations of all generators of $SO_a(n)$ are the same as in the undeformed inhomogeneous $SO(n)$. Therefore the generators are $\hat{M}^{\mu\nu}$, but not the derivatives $\hat{\partial}_\mu$ introduced in (4.3) as a minimal, linear deformation of commutative partial derivatives. These $\hat{\partial}_\mu$ are a module of $SO_a(n)$ rotations, i.e. they are consistent with (4.25) under

$$\begin{aligned} [\hat{M}^{rs}, \hat{\partial}_n] &= 0, \qquad [\hat{M}^{rs}, \hat{\partial}_j] = \delta^r_j \hat{\partial}_s - \delta^s_j \hat{\partial}_r, \\ [\hat{N}^i, \hat{\partial}_n] &= \hat{\partial}_i, \qquad [\hat{N}^i, \hat{\partial}_j] = \delta^i_j \frac{1 - e^{2ia\hat{\partial}_n}}{2ia} - \delta^i_j \frac{ia}{2} \hat{\partial}_k \hat{\partial}_k + ia\hat{\partial}_i \hat{\partial}_j \, . \end{aligned} \tag{4.33}$$

These commutation relations are deformed in comparison with the commutation relations of the ordinary inhomogeneous $SO(n)$. The commutation relations (4.33) therefore enforce the definition of other derivatives, called Dirac derivatives, as generators of translations in $SO_a(n)$.

The algebra of $SO_a(n)$ rotations is undeformed, therefore the action on the index part of vector and spinors is as usual (see below). However, the action of $\hat{M}^{\mu\nu}$ on $\hat{x}^\mu$-dependent functions and on the densities of a vector or a spinor is deformed. The deformed action on functions and on vector/spinor densities is described by a deformed representation of the orbital rotations $M^{\mu\nu}_{\text{orb}} = x^\nu \partial_\mu - x^\mu \partial_\nu$. Represented in terms of $\hat{x}^\mu$ and $\hat{\partial}_\mu$, this deformed orbital rotation reads:

$$\hat{M}^{rs}_{\text{orb}} = \hat{x}^s \hat{\partial}_r - \hat{x}^r \hat{\partial}_s, \qquad \hat{N}^i_{\text{orb}} = \hat{x}^i \frac{e^{2ia\hat{\partial}_n} - 1}{2ia} - \hat{x}^n \hat{\partial}_i + \frac{ia}{2} \hat{x}^i \hat{\partial}_k \hat{\partial}_k. \tag{4.34}$$

We remark that the other sets of linear derivatives $\hat{\partial}^{c_i}_\mu$ (4.3) are modules of $SO_a(n)$ rotations as well, e.g. for $\hat{\partial}^{c_1}_\mu$ we obtain:

$$\begin{aligned} [\hat{M}^{rs}, \hat{\partial}^{c_1}_n] &= 0, \qquad [\hat{M}^{rs}, \hat{\partial}^{c_1}_j] = \delta^r_j \hat{\partial}^{c_1}_s - \delta^s_j \hat{\partial}^{c_1}_r, \\ [\hat{N}^i, \hat{\partial}^{c_1}_n] &= \hat{\partial}^{c_1}_i (1 + iac_1 \hat{\partial}^{c_1}_n), \quad [\hat{N}^i, \hat{\partial}^{c_1}_j] = \delta^i_j \frac{1 - (1 + iac_1 \hat{\partial}^{c}_n)^{\frac{2}{c_1}}}{2ia} - \delta^i_j \frac{ia}{2} \hat{\partial}^{c_1}_k \hat{\partial}^{c_1}_k + ia\hat{\partial}^{c_1}_i \hat{\partial}^{c_1}_j \, . \end{aligned} \tag{4.35}$$

Similarly the module properties of $\hat{\partial}^{c_2}_\mu$ and $\hat{\partial}^{c_3}_\mu$ can be determined. The orbital part of the generators of rotations can be rewritten in terms of $\hat{\partial}^{c_i}_\mu$ as well, using (4.12), (4.14) and (4.15).

$SO_a(n)$ is a Hopf algebra over $\mathbb{C}$, the definition of the conjugation † defined in section 2.4 on $\mathcal{A}_{\hat{x}}$ can be generalised to $SO_a(n)$. For the moment we only announce that this generalisation is possible, details are spelt out in section 5.2.

4.3 Invariants and Dirac derivative

The lowest order polynomial in the coordinates which is invariant under $SO_a(n)$ rotations is not $\hat{x}^\mu\hat{x}^\mu$ but

$$\hat{I}_1 = \hat{x}^\mu\hat{x}^\mu - ia(n-1)\hat{x}^n. \tag{4.36}$$

This is a familiar result [28], the polynomial (4.36) is not invariant in the sense $[\hat{N}^i, \hat{I}_1] = 0$, but

$$\begin{aligned} [\hat{N}^i, \hat{I}_1] &= 2ia\hat{x}^\mu\hat{M}^{\mu i} + a^2(n-2)\hat{N}^i, \\ [\hat{M}^{rs}, \hat{I}_1] &= 0. \end{aligned} \tag{4.37}$$

The polynomial (4.36) can meaningfully be interpreted as an invariant, since another invariant (in the sense of (4.37)) is obtained multiplying it with any $SO_a(n)$-invariant expression from the right.

Equation (4.36) is the lowest order invariant in the coordinates alone. The Laplace operator $\hat{\Box}$ is the lowest order invariant constructed from derivatives[3] and it is invariant under $SO_a(n)$ rotations in the usual sense [26], [27]:

$$\hat{\Box} = \hat{\partial}_k\hat{\partial}_k e^{-ia\hat{\partial}_n} + \frac{2}{a^2}(1-\cos(a\hat{\partial}_n)), \quad \text{with} \quad [\hat{N}^i, \hat{\Box}] = 0, \quad [\hat{M}^{rs}, \hat{\Box}] = 0. \tag{4.38}$$

All functions $\hat{\Box} f(a^2\hat{\Box})$ of the Laplace operator are invariant as well, have the proper dimensionality and are consistent with the commutative limit $\hat{\Box} = \hat{\partial}_\mu\hat{\partial}_\mu + \mathcal{O}(a)$.

The Dirac operator $\hat{D}$ is defined as the invariant under

$$[\hat{N}^i, \hat{D}] + [n^i, \hat{D}] = 0, \qquad [\hat{M}^{rs}, \hat{D}] + [m^{rs}, \hat{D}] = 0, \tag{4.39}$$

where n^i and m^{rs} are the generators of rotations on spinorial degrees of freedom:

$$n^i = \frac{1}{4}[\gamma^i, \gamma^n], \qquad m^{rs} = \frac{1}{4}[\gamma^s, \gamma^r], \tag{4.40}$$

with Euclidean γ-matrices $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. The components of the Dirac operator $\hat{D} = \gamma^\mu\hat{D}_\mu$ are called Dirac derivatives [115], [116]. These derivatives transform linearly under $SO_a(n)$ rotations:

$$\begin{aligned} [\hat{N}^i, \hat{D}_n] &= \hat{D}_i, & [\hat{N}^i, \hat{D}_j] &= -\delta^{ij}\hat{D}_n, \\ [\hat{M}^{rs}, \hat{D}_n] &= 0, & [\hat{M}^{rs}, \hat{D}_j] &= \delta^r_j\hat{D}_s - \delta^s_j\hat{D}_r. \end{aligned} \tag{4.41}$$

Suppose there is a solution $\hat{D}_\mu$ of (4.41), then $\hat{D}_\mu f(a^2\hat{\Box})$ with an arbitrary function of the Laplace operator is a solution as well. A solution of (4.41) expressed in terms of $\hat{\partial}_\mu$ is:

$$\begin{aligned} \hat{D}_n &= \left(\frac{1}{a}\sin(a\hat{\partial}_n) + \frac{ia}{2}\hat{\partial}_k\hat{\partial}_k e^{-ia\hat{\partial}_n}\right) f(a^2\hat{\Box}), \\ \hat{D}_j &= \hat{\partial}_j e^{-ia\hat{\partial}_n} f(a^2\hat{\Box}). \end{aligned} \tag{4.42}$$

[3] We represent the Laplace operator and the Dirac derivative in terms of $\hat{\partial}_\mu$, they can equally well be represented using $\hat{\partial}^{c_i}_\mu$.

The simplest solution of (4.41) is the one with $f(a^2\hat{\Box}) = 1$. We choose this solution $\hat{D}_\mu$ to be *the* Dirac derivative, it is a nonlinear derivative in the sense of (4.2):

$$\begin{aligned}
[\hat{D}_n, \hat{x}^i] &= ia\hat{D}_i, \\
[\hat{D}_n, \hat{x}^n] &= \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu} = 1 - \frac{a^2}{2}\hat{\Box}, \\
[\hat{D}_j, \hat{x}^i] &= \delta^i_j\Big(- ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\Big) = \delta^i_j\Big(1 - ia\hat{D}_n - \frac{a^2}{2}\hat{\Box}\Big), \\
[\hat{D}_j, \hat{x}^n] &= 0.
\end{aligned} \tag{4.43}$$

Its coproduct is given by

$$\begin{aligned}
\Delta\hat{D}_n &= \hat{D}_n \otimes e^{-ia\hat{\partial}_n} + e^{ia\hat{\partial}_n} \otimes \hat{D}_n + ia\hat{D}_i e^{ia\hat{\partial}_n} \otimes \hat{D}_i, \\
\Delta\hat{D}_j &= \hat{D}_j \otimes e^{-ia\hat{\partial}_n} + 1 \otimes \hat{D}_j.
\end{aligned} \tag{4.44}$$

The map (4.42) from $\hat{D}_\mu$ to $\hat{\partial}_\mu$ can be inverted, using the identities (found independently in [117]):

$$\begin{aligned}
e^{-ia\hat{\partial}_n} &= -ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu} = 1 - ia\hat{D}_n - \frac{a^2}{2}\hat{\Box}, \\
e^{ia\hat{\partial}_n} &= \frac{ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_j\hat{D}_j} = \frac{1 + ia\hat{D}_n - \frac{a^2}{2}\hat{\Box}}{1 - a^2\hat{D}_j\hat{D}_j}.
\end{aligned} \tag{4.45}$$

We obtain the map inverse to (4.42)

$$\begin{aligned}
\hat{\partial}_n &= -\frac{1}{ia}\ln\left(-ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right) = -\frac{1}{ia}\ln\left(1 - ia\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right), \qquad (4.46) \\
\hat{\partial}_j &= \frac{\hat{D}_i}{1 - a^2\hat{D}_k\hat{D}_k}\left(ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right) = \frac{\hat{D}_i}{1 - a^2\hat{D}_k\hat{D}_k}\left(1 + ia\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right).
\end{aligned}$$

The Dirac and the linear derivative are different bases in the abstract algebra of derivatives, which are modules of $SO_a(n)$ and consistent with the coordinate algebra $\mathcal{A}_{\hat{x}}$. We could construct maps $\hat{D}_\mu \leftrightarrow \hat{\partial}^{c_i}_\mu$ as well.

The Dirac derivative $\hat{D}_\mu$ together with $\hat{M}^{\mu\nu}$, the generators of $SO_a(n)$ rotations, form the particular κ-deformed Euclidean Hopf algebra which is undeformed in the algebra sector, (4.25) and (4.41). The deformation is purely in the coalgebra sector, (4.27) and (4.44). We will refer to this special basis of $SO_a(n)$ in the following as *the* $SO_a(n)$. Recall that it is not unique (4.42). We now quote once more all relations of the full $SO_a(n)$, without using $\hat{\partial}_\mu$ as a shorthand. The algebra relations are undeformed,

$$\begin{aligned}
[\hat{M}^{\mu\nu}, \hat{M}^{\kappa\lambda}] &= \delta^{\mu\kappa}\hat{M}^{\nu\lambda} + \delta^{\nu\lambda}\hat{M}^{\mu\kappa} - \delta^{\mu\lambda}\hat{M}^{\nu\kappa} - \delta^{\nu\kappa}\hat{M}^{\mu\lambda}, \\
[\hat{M}^{\mu\nu}, \hat{D}_\lambda] &= \delta^\mu_\lambda\hat{D}_\nu - \delta^\nu_\lambda\hat{D}_\mu,
\end{aligned}$$

while the coalgebra is deformed

$$\begin{aligned}
\Delta\hat{M}^{rs} &= \hat{M}^{rs}\otimes 1 + 1\otimes\hat{M}^{rs},\\
\Delta\hat{N}^{i} &= \hat{N}^{i}\otimes 1 + \frac{ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_j\hat{D}_j}\otimes\hat{N}^{i}\\
&\quad -\frac{ia\hat{D}_k}{1-a^2\hat{D}_j\hat{D}_j}\left(ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\otimes\hat{M}^{ik},\\
\Delta\hat{D}_n &= \hat{D}_n\otimes\left(-ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)+\frac{ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_j\hat{D}_j}\otimes\hat{D}_n\\
&\quad +ia\frac{\hat{D}_k}{1-a^2\hat{D}_j\hat{D}_j}\left(ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)\otimes\hat{D}_k, \qquad (4.47)\\
\Delta\hat{D}_j &= \hat{D}_j\otimes\left(-ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\right)+1\otimes\hat{D}_j.
\end{aligned}$$

Together with the counit and the antipode of the Dirac derivative

$$\begin{aligned}
\epsilon(\hat{D}_n) &= 0, \qquad S(\hat{D}_n) = -\hat{D}_n + ia\hat{D}_l\hat{D}_l\frac{ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k},\\
\epsilon(\hat{D}_j) &= 0, \qquad S(\hat{D}_j) = -\hat{D}_j\frac{ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k}, \qquad (4.48)
\end{aligned}$$

and the property $S^2(\hat{D}_\mu) = \hat{D}_\mu$, all operations of the Euclidean Hopf algebra $SO_a(n)$ generated by $\hat{M}^{\mu\nu}$ and $\hat{D}_\mu$ have been defined.

The antipode of the Laplace operator is $S(\hat{\Box}) = \hat{\Box}$, its commutator with coordinates

$$\begin{aligned}
[\hat{\Box},\hat{x}^i] &= 2\hat{D}_i,\\
[\hat{\Box},\hat{x}^n] &= 2\hat{D}_n, \qquad (4.49)
\end{aligned}$$

and its coproduct

$$\begin{aligned}
\hat{\Box}\,(\hat{f}\cdot\hat{g}) &= (\hat{\Box}\hat{f})\cdot(e^{-ia\hat{\partial}_n}\hat{g}) + (e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{\Box}\hat{g}) +\\
&\quad +2(\hat{D}_i e^{ia\hat{\partial}_n}\hat{f})\cdot(\hat{D}_i\hat{g}) + \frac{2}{a^2}((1-e^{ia\hat{\partial}_n})\hat{f})\cdot((1-e^{-ia\hat{\partial}_n})\hat{g}). \qquad (4.50)
\end{aligned}$$

The square of the Dirac derivative is not the Laplace operator, but

$$\hat{D}_\mu\hat{D}_\mu = \hat{\Box}(1-\frac{a^2}{4}\hat{\Box}), \qquad (4.51)$$

and therefore

$$\hat{\Box} = \frac{2}{a^2}\Big(1-\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\Big). \qquad (4.52)$$

The relation (4.51) can potentially be very troublesome. It implies that the Dirac derivative used as the derivative operator in physical equations of motion has a non-trivial pole structure. The mass of a particle would be equivocal. However, having in mind the caveats below equations (4.38) and (4.42) we could rescale the Dirac derivative $\hat{D}'_\mu = \frac{\hat{D}_\mu}{1-\frac{a^2}{4}\hat{\Box}}$ and the Laplace operator $\hat{\Box}' = \frac{\hat{\Box}}{1-\frac{a^2}{4}\hat{\Box}}$ such that $\hat{D}'_\mu \hat{D}'_\mu = \hat{\Box}'$. Alternatively, we could rescale only the Dirac derivative $\hat{D}''_\mu = \frac{\hat{D}_\mu}{\sqrt{1-\frac{a^2}{4}\hat{\Box}}}$, such that $\hat{D}''_\mu \hat{D}''_\mu = \hat{\Box}$.

We will find in section 4.6, that the (antipode of) $\hat{D}'_\mu$ appears naturally in the context of frame one-forms. Attractive as these derivatives $\hat{D}'_\mu$ and $\hat{D}''_\mu$ operators may be, they are very complicated to handle technically. The (square-root of the) Laplace operator in the denominator can only be analysed perturbatively, it seems to be impossible to formulate a closed expression for the coproduct of $\hat{D}'_\mu$ or $\hat{D}''_\mu$. We postpone the analysis of these derivatives to future research.

There are also further invariants, such as in four-dimensional κ-Minkowski spacetime the Pauli-Lubanski vector, which has been discussed in [27] and in [108] in the bicrossproduct basis. From (4.27) and (4.41) the generalisation of the Pauli-Lubanski vector in $n = 2m$ Euclidean dimensions can be deduced:

$$\begin{aligned} W^2_{i+1} &= W_{\mu_1\dots\mu_{2i-1}} W_{\mu_1\dots\mu_{2i-1}}, \qquad W^2_1 = \hat{D}_\mu \hat{D}_\mu, \qquad i = 1, \dots \frac{n-2}{2}, \\ W_{\mu_1\dots\mu_{2i-1}} &= \epsilon_{\mu_1\dots\mu_n} M^{\mu_{2i}\mu_{2i+1}} \dots M^{\mu_{n-2}\mu_{n-1}} \hat{D}_{\mu_n}. \end{aligned} \tag{4.53}$$

All other invariants should be identical to their undeformed counterparts, if we exchange ordinary with the Dirac derivatives, since the algebra sector in the basis of $SO_a(n)$ generated by $\hat{M}^{\mu\nu}$ is undeformed.

4.4 κ-deformed transformations of fields

This action of symmetry generators on products of functions has to be discussed more carefully, since this issue lies at the heart of deformed field theory. We analyse in this section the notion of a transformed field and the properties of a covariant field equation.

A scalar field on commutative space transforming invariantly under a symmetry is defined by:

$$\phi'(x') = \phi(x), \tag{4.54}$$

$$\text{if} \quad x \xrightarrow{\epsilon} x' \quad \Rightarrow \quad \phi(x) \xrightarrow{\epsilon} \phi'(x). \tag{4.55}$$

For example, let ϵ be a finite translation: $x^\mu \to x^{\mu'} = x^\mu + \epsilon^\mu$. Taylor expanding $\phi'(x')$ in terms of x, the transformation law can be rewritten as

$$\delta_\epsilon \phi(x) = \phi'(x) - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x) + \mathcal{O}(\epsilon^2) \tag{4.56}$$

Rotations and other transformations are treated analogously. If the field is in addition in a non-scalar representation of the symmetry group, e.g. a spinor or a tensor field, it has additional degrees of freedom, these have to be transformed accordingly.

We now show that the covariance condition (4.55) cannot be generalised in a straightforward way to NC spaces. We have introduced fields as formal power series in the abstract coordinate algebra in 2.1:

$$\hat{\phi}(\hat{x}) = \sum_{n=0}^{\infty} \sum_{(\mu_1,\ldots,\mu_n)} \phi_{\mu_1\ldots\mu_n} : (\hat{x}^1)^{\mu_1} \ldots (\hat{x}^n)^{\mu_i} : . \tag{4.57}$$

Fields are elements of $\mathcal{A}_{\hat{x}}$, therefore they can be added, they can be multiplied by complex numbers and they can be multiplied. In addition this algebra of fields is acted upon by the symmetry Hopf algebra, it induces maps $SO_a(n) : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$ through the symmetry generators with a deformed coproduct. The action of a symmetry generator on a coordinate is in Hopf algebra notation written as $\hat{M}^{\mu\nu} \triangleright \hat{x}^{\lambda}$. This notation means that all terms of a commutation relation $[\hat{M}^{\mu\nu}, \hat{x}^{\lambda}] = \ldots$, where $\hat{M}^{\mu\nu}$ acts to the right, are omitted:

$$\hat{x}'^{\mu} = \hat{x}^{\mu} + \epsilon_l \hat{N}^l \triangleright \hat{x}^{\mu} \equiv \hat{x}^{\mu} + \epsilon_l \left(N^l \hat{x}^{\mu}\right). \tag{4.58}$$

We prefer the notation using brackets instead of $\triangleright$. The nontrivial commutation relation of $\hat{N}^l$ with a coordinate (4.47) forbids to write expressions involving products of transformed coordinates:

$$\hat{x}'^{\mu}\hat{x}'^{\nu} = (1 + \epsilon_l \hat{N}^l)\hat{x}^{\mu}(1 + \epsilon_l \hat{N}^l)\hat{x}^{\nu} \neq (1 + \epsilon_l \hat{N}^l)\hat{x}^{\mu}\hat{x}^{\nu}.$$

Therefore $\hat{\phi}'(\hat{x}') \neq (1+\epsilon_l \hat{N}^l) \sum_{n=0}^{\infty} \sum_{(\mu_1,\ldots,\mu_n)} \phi_{\mu_1\ldots\mu_n} : (\hat{x}^1)^{\mu_1} \ldots (\hat{x}^n)^{\mu_i} :$. Instead of $\phi(\hat{x}) = \hat{\phi}'(\hat{x}')$ we use the following definition to compare the transformed and the untransformed field

$$\phi(\hat{x}) = (1 + \epsilon_l \hat{N}^l)\phi'(\hat{x}), \tag{4.59}$$

it defines a scalar field transforming covariantly under the symmetry generated by the rotation $\hat{N}^l$. Similarly the covariance of a field under a translation is treated

$$(1 + \epsilon^{\mu}\hat{D}_{\mu})\hat{\phi}'(\hat{x}) = \hat{\phi}(\hat{x}). \tag{4.60}$$

Having defined the transformation behaviour of a scalar field, κ-covariant field equations can be defined. They are implemented by acting on the covariant field with a derivative operator which itself is invariant under $SO_a(n)$ transformations. For example, the Laplace operator (4.38) defines the Laplace equation for a covariant field

$$\left(\hat{\Box} + m^2\right) \hat{\phi}(\hat{x}) = 0. \tag{4.61}$$

This is an adequate covariant field equation, since

$$(1 + \epsilon_l \hat{N}^l) \left(\hat{\Box} + m^2\right) \hat{\phi}'(\hat{x}) = \left(\hat{\Box} + m^2\right) (1 + \epsilon_l \hat{N}^l)\hat{\phi}'(\hat{x}) = \left(\hat{\Box} + m^2\right) \hat{\phi}(\hat{x}). \tag{4.62}$$

Physical fields with additional spacetime degrees of freedom have to be transformed according to the representation that they belong to. Since the algebraic properties of the $SO_a(n)$ are undeformed, the representations of the symmetry algebra (to which the physical fields belong) are the same as for the commutative theory. Vector fields can be defined in analogy to the Dirac derivative, as well as tensor and spinor fields[4]. Therefore the particle content of a field theory on κ-deformed space is identical to that of the commutative theory. This was the reason why we emphasised to choose a basis of $SO_a(n)$ with an undeformed algebra sector.

The transformation behaviour of fields with additional structure, e.g. of a spinor field, can be written in the following way:

$$(1+\epsilon_l \hat{N}^l)\hat{\psi}'_\sigma(\hat{x}) = (1+\epsilon_l N^l_{\text{spin}})_{\sigma\rho}\hat{\psi}_\rho(\hat{x}), \tag{4.63}$$

where N^l_{spin} is a representation of N^l acting on the coordinate independent (spinorial) part of ψ. The generalisation to vector fields or tensor fields is obvious.

In chapter 6, we will formulate gauge theories by means of covariant derivatives, constructed from the Dirac derivative. This will render the theory covariant, provided $\hat{D}_\mu\hat{\psi}(\hat{x})$ transforms appropriately:

$$(1+\epsilon_l \hat{N}^l)\hat{D}_\mu\hat{\phi}'(\hat{x}) \quad \text{replaces} \quad \hat{D}'_\mu\hat{\phi}'(\hat{x}'). \tag{4.64}$$

Therefore

$$\begin{aligned}(1+\epsilon_l \hat{N}^l)\hat{D}_\mu\hat{\phi}'(\hat{x}) &= \hat{D}_\mu(1+\epsilon_l\hat{N}^l)\hat{\phi}'(\hat{x}) + \epsilon_l[\hat{N}^l,\hat{D}_\mu]\hat{\phi}'(\hat{x}) \\ &= \hat{D}_\mu\hat{\phi}(\hat{x}) + \epsilon_l(\delta^n_\mu\hat{D}_l - \delta^l_\mu\hat{D}_n)\hat{\phi}(\hat{x}).\end{aligned} \tag{4.65}$$

It is possible to define a Euclidean Dirac equation with the Dirac derivative :

$$\left(i\gamma^\lambda\hat{D}_\lambda - m\right)\hat{\psi}(\hat{x}) = 0, \tag{4.66}$$

which is covariant in the same sense as (4.62):

$$(1+\epsilon_l\hat{N}^l)\left(i\gamma^\lambda_{\rho\sigma}\hat{D}_\lambda - m\delta_{\rho\sigma}\right)\psi'_\sigma = (1+\epsilon_l\hat{N}^l_{\text{spin}})_{\phi\rho}\left(i\gamma^\lambda_{\rho\sigma}\hat{D}_\lambda - m\delta_{\rho\sigma}\right)\hat{\psi}_\sigma(\hat{x}). \tag{4.67}$$

The transformation law of the derivative of a scalar field is used to define the transformation law of a vector field $\hat{V}_\mu$:

$$(1+\epsilon_l\hat{N}^l)\hat{V}'_\mu(\hat{x}) = \hat{V}_\mu(\hat{x}) + \epsilon_l(\delta^n_\mu\hat{V}_l - \delta^l_\mu\hat{V}_n). \tag{4.68}$$

Gauge potentials will be introduced as vector fields $\hat{V}_\mu$ with gauge degrees of freedom, which transform like the Dirac derivative under $SO_a(n)$. Thus, the gauge-covariant derivative

$$\hat{\mathcal{D}}_\mu = \hat{D}_\mu - i\hat{V}_\mu, \tag{4.69}$$

is covariant under κ-deformed rotations.

[4]Spinors are usually defined as representations of the spin covering group. This structure has not been defined yet for the deformed symmetry Hopf algebra, therefore we have to warn that this loose definition might have some yet unknown problems.

4.5 Vector-like transforming one-forms

A crucial ingredient of a geometric approach towards the κ-deformed space is the exterior differential, denoted by d. In order to find a representation of d, a working definition of a one-form is needed.

The expected properties of d are:

- nilpotency: $\mathrm{d}^2 = 0$;
- application of d to a coordinate gives a one-form $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$;
- invariance under $SO_a(n)$: $[\hat{M}^{rs}, \mathrm{d}] = 0$, $[\hat{N}^l, \mathrm{d}] = 0$;
- undeformed Leibniz rule: $\mathrm{d}(\hat{f} \cdot \hat{g}) = (\mathrm{d}\hat{f}) \cdot \hat{g} + \hat{f} \cdot \mathrm{d}\hat{g}$.

Demanding invariance of d under $SO_a(n)$, a natural ansatz is that the Dirac derivative $\hat{D}_\mu$ is the convenient derivative dual to a set of vector-like transforming one-forms $\hat{\xi}^\mu$, d$= \hat{\xi}^\mu \hat{D}_\mu$:

$$[\hat{M}^{rs}, \hat{\xi}^\mu] = \delta^{r\mu}\hat{\xi}^s - \delta^{s\mu}\hat{\xi}^r, \qquad [\hat{N}^l, \hat{\xi}^\mu] = \delta^{n\mu}\hat{\xi}^l - \delta^{l\mu}\hat{\xi}^n. \tag{4.70}$$

The nilpotency of $\mathrm{d}^2 = 0$ can be achieved requiring that one-forms $\hat{\xi}^\mu$ commute with derivatives and anti-commute among themselves $\{\hat{\xi}^\mu, \hat{\xi}^\nu\} = 0$.

Demanding that the commutator of d with a coordinate is a one-form, $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$ is a sufficient condition for an undeformed Leibniz rule of d.

If we add the condition that the commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ close in the space of one-forms alone, there is no differential calculus consisting of n one-forms fulfilling all these conditions simultaneously. Under this additional condition, a familiar result (e.g. [118]) states that the basis of one-forms is $(n+1)$-dimensional.

There have been hints towards this result in our discussion of the Dirac operator (4.43). Its commutator with the coordinates $[\hat{D}_\mu, \hat{x}^\nu]$ is an infinite power series in the Dirac derivative alone, but it is linear adding the Laplace operator $\hat{\Box}$. The $(n+1)$-dimensional set of derivatives $(\hat{D}_\mu, \hat{\Box})$ is the dual of the $(n+1)$-dimensional set of one-forms $(\widehat{\mathrm{d}x}^\mu, \widehat{\phi})$ introduced in [118]

$$\begin{aligned}
\mathrm{d} &= \widehat{\mathrm{d}x}^n \hat{D}_n + \widehat{\mathrm{d}x}^j \hat{D}_j - \frac{a^2}{2}\widehat{\phi}\,\hat{\Box}, && [\mathrm{d}, \hat{x}^\mu] = \widehat{\mathrm{d}x}^\mu, \\
[\widehat{\mathrm{d}x}^n, \hat{x}^n] &= a^2\widehat{\phi}, \qquad [\widehat{\mathrm{d}x}^j, \hat{x}^n] = 0, && [\widehat{\phi}, \hat{x}^n] = -\widehat{\mathrm{d}x}^n, \\
[\widehat{\mathrm{d}x}^n, \hat{x}^i] &= ia\widehat{\mathrm{d}x}^i, \qquad [\widehat{\mathrm{d}x}^j, \hat{x}^i] = -ia\delta^{ij}\widehat{\mathrm{d}x}^n + a^2\delta^{ij}\widehat{\phi}, && [\widehat{\phi}, \hat{x}^i] = -\widehat{\mathrm{d}x}^i.
\end{aligned} \tag{4.71}$$

It is a general observation in NC geometry [119] that the set of linear (bicovariant) one-forms on a particular space has one element more than in the commutative setting. In this case it is acceptable at first sight since for $a \to 0$, $\mathrm{d} \to \mathrm{d}_{\text{class}}$. But several problems remain. Only n one-forms can be obtained by applying d to the coordinates. A gauge theory with gauge potentials as one-forms would result in an additional degree of freedom in the gauge potentials. The cohomology of the differential calculus has an entirely different structure than in the commutative case.

We will therefore follow a different strategy and demand as a central condition that there are only n one-forms on the NC spacetime. Of course we will not get this condition for free, there will be a trade-off of the kind that the one-forms $\hat{\xi}^\mu$ will have derivative-valued commutation relations with the coordinates.

The one-forms $\hat{\xi}^\mu$ are characterised by their commutation relations with the coordinates:

$$
\begin{aligned}
\hat{\xi}^\nu \stackrel{!}{=} [\mathrm{d}, \hat{x}^\nu] &= [\hat{\xi}^\mu \hat{D}_\mu, \hat{x}^\nu] = \\
&= [\hat{\xi}^n, \hat{x}^\nu]\hat{D}_n + \hat{\xi}^n\Big(ia\hat{D}_\nu + \delta^{\nu n}\big(-ia\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\big)\Big) \\
&\quad +[\hat{\xi}^i, \hat{x}^\nu]\hat{D}_i + \hat{\xi}^i\delta^{\nu i}\big(-ia\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}\big).
\end{aligned} \tag{4.72}
$$

To calculate the commutator $[\hat{\xi}^\mu, \hat{x}^\nu]$ we have made a general ansatz with derivative-valued commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ involving all terms compatible with the index structure. For example, up to second order a general ansatz reads (α, β, γ are constants to be determined):

$$
\begin{aligned}
[\hat{\xi}^n, \hat{x}^n]|_{\mathcal{O}(a^2)} &= (\alpha + \frac{1}{2})a^2\hat{\xi}^n\hat{D}_n + \gamma a^2\hat{\xi}^j\hat{D}_j, \\
[\hat{\xi}^i, \hat{x}^n]|_{\mathcal{O}(a^2)} &= (\beta + \frac{1}{2})a^2\hat{\xi}^n\hat{D}_i - \gamma a^2\hat{\xi}^i\hat{D}_n.
\end{aligned}
$$

The solution to this ansatz is derived requiring that the commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ are compatible with (4.1). Invariance under $SO_a(n)$ rotations does not add further constraints and we find the unique solution:

$$
[\hat{\xi}^\mu, \hat{x}^\nu] = ia(\delta^{\mu n}\hat{\xi}^\nu - \delta^{\mu\nu}\hat{\xi}^n) + (\hat{\xi}^\mu\hat{D}_\nu + \hat{\xi}^\nu\hat{D}_\mu - \delta^{\mu\nu}\hat{\xi}^\rho\hat{D}_\rho)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}. \tag{4.73}
$$

In components, (4.73) reads:

$$
\begin{aligned}
[\hat{\xi}^n, \hat{x}^n] &= (\hat{\xi}^n\hat{D}_n - \hat{\xi}^k\hat{D}_k)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\sigma\hat{D}_\sigma}, \\
[\hat{\xi}^i, \hat{x}^n] &= (\hat{\xi}^i\hat{D}_n + \hat{\xi}^n\hat{D}_i)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\sigma\hat{D}_\sigma}, \\
[\hat{\xi}^n, \hat{x}^j] &= ia\hat{\xi}^j + (\hat{\xi}^n\hat{D}_j + \hat{\xi}^j\hat{D}_n)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\sigma\hat{D}_\sigma}, \\
[\hat{\xi}^i, \hat{x}^j] &= -ia\delta^{ij}\hat{\xi}^n + (\hat{\xi}^i\hat{D}_j + \hat{\xi}^j\hat{D}_i - \delta^{ij}\hat{\xi}^\sigma\hat{D}_\sigma)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\sigma\hat{D}_\sigma}.
\end{aligned} \tag{4.74}
$$

To derive (4.73), we have used the following formulae:

$$
\begin{aligned}
[\hat{D}_\sigma\hat{D}_\sigma, \hat{x}^\nu] &= 2\hat{D}_\nu\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma},\\
[\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}, \hat{x}^\nu] &= -a^2\hat{D}_\nu,\\
[\Big(\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}\Big), \hat{x}^\nu] &= \Big(\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}\Big)^2\hat{D}_\nu.
\end{aligned}
\tag{4.75}
$$

As an aside note that $\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda} = \frac{a^2}{2}\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$.

The reduction of the number of one-forms from $(n+1)$ to n has the drawback that the commutator (4.73) is highly nonlinear in the Dirac derivative. But be aware that it was not clear from the outset that (4.72) can be solved at all.

It is possible to generalise one of the conditions for the differential calculus, the undeformed Leibniz rule $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$. We define commutation relations between a second set of one-forms $\widetilde{\xi}^\mu$ and coordinates $\hat{x}^\nu$, consistent with (4.1) and (4.25). For these one-forms $\widetilde{\xi}^\mu$ the application of d to a coordinate does not return the one-form, but a derivative-valued expression

$$
[\widetilde{\xi}^n\hat{D}_n + \widetilde{\xi}^j\hat{D}_j, \hat{x}^\nu] = [\mathrm{d}, \hat{x}^\nu] = (\mathrm{d}\hat{x})^\nu = \widetilde{\xi}^\nu \cdot f(\hat{D}_n, \hat{D}_j\hat{D}_j), \tag{4.76}
$$

with a suitable function of the Dirac derivative $f(\hat{D}_n, \hat{D}_j\hat{D}_j)$.

The most general solution for (4.76) is

$$
\begin{aligned}
[\mathrm{d}, \hat{x}^\nu] &= \widetilde{\xi}^\nu + c'\,\widetilde{\xi}^\nu\Big(\frac{1}{\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}} - 1\Big),\\
[\widetilde{\xi}^\mu, \hat{x}^\nu] &= ia(\delta^{\mu n}\widetilde{\xi}^\nu - \delta^{\mu\nu}\widetilde{\xi}^n) + (1-c')\big(\widetilde{\xi}^\mu\hat{D}_\nu + \widetilde{\xi}^\nu\hat{D}_\mu - \delta^{\mu\nu}\widetilde{\xi}^\rho\hat{D}_\rho\big)\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}\\
&\quad + c'\,\widetilde{\xi}^\nu\hat{D}_\mu\frac{a^2}{\sqrt{1-a^2\hat{D}_\lambda\hat{D}_\lambda}},
\end{aligned}
\tag{4.77}
$$

for an arbitrary constant c'. The solution (4.73) corresponding to $c' = 0$ will be used exclusively in the following.

We have not been able to calculate the coproduct of the $\hat{\xi}^\mu$ so far. Therefore we are not able to calculate the action of d on a general x-dependent one-form $\alpha_\mu(\hat{x})\hat{\xi}^\mu$:

$$
\mathrm{d}\alpha = \mathrm{d}(\alpha_\mu(\hat{x})\hat{\xi}^\mu) = \hat{\xi}^\nu(\hat{D}_\nu\alpha_\mu(\hat{x}))\hat{\xi}^\mu \neq (\hat{D}_\nu\alpha_\mu(\hat{x}))\hat{\xi}^\nu\hat{\xi}^\mu. \tag{4.78}
$$

However, a general one-form may be defined in such a way that $\hat{\xi}^\mu$ stands to the left of the coefficient function:

$$
\mathrm{d}\alpha = \mathrm{d}(\hat{\xi}^\mu\alpha_\mu(\hat{x})) = \hat{\xi}^\nu\hat{\xi}^\mu(\hat{D}_\nu\alpha_\mu(\hat{x})). \tag{4.79}
$$

Still it is interesting to see whether there are one-forms which allow an action of d as in (4.78), independent of the order. This motivates the introduction of a second basis of one-forms, which we call frame.

4.6 Frame one-forms

We have defined the one-forms $\hat{\xi}^\mu$ based on their vector-like transformation behaviour under $SO_a(n)$. Alternatively we can define one-forms $\hat{\omega}^\mu$, called frame one-forms in the spirit of [119], starting from the condition that they should commute with coordinates $[\hat{\omega}^\mu, \hat{x}^\nu] = 0$ and therefore with all functions. We make the ansatz

$$\begin{aligned}
\hat{\xi}^n &= \hat{\omega}^n g_1(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^j \hat{D}_j g_2(\hat{D}_n, \hat{D}_l\hat{D}_l), \\
\hat{\xi}^i &= \hat{\omega}^n \hat{D}_i h_1(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^i h_2(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^j \hat{D}_j \hat{D}_i h_3(\hat{D}_n, \hat{D}_l\hat{D}_l),
\end{aligned} \tag{4.80}$$

with functions of the Dirac derivative with appropriate index structure and expand (4.73). Since we assume that $\hat{\omega}^\mu$ commute with the coordinates, we can eliminate them from the result of this expansion. Thus, we are left with commutation relations between functions of derivatives g_a and h_a with the coordinates. Because of the commutators (4.43) and

$$\begin{aligned}
[g_a, \hat{x}^n] &= \frac{\partial g_a}{\partial \hat{D}_n}\sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}, \\
[g_a, \hat{x}^j] &= \frac{\partial g_a}{\partial \hat{D}_n} ia\hat{D}_j + \frac{\partial g_a}{\partial \hat{D}_l\hat{D}_l}\hat{D}_j\left(-ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right),
\end{aligned} \tag{4.81}$$

we obtain coupled differential equations for g_a and h_a. With the abbreviations

$$\zeta_1 = \frac{1}{\sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}, \qquad \zeta_2 = \frac{1 - \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{\hat{D}_\sigma\hat{D}_\sigma},$$

$$\zeta_3 = -ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu},$$

these differential equations are

$$\begin{aligned}
\frac{\partial g_1}{\partial \hat{D}_n} &= (g_1\hat{D}_n - h_1\hat{D}_j\hat{D}_j)\zeta_1\zeta_2, & ia\frac{\partial g_1}{\partial \hat{D}_n} + 2\frac{\partial g_1}{\partial \hat{D}_j\hat{D}_j}\zeta_3 &= iah_1 + (g_1 + h_1\hat{D}_n)\zeta_2, \\
\frac{\partial g_2}{\partial \hat{D}_n} &= (g_2\hat{D}_n - h_2 - h_3\hat{D}_j\hat{D}_j)\zeta_1\zeta_2, & ia\frac{\partial g_2}{\partial \hat{D}_n} + 2\frac{\partial g_2}{\partial \hat{D}_j\hat{D}_j}\zeta_3 &= iah_3 + (g_2 + h_3\hat{D}_n)\zeta_2, \\
\frac{\partial h_1}{\partial \hat{D}_n} &= (h_1\hat{D}_n + g_1)\zeta_1\zeta_2, & ia\frac{\partial h_1}{\partial \hat{D}_n} + 2\frac{\partial h_1}{\partial \hat{D}_j\hat{D}_j}\zeta_3 &= 2h_1\zeta_2, \\
\frac{\partial h_2}{\partial \hat{D}_n} &= h_2\hat{D}_n\zeta_1\zeta_2, & ia\frac{\partial h_2}{\partial \hat{D}_n} + 2\frac{\partial h_3}{\partial \hat{D}_j\hat{D}_j}\zeta_3 &= h_2\zeta_2, \\
\frac{\partial h_3}{\partial \hat{D}_n} &= (h_3\hat{D}_n + g_2)\zeta_1\zeta_2, & ia\frac{\partial h_3}{\partial \hat{D}_n} + 2\frac{\partial h_3}{\partial \hat{D}_j\hat{D}_j}\zeta_3 &= 2h_3\zeta_2.
\end{aligned}$$

In addition we obtain the identities

$$\begin{aligned}
\frac{\partial}{\partial \hat{D}_n}\zeta_2 &= \hat{D}_n\zeta_1\zeta_2^2, & \frac{\partial}{\partial \hat{D}_j\hat{D}_j}\zeta_2 &= \frac{1}{2}\zeta_1\zeta_2^2, \\
\frac{\partial}{\partial \hat{D}_n}\zeta_3 &= -ia\zeta_1\zeta_3, & \frac{\partial}{\partial \hat{D}_n}\zeta_3^{-1} &= ia\zeta_1\zeta_3^{-1},
\end{aligned}$$

The unique solution of these differential equations is:

$$\begin{aligned} g_1 &= (1+\hat{D}_j\hat{D}_j\zeta_2\zeta_3^{-1})\frac{a^2}{2}\zeta_2, \qquad g_2 = (ia+\hat{D}_n\zeta_2)\frac{a^2}{2}\zeta_2\zeta_3^{-1}, \\ h_1 &= (-ia-\hat{D}_n\zeta_2)\frac{a^2}{2}\zeta_2\zeta_3^{-1}, \qquad h_2 = \frac{a^2}{2}\zeta_2, \qquad h_3 = \frac{a^2}{2}\zeta_2^2\zeta_3^{-1}. \end{aligned} \tag{4.82}$$

Writing d in terms of the frame one-forms $\hat{\omega}^\mu$ we obtain

$$\begin{aligned} \mathrm{d} &= \hat{\xi}^\mu\hat{D}_\mu = \Big(\hat{\omega}^n\hat{D}_n - ia\hat{\omega}^n\hat{D}_l\hat{D}_l\zeta_3^{-1} + \hat{\omega}^j\hat{D}_j\zeta_3^{-1}\Big)\frac{a^2}{2}\zeta_2 \\ &= \Big(\hat{\omega}^n\hat{D}_n + \frac{\hat{\omega}^j\hat{D}_j - ia\hat{\omega}^n\hat{D}_l\hat{D}_l}{-ia\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}\Big)\frac{a^2}{2}\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}\ . \end{aligned} \tag{4.83}$$

We simplify this result using the Laplace operator $\hat{\Box}$ and the derivatives $\hat{\partial}_\mu$

$$\mathrm{d} = \Big(\hat{\omega}^n(\frac{1}{a}\sin(a\hat{\partial}_n) - \frac{ia}{2}\hat{\partial}_l\hat{\partial}_l e^{-ia\hat{\partial}_n}) + \hat{\omega}^j\hat{\partial}_j\Big)\frac{1}{1-\frac{a^2}{4}\hat{\Box}}\ . \tag{4.84}$$

To determine the transformation behaviour of $\hat{\omega}^\mu$ under $SO_a(n)$-rotations, we first consider the derivative operators dual to $\hat{\omega}^\mu$. The factor $\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$ is an invariant under $SO_a(n)$-rotations by itself. We define

$$\widetilde{\partial}_j = \hat{\partial}_j, \qquad \widetilde{\partial}_n = \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{ia}{2}\hat{\partial}_j\hat{\partial}_j e^{-ia\hat{\partial}_n}. \tag{4.85}$$

By means of (4.33) we determine the transformation behaviour of $\widetilde{\partial}_\mu$

$$\begin{aligned} [\hat{M}^{rs},\widetilde{\partial}_j] &= \delta^r_j\widetilde{\partial}_s - \delta^s_j\widetilde{\partial}_r, \\ [\hat{M}^{rs},\widetilde{\partial}_n] &= 0, \\ [\hat{N}^l,\widetilde{\partial}_j] &= -\delta^l_j\widetilde{\partial}_n\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} - ia\delta^l_j\widetilde{\partial}_\mu\widetilde{\partial}_\mu + ia\widetilde{\partial}_j\widetilde{\partial}_l, \\ [\hat{N}^l,\widetilde{\partial}_n] &= \widetilde{\partial}_l(ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}). \end{aligned} \tag{4.86}$$

The derivatives $\widetilde{\partial}_\mu$ form a module under $SO_a(n)$ and $[\hat{N}^l,\widetilde{\partial}_\mu\widetilde{\partial}_\mu] = 0$. Comparing with (4.45) we find:

$$\begin{aligned} e^{-ia\hat{\partial}_n} &= \frac{-ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1-a^2\widetilde{\partial}_k\widetilde{\partial}_k}, \\ e^{ia\hat{\partial}_n} &= ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}, \end{aligned} \tag{4.87}$$

and the coproducts are

$$\begin{aligned} \widetilde{\partial}_j(\hat{f}\cdot\hat{g}) &= \widetilde{\partial}_j(\hat{f})\cdot\hat{g} + \big(e^{ia\hat{\partial}_n}\hat{f}\big)\cdot(\widetilde{\partial}_j\hat{g}), \\ \widetilde{\partial}_n(\hat{f}\cdot\hat{g}) &= \widetilde{\partial}_n(\hat{f})\cdot(e^{-ia\hat{\partial}_n}\hat{g}) + \big(e^{ia\hat{\partial}_n}\hat{f}\big)\cdot(\widetilde{\partial}_n\hat{g}) - ia(\widetilde{\partial}_k\hat{f})\cdot(e^{-ia\hat{\partial}_n}\widetilde{\partial}_k\hat{g}). \end{aligned} \tag{4.88}$$

For compactness, we have used (4.87) to write (4.88). The Dirac derivative $\hat{D}_\mu$ and $\widetilde{\partial}_\mu$ are closely related, we find $\widetilde{\partial}_\mu\widetilde{\partial}_\mu = \hat{D}_\mu\hat{D}_\mu$. Therefore the Laplace operator $\hat{\Box}$ can be written in terms of $\widetilde{\partial}_\mu$ as $\hat{\Box} = \frac{2}{a^2}(1 - \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu})$. Thus, we can write (4.84) purely in terms of $\hat{\omega}^\mu$ and $\widetilde{\partial}_\mu$:

$$\mathrm{d} = \left(\hat{\omega}^n\widetilde{\partial}_n + \hat{\omega}^j\widetilde{\partial}_j\right)\frac{2}{1 + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}. \tag{4.89}$$

Comparing with (4.78), we can evaluate the action of the differential d on a one-form with an arbitrary ordering of the one-forms w.r.t. the coefficient functions

$$\mathrm{d}\alpha = \mathrm{d}(\alpha_\mu(\hat{x})\hat{\omega}^\mu) = \hat{\omega}^\nu\Big(\frac{2\widetilde{\partial}_\nu}{1 + \sqrt{1 - a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}\alpha_\mu(\hat{x})\Big)\hat{\omega}^\mu = \Big(\frac{2\widetilde{\partial}_\nu}{1 + \sqrt{1 - a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}\alpha_\mu(\hat{x})\Big)\hat{\omega}^\nu\hat{\omega}^\mu. \tag{4.90}$$

From (4.86) and the requirement that d is an invariant, we can determine the transformation behaviour of $\hat{\omega}^\mu$:

$$\begin{aligned}
[\hat{M}^{rs}, \hat{\omega}^n] &= 0, \qquad [\hat{M}^{rs}, \hat{\omega}^j] = \delta^{rj}\hat{\omega}^s - \delta^{sj}\hat{\omega}^r, \\
[\hat{N}^l, \hat{\omega}^n] &= \hat{\omega}^l\sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + ia(\hat{\omega}^l\widetilde{\partial}_n - \hat{\omega}^n\widetilde{\partial}_l), \\
[\hat{N}^l, \hat{\omega}^j] &= -\delta^{lj}\hat{\omega}^n\sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + ia(\hat{\omega}^l\widetilde{\partial}_j - \hat{\omega}^j\widetilde{\partial}_l).
\end{aligned} \tag{4.91}$$

The frame one-forms form a module under $SO_a(n)$ rotations.

The commutation relations between derivatives $\widetilde{\partial}_\mu$ and coordinates are

$$\begin{aligned}
[\widetilde{\partial}_j, \hat{x}^n] &= ia\widetilde{\partial}_j, \qquad [\widetilde{\partial}_n, \hat{x}^n] = \frac{-ia^3\widetilde{\partial}_s\widetilde{\partial}_s\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1 - a^2\widetilde{\partial}_k\widetilde{\partial}_k}, \\
[\widetilde{\partial}_j, \hat{x}^i] &= \delta^i_j, \qquad [\widetilde{\partial}_n, \hat{x}^i] = -ia\widetilde{\partial}_i\frac{-ia\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1 - a^2\widetilde{\partial}_k\widetilde{\partial}_k}.
\end{aligned} \tag{4.92}$$

Taking into account the factor $\frac{2}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}$ in the commutator with the coordinates, we define $\eth_\nu = \frac{2\widetilde{\partial}_\nu}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}$ as the derivatives dual to $\hat{\omega}^\mu$, d$= \hat{\omega}^\mu\eth_\mu$.

An interesting observation is that $\widetilde{\partial}_\mu$ are the antipodes of the Dirac derivative $\widetilde{\partial}_\mu = S(\hat{D}_\mu)$. Since $S(\hat{\Box}) = \hat{\Box}$, we obtain that $\eth_\mu = \frac{\widetilde{\partial}_\mu}{1-\frac{a^2}{4}\hat{\Box}} = \frac{S(\hat{D}_\mu)}{1-\frac{a^2}{4}S(\hat{\Box})}$. We are not yet in the position to understand potential benefits of this observation.

Using $\frac{2}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}} = 1 + \frac{a^2}{4}\eth_\mu\eth_\mu$ we obtain the commutation relations of $\eth_\mu$ with coordinates:

$$
\begin{aligned}
[\eth_j, \hat{x}^n] &= \frac{ia}{2}\eth_j(1+\frac{a^2}{4}\eth_\mu\eth_\mu)\Big(1+\frac{(1-ia\eth_n-\frac{a^2}{4}\eth_\nu\eth_\nu)(1+\frac{a^2}{4}\eth_\lambda\eth_\lambda)}{1+\frac{a^2}{4}\eth_\rho\eth_\rho-a^2\eth_k\eth_k}\Big),\\
[\eth_j, \hat{x}^i] &= \delta^i_j(1+\frac{a^2}{4}\eth_\mu\eth_\mu)+\frac{a^2}{2}\eth_i\eth_j\cdot\frac{1-ia\eth_n-\frac{a^2}{4}\eth_\nu\eth_\nu}{1+\frac{a^2}{4}\eth_\rho\eth_\rho-a^2\eth_k\eth_k},\\
[\eth_n, \hat{x}^i] &= -\frac{ia}{2}\eth_i\Big(1+\frac{1+\frac{ia}{2}\eth_n-\frac{a^2}{4}\eth_\nu\eth_\nu}{1+\frac{a^2}{4}\eth_\lambda\eth_\lambda}\Big), \qquad (4.93)\\
[\eth_n, \hat{x}^n] &= (1+\frac{a^2}{4}\eth_\mu\eth_\mu)\cdot\Big(1+\frac{-\frac{ia^3}{8}\eth_s\eth_s\eth_n+(1-\frac{a^2}{4}\eth_\mu\eth_\mu)(1+\frac{a^2}{4}\eth_\nu\eth_\nu)^2}{(1+\frac{a^2}{4}\eth_\rho\eth_\rho)^2(1+(\frac{a^2}{4}\eth_\sigma\eth_\sigma)^2+\frac{a^2}{2}(\eth_n\eth_n-\eth_k\eth_k)}\Big).
\end{aligned}
$$

These complicated commutators are the price we have to pay for the fact that frame one-forms commute with all functions. It seems impossible to calculate Leibniz rules for $\eth_\mu$ from (4.93).

4.7 Vector fields

Vector fields that have the same transformation properties as derivatives under $SO_a(n)$ are a necessary ingredient for the definition of gauge theories. This has been argued for in section 4.4. Here we derive the properties of several vector fields under symmetry transformations, treating them as elements of an abstract algebra, not as $\hat{x}$-dependent densities. Therefore derivatives are not evaluated on $\hat{A}_\mu$ in terms of the coproduct.

We assume that the vector fields appear linearly on the right hand side of the commutation relations with the symmetry generators (we made the same assumption for one forms). The vector fields have to form a module of $SO_a(n)$.

Vector fields corresponding to the vector-like transforming Dirac derivative which fulfil these requirements are easily constructed:

$$
\begin{aligned}
[\hat{M}^{rs}, \hat{V}_n] &= 0, \qquad [\hat{M}^{rs}, \hat{V}_i] = \delta^r_i\hat{V}^s - \delta^s_i\hat{V}^r,\\
[\hat{N}^l, \hat{V}_n] &= \hat{V}^l, \qquad [\hat{N}^l, \hat{V}_i] = -\delta^l_i\hat{V}_n, \qquad (4.94)
\end{aligned}
$$

these vector fields $\hat{V}_\mu$ are a module of $SO_a(n)$ rotations.

It is more difficult to construct vector fields with transformation properties analogous to the other derivatives defined in previous sections. Although the derivatives $\hat{\partial}_\mu$ are not treated as conceptually fundamental objects, they establish the contact with the commutative regime, since they are very similar to ordinary derivatives, cp. section 4.8. This motivates the definition of vector fields $\hat{A}_\mu$ whose transformation behaviour corresponds to that of $\hat{\partial}_\mu$.

On the right hand side of (4.33) appears a complicated expression in terms of $\hat{\partial}_\mu$. The task is to construct the transformation law of a vector field $\hat{A}_\mu$ that agrees with (4.33), when $\hat{A}_\mu$ is re-substituted with $\hat{\partial}_\mu$. We make the choice that derivatives are always to the left of the vector field $\hat{A}_\mu$ in nonlinear expressions such as the vector field analog of (4.33). The problem can be solved in a power series expansion in a. It leads to a recursion formula with the solution [120]:

$$\begin{aligned}
[\hat{M}^{rs}, \hat{A}_i] &= \delta_i^r \hat{A}_s - \delta_i^s \hat{A}_r, \qquad [\hat{M}^{rs}, \hat{A}_n] = 0, \\
[\hat{N}^l, \hat{A}_i] &= \delta_i^l \frac{1 - e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n} \hat{A}_n - \frac{ia}{2} \delta_i^l \hat{\partial}_j \hat{A}_j + \frac{ia}{2} \big(\hat{\partial}_l \hat{A}_i + \hat{\partial}_i \hat{A}_l\big) \\
&\quad - \delta_i^l \frac{a}{2\hat{\partial}_n} \tan\big(\frac{a\hat{\partial}_n}{2}\big) \big(\hat{\partial}_n \hat{\partial}_j \hat{A}_j - \hat{\partial}_j \hat{\partial}_j \hat{A}_n\big) \\
&\quad + \Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big) \big(\hat{\partial}_n \hat{\partial}_i \hat{A}_l + \hat{\partial}_n \hat{\partial}_l \hat{A}_i - 2\hat{\partial}_l \hat{\partial}_i \hat{A}_n\big), \\
[\hat{N}^l, \hat{A}_n] &= \hat{A}_l.
\end{aligned} \tag{4.95}$$

The square of the vector field corresponding to the Dirac derivative is an invariant $[\hat{M}^{\mu\nu}, \hat{V}_\lambda \hat{V}_\lambda] = 0$. To form an invariant out of the vector field $\hat{A}_\mu$, we have to define a vector field $\breve{A}_\mu$ with transformation laws in which the derivatives are to the right of the vector field $\breve{A}_\mu$. We demand

$$[\hat{M}^{rs}, \breve{A}_\lambda \hat{A}_\lambda] = 0, \quad \text{and} \quad [\hat{N}^l, \breve{A}_\lambda \hat{A}_\lambda] = 0. \tag{4.96}$$

From (4.95) we construct the transformation laws for $\breve{A}_\mu$ such that (4.96) is fulfilled and such that $\breve{A}_\mu$ are a module of $SO_a(n)$ rotations:

$$\begin{aligned}
[\hat{M}^{rs}, \breve{A}_i] &= \delta_i^r \breve{A}_s - \delta_i^s \breve{A}_r, \qquad [\hat{M}^{rs}, \breve{A}_n] = 0, \\
[\hat{N}^l, \breve{A}_i] &= -\delta_i^l \breve{A}_n + \frac{ia}{2} \breve{A}_l \hat{\partial}_i - \frac{ia}{2} \breve{A}_i \hat{\partial}_l - \frac{ia}{2} \delta_i^l \breve{A}_j \hat{\partial}_j \\
&\quad + \frac{a}{2} \breve{A}_l \tan\big(\frac{a\hat{\partial}_n}{2}\big) \hat{\partial}_i - (\delta_i^l \breve{A}_j \hat{\partial}_j + \breve{A}_i \hat{\partial}_l) \Big(\frac{1}{\hat{\partial}_n} - \frac{a}{2} \cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big), \\
[\hat{N}^l, \breve{A}_n] &= \breve{A}_l \frac{e^{2ia\hat{\partial}_n} - 1}{2ia\hat{\partial}_n} - \breve{A}_l \frac{a}{2\hat{\partial}_n} \tan\big(\frac{a\hat{\partial}_n}{2}\big) \hat{\partial}_j \hat{\partial}_j + 2\breve{A}_j \Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big) \hat{\partial}_l \hat{\partial}_j.
\end{aligned} \tag{4.97}$$

Comparing (4.95) and (4.97), we see that $\breve{A}_\mu$ transforms with the derivatives on the right hand side. But it transforms in a different way than $\hat{A}_\mu^\dagger$, the conjugate of $\hat{A}_\mu$. The order of derivatives and vector fields in $\hat{A}_\mu^\dagger$ is simply reversed in comparison with (4.95):

$$\begin{aligned}
[\hat{M}^{rs}, \hat{A}_i^\dagger] &= \delta_i^r \hat{A}_s^\dagger - \delta_i^s \hat{A}_r^\dagger, \qquad [\hat{M}^{rs}, \hat{A}_n^\dagger] = 0, \\
[\hat{N}^l, \hat{A}_i^\dagger] &= \delta_i^l \hat{A}_n^\dagger \frac{1 - e^{2ia\hat{\partial}_n}}{2ia\hat{\partial}_n} - \frac{ia}{2} \delta_i^l \hat{A}_j^\dagger \hat{\partial}_j + \frac{ia}{2} \big(\hat{A}_i^\dagger \hat{\partial}_l + \hat{A}_l^\dagger \hat{\partial}_i\big) \\
&\quad - \delta_i^l \big(\hat{A}_j^\dagger \hat{\partial}_j \hat{\partial}_n - \hat{A}_n^\dagger \hat{\partial}_j \hat{\partial}_j\big) \frac{a}{2\hat{\partial}_n} \tan\big(\frac{a\hat{\partial}_n}{2}\big) \\
&\quad + \big(\hat{A}_l^\dagger \hat{\partial}_n \hat{\partial}_i + \hat{A}_i^\dagger \hat{\partial}_n \hat{\partial}_l - 2\hat{A}_n^\dagger \hat{\partial}_l \hat{\partial}_i\big) \Big(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n} \cot\big(\frac{a\hat{\partial}_n}{2}\big)\Big), \\
[\hat{N}^l, \hat{A}_n^\dagger] &= \hat{A}_l^\dagger.
\end{aligned} \tag{4.98}$$

Because of the different transformation behaviour $\check{A}^\dagger_\mu \neq \hat{A}_\mu$. The dual of $\hat{A}^\dagger_\mu$ is $\check{A}^\dagger_\mu$, $[\hat{N}^l, \hat{A}^\dagger_\mu \check{A}^\dagger_\mu] = [\hat{M}^{rs}, \hat{A}^\dagger_\mu \check{A}^\dagger_\mu] = 0$, its transformation behaviour is obtained by conjugation of (4.97).

The vector fields $\hat{A}_\mu$, $\check{A}_\mu$, $\check{A}^\dagger_\mu$ and $\hat{A}^\dagger_\mu$ can be obtained from the vector field $\hat{V}_\mu$ by a derivative-valued map $\hat{e}_{\mu\nu} = \hat{e}_{\mu\nu}(\partial)$

$$\hat{V}_\mu = \hat{e}_{\mu\nu} \hat{A}_\nu, \qquad \hat{A}_\mu = (\hat{e}^{-1})_{\mu\nu} \hat{V}_\nu. \tag{4.99}$$

We know the transformation properties of $\hat{V}_\mu$, $\hat{A}_\mu$ and $\hat{\partial}_\mu$, (4.94), (4.95) and (4.33). We expand these in powers of a, at zeroth order we assume that $\hat{V}_\mu\big|_{\mathcal{O}(a^0)} = \hat{A}_\mu\big|_{\mathcal{O}(a^0)}$ are the same vector field. This leads to a solvable recursion formula in a [120]:

$$\begin{aligned}
\hat{e}_{nn} &= \frac{1}{a\hat{\partial}_n}\sin(a\hat{\partial}_n) + e^{-ia\hat{\partial}_n}\Big(\frac{ia}{2} - \frac{i}{\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\Big)\frac{\hat{\partial}_k\hat{\partial}_k}{\hat{\partial}_n},\\
\hat{e}_{nj} &= \frac{i}{\hat{\partial}_n}e^{-ia\hat{\partial}_n}\tan\big(\frac{a\hat{\partial}_n}{2}\big)\hat{\partial}_j,\\
\hat{e}_{ln} &= \Big(e^{-ia\hat{\partial}_n} - \frac{1-e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}\Big)\frac{\hat{\partial}_l}{\hat{\partial}_n},\\
\hat{e}_{lj} &= \frac{1-e^{-ia\hat{\partial}_n}}{ia\hat{\partial}_n}\delta_{lj}.
\end{aligned} \tag{4.100}$$

To find the inverse of the matrix $\hat{e}_{\mu\nu}$, we have to take care to single out the right partial derivatives. The result is:

$$\begin{aligned}
(\hat{e}^{-1})_{nn} &= F^{-1}(\hat{\partial}_\mu)\frac{e^{-ia\hat{\partial}_n}-1}{-ia\hat{\partial}_n},\\
(\hat{e}^{-1})_{nj} &= F^{-1}(\hat{\partial}_\mu)\left(-\frac{i}{\hat{\partial}_n}e^{-ia\hat{\partial}_n}\tan(\frac{a\hat{\partial}_n}{2})\right)\hat{\partial}_j,\\
(\hat{e}^{-1})_{ln} &= F^{-1}(\hat{\partial}_\mu)\left(\frac{e^{-ia\hat{\partial}_n}-1}{-ia\hat{\partial}_n} - e^{-ia\hat{\partial}_n}\right)\hat{\partial}_l,\\
(\hat{e}^{-1})_{lj} &= \frac{-ia\hat{\partial}_n}{e^{-ia\hat{\partial}_n}-1}\delta_{lj} + \frac{\frac{i}{\hat{\partial}_n^2}e^{-ia\hat{\partial}_n}\tan(\frac{a\hat{\partial}_n}{2})\left(e^{-ia\hat{\partial}_n} - \frac{e^{-ia\hat{\partial}_n}-1}{-ia\hat{\partial}_n}\right)}{F(\hat{\partial}_\mu)\left(\frac{e^{-ia\hat{\partial}_n}-1}{-ia\hat{\partial}_n}\right)}\hat{\partial}_l\hat{\partial}_j,\\
F(\hat{\partial}_\mu) &= \left(\frac{1}{ia^2\hat{\partial}_n^2}\sin(a\hat{\partial}_n)\left(1-e^{-ia\hat{\partial}_n}\right) - \frac{\hat{\partial}_k\hat{\partial}_k}{2i\hat{\partial}_n^2}\tan(\frac{a\hat{\partial}_n}{2})e^{-ia\hat{\partial}_n}\left(1-e^{-ia\hat{\partial}_n}\right)\right).
\end{aligned} \tag{4.101}$$

The vector field $\check{A}_\mu$ is defined by the transformation behaviour that was derived from (4.96). As the derivatives are on the right of $\check{A}_\mu$ we make the ansatz $\check{A}_\nu = \hat{V}_\mu(\check{e}^{-1}_{\mu\nu})$ and insert it into (4.96):

$$\check{A}_\mu \hat{A}_\mu = \hat{V}_\rho(\check{e}^{-1})_{\rho\mu}(\hat{e}^{-1})_{\mu\nu}\hat{V}_\nu = \hat{V}_\nu\hat{V}_\nu, \tag{4.102}$$

therefore we conclude that

$$(\breve{e}^{-1})_{\rho\mu} = \hat{e}_{\rho\mu}. \tag{4.103}$$

The formulae for $\breve{A}^\dagger_\mu$ and $\hat{A}^\dagger_\mu$ are obtained by conjugation.

In the same manner we determine vector fields $\widetilde{A}_\mu$, corresponding to $\widetilde{\partial}_\mu$, the derivative dual to the frame one-forms up to the factor $\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$. The calculation is much simpler and we obtain:

$$\begin{aligned}
[\hat{M}^{rs}, \widetilde{A}_i] &= \delta^r_i \widetilde{A}_s - \delta^s_i \widetilde{A}_r, \qquad [\hat{M}^{rs}, \widetilde{A}_n] = 0, \\
[\hat{N}^l, \widetilde{A}_i] &= -\delta^l_i \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}\,\widetilde{A}_n + ia\widetilde{\partial}_i\widetilde{A}_l - ia\delta^l_i\widetilde{\partial}_\mu\widetilde{A}_\mu, \\
[\hat{N}^l, \widetilde{A}_n] &= \Big(ia\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}\Big)\widetilde{A}_l.
\end{aligned} \tag{4.104}$$

From (4.104) we could read off immediately the transformation behaviour of $\widetilde{A}^\dagger_\mu$, but comparing with $\breve{\widetilde{A}}_\mu$, which can be obtained from the invariant

$$[\hat{M}^{rs}, \breve{\widetilde{A}}_\mu\widetilde{A}_\mu] = 0, \quad \text{and} \quad [\hat{N}^l, \breve{\widetilde{A}}_\mu\widetilde{A}_\mu] = 0, \tag{4.105}$$

we find that $\widetilde{A}^\dagger_\mu = \breve{\widetilde{A}}_\mu$, the vector field $\widetilde{A}_\mu$ is self-conjugate:

$$\begin{aligned}
[\hat{M}^{rs}, \widetilde{A}^\dagger_i] &= \delta^r_i \widetilde{A}^\dagger_s - \delta^s_i \widetilde{A}^\dagger_r, \qquad [\hat{M}^{rs}, \widetilde{A}^\dagger_n] = 0, \\
[\hat{N}^l, \widetilde{A}^\dagger_i] &= -\delta^l_i \widetilde{A}^\dagger_n \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + ia\widetilde{A}^\dagger_l\widetilde{\partial}_i - ia\delta^l_i\widetilde{A}^\dagger_\mu\widetilde{\partial}_\mu, \\
[\hat{N}^l, \widetilde{A}^\dagger_n] &= \widetilde{A}^\dagger_l\Big(ia\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}\Big).
\end{aligned} \tag{4.106}$$

There is also a transformation matrix $\widetilde{e}_{\mu\nu}$ from $\hat{V}_\mu$ to $\widetilde{A}_\nu$ (respectively $\breve{\widetilde{A}}_\nu = \widetilde{A}^\dagger_\nu$) :

$$\widetilde{A}_\mu = \widetilde{e}_{\mu\nu}\hat{V}_\nu, \qquad \widetilde{A}^\dagger_\mu = \hat{V}_\nu\widetilde{e}_{\mu\nu}, \tag{4.107}$$

which is

$$\begin{aligned}
\widetilde{e}_{nn} &= 1, \qquad \widetilde{e}_{nj} = -ia\hat{D}_j \frac{ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k}, \\
\widetilde{e}_{ln} &= ia\hat{D}_l, \qquad \widetilde{e}_{lj} = \delta_{lj}\sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu} + a^2\hat{D}_j\hat{D}_l \frac{ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k}.
\end{aligned} \tag{4.108}$$

The inverse matrix $\hat{V}_\mu = \widetilde{A}^\dagger_\nu (\widetilde{e}^{-1})_{\mu\nu}$ with $(\widetilde{e}^{-1})_{\lambda\nu}\widetilde{e}_{\nu\mu} = \delta_{\lambda\mu}$ is

$$
\begin{aligned}
(\widetilde{e}^{-1})_{nn} &= 1 + \frac{a^2\hat{D}_k\hat{D}_k}{\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}} \frac{ia\hat{D}_n + \sqrt{1-a^2\hat{D}_\lambda\hat{D}_\lambda}}{1-a^2\hat{D}_s\hat{D}_s} = 1 + \frac{a^2\widetilde{\partial}_k\widetilde{\partial}_k}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}} \frac{-ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}{1-a^2\widetilde{\partial}_s\widetilde{\partial}_s}, \\
(\widetilde{e}^{-1})_{nj} &= \frac{ia\hat{D}_j}{\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}} \frac{ia\hat{D}_n + \sqrt{1-a^2\hat{D}_\lambda\hat{D}_\lambda}}{1-a^2\hat{D}_k\hat{D}_k} = \frac{ia\widetilde{\partial}_j}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}, \\
(\widetilde{e}^{-1})_{ln} &= \frac{-ia\hat{D}_l}{\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}} = \frac{-ia\widetilde{\partial}_l}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}} \frac{-ia\widetilde{\partial}_n + \sqrt{1-a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}{1-a^2\widetilde{\partial}_k\widetilde{\partial}_k}, \\
(\widetilde{e}^{-1})_{lj} &= \delta_{lj}\frac{1}{\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}} = \delta_{lj}\frac{1}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}.
\end{aligned}
\tag{4.109}
$$

4.8 $\star$-representations

In chapter 2 we have discussed that the algebra $\mathcal{A}_{\hat{x}}$ of functions on coordinate space can be mapped isomorphically to an algebra of functions defined on commutative space $\mathcal{A}_x[[\hbar]]$ by means of a $\star$-product. In section 4.4 we have discussed the generators of symmetry transformations $\hat{\partial}_\mu$, $\hat{M}^{rs}$ and $\hat{N}^l$ on the abstract coordinate algebra. In this section we show that the deformed action of symmetry algebras on abstract spaces and the representation of an abstract space on an ordinary function space can be combined.

We construct representations $M^{*\mu\nu}, D^*_\mu \in SO(n)[[a\hbar]]$ of the abstract generators of rotations and derivatives on the algebra of function $\mathcal{A}_x[[\hbar]]$, such that they constitute maps $SO(n)[[a\hbar]] : \mathcal{A}_x[[\hbar]] \to \mathcal{A}_x[[\hbar]]$. This will be done in such a way that the Hopf algebraic properties can be realised purely by nonlinear operators on functions of commutative spacetime. The realisation of the antipode of the Hopf algebra will be discussed in the subsequent chapter.

These representations will use only ordinary coordinates x^μ and ordinary derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$. The same approach with similar results has been pursued by several groups [55], [56], [112], [54], [121], [122].

First we consider the derivatives $\hat{\partial}_\mu$ (4.3). We want to construct a "$\star$-representation"

$$
\hat{\partial}_\mu \to \partial^*_\mu,
\tag{4.110}
$$

such that ∂^*_μ is a map of the space of functions of commuting variables into itself, $\partial^*_\mu : \mathcal{A}_x[[\hbar]] \to \mathcal{A}_x[[\hbar]]$. To zeroth order in ($\hbar a \equiv a$), ∂^*_μ has to coincide with the ordinary ∂_μ. The crucial condition is that the $\star$-representation ∂^*_μ has to fulfil the deformed Leibniz rules

$$
\begin{aligned}
\partial^*_n(f(x)\star g(x)) &= (\partial^*_n f(x))\star g(x) + f(x)\star(\partial^*_n g(x)), \\
\partial^*_j(f(x)\star g(x)) &= \big(\partial^*_j f(x)\big)\star g(x) + (e^{ia\partial^*_n} f(x))\star\big(\partial^*_j g(x)\big).
\end{aligned}
\tag{4.111}
$$

Note that formulae like (4.111) motivate the notation $f(x) \star g(x)$, cp. section 2.2. In a mathematically correct way, we should use the coproduct symbol, e.g.:

$$\Delta\partial_j^*(f \star g)(x) = \left(\partial_j^* f(x)\right) \star g(x) + (e^{ia\partial_n^*} f(x)) \star \left(\partial_j^* g(x)\right).$$

If the proper definition is kept in mind, the notation (4.111) is more intuitive in our opinion.

Equation (4.111) has to be valid for all $\star$-products representing different ordering prescriptions on the abstract algebra. Therefore the $\star$-representation ∂_μ^* may have different forms depending on the actual $\star$-product.

For the symmetric $\star$-product we find that[5]

$$\begin{aligned} \partial_n^* f(x) &= \partial_n f(x), \\ \partial_j^* f(x) &= \partial_j \frac{e^{ia\partial_n} - 1}{ia\partial_n} f(x), \end{aligned} \tag{4.112}$$

fulfils (4.111). This $\star$-representation can be derived in a perturbation expansion on symmetrised monomials multiplied with the $\star$-product. However, it is easier to derive it using the $\star$-product of a coordinate with a function $x^\mu \star f(x)$ in (2.29). Rewriting this $\star$-product symbolically as $x^{*\mu} f(x)$, relations such as

$$[\partial_j^*, x^{*n}] f(x) = \partial_j^* x^{*n} f(x) - x^{*n} \partial_j^* f(x) \stackrel{!}{=} ia\partial_j^* f(x), \tag{4.113}$$

have to be fulfilled for arbitrary $f(x)$, therefore $[\partial_j^*, x^{*n}] = ia\partial_j^*$. This equation results in differential equations for ∂_μ^*, whose unique solution consistent with the commutative limit is (4.112). This scheme allows to efficiently calculate $\star$-representations.

$\star$-representations of $\hat{N}^l \to N^{*l}$, $\hat{M}^{rs} \to N^{*rs}$ and of $\hat{D}_\mu \to D_\mu^*$ have to fulfil the Leibniz rules

$$\begin{aligned} M^{*rs}\left(f(x) \star g(x)\right) &= (M^{*rs} f(x)) \star g(x) + f(x) \star (M^{*rs} g(x)), \\ N^{*l}\left(f(x) \star g(x)\right) &= \left(N^{*l} f(x)\right) \star g(x) + \left(e^{ia\partial_n^*} f(x)\right) \star \left(N^{*l} g(x)\right) \\ &\quad - ia\left(\partial_b^* f(x)\right) \star \left(M^{*lb} g(x)\right), \\ D_n^*(f(x) \star g(x)) &= (D_n^* f(x)) \star (e^{-ia\partial_n^*} g(x)) + (e^{ia\partial_n^*} f(x)) \star (D_n^* g(x)) \\ &\quad + ia\left(D_j^* e^{ia\partial_n^*} f(x)\right) \star (D_j^* g(x)), \\ D_j^*(f(x) \star g(x)) &= (D_j^* f(x)) \star (e^{-ia\partial_n^*} g(x)) + f(x) \star (D_j^* g(x)). \end{aligned} \tag{4.114}$$

If the $\star$-representations can be constructed such that these coproducts are fulfilled[6], the algebra of functions of commutative variables with the $\star$-product as multiplication is a

[5] This form has been given first in [123].

[6] In addition, the antipode has to be fixed. This issue will be discussed in chapter 5.

module of the a-Euclidean Hopf algebra $SO_a(n)$. The solution is

$$
\begin{aligned}
\hat{N}^i\hat{f} \longrightarrow N^{*i}f(x) &= \Big(x^i\partial_n - x^n\partial_i + x^i\partial_\mu\partial_\mu\frac{e^{ia\partial_n}-1}{2\partial_n} - x^\mu\partial_\mu\partial_i\frac{e^{ia\partial_n}-1-ia\partial_n}{ia\partial_n^2}\Big)f(x),\\
\hat{M}^{rs}\hat{f} \longrightarrow M^{*rs}f(x) &= (x^s\partial_r - x^r\partial_s)f(x),\\
\hat{D}_n\hat{f} \longrightarrow D^*_n f(x) &= \Big(\frac{1}{a}\sin(a\partial_n) - \frac{1}{ia\partial_n\partial_n}\partial_k\partial_k(\cos(a\partial_n)-1)\Big)f(x),\\
\hat{D}_j\hat{f} \longrightarrow D^*_j f(x) &= \partial_j\Big(\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\Big)f(x),\\
\hat{\Box}\hat{f} \longrightarrow \Box^* f(x) &= \partial_\mu\partial_\mu\,\frac{2(1-\cos(a\partial_n))}{a^2\partial_n\partial_n}f(x).
\end{aligned}
\tag{4.115}
$$

For other $\star$-products, the $\star$-representations differ from (4.115). For the left ordered normal $\star$-product ($\star_L$) we find:

$$
\begin{aligned}
\partial_n^{*_L}f(x) &= \partial_n f(x),\\
\partial_i^{*_L}f(x) &= \partial_i e^{ia\partial_n}f(x),\\
N^{*_L\,l}f(x) &= \Big(x^l\frac{1}{a}\sin(a\partial_n) - x^n\partial_l e^{ia\partial_n} + \frac{ia}{2}x^l\partial_k\partial_k e^{ia\partial_n}\Big)f(x),\\
M^{*_L\,rs}f(x) &= (x^s\partial_r - x^r\partial_s)\,f(x),\\
D_n^{*_L}f(x) &= \left(\frac{1}{a}\sin(a\partial_n) + \frac{ia}{2}\partial_k\partial_k e^{ia\partial_n}\right)f(x),\\
D_j^{*_L}f(x) &= \partial_j f(x),\\
\Box^{*_L}f(x) &= \left(-\frac{2}{a^2}(\cos(a\partial_n)-1) + \partial_k\partial_k e^{ia\partial_n}\right)f(x).
\end{aligned}
\tag{4.116}
$$

The result for the right ordered $\star$-product ($\star_R$) is:

$$
\begin{aligned}
\partial_n^{*_R}f(x) &= \partial_n f(x),\\
\partial_i^{*_R}f(x) &= \partial_i f(x),\\
N^{*_R\,l}f(x) &= \Big(x^l\frac{1}{2ia}(e^{2ia\partial_n}-1) - x^n\partial_l - iax^k\partial_k\partial_l + \frac{ia}{2}x^l\partial_k\partial_k\Big)f(x),\\
M^{*_R\,rs}f(x) &= (x^s\partial_r - x^r\partial_s)\,f(x),\\
D_n^{*_R}f(x) &= \left(\frac{1}{a}\sin(a\partial_n) + \frac{ia}{2}\partial_k\partial_k e^{-ia\partial_n}\right)f(x),\\
D_j^{*_R}f(x) &= \partial_j e^{-ia\partial_n}f(x),\\
\Box^{*_R}f(x) &= \left(-\frac{2}{a^2}(\cos(a\partial_n)-1) + \partial_k\partial_k e^{-ia\partial_n}\right)f(x).
\end{aligned}
\tag{4.117}
$$

The representations (4.115), (4.116) and (4.117) represent the same abstract $SO_a(n)$ on different $\star$-products, i.e. they are representations compatible with different orderings. Especially their commutation relations are those of the abstract algebra. We have noted in section 2.4 that the symmetric and the normal-ordered $\star$-products are equivalent and that they can be related with the equivalence operator T. Therefore we can also relate the different $\star$-representations [49], e.g.

$$\begin{aligned} \partial_j^{*L} &= T^{-1}\partial_j^* T \\ \Leftrightarrow \quad \partial_j e^{ia\partial_n} &= \lim_{y\to x} \exp\left(y^i\partial_{x^i}\left(\frac{e^{-ia\partial_{x^n}}-1}{-ia\partial_{x^n}}-1\right)\right)\cdot\partial_j\frac{e^{ia\partial_n}-1}{ia\partial_n}\cdot \\ & \quad \lim_{z\to x}\exp\left(x^i\partial_{z^i}\left(\frac{-ia\partial_{z^n}}{e^{-ia\partial_{z^n}}-1}-1\right)\right), \end{aligned} \tag{4.118}$$

which can be checked explicitly.

In section 4.1 we have defined three one-parameter sets of linear derivatives $\hat{\partial}_\mu^{c_i}$. For the symmetric $\star$-product their representation in terms of ordinary derivatives reads

$$\begin{aligned} \partial_n^{*c_1} f(x) &= \frac{e^{iac_1\partial_n}-1}{iac_1} f(x), & \partial_i^{*c_1} f(x) &= \partial_i \frac{e^{ia\partial_n}-1}{ia\partial_n} f(x), \\ \partial_n^{*c_2} f(x) &= \frac{e^{iac_2\partial_n}-1}{iac_2} f(x), & \partial_i^{*c_2} f(x) &= \partial_i e^{iac_2\partial_n}\frac{e^{ia\partial_n}-1}{ia\partial_n} f(x), \\ \partial_n^{*c_3} f(x) &= \frac{e^{2ia\partial_n}-1}{2ia} + \frac{iac_3}{2}\partial_k\partial_k\Big(\frac{e^{ia\partial_n}-1}{ia\partial_n}\Big)^2 f(x), & \partial_i^{*c_3} f(x) &= \partial_i \frac{e^{ia\partial_n}-1}{ia\partial_n} f(x). \end{aligned} \tag{4.119}$$

Although these $\star$-representations differ from each other, the $\star$-representations of the Dirac derivative, the Laplace operator and the generators of rotations in terms of commutative ∂_μ and x^ν are unique (up to representations on different $\star$-products). This is independent of which intermediate representation $\hat{\partial}_\mu^{c_i}$ is chosen, since there is effectively only one set of derivatives.

4.9 Representation of forms and volume form

In this section we represent the one-forms $\hat{\xi}^\mu$ on $\star$-product spaces $\mathcal{A}_x[[\hbar]]$ as well, $\hat{\xi}^\mu \to \xi^{*\mu}$. We assume that the $\xi^{*\mu}$ can be written as formal power series of the commutative derivatives ∂_ν, being at most linear in the commutative one-forms $\mathrm{d}x^\mu$. These commute with ∂_ν and functions, while they anti-commute among themselves.

The starting point is the commutator (4.73), a power series expansion in the derivatives. The most general ansatz to solve (4.73), compatible with the index structure, is:

$$\begin{aligned} \xi^{*n} &= \mathrm{d}x^n e_1(\partial_i\partial_i,\partial_n) + \mathrm{d}x^k\partial_k e_2(\partial_i\partial_i,\partial_n), \\ \xi^{*j} &= \mathrm{d}x^j f_1(\partial_i\partial_i,\partial_n) + \mathrm{d}x^n\partial_j f_2(\partial_i\partial_i,\partial_n) + \mathrm{d}x^k\partial_k\partial_j f_3(\partial_i\partial_i,\partial_n). \end{aligned} \tag{4.120}$$

The scheme used to calculate the $\star$-representation of a derivative operator from its commutator with a coordinate in (4.113) can also be used to calculate the $\star$-representations of forms. We collect the terms proportional to different one-forms $\mathrm{d}x^\mu$ and different combinations of derivatives, and use

$$\begin{aligned}[f(\partial_i\partial_i,\partial_n),x^k] &= 2\partial_k\frac{\partial f(\partial_i\partial_i,\partial_n)}{\partial(\partial_i\partial_i)},\\ [f(\partial_i\partial_i,\partial_n),x^n] &= \frac{\partial f(\partial_i\partial_i,\partial_n)}{\partial(\partial_n)}.\end{aligned}$$

We obtain an over-determined system of equations which can be solved consistently. With the abbreviation

$$\gamma = \frac{1}{1+\frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n)-1)}. \tag{4.121}$$

we obtain

$$\begin{aligned} f_1 &= \gamma, \qquad f_2 = -\frac{2i}{\partial_n}\sin(a\partial_n)\gamma^2, \qquad f_3 = -\frac{1}{\partial_n^2}(\cos(a\partial_n)-1)\gamma^2,\\ e_1 &= \Big(1+\cos(a\partial_n)-\frac{\partial_k\partial_k}{\partial_n^2}(\cos(a\partial_n)-1)\Big)\gamma^2, \qquad e_2 = \frac{2i}{\partial_n}\sin(a\partial_n)\gamma^2, \end{aligned} \tag{4.122}$$

or

$$\xi^{*n} = \Big(\mathrm{d}x^n(1+\cos(a\partial_n)-\frac{\partial_k\partial_k}{\partial_n^2}(\cos(a\partial_n)-1)+\mathrm{d}x^k\frac{2i\partial_k}{\partial_n}\sin(a\partial_n)\Big)\gamma^2, \tag{4.123}$$

$$\xi^{*j} = \Big(\mathrm{d}x^j(1+\frac{\partial_\mu\partial_\mu}{2\partial_n\partial_n}(\cos(a\partial_n)-1))-\mathrm{d}x^n\frac{2i\partial_j}{\partial_n}\sin(a\partial_n)-\mathrm{d}x^k\frac{2\partial_k\partial_j}{\partial_n^2}(\cos(a\partial_n)-1)\Big)\gamma^2.$$

The more general differential calculus (4.77) has a particularly simple solution for $c'=1$. The one-forms $\widetilde{\xi}^\mu$ for $c'=1$ have the following $\star$-representation $\widetilde{\xi}^{*\mu}$:

$$\widetilde{\xi}^{*n} = \Big(\mathrm{d}x^n+\mathrm{d}x^l\frac{\partial_l}{\partial_n}(1-e^{-ia\partial_n})\Big)\frac{1}{1+\frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n)-1)}, \tag{4.124}$$

$$\widetilde{\xi}^{*j} = \Big(-\mathrm{d}x^l\frac{\partial_l\partial_j}{\partial_n\partial_n}(\cos(a\partial_n)-1)+\mathrm{d}x^n\frac{\partial_j}{\partial_n}(1-e^{-ia\partial_n})\Big)\frac{1}{1+\frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n)-1)}.$$

It is interesting to note that for this specific set of differentials $\widetilde{\xi}^{*\mu}$ with $c'=1$ we obtain a $\star$-representation, in which $\widetilde{\xi}^j$ is not proportional to $\mathrm{d}x^j$.

The result (4.73) allows to determine the commutation relations of higher-order forms with the coordinates. We can determine them for two-forms, three-forms etc. up to n-forms. Since we know that the $\hat{\xi}^\mu$ anti-commute among themselves, the dimension of the set of j-forms is $\binom{n}{j}$. This is the ordinary exterior calculus of commutative spacetime, allowing the application of all tools of deRham cohomology, especially the Hodge-$*$. Specifically, there is only one n-form, which should be a NC analog of the volume form. The volume form $\hat{\xi}^1\hat{\xi}^2\ldots\hat{\xi}^n$ has particularly simple commutation properties. From (4.73) and (4.75) we determine

$$[\hat{\xi}^1\hat{\xi}^2\ldots\hat{\xi}^n,\hat{x}^\mu] = n\hat{\xi}^1\hat{\xi}^2\ldots\hat{\xi}^n\hat{D}_\mu\frac{1-\sqrt{1-a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}. \tag{4.125}$$

The vector-like transformation behaviour of $\hat{\xi}^\mu$ (4.70) implies that j-forms transform as j-tensors and that the volume form $(\hat{\xi}^1 \dots \hat{\xi}^n)$ is an invariant under $SO_a(n)$ rotations:

$$[\hat{M}^{\rho\sigma}, \hat{\xi}^1 \dots \hat{\xi}^n] = 0. \tag{4.126}$$

The representation of $(\hat{\xi}^1 \dots \hat{\xi}^n)$ on functions multiplied with the $\star$-product is:

$$\left(\xi^1 \xi^2 \dots \xi^n\right)^* = \frac{\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^n}{\left(1 + \frac{\partial_\mu \partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)\right)^n}. \tag{4.127}$$

The $\star$-representation of a frame one-form is just the ordinary commutative one-form, since it commutes with all x-dependent functions and all derivatives. Similarly the frame volume form is just the commutative volume form:

$$\omega^\mu = \mathrm{d}x^\mu, \qquad (\omega^1 \dots \omega^n)^* = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^n = \mathrm{d}^n x. \tag{4.128}$$

The volume form constructed from the frame one-forms is *not* an $SO_a(n)$-invariant. From (4.91) we obtain:

$$[\hat{M}^{rs}, \hat{\omega}^1 \dots \hat{\omega}^n] = 0, \qquad [\hat{N}^l, \hat{\omega}^1 \dots \hat{\omega}^n] = -ia(n-1)\hat{\omega}^1 \dots \hat{\omega}^n \hat{\partial}_l. \tag{4.129}$$

That the commutative volume form $\mathrm{d}x^n$ transforms non-trivially under $SO_a(n)$ (4.129) is a crucial observation for the following chapter.

Chapter 5

κ-deformed field theory and integration

5.1 Variational principle

In section 4.4 we have defined the notion of a field covariant under the generators of the deformed symmetry. We have also defined invariant derivative operators, such as the Laplace operator and the Dirac operator and based on them field equations in the abstract algebra.

Using the $\star$-representation introduced in section 4.8 we obtain field equations on functions of commutative variables, multiplied with the $\star$-product. Examples are the deformed massive Laplace equation

$$\left(\Box^* + m^2\right)\phi(x) = \left(\partial_\mu\partial_\mu \frac{2(1-\cos(a\partial_n))}{a^2\partial_n^2} + m^2\right)\phi(x) = 0 \tag{5.1}$$

and the deformed massive Dirac equation

$$\left(i\gamma^\lambda D^*_\lambda - m\right)\psi(x) = \left(\gamma^n\left(\frac{i}{a}\sin(a\partial_n) + \frac{\partial_j\partial_j}{a\partial_n^2}(\cos(a\partial_n)-1)\right) + i\gamma^j\partial_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n} - m\right)\psi(x) = 0. \tag{5.2}$$

Because of the suitably chosen abstract algebraic definition of derivative operators, these equations are by definition covariant under the κ-deformed symmetry transformations. In this chapter these field equations will be derived by means of a variational principle, the Hamiltonian principle of extremal action. This necessitates the definition of an action that can be varied and from which these field equations can be derived. The formulation of an action functional is expected to be the most direct way towards path integral quantisation and therefore towards a quantum field theory on κ-deformed space.

Yet the formulation of the Lagrangian formalism and of an action functional presupposes the existence of an integral with suitable properties. For the κ-deformed space, a definition with only minimal desirable properties is much less straightforward then in the canonically NC space. Therefore the subsequent arguments are given in great detail.

Algebraically an integral is a linear map of the algebra into the number field on which it is defined:

$$\int : \mathcal{A}_{\hat{x}} \longrightarrow \mathbb{C}, \tag{5.3}$$

$$\int (c_1\hat{\psi} + c_2\hat{\phi}) = c_1 \int \hat{\psi} + c_2 \int \hat{\phi}, \qquad \forall \hat{\psi}, \hat{\phi} \in \mathcal{A}_{\hat{x}}, \;\; c_i \in \mathbb{C}. \tag{5.4}$$

In addition we would like to demand the trace property:

$$\int \hat{\psi}\hat{\phi} = \int \hat{\phi}\hat{\psi}. \tag{5.5}$$

The trace property implies that the integral is cyclic $\int \hat{\psi}\hat{\phi}\hat{\chi} = \int \hat{\phi}\hat{\chi}\hat{\psi}$. It seems to be impossible to define gauge-*invariant* quantities from gauge-*covariant* ones without the trace property. Also the variational principle seems to require the trace property.

A purely algebraic definition of the integral is not sufficient for our purposes. The integral on the abstract algebra has to be realised in terms of an integral over commutative space, e.g. the Lebesgue integral. Therefore we need a definition of the integral in the $\star$-product formalism. The realisation of the algebraic integral (5.3) in the $\star$-product setting allows to perform integration explicitly. Such an integral will certainly be linear (5.4).

An essential property of the integral is that it allows the use of Stokes' theorem, i.e. that it allows partial integration. We will assume that all total derivatives vanish, thereby increasing the number of constraints mentioned in chapter 2, concerning the space of functions of commuting variables, on which the theory is defined.

Provided that all derivatives of the expanded $\star$-product could be eliminated by partial integration at every finite order:

$$\int \mathrm{d}^n x \; \psi(x) \star \phi(x) \overset{\text{part. int.}}{\longrightarrow} \int \mathrm{d}^n x \; \psi(x)\phi(x) = \int \mathrm{d}^n x \; \phi(x)\psi(x),$$

the $\star$-product reduces to point-wise multiplication, which of course has the trace property (5.5).

It is possible to eliminate with such an ansatz the Moyal-Weyl $\star$-product (3.135) , but not an arbitrary, x-dependent $\star$-product. When we partially integrate an x-dependent $\star$-product, new, non-vanishing terms appear, e.g. for κ-deformed space in first order in a:

$$\begin{aligned} &\int \mathrm{d}^n x \; \frac{ia}{2}\big((\partial_n\psi(x))(x^j\partial_j\phi(x)) - (x^j\partial_j\psi(x))(\partial_n\phi(x))\big) \\ &\qquad \overset{\text{part. int.}}{\longrightarrow} \frac{ia}{2}\int \mathrm{d}^n x \; \Big(\psi(x)\phi(x) - (n-1)(\partial_n\psi(x))\phi(x)\Big). \end{aligned} \tag{5.6}$$

The additional terms appear since the explicit coordinate x^j in the $\star$-product has to be differentiated as well under partial integration.

It has been shown in the framework of deformation quantisation of Poisson manifolds [124] that it is always possible to define a measure function $\mu(x)$ such that the integral of two functions multiplied with the ⋆-product is cyclic. This has been shown in [125] in a constructive way for quantum spaces. The measure for the κ-deformed space has been discussed first in [37] and then in [69] from the deformation quantisation perspective. For an x-dependent ⋆-product $\theta^{\rho\sigma}(x)$ the measure function $\mu(x)$ has to fulfil the condition:

$$\partial_\rho\big(\mu(x)\theta^{\rho\sigma}(x)\big) = 0. \tag{5.7}$$

For κ-deformed space (5.7) entails the following conditions on $\mu(x)$:

$$\partial_\rho\big(\mu(x)a(\delta_n^\rho x^\sigma - \delta_n^\sigma x^\rho)\big) = 0 \quad \Rightarrow \quad \partial_n\mu(x) = 0, \qquad x^j\partial_j\mu(x) = -(n-1)\mu(x). \tag{5.8}$$

Examples of measures $\mu(x)$ fulfilling (5.8) are

$$\mu_1(x) = \Big(\prod_{i=1}^{n-1} x^i\Big)^{-1}, \quad \mu_2(x) = \Big(\sum_{i=1}^{n-1}(x^ix^i)\Big)^{-\frac{(n-1)}{2}}, \quad \mu_k(x) = \Big(\sum_{i=1}^{n-1}(x^i)^k\Big)^{-\frac{(n-1)}{k}}, \forall k \in \mathbb{N}. \tag{5.9}$$

If $\mu(x)$ is given, the integral over the ⋆-product of two functions has the trace property:

$$\int \mathrm{d}^n x\ \mu(x)\ (\psi(x) \star \phi(x)) = \int \mathrm{d}^n x\ \mu(x)\ (\phi(x) \star \psi(x)) = \int \mathrm{d}^n x\ \mu(x)\ \psi(x)\phi(x). \tag{5.10}$$

Note that $\mu(x)$ is not ⋆-multiplied with the other functions under the integral, it is part of the volume element.

The measure $\mu(x)$ allows to eliminate any one of the ⋆-products from the ⋆-product of several functions, because of associativity. This allows to cyclically permute under the integral an arbitrary number of ⋆-multiplied functions

$$\int \mathrm{d}^n x\ \mu(x)\ (\psi_1(x) \star \cdots \star \psi_k(x)) = \int \mathrm{d}^n x\ \mu(x)\ (\psi_k(x) \star \psi_1(x) \star \cdots \star \psi_{k-1}(x)). \tag{5.11}$$

Thus, any function can be brought to the furthest left or furthest right of a multiple ⋆-product. This allows a formulation of the variational principle. Varying a product of several functions under the integral, we cyclically permute the function to be varied to one side, eliminate one of the ⋆-products and finally perform the variation:

$$\begin{aligned} \frac{\delta}{\delta\phi(x)}\int \mathrm{d}^n x\ \mu(x)\ \big(\psi(x) \star \phi(x) \star \zeta(x)\big) &= \frac{\delta}{\delta\phi(x)}\int \mathrm{d}^n x\ \mu(x)\ \phi(x)\big(\zeta(x) \star \psi(x)\big) \\ &= \mu(x)\ \ \zeta(x) \star \psi(x). \end{aligned} \tag{5.12}$$

5.2 Hermitian derivative operators

Conjugation can be defined on $\mathcal{A}_{\hat{x}}$ and also on its symmetry Hopf algebra $SO_a(n)$ as a formal involution $^\dagger : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$ and $^\dagger : SO_a(n) \to SO_a(n)$. Since $\mathcal{A}_{\hat{x}}$ and $SO_a(n)$ are constructed from vector spaces over $\mathbb{C}$, it is important to discuss the effect of complex conjugating the underlying number field. We demand that

- Conjugation has to be consistent with the algebraic structure, i.e. $([\mathcal{V},\mathcal{W}]-\mathcal{U})^\dagger = 0$, if $[\mathcal{V},\mathcal{W}]-\mathcal{U}=0$.
- Conjugation has to send a complex number to its complex conjugate.
- Conjugation is an involution, i.e. applied on products of elements of $\mathcal{A}_{\hat{x}}$ and $SO_a(n)$, conjugation reverses the order $(\mathcal{V}\mathcal{W})^\dagger = \mathcal{W}^\dagger\mathcal{V}^\dagger$.

We call an operator hermitian if it fulfils $\mathcal{V}^\dagger = \mathcal{V}$. Since all elements of the abstract $\mathcal{A}_{\hat{x}}$ and $SO_a(n)$ have a well-defined commutative limit, they need to behave as their commutative counterparts under conjugation. Especially coordinates should be hermitian and derivatives anti-hermitian. These conditions are fulfilled for the following conjugated operators:

$$\begin{aligned}
(\hat{x}^\mu)^\dagger &= \hat{x}^\mu, \qquad (\hat{\partial}_n)^\dagger = -\hat{\partial}_n, \qquad (\hat{\partial}_j)^\dagger = -\hat{\partial}_j,\\
(\hat{D}_j)^\dagger &= \left(\hat{\partial}_j e^{-ia\hat{\partial}_n}\right)^\dagger = \left(e^{ia\hat{\partial}_n^\dagger}\hat{\partial}_j^\dagger\right) = -\hat{D}_j,\\
(\hat{D}_n)^\dagger &= \left(\frac{1}{a}\sin(a\hat{\partial}_n) + \frac{ia}{2}\hat{\partial}_k\hat{\partial}_k e^{-ia\hat{\partial}_n}\right)^\dagger = -\hat{D}_n,\\
(\hat{M}^{rs})^\dagger &= (\hat{x}^s\hat{\partial}_r - \hat{x}^r\hat{\partial}_s)^\dagger = (\hat{\partial}_r^\dagger(\hat{x}^s)^\dagger - \hat{\partial}_s^\dagger(\hat{x}^r)^\dagger) = -\hat{M}^{rs},\\
(\hat{N}^l)^\dagger &= \frac{e^{2ia\hat{\partial}_n}-1}{-2ia}\hat{x}^l + \hat{\partial}_l\hat{x}^n - \frac{ia}{2}\hat{\partial}_k\hat{\partial}_k\hat{x}^l\\
&= -\hat{x}^l\frac{e^{2ia\hat{\partial}_n}-1}{2ia} + \hat{x}^n\hat{\partial}_l - \frac{ia}{2}\hat{x}^l\hat{\partial}_k\hat{\partial}_k + ia\hat{\partial}_l - ia\hat{\partial}_l = -\hat{N}^l.
\end{aligned} \tag{5.13}$$

Thus, formal conjugation can be defined consistently in the abstract algebra. But we also need the conjugation behaviour of the $\star$-representations of the abstract algebra elements. These derivative operators should not only be formally conjugated, but in a concrete sense, using hermitian conjugation. Hermitian conjugation should be implemented by partial integration under the integral (cp. the definition of a hermitian operator in wave mechanics)[1]. We we call a derivative operator $\mathcal{V}^*$ hermitian if

$$\int \mathrm{d}^n x\, \mu\, \overline{\psi} \star \mathcal{V}^*\phi = \int \mathrm{d}^n x\, \mu\, \overline{\mathcal{V}^*\psi} \star \phi, \tag{5.14}$$

under partial integration. A quick look at the partial integration of the two simplest derivative operators ∂_n^* and ∂_j^* shows that although $\hat{\partial}_\mu^\dagger = -\hat{\partial}_\mu$, we obtain that $\overline{\partial_n^*} = -\partial_n^*$, but

$$\begin{aligned}
\int \mathrm{d}^n x\, \mu\, \overline{\psi} \star (\partial_i^*\phi) &= \int \mathrm{d}^n x\, \mu\, \overline{\psi}\, (\partial_i^*\phi) \overset{p.i.}{\to} \int \mathrm{d}^n x\, \mu\, \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\overline{\psi}\, \partial_i\phi\\
&\overset{p.i.}{\to} -\int \mathrm{d}^n x\, \mu\, \overline{\partial_i^*\psi}\, \phi - \int \mathrm{d}^n x\, \partial_i\mu\, \overline{\frac{e^{ia\partial_n}-1}{ia\partial_n}\psi}\, \phi.
\end{aligned} \tag{5.15}$$

[1]The notion of selfadjointness requires a very careful definition of the domain of the operators. This topic goes beyond the discussion in this thesis.

The derivative ∂_j^* is not anti-hermitian, since ∂_j also acts on the measure μ. The Dirac derivatives D_μ^* are not anti-hermitian by this definition either.

The solution to this problem is familiar from three-dimensional quantum mechanics of a central potential. Here the radial momentum (in spherical coordinates) $p_r = i\frac{\partial}{\partial r}$ is not a hermitian operator, it has to be rescaled because of the spherical measure $\mu = r^2 \sin\theta$. The hermitian radial momentum is

$$p_r = i\frac{\partial}{\partial r} + \rho_r, \quad \text{with} \quad \rho_r = \frac{1}{2\mu}\frac{\partial\mu}{\partial r} = \frac{1}{r}. \tag{5.16}$$

Mimicking this trick, we perform a rescaling of the derivative ∂_j

$$\partial_j \to \tilde{\partial}_j = \partial_j + \rho_j = \partial_j + \frac{(\partial_j\mu)}{2\mu}. \tag{5.17}$$

It inherits from μ the properties:

$$x^l\partial_l\rho_j = -\rho_j \quad \text{and} \quad \partial_n\rho_j = 0. \tag{5.18}$$

For the choices of μ presented in (5.9), we would obtain:

$$\rho_j(\mu_1) = -\frac{1}{2x^j}, \quad \rho_j(\mu_2) = -\frac{n-1}{2}\frac{x^j}{\sum_{i=1}^{n-1} x^i x^i}, \quad \rho_j(\mu_3) = -\frac{n-1}{2}\frac{(x^j)^{k-1}}{\sum_{i=1}^{n-1}(x^i)^k}. \tag{5.19}$$

However, it is not necessary in any application to specify a particular form neither for μ nor for ρ_j. The derivative ∂_n is not rescaled since $\partial_n\mu = 0$.

With the rescaled derivative $\tilde{\partial}_j$, anti-hermitian derivative operators can be constructed such as $\tilde{\partial}_j^*$:

$$\tilde{\partial}_j^* = (\partial_j + \rho_j)\frac{e^{ia\partial_n} - 1}{ia\partial_n}. \tag{5.20}$$

This derivative operator $\tilde{\partial}_j^*$ is anti-hermitian in the sense of (5.14):

$$\int \mathrm{d}^n x\, \mu\, \overline{\psi}\ (\partial_j + \rho_j)\frac{e^{ia\partial_n} - 1}{ia\partial_n}\phi = -\int \mathrm{d}^n x\, \mu\, \overline{(\partial_j + \rho_j)\frac{e^{ia\partial_n} - 1}{ia\partial_n}\psi}\ \phi. \tag{5.21}$$

Similarly, D_μ^* is rescaled:

$$\begin{aligned} D_j^* &\longrightarrow \tilde{D}_j^* = (\partial_j + \rho_j)\frac{e^{-ia\partial_n} - 1}{-ia\partial_n}, \\ D_n^* &\longrightarrow \tilde{D}_n^* = \frac{1}{ia\partial_n^2}(\partial_k + \rho_k)(\partial_k + \rho_k)(\cos(a\partial_n) - 1) + \frac{1}{a}\sin(a\partial_n). \end{aligned} \tag{5.22}$$

These $\tilde{D}_\mu^*$ are also anti-hermitian in the sense of (5.14).

The rescaling with ρ_j is an algebraically consistent operation, since ρ_j is a function of the coordinates and coordinates commute among each other. Therefore the commutator $[(\partial_j + \rho_j), x^\mu] = \delta_j^\mu$ is unchanged and due to antisymmetry also $[(\partial_i + \rho_i), (\partial_j + \rho_j)] = 0$.

In [37] we have argued that the rescaling leaves the algebra of commutative quantities invariant. This rescaling can formally be lifted into the abstract algebra and does not change the commutation relations of the abstract quantities. For this to be true, the representation of all operators $M^{*\mu\nu} \to \tilde{M}^{*\mu\nu}$ and $x^* \to \tilde{x}^{*\mu}$ has to be changed as well:

$$\begin{aligned}
\tilde{N}^{*l} &= x^l \partial_n \frac{e^{ia\partial_n}-1}{2} + x^l \tilde{\partial}_j \tilde{\partial}_j \frac{e^{ia\partial_n}-1}{2\partial_n} - x^n \tilde{\partial}_l \frac{e^{ia\partial_n}-1}{ia\partial_n} - x^j \tilde{\partial}_j \tilde{\partial}_l \frac{e^{ia\partial_n}-1-ia\partial_n}{ia\partial_n^2}, \\
\tilde{M}^{*rs} &= x^s \tilde{\partial}_r - x^r \tilde{\partial}_s, \\
\tilde{x}^{*n} &= \tilde{x}^{\overrightarrow{*}n} = x^n - x^k \frac{\tilde{\partial}_k}{\partial_n}\Big(\frac{ia\partial_n}{e^{ia\partial_n}-1} - 1\Big), \\
\tilde{x}^{*j} &= \tilde{x}^{\overrightarrow{*}j} = x^{*j}.
\end{aligned} \tag{5.23}$$

We have introduced the notation $\tilde{x}^{\overrightarrow{*}\mu}$ for a derivative multiplied from the *left* to a function. We find that $[\tilde{M}^{*\mu\nu}, \tilde{\partial}_j^*] = [M^{*\mu\nu}, \partial_j^*]$ etc. and we can lift this rescaling into the abstract algebra.

Unfortunately, there is still one problem, i.e. that $\tilde{N}^{*l}$ is not hermitian under this definition. This problem arises from the term in $\tilde{N}^{*l}$ proportional to $x^j \tilde{\partial}_j \tilde{\partial}_l$. Under partial integration with μ a term proportional to $x^j \partial_j \tilde{\partial}_l$ could be partially integrated in a proper way, because of the properties of μ. The additional term ρ_j spoils the hermiticity.

We do not discuss in this section how N^{*l} acts on $\mathrm{d}^n x$ (4.129). Without going into details (cp. the next section), we state that including the action on $\mathrm{d}^n x$ does not solve any of the problems presented in the rest of this section.

The disturbing term arises because of the derivative representation of $\tilde{x}^{*n}$. Therefore let us look more carefully at the hermiticity of coordinates. If we treat the coordinates as rescaled derivative operators, whose hermiticity is checked by partial integration, we obtain

$$\begin{aligned}
\int \mathrm{d}^n x\, \mu\ \overline{\psi} \star (x^{\overrightarrow{*}l}\phi) &\xrightarrow{p.i.} \int \mathrm{d}^n x\, \mu\ x^l \frac{-ia\partial_n}{e^{-ia\partial_n}-1}\overline{\psi}\ \phi = \int \mathrm{d}^n x\, \mu\ \overline{x^{\overrightarrow{*}l} \star \psi}\ \phi, \\
\int \mathrm{d}^n x\, \mu\ \overline{\psi} \star (\tilde{x}^{\overrightarrow{*}n}\phi) &\xrightarrow{p.i.} \int \mathrm{d}^n x\, \mu\ \Big(\overline{\tilde{x}^{\overrightarrow{*}n}\psi}\ \phi + (n-1)\overline{\psi}\big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big)\phi\Big).
\end{aligned} \tag{5.24}$$

While x^{*j} is a hermitian coordinate as expected (x^{*j} is not rescaled, it depends on ∂_n only), , the coordinate $\tilde{x}^{*n}$ obviously cannot be treated in the above way as a derivative operator. Of course, the ansatz treating $\tilde{x}^{*\mu}$ as a derivative operator is questionable from the outset and its limitations become obvious here. We have to introduce a more careful notation.

Coordinates are $\star$-multiplied to a function, so far we have always treated $\star$-multiplication from the left. Let us also take into account $\star$-multiplication from the right:

$$\begin{aligned}
f(x)x^{\overleftarrow{*}l} &= f(x) \star x^l = x^l \frac{-ia\partial_n}{e^{-ia\partial_n}-1} f(x), \\
f(x)x^{\overleftarrow{*}n} &= f(x) \star x^n = \Big(x^n - \frac{x^k\partial_k}{\partial_n}\big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1} - 1\big)\Big) f(x), \\
x^{\overrightarrow{*}l} f(x) &= (e^{-ia\partial_n} f(x)) x^{\overleftarrow{*}l}, \qquad x^{\overrightarrow{*}n} f(x) = f(x)x^{\overleftarrow{*}n} - iax^k\partial_k f(x).
\end{aligned} \tag{5.25}$$

We observe that

$$\overline{x^{\overrightarrow{*}\mu}} = x^{\overleftarrow{*}\mu}, \tag{5.26}$$

where the bar means complex conjugation (conjugating complex numbers, not operators).

Interpreting the left multiplication of a coordinate as a left action, we can use the associativity of the $\star$-product and obtain:

$$\begin{aligned} \int \mathrm{d}^n x\,\mu\,\overline{\psi}\big(x^{\overrightarrow{*}\mu}\phi\big) &= \int \mathrm{d}^n x\,\mu\,\overline{\psi}\big(x^\mu \star \phi\big) = \int \mathrm{d}^n x\,\mu\,\overline{\psi} \star \big(x^\mu \star \phi\big) \\ &= \int \mathrm{d}^n x\,\mu\,\big(\overline{\psi} \star x^\mu\big) \star \phi = \int \mathrm{d}^n x\,\mu\,\big(\overline{\psi} \star x^\mu\big)\phi \\ &= \int \mathrm{d}^n x\,\mu\,\big(\overline{\psi}(x^{\overleftarrow{*}\mu})\big)\phi = \int \mathrm{d}^n x\,\mu\,\big(\overline{x^{\overrightarrow{*}\mu}\psi}\big)\phi. \end{aligned} \tag{5.27}$$

The first three identities are simple re-formulations according to definitions. The last identity is due to (5.26). The same manipulations can be performed for $\tilde{x}^{*\mu}$.

We may try to generalise the interpretation of right multiplication of coordinates as right action to the right action of an arbitrary operator. Throughout this thesis we have only considered left actions of operators. Derivatives have been defined as acting from the left on functions, as do generators of rotation etc.

In a nutshell, the right action of operators in the abstract algebra (this is also called the opposite algebra) can be obtained by reinterpreting the defining commutators, for example

$$[\hat{x}^i_R, \hat{\partial}_{jR}] = -\delta^i_j, \qquad [\hat{x}^n_R, \hat{\partial}_{jR}] = -ia\hat{\partial}_{jR}, \tag{5.28}$$

etc. The concept of the opposite algebra can be given a very concrete meaning in terms of $\star$-representations. Just as we have defined the left $\star$-representations in section 4.8 from commutators with $x^{\overrightarrow{*}\mu}$, we can define right $\star$-representations from the commutators of the opposite algebra using $x^{\overleftarrow{*}\mu}$. This gives a $\star$-representation of the right action which only has to be *complex* conjugated afterwards to obtain a formula for all operators considered so far:

$$\int \mathrm{d}^n x\,\mu\,\overline{\psi}\big(\mathcal{V}^{\overrightarrow{*}}\phi\big) = \int \mathrm{d}^n x\,\mu\,\big(\overline{\psi}\mathcal{V}^{\overleftarrow{*}}\big)\phi = \int \mathrm{d}^n x\,\mu\,\overline{\mathcal{V}^{\overrightarrow{*}}\psi}\phi. \tag{5.29}$$

The same is valid for all rescaled operators $\tilde{\mathcal{V}}^{\overrightarrow{*}}$.

This interpretation is perfectly viable and gives consistent results for all operators. However, while this interpretation is natural for the coordinates using the associativity of the $\star$-product, it is entirely ad hoc for all operators. It is not clear how to get technically from the left action on ϕ to the right action on ψ. Certainly this cannot be done by partial integration because of the problems stated above. It is also not an option to treat the composite operator $\tilde{N}^{*l}$ in such a way that the part of the derivative representation stemming from x^{*n} according to (5.27), while all other derivatives are partially integrated. Ignoring the origin in the abstract algebra, we are unable to distinguish which derivative belongs to the representation of one part of a composite operator and which to another.

All the approaches discussed so far are unsatisfactory. Brute force partial integration does not work for x^{*n} and N^{*l}, considered as derivative operators. Using the associativity of the $\star$-product to conjugate the coordinates has the disadvantage that operators such as N^{*l} have to remember their abstract algebraic structure. Finally right action is a possible definition for hermitian conjugation, but here the concept of partial integration is completely given up on.

To summarise, with a cyclic integral involving μ, the integral cannot be defined such that N^{*l} has acceptable properties under conjugation. Especially, it looks as if we cannot define the integral in an invariant way. N^{*l} does not commute with μ under partial integration, it also does not commute with the volume element (see below); the effects do not cancel.

In the subsequent sections we will discuss another approach, using Hopf algebraic language. The antipode operation is closely related to conjugation, but algebraically more appropriate. This approach will have a different drawback, since the integral thus defined is not cyclic, at least not at face value. So far we have not been able to combine the two features of symmetry-invariance and gauge-invariance. The difficulties to combine gauge-invariance and invariance under symmetry transformations haunt the construction of many NC field theoretical models [127].

Before turning to this alternative approach we finish the derivation of equations of motion from the cyclic action. An action for a spinor field $\tilde{\psi}$ with the rescaled anti-hermitian Dirac operator is:

$$\mathcal{S} = \int \mathrm{d}^n x \, \mu \, \overline{\tilde{\psi}} \star (i\gamma^\lambda \tilde{D}^*_\lambda - m)\tilde{\psi}. \tag{5.30}$$

Varying with respect to $\overline{\tilde{\psi}}$ we obtain an equation of motion involving the rescaled Dirac derivative $\tilde{D}^*_\lambda$ and the measure function:

$$\mu(i\gamma^\lambda \tilde{D}^*_\lambda - m)\tilde{\psi} = 0. \tag{5.31}$$

Commuting a factor of $\mu^{-\frac{1}{2}}$ with the Dirac derivative, the rescaling is eliminated:

$$\tilde{D}^*_\lambda\big(\mu^{-\frac{1}{2}}\psi\big) = \mu^{-\frac{1}{2}} D^*_\lambda \psi. \tag{5.32}$$

Therefore by a redefinition of $\tilde{\psi} = \mu^{-\frac{1}{2}}\psi$, which also means $\overline{\tilde{\psi}} = \mu^{-\frac{1}{2}}\overline{\psi}$ we can obtain an action with the rescaling and the measure function eliminated:

$$\mathcal{S} = \int \mathrm{d}^n x \, \overline{\psi}(i\gamma^\lambda D^*_\lambda - m)\psi, \tag{5.33}$$

Performing the redefinition $\tilde{\psi} = \mu^{-\frac{1}{2}}\psi$ *after* the removal of the one $\star$-product with μ, the action (5.33) is equivalent to (5.30). It would be inconsistent or at least inconvenient if the equivalence of these two actions depends on the order in which the two operations *partial integration* and *field redefinition* is performed. Therefore we have checked (up to second order) that it is also true that

$$\int \mathrm{d}^n x \, \mu(\tilde{f} \star \tilde{g}) = \int \mathrm{d}^n x \, \mu\Big((\mu^{-\frac{1}{2}} f) \star (\mu^{-\frac{1}{2}} g)\Big) = \int \mathrm{d}^n x \, f \cdot g, \tag{5.34}$$

where *first* the two factors of $\mu^{-\frac{1}{2}}$ are extracted from under the $\star$-product and *then* the partial integration is performed. We have used that under partial integration (without μ) the $\star$-product gives

$$\int \mathrm{d}^n x\; f \star g = \int \mathrm{d}^n x\; \Big(f\big(1 + \frac{ia}{2}(n-1)\partial_n + \frac{a^2}{24}(4(n-1) - 3(n-1)^2)\partial_n^2 + \dots\big)g\Big), \tag{5.35}$$

The action (5.33) gives after variation w.r.t. $\overline{\psi}$ the equation of motion

$$(i\gamma^\lambda D^*_\lambda - m)\psi = 0. \tag{5.36}$$

This is the result that we expected in (5.2). This equation of motion does not involve any remnant of the $\star$-product with $\overline{\psi}$, neither any remnant of the integral of the action. It is the classical Dirac equation with a nonlinearly deformed Dirac derivative. This equation of motion can be used to define the propagator for a fermionic quantum field by Fourier transformation.

5.3 Integration of forms

In this and the next section we present an attempt based on algebraic/geometric considerations to solve the not yet fully satisfactory integral of the previous section. We do not claim that this approach is better than the previous one. In fact in its current shape it has a very serious disadvantage for physical applications, since it does not allow the formulation of gauge invariant actions.

In this ansatz, action integrals are formulated as inner products of forms. In commutative physics, actions are often written in terms of the inner product of two differential r-forms ψ and ϕ, using the Hodge-$*$ operator (note the different symbols for the $\star$-product and the Hodge-$*$). In an n-dimensional commutative manifold the Hodge-$*$ is defined on an r-form[2]

$$\phi = \frac{1}{r!}\phi_{\mu_1\dots\mu_r}\mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_r}, \tag{5.37}$$

as

$$*\phi = \frac{\sqrt{\det g}}{r!(n-r)!}\phi_{\mu_1\dots\mu_r}\epsilon^{\mu_1\dots\mu_r}{}_{\nu_{r+1}\dots\nu_n}\mathrm{d}x^{\nu_{r+1}} \wedge \dots \wedge \mathrm{d}x^{\nu_n}. \tag{5.38}$$

Here g is the metric defined on the commutative manifold. Recall the identities $*1 = \sqrt{\det g}\;\mathrm{d}^n x$ and $*^2\omega = (-1)^{r(n-r)}\omega$. The inner product of two r-forms is the integral over the full spacetime times a measure:

$$(\psi, \phi) = \int \psi \wedge *\phi = \frac{1}{r!}\int \mathrm{d}^n x \sqrt{\det g}\;\; \psi_{\mu_1\dots\mu_r}\phi^{\mu_1\dots\mu_r}. \tag{5.39}$$

All actions considered so far such as the Yang-Mills action and the minimally coupled massive fermionic action can be formulated in the language of forms.

[2]Conventions are according to [126].

In this geometric setting, gauge potentials are components of the connection on the associated vector bundle of a principle bundle. Locally, gauge potentials are Lie algebra-valued one-forms $A^0 = iA^0_\mu \mathrm{d}x^\mu$. The field strength $F^0_{\mu\nu}$ are components of a Lie algebra-valued two-form, $F^0 = \mathrm{d}A^0 + A^0 \wedge A^0 = iF^0_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, fulfilling the Bianchi identity $\mathrm{d}F^0 + F^0 \wedge A^0 + A^0 \wedge F^0 = 0$.

To be more specific, the Yang-Mills action is of the form:

$$\begin{aligned}(F^0, F^0) &= \mathrm{Tr}\int (iF^0_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \wedge *(iF^0_{\rho\sigma}\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \\ &= -\frac{1}{2}\mathrm{Tr}\int \mathrm{d}^n x\sqrt{\det g}\ F^0_{\mu\nu}F^{0\mu\nu}. \end{aligned} \tag{5.40}$$

Similarly the minimally coupled massive fermionic action can be rewritten as the inner product of spinor fields, which are forms of degree 0. The Dirac derivative is the sum of two Dirac operators acting on the two spin bundles which together make up the exterior bundle. The intricacies of the exact definition are however not important, since the κ-deformed space in our ansatz is considered to be flat. It is therefore sufficient to treat spinor fields as fields of form degree zero.

Analogously, we want to formulate NC field theories in the language of forms. According to the prescription given in section 3.5, we can try to replace all point-wise products with $\star$-products. We see from equation (5.40) that this not enough. We also need a suitable definition of differential forms which can be combined into a volume form. For example in the Yang-Mills action, one of the two two-forms has to be commuted through the field-components $F^0_{\mu\nu}$ in order to be combined into a volume form. Frame one-forms $\hat{\omega}^\mu$ have been defined in section 4.6 such that they commute with functions, they can be identified with the commutative one-forms $\hat{\omega}^\mu \longrightarrow \omega^\mu = \mathrm{d}x^\mu$.

This means that the NC Yang-Mills action may be written in the following way, commuting the frame one-forms to the furthest left:

$$\begin{aligned}(F, F) &= \mathrm{Tr}\int (iF_{\mu\nu}\omega^\mu\omega^\nu) \overset{\star}{\wedge} *(iF_{\rho\sigma}\omega^\rho\omega^\sigma) \\ &= -\frac{1}{2}\mathrm{Tr}\int \omega^{\mu_1}\dots\omega^{\mu_n}\ F_{\mu\nu} \star \big(\sqrt{\det g}F^{\mu\nu}\big). \end{aligned} \tag{5.41}$$

The Hodge-$*$ applied to the field strength tensor is proportional to $\sqrt{\det g}$. The authors of [68] have found that $\sqrt{\det g}$ can be identified with the measure μ of section 5.1. The measure μ is the Pfaffian of the NC structure, given by

$$\mu = \det{}^{-\frac{1}{2}}(x^\lambda C^{\mu\nu}_\lambda) = \frac{1}{n!2^n}\epsilon_{\mu_1\mu_2\dots\mu_{2n}}(x^\lambda C^{\mu_1\mu_2}_\lambda)\dots(x^\lambda C^{\mu_{2n-1}\mu_{2n}}_\lambda). \tag{5.42}$$

Since $x^\lambda C^{\mu\nu}_\lambda$ is degenerate at the origin and not invertible there, the origin has to be excluded for defining μ. Defining [69] in the abstract algebra a radius $\hat{r}$ in the $(n-1)$-dimensional subspace as $\hat{r} = \sqrt{\sum_{i=1}^{n-1}\hat{x}^i\hat{x}^i}$, then the derivations $\hat{r}^j\hat{\partial}_j$ and $\hat{\partial}_n$ have ordinary Leibniz rules (cp. (4.20)). These derivations are identical to the commutative $r^j\partial_j$ and ∂_n

(cp. section 4.8). These commutative derivations can be used to construct a commutative metric

$$g = r^{-2}\sum_{i=1}^{n-1}(\mathrm{d}x^i)^2 + (\mathrm{d}x^n)^2 = (\mathrm{d}\ln r)^2 + \mathrm{d}\Omega^2_{n-2} + (\mathrm{d}x^n)^2, \tag{5.43}$$

with $\mathrm{d}\Omega^2_{n-2}$ the $(n-2)$-dimensional spherical volume element. Therefore $\sqrt{\det g} = r^{-(n-1)} = \mu_2$. Note that in spite of this preference for μ_2, we will continue to use only the properties (5.8) for characterising μ.

The measure $\sqrt{\det g} = \mu$ appears as part of the action of the Hodge-$*$

$$*(iF_{\rho\sigma}\omega^\rho\omega^\sigma) = \frac{\mu}{2!(n-2)!}F_{\alpha\beta}\epsilon^{\alpha\beta}{}_{\nu_3\ldots\nu_n}\omega^{\nu_3}\ldots\omega^{\nu_n}. \tag{5.44}$$

If μ should play the role of a measure as in section 5.1, it should multiply the volume element. It can be extracted from within the $\star$-product because of the properties of μ, $x^j\partial_j\mu = -(n-1)\mu$, $\partial_n\mu = 0$, but this leaves additional derivatives ∂_n acting on the two factors of the field strength. We expand up to second order (for two arbitrary r-forms ψ and ϕ):

$$\begin{aligned}\psi\star(\mu\phi) \;=\;& \mu\psi\phi + \frac{ia}{2}\mu\big(\partial_n\psi x^j\partial_j\phi - x^j\partial_j\psi\partial_n\phi\big) - \frac{ia}{2}(n-1)\mu\partial_n\psi\phi \\ &-\frac{a^2}{8}\mu\big(\partial_n^2\psi x^j x^k\partial_j\partial_k\phi - x^j\partial_j\partial_n\psi x^k\partial_k\partial_n\phi + x^j x^k\partial_j\partial_k\psi\partial_n^2\phi\big) \\ &+\frac{a^2}{4}(n-1)\mu\big(\partial_n^2\psi x^j\partial_j\phi - x^j\partial_j\partial_n\psi\partial_n\phi\big) \\ &-\frac{a^2}{8}(n-1)n\mu\partial_n^2\psi\phi + \frac{a^2}{12}(n-1)\mu\partial_n^2\psi\phi - \frac{a^2}{12}(n-1)\mu\partial_n\psi\partial_n\phi + \ldots\ .\end{aligned} \tag{5.45}$$

Under an integral allowing partial integration, the derivatives ∂_n can be combined into one derivative operator (∂_n commutes with the $\star$-product and μ), which we call K:

$$\int \mathrm{d}^n x\ \ \psi\star(\mu\phi) = \int \mathrm{d}^n x\mu\ \ \psi\star(K\phi). \tag{5.46}$$

Up to second order we find:

$$\begin{aligned}K \;=\;& 1 + \frac{ia}{2}(n-1)\partial_n - \frac{a^2(n-1)(n-2)}{8}\partial_n^2 - \frac{a^2}{12}(n-1)\partial_n^2 + \ldots \\ =\;& \big(1 + \frac{ia}{2}\partial_n - \frac{a^2}{12}\partial_n^2 - \ldots\big)^{n-1} = \big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big)^{n-1}.\end{aligned} \tag{5.47}$$

Why we have dared to identify an expansion up to second order with an all orders expression might be surprising at this stage. Continuing the formulation of the action in terms of forms we will rediscover this derivative operator from an entirely different argument.

Thus, we have constructed an expression of the action, in which the measure function appears naturally, outside of the $\star$-product (using $\omega^1 \dots \omega^n = \mathrm{d}^n x$):

$$\begin{aligned}
(F,F) &= \mathrm{Tr}\int (iF_{\mu\nu}\omega^\mu\omega^\nu) \star *(iF_{\rho\sigma}\omega^\rho\omega^\sigma) \\
&= \mathrm{Tr}\int (iF_{\mu\nu}\omega^\mu\omega^\nu) \star \Big(\frac{\mu}{2!(n-2)!}F_{\alpha\beta}\epsilon^{\alpha\beta}{}_{\nu_3\dots\nu_n}\omega^{\nu_3}\dots\omega^{\nu_n}\Big) \qquad (5.48)\\
&= -\frac{1}{2}\mathrm{Tr}\int \omega^1\dots\omega^n\mu\ F_{\mu\nu}\star\big(KF^{\mu\nu}\big) = -\frac{1}{2}\mathrm{Tr}\int \mathrm{d}^n x\mu\ F_{\mu\nu}\big(KF^{\mu\nu}\big),
\end{aligned}$$

since μ allows to eliminate one $\star$-product. Note that the derivative factors that appeared when we defined the forms $\xi^{*\mu}$ do not contribute to this definition of the form. The volume form $\xi^{*1}\dots\xi^{*n}$ defined at the end of chapter 4.9 involved a derivative operator $\frac{1}{(1-\frac{a^2}{4}\Box^*)^n}$. This derivative operator was the reason to introduce the frame one-forms and the derivative operator was alloted in $\eth_\mu$ dual to ω^μ in d$=\eth_\mu\omega^\mu$. Therefore these derivatives are not a problem provided that the frame one-forms are used.

5.4 Invariance of the integral over forms

The definition of the integral in section 5.1 is based on the measure μ defined as part of the volume element. This definition is motivated to achieve the trace property, invariance under $SO_a(n)$ rotations has not been a guiding principle in the construction.

Therefore we will now focus on formulating an integral, such that it is $SO_a(n)$-invariant by definition. Since $SO_a(n)$ is a Hopf algebra, we have to adapt the notion of invariance used in the context of integrals invariant under symmetry *groups*. Invariance can be formulated in such a way that the action of an operator $\mathcal{V}$ on the integral is the same as the action of $\mathcal{V}$ on the trivial one-dimensional representation $\mathbb{C}$. This means that an invariant action transforms like a complex number.

With this notion of invariance, we can construct an action from fields which are modules of $SO_a(n)$ from the inner product introduced in the previous section. If the field $\hat\psi$ transforms under $\hat M^{\mu\nu}$, then the dual space, i.e. the linear form mapping $\hat\psi$ into the complex numbers, has to transform under the antipode $S(\hat M^{\mu\nu})$. The condition that the antipode of an arbitrary Hopf algebra has to fulfil is

$$\mathsf{m}(S\otimes 1)\Delta = \eta\epsilon, \quad \text{and} \quad \mathsf{m}(1\otimes S)\Delta = \eta\epsilon. \qquad (5.49)$$

Here m denotes the multiplication of two factors of a tensor product, η is the unit embedding $\mathbb{C}$ into $SO_a(n)$, Δ the coproduct, and ϵ the counit (cp. section 4.2). This condition states the following: assume that an element of $SO_a(n)$ acts on a tensor product of functions (equivalently forms) according to the coproduct. Regarding one of the two factors of the tensor product as a dual space, the action on this factor in the tensor product is dualised with the antipode. Multiplying the resulting expression, the product transforms like a trivial one-dimensional representation would transform. In other words, the product would be invariant.

We can therefore prove the invariance of an action integral under $SO_a(n)$. We have to verify that (we choose the convention that the dual space is the factor on the right hand side of the inner product)

$$(\hat{M}^{\mu\nu}\hat{\psi}, \hat{\phi}) = (\hat{\psi}, S(\hat{M}^{\mu\nu})\hat{\phi}). \tag{5.50}$$

Writing the inner product for two r-forms ψ and ϕ explicitly, we obtain the condition that (with the Hodge-dual form on the right in the inner product):

$$\int \big(M^{*\mu\nu}\psi\big) \star (*\phi) = \int \psi \star (S(M^{*\mu\nu}) * \phi). \tag{5.51}$$

Note that in (5.51) the volume element $\mathrm{d}^n x$ is still split up among the forms ψ and ϕ. In the following, we want to check explicitly that this condition is fulfilled for our choice of inner product. We can perform this check by partial integration.

First we repeat the definition of the antipode on derivatives and generators of $SO_a(n)$:

$$\begin{aligned} S(\hat{\partial}_j) &= -e^{-ia\hat{\partial}_n}\hat{\partial}_j, \qquad S(\hat{\partial}_n) = -\hat{\partial}_n, \qquad S(e^{ia\hat{\partial}_n}) = e^{-ia\hat{\partial}_n}, \\ S(\hat{D}_j) &= -e^{ia\hat{\partial}_n}\hat{D}_j, \qquad S(\hat{D}_n) = -\hat{D}_n + ia\hat{D}_j\hat{D}_j e^{ia\hat{\partial}_n}, \\ S(\hat{M}^{rs}) &= -\hat{M}^{rs}, \qquad S(\hat{N}^l) = -\hat{N}^l e^{-ia\hat{\partial}_n} - ia\hat{M}^{lk}\hat{\partial}_k e^{-ia\hat{\partial}_n} - ia(n-1)\hat{\partial}_l e^{-ia\hat{\partial}_n}. \end{aligned} \tag{5.52}$$

The antipode of the coordinates $\hat{x}^\mu \in \mathcal{A}_{\hat{x}}$ is not defined. In the approach of this thesis the coordinates are not regarded as finite translations, i.e. as elements of the κ-deformed Euclidean/Poincaré group, the dual Hopf algebra of $SO_a(n)$. The coordinates in our definition therefore do not have a coproduct, but the commutation relations of $\hat{x}^\mu$ with an arbitrary function can be considered formally as a coproduct (this leads to the same result for the antipode as in the framework of the κ-deformed group):

$$\begin{aligned} \hat{x}^j \hat{f}(\hat{x}) &= (e^{-ia\hat{\partial}_n}\hat{f}(\hat{x}))\hat{x}^j, \qquad \longrightarrow \ \hat{x}^j \otimes 1 - e^{-ia\hat{\partial}_n} \otimes \hat{x}^j = 0, \\ \hat{x}^n \hat{f}(\hat{x}) &= \hat{f}(\hat{x})\hat{x}^n + (ia\hat{x}^k\hat{\partial}_k\hat{f}(\hat{x})), \qquad \longrightarrow \ \hat{x}^n \otimes 1 - 1 \otimes \hat{x}^n - ia\hat{x}^k\hat{\partial}_k \otimes 1 = 0, \\ S(\hat{x}^j) &= \hat{x}^j e^{ia\hat{\partial}_n}, \\ S(\hat{x}^n) &= \hat{x}^n - ia\hat{\partial}_k\hat{x}^k = \hat{x}^n - ia\hat{x}^k\hat{\partial}_k - ia(n-1). \end{aligned} \tag{5.53}$$

We stress that these relations have to be taken with a grain of salt. We will use this Hopf algebraic discussion of the coordinates only in this chapter.

The two operators which worried us in section 5.2, $\hat{N}^l$ and $\hat{x}^n$, are the ones whose antipode involve factors proportional to $(n-1)$. The problem gets more obvious in the

$\star$-representation:

$$
\begin{aligned}
S(\partial_j^*) &= S\big(\partial_j \frac{e^{ia\partial_n}-1}{ia\partial_n}\big) = -\partial_j \frac{e^{-ia\partial_n}-1}{-ia\partial_n}, \qquad S(\partial_n) = -\partial_n, \qquad S(e^{ia\partial_n}) = e^{-ia\partial_n}, \\
S(D_j^*) &= S\big(\partial_j \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\big) = -\partial_j \frac{e^{ia\partial_n}-1}{ia\partial_n}, \\
S(D_n^*) &= S\big(\frac{1}{a}\sin(a\partial_n) + \frac{\partial_k\partial_k}{ia\partial_n\partial_n}(\cos(a\partial_n)-1)\big) = -\frac{1}{a}\sin(a\partial_n) + \frac{\partial_k\partial_k}{ia\partial_n\partial_n}(\cos(a\partial_n)-1), \\
S(M^{*rs}) &= S(x^s\partial_r - x^r\partial_s) = -x^s\partial_r + x^r\partial_s, \qquad (5.54) \\
S(N^{*l}) &= S\Big(x^l\partial_n\frac{e^{ia\partial_n}+1}{2} - x^n\partial_l\frac{e^{ia\partial_n}-1}{ia\partial_n} + x^l\partial_k\partial_k\frac{e^{ia\partial_n}-1}{2\partial_n} - x^k\partial_k\partial_l\frac{e^{ia\partial_n}-1-ia\partial_n}{ia\partial_n^2}\Big) \\
&= -x^l\partial_n\frac{e^{-ia\partial_n}+1}{2} + x^n\partial_l\frac{e^{-ia\partial_n}-1}{-ia\partial_n} + x^l\partial_k\partial_k\frac{e^{-ia\partial_n}-1}{-2\partial_n} - x^k\partial_k\partial_l\frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2} \\
&\quad +(n-1)\partial_l\frac{e^{-ia\partial_n}-1}{\partial_n}.
\end{aligned}
$$

The $\star$-representations $S(\mathcal{V}^*)$ are actually the $\star$-representations $S(\mathcal{V})^*$, we have calculated them from (5.52).

The $\star$-representations $S(\mathcal{V})^*$ are almost identical to the the result of partially integrating $\mathcal{V}^*$ under an integral fulfilling Stokes' law. The result $S(\mathcal{V})^*$ is therefore almost identical to the result of conjugating $\mathcal{V}^* \to \overline{\mathcal{V}^*}$ (by partial integration). The difference is that the antipode does not involve complex conjugation of $i \to -i$. Of course, for this partial integration we have to employ the integral definition involving the measure μ and the rescaling $\partial_j \to \tilde{\partial}_j = \partial_j + \rho_j$.

We give an example:

$$
\begin{aligned}
\int \mu\,(\tilde{D}_j^*\tilde{\psi}) \star \tilde{\phi} &= \int \mu\,\Big(\tilde{\partial}_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\tilde{\psi}\Big) \star \tilde{\phi} = \int \mu\tilde{\psi} \star \Big(-\tilde{\partial}_j\frac{e^{ia\partial_n}-1}{ia\partial_n}\tilde{\phi}\Big) \\
&= \int \mu\,\tilde{\psi} \star \Big(-\tilde{D}_j^* e^{ia\partial_n}\tilde{\phi}\Big) = \int \mu\,\tilde{\psi} \star \Big(S(\tilde{D}_j^*)\tilde{\phi}\Big). \qquad (5.55)
\end{aligned}
$$

Although with this definition we can treat in a satisfactory way almost all operators, $\tilde{N}^{*l}$ and $\tilde{x}^{*n}$ again do not fit into this framework. The problematic piece is the factor proportional to $(n-1)$:

$$
S(\tilde{N}^l)^* \sim (n-1)\partial_l\frac{e^{-ia\partial_n}-1}{\partial_n} = (n-1)\big(-ia\partial_l - \frac{a^2}{2}\partial_n\partial_l + \ldots\big). \qquad (5.56)
$$

Although we obtain a factor proportional to $(n-1)$ from partially integrating $\tilde{N}^{*l}$ (from the term proportional to $x^j\tilde{\partial}_j$)

$$
\tilde{N}^{*l} \xrightarrow{\text{p.i.}} -(n-1)\partial_l\frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2} = (n-1)\big(-\frac{ia}{2}\partial_l - \frac{a^2}{6}\partial_l\partial_n + \ldots\big), \qquad (5.57)
$$

this is not the right term for $S(\tilde{N}^l)^*$. Changing the definition of μ or the rescaling ρ_j to account for the additional terms does not work, since this would spoil the behaviour of other operators under partial integration.

The only handle that we have to obtain new terms proportional to $(n-1)$ to fix the antipode $S(N^l)^*$, is to introduce a derivative operator which acts on the coordinate x^n, i.e. an asymmetrically acting operator K, which is a power series in the derivatives ∂_n (it does not depend on coordinates x^μ or on ∂_j). We define K such that for all $\mathcal{V} \in SO_a(n)$ the following equation is valid, for two r-forms $\tilde{\psi}$ and $\tilde{\phi}$

$$\int \mu\, (\tilde{\mathcal{V}}^* \tilde{\psi}) \star (K\tilde{\phi}) \equiv \int \mu\, \tilde{\psi} \star \Big(K\big(S(\tilde{\mathcal{V}}^*)\tilde{\phi}\big)\Big). \tag{5.58}$$

To simplify the calculation, we eliminate on both sides the measure, the $\star$-product and the rescaling $\tilde{\mathcal{V}}^* \to \mathcal{V}^*$ by the field redefinition $\tilde{\phi} = \mu^{-\frac{1}{2}}\phi$ according to the prescription in section 5.2:

$$\int (\mathcal{V}^*\psi)(K\phi) \equiv \int \psi\Big(K\big(S(\mathcal{V}^*)\phi\big)\Big). \tag{5.59}$$

Note that $\mu^{-\frac{1}{2}}$ commutes with K. The result of the calculation does not depend on whether this redefinition is performed or not.

The equation that K has to satisfy is therefore

$$\begin{aligned}
[K, \big(-x^n\partial_l \frac{e^{-ia\partial_n}-1}{-ia\partial_n}\big)] &\overset{!}{=} (n-1)\partial_l\Big(\frac{e^{-ia\partial_n}-1}{\partial_n} + \frac{e^{-ia\partial_n}-1+ia\partial_n}{ia\partial_n^2}\Big)K,\\
\Leftrightarrow \quad \frac{\partial K}{\partial \partial_n} &\overset{!}{=} -(n-1)\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big(\frac{ia\partial_n e^{-ia\partial_n} + e^{-ia\partial_n} - 1}{ia\partial_n^2}\big)K, \\
\Leftrightarrow \quad K &= c\big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big)^{n-1}.
\end{aligned} \tag{5.60}$$

The solution is unique up to a complex multiplicative factor c which we fix $c = 1$, such that $K = 1 + \mathcal{O}(a)$, i.e. a well-behaved commutative limit.

This operator K is the same derivative operator that we have guessed as the remnant of extracting the measure μ from one of the two factors of the $\star$-product. This means that by constructing an action in terms of differential forms with the Hodge-$*$ we have found an action which is at the same time invariant under all $\mathcal{V} \in SO_a(n)$

$$(\hat{\mathcal{V}}\hat{\psi}, \hat{\phi}) = (\hat{\psi}, S(\hat{\mathcal{V}})\hat{\phi}),$$

since

$$\begin{aligned}
&\int \Big(\tilde{\mathcal{V}}^*\big(\tilde{\psi}_{\mu_1\ldots\mu_r}\omega^{\mu_1}\ldots\omega^{\mu_r}\big)\Big) \star *\big(\tilde{\phi}_{\nu_1\ldots\nu_r}\omega^{\nu_1}\ldots\omega^{\nu_r}\big) = \\
&\quad = \int (\tilde{\psi}_{\mu_1\ldots\mu_r}\omega^{\mu_1}\ldots\omega^{\mu_r}) \star \Big(S(\tilde{\mathcal{V}}^*) * \big(\tilde{\phi}_{\nu_1\ldots\nu_r}\omega^{\nu_1}\ldots\omega^{\nu_r}\big)\Big),\\
\Leftrightarrow &\int \mu\ \Big(\tilde{\mathcal{V}}^*\big(\tilde{\psi}_{\mu_1\ldots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r}\big)\Big) \star \big(K\tilde{\phi}_{\nu_1\ldots\nu_r}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\big) = \\
&\quad = \int \mu\ (\tilde{\psi}_{\mu_1\ldots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r}) \star \Big(KS(\tilde{\mathcal{V}}^*)\big(\tilde{\phi}_{\nu_1\ldots\nu_r}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\big)\Big),\\
\Leftrightarrow &\int \Big(\mathcal{V}^*\big(\psi_{\mu_1\ldots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r}\big)\Big)\big(K(\phi_{\nu_1\ldots\nu_r}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\ldots\omega^{\mu_n})\big) = \\
&\quad = \int (\psi_{\mu_1\ldots\mu_r}\frac{1}{r!}\omega^{\mu_1}\ldots\omega^{\mu_r})\Big(KS(\mathcal{V}^*)\big(\phi_{\nu_1\ldots\nu_r}\frac{\epsilon^{\nu_1\ldots\nu_r}{}_{\mu_{r+1}\ldots\mu_n}}{r!(n-r)!}\omega^{\mu_{r+1}}\ldots\omega^{\mu_n}\big)\Big).
\end{aligned} \tag{5.61}$$

The same is valid for the coordinates in the ad-hoc definition (5.53)

$$\int (\hat{x}^{\overrightarrow{*}\mu}\psi)(K\phi) = \int \psi\big(KS(\hat{x}^{\overrightarrow{*}\mu})\phi\big). \tag{5.62}$$

The last step in the derivation of an invariant integral is to extract from formulae such as (5.61) the one-forms ω^μ and to combine them into the volume form. We have to be careful in performing this step, since N^{*l} acts non-trivially on the frame one-forms (4.129). We derive the final result in two steps: first we treat the special case of the inner product of two functions, i.e. two zero-forms. The Hodge dual of a function is proportional to the volume form $\mathrm{d}^n x$. According to (4.129) $\mathrm{d}^n x = \hat{\omega}^1 \dots \hat{\omega}^n$ transforms as

$$[\hat{N}^l, \hat{\omega}^1 \dots \hat{\omega}^n] = -ia(n-1)\hat{\partial}_l.$$

On the other hand

$$S(\hat{N}^l) = -\hat{N}^l e^{-ia\hat{\partial}_n} - ia\hat{M}^{lk}\partial_k e^{-ia\hat{\partial}_n} - ia(n-1)\hat{\partial}_l e^{-ia\hat{\partial}_n}. \tag{5.63}$$

Since $[\hat{M}^{rs}, \hat{\omega}^1 \dots \hat{\omega}^n] = 0$ and $[\hat{\partial}_\mu, \hat{\omega}^\nu] = 0$, we obtain

$$S(\hat{N}^l)\hat{\omega}^1 \dots \hat{\omega}^n = \hat{\omega}^1 \dots \hat{\omega}^n(-\hat{N}^l e^{-ia\hat{\partial}_n} - ia\hat{M}^{lk}\hat{\partial}_k e^{-ia\hat{\partial}_n}), \tag{5.64}$$

The term appearing at the right hand side of (5.64) is

$$-N^{*l}e^{-ia\partial_n} - iaM^{*lk}\partial_k^* e^{-ia\partial_n} = -\overline{N^{*l}}, \tag{5.65}$$

where the bar denotes complex conjugation. Therefore we can equivalently rewrite (5.61) for the case in which ψ and ϕ are two *complex valued* zero-forms:

$$\begin{aligned}\int \mathrm{d}^n x (N^{*l}\psi)(K\overline{\phi}) = \int (N^{*l}\psi)\big(K(\overline{\phi}\ \mathrm{d}^n x)\big) \ &= \ \int \psi\ \Big(KS(N^{*l})(\phi\ \mathrm{d}^n x)\Big) \\ &= \ -\int \mathrm{d}^n x\ \psi\ \Big(K\overline{N^{*l}\phi)}\Big).\end{aligned} \tag{5.66}$$

The same steps can be repeated, if ψ and ϕ are r-forms. We may commute ω^μ with the coefficient functions (we regard the case, in which ω^l is in the first factor, the other case is analogous):

$$\begin{aligned}
&\int \Big(N^{*l}\big(\omega^{\mu_1}\dots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\dots\mu_r}\big)\Big)\big(K\omega^{\mu_{r+1}}\dots\omega^{\mu_n}\frac{\epsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_n}}{r!(n-r)!}\phi_{\nu_1\dots\nu_r}\big) \\
&= \int \Big(\omega^{\mu_1}\dots\omega^{\mu_r}\frac{1}{r!}\big(N^{*l}\psi_{\mu_1\dots\mu_r} - ia(r-1)\partial_l^*\psi_{\mu_1\dots\mu_r}\big)\Big)\big(K\omega^{\mu_{r+1}}\dots\omega^{\mu_n}\frac{\epsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_n}}{r!(n-r)!}\phi_{\nu_1\dots\nu_r}\big) \\
&= \int \mathrm{d}^n x\ \ \big(N^{*l}\psi_{\mu_1\dots\mu_r}\big)\big(K\phi^{\mu_1\dots\mu_r}\big) - ia(r-1)\int \mathrm{d}^n x\ \ \big(\partial_l^*\psi_{\mu_1\dots\mu_r}\big)\big(K\phi^{\mu_1\dots\mu_r}\big) \stackrel{!}{=} \\
&\stackrel{!}{=} \int \big(\omega^{\mu_1}\dots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\dots\mu_r}\big)\big(KS(N^{*l})\omega^{\mu_{r+1}}\dots\omega^{\mu_n}\frac{\epsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_n}}{r!(n-r)!}\phi_{\nu_1\dots\nu_r}\big) \\
&= \int \big(\omega^{\mu_1}\dots\omega^{\mu_r}\frac{1}{r!}\psi_{\mu_1\dots\mu_r}\big)\Big(K\omega^{\mu_{r+1}}\dots\omega^{\mu_n}\frac{\epsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_n}}{r!(n-r)!}\cdot \\
&\qquad\qquad \cdot\big(\overline{(-N^{*l})}\phi_{\nu_1\dots\nu_r} + ia\big((n-1)-(n-r)\big)\partial_l^* e^{-ia\partial_n}\phi_{\nu_1\dots\nu_r}\big)\Big) \\
&= \int \mathrm{d}^n x\ \ \psi_{\mu_1\dots\mu_r}\big(K\overline{(-N^{*l})}\phi^{\mu_1\dots\mu_r}\big) + ia(r-1)\int \mathrm{d}^n x\ \ \psi_{\mu_1\dots\mu_r}\big(K\partial_l^* e^{-ia\partial_n}\phi^{\mu_1\dots\mu_r}\big).
\end{aligned} \tag{5.67}$$

Partially integrating the term proportionally to $(r-1)$, the result for complex valued forms is:

$$\int \mathrm{d}^n x \ \big(N^{*l}\psi_{\mu_1\cdots\mu_r}\big)\big(K\overline{\phi^{\mu_1\cdots\mu_r}}\big) = -\int \mathrm{d}^n x \ \psi_{\mu_1\cdots\mu_r}\big(K\overline{N^{*l}\phi^{\mu_1\cdots\mu_r}}\big). \tag{5.68}$$

This identity is valid by partial integration and taking into account the action on the volume element and the commutation relation with K. From an abstract definition of inner product we have derived a hermitian representation of N^{*l}. More importantly, the identity (5.68) shows that the action defined in terms of forms is invariant under N^{*l}.

All other operators M^{*rs} and the derivatives D^*_μ and ∂^*_μ (no tilde) can be treated analogously. The discussion of these operators is straightforward since they commute with K and with the volume element $\mathrm{d}^n x$ and they be partially integrated without harm (since μ has been eliminated).

We have achieved the definition of an invariant integral as an inner product of two r-forms. This definition leads to an integral where the $\star$-product between the two r-forms is removed, but which has a derivative operator K acting on the Hodge-dualised form. Under the resulting integral, all symmetry generators are hermitian by partial integration.

However, the integral just defined is obviously not cyclic, since from the outset we have discussed an asymmetric setting: the $\star$-product is not commutative and therefore it matters whether the Hodge-dual form is in the first or in the second place of the inner product. For the Hopf algebra setting, this however is essential: the order in the inner product *must not* be reversible, since the module space and its second dual space, i.e. the dual of the dual space, are not identical. We recall the result of section 4.2 that the square of the antipode is not the identity:

$$S^2(\hat{N}^i) = \hat{N}^i + ia(n-1)\hat{\partial}_i \neq \hat{N}^i.$$

The generator N^{*l} acts in different ways on a space and its second dual. Therefore it is clear that in formulae such as (5.61) we cannot simply partially integrate once more to obtain the action on the second dual space. The construction of the bidual space has to be redone from scratch. We will not perform the calculations once more, but they result in an expression in which we have to partially integrate (ψ and ϕ arbitrary r-forms)

$$\int \big(S(N^{*l})\psi\big)\big(K\phi\big) = \int \psi\big(KS^2(N^{*l})\phi\big). \tag{5.69}$$

This indeed gives the correct result for the algebraic expression of the square of the antipode. Because of this property, derivative operators such as K generally occur for traces for general Hopf algebras [114]. The integral defined with such an operator is called the *quantum trace*.

Note that the definition of this integral presupposes the interpretation of an integral as an inner product. The integral over a field $\int \psi(x)$ cannot sensibly analysed in this way, especially the question of its invariance is not well posed.

We have not been able yet to fully understand the usefulness of the quantum trace. Variation in principle is possible with this definition of integral, since the action can be varied w.r.t. one of the coefficient functions of the forms. In addition, the derivative operator K can be partially integrated onto the other form:

$$\int \mathrm{d}^n x \;\; \psi\Big(\big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big)^{n-1}\phi\Big) = \int \mathrm{d}^n x \;\; \Big(\big(\frac{ia\partial_n}{e^{ia\partial_n}-1}\big)^{n-1}\psi\Big)\phi. \tag{5.70}$$

It is not clear how to define the products of several fields in this language. Most importantly, a priori the quantum trace does not allow to formulate gauge invariant actions from gauge covariant Lagrangians, since it is not cyclic. We believe that it may be possible to formulate a gauge-covariantised version of the quantum trace (see section 6.4, so far we have not been able to find a satisfactory solution).

The upshot of the discussion of this section is that we have to choose between formulations of the integral which are either not invariant under symmetry transformations (at least at face value) or not gauge invariant (at least at face value).

Chapter 6

κ-deformed gauge theory

In this chapter, we will discuss gauge theory for κ-deformed space, generalising the results found for canonical NC space. Since the discussion of chapter 3 has been in-depth, we will be brief here and emphasise only new features. Those properties of NC gauge theory which arise only because of the $\star$-product can be formulated immediately for arbitrary Lie algebra NC spaces, this will be the content of the first section. The truly new feature is that gauge potentials become derivative-valued. For discussing this issue, we specialise to gauging the Dirac derivatives on κ-deformed space. In this chapter we will also discuss how to build actions on κ-deformed space and we will gauge other derivatives to understand generic features of derivative-valued gauge potentials.

6.1 Gauge theories on Lie algebra NC spaces

Every Lie algebra NC space with symmetric ordering can be represented in terms of the BCH $\star$-product. Expanding the BCH $\star$-product in terms of the structure constants $C_\lambda^{\mu\nu}$

$$\begin{aligned} f \star g\,(x) \;&=\; \mathsf{m}\Big(\exp\big(\frac{i\hbar}{2} x^\lambda C_\lambda^{\mu\nu} \partial_\mu \otimes \partial_\nu + \frac{\hbar^2}{12} x^\lambda C_\lambda^{\rho\sigma} C_\rho^{\mu\nu} (\partial_\sigma \otimes 1 - 1 \otimes \partial_\sigma) \partial_\mu \otimes \partial_\nu + \\ &\quad + \frac{i\hbar^3}{24} x^\lambda C_\lambda^{\alpha\beta} C_\alpha^{\rho\sigma} C_\rho^{\mu\nu} \partial_\beta \partial_\mu \otimes \partial_\sigma \partial_\nu + \dots \big)\; f(y) \otimes g(z) \Big)_{y,z\to x}, \end{aligned} \tag{6.1}$$

the enveloping algebra gauge theory is treated in analogy to the discussion in section 3.3. We expand the gauge parameter Λ_α and all other elements of the gauge theory as formal power series in the dimensionless expansion parameter $\hbar$ of the $\star$-product[1]. The gauge parameter Λ_α is a tower in the enveloping algebra that coincides to zeroth order with the Lie algebra gauge parameter:

$$\Lambda_\alpha = \alpha + \hbar \Lambda_\alpha^1 + \hbar^2 \Lambda_\alpha^2 + \cdots \quad . \tag{6.2}$$

We start from the consistency condition

$$i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{\alpha\times\beta}. \tag{6.3}$$

[1] For κ-deformed space we replace $a \to \hbar a$.

At first order in $\hbar$, the consistency condition is almost identical to the consistency condition for constant $\theta^{\mu\nu}$ (compare (3.55)):

$$\Delta\Lambda^1 = i\delta_\alpha\Lambda^1_\beta - i\delta_\beta\Lambda^1_\alpha + [\alpha, \Lambda^1_\beta] - [\beta, \Lambda^1_\alpha] - \Lambda^1_{[\alpha,\beta]} = -[\alpha \overset{1}{\star} \beta] = -\frac{i\hbar}{2} x^\lambda C^{\mu\nu}_\lambda \{\partial_\mu\alpha, \partial_\nu\beta\}. \quad (6.4)$$

The shorthand for the first order of the $\star$-product expansion: $f \star^1 g = \frac{i\hbar}{2} x^\lambda C^{\mu\nu}_\lambda \partial_\mu f \partial_\nu g$ has been defined in section 3.4. We obtain a solution of (6.4) replacing in (3.56) $\theta^{\mu\nu} = x^\lambda C^{\mu\nu}_\lambda$:

$$\Lambda^1_\alpha = -\frac{1}{4} x^\lambda C^{\mu\nu}_\lambda \{A^0_\mu, \partial_\nu\alpha\}. \quad (6.5)$$

This is the usual hermitian solution, we shortly discuss the freedom at the end of this section.

Also in higher orders in $\hbar$ certain terms in the solution of the consistency conditions are analogous to the canonical NC space, replacing $\theta^{\mu\nu}$ with $x^\lambda C^{\mu\nu}_\lambda$. We will call such terms $\Lambda^{k\theta}_\alpha$. All terms analogous to the canonical case in Λ^k_α are of order k in the explicit appearance of coordinates x. All terms in Λ^k_α with explicit x-dependence of order $j < k$ have no analogon in the canonical case.

Such new terms appear at second order in $\hbar$:

$$\begin{aligned} \Delta\Lambda^2 &= i\delta_\alpha\Lambda^2_\beta - i\delta_\beta\Lambda^2_\alpha + [\alpha, \Lambda^2_\beta] - [\beta, \Lambda^2_\alpha] - \Lambda^2_{[\alpha,\beta]} = \\ &+\frac{1}{8} x^\lambda x^\kappa C^{\mu\nu}_\lambda C^{\rho\sigma}_\kappa [\partial_\mu\partial_\rho\alpha, \partial_\nu\partial_\sigma\beta] - \frac{1}{12} x^\kappa C^{\rho\sigma}_\kappa C^{\mu\nu}_\rho \Big([\partial_\sigma\partial_\mu\alpha, \partial_\nu\beta] - [\partial_\sigma\partial_\mu\beta, \partial_\nu\alpha]\Big) \quad (6.6) \\ &-\frac{i}{2} x^\lambda C^{\mu\nu}_\lambda \Big(\{\partial_\mu\Lambda^1_\alpha, \partial_\nu\beta\} - \{\partial_\mu\Lambda^1_\beta, \partial_\nu\alpha\}\Big) - [\Lambda^1_\alpha, \Lambda^1_\beta]. \end{aligned}$$

The terms quadratic in the explicit x-dependence are solved by terms $\Lambda^{2\theta}_\alpha$ (3.60):

$$\begin{aligned} \Lambda^{2\theta}_\alpha &= \frac{1}{32} x^\rho x^\sigma C^{\mu\nu}_\rho C^{\kappa\lambda}_\sigma \Big(\{A^0_\mu, \{\partial_\nu A^0_\kappa, \partial_\lambda\alpha\}\} + \{A^0_\mu, \{A^0_\kappa, \partial_\nu\partial_\lambda\alpha\}\} \\ &+ \{\{A^0_\mu, \partial_\nu A^0_\kappa\}, \partial_\lambda\alpha\}\} - \{\{F^0_{\mu\kappa}, A^0_\nu\}, \partial_\lambda\alpha\} - 2i[\partial_\mu A^0_\kappa, \partial_\nu\partial_\lambda\alpha]\Big). \quad (6.7) \end{aligned}$$

There is a new term proportional to $\frac{1}{12}$ from the second order of the BCH expansion of the symmetric $\star$-product. In addition, there is a term in which a derivative is acting on Λ^1_α:

$$\partial_\sigma\Lambda^1_\alpha = -\frac{1}{4} x^\lambda C^{\mu\nu}_\lambda (\{\partial_\sigma A^0_\mu, \partial_\nu\alpha\} + \{A^0_\mu, \partial_\sigma\partial_\nu\alpha\}) - \frac{1}{4} C^{\mu\nu}_\sigma \{A^0_\mu, \partial_\nu\alpha\}. \quad (6.8)$$

The last term is not present in the canonical case. Explicitly using the Jacobi identity $C^{\rho\sigma}_\kappa C^{\mu\nu}_\rho + C^{\rho\mu}_\kappa C^{\nu\sigma}_\rho + C^{\rho\nu}_\kappa C^{\sigma\mu}_\rho = 0$ the hermitian solution for the consistency condition including these additional terms is:

$$\Lambda^2_\alpha = \Lambda^{2\theta}_\alpha - \frac{1}{24} x^\kappa C^{\rho\sigma}_\kappa C^{\mu\nu}_\rho \Big(\{A^0_\sigma\{A^0_\mu, \partial_\nu\alpha\}\} - 2i[\partial_\sigma A^0_\mu, \partial_\nu\alpha]\Big). \quad (6.9)$$

Similarly, we obtain at third order in $\hbar$ terms which are linear, quadratic and cubic in explicit x (the cubic terms are $\Lambda^{3\theta}_\alpha$).

We use (6.5) and (6.9) to construct a field ψ in a representation of the gauge enveloping algebra, transforming from the left. This field ψ is a functional of the fields ψ^0 and A^0_μ, it can be expanded in $\hbar$:

$$\psi = \psi^0 + \hbar\psi^1 + \hbar^2\psi^2 + \cdots \quad . \tag{6.10}$$

The defining equation for ψ is to first order

$$\delta_\alpha\psi^1 = i\Lambda^1_\alpha\psi^0 + i\alpha\psi^1 - \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda \partial_\mu\alpha\, \partial_\nu\psi^0, \tag{6.11}$$

and second order in $\hbar$

$$\begin{aligned} \delta_\alpha\psi^2 \;=\;& i\Lambda^2_\alpha\psi^0 + i\Lambda^1_\alpha\psi^1 + i\alpha\psi^2 - \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda \partial_\mu\Lambda^1_\alpha\, \partial_\nu\psi^0 - \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda \partial_\mu\alpha\, \partial_\nu\psi^1 \\ & -\frac{i}{8}x^\lambda x^\kappa C^{\mu\nu}_\lambda C^{\rho\sigma}_\kappa \partial_\mu\partial_\rho\alpha\, \partial_\sigma\partial_\nu\psi^0 + \frac{i}{12}x^\lambda C^{\rho\sigma}_\lambda C^{\mu\nu}_\rho \Big(\partial_\sigma\partial_\mu\alpha\, \partial_\nu\psi^0 - \partial_\mu\alpha\, \partial_\sigma\partial_\nu\psi^0\Big). \end{aligned} \tag{6.12}$$

The terms analogous to the canonical case ($\theta^{\mu\nu} \to x^\lambda C^{\mu\nu}_\lambda$) are called $\psi^{k\theta}$.

The first order solution is:

$$\psi^1 = \psi^{1\theta} = -\frac{1}{2}x^\lambda C^{\mu\nu}_\lambda A^0_\mu\partial_\nu\psi^0 + \frac{i}{4}x^\lambda C^{\mu\nu}_\lambda A^0_\mu A^0_\nu\psi^0. \tag{6.13}$$

In second order, terms linear in explicit x arise from the BCH $\star$-product and the action of derivatives on explicit x-dependent first order solutions:

$$\partial_\sigma\Lambda^1_\alpha = -\frac{1}{4}C^{\mu\nu}_\sigma\{A^0_\mu, \partial_\nu\alpha\} + \ldots \qquad \partial_\sigma\psi^1 = -\frac{1}{2}C^{\mu\nu}_\sigma A^0_\mu\partial_\nu\psi^0 + \ldots \quad . \tag{6.14}$$

Working with the hermitian solution for Λ^k_α we obtain:

$$\begin{aligned} \psi^2 \;=\;& \psi^{2\theta} + \frac{1}{24}x^\kappa C^{\rho\sigma}_\kappa C^{\mu\nu}_\rho \Big(2i\partial_\sigma A^0_\mu\partial_\nu\psi^0 - 2iA^0_\mu\partial_\sigma\partial_\nu\psi^0 - A^0_\sigma A^0_\mu\partial_\nu\psi^0 - 3A^0_\mu A^0_\sigma\partial_\nu\psi^0 \\ & -2A^0_\mu\partial_\sigma A^0_\nu\psi^0 + 3A^0_\mu A^0_\sigma A^0_\nu\psi^0 - 2A^0_\mu A^0_\nu A^0_\sigma\psi^0\Big). \end{aligned} \tag{6.15}$$

We have used $\psi^{2\theta}$ (3.67):

$$\begin{aligned} \psi^{2\theta} \;=\;& -\frac{i}{8}x^\rho x^\sigma C^{\kappa\lambda}_\rho C^{\mu\nu}_\sigma \Big(\partial_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 + iA^0_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 - i\partial_\kappa A^0_\mu A^0_\nu\partial_\lambda\psi^0 + iF^0_{\kappa\mu}A^0_\nu\partial_\lambda\psi^0 \\ & -iA^0_\nu\partial_\kappa A^0_\mu\partial_\lambda\psi^0 + 2iA^0_\nu F^0_{\kappa\mu}\partial_\lambda\psi^0 + 2A^0_\mu A^0_\kappa A^0_\nu\partial_\lambda\psi^0 - A^0_\mu A^0_\nu A^0_\kappa\partial_\lambda\psi^0\Big) \\ & -\frac{1}{32}x^\rho x^\sigma C^{\kappa\lambda}_\rho C^{\mu\nu}_\sigma \Big(2\partial_\kappa A^0_\mu\partial_\lambda A^0_\nu\psi^0 - 2i\partial_\kappa A^0_\mu A^0_\lambda A^0_\nu\psi^0 + 2iA^0_\nu A^0_\lambda\partial_\kappa A^0_\mu\psi^0 \\ & +i[[\partial_\kappa A^0_\mu, A^0_\nu], A^0_\lambda]\psi^0 + 4iA^0_\nu F^0_{\kappa\mu}A^0_\lambda\psi^0 - A^0_\kappa A^0_\lambda A^0_\mu A^0_\nu\psi^0 + 2A^0_\kappa A^0_\mu A^0_\nu A^0_\lambda\psi^0\Big). \end{aligned}$$

The adjoint field $\overline{\psi}$ is obtained by conjugation.

Still we need to discuss the ambiguities of the construction. We could start to discuss the ambiguities systematically order by order as in section 3.6. For the main part, the analysis would provide the same ambiguities like for the canonically NC space. For example, there will be the usual first order non-covariant ambiguity $\Lambda_\alpha^{1,c_1} = ic_1 x^\lambda C_\lambda^{\mu\nu}[A_\mu^0, \partial_\nu \alpha]$. There is an additional freedom for Lie algebras, since in the expansion of Λ_α^2 there are terms proportional to a linear appearance of x. These terms do not have a counterpart in the canonical case.

Instead of a cumbersome systematic analysis, we use the lessons learnt in section 3.7. The non-covariant or gauge ambiguities are of the form $\Delta_\alpha \xi^2$ with $\Delta\cdot = \delta_\alpha \cdot -i[\alpha, \cdot]$. Demanding that the ambiguities should be hermitian, there are two possible non-covariant ambiguities:

$$\begin{aligned}
\Delta_\alpha \xi^{2,c_{12}} &= \Delta_\alpha c_{12} x^\kappa C_\kappa^{\rho\sigma} C_\rho^{\mu\nu}\{\partial_\sigma A_\mu^0, A_\nu^0\} \\
&= c_{12} x^\kappa C_\kappa^{\rho\sigma} C_\rho^{\mu\nu}\big(\{\partial_\sigma \partial_\mu \alpha, A_\nu^0\} + \{\partial_\sigma A_\mu^0, \partial_\nu \alpha + i\{[\partial_\sigma \alpha, A_\mu^0], A_\nu^0\}\big), \\
\Delta_\alpha \xi^{2,c_{13}} &= \Delta_\alpha c_{13} x^\kappa C_\kappa^{\rho\sigma} C_\rho^{\mu\nu}\{A_\sigma^0, [A_\mu^0, A_\nu^0]\} \\
&= c_{13} x^\kappa C_\kappa^{\rho\sigma} C_\rho^{\mu\nu}\big(\{\partial_\sigma \alpha, [A_\mu^0, A_\nu^0]\} + 2\{A_\sigma^0, [\partial_\mu \alpha, A_\nu^0]\}\big).
\end{aligned} \tag{6.16}$$

These two new non-covariant ambiguities in a^2 and linear in x exhaust all hermitian ambiguities because of the Jacobi identity.

6.2 Gauge theory and deformed symmetries

In κ-deformed space, there is not only a deformed product between fields ($\star$-product), but in addition nonlinear symmetry generators act in a deformed way on products of functions. Since the coproduct of symmetry generators is cooked up such that it respects the $\star$-product, it is clear that NC gauge theory (realized by $\star$-multiplying a gauge parameter to a field) is consistent with a deformed symmetry. However, a priori it is not clear that this deformed action is consistent with an explicitly θ-expanded enveloping algebra-valued gauge theory.

Gauge transformations induce a map

$$\psi \longrightarrow \psi' = \psi + \delta_\alpha \psi = \psi + i\Lambda_\alpha \star \psi. \tag{6.17}$$

The enveloping algebra gauge parameter Λ_α depends on α, A_μ^0 and their derivatives. The solution of the consistency condition, expanded in Lie algebra quantities, has to transform properly under the deformed symmetry. We check that it indeed does by discussing the transformation behaviour of the consistency condition (6.3) and of the defining equations (e.g. (6.17)). If these equations transform in a covariant way, the quantities derived from them will transform covariantly as well.

The symmetry generators act with a deformed coproduct on the right hand side of (6.17). We show that a field acted upon by a symmetry generator still fulfils (6.3). The rotation N^{*l} acts on the field ψ as follows:

$$\tilde{\psi} = \psi - \epsilon_l N^{*l} \psi, \tag{6.18}$$

The inner degrees of freedom (in the gauge group) of ψ and Λ_α do not transform.

Applying first a gauge transformation (6.17) and second a rotation we obtain $\widetilde{\psi'}$:

$$\widetilde{\psi'} = \psi + i\Lambda_\alpha \star \psi - (\epsilon_l N^{*l}\psi) - i\epsilon_l N^{*l}(\Lambda_\alpha \star \psi). \tag{6.19}$$

The result of first rotating ψ and afterwards performing a gauge transformation (6.17) is called $\tilde{\psi}\,'$:

$$\tilde{\psi}\,' = \psi - (\epsilon_l N^{*l}\psi) + i\Lambda_\alpha \star \psi - i\epsilon_l N^{*l}(\Lambda_\alpha \star \psi). \tag{6.20}$$

The two transformations commute:

$$\begin{array}{ccc} \psi & \xrightarrow{\alpha} & \psi' \\ \epsilon \downarrow & & \downarrow \epsilon \\ \tilde{\psi} & \xrightarrow{\alpha} & \widetilde{\psi'} \equiv \tilde{\psi}\,' \end{array} . \tag{6.21}$$

The gauge transformation (6.19) can be written as a gauge transformation on $\tilde{\psi}$:

$$\delta_\alpha \tilde{\psi} = i\Lambda_\alpha \star \tilde{\psi} + i\Lambda_\alpha \star (\epsilon_l N^{*l}\tilde{\psi}) - i\epsilon_l N^{*l}(\Lambda_\alpha \star \tilde{\psi}). \tag{6.22}$$

If the gauge transformation on the rotated field $\tilde{\psi}$ is again an enveloping algebra-valued gauge transformation, it has to fulfil the consistency condition (6.3). Using (6.17) and (4.114) and computing $\delta_\beta\delta_\alpha\tilde{\psi}$ from (6.20) we obtain:

$$\begin{aligned}(\delta_\beta\delta_\alpha - \delta_\alpha\delta_\beta)\tilde{\psi} \;=\;& \Big(i(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\Big) \star \psi \\ & -\epsilon_l N^{*l}\Big(i(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\Big) \star \psi \\ & -e^{ia\partial_n^*}\Big(i(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\Big) \star \epsilon_l N^{*l}\psi \\ & +ia\partial_j^*\Big(i(\delta_\beta\Lambda_\alpha - \delta_\alpha\Lambda_\beta) - (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha)\Big) \star \epsilon_l M^{lj}\psi.\end{aligned} \tag{6.23}$$

The consistency condition (6.3) is fulfilled on the individual components and therefore gauge transformations on the rotated field $\tilde{\psi}$ are again enveloping algebra-valued. Thus, the multiplicative structure of the enveloping algebra-valued gauge theory is covariant under deformed rotations.

Derivatives have to be defined in such a way that they are both covariant under gauge transformations and under deformed $SO_a(n)$ rotations. Since we have introduced the Dirac derivative $\hat{D}_\mu$ in section 4.2 such that it fulfils the requirement of covariance under $SO_a(n)$ rotations, it is a suitable candidate for a derivative. We discuss general issues of the covariant derivative for clarity first in the abstract algebra.

To fulfil the second requirement and render the Dirac derivative covariant under gauge transformations, we proceed as in ordinary commutative gauge theory. The derivative of a field $\hat{D}_\mu\hat{\psi}$ has a different transformation property under gauge transformations than the field $\hat{\psi}$. Therefore we add as usual a gauge potential $\hat{V}_\mu$ to obtain a derivative, which is both covariant under $SO_a(n)$ and under gauge transformations. This covariant derivative is denoted $\hat{\mathcal{D}}_\mu$:

$$\hat{\mathcal{D}}_\mu \hat{\psi} = (\hat{D}_\mu - i\hat{V}_\mu)\hat{\psi}, \quad \text{such that} \quad \delta_\alpha \hat{\mathcal{D}}_\mu \hat{\psi} = i\hat{\Lambda}_\alpha \hat{\mathcal{D}}_\mu \hat{\psi}, \quad \text{if} \quad \delta_\alpha \hat{\psi} = i\hat{\Lambda}_\alpha \hat{\psi}. \tag{6.24}$$

In order to determine how the gauge potential $\hat{V}_\mu$ transforms under gauge transformations, we have to express the previous equation without the field $\hat{\psi}$ on the right hand side. But there is a difficulty, which can be seen writing (6.24) with brackets:

$$(\delta_{\hat{\Lambda}} \hat{V}_\mu)\hat{\psi} = \hat{D}_\mu(\hat{\Lambda}\hat{\psi}) - \hat{\Lambda}(\hat{D}_\mu \hat{\psi}) - i\hat{V}_\mu(\hat{\Lambda}\hat{\psi}) + i\hat{\Lambda}_\alpha(\hat{V}_\mu \hat{\psi}). \tag{6.25}$$

Since the Dirac derivative $\hat{D}_\mu$ has a nontrivial Leibniz rule, in components

$$\begin{aligned} \hat{D}_j(\hat{\Lambda}\ \cdot) - \hat{\Lambda}(\hat{D}_j\ \cdot) &= (\hat{D}_j \hat{\Lambda})\ e^{-ia\hat{\partial}_n} \cdot, \qquad (6.26)\\ \hat{D}_n(\hat{\Lambda}\ \cdot) - \hat{\Lambda}(\hat{D}_n\ \cdot) &= (\hat{D}_n \hat{\Lambda})\ e^{-ia\hat{\partial}_n} \cdot + ((e^{ia\hat{\partial}_n} - 1)\hat{\Lambda})\ \hat{D}_n \cdot + ia(\hat{D}_j e^{ia\hat{\partial}_n}\hat{\Lambda})\ \hat{D}_j \cdot, \end{aligned}$$

we can *not* reduce the first two terms of (6.25) to $(\hat{D}_\mu \hat{\Lambda})$ and therefore

$$\delta_{\hat{\Lambda}} \hat{V}_\mu \neq \hat{D}_\mu \hat{\Lambda} - i[\hat{V}_\mu, \hat{\Lambda}].$$

Equation (6.25) together with the identities (6.26) can only accommodate gauge potentials $\hat{V}_\mu$ if we define them as being *derivative-valued*. This means: If a derivative-valued gauge potential $\hat{V}_\mu = \hat{A}_\lambda \zeta_\mu^{\lambda\nu} \hat{\partial}_\nu$ ($\zeta_\mu^{\lambda\nu}$ some complex constants) is multiplied from the right hand side with another field, the derivatives are evaluated on the field $\hat{V}_\mu \psi = \hat{A}_\lambda \rho_\mu^{\lambda\nu}(\hat{\partial}_\nu \psi)$. If the gauge potential is multiplied from the right with products of fields, the (deformed) coproduct has to be used.

The two components $\hat{V}_j$ and $\hat{V}_n$ have to be treated separately because of their different coproducts. First we examine the simpler case $\hat{V}_j$. Using (6.26) we introduce the physical gauge potentials $\hat{A}_j$

$$\hat{V}_j = \hat{A}_j e^{-ia\hat{\partial}_n}, \qquad \text{that is} \qquad \hat{V}_j \cdot \hat{\psi} = \hat{A}_j \cdot e^{-ia\hat{\partial}_n} \hat{\psi}. \tag{6.27}$$

The shift operator acting to the right constitutes a new interaction. Since the gauge potential $\hat{A}_j$ is accompanied with this derivative-valued operator, ψ is shifted by $e^{-ia\hat{\partial}_n}$ in all terms on the left and right hand side of (6.25). Therefore $e^{-ia\hat{\partial}_n}\hat{\psi}$ can be eliminated altogether. Thus, the physical gauge potential $\hat{A}_j$ has the following behaviour under gauge transformations:

$$\delta_{\hat{\Lambda}} \hat{A}_j = \hat{D}_j \hat{\Lambda} - i\hat{A}_j e^{-ia\hat{\partial}_n}\hat{\Lambda} + i\hat{\Lambda}\hat{A}_j. \tag{6.28}$$

The Leibniz rule of the Dirac derivative $\hat{D}_n$ is more complicated than the one for $\hat{D}_j$, therefore the gauge potential $\hat{V}_n$ is suitably split $\hat{V}_n$ into several distinct parts. Because of (6.26) a splitting of $\hat{V}_n$ into three different pieces (not three different degrees of freedom) is convenient[2]:

$$\hat{V}_n \cdot = \hat{A}_{n,1} e^{-ia\hat{\partial}_n} \cdot + \hat{A}^j_{n,2} \hat{D}_j \cdot + \hat{A}_{n,3} \hat{D}_n \cdot \quad . \tag{6.29}$$

[2]Later we find that it is more convenient to split the definition of $\hat{V}_n$ into five pieces for practical calculations.

With the help of the definition (6.29) we can rewrite (6.25), carefully keeping track of the Leibniz rules of $\hat{D}_\mu$:

$$\begin{aligned}
\delta_{\hat{\Lambda}}\hat{A}_{n,1} &= \hat{D}_n\hat{\Lambda} - i\hat{A}_{n,1}e^{-ia\hat{\partial}_n}\hat{\Lambda} + i\hat{\Lambda}\hat{A}_{n,1} - i\hat{A}^j_{n,2}\hat{D}_j\hat{\Lambda} - i\hat{A}_{n,3}\hat{D}_n\hat{\Lambda}, \\
\delta_{\hat{\Lambda}}\hat{A}^j_{n,2} &= ia\hat{D}_j e^{ia\hat{\partial}_n}\hat{\Lambda} - i\hat{A}^j_{n,2}\hat{\Lambda} + i\hat{\Lambda}\hat{A}^j_{n,2} + a\hat{A}_{n,3}\hat{D}_j e^{ia\hat{\partial}_n}\hat{\Lambda}, \\
\delta_{\hat{\Lambda}}\hat{A}_{n,3} &= (e^{ia\hat{\partial}_n} - 1)\hat{\Lambda} - i\hat{A}_{n,3}e^{ia\hat{\partial}_n}\hat{\Lambda} + i\hat{\Lambda}\hat{A}_{n,3}.
\end{aligned} \tag{6.30}$$

The equations for $\hat{A}_{n,1}$ and $\hat{A}^j_{n,2}$ do not close, the three relations (6.30) have to be solved in parallel. The additional index j in $\hat{A}^j_{n,2}$ signals that this j explicitly appears in the defining equation for $\hat{A}^j_{n,2}$.

Before we solve these relations, some conceptual issues need to be discussed. The gauge potential $\hat{V}_\mu$ has been defined as a vector field transforming vector-like under $SO_a(n)$, like the Dirac derivative. In section 4.7 we have shown that vector fields corresponding to the Dirac derivative can be defined easily. The problem arises comparing the defining equations (6.28) and (6.30). Transforming for example $[\hat{N}^l, (\hat{D}_j\hat{\Lambda})] \neq \hat{D}_n\hat{\Lambda}$, we see that we do not obtain the vector index structure of the derivative on $\hat{\Lambda}$ that might have been expected. Since $\hat{V}_\mu$ is a vector field, the first expectation would have been that $\hat{A}_\mu$ is a vector-like transforming field as well. This need not be the case, since we have seen in many examples, that a composite operator may have linear transformation properties (e.g. the Dirac derivative can be expanded in terms of $\hat{\partial}_\mu$), while the components (e.g. $\hat{\partial}_\mu$) have a complicated, nonlinear transformation behaviour.

Of course, we expect a well-defined transformation behaviour of the fields A_μ defined here, but it is probably not the best strategy to derive it from the defining equations (6.28) and (6.30). We defer a discussion of the transformation of the physical field for later research, this topic can certainly be discussed easier in the generic κ-deformed space.

In addition it may be asked whether the particular derivative operators acting to the right are stable under additional gauge transformations. To analyse this question, we perform a second gauge transformation on $\hat{V}_\mu$ (we only treat $\hat{V}_j$). Concretely we perform a second gauge transformation on both sides of

$$(\delta_{\hat{\Lambda}_1}\hat{V}_j)\hat{\psi} = \hat{D}_j(\hat{\Lambda}_1\hat{\psi}) - \hat{\Lambda}_1\hat{D}_j\hat{\psi} - i\hat{V}_j\hat{\Lambda}_1\hat{\psi} + i\hat{\Lambda}_1\hat{V}_j\hat{\psi}$$

and obtain

$$\begin{aligned}
\big((\delta_{\hat\Lambda_1}\delta_{\hat\Lambda_2} - \delta_{\hat\Lambda_2}\delta_{\hat\Lambda_1})\hat V_j\big)\hat\psi + i(\delta_{\hat\Lambda_2}\hat V_j)(\hat\Lambda_1\hat\psi) - i(\delta_{\hat\Lambda_1}\hat V_j)(\hat\Lambda_2\hat\psi) = \\
\hat D_j\big((\delta_{\hat\Lambda_1}\hat\Lambda_2 - \delta_{\hat\Lambda_2}\hat\Lambda_1)\hat\psi\big) + (\delta_{\hat\Lambda_1}\hat\Lambda_2 - \delta_{\hat\Lambda_2}\hat\Lambda_1)\big(\hat D_j\hat\psi\big) \\
-i\hat V_j(\delta_{\hat\Lambda_1}\hat\Lambda_2 - \delta_{\hat\Lambda_2}\hat\Lambda_1)\hat\psi + i(\delta_{\hat\Lambda_1}\hat\Lambda_2 - \delta_{\hat\Lambda_2}\hat\Lambda_1)\big(\hat V_j\hat\psi\big) \qquad (6.31)\\
+i\hat D_j\big(\hat\Lambda_2(\hat\Lambda_1\hat\psi)\big) - i\hat\Lambda_2\hat D_j(\hat\Lambda_1\hat\psi) + \hat V_j\big(\hat\Lambda_2(\hat\Lambda_1\hat\psi)\big) - \hat\Lambda_2\hat V_j(\hat\Lambda_1\hat\psi) \\
-i\hat D_j\big(\hat\Lambda_1(\hat\Lambda_2\hat\psi)\big) + i\hat\Lambda_1\hat D_j(\hat\Lambda_2\hat\psi) - \hat V_j\big(\hat\Lambda_1(\hat\Lambda_2\hat\psi)\big) + \hat\Lambda_1\hat V_j(\hat\Lambda_2\hat\psi) \\
-i(\delta_{\hat\Lambda_1}\hat V_j)(\hat\Lambda_2\hat\psi) + i(\delta_{\hat\Lambda_2}\hat V_j)(\hat\Lambda_1\hat\psi) + i\hat\Lambda_2(\delta_{\hat\Lambda_1}\hat V_j)\hat\psi - i\hat\Lambda_1(\delta_{\hat\Lambda_2}\hat V_j)\hat\psi \\
= \hat D_j(\hat\Lambda_{1\times 2}\hat\psi) - \hat\Lambda_{1\times 2}\hat D_j\hat\psi - i\hat V_j\hat\Lambda_{1\times 2}\hat\psi + i\hat\Lambda_{1\times 2}\hat V_j\hat\psi \\
+i(\delta_{\hat\Lambda_2}\hat V_j)(\hat\Lambda_1\hat\psi) - i(\delta_{\hat\Lambda_1}\hat V_j)(\hat\Lambda_2\hat\psi) \\
= (\delta_{\hat\Lambda_1\times\hat\Lambda_2}\hat V_j)\hat\psi + i(\delta_{\hat\Lambda_2}\hat V_j)(\hat\Lambda_1\hat\psi) - i(\delta_{\hat\Lambda_1}\hat V_j)(\hat\Lambda_2\hat\psi).
\end{aligned}$$

Thus, it has been shown that two consecutive gauge transformations close on a gauge potential multiplied from the right with a field. Therefore the derivative operator acting to the right is left unchanged. For $\hat V_n$, the analysis is analogous and the result identical.

In addition note that the coproducts of the derivative operators appearing on the right close in the derivative operators present after one gauge transformation: the shift operator of $\hat V_j$ has a group-like coproduct $\Delta e^{-ia\hat\partial_n} = e^{-ia\hat\partial_n}\otimes e^{-ia\hat\partial_n}$, applied on arbitrary functions to the right, no new derivative dependence appears. For $\hat V_n$, an arbitrary number of coproducts of the three derivatives $\hat D_n$, $\hat D_j$ and $e^{-ia\hat\partial_n}$ is again expressible by these three operators alone.

6.3 Gauge potentials expanded up to second order

First we examine the vector field component $\hat V_j$. According to (6.27), it acts with a shift operator to the right:

$$\hat V_j\hat\psi = \hat A_j e^{-ia\hat\partial_n}\hat\psi \longrightarrow V_j\star\psi = A_j\star e^{-ia\partial_n}\psi. \qquad (6.32)$$

The gauge potential A_j is the enveloping algebra-valued, physical degree of freedom and terms with derivatives acting to the right must not be counted doubly. In addition, the $\star$-multiplication between A_j and a field on the right is of course not expanded in determining the enveloping algebra gauge potential A_j.

We expand the $\star$-representation of $\hat D_j$(4.43), with the replacement $a\to\hbar a$:

$$\hat D_j\hat\Lambda_\alpha \longrightarrow D_j^*\Lambda_\alpha = \partial_j\frac{e^{-i\hbar a\partial_n}-1}{-i\hbar a\partial_n}\Lambda_\alpha. \qquad (6.33)$$

The enveloping algebra gauge potential which in zeroth order is the Lie algebra gauge potential A_μ^0

$$A_j = A_j^0 + \hbar A_j^1 + \hbar^2 A_j^2 + \dots\ , \qquad (6.34)$$

is constructed as a solution of the following equation:

$$\delta_\alpha A_j = \partial_j \frac{e^{-i\hbar a\partial_n} - 1}{-ia\partial_n}\Lambda_\alpha - iA_j \star e^{-i\hbar a\partial_n}\Lambda_\alpha + i\Lambda_\alpha \star A_j. \tag{6.35}$$

We solve this equation order by order in $\hbar$ to determine the dependence of V_j on A^0_μ, in first order in $\hbar$:

$$\delta_\alpha A^1_j = \partial_j \Lambda^1_\alpha - \frac{ia}{2}\partial_n\partial_j\alpha - aA^0_j\partial_n\alpha - i[A^0_j, \Lambda^1_\alpha] + \frac{1}{2}x^\lambda C^{\mu\nu}_\lambda\{\partial_\mu A^0_j, \partial_\nu\alpha\}. \tag{6.36}$$

Already in first order, there are terms different from the canonical case (compare section 3.3):

$$\begin{aligned} A^1_j &= -\frac{ia}{2}\partial_n A^0_j - \frac{a}{4}\{A^0_n, A^0_j\} + \frac{1}{4}x^\lambda C^{\mu\nu}_\lambda\Big(\{F^0_{\mu j}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_j\}\Big) \\ &= -\frac{ia}{2}\partial_n A^0_j - \frac{a}{4}\{A^0_n, A^0_j\} + A^{1\theta}_j. \end{aligned} \tag{6.37}$$

$A^{1\theta}_j$ again is a shorthand for the part of A^1_j analogous to the canonical case, replacing $\theta^{\mu\nu}$ with $x^\lambda C^{\mu\nu}_\lambda$ (3.70). Also recall that $C^{\mu\nu}_\lambda$ is proportional to a, all terms in A^1_j are a-linear. For A^2_j, we have to solve the defining equation:

$$\begin{aligned} \delta_\alpha A^2_j =& -\frac{a^2}{6}\partial_j\partial^2_n\alpha - \frac{ia}{2}\partial_j\partial_n\Lambda^1_\alpha + \partial_j\Lambda^2_\alpha + \frac{ia^2}{2}A^0_j\partial^2_n\alpha - aA^1_j\partial_n\alpha - aA^0_j\partial_n\Lambda^1_\alpha - aA^0_j \star^1 \partial_n\alpha \\ & -i[A^0_j, \Lambda^2_\alpha] - i[A^0_j \stackrel{\star^1}{,} \Lambda^1_\alpha] - i[A^0_j \stackrel{\star^2}{,} \alpha] - i[A^1_j, \Lambda^1_\alpha] - i[A^1_j \stackrel{\star^1}{,} \alpha] - i[A^2_j, \alpha]. \end{aligned} \tag{6.38}$$

A hermitian solution of (6.38) is:

$$\begin{aligned} A^2_j =& A^{2\theta}_j - \frac{a^2}{6}\partial_n\partial_n A^0_j + \frac{ia^2}{8}\partial_n\{A^0_n, A^0_j\} - \frac{ia^2}{12}[\partial_n A^0_n, A^0_j] + \frac{a^2}{24}\{A^0_n, \{A^0_n, A^0_j\}\} \\ & -\frac{ia}{16}x^\lambda C^{\mu\nu}_\lambda\big(2\partial_n\left(\{F^0_{\mu j}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_j\}\right) - [\partial_n A^0_\nu, F^0_{\mu j}] - [(\partial_n A^0_\nu + \partial_\nu A^0_n), \partial_\mu A^0_j]\big) \\ & +\frac{ia}{12}x^\lambda C^{\mu\nu}_\lambda\left([\partial_n F^0_{\mu j}, A^0_\nu] - [\partial_n A^0_\mu, \partial_\nu A^0_j]\right) \\ & -\frac{a}{24}x^\lambda C^{\mu\nu}_\lambda\left(\{A^0_n, \{F^0_{\mu j}, A^0_\nu\}\} - \{A^0_n, \{A^0_\mu, \partial_\nu A^0_j\},\}\} + \{F^0_{\mu j}, \{A^0_n, A^0_\nu\}\}\right) \\ & -\frac{a}{32}x^\lambda C^{\mu\nu}_\lambda\Big(\{\{F^0_{\mu n}, A^0_\nu\}, A^0_j\} - \{\{A^0_\mu, \partial_\nu A^0_n\}, A^0_j\} + \{\{F^0_{\mu j}, A^0_\nu\}\}, A^0_n\} \\ & \qquad -\{\{A^0_\mu, \partial_\nu A^0_j\}, A^0_n\} + \{\partial_\mu\{A^0_n, A^0_j\}, A^0_\nu\} + 2\{\{F^0_{\mu j}, A^0_n\}, A^0_\nu\}\Big). \end{aligned} \tag{6.39}$$

We conclude that it is possible to expand the gauge potential A_j in the enveloping algebra up to second order. There are no hints that equation (6.35) could not be solved to all orders, although we do not yet have an all-orders scheme available like in the canonical case.

The coproduct of D^*_n in turn has made it necessary to split the gauge potential $\hat{V}_n$ into several parts (6.29). The splitting is only a tool, important for explicit calculations. After determining the distinct parts of $\hat{V}_n$ perturbatively, they have to be joined together into one expression.

Expanding the Dirac derivative $\hat{D}_n \to D_n^*$ in terms of the classical partial derivatives ∂_μ, we see that D_n^* is made up of two different derivative operators, one proportional to $\partial_k\partial_k$, and another a function of ∂_n alone. This motivates a further splitting in comparison with (6.29). We find that the most suitable choice for efficiently calculating V_n is to split it into a sum of five physical gauge potentials $\mathbb{A}\ldots\mathbb{E}$, each with different derivatives acting to the right:

$$\begin{aligned}\hat{V}_n\cdot &= \hat{A}_{n,\mathbb{A}}\hat{\partial}_j\hat{\partial}_j e^{-ia\hat{\partial}_n}\cdot + \hat{A}^j_{n,\mathbb{B}}\hat{\partial}_j e^{-ia\hat{\partial}_n}\cdot + \hat{A}^{jj}_{n,\mathbb{C}} e^{-ia\hat{\partial}_n}\cdot + \hat{A}_{n,\mathbb{D}}\frac{1}{a}\sin(a\hat{\partial}_n)\cdot + \hat{A}_{n,\mathbb{E}}\cos(a\hat{\partial}_n)\cdot \\ V_n\star\cdot &= A_{n,\mathbb{A}}\star\partial_j\partial_j\frac{-2}{a^2\partial_n}(\cos(a\partial_n)-1)\cdot + A^j_{n,\mathbb{B}}\star\partial_j\frac{e^{-ia\partial_n}-1}{-ia\partial_n}\cdot + \\ &\quad + A^{jj}_{n,\mathbb{C}}\star e^{-ia\partial_n}\cdot + A_{n,\mathbb{D}}\star\frac{1}{a}\sin(a\partial_n)\cdot + A_{n,\mathbb{E}}\star\cos(a\partial_n)\cdot\quad .\end{aligned} \tag{6.40}$$

To evaluate an expression of the gauge potential multiplied on the right with a field order by order in $\hbar$, the derivative operators acting to the right need to be expanded as well. For calculating the physical degrees of freedom $A_{n,\mathbb{I}}$ these derivative are not used, neither are contributions included from the $\star$-product which stands to the right in (6.40). There is no double counting: $A^{jj}_{n,\mathbb{C}}$ derives from the part of D_n^* proportional to $\partial_j\partial_j$, $A_{n,\mathbb{D}}$ and $A_{n,\mathbb{E}}$ from the part proportional to $\sin(a\hat{\partial}_n)$. $A^{jj}_{n,\mathbb{C}}$ carries upper index jj indicating the sum over the $n-1$ undeformed directions. Similarly, $A^j_{n,\mathbb{B}}$ has one additional upper index j summed over with the index in ∂_j acting to the right.

The defining equations for $A_{n,\mathbb{I}}$, $\mathbb{I}=\mathbb{A}\ldots\mathbb{E}$ are not independent, since the derivatives in $V_n\star(\Lambda_\alpha\star\cdot)$ act according to non-trivial Leibniz rules. The defining equations for $A_{n,\mathbb{B}}$ and $A_{n,\mathbb{C}}$ depend on $A_{n,\mathbb{A}}$, and $A_{n,\mathbb{D}}$ and $A_{n,\mathbb{E}}$ mutually depend on each other. We obtain:

$$\begin{aligned}\delta_\alpha A_{n,\mathbb{A}} &= \frac{ia}{2}(e^{i\hbar a\partial_n}-1)\Lambda_\alpha - iA_{n,\mathbb{A}}\star e^{i\hbar a\partial_n}\Lambda_\alpha + i\Lambda_\alpha\star A_{n,\mathbb{A}}, \\ \delta_\alpha A^j_{n,\mathbb{B}} &= \partial_j\frac{e^{i\hbar a\partial_n}-1}{\partial_n}\Lambda_\alpha - i[A^j_{n,\mathbb{B}}\stackrel{\star}{,}\Lambda_\alpha] - 2iA_{n,\mathbb{A}}\star\partial_j\frac{e^{i\hbar a\partial_n}-1}{i\hbar a\partial_n}\Lambda_\alpha, \\ \delta_\alpha A^{jj}_{n,\mathbb{C}} &= \frac{i}{\hbar a\partial_n^2}\partial_j\partial_j(1-\cos(\hbar a\partial_n))\Lambda_\alpha - iA^{jj}_{n,\mathbb{C}}\star e^{-ia\partial_n}\Lambda_\alpha + i\Lambda_\alpha\star A^{jj}_{n,\mathbb{C}} \\ &\quad - iA_{n,\mathbb{A}}\star\frac{2\partial_j\partial_j}{\hbar^2a^2\partial_n^2}(1-\cos(\hbar a\partial_n))\Lambda_\alpha - iA^j_{n,\mathbb{B}}\star\partial_j\frac{e^{-i\hbar a\partial_n}-1}{-i\hbar a\partial_n}\Lambda_\alpha, \\ \delta_\alpha A_{n,\mathbb{D}} &= (\cos(\hbar a\partial_n)-1)\Lambda_\alpha - iA_{n,\mathbb{D}}\star\cos(\hbar a\partial_n)\Lambda_\alpha + i\Lambda_\alpha\star A_{n,\mathbb{D}} \\ &\quad + iA_{n,\mathbb{E}}\star a\sin(\hbar a\partial_n)\Lambda_\alpha, \\ \delta_\alpha A_{n,\mathbb{E}} &= \frac{1}{\hbar a}\sin(\hbar a\partial_n)\Lambda_\alpha - iA_{n,\mathbb{E}}\star\cos(\hbar a\partial_n)\Lambda_\alpha + i\Lambda_\alpha\star A_{n,\mathbb{E}} \\ &\quad - iA_{n,\mathbb{D}}\star\frac{1}{a}\sin(\hbar a\partial_n)\Lambda_\alpha.\end{aligned} \tag{6.41}$$

The Lie algebra gauge potential A_n^0 is the zeroth order of $A^0_{n,\mathbb{E}}$, $A^0_{n,\mathbb{I}}=0\ \ \forall\mathbb{I}\neq\mathbb{E}$. This can be seen from the dimensionality of the equations. The expansion

$$A_n = A_n^0 + \hbar(A_{n,\mathbb{A}}^1 + A_{n,\mathbb{B}}^1 + A_{n,\mathbb{C}}^1 + A_{n,\mathbb{D}}^1 + A_{n,\mathbb{E}}^1) + \dots \ , \tag{6.42}$$

can be solved order by order. To first order in $\hbar$ we obtain:

$$\begin{aligned}
\delta_\alpha A_{n,\mathbb{A}}^1 &= -i[A_{n,\mathbb{A}}^1, \alpha], \qquad\qquad \delta_\alpha A_{n,\mathbb{B}}^{1,j} = ia\partial_j\alpha - i[A_{n\mathbb{B}}^{1,j}, \alpha], \\
\delta_\alpha A_{n,\mathbb{C}}^{1,jj} &= \frac{ia}{2}\partial_j\partial_j\alpha - i[A_{n,\mathbb{C}}^{1,jj}, \alpha] - iA_{n,\mathbb{B}}^{1,j}\partial_j\alpha, \qquad \delta_\alpha A_{n,\mathbb{D}}^1 = -i[A_{n,\mathbb{D}}^1, \alpha], \\
\delta_\alpha A_{n,\mathbb{E}}^1 &= \partial_n\Lambda_\alpha^1 - i[A_{n,\mathbb{E}}^1, \alpha] - i[A_n^0, \Lambda_\alpha^1] - i[A_n^0 \stackrel{\star^1}{,} \alpha] - iA_{n,\mathbb{D}}^1\partial_n\alpha \ .
\end{aligned} \tag{6.43}$$

The equations (6.43) have the solutions (with Λ_α as in (6.5)):

$$\begin{aligned}
A_{n,\mathbb{A}}^1 &= 0, \qquad A_{n,\mathbb{B}}^{1,j} = iaA_j^0, \qquad A_{n,\mathbb{C}}^{1,jj} = \frac{ia}{2}\partial_j A_j^0 + \frac{a}{2}A_j^0 A_j^0, \qquad A_{n,\mathbb{D}}^1 = 0, \\
A_{n,\mathbb{E}}^1 &= \frac{1}{4}x^\lambda C_\lambda^{\mu\nu}\Big(\{F_{\mu n}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_n^0\}\Big) = A_n^{1\theta}.
\end{aligned} \tag{6.44}$$

The equations for $A_{n,\mathbb{I}}^2$ are:

$$\begin{aligned}
\delta_\alpha A_{n,\mathbb{A}}^2 &= -\frac{a^2}{2}\partial_n\alpha - i[A_{n,\mathbb{A}}^2, \alpha], \\
\delta_\alpha A_{n,\mathbb{B}}^{2,j} &= -\frac{a^2}{2}\partial_n\partial_j\alpha + ia\partial_j\Lambda_\alpha^1 - i[A_{n\mathbb{B}}^{2,j}, \alpha] - i[A_{n\mathbb{B}}^{1,j}, \Lambda_\alpha^1] - i[A_{n\mathbb{B}}^{1,j} \stackrel{\star^1}{,} \alpha] - 2iA_{n,\mathbb{A}}^2\partial_j\alpha, \\
\delta_\alpha A_{n,\mathbb{C}}^{2,jj} &= \frac{ia}{2}\partial_j\partial_j\Lambda_\alpha^1 - i[A_{n,\mathbb{C}}^{2,jj}, \alpha] - i[A_{n,\mathbb{C}}^{1,jj} \stackrel{\star^1}{,} \alpha] - i[A_{n,\mathbb{C}}^{1,jj}, \Lambda_\alpha^1] - aA_{n,\mathbb{C}}^{1,jj}\partial_n\alpha \\
&\quad -iA_{n,\mathbb{A}}^2\partial_j\partial_j\alpha - \frac{a}{2}A_{n,\mathbb{B}}^{1,j}\partial_j\partial_n\alpha - iA_{n,\mathbb{B}}^{1,j} \star^1 \partial_j\alpha - iA_{n,\mathbb{B}}^{1,j}\partial_j\Lambda_\alpha^1 - iA_{n,\mathbb{B}}^{2,j}\partial_j\alpha, \\
\delta_\alpha A_{n,\mathbb{D}}^2 &= -\frac{a^2}{2}\partial_n\partial_n\alpha - i[A_{n,\mathbb{D}}^2, \alpha] + ia^2 A_n^0\partial_n\alpha, \\
\delta_\alpha A_{n,\mathbb{E}}^2 &= \partial_n\Lambda_\alpha^2 - \frac{a^2}{6}\partial_n\partial_n\partial_n\alpha - i[A_{n,\mathbb{E}}^2, \alpha] - i[A_{n,\mathbb{E}}^1, \Lambda_\alpha^1] - i[A_n^0, \Lambda_\alpha^2] \\
&\quad -i[A_n^0 \stackrel{\star^1}{,} \Lambda_\alpha^1] - i[A_{n,\mathbb{E}}^1 \stackrel{\star^1}{,} \alpha] - i[A_n^0 \stackrel{\star^2}{,} \alpha] + \frac{ia^2}{2}A_n^0\partial_n\partial_n\alpha - iA_{n,\mathbb{D}}^2\partial_n\alpha.
\end{aligned} \tag{6.45}$$

Using Λ^2_α as in (6.9), we obtain the following result for $A^2_{n,\mathbb{I}}$:

$$
\begin{aligned}
A^2_{n,\mathbb{A}} &= -\frac{a^2}{2}A^0_n, \\
A^{2,j}_{n,\mathbb{B}} &= -\frac{a^2}{2}\partial_j A^0_n + \frac{ia^2}{4}\{A^0_j, A^0_n\} + \frac{ia}{4}x^\lambda C^{\mu\nu}_\lambda\Big(\{F^0_{\mu j}, A^0_\nu\} - \{A^0_\mu, \partial_\nu A^0_j\}\Big), \\
A^{2,jj}_{n,\mathbb{C}} &= \frac{ia^2}{4}C^{\mu\nu}_j[A^0_\mu, \partial_j A^0_\nu] + \frac{a^2}{4}A^0_j[A^0_n, A^0_j] + \frac{a^2}{8}x^\lambda C^{\mu\nu}_\lambda\Big(\{\{F^0_{\mu j}, A^0_j\}A^0_\nu\} + \{\{\partial_\mu A^0_j, A^0_j\}A^0_\nu\}\Big), \\
&\quad + \frac{ia^2}{8}x^\lambda C^{\mu\nu}_\lambda\Big(\{\partial_j F^0_{\mu j}, A^0_\nu\} - \{A^0_\mu, \partial_\nu\partial_j A^0_j\} + 2F^0_{\mu j}\partial_j A^0_\nu + 2\partial_\mu A^0_j \partial_j A^0_\nu\Big) \\
A^2_{n,\mathbb{D}} &= -\frac{a^2}{2}\partial_n A^0_n + \frac{ia^2}{2}A^0_n A^0_n, \qquad (6.46) \\
A^2_{n,\mathbb{E}} &= -\frac{a^2}{6}\partial_n\partial_n A^0_n + \frac{ia^2}{6}\partial_n A^0_n A^0_n + \frac{ia^2}{3}A^0_n\partial_n A^0_n + \frac{a^2}{6}A^0_n A^0_n A^0_n \\
&\quad + \frac{1}{24}x^\lambda C^{\rho\sigma}_\lambda C^{\mu\nu}_\rho\Big(-2i[\partial_\sigma F^0_{\mu n}, A^0_\nu] + 2i[\partial_\sigma A^0_\mu, \partial_\nu A^0_n] + \{A^0_\sigma, \{F^0_{\mu n}, A^0_\nu\}\} \\
&\qquad - \{A^0_\sigma, \{A^0_\mu, \partial_\nu A^0_n\}\} + \{F^0_{\mu n}, \{A^0_\nu, A^0_\sigma\}\}\Big) + A^{2,\theta}_n.
\end{aligned}
$$

With these solutions for V_n and V_j to first and second order we calculate the field strengths $\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu, \mathcal{D}_\nu]$, they are derivative-valued again. We consider separately the two cases $\mathcal{F}_{ij}$ and $\mathcal{F}_{nj}$, of course $\mathcal{F}_{nn} \equiv 0$. The separation of the derivatives

$$\mathcal{F}_{ij} \star \cdot = F_{ij} \star e^{-2ia\partial_n} \cdot \quad . \qquad (6.47)$$

leads to the physical field strength:

$$F_{ij} = D^*_i A_j - D^*_j A_i - iA_i \star e^{-ia\partial_n} A_j + iA_j \star e^{-ia\partial_n} A_i. \qquad (6.48)$$

We have explicitly checked that this derivative-valued field strength $\mathcal{F}_{ij}$ (6.47) fulfils the Bianchi identity $\mathcal{D}_k\mathcal{F}_{ij}$+(cycl.)= 0, acting with the covariant derivative on (6.48)[3].

The first order in $\hbar$ is obtained inserting the solution for A_i:

$$
\begin{aligned}
F^1_{ij} &= x^\lambda C^{\mu\nu}_\lambda\Big(\frac{1}{2}\{F^0_{\mu i}, F^0_{\nu j}\} + \frac{1}{4}\{\mathcal{D}^0_\mu F^0_{ij}, A^0_\nu\} - \frac{1}{4}\{A^0_\mu, \partial_\nu F^0_{ij}\}\Big) \\
&\quad - ia\partial_n F^0_{ij} - a\{A^0_n, F^0_{ij}\}. \qquad (6.49)
\end{aligned}
$$

Now we include the derivative action of $\mathcal{F}_{ij}$ to the right, i.e. in first order in $\hbar$ the term $-2iaF^0_{ij}\partial_n$. The $\star$-product which is standing to the right of $\mathcal{F}_{ij}$ is not expanded, of course. We can combine the derivative-valued term with the terms in (6.49) into

$$\mathcal{F}_{ij} = F^{1\theta}_{ij} - 2iaF^0_{ij}\mathcal{D}^0_n - ia\mathcal{D}^0_n F^0_{ij}. \qquad (6.50)$$

We see that we can successfully rewrite all derivatives acting to the right into covariant derivatives. The terms without covariant derivatives to the right are the sum of a term analogous to the case $\theta^{\mu\nu} = const$ and a fully covariant term.

[3] We have not checked explicitly the Bianchi identity with n as part of the index structure.

In general relativity, the commutator of two covariant derivatives is the curvature plus a term with a covariant derivative acting to the right, the torsion. General relativity uses only the torsion-less Levi-Civita connections, demanding explicitly that torsion vanishes. A similar situation arises in κ-deformed gauge theory. However, there is an infinite number of torsion-like terms (we have only looked at first order in (6.50)). We can impose the same condition as in general relativity and extract from $\mathcal{F}_{ij}$ the curvature term F_{ij}, demanding that all torsion-like terms explicitly vanish. Of course the curvature term F_{ij} is itself enveloping algebra-valued and has contributions from all orders in $\hbar$.

However, the splitting into curvature- and torsion-like terms is not trivial. The curvature and the torsion term (at first order) are hermitian taken together, but not the curvature term without the torsion ($F^{1\theta}_{ij}$ is hermitian by itself):

$$\overline{-2iaF^0_{ij}\mathcal{D}^0_n - ia\mathcal{D}^0_n F^0_{ij}} = 2ia\overleftarrow{\mathcal{D}^0_n}F^0_{ij} + ia\mathcal{D}^0_n F^0_{ij} \overset{\text{part. int.}}{\longrightarrow} -2iaF^0_{ij}\mathcal{D}^0_n - ia\mathcal{D}^0_n F^0_{ij}, \tag{6.51}$$

in contrast to

$$\overline{-ia\mathcal{D}^0_n F^0_{ij}} = ia\mathcal{D}^0_n F^0_{ij}. \tag{6.52}$$

While the splitting of $\mathcal{F}_{ij}$ into curvature- and torsion-terms is a promising ansatz, the full Einstein-Cartan field strength $\mathcal{F}_{ij}$ could be investigated as well. We will not formulate this theory, since at present we are not able to formulate it properly in a Lagrangian setting.

The second order field strength F^2_{ij} has already a quite rich structure. We have not managed to calculate the x-linear terms $F^2_{ij}|_{\mathcal{O}(x^1)}$ so far, since the number of contributions is very large. The x^2-dependent terms are analogous to the canonical case. The terms in F^2_{ij} which do not depend explicitly on x are:

$$\begin{aligned} F^2_{ij}|_{\mathcal{O}(x^0)} &= -\frac{7a^2}{12}\partial_n\partial_n F^0_{ij} + ia^2\partial_n\{A^0_n, F^0_{ij}\} + \frac{ia^2}{6}[A^0_n, \partial_n F^0_{ij}] \\ &\quad -\frac{5ia^2}{12}[\partial_n A^0_n, F^0_{ij}] + \frac{5a^2}{12}\{\{F^0_{ij}, A^0_n\}, A^0_n\} + \frac{a^2}{12}\{F^0_{ij}, \{A^0_n, A^0_n\}\}. \end{aligned} \tag{6.53}$$

We have not managed to calculate the x-linear terms $F^2_{ij}|_{\mathcal{O}(x^1)}$ so far, since the number of contributions is very large.

The terms with derivatives acting to the right hand side are the following

$$\begin{aligned} -2a^2F^0_{ij}\partial_n\partial_n - 2iaF^1_{ij}\partial_n &\equiv -2a^2F^0_{ij}\mathcal{D}^0_n\mathcal{D}^0_n - 2a^2(\mathcal{D}^0_n F^0_{ij})\mathcal{D}^0_n - 2iaF^{1\theta}_{ij}\mathcal{D}^0_n \\ &\quad -2ia^2F^0_{ij}(\partial_n A^0_n) - 2ia^2(\partial_n F^0_{ij})A^0_n - 2a^2A^0_nF^0_{ij}A^0_n + 2aF^{1\theta}_{ij}A^0_n, \end{aligned} \tag{6.54}$$

which we can combine with (6.53):

$$\mathcal{F}^2_{ij}|_{\mathcal{O}(x^0)} = -2a^2F^0_{ij}\mathcal{D}^0_n\mathcal{D}^0_n - 2a^2(\mathcal{D}^0_n F^0_{ij})\mathcal{D}^0_n - \frac{7a^2}{12}\mathcal{D}^0_n\mathcal{D}^0_n F^0_{ij}.$$

There will also be a torsion term linear in explicit x at second order in a.

The field strength involving an n-index $\mathcal{F}_{nj}$

$$\hat{F}_{nj} = i\Big((\hat{D}_n - i\hat{V}_n)(\hat{D}_j - i\hat{V}_j) - (\hat{D}_j - i\hat{V}_j)(\hat{D}_n - i\hat{V}_n)\Big), \tag{6.55}$$

can technically be treated in the best way performing the splitting according to (6.40). Therefore we also introduce field strengths $\mathcal{F}_{nj\mathbb{A}}, \ldots, \mathcal{F}_{nj\mathbb{E}}$, such that

$$\begin{aligned}\mathcal{F}_{nj} &= F_{nj\mathbb{A}} \star D_k^* D_k^* \cdot + F_{nj\mathbb{B}}^k \star D_k^* e^{-ia\partial_n} \cdot + F_{nj\mathbb{C}} \star e^{-2ia\partial_n} \cdot \\ &\quad + F_{nj\mathbb{D}} \star \big(\frac{1}{a}\sin(a\partial_n) e^{-ia\partial_n} \cdot\big) + F_{nj\mathbb{E}} \star \big(\cos(a\partial_n) e^{-ia\partial_n} \cdot\big). \end{aligned} \tag{6.56}$$

The individual components can be calculated from the following equations:

$$\begin{aligned}F_{nj\mathbb{A}} &= -\frac{ia}{2}(e^{ia\partial_n} - 1)A_j - D_j^* A_{n\mathbb{A}} - iA_{n\mathbb{A}} \star e^{ia\partial_n} A_j + iA_j \star e^{-ia\partial_n} A_{n\mathbb{A}}, \\ F_{nj\mathbb{B}}^k &= -ia(D_k^* A_j) - (D_j^* A_{n\mathbb{B}}^k) - 2iA_{n\mathbb{A}} \star (D_k^* e^{ia\partial_n} A_j) - iA_{n\mathbb{B}}^k \star A_j + iA_j \star e^{-ia\partial_n} A_{n\mathbb{B}}^k, \\ F_{nj\mathbb{C}}^{kk} &= -\frac{ia}{2} D_k^* D_k^* e^{ia\partial_n} A_j - (D_j^* A_{n\mathbb{C}}^{kk}) - iA_{n\mathbb{A}} \star (D_k^* D_k^* e^{ia\partial_n} A_j) - iA_{n\mathbb{B}}^k \star (D_k^* A_j) \\ &\quad -i(A_{n\mathbb{C}}^{kk} \star e^{-ia\partial_n} A_j) + i(A_j \star e^{-ia\partial_n} A_{n\mathbb{C}}^{kk}), \\ F_{nj\mathbb{D}} &= (\cos(a\partial_n) - 1)A_j - D_j^* A_{n\mathbb{D}} - iA_{n\mathbb{D}} \star \cos(a\partial_n) A_j \\ &\quad -iaA_{n\mathbb{E}} \star \sin(a\partial_n) A_j + iA_j \star e^{-ia\partial_n} A_{n\mathbb{D}}, \\ F_{nj\mathbb{E}} &= \frac{1}{a}\sin(a\partial_n) A_j - D_j^* A_{n\mathbb{E}} - iA_{n\mathbb{D}} \star \frac{1}{a}\sin(a\partial_n) A_j \\ &\quad -iA_{n\mathbb{E}} \star \cos(a\partial_n) A_j + iA_j \star e^{-ia\partial_n} A_{n\mathbb{E}}, \end{aligned} \tag{6.57}$$

The result is to first order in $\hbar$:

$$\begin{aligned}F_{nj\mathbb{A}}|_{\mathcal{O}(a)} &= 0, \qquad\qquad F_{nj\mathbb{B}}^k|_{\mathcal{O}(a)} = -iaF_{kj}^0, \\ F_{nj\mathbb{C}}^{kk}|_{\mathcal{O}(a)} &= -\frac{ia}{2}\partial_k F_{kj}^0 - \frac{a}{2}\{A_k^0, F_{kj}^0\}, \qquad F_{nj\mathbb{D}}|_{\mathcal{O}(a)} = 0, \\ F_{nj\mathbb{E}}|_{\mathcal{O}(a)} &= -\frac{ia}{2}\partial_n F_{nj}^0 - \frac{a}{2}\{A_n^0, F_{nj}^0\} \\ &\quad + x^\lambda C_\lambda^{\mu\nu}\Big(\frac{1}{2}\{F_{\mu n}^0, F_{\nu j}^0\} + \frac{1}{4}\{\mathcal{D}_\mu^0 F_{nj}^0, A_\nu^0\} - \frac{1}{4}\{A_\mu^0, \partial_\nu F_{nj}^0\}\Big). \end{aligned} \tag{6.58}$$

Finally, we combine the different parts of $F_{nj\mathbb{I}}$, $\mathbb{I} = \mathbb{A}, \ldots \mathbb{E}$ into one expression. Adding the terms with derivatives on the right hand side and linear in $\hbar$ we obtain the following expression:

$$\begin{aligned}\mathcal{F}_{nj}^1 &= F_{nj}^{1\theta} - \frac{ia}{2}\partial_n F_{nj}^0 + \frac{ia}{2}\partial_k F_{kj}^0 - \frac{a}{2}\{A_n^0, F_{jn}^0\} + \frac{a}{2}\{A_k^0, F_{kj}^0\} \\ &\quad -iaF_{nj}^0 \partial_n + iaF_{kj}^0 \partial_k \\ &= F_{nj}^{1\theta} - iaF_{nj}^0 \mathcal{D}_n - \frac{ia}{2}\mathcal{D}_n F_{nj}^0 + iaF_{kj}^0 \mathcal{D}_k + \frac{ia}{2}\mathcal{D}_k F_{kj}^0. \end{aligned} \tag{6.59}$$

Again there are curvature terms and torsion terms. Following the approach mentioned above, we ignore the torsion terms.

The second order terms for $\mathcal{F}_{nj}^2$ have not been calculated yet because of the large number of contributions.

6.4 Gauging other derivatives

In the previous section we have gauged the Dirac derivative $\hat{D}_\mu$. For physical applications, this is the right choice of derivative operator because of its transformation property under $SO_a(n)$ rotations. However, studying enveloping algebra-valued gauge potentials corresponding to derivatives $\hat{\partial}_\mu$, we may learn also more about the structure of the Dirac gauge potential, since the two derivatives are related by a change of basis (4.46). This study is the content of this section. Since the ansatz is identical to the previous section, we concentrate on the results. The gauge potential corresponding to $\hat{\partial}_\mu$ will be called ν_μ, the physical gauge potential without derivatives acting to the right $\mathcal{A}_\mu$.

The derivative $\hat{\partial}_n$ is the simplest derivative which can be defined on the κ-deformed spacetime, because it has an undeformed Leibniz rule and its $\star$-representation ∂_n^* is just the commutative ∂_n. Gauging ∂_n by expanding the enveloping algebra-valued gauge potential ν_n, we therefore expect a particularly simple expression. In particular, ν_n will be different from a gauge potential for a derivative in the case $\theta^{\mu\nu} = \text{const}$ only w.r.t. the higher orders in the BCH $\star$-product. Because of the undeformed Leibniz rule $\nu_n = \mathcal{A}_n$ and

$$\delta_\alpha \mathcal{A}_n = \partial_n \Lambda_\alpha - i[\mathcal{A}_n \stackrel{\star}{,} \Lambda_\alpha], \tag{6.60}$$

with the following first and second order solutions:

$$\begin{aligned}
\mathcal{A}_n^1 &= \frac{1}{4} x^\lambda C_\lambda^{\mu\nu}\big(\{F_{\mu n}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_n^0\}\big) = A_n^{1\theta}, \\
\mathcal{A}_n^2 &= A_n^{2\theta} - \frac{i}{12} x^\lambda C_\lambda^{\rho\sigma} C_\rho^{\mu\nu}\big([\partial_\sigma F_{\mu n}^0, A_\nu^0] - [\partial_\sigma A_\mu^0, \partial_\nu A_n^0]\big) \\
&\quad + \frac{1}{24} x^\lambda C_\lambda^{\rho\sigma} C_\rho^{\mu\nu}\big(\{A_\sigma^0, \{F_{\mu n}^0, A_\nu^0\}\} + \{F_{\mu n}^0, \{A_\sigma^0, A_\nu^0\}\} - \{A_\sigma^0, \{A_\mu^0, \partial_\nu A_n^0\}\}\big).
\end{aligned} \tag{6.61}$$

In first order, $\mathcal{A}_n$ coincides with the gauge potential found in the canonical case. There are new terms in second order, linear in the explicit x-dependence. The solution (6.61) is a generic result for an arbitrary Lie algebra space, provided that an exterior derivative with undeformed Leibniz rule can be defined on it.

The gauge potential ν_j corresponding to $\partial_j^* = \partial_j \frac{e^{ia\partial_n} - 1}{ia\partial_n}$ has to be split into two components (cp. the discussion of D_n^*), because of its nontrivial Leibniz rule:

$$\nu_j \star \cdot = \mathcal{A}_{j1} \star \cdot + \mathcal{A}_{(j)2} \star \partial_j^* \cdot \quad . \tag{6.62}$$

This splitting leads to the two equations

$$\delta_\alpha \mathcal{A}_{j1} = \partial_j^* \Lambda_\alpha - i[\mathcal{A}_{j1} \stackrel{\star}{,} \Lambda_\alpha] - i\mathcal{A}_{(j)2} \star \partial_j^* \Lambda_\alpha \tag{6.63}$$

$$\delta_\alpha \mathcal{A}_{(j)2} = (e^{ia\partial_n} - 1)\Lambda_\alpha - i\mathcal{A}_{(j)2} \star e^{ia\partial_n} \Lambda_\alpha + i\Lambda_\alpha \star \mathcal{A}_{(j)2} \,. \tag{6.64}$$

We solve these two defining equations separately. $\mathcal{A}_{(j)2}$ does not depend on the index j at all, it has the following first, second and third order solution

$$
\begin{aligned}
\mathcal{A}^1_{(j)2} &= iaA^0_n, \qquad\qquad \mathcal{A}^2_{(j)2} = -\frac{a^2}{2}\partial_n A^0_n + \frac{ia^2}{2}A^0_n A^0_n + ia\mathcal{A}^1_n, \\
\mathcal{A}^3_{(j)2} &= -\frac{ia^3}{6}\partial_n\partial_n A^0_n - \frac{a^3}{6}\partial_n A^0_n A^0_n - \frac{a^3}{3}A^0_n\partial_n A^0_n + \frac{ia^3}{6}A^0_n A^0_n A^0_n \\
&\quad -\frac{a^2}{8}x^\lambda C^{\mu\nu}_\lambda\Big(\{\partial_n F^0_{\mu n}, A^0_\nu\} - \{A^0_\mu, \partial_n\partial_\nu A^0_n\} - 2F^0_{\mu n}\partial_n A^0_\nu - 2\partial_\mu A^0_n\partial_n A^0_\nu\Big) \\
&\quad +\frac{ia^2}{8}x^\lambda C^{\mu\nu}_\lambda\Big(\{A^0_\nu, \{A^0_n, F^0_{\mu n}\}\} + \{A^0_\nu, \{\partial_\mu A^0_n, A^0_n\}\}\Big) + ia\mathcal{A}^2_n \\
&= -\frac{ia^3}{6}\partial_n\partial_n A^0_n - \frac{a^3}{6}\partial_n A^0_n A^0_n - \frac{a^3}{3}A^0_n\partial_n A^0_n + \frac{ia^3}{6}A^0_n A^0_n A^0_n \\
&\quad +\frac{ia^2}{2}\Big(\mathcal{A}^1_n A^0_n + A^0_n\mathcal{A}^1_n + A^0_n \star^1 A^0_n\Big) - \frac{a^2}{2}\partial_n\mathcal{A}^1_n + ia\mathcal{A}^2_n \ .
\end{aligned}
\tag{6.65}
$$

The second component of ν_j, $\mathcal{A}_{j1}$, has been calculated up to second order:

$$
\begin{aligned}
\mathcal{A}^1_{j1} &= \frac{ia}{2}\partial_j A^0_n + \frac{a}{4}\{A^0_n, A^0_j\} + A^{1\theta}_j, \\
\mathcal{A}^2_{j1} &= -\frac{a^2}{6}\partial_n\partial_j A^0_n + \frac{ia^2}{8}\partial_n\{A^0_n, A^0_j\} + \frac{ia^2}{12}[\partial_n A^0_j, A^0_n] \\
&\quad -\frac{ia^2}{6}[\partial_j A^0_n, A^0_n] + \frac{3a^2}{24}A^0_n A^0_n A^0_j + \frac{a^2}{24}A^0_j A^0_n A^0_n \\
&\quad +x^\lambda C^{\mu\nu}_\lambda\Big(\frac{ia}{12}[\partial_n F^0_{\mu j}, A^0_\nu] - \frac{ia}{12}[\partial_\mu A^0_j, \partial_n A^0_\nu] + \frac{ia}{8}\{\partial_j F^0_{\mu n}, A^0_\nu\} - \frac{ia}{8}\{A^0_\mu, \partial_\nu\partial_j A^0_n\} \\
&\quad +\frac{3ia}{16}F^0_{\mu n}\partial_j A^0_\nu + \frac{3ia}{16}\partial_\mu A^0_n\partial_j A^0_\nu + \frac{ia}{16}\partial_\mu A^0_n\partial_\nu A^0_j + \frac{ia}{16}F^0_{\mu j}\partial_n A^0_\nu + \frac{ia}{16}\partial_j A^0_\mu F^0_{n\nu} \\
&\quad +\frac{a}{32}\{\{F^0_{\mu n}, A^0_\nu\}, A^0_j\} + \frac{a}{16}\{\{F^0_{\mu n}, A^0_j\}, A^0_\nu\} + \frac{a}{32}\{[A^0_\nu, A^0_j], \partial_\mu A^0_n\} \\
&\quad +\frac{a}{16}\{\{\partial_\mu A^0_n, A^0_j\}, A^0_\nu\} - \frac{3a}{32}\{\{F^0_{\mu j}, A^0_\nu\}, A^0_n\} + \frac{a}{16}F^0_{\mu j}[A^0_n, A^0_\nu] \\
&\quad +\frac{a}{32}\{\{\partial_\mu A^0_j, A^0_n\}, A^0_\nu\} + \frac{a}{32}\{\{\partial_\mu A^0_j, A^0_\nu\}, A^0_n\} \\
&\quad -\frac{a}{24}\{\{\partial_j A^0_\mu, A^0_\nu\}, A^0_n\} - \frac{a}{24}[[F^0_{\mu j}, A^0_n], A^0_\nu] + \frac{ia}{24}\{\{[A^0_j, A^0_\mu], A^0_\nu\}, A^0_n\}\Big) + A^{2\theta}_j \ .
\end{aligned}
\tag{6.66}
$$

The commutative derivative ∂_n is a very suitable basis for the gauge theory because of the undeformed Leibniz rule. It also does not act on the explicit x-dependence of expanded solutions, $x^\lambda C^{\mu\nu}_\lambda \not\sim x^n$, it acts only on the Lie algebra fields A^0_μ. Therefore we may immediately conclude that products of covariant derivatives $\mathcal{D}_n = \partial_n - i\mathcal{A}_n$ are covariant under gauge transformations as well:

$$
\mathcal{D}_n \star \mathcal{D}_n = (\partial_n - i\mathcal{A}_n)\star(\partial_n - i\mathcal{A}_n) = \partial_n^2 - i\partial_n\mathcal{A}_n - 2i\mathcal{A}_n\partial_n - \mathcal{A}_n \star \mathcal{A}_n. \tag{6.67}
$$

The gauge potential $\mathcal{A}_n$ is only sensitive to the noncommutativity due to the $\star$-product. This motivates an attempt to rewrite the complicated expansions of enveloping algebra-valued gauge potentials derived in this and the previous section in terms of $\star$-multiplied functions of $\mathcal{D}_n$.

So far this strategy is quite speculative, we have not been able to derive results beyond second non-trivial order. Future research has to show whether this programme can be put into practice. However, it is clear that we may formally covariantise derivative operators like the shift operator $e^{\pm ia\partial_n} \to e_\star^{\pm ia\mathcal{D}_n}$. This implies a covariant shift operator:

$$\delta_\alpha\big(e_\star^{\pm ia\mathcal{D}_n} \star \psi\big) = i\Lambda_\alpha \star \big(e_\star^{\pm ia\mathcal{D}_n} \star \psi\big). \tag{6.68}$$

Expanding the exponential, we may check that this identity holds order by order. Similarly, we may gauge other derivative operators which only involve ∂_n. In section 5.3 the operator $K = \big(\frac{-ia\partial_n}{e^{-ia\partial_n}-1}\big)^{n-1}$ has been defined, which is the result of extracting $\mu(x)$ from under the $\star$-product. We replace ∂_n by $\mathcal{D}_n$ and see that K can be gauge covariantised:

$$\delta_\alpha\Big(\big(\frac{-ia\mathcal{D}_n}{e_\star^{-ia\mathcal{D}_n}-1}\big)^{n-1} \star \psi\Big) = i\Lambda_\alpha \star \big(\frac{-ia\mathcal{D}_n}{e_\star^{-ia\mathcal{D}_n}-1}\big)^{n-1} \star \psi.. \tag{6.69}$$

We may ask whether this gauge covariantised operator is the solution for the problem of formulating a simultaneously gauge- and symmetry-covariant integral. While we believe that (6.69) indeed solves the problem of the missing gauge covariance of K, it seems that we are losing at the same time the essential property of K. We introduced it such that commuting N^{*l} through K results in the antipode $S(N^{*l})$. We are not aware, how to accommodate this covariantised factor (6.69) without having to cope with an infinite number of $\star$-multiplied gauge potentials. Of course we could always limit ourselves to discuss only expansions in a, but still the underlying structure should be rigid.

Another important structure arises, if we compare (6.65) with (6.61). Since (6.64) is an exponentiated version of (6.60) involving only derivatives ∂_n, it is reasonable to expect that (6.65) can be formulated as an exponentiated version of $\mathcal{A}_n$ (6.61) as well. However, this exponentiation involves both the physical gauge potential and the derivatives. The derivatives must not act to the right. We may multiply the inverse shift operator from the right to subtract the freely acting derivatives

$$A_{(j)2} \star \cdot = i\big(e_\star^{ia(\partial_n - i\mathcal{A}_n)} \star e^{-ia\partial_n} - 1\big)\cdot \ , \tag{6.70}$$

where $\mathcal{A}_n$ is the gauge potential corresponding to ∂_n. Note that an expression similar to (6.70) appeared in section 3.1 as the gauge potential for covariant coordinates. This object has been derived in (3.29) as the quantisation of the flow in the Poisson manifold which turns an Abelian gauge potential into a non-Abelian one. We expand (6.70) to verify its structure:

$$\begin{aligned} e_\star^{ia(\partial_n - i\mathcal{A}_n)} e^{-ia\partial_n} &= \Big(1 + ia\partial_n + a\mathcal{A}_n - \frac{a^2}{2}(\partial_n - i\mathcal{A}_n)_\star^2 + \frac{(ia)^3}{6}(\partial_n - i\mathcal{A}_n)_\star^3 + \dots\Big)\cdot \\ &\quad \cdot(1 - ia\partial_n - \frac{a^2}{2}\partial_n^2 + \frac{ia^3}{6}\partial_n^3 + \dots) \\ &= 1 + a\mathcal{A}_n + \frac{a^2}{2}\mathcal{A}_n \star \mathcal{A}_n + \frac{ia^2}{2}(\partial_n\mathcal{A}_n) + \frac{a^3}{6}\mathcal{A}_n \star \mathcal{A}_n \star \mathcal{A}_n \\ &\quad + \frac{ia^3}{3}\mathcal{A}_n \star (\partial_n\mathcal{A}_n) + \frac{ia^3}{6}(\partial_n\mathcal{A}_n) \star \mathcal{A}_n - \frac{a^3}{6}(\partial_n\partial_n\mathcal{A}_n) \dots \ . \end{aligned} \tag{6.71}$$

We see that up to third order, the terms of (6.71) coincide with those of (6.65).

The combined gauge potential ν_j is $\nu_j \star \cdot = \nu_{j1} \star \cdot + \nu_{(j)2} \star \partial_j^* \cdot$. Using the explicit expressions (6.70) and (6.66) and adding ∂_j^* we obtain the covariant derivative

$$\mathcal{D}_j \cdot = \partial_j^* \cdot - i\nu_j \star \cdot = e_\star^{ia(\partial_n - i\mathcal{A}_n)} \star \partial_j^* e^{-ia\partial_n} \cdot - i\nu_{j1} \star \cdot \tag{6.72}$$

The formula (6.72) seems to be a good starting point for generalising the transformation behaviour of derivatives ∂_μ^* (4.33) to the transformation behaviour of covariant derivatives $\mathcal{D}_\mu$. The simplest relation to check would be the generalisation of the transformation

$$[N^{*l}, \partial_n^*] = \partial_l^*, \qquad \overset{?}{\Rightarrow} \qquad [N^{*l}, \mathcal{D}_n] \overset{?}{=} \mathcal{D}_l. \tag{6.73}$$

Note that we have already discussed in section 4.7 the transformation behaviour of vector fields. But there we assumed that vector fields appear linearly in the transformation formulae. Presupposing that formula (6.72) is not only a pathological coincidence, we would have to analyse a transformation behaviour, where the vector fields $\mathcal{A}_\mu$ appear non-linearly on the right hand side. This question is still under research.

Functions of the NC gauge potential $\mathcal{A}_n$ appear also for the covariant coordinates of the κ-deformed space, cp. section 3.1. The gauge potentials A_x^μ are calculated from

$$\delta_\alpha A_x^\mu = -i[x^\mu \overset{\star}{,} \Lambda_\alpha] - i[A_x^\mu \overset{\star}{,} \Lambda_\alpha]. \tag{6.74}$$

For x-dependent $\star$-products it is more difficult to calculate the potential, $A_x^\mu \neq x^\lambda C_\lambda^{\mu\nu} A_\nu$. Although in the κ-deformed case $[x^j \overset{\star}{,} \Lambda_\alpha] = -iax^j\partial_n\Lambda_\alpha$ and $[x^n \overset{\star}{,} \Lambda_\alpha] = -iax^k\partial_k\Lambda_\alpha$, the explicit x-dependence carries over to the potential and $[x^\nu A_{x,\nu}^\mu \overset{\star}{,} \Lambda_\alpha] \neq x^\nu [A_{x,\nu}^\mu \overset{\star}{,} \Lambda_\alpha]$. This is a similar effect like the one for covariant *functions* for canonical NC space. Up to third order we find:

$$\begin{aligned} X^j = x^j + A_x^j &= x^j\big(1 - aA_n^0 - a\mathcal{A}_n^1 + \frac{a^2}{2}A_n^0A_n^0 - a\mathcal{A}_n^2 + \frac{a^2}{2}(A_n^1A_n^0 + A_n^0\mathcal{A}_n^1 + A_n^0 \star^1 A_n^0) \\ &\quad - \frac{a^3}{6}A_n^0A_n^0A_n^0 + \frac{ia^3}{12}[\partial_nA_n^0, A_n^0] + \mathcal{O}(a^4)\big) \ , \end{aligned} \tag{6.75}$$

$$X^n = x^n + A_x^n = x^n - iax^kA_k^0 - iax^kA_k^{1\theta} + \frac{ia^2}{4}x^k\{A_k^0, A_n^0\} + \mathcal{O}(a^3) \ . \tag{6.76}$$

The covariant coordinate X^j has an interesting structure. The first three orders seem to indicate the following expansion:

$$X^j = x^j e_\star^{-a\mathcal{A}_n} + \mathcal{O}(a^3). \tag{6.77}$$

We have to include higher order terms, since the last term in the third order expansion (6.75), which arises due to the second order BCH $\star$-product, is not covered by the symbolic notation (6.77). The covariant coordinate X^n has a complicated structure already at second order, there is a symmetrisation $\{A_k^0, A_n^0\}$.

As a final application of these speculative considerations we analyse possible relations between covariant derivatives $\mathcal{D}_\mu$ found in this chapter and the covariant derivatives $\mathcal{D}_\mu$ corresponding to the Dirac derivative D^*_μ.

This is necessary not only for aesthetic reasons, but because of assumptions made in section 6.3. We have chosen the Dirac derivative as the basis for the physical generators of translations. However, among the derivatives acting to the right in the definitions of the gauge potentials V_μ, there are also shift operators $e^{-ia\partial_n}$, expressed in terms of ∂_n.

We assumed that the field strength corresponding to the commutator of two Dirac covariant derivatives $\mathcal{F}_{\mu\nu} = i[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu]$ can be split into a curvature-like term and an (infinite) power series of torsion terms. We have shown that this splitting works to first order. But the infinite series of torsion terms should of course be expressed in terms of the covariant Dirac derivative.

While ordinary shift operators $e^{-ia\partial_n}$ can be written in terms of Dirac derivatives according to the formula (4.45)

$$e^{-ia\hat{\partial}_n} = -ia\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu},$$

we have to verify the corresponding statement for covariant derivatives. We have to check that the following identity is true:

$$e_\star^{-ia\mathcal{D}_n} = -ia\mathcal{D}_n + \sqrt{1 - a^2\mathcal{D}_\mu \star \mathcal{D}_\mu}|_\star, \tag{6.78}$$

where $\mathcal{D}_n = \partial_n - i\mathcal{A}_n$ and $\mathcal{D}_\mu = \hat{D}^*_\mu - iV_\mu$. We have checked this identity up to the second (nontrivial) order, i.e. the third order in a of (6.78).

Whether the speculations of this section about the possibility of rewriting formal expressions valid for the κ-deformed space in a gauged way can be put into practice is not clear yet. Certainly it would be an attractive strategy to be able to replace ordinary derivatives by covariant derivatives everywhere (using $\star$-exponentials). In some cases this may be possible, in particular concerning all derivative operators involving only ∂_n. But still this approach is probably very limited, since for all formulae involving other derivatives, we need at least a strategy to coherently symmetrise different vector fields.

6.5 κ-deformed gauge theories and the action

With the results of sections 6.1 and 6.3 we have almost all ingredients available to construct an action functional with the help of the integral defined in section 5, using the measure function $\mu(x)$. The Lagrangians which we would like to consider are the same that have been discussed in section 3.5, but here the derivative is the Dirac derivative:

$$\text{Yang-Mills} \qquad \mathcal{L}_{\text{YM}} = \tilde{c}\,\text{Tr}(F_{\mu\nu} \star F^{\mu\nu}), \tag{6.79}$$

$$\text{Minimally coupled fermions} \qquad \mathcal{L}_{\text{MCF}} = i\overline{\psi} \star \gamma^\mu \mathcal{D}_\mu \psi, \tag{6.80}$$

$$\text{Fermion masses} \qquad \mathcal{L}_{\text{MF}} = m\overline{\psi} \star \psi, \tag{6.81}$$

and the action functionals

$$\mathcal{S}_{\text{YM}} = \tilde{c}\text{Tr}\int \mathrm{d}^n x\, \mu(x)\, (F_{\mu\nu} \star F^{\mu\nu}), \tag{6.82}$$

$$\mathcal{S}_{\text{MCF}} = i\int \mathrm{d}^n x\, \mu(x)\, \overline{\psi} \star \gamma^\mu \mathcal{D}_\mu \psi, \tag{6.83}$$

$$\mathcal{S}_{\text{MF}} = m\int \mathrm{d}^n x\, \mu(x)\, \overline{\psi} \star \psi, \tag{6.84}$$

Still it has to be checked that the integral formalism developed in sections 5.1 and 5.2 is compatible with the gauge theory setting.

First of all, the introduction of the measure function made it necessary to redefine all derivatives $D^*_\mu \to \tilde{D}^*_\mu$ in such a way that they are hermitian w.r.t. $\mu(x)$:

$$\int \mathrm{d}^n x\, \mu(x)\, \overline{\psi}(x) \star i\gamma^\lambda \tilde{D}^*_\mu \psi(x),$$

with $\tilde{D}^*_j = (\partial_j + \rho_j)\frac{e^{-ia\partial_n}-1}{-ia\partial_n}$ and $\tilde{D}^*_n = \frac{1}{a}\sin(a\partial_n) + \frac{i}{a\partial_n^2}(\partial_j+\rho_j)^2(1-\cos(a\partial_n))$. An obvious question is how this redefinition of $\partial_j \to (\partial_j + \rho_j)$ affects the gauge theory that we have just defined. Using the defining equations for ρ_j, $x^k\partial_k\rho_j = -\rho_j$ and $\partial_n\rho_j = 0$ it is immediate to see that

$$\rho_j \frac{e^{ia\partial_n}-1}{ia\partial_n}\big(f(x)\star g(x)\big) = (e^{ia\partial_n} f(x)) \star \big(\rho_j \frac{e^{ia\partial_n}-1}{ia\partial_n} g(x)\big), \tag{6.85}$$

and therefore

$$\tilde{\partial}^*_j\big(f(x)\star g(x)\big) = (\partial^*_j f(x))\star g(x) + (e^{ia\partial_n} f(x)) \star (\tilde{\partial}^*_j(x)). \tag{6.86}$$

The redefinition factor ρ_j can always be brought to the second term in a coproduct, since coordinates x (ρ_j is an x-dependent function) do not have a coproduct. Therefore the factor ρ_j redefines only the derivatives acting to the right in a derivative-valued gauge potential. The gauge potential V_j is not redefined at all, while V_n has redefined derivatives acting to the right in comparison with (3.11):

$$\tilde{V}_n \cdot = A_{n,1} e^{-ia\partial_n} \cdot + A^j_{n,2} \tilde{D}^*_j \cdot + A_{n,3} \tilde{D}^*_n \cdot \quad . \tag{6.87}$$

The field strength is also redefined only through the derivatives acting to the right. Therefore the redefinition $D^*_\mu \to \tilde{D}^*_\mu$ is entirely harmless for the presented setting of gauge theories.

In addition we have to analyse whether the equations of motions of the gauge theory, which are derived from the action, have the proper classical limit. The measure $\mu(x)$ allows us to vary the Lagrangian (w.r.t. $\overline{\psi}$ or A_μ) and to derive the equations of motion of the enveloping algebra-valued fields ψ and A_μ, e.g.

$$\mu(x) i\gamma^\mu \mathcal{D}_\mu \psi = 0 \tag{6.88}$$

The equations of motion are multiplied with the measure $\mu = \mu(x)$. For field theories without gauge degrees of freedom we have chosen a rescaling of the fermionic fields by $\mu^{-\frac{1}{2}}$

that eliminated the measure function from the equations of motion. With the new gauge degrees of freedom, this recipe is not as straightforward. Rescaling $\psi \to (\mu^{-\frac{1}{2}}\psi')$ is not gauge covariant operation, in general $\delta_\alpha\psi' \neq i\Lambda_\alpha\star\psi'$, if ψ is transforming as $\delta_\alpha\psi = i\Lambda_\alpha\star\psi$. The reason is that extracting $\mu^{-\frac{1}{2}}$ from one of the factors of the $\star$-product, new derivatives ∂_n acting on Λ_α and ψ' remain.

How this problem can be circumvented is not clear yet. A possible ansatz uses $\mu(X)$, the gauge-covariantised version of μ, with all ordinary coordinates replaced by covariant coordinates (6.77). One may use that μ is proportional to $(x^j)^{-(n-1)}$ and obtain

$$\mu(X) \sim \mu(e^{(n-1)a\mathcal{A}_n} + \dots). \tag{6.89}$$

With such gauge covariantised measures μ may be eliminated from the equation of motion. The covariant coordinates give a contribution at first order in a only. The commutative limit is rescued.

The equations of motion for the gauge potential cannot be brought to the classical form by rescaling A_μ with $\mu^{-\frac{1}{2}}$. A_μ appears in the minimally coupled Lagrangian and in the Yang-Mills Lagrangian, to the powers two, three and four. The covariantised measure $\mu(X)$ has to be inserted by hand into the Yang-Mills action. How these gauged redefinitions really work in practice has not been understood yet in a satisfactory manner.

Thus, the following results for the fermionic action and the Yang-Mills action have to be used with a caveat: they may be correct only up to additional covariant derivatives $\partial_n - i\mathcal{A}_n$, which may stem from a redefinition of μ through covariant coordinates or from a gauged quantum trace K. In addition, at second order, new couplings might appear for the fermionic kinetic term, if a definition of the Dirac derivative is used which is different than the minimal one (4.52).

Expanded in a, the Lagrangian of minimally coupled massive fermions reads

$$\begin{aligned}\mathcal{L}_{\text{MCF, MCF}}\big|_{\mathcal{O}(a)} &= \frac{i}{2}x^\nu C_\nu^{\rho\sigma}\overline{\mathcal{D}_\rho^0\psi^0}\,\mathcal{D}_\sigma^0(i\gamma^\mu\mathcal{D}_\mu^0 - m)\psi^0 - \frac{i}{2}x^\nu C_\nu^{\rho\sigma}\overline{\psi^0}\gamma^\mu F_{\mu\rho}^0\mathcal{D}_\sigma^0\psi^0 \\ &\quad + \frac{a}{2}\overline{\psi^0}\gamma^j\mathcal{D}_n^0\mathcal{D}_j^0\psi^0 - \frac{a}{2}\overline{\psi^0}\gamma^n\mathcal{D}_j^0\mathcal{D}_j^0\psi^0,\end{aligned} \tag{6.90}$$

while the Yang-Mills Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{YM}}\big|_{\mathcal{O}(a)} &= \frac{i}{2}x^\nu C_\nu^{\rho\sigma}\text{Tr}\Big(\mathcal{D}_\rho^0 F^{0\mu\nu}\mathcal{D}_\sigma^0 F_{\mu\nu}^0 + \frac{i}{2}\{A_\rho^0, (\partial_\sigma + \mathcal{D}_\sigma^0)(F^{0\mu\nu}F_{\mu\nu}^0)\} \\ &\qquad - i\{F^{0\mu\nu}, \{F_{\mu\rho}^0, F_{\nu\sigma}^0\}\}\Big) \\ &\quad - ia\text{Tr}\Big(\mathcal{D}_n^0(F^{0\mu\nu}F_{\mu\nu}^0) - \{\mathcal{D}_\mu^0 F^{0\mu j}, F_{nj}^0\}\Big).\end{aligned} \tag{6.91}$$

Note that the fermion mass Lagrangian is identical to the canonical case under $\theta^{\mu\nu} \to x^\lambda C_\lambda^{\mu\nu}$, while new terms appear for the minimally coupled Lagrangian and the Yang-Mills Lagrangian.

Using the integral, we obtain the action (the Yang-Mills action is multiplied with a constant $\tilde{c}$ indicating the ambivalence of the trace):

$$\mathcal{S}_{\mathrm{MF}}|_{\mathcal{O}(a)} = -\frac{1}{4}\int \mathrm{d}^n x\ \mu(x)\ x^\lambda C_\lambda^{\rho\sigma}\overline{\psi}^0 F^0_{\rho\sigma}\psi^0, \tag{6.92}$$

$$\begin{aligned}\mathcal{S}_{\mathrm{MCF}}|_{\mathcal{O}(a)} &= -\frac{i}{4}\int \mathrm{d}^n x\ \mu(x)\ x^\nu C_\nu^{\rho\sigma}\big(\overline{\psi}^0 F^0_{\rho\sigma}\gamma^\mu\mathcal{D}^0_\mu\psi^0 + 2\overline{\psi^0}\gamma^\mu F^0_{\mu\rho}\mathcal{D}^0_\sigma\psi^0\big)\\ &\quad +\frac{a}{2}\int \mathrm{d}^n x\ \mu(x)\ \big(\overline{\psi^0}\gamma^j\mathcal{D}^0_n\mathcal{D}^0_j\psi^0 - \overline{\psi^0}\gamma^n\mathcal{D}^0_j\mathcal{D}^0_j\psi^0\big),\end{aligned} \tag{6.93}$$

$$\begin{aligned}\mathcal{S}_{\mathrm{YM}}|_{\mathcal{O}(a)} &= \tilde{c}\mathrm{Tr}\int \mathrm{d}^n x\ \mu(x)\ \Big(-\frac{1}{2}F^{0\mu\nu}F^0_{\rho\sigma}F^0_{\mu\nu} + 2F^{0\mu\nu}F^0_{\mu\rho}F^0_{\nu\sigma}\Big)\\ &\quad -iac\mathrm{Tr}\int \mathrm{d}^n x\ \mu(x)\ \Big(\mathcal{D}^0_n(F^{0\mu\nu}F^0_{\mu\nu}) - 2(\mathcal{D}^0_\mu F^{0\mu j}), F^0_{nj}\Big).\end{aligned} \tag{6.94}$$

Concerning the ambiguities of the enveloping algebra construction, we should treat several terms of these actions as unphysical field redefinitions. For example, as discussed in section 3.6, only one of the x-linear terms of the fermionic action survives this ambiguity analysis. In contrast, it seems that the new x-independent terms are not due to ambiguities. Since these terms in (6.93) are contracted with the γ-matrices, these terms cannot be obtained from redefining ψ^0. We have not been able yet to finally decide whether a redefinition of A^0_μ in terms of two covariant derivatives acting to the right is allowed by the enveloping algebra structure of the gauge potentials (cp. the discussion in 3.6), but it seems quite unlikely that this is the case.

Therefore we dare to declare that the new x-independent terms in the self-coupling of gauge bosons and in the coupling of fermions to gauge bosons are true new physical interaction terms. These new interaction terms have quite a different structure than those in the canonical spacetime. Therefore, they may lead to new predictions.

The construction of a Standard Model on κ-deformed along the lines of section 3.8 would be an obvious next step. However, the open questions in properly defining an integral, which is at the same time gauge-invariant and invariant under symmetry transformations, have to be discussed in advance. The reason is that the solution of these questions, e.g. by introducing a covariantised quantum trace or a covariantised measure, could result in additional interaction terms already at first order in a.

Appendix A

The generic κ-deformed space

In this appendix we state generalisations of the most important formulae of the main body of the text to the more general commutation relations of κ-deformed space, where the vector of noncommutativity a^μ is not aligned with the n-th direction:

$$[x^\mu \stackrel{\star}{,} x^\nu] = ia^\mu x^\nu - ia^\nu x^\mu. \tag{A.1}$$

In this appendix, all formulae are valid for arbitrary signature of the spacetime, for non-Euclidean spacetime a^μ can be space-, light- or time-like. For this case the symmetric $\star$-product (2.43) takes the form

$$\begin{aligned} f(x) \star g(x) \ &= \ \lim_{\substack{y\to x\\ z\to x}} \exp\left(x^\mu \partial_{y^\mu} \left(e^{-ia^\lambda \partial_{z^\lambda}} \frac{-ia^\nu \partial_{(y^\nu+z^\nu)}}{e^{-ia^\gamma \partial_{(y^\gamma+z^\gamma)}}-1} \frac{e^{-ia^\sigma \partial_{y^\sigma}}-1}{-ia^\rho \partial_{y^\rho}} - 1\right)\right. \\ &\qquad \left. + x^\mu \partial_{z^\mu} \left(\frac{-ia^\nu \partial_{(y^\nu+z^\nu)}}{e^{-ia^\gamma \partial_{(y^\gamma+z^\gamma)}}-1} \frac{e^{-ia^\sigma \partial_{z^\sigma}}-1}{-ia^\rho \partial_{z^\rho}} - 1\right)\right) f(y) g(z), \end{aligned} \tag{A.2}$$

since the structure constants $C^{\mu\nu}_\lambda = a^\mu \delta^\nu_\lambda - a^\nu \delta^\mu_\lambda$ fulfil

$$C^{\mu_1\nu_1}_\lambda C^{\mu_2\nu_2}_{\mu_1} C^{\mu_3\nu_3}_{\mu_2} \dots C^{\mu_k\nu_k}_{\mu_{k-1}} = (-1)^{k-1} a^{\nu_1} a^{\nu_2} \dots a^{\nu_{k-1}} C^{\mu_k\nu_k}_\lambda. \tag{A.3}$$

We have guessed formula (A.2) from analogy with (2.43). The crucial input for such a formula is that products of the structure constants simplify as in (A.3). We have checked the validity of this guess up to second order.

It is possible to derive from (A.2) closed expressions for the symmetric $\star$-product between an arbitrary function and one coordinate, analogously to (A.3):

$$\begin{aligned} x^\rho \star f(x) \ &= \ \Big(x^\rho \frac{ia^\alpha \partial_\alpha}{e^{ia^\beta \partial_\beta}-1} - a^\rho \frac{x^\alpha \partial_\alpha}{a^\beta \partial_\beta} \big(\frac{ia^\gamma \partial_\gamma}{e^{ia^\delta \partial_\delta}-1} - 1\big)\Big) f(x), \\ f(x) \star x^\rho \ &= \ \Big(x^\rho \frac{-ia^\alpha \partial_\alpha}{e^{-ia^\beta \partial_\beta}-1} - a^\rho \frac{x^\alpha \partial_\alpha}{a^\beta \partial_\beta} \big(\frac{-ia^\gamma \partial_\gamma}{e^{-ia^\delta \partial_\delta}-1} - 1\big)\Big) f(x). \end{aligned} \tag{A.4}$$

These formulae follow from the properties of the BCH formula.

Linear derivatives can also be formulated on the generic κ-deformed space. There are three possibilities for such linear derivatives. In contrast to the case $a^\mu = a\delta^\mu_n$ there is no additional parametric freedom, if a covariant ansatz is made $[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + ic_1 a^\mu \widehat{\partial}_\nu + ic_2 a^\nu \widehat{\partial}_\mu + ic_3 \delta^\nu_\mu a^\lambda \widehat{\partial}_\lambda$:

$$\begin{aligned}
[\hat{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu + ia^\nu \hat{\partial}_\mu, \\
[\check{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu (1 - ia^\lambda \check{\partial}_\lambda), \\
[\tilde{\partial}_\mu, \hat{x}^\nu] &= \delta^\nu_\mu + ia^\nu \tilde{\partial}_\mu + i\eta_{\mu\rho}\eta^{\nu\kappa} a^\rho \tilde{\partial}_\kappa.
\end{aligned} \tag{A.5}$$

While the derivatives $\hat{\partial}_\mu$ and $\check{\partial}_\mu$ correspond to $\rho^{\mu\nu}_\lambda = a^\mu \delta^\nu_\lambda$ and $\rho^{\mu\nu}_\lambda = -a^\nu \delta^\mu_\lambda$ respectively (4.6), $\tilde{\partial}_\mu$ corresponds to $\rho^{\mu\nu}_\lambda = a^\mu \delta^\nu_\lambda + \eta_{\lambda\rho}\eta^{\mu\nu} a^\rho$; because of the symmetry of the metric $\eta^{\mu\nu}$, (4.6) is fulfilled also for $\tilde{\partial}_\mu$. These linear derivatives can be mapped into each other in the following way:

$$\check{\partial}_\mu = \frac{\hat{\partial}_\mu}{1 + ia^\nu \hat{\partial}_\nu}, \qquad \tilde{\partial}_\mu = \hat{\partial}_\mu + \frac{i}{2}\eta_{\mu\rho}\eta^{\kappa\lambda} a^\rho \hat{\partial}_\kappa \hat{\partial}_\lambda. \tag{A.6}$$

The commutation relations (A.5) can be generalised to Leibniz rules:

$$\begin{aligned}
\hat{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\hat{\partial}_\mu \hat{f}) \cdot \hat{g} + ((1 + ia^\nu \hat{\partial}_\nu)\hat{f}) \cdot (\hat{\partial}_\mu \hat{g}), \\
\check{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\check{\partial}_\mu \hat{f}) \cdot ((1 - ia^\nu \check{\partial}_\nu)\hat{g}) + \hat{f} \cdot (\check{\partial}_\mu \hat{g}), \\
\tilde{\partial}_\mu (\hat{f} \cdot \hat{g}) &= (\tilde{\partial}_\mu \hat{f}) \cdot \hat{g} + ((1 + ia^\nu \tilde{\partial}_\nu)\hat{f}) \cdot (\tilde{\partial}_\mu \hat{g}) + i\eta_{\mu\rho} a^\rho \eta^{\kappa\lambda} (\tilde{\partial}_\kappa \hat{f}) \cdot (\tilde{\partial}_\lambda \hat{g}).
\end{aligned} \tag{A.7}$$

These linear derivatives can also be represented in terms of ordinary derivatives ∂_ν on the algebra of functions multiplied with the symmetric $\star$-product (A.2). The linear derivatives $\hat{\partial}_\mu$ corresponding to $\rho^{\mu\nu}_\lambda = a^\mu \delta^\nu_\lambda$ have the $\star$-representation

$$\partial^*_\mu = \partial_\mu \frac{e^{ia^\nu \partial_\nu} - 1}{ia^\lambda \partial_\lambda}, \tag{A.8}$$

leading to the Leibniz rule expressed in terms of ordinary derivatives

$$\partial^*_\mu (f(x) \star g(x)) = \left(\partial_\mu \frac{e^{ia^\nu \partial_\nu} - 1}{ia^\lambda \partial_\lambda} f(x)\right) \star g(x) + \left(e^{ia^\sigma \partial_\sigma} f(x)\right) \star \left(\partial_\mu \frac{e^{ia^\nu \partial_\nu} - 1}{ia^\lambda \partial_\lambda} g(x)\right). \tag{A.9}$$

The linear derivatives $\check{\partial}_\mu$ corresponding to $\rho^{\mu\nu}_\lambda = -a^\nu \delta^\mu_\lambda$ have the $\star$-representation

$$\partial^{*\vee}_\mu = \partial_\mu \frac{e^{-ia^\nu \partial_\nu} - 1}{-ia^\lambda \partial_\lambda}, \tag{A.10}$$

leading to the Leibniz rule expressed in terms of ordinary derivatives

$$\partial^{*\vee}_\mu (f(x) \star g(x)) = \left(\partial_\mu \frac{e^{-ia^\nu \partial_\nu} - 1}{-ia^\lambda \partial_\lambda} f(x)\right) \star \left(e^{-ia^\sigma \partial_\sigma} g(x)\right) + f(x) \star \left(\partial_\mu \frac{e^{-ia^\nu \partial_\nu} - 1}{-ia^\lambda \partial_\lambda} g(x)\right). \tag{A.11}$$

That this is true can be checked (4.113), using (A.4). The derivative $\tilde{\partial}_\mu$ can be expressed in terms of commutative derivatives using (A.6) and (A.8).

In addition, we can define generators of rotations on the generic κ-deformed space via their commutation relations with the coordinates. We obtain the result:

$$\begin{aligned}[\hat{M}^{\rho\sigma}, \hat{x}^\mu] &= \eta^{\sigma\mu}\hat{x}^\rho - \eta^{\rho\mu}\hat{x}^\sigma + ia^\rho\hat{M}^{\mu\sigma} - ia^\sigma\hat{M}^{\mu\rho}, \\ [\hat{M}^{\rho\sigma}, \hat{M}^{\kappa\lambda}] &= \eta^{\rho\lambda}\hat{M}^{\sigma\kappa} + \eta^{\sigma\kappa}\hat{M}^{\rho\lambda} - \eta^{\rho\kappa}\hat{M}^{\sigma\lambda} - \eta^{\sigma\lambda}\hat{M}^{\rho\kappa}. \end{aligned} \tag{A.12}$$

The derivatives $\hat{\partial}_\mu$ are a module of $\hat{M}^{\rho\sigma}$ ($\check{\partial}_\mu$ and $\tilde{\partial}_\mu$ as well) with commutation relations:

$$[\hat{M}^{\rho\sigma}, \hat{\partial}_\mu] = \delta^\sigma_\mu\eta^{\rho\lambda}\hat{\partial}_\lambda - \delta^\rho_\mu\eta^{\sigma\lambda}\hat{\partial}_\lambda + ia^\sigma\hat{\partial}_\mu\eta^{\rho\lambda}\hat{\partial}_\lambda - ia^\rho\hat{\partial}_\mu\eta^{\sigma\lambda}\hat{\partial}_\lambda - \frac{i}{2}(\delta^\rho_\mu a^\sigma - \delta^\sigma_\mu a^\rho)\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda. \tag{A.13}$$

Therefore the coproduct of the generators of rotations is:

$$\hat{M}^{\rho\sigma}(\hat{f}\cdot\hat{g}) = (\hat{M}^{\rho\sigma}\hat{f})\cdot\hat{g} + \hat{f}\cdot(\hat{M}^{\rho\sigma}\hat{g}) + (ia^\rho\hat{\partial}_\lambda\hat{f})\cdot(\hat{M}^{\lambda\sigma}\hat{g}) - (ia^\sigma\hat{\partial}_\lambda\hat{f})\cdot(\hat{M}^{\lambda\rho}\hat{g}). \tag{A.14}$$

The orbital part of the generators of rotations can be expressed in terms of $\hat{x}^\mu$ and $\hat{\partial}_\nu$:

$$\hat{M}^{\rho\sigma}_{\text{orb}} = \hat{x}^\rho\eta^{\sigma\lambda}\hat{\partial}_\lambda - \hat{x}^\sigma\eta^{\rho\lambda}\hat{\partial}_\lambda - \frac{i}{2}(a^\rho\hat{x}^\sigma - a^\sigma\hat{x}^\rho)\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda. \tag{A.15}$$

That this representation of the orbital part fulfils the Leibniz rule, can be seen using the following identification, cp. (A.4):

$$\hat{f}(\hat{x})\hat{x}^\mu = \hat{x}^\mu\hat{f}(\hat{x}) + ia^\lambda\hat{x}^\mu\hat{\partial}_\lambda\hat{f}(\hat{x}) - ia^\mu\hat{x}^\lambda\hat{\partial}_\lambda\hat{f}(\hat{x}). \tag{A.16}$$

The Dirac derivative for the generic κ-deformed space can be derived, using the most general ansatz compatible with the index structure:

$$\hat{D}_\mu = \hat{\partial}_\mu f(\hat{\partial}_\lambda\hat{\partial}_\lambda, a^\kappa\hat{\partial}_\kappa) + a^\mu g(\hat{\partial}_\lambda\hat{\partial}_\lambda, a^\kappa\hat{\partial}_\kappa), \tag{A.17}$$

with the solution:

$$\hat{D}_\mu = \frac{\hat{\partial}_\mu + \frac{i}{2}\eta_{\mu\rho}a^\rho\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda}{1 + ia^\kappa\hat{\partial}_\kappa}, \quad \text{with} \quad [\hat{M}^{\rho\sigma}, \hat{D}_\mu] = \delta^\rho_\mu\hat{D}_\sigma - \delta^\sigma_\mu\hat{D}_\rho. \tag{A.18}$$

Note that although a^μ in (A.18) carries a vector index, it is not rotated, it is a fixed vector. Again the shift operators can be expressed in terms of the Dirac operator alone:

$$\begin{aligned}\frac{1}{1 + ia^\nu\hat{\partial}_\nu} &= 1 - ia^\nu\hat{D}_\nu - \frac{a^\kappa\eta_{\kappa\lambda}a^\lambda}{2}\hat{\Box} = -ia^\nu\hat{D}_\nu + \sqrt{1 - a^\kappa\eta_{\kappa\lambda}a^\lambda\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu}, \qquad \text{(A.19)} \\ 1 + ia^\nu\hat{\partial}_\nu &= \frac{1 + ia^\nu\hat{D}_\nu - \frac{a^\kappa\eta_{\kappa\lambda}a^\lambda}{2}\hat{\Box}}{1 + a^\rho a^\sigma\hat{D}_\rho\hat{D}_\sigma - a^\rho\eta_{\rho\sigma}a^\sigma\hat{D}_\alpha\eta^{\alpha\beta}\hat{D}_\beta} = \frac{ia^\nu\hat{D}_\nu + \sqrt{1 - a^\kappa\eta_{\kappa\lambda}a^\lambda\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu}}{1 + a^\rho a^\sigma\hat{D}_\rho\hat{D}_\sigma - a^\rho\eta_{\rho\sigma}a^\sigma\hat{D}_\alpha\eta^{\alpha\beta}\hat{D}_\beta}.\end{aligned}$$

The commutation relations of the Dirac derivative with a coordinate are:

$$[\hat{D}_\mu, \hat{x}^\nu] = \delta^\nu_\mu\big(- ia^\lambda\hat{D}_\lambda + \sqrt{1 - a^\kappa\eta_{\kappa\lambda}a^\lambda\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu}\big) + ia^\nu\hat{D}_\mu, \tag{A.20}$$

while the coproduct of the Dirac derivative reads

$$\begin{aligned}\hat{D}_\mu(\hat{f}\cdot\hat{g}) &= (\hat{D}_\mu\hat{f})\cdot\frac{1}{1+ia^\nu\hat{\partial}_\nu}\hat{g}+((1+ia^\nu\hat{\partial}_\nu)\hat{f})\cdot(\hat{D}_\mu\hat{g})\\ &\quad +(ia^\mu\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{f})\cdot\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu}\hat{g}-(ia^\lambda\hat{\partial}_\lambda\hat{f})\cdot\frac{\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu}\hat{g}. \end{aligned}\tag{A.21}$$

The Klein-Gordon operator is

$$\hat{\Box}=\frac{\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda}{1+ia^\kappa\hat{\partial}_\kappa},\quad \text{with}\quad [\hat{M}^{\rho\sigma},\hat{\Box}]=0. \tag{A.22}$$

Again the square of the Dirac derivative can be expressed in terms of the Klein-Gordon operator

$$\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu=\hat{\Box}(1-\frac{a^\mu\eta_{\mu\nu}a^\nu}{4}\hat{\Box}),\quad \text{or}\quad \frac{a^\mu\eta_{\mu\nu}a^\nu}{2}\hat{\Box}=1-\sqrt{1-a^\kappa\eta_{\kappa\lambda}a^\lambda\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu}. \tag{A.23}$$

The $\star$-representations of all these operators can be obtained by inserting the expressions (A.4) and (A.8). For example the Laplace operator and the Dirac derivative are

$$\begin{aligned}\Box^* &= \partial_\mu\eta^{\mu\nu}\partial_\nu\,\frac{2(1-\cos(a^\nu\partial_\nu))}{a^\kappa a^\lambda\partial_\kappa\partial_\lambda},\\ D^*_\mu &= \partial_\mu\frac{e^{-ia^\kappa\partial_\kappa}-1}{-ia^\kappa\partial_\kappa}+\frac{ia^\mu}{2}\partial_\alpha\eta^{\alpha\beta}\partial_\beta\,\frac{2(1-\cos(a^\nu\partial_\nu))}{a^\kappa a^\lambda\partial_\kappa\partial_\lambda}.\end{aligned}\tag{A.24}$$

The antipodes of the most important operators are:

$$\begin{aligned}S(\hat{\partial}_\mu) &= -\frac{\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu},\\ S(\hat{D}_\mu) &= -\hat{D}_\mu+\frac{ia^\mu\hat{\partial}_\kappa\eta^{\kappa\lambda}\hat{\partial}_\lambda-ia^\lambda\hat{\partial}_\lambda\hat{\partial}_\mu}{1+ia^\nu\hat{\partial}_\nu},\\ S(\hat{M}^{\rho\sigma}) &= -\hat{M}^{\rho\sigma}+ia^\rho\hat{M}^{\lambda\sigma}\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu}-ia^\sigma\hat{M}^{\lambda\rho}\frac{\hat{\partial}_\lambda}{1+ia^\nu\hat{\partial}_\nu}-i(n-1)\frac{(a^\rho\hat{\partial}_\sigma-a^\sigma\hat{\partial}_\rho)}{1+ia^\nu\hat{\partial}_\nu}.\end{aligned}\tag{A.25}$$

The counits are trivial.

There is also a straightforward generalisation of the commutation relations of the vector-like transforming one-forms $\hat{\xi}^\mu$ with coordinates:

$$[\hat{\xi}^\mu,\hat{x}^\nu]=ia^\mu\hat{\xi}^\nu-ia^\nu\hat{\xi}^\mu+(\hat{\xi}^\mu\hat{D}_\nu+\hat{\xi}^\nu\hat{D}_\mu-\delta^{\mu\nu}\hat{\xi}^\lambda\hat{D}_\lambda)\frac{1-\sqrt{1-a^\kappa\eta_{\kappa\lambda}a^\lambda\hat{D}_\mu\eta^{\mu\nu}\hat{D}_\nu}}{\hat{D}_\alpha\eta^{\alpha\beta}\hat{D}_\beta}. \tag{A.26}$$

All formulae in this appendix correspond to those for $a^\mu=a\delta^\mu_n$, replacing the derivatives $\hat{\partial}^{c=1}_\mu$ with $\hat{\partial}_\mu$.

Bibliography

[1] *Letter of Heisenberg to Peierls* (1930), in: Wolfgang Pauli, Scientific Correspondence, vol. II, 15, Ed. Karl von Meyenn, Springer-Verlag 1985.

[2] H. S. Snyder, *Quantized spacetime*, Phys.Rev. **71**, 38 (1947).

[3] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995).

[4] S. Doplicher, K. Fredenhagen and J. E. Roberts, *The quantum structure of spacetime at the Planck scale and quantum fields*, Commun. Math. Phys. **172**, 187 (1995) [hep-th/0303037].

[5] I. M. Gel'fand and M. A. Naimark, *On the embedding of normed linear rings into the ring of operators in Hilbert space*, Mat. Sbornik. **12**, 197 (1947).

[6] A. Connes, *Noncommutative Geometry*, Academic Press (1994).

[7] H. Hopf, *Über die Topologie der Gruppenmannigfaltigkeiten und ihre Verallgemeinerungen*, Ann. Math. **42**, 22 (1941).

[8] M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10**, 63 (1985).

[9] V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Sov. Math. Dokl. **32**, 254 (1985).

[10] S. L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. **111**, 613 (1987).

[11] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtadzhyan, *Quantisation of Lie groups and Lie algebras*, Leningrad Math. J. **1**, 193 (1990).

[12] B. L. Cerchiai and J. Wess, *q-deformed Minkowski space based on a q-Lorentz algebra*, Eur. Phys. J. C **5**, 553 (1998) [math.qa/9801104].

[13] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, Ann. Phys. **111**, 61 (1978).

[14] D. Sternheimer, *Deformation quantization: Twenty years after*, AIP Conf. Proc. **453**, 107 (1998) [math.qa/9809056].

[15] Maxim Kontsevich, *Deformation quantization of Poisson manifolds, I*, [q-alg/9709040].

[16] C. S. Chu and P. M. Ho, *Noncommutative open string and D-brane*, Nucl. Phys. B **550**, 151 (1999) [hep-th/9812219].

[17] V. Schomerus, *D-branes and deformation quantization*, JHEP **9906**, 030 (1999) [hep-th/9903205].

[18] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **9909**, 032 (1999) [hep-th/9908142].

[19] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, Rev. Mod. Phys. **73**, 977 (2001) [hep-th/0106048].

[20] R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rept. **378**, 207 (2003) [hep-th/0109162].

[21] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, *Noncommutative field theory and Lorentz violation*, Phys. Rev. Lett. **87**, 141601 (2001) [hep-th/0105082].

[22] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, JHEP **0002**, 020 (2000) [hep-th/9912072].

[23] M. Van Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, JHEP **0003**, 035 (2000) [hep-th/0002186].

[24] L. Alvarez-Gaume and M. A. Vazquez-Mozo, *Comments on noncommutative field theories*, [hep-th/0311244].

[25] R. Oeckl, *Braided quantum field theory*, Commun. Math. Phys. **217**, 451 (2001) [hep-th/9906225].

[26] J. Lukierski, A. Nowicki, H. Ruegg and V. N. Tolstoy, *Q-deformation of Poincaré algebra*, Phys. Lett. **B264**, 331 (1991).

[27] J. Lukierski, A. Nowicki and H. Ruegg, *New quantum Poincaré algebra and κ-deformed field theory*, Phys. Lett. **B293**, 344 (1992).

[28] S. Majid and H. Ruegg, *Bicrossproduct structure of κ-Poincaré group and noncommutative geometry*, Phys. Lett. **B334**, 348 (1994) [hep-th/9405107].

[29] I. Bars, M. M. Sheikh-Jabbari and M. A. Vasiliev, *Noncommutative $o^*(N)$ and $usp^*(2N)$ algebras and the corresponding gauge field theories*, Phys. Rev. D **64**, 086004 (2001) [hep-th/0103209].

[30] L. Bonora, M. Schnabl, M. M. Sheikh-Jabbari and A. Tomasiello, *Noncommutative $SO(n)$ and $Sp(n)$ gauge theories*, Nucl. Phys. B **589**, 461 (2000) [hep-th/0006091].

[31] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. **C16**, 161 (2000) [hep-th/0001203].

[32] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-Abelian gauge groups on non-commutative spaces*, Eur. Phys. J. **C17**, 521 (2000) [hep-th/0006246].

[33] B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. **C21**, 383 (2001) [hep-th/0104153].

[34] C. P. Martín, *The gauge anomaly and the Seiberg-Witten map*, Nucl. Phys. B **652**, 72 (2003) [hep-th/0211164].

[35] F. Brandt, C. P. Martín and F. R. Ruiz, *Anomaly freedom in Seiberg-Witten noncommutative gauge theories*, JHEP **0307**, 068 (2003) [hep-th/0307292].

[36] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, *The Standard Model on noncommutative spacetime*, Eur. Phys. J. **C23**, 363 (2002) [hep-ph/0111115].

[37] M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess and M. Wohlgenannt, *Deformed field theories on κ-spacetime*, Eur. Phys. J. **C31**, 129 (2003) [hep-th/0307149].

[38] M. Dimitrijević, F. Meyer, L. Möller and J. Wess, *Gauge theories on the κ-Minkowski spacetime*, accepted for publication by Eur. Phys. J. [hep-th/0310116].

[39] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. **79**, 59 (1964).

[40] W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, *The $Z \to$ gamma gamma, g g decays in the noncommutative standard model*, Eur. Phys. J. C **29**, 441 (2003) [hep-ph/0202121].

[41] S. Majid, *Noncommutative physics on Lie algebras, $(Z(2))^{**n}$ lattices and Clifford algebras*, [hep-th/0302120].

[42] M. Dimitrijević, L. Möller and E. Tsouchnika, *Derivatives, forms and vector fields on the κ-deformed Euclidean space*, submitted to J. Phys. A [hep-th/0404224].

[43] Y. I. Manin, *Multiparametric quantum deformation of the general linear supergroup*, Commun. Math. Phys. **123**, 163 (1989).

[44] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. Proc. Suppl. **18B**, 302-312 (1991).

[45] S. L. Woronowicz, *Differential calculus on compact matrix pseudogroups (Quantum Groups)*, Commun. Math. Phys. **122**, 125-170 (1989).

[46] S. Waldmann, *Deformation quantization: Observable algebras, states and representation theory*, [hep-th/0303080].

[47] R. Peierls, Z. Phys. **80**, 763 (1933).

[48] R. Jackiw, *Physical instances of noncommuting coordinates*, in: J. Trampetic, J. Wess (eds.) Paricle Physics in the new millennium, Proc. 8^{th} Adriatic Meeting, 294 (2003).

[49] F. Meyer, *Models of gauge field theory on noncommutative spaces*, Diploma thesis at the University of München (2003).

[50] H. Weyl, *Quantenmechanik und Gruppentheorie*, Z. Phys. **46**,1 (1927).

[51] M. Bertelson, M. Cahen and S. Gutt, *Equivalence of star products*, Class. Quant. Grav. **14**, A93 (1997).

[52] V. Kathotia, *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula, I*, UC Davis Math 1998-16 [math.qa/9811174].

[53] B. Shoikhet, *On the Kontsevich and the Campbell-Baker-Hausdorff deformation quantizations of a linear Poisson structure*, [math.qa/9903036].

[54] G. Amelino-Camelia and M. Arzano, *Coproduct and star-product in field theories on Lie algebra noncommutative spacetimes*, Phys. Rev. D **65**, 084044 (2002) [hep-th/0105120].

[55] P. Kosiński, J. Lukierski and P. Maślanka, *Local $D = 4$ field theory on κ-deformed Minkowski space*, Phys. Rev. D **62**, 025004 (2000) [hep-th/9902037].

[56] P. Kosiński, J. Lukierski and P. Maślanka, *κ-deformed Wigner construction of relativistic wave functions and free fields on κ-Minkowski space*, Nucl.Phys.Proc.Suppl. **102**, 161-168 (2001) [hep-th/0103127].

[57] K. Kosiński, J. Lukierski and P. Maślanka, *Local field theory on κ-Minkowski space, $\star$-products and noncommutative translations*, Czech.J.Phys. **50**, 1283 (2000) [hep-th/0009120].

[58] A. Agostini, F. Lizzi and A. Zampini, *Generalized Weyl systems and κ-Minkowski space*, Mod. Phys. Lett. A **17**, 2105 (2002) [hep-th/0209174].

[59] A. Connes, M. R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori*, JHEP **9802**, 003 (1998) [hep-th/9711162].

[60] B. Jurčo and P. Schupp, *Noncommutative Yang-Mills from equivalence of star products*, Eur. Phys. J. C **14**, 367 (2000) [hep-th/0001032].

[61] B. Jurčo, P. Schupp and J. Wess, *Noncommutative gauge theory for Poisson manifolds*, Nucl. Phys. B **584**, 784 (2000) [hep-th/0005005].

[62] B. Jurčo, P. Schupp and J. Wess, *Nonabelian noncommutative gauge theory via noncommutative extra dimensions*, Nucl. Phys. B **604**, 148 (2001) [hep-th/0102129].

[63] B. Jurčo, P. Schupp and J. Wess, *Noncommutative line bundle and Morita equivalence*, Lett. Math. Phys. **61**, 171 (2002) [hep-th/0106110].

[64] T. Mehen and M. B. Wise, *Generalized $\star$-products, Wilson lines and the solution of the Seiberg-Witten equations*, JHEP **0012**, 008 (2000) [hep-th/0010204].

[65] A. S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, Comm. Math. Phys. 212, 591 (2000) [math.qa/9902090].

[66] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, Commun. Math. Phys. **225**, 33 (2002) [hep-th/0101219].

[67] X. Calmet and M. Wohlgenannt, *Effective field theories on noncommutative spacetime*, Phys. Rev. D **68**, 025016 (2003) [hep-ph/0305027].

[68] W. Behr and A. Sykora, *Construction of gauge theories on curved noncommutative spacetime*, [hep-th/h0309145].

[69] W. Behr and A. Sykora, *NC Wilson lines and the inverse Seiberg-Witten map for non-degenerate star products*, [hep-th/h0312138].

[70] C. N. Yang and R. L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96**, 191 (1954).

[71] S. Goto and H. Hata, *Noncommutative monopole at the second order in theta*, Phys. Rev. D **62**, 085022 (2000) [hep-th/0005101].

[72] T. Asakawa and I. Kishimoto, *Noncommutative gauge theories from deformation quantization*, Nucl. Phys. B **591**, 611 (2000) [hep-th/0002138].

[73] K. Okuyama, *Comments on open Wilson lines and generalized star products*, Phys. Lett. B **506**, 377 (2001) [hep-th/0101177].

[74] H. Liu, **-Trek II: *n operations, open Wilson lines and the Seiberg-Witten map*, Nucl. Phys. B **614**, 305 (2001) [hep-th/0011125].

[75] B. L. Cerchiai, A. F. Pasqua and B. Zumino, *The Seiberg-Witten map for noncommutative gauge theories*, [hep-th/0206231].

[76] S. Fidanza, *Towards an explicit expression of the Seiberg-Witten map at all orders*, JHEP **0206**, 016 (2002) [hep-th/0112027].

[77] E. P. Wigner, Ann. Math. **40**, 140 (1939).

[78] J. Mund, B. Schroer and J. Yngvason, *String-localized quantum fields from Wigner representations*, [math-ph/0402043].

[79] C. Blohmann, *Spin representations of the q-Poincare algebra*, PhD thesis at the University of München (2001) [math.qa/0110219].

[80] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, *Deformed QED via Seiberg-Witten map*, [hep-th/0102103].

[81] J. M. Grimstrup and R. Wulkenhaar, *Quantisation of theta-expanded noncommutative QED*, Eur. Phys. J. C **26**, 139 (2002) [hep-th/0205153].

[82] A. H. Chamseddine, *Deforming Einstein's gravity*, Phys. Lett. B **504**, 33 (2001) [hep-th/0009153].

[83] D. Mikulovic, *Seiberg-Witten map for superfields on canonically deformed $N = 1$, $d = 4$ superspace*, JHEP **0401**, 063 (2004) [hep-th/0310065].

[84] P. Aschieri, B. Jurčo, P. Schupp and J. Wess, *Noncommutative GUTs, standard model and C, P, T*, Nucl. Phys. B **651**, 45 (2003) [hep-th/0205214].

[85] F. Brandt, *Seiberg-Witten maps and anomalies in noncommutative Yang-Mills theories*, [hep-th/0403143].

[86] T. Asakawa and I. Kishimoto, *Comments on gauge equivalence in noncommutative geometry*, JHEP **9911**, 024 (1999) [hep-th/9909139].

[87] D. Brace, B. L. Cerchiai and B. Zumino, *Non-Abelian gauge theories on noncommutative spaces*, Int. J. Mod. Phys. A **17S1**, 205 (2002) [hep-th/0107225].

[88] G. Barnich, F. Brandt and M. Grigoriev, *Local BRST cohomology and Seiberg-Witten maps in noncommutative Yang-Mills theory*, [hep-th/0308092].

[89] K. Okuyama, *A path integral representation of the map between commutative and noncommutative gauge fields*, JHEP **0003**, 016 (2000) [hep-th/9910138].

[90] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, *Renormalization of the noncommutative photon self-energy to all orders via Seiberg-Witten map*, JHEP **0106**, 013 (2001) [hep-th/0104097].

[91] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, *Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map*, Eur. Phys. J. C **24**, 165 (2002) [hep-th/0108045].

[92] D. Brace, B. L. Cerchiai, A. F. Pasqua, U. Varadarajan and B. Zumino, *A cohomological approach to the non-Abelian Seiberg-Witten map*, JHEP **0106**, 047 (2001) [hep-th/0105192].

[93] M. Picariello, A. Quadri and S. P. Sorella, *Chern-Simons in the Seiberg-Witten map for noncommutative Abelian gauge theories in 4D*, JHEP **0201**, 045 (2002) [hep-th/0110101].

[94] G. Barnich, F. Brandt and M. Grigoriev, *Seiberg-Witten maps and noncommutative Yang-Mills theories for arbitrary gauge groups*, JHEP **0208**, 023 (2002) [hep-th/0206003].

[95] M. Hayakawa, *Perturbative analysis on infrared aspects of noncommutative QED on* R^4, Phys. Lett. B **478**, 394 (2000) [hep-th/9912094].

[96] M. Hayakawa, *Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on* R^4, [hep-th/9912167].

[97] F. J. Petriello, *The Higgs mechanism in noncommutative gauge theories*, Nucl. Phys. B **601**, 169 (2001) [hep-th/0101109].

[98] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari and A. Tureanu, *Noncommutative standard model: Model building*, Eur. Phys. J. C **29**, 413 (2003) [hep-th/0107055].

[99] M. Chaichian, P. Prešnajder, M. M. Sheikh-Jabbari and A. Tureanu, *Noncommutative gauge field theories: A no-go theorem*, Phys. Lett. B **526**, 132 (2002) [hep-th/0107037].

[100] H. Grosse and R. Wulkenhaar, *Power-counting theorem for non-local matrix models and renormalisation*, [hep-th/0305066].

[101] H. Grosse and R. Wulkenhaar, *Renormalisation of* phi^4 *theory on noncommutative* R^4 *in the matrix base*, [hep-th/0401128].

[102] P. Schupp, J. Trampetic, J. Wess and G. Raffelt, *The photon neutrino interaction in non-commutative gauge field theory and astrophysical bounds*, [hep-ph/0212292].

[103] P. Minkowski, P. Schupp and J. Trampetic, *Non-commutative '*-charge radius' and '*-dipole moment' of the neutrino*, [hep-th/0302175].

[104] G. Duplancic, P. Schupp and J. Trampetic, *Comment on triple gauge boson interactions in the non-commutative electroweak sector*, Eur. Phys. J. C **32**, 141 (2003) [hep-ph/0309138].

[105] X. Calmet, *What are the bounds on space-time noncommutativity?*, [hep-ph/0401097].

[106] T. G. Rizzo, *Signals for noncommutative QED at future e+ e- colliders*, Int. J. Mod. Phys. A **18**, 2797 (2003) [hep-ph/0203240].

[107] C. E. Carlson, C. D. Carone and R. F. Lebed, *Bounding noncommutative QCD*, Phys. Lett. B **518**, 201 (2001) [hep-ph/0107291].

[108] P. Kosiński, J. Lukierski, P. Maślanka and J. Sobczyk, *The classical basis for* κ*-deformed Poincaré (super)algebra and the second* κ*-deformed supersymmetric Casimir*, Mod. Phys. Lett. A **10**, 2599 (1995) [hep-th/9412114].

[109] P. Kosiński and P. Maślanka, *The duality between* κ*-Poincaré algebra and* κ*-Poincaré group*, [hep-th/9411033].

[110] J. Lukierski and H. Ruegg, *Quantum* κ*-Poincaré in any dimension*, Phys. Lett. B **329**, 189 (1994) [hep-th/9310117].

[111] A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander, *A new "null-plane" quantum Poincaré algebra*, Phys. Lett. B **351**, 137-145 (1995).

[112] P. Kosiński, P. Maślanka, J. Lukierski and A. Sitarz, *Generalized* κ*-deformations and deformed relativistic scalar fields on noncommutative Minkowski space*, [hep-th/0307038].

[113] C. Blohmann, *Perturbative Symmetries on Noncommutative Spaces*, [math.qa/0402200].

[114] J. C. Jantzen, *Lectures on quantum groups*, American Mathematical Society (1996).

[115] A. Nowicki, E. Sorace and M. Tarlini, *The quantum deformed Dirac equation from the* κ*-Poincaré algebra*, Phys. Lett. **B302**, 419-422 (1993) [hep-th/9212065].

[116] J. Lukierski, H. Ruegg and W. Rühl, *From* κ*-Poincaré algebra to* κ*-Lorentz quasi-group: A deformation of relativistic symmetry*, Phys. Lett. B **313**, 357 (1993).

[117] J. Kowalski-Glikman and S. Nowak, *Doubly Special Relativity theories as different bases of* κ*-Poincaré algebra*, Phys. Lett. **B539**, 126-132 (2002) [hep-th/0203040].

[118] A. Sitarz, *Noncommutative differential calculus on the* κ*-Minkowski space*, Phys. Lett. **B349**, 42-48 (1995) [hep-th/9409014].

[119] J. Madore, *An Introduction to noncommutative differential geometry and its physical applications*, Cambridge University Press (1995).

[120] E. Tsouchnika, *Field theories in noncommutative spacetime*, Diploma thesis at the University of Munich (2003).

[121] G. Amelino-Camelia, F. D'Andrea and G. Mandanici, *Group velocity in noncommutative spacetime*, [hep-th/0211022].

[122] A. Agostini, G. Amelino-Camelia and F. D'Andrea, *Hopf-algebra description of noncommutative-spacetime symmetries*, [hep-th/0306013].

[123] P. Podleś, *Solutions of Klein-Gordon and Dirac equations on quantum Minkowski spaces*, Commun. Math. Phys. **181**, 569 (1996) [q-alg/9510019].

[124] G. Felder and B. Shoikhet, *Deformation quantization with traces*, [math.qa/0002057].

[125] M. A. Dietz, *Symmetrische Formen auf Quantenalgebren*, Diploma thesis at the University of Hamburg (2001).

[126] M. Nakahara, *Geometry, topology and physics*, Institute of physics publishing (1990).

[127] F. Meyer and H. Steinacker, *Gauge field theory on the* $E_q(2)$*-covariant plane*, [hep-th/0309053].

LEBENSLAUF

Persönliche Daten	Lutz Möller Schellingstr. 20 80799 München geboren am 21. Sep. 1973 in München Staatsangehörigkeit deutsch ledig

Schule und Universität

Juni 1993	Abitur am Thomas-Mann-Gymnasium, München (Note: 1.0)
Nov. 1994 - Sep. 2000	Studium der Physik an der Ludwig-Maximilians-Universität München, Stipendium gemäß BayBFG §10.3, Abschluss mit Diplom (Note: 1.1), Betreuung der Diplomarbeit durch Prof. Julius Wess
Nov. 1995 - Juli 1997	Studium der Philosophie an der Hochschule für Philosophie München, Abschluss mit Bakkalaureat (Note: 1.4)
Okt. 1997 - Juli 1998	Auslandsjahr an der University of Oxford
Nov. 2000 - März 2001	Gast an der Humboldt-Universität in Berlin
Nov. 2000 - März 2004	Doktorandenstudium Physik an der LMU München Betreuung der Doktorarbeit durch Prof. Julius Wess

Praktika und Beschäftigungsverhältnisse

April 1997	Werkstudent am MPI für Plasmaphysik, Garching
Aug. 1997 + Sep. 1998	Werkstudent am MPI für Quantenoptik, Garching
Nov. 1998 - Feb. 1999	Werkstudent bei der Sektion Physik der LMU München
Mai 1999 - Okt. 2000	Praktikant und Werkstudent bei KPMG Consulting, München
Mai 2000 - Okt. 2000	Werkstudent am MPI für Physik, München, Initiative „WIGV“
Nov. 2000 - Okt. 2003	Wissenschaftlicher Mitarbeiter am MPI für Physik
Nov. 2003 - Feb. 2004	Wissenschaftlicher Mitarbeiter an der LMU München

Zivildienst

Aug.1993 - Okt. 1994	Zivildienst im Kinderzentrum München

Wissenschaftliche Veröffentlichungen

- M. Dimitrijević, L. Möller, E. Tsouchnika;
  eingereicht bei J.Phys.A, Preprint [hep-th/0404224].
- M. Dimitrijević, F. Meyer, L. Möller, J. Wess;
  angenommen durch Eur. Phys. J., Preprint [hep-th/0310116].
- M. Dimitrijević, L. Jonke, L. Möller, E. Tsouchnika, J. Wess, M. Wohlgenannt;
  Eur.Phys.J. **C31** (2003) 129, Preprint [hep-th/0307149].
- B. Jurčo, L. Möller, S. Schraml, P. Schupp, J. Wess;
  Eur.Phys.J. **C21** (2001) 383, Preprint [hep-th/0104153].

München, den 20. Juli 2004