Spin Representations of the q-Poincaré Algebra

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Zusammenfassung

In der Quantenmechanik können freie Elementarteilchen durch irreduzible Darstellungen der Poincaré-Algebra beschrieben werden. Im Rahmen der Darstellungtheorie der q-deformierten Poincaré-Algebra untersucht diese Arbeit den Spin von Teilchen auf einer nichtkommutativen Geometrie.

Zunächst wird eine Übersicht über die Konstruktion der q-Lorentz-Algebra gegeben. Ausgehend von q-Spinoren, wird die q-Lorentz-Gruppe und die zu ihr duale q-Lorentz-Algebra konstruiert. Dabei soll gezeigt werden, dass die q-Lorentz-Algebra weitgehend durch mathematische Konsistenzbedingungen festgelegt ist.

Anschließend wird die Struktur der q-Lorentz-Algebra untersucht. Ihre Darstellungstheorie einschließlich expliziter Formeln für die q-Clebsch-Gordan-Koeffizienten wird zusammengefasst. Nach einer allgemeinen Betrachtung von Tensor-Operatoren in Hopf-Algebren werden die Vektorgeneratoren der Quantenalgebra der Drehungen berechnet. Zwei weitere Formen der q-Lorentz-Algebra, die vektorielle oder RS-Form (Wess) und die Quantendoppel-Form (Woronowicz), werden vorgestellt. Ein Isomorphismus zwischen beiden Formen wird gefunden.

Die Darstellungstheorie der q-Lorentz-Algebra wird verwendet, um die Algebra des q-Minkowski-Raumes zu konstruieren. Vertauschungsregeln zwischen den Erzeugern der q-Minkowski-Algebra und den verschiedenen Formen der q-Lorentz-Algebra werden angegeben. Die Struktur der von Rotationen und Translationen erzeugten q-Euklidischen Algebra wird eingehend untersucht und dadurch ihr Zentrum bestimmt. Daraus können zunächst die nullte Komponente und schließlich alle Komponenten des q-Pauli-Lubanski-Vektor bestimmt werden. Mit dem q-Pauli-Lubanski-Vektor können die Algebren der Spin-Symmetrie, die kleinen Algebren, berechnet werden, sowohl für den massiven als auch den masselosen Fall.

Irreduzible Spin-Darstellungen der q-Poincaré-Algebra werden konstruiert. Zunächst werden Darstellungen in einer physikalisch interpretierbaren Drehimpuls-Basis berechnet. Die Berechnungen werden dabei durch die Verwendung des q-Wigner-Eckart-Theorems stark vereinfacht. Anschließend wird gezeigt, wie Darstellungen durch die Methode der Induktion gewonnen werden können.

Ausgehend von einer darstellungstheoretischen Interpretation von Wellengleichungen werden schließlich freie q-relativistische Wellengleichungen bestimmt. Dazu werden zunächst allgemeine Betrachtungen zu q-Lorentz-Spinoren, konjugierten Spinoren und dem Verhältnis von q-Impulsen und q-Ableitungen auf den Spinor-Darstellungen angestellt. Als Beispiele werden die q-Dirac-Gleichung, die q-Weyl-Gleichungen und die q-Maxwell-Gleichungen eindeutig bestimmt.

Contents

Introduction

Motivation From the beginnings of quantum field theory it has been argued that the pathological ultraviolet divergences should be remedied by limiting the precision of position measurements by a fundamental length [2–5]. In view of how position-momentum uncertainty enters into quantum mechanics, a natural way to integrate such a position uncertainty in quantum theory would have been to replace the commutative algebra of space observables with a non-commutative one [6]. However, deforming the space alone will in general break the symmetry of spacetime.

In order to preserve a background symmetry the symmetry group must be deformed together with the space it acts on. It is clear that Lie groups cannot be continuously deformed within their proper category: From the classification of semi-simple Lie groups we know that they form a countable and hence discrete set. Being manifolds, however, they can be naturally embedded in the category of algebras by the Gelfand-Neumark map [7], the additional group structure on the manifold side translating into a Hopf structure on the algebra side. But although Hopf algebras in general had been familiar to mathematicians for some time [8–10], hardly any non-trivial examples of Hopf algebras were known [11]. This situation changed with the discovery of quantum groups [12], that is, with the discovery of generic methods to continuously deform Lie algebras [13, 14] and matrix groups [15–17] within the category of Hopf algebras.

Quantum groups now provided a consistent mathematical framework to formulate physical theories on non-commutative spaces. Beginning with the non-commutative plane [18], q-deformations of a variety of objects have since been constructed: differential calculi on non-commutative spaces [19], Euclidean space [16], Minkowski-Space [20], the Lorentz group and the Lorentz algebra [21–24], the Poincaré algebra [25], to name a few. The study of these objects has produced interesting results. For example, it has been found that free theories on non-commutative spaces can be viewed as theories on ordinary commutative spaces with complicated interactions [26–28].

Another result is, that q-deformation will in general discretize the spectra of spacetime observables [29–31], that is, q-deformation puts physics on a spacetime lattice. This nourishes the hope that q-deformed field theories might be regularized, one of the original motivations to consider non-commutative geometries. It is not new that deformation is a method to regularize field theory — at least, it is one way to look at the first step of renormalization: In the loop expansion of transition probabilities some terms turn out to be infinite, so we regularize them by a sort of deformation process in order to classify the divergences. In this sense q-deformation can be viewed as attempt to shift the deformation from the end of the construction of field theory (perturbative expansion) to the beginning (symmetry structures).

Aims Given the q-Poincaré algebra as background symmetry, how can we construct a quantum theory upon it? If states continue to be described by vectors of a Hilbert space, it must be specified how the q-Poincaré algebra acts on them, that is, we must construct representations of the q-Poincare algebra. If we further want to describe *elementary* particles the representations must be *irreducible* [1]. If we want to use reducible representations such as the Dirac spinor representation, we need additional constraints to eliminate the redundant degrees of freedom. These constraints are the wave equations. The interpretation of this quantum theory forces us to consider multi-particle states. These are described as properly symmetrized (or anti-symmetrized) tensor product representations. Symmetrization or anti-symmetrization means that we need a ray representation of the permutation group on the tensor product space which is compatible with (intertwines with) the action of the q-Poincaré algebra. The physical states are the orbits of this action of the permutation group, while the direct sum of all such multi-particle spaces is the Fock space. We summarize:

(i) Elementary q-particles are irreducible representations of the q-Poincaré algebra.

(ii) Wave equations are the constraints to eliminate the redundant degrees of freedom of a reducible representation.

(iii) q-Fields are symmetrized or anti-symmetrized multi-particle states.

In the undeformed case these principles completely determine the free relativistic quantum theory. Therefore, it is reasonable to use them as program to construct the deformed theory.

This program has been pursued in previous work [31–40]. In [31–35] irreducible spin zero representations of the q-Poincaré algebra were constructed. While in [31, 32] the realization of the q-Poincaré algebra within q-Minkowski phase space was considered, such that the representations were naturally limited to orbital angular momentum,[1] it is possible to extend [34, 35] to include spin representations (Sec. 4.1.1). Various methods to construct wave equations have been proposed, based on q-Clifford algebras [36], q-deformed co-spinors [37], or differential calculi on quantum spaces [38–40], leading to mutually different results. This is unsatisfactory since the construction of wave equations according to (ii) should determine the wave equations uniquely as in the undeformed

[1] See Eq. (3.57).

case [41] and should not demand any additional mathematical structure besides the q-Poincaré algebra and the basic apparatus of quantum mechanics.

The aim of the present work is to investigate the nature of spin within the representation theory of the q-Poincaré algebra.

Results Our main results are:

- The q-deformed Pauli-Lubanski vector is computed (Sec. 3.2), from which the spin Casimir and the little algebras can be determined (Sec. 3.3).
- Irreducible representations with spin are constructed (Chap. 4).
- A practical method to uniquely compute the wave equations is developed (Sec. 5.1). As examples the q-Dirac equation (Sec. 5.2) and the q-Maxwell equations (Sec. 5.3) are computed.

To give a more detailed overview:

In chapter 1 we review the construction of the q-Lorentz algebra. We start with the quantum plane, $xy = qyx$, derive the algebra of coacting quantum matrices $M_q(2)$, introduce the q-spinor metric, the quantum special linear group $SL_q(2)$ and its real form $SU_q(2)$. We introduce dotted spinors, join an undotted and a dotted corepresentation to form the quantum Lorentz group $SL_q(2, \mathbb{C})$. Using the duality between $SL_q(2)$ and $\mathcal{U}_q(\mathrm{sl}_2)$ we compute the quantum Lorentz algebra $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ by dualizing $SL_q(2, \mathbb{C})$ [18, 20, 22, 42]. The presentation puts emphasis on the fact that in the construction of the q-Lorentz algebra as it is understood now, hardly any arbitrariness is involved.

Chapter 2 explores the structure of the q-Lorentz algebra. The representation theory of the q-Lorentz algebra is reviewed, explicit formulas for the q-Clebsch-Gordan coefficients are given. After a general consideration of the different sorts of tensor operators, the vectorial generators of $\mathcal{U}_q(\mathrm{sl}_2)$ are determined. Three different forms of the q-Lorentz algebra are related by explicit formulas: the dual of the q-Lorentz group [42], the quantum double form [21], and the vectorial or RS-form [23, 44, 52]. The isomorphism between the quantum double form and the vectorial form that is found (Sec. 2.4.3) relates the work of the Warsaw and the Munich group.

In chapter 3 the results of chapter 2 are used to construct the algebra of q-Minkowski space [20]. Commutation relations of the generators of different forms of the q-Lorentz algebra with the spacetime generators are given. We study the structure of the q-Euclidean algebra consisting of rotations and translations in order to find a good zero component of the q-Pauli-Lubanski vector. A technique of boosting is used to calculate the other components (Sec. 3.2). The q-Pauli-Lubanski vector is used to compute the little algebras for the massive and the massless case (Sec. 3.3).

Chapter 4 contains the construction of massive spin representations of the q-Poincaré algebra. In the first part we construct irreducible representations in an angular momentum basis, which is accessible to physical interpretation. The calculations are considerably simplified by the q-Wigner-Eckart theorem. In the second part we briefly show how representations of the q-Poincaré algebra can be constructed using the method of induced representations.

In chapter 5 we calculate free wave equations. We start with the representation theoretic interpretation of free wave equations. Then we consider the generalities of q-Lorentz spinor representations, conjugate spinors, and the relation between momenta and derivations. Finally, we put things together and uniquely determine the q-Dirac equation including q-gamma matrices and their q-Clifford algebra, the q-Weyl equations, and the q-Maxwell equations.

Outlook While our approach to the q-Poincaré algebra was representation theoretic, the problems we had to overcome were mostly on the algebraic side: A method to boost vector operators, complete sets of commuting observables, the spin Casimir, the spin symmetry algebras, spinor conjugation — all this had to be found before spin representations and spinorial wave equations could be computed. Now, that the algebraic tool set is more complete, we are prepared for the next steps towards a q-deformed relativistic quantum theory.

One promising way to continue this work would be to couple the q-Dirac and the q-Maxwell field, for which the mathematical setting has been provided in chapter 5.

Notation Throughout this work, the deformation parameter q is assumed real, $q > 1$. We frequently use the abbreviations

$$\lambda := q - q^{-1}\,, \qquad\qquad [j] := \frac{q^j - q^{-j}}{q - q^{-1}}\,, \tag{1}$$

where j is a number. In particular, we have $[2] = q + q^{-1}$. Spinor indices running through $\{-,+\} = \{-\frac{1}{2}, +\frac{1}{2}\}$ are denoted by lower case Roman letters (a, b, c, d), 3-vector indices running through $\{-,3,+\} = \{-1,0,+1\}$ by upper case Roman letters (A, B, C), and 4-vector indices running through $\{0,-,+,3\}$ by lower case Greek letters (μ, ν, σ, τ). Quantum Lie groups are written with a subscript q like $SL_q(2)$, quantum enveloping algebras like $\mathcal{U}_q(\mathrm{sl}_2)$.

Chapter 1

Construction of the q-Lorentz Algebra

In undeformed quantum mechanics we can represent a state by a wave function $\psi : \mathbb{R}^n \to \mathbb{C}$. In this representation, the observables x_i, which describe the measurement of the position of the particle, act on ψ by multiplications with the functions $x_i : \mathbb{R}^n \to \mathbb{C}$, $x_i(\vec{r}) = r_i$. In this sense, geometry is described by the algebra of functions over a space, $\mathcal{F}(\mathbb{R}^n)$, rather than by the space $\mathbb{R}^n$ itself. Replacing a space by its function algebra, it is natural to replace an endomorphism f by its pullback f^*,

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n \quad \Rightarrow \quad \mathcal{F}(\mathbb{R}^n) \xleftarrow{f^*} \mathcal{F}(\mathbb{R}^n)\,, \quad \text{where} \quad (f^* x_i)(\vec{r}) := x_i(f\vec{r})\,, \tag{1.1}$$

yielding a recipe to translate spaces and homomorphisms of spaces into algebras and homomorphisms of algebras. In the language of category theory $\mathcal{F}$ is called a cofunctor [45], the prefix "co" reminding us that we have to reverse arrows.

For a consistent mathematical framework we must extend this method of algebraization to any additional structure on $\mathbb{R}^n$. If there is for example the action ϕ of a group G on the space we get

$$G \otimes \mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^n \quad \Rightarrow \quad \mathcal{F}(G) \otimes \mathcal{F}(\mathbb{R}^n) \xleftarrow{\phi^*} \mathcal{F}(\mathbb{R}^n)\,, \tag{1.2}$$

where $\mathcal{F}(G)$ is the algebra of functions over the group and the homomorphism of algebras $\rho := \phi^*$ is called the coaction. The structure maps of the group, multiplication μ, unit η, and inverse, translate into comultiplication $\Delta = \mu^*$, counit $\varepsilon = \eta^*$, and coinverse or antipode S. The group axioms translate into axioms of this co-structure [8]. An algebra equipped with this co-structure is called a Hopf algebra [9, 10].

So far, the structure of spaces and groups acting on them has only been rephrased in a more algebraic but equivalent language. But unlike the category of Lie groups, the category of algebras allows for continuous deformation: We can replace the trivial commutation relations of the algebra of space functions by non-trivial ones, which depend on a real parameter q. This q-deformation of the space algebra forces us to q-deform any Hopf algebra coacting on it, as well. Reminiscent of their relation to quantum theory, these deformed algebras are called quantum spaces and quantum groups. Instead of quantum groups we can consider their Hopf duals [46, 47], the quantum algebras, which are deformations of the enveloping Lie algebras. Since quantum algebras have a familiar undeformed counterpart, they become directly accessible to physical interpretation. For example, the generators of the quantum algebra of rotations are the q-deformed angular momentum operators.

1.1 q-Spinors and $SU_q(2)$

1.1.1 q-Spinors and Their Cotransformations

The simplest quantum space is the deformation of the algebra $\mathcal{F}(\mathbb{C}^2) = \mathbb{C}[x, y]$ of polynomial spinor functions. We replace the trivial commutation relations $xy = yx$ with $xy = qyx$, where q is a real parameter $q > 1$, and call the resulting algebra

$$\mathbb{C}_q^2 := \mathbb{C}\langle x, y\rangle / \langle xy = qyx\rangle \tag{1.3}$$

the algebra of q-spinors or the quantum plane [18].

As in the undeformed case we want the spinor algebra to carry a left and a right matrix corepresentation. We define the vector of spinor generators

$$\psi_a = (\psi_-, \psi_+) := (x, y) \tag{1.4}$$

a matrix of generators of the algebra $M_q(2)$ of 2×2-matrices

$$M^a{}_b = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.5}$$

with respect to the indices $\{-, +\} = \{-\frac{1}{2}, +\frac{1}{2}\}$ and the left and right coaction of this matrix on the spinor

$$\rho_{\mathrm{L}}(\psi_a) := M^a{}_{a'} \otimes \psi_{a'}\,, \qquad \rho_{\mathrm{R}}(\psi_a) := \psi_{a'} \otimes M^{a'}{}_a\,, \tag{1.6}$$

where we sum over repeated indices, and where the coproduct of $M_q(2)$ is defined by $\Delta(M^a{}_c) = M^a{}_b \otimes M^b{}_c$.

We want the deformed commutation relations between the generators of $M_q(2)$ to be consistent with those of the q-spinor, $xy = qyx$, that is, the coaction maps must be algebra homomorphisms. This uniquely determines the relations

$$\begin{gathered} ab = qba\,, \quad ac = qca\,, \quad bd = qdb\,, \quad cd = qdc \\ bc = cb\,, \quad ad - da = (q - q^{-1})bc\,. \end{gathered} \tag{1.7}$$

The algebra freely generated by a, b, c, d modulo these relations (1.7) is called $M_q(2)$ the algebra of $2{\times}2$ quantum matrices. Introducing the R-Matrix

$$R^{ab}{}_{cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{1.8}$$

with respect to the indices $\{--,-+,+-,++\}$, Eqs. (1.7) can be written in the compact form

$$R^{ab}{}_{c'd'}M^{c'}{}_{c}M^{d'}{}_{d} = M^{b}{}_{b'}M^{a}{}_{a'}R^{a'b'}{}_{cd}\,, \tag{1.9}$$

the famous FRT-relations of matrix quantum groups [16].

1.1.2 The q-Spinor Metric and $SL_q(2)$

With the spinor metric

$$\varepsilon_{ab} = -\varepsilon^{ab} = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}, \quad \text{with} \quad \varepsilon_{ab}\varepsilon^{bc} = \delta_a^c \tag{1.10}$$

we can write $xy = qyx$ as $\psi_a\psi_b\varepsilon^{ab} = 0$. In analogy to the undeformed case the spinor metric must thus be invariant under $M_q(2)$-transformations up to a factor. Indeed, we find

$$M^{a}{}_{a'}M^{b}{}_{b'}\varepsilon^{a'b'} = (\det{}_q M)\,\varepsilon^{ab}\,, \tag{1.11}$$

where $\det_q M = ad - qbc$ is central in $M_q(2)$. Constraining the transformations to leave the scalar product $\psi_a\phi_b\varepsilon^{ab}$ of two spinors strictly invariant, we obtain $SL_q(2) := M_q(2)/\langle \det_q M = 1\rangle$ the deformation of the function algebra of the group of special linear transformations.

Finally, Eq. (1.11) can be contracted with the metric from the right. From the resulting equation

$$M^{a}{}_{a'}(M^{b}{}_{b'}\varepsilon^{a'b'}\varepsilon_{bc}) = \delta_c^a \tag{1.12}$$

we can read off the antipode

$$S(M^{a}{}_{b}) := \varepsilon^{aa'}M^{b'}{}_{a'}\varepsilon_{b'b} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}, \tag{1.13}$$

playing the role of the inverse, $(M^{-1})^{a}{}_{b} = S(M^{a}{}_{b})$. This completes the Hopf algebra structure of $SL_q(2)$.

1.1.3 Upper Spinor Indices, Conjugation, and $SU_q(2)$

Defining a transposition on $SL_q(2)$ by

$$T(M^{a}{}_{b}) = (M^T)^{a}{}_{b} := M^{b}{}_{a}\,, \tag{1.14}$$

we can consider now a spinor transforming under the congredient representation $(M^T)^{-1}$. As in the undeformed case we indicate this transformation property by an upper index.

$$\rho_{\mathrm{R}}(\psi^a) = \psi^b \otimes ((M^T)^{-1})^{b}{}_{a} = \psi^b \otimes S(M^{a}{}_{b}) = \psi^b \otimes \varepsilon^{aa'}M^{b'}{}_{a'}\varepsilon_{b'b} \tag{1.15}$$

Contracting this equation with the spinor metric we find

$$\rho_{\mathrm{R}}(\varepsilon_{aa'}\psi^{a'}) = (\varepsilon_{bb'}\psi^{b'}) \otimes M^b{}_a \,, \tag{1.16}$$

telling us that $\varepsilon_{aa'}\psi^{a'}$ transforms as a spinor with lower index. We conclude that we can raise and lower indices by

$$\psi^a = \varepsilon^{aa'}\psi_{a'} \,, \qquad\qquad \psi_a = \varepsilon_{aa'}\psi^{a'} \,. \tag{1.17}$$

When we rewrite the spinor commutation relations as

$$0 = \psi_a\psi_b\varepsilon^{ab} = \varepsilon_{aa'}\varepsilon_{bb'}\psi^{a'}\psi^{b'}\varepsilon^{ab} = \psi^a\psi^b\varepsilon_{ba} = -\psi^a\psi^b\varepsilon^{ba} \,, \tag{1.18}$$

we see that a spinor with upper index satisfies commutation relations opposite to a spinor with lower index. Thus, we can define a $*$-structure on the spinor algebra $\mathbb{C}^2_q$ by $(\psi_a)^* := \psi^a$. This induces a $*$-structure on $SL_q(2)$ as well, by demanding the stars to be compliant with the coaction, $\rho_{\mathrm{R}} \circ * = (* \otimes *) \circ \rho_{\mathrm{R}}$.

A stared spinor transforms as a spinor with upper index, that is, by the congredient representation. We conclude that the induced $*$-operation on $SL_q(2)$ is given by

$$(M^a{}_b)^* = S(M^b{}_a) \,. \tag{1.19}$$

In other words, we have $(M^T)^* = M^{-1}$, which can be viewed as a quantum group analogue of a unitarity condition. Therefore, $SL_q(2)$ with this $*$-structure is called $SU_q(2)$.

1.2 The q-Lorentz Group

1.2.1 Dotted Spinors

We want to construct a deformation of the Lorentz group $SL(2,\mathbb{C})$, which is, viewed as real manifold, 6-dimensional, having 6 independent infinitesimal generators. Now, a spinor and its complex conjugate and thus the corepresentation matrix and its conjugate are no longer linearly dependent. This means that we have to add the conjugates $\bar{M}^a{}_b := (M^a{}_b)^*$ and $\bar{\psi}_a := (\psi_a)^*$ as extra generators. Of course, the conjugate spinor cotransforms under the conjugate matrix. As in the undeformed case, we will indicate that a quantity transforms like a conjugate spinor by a dotted index. Thus, writing $\psi_{\dot{a}}$ implies

$$\rho_{\mathrm{R}}(\psi_{\dot{a}}) = \psi_{\dot{b}} \otimes \bar{M}^b{}_a \,, \tag{1.20}$$

where we think of the dot as belonging to ψ rather than to the index itself. Since the $*$-operation is by definition an algebra anti-homomorphism (and a coalgebra homomorphism), the conjugate generators satisfy the opposite commutation relations of their pre-images. However, it is more convenient to combine the conjugate generators $\bar{M}$ linearly to form another matrix M_2 defined implicitly by $(M_2^T)^{-1} := \bar{M}$, that is,

$$S(M_{2}{}^{b}{}_{a}) := \bar{M}^{a}{}_{b} \,. \tag{1.21}$$

$S \circ T$ is an algebra anti-homomorphism (and a coalgebra homomorphism), so M_2 naturally generates a $SL_q(2)$ Hopf algebra. We now have two sets of generators generating two copies of $SL_q(2)$. For a consistent notation we will subscript the first set $M = M_1$ as well. The $*$-operation can then be written as

$$(M_{1}{}^{a}{}_{b})^{*_{SL_q(2,\mathbb{C})}} = (M_{2}{}^{a}{}_{b})^{*_{SU_q(2)}} \,. \tag{1.22}$$

Finally, we introduce upper dotted indices by demanding them to transform according to

$$\rho_{\mathrm{R}}(\psi^{\dot{a}}) = \psi^{\dot{b}} \otimes M_{2}{}^{b}{}_{a} \,. \tag{1.23}$$

This leads to formulas for raising and lowering dotted indices

$$\psi^{\dot{a}} = \psi_{\dot{b}} \varepsilon^{ba} \,, \qquad\qquad \psi_{\dot{a}} = \psi^{\dot{b}} \varepsilon_{ba} \,. \tag{1.24}$$

1.2.2 Commutation Relations of the q-Lorentz Group

So far, we know that the q-Lorentz group must be generated by two copies of $SL_q(2)$, generated by two sets of generators $M_{1}{}^{a}{}_{b}$ and $M_{2}{}^{a}{}_{b}$, respectively. The only thing we do not know yet are the commutation relations between M_1 and M_2. A priori, there are several choices of commutation relations, from which we will select one by an additional requirement: We will demand $SL_q(2,\mathbb{C})$ to possess a substructure of rotational symmetry, that is, we are looking for a homomorphism of Hopf-$*$ algebras[1] $\mu : SL_q(2,\mathbb{C}) \to SU_q(2)$.

Embedding the generators by $M_{1}{}^{a}{}_{b} \hookrightarrow M^{a}{}_{b} \otimes 1$ and $M_{2}{}^{a}{}_{b} \hookrightarrow 1 \otimes M^{a}{}_{b}$ in a tensor product of two $SL_q(2)$, the multiplication map

$$\mu : SL_q(2) \otimes SL_q(2) \to SU_q(2) \tag{1.25}$$

is the obvious choice. Note, that according to the preceding section $g \otimes h \in SL_q(2) \otimes SL_q(2)$ is to be equipped with the $*$-structure $(g \otimes h)^* := h^* \otimes g^*$, such that $\mu((g \otimes h)^*) = \mu(h^* \otimes g^*) = h^* g^* = (gh)^* = \mu(g \otimes h)^*$. In other words, μ is already compliant with the $*$-structures.

For μ to be a homomorphism of algebras, the images of the generators, $\mu(M_{1}{}^{a}{}_{b}) = \mu(M_{2}{}^{a}{}_{b}) = M^{a}{}_{b}$, must satisfy the $SL_q(2)$ commutation relations (1.9). This means that the generators have to satisfy

$$R^{ab}{}_{c'd'} M_{2}{}^{c'}{}_{c} M_{1}{}^{d'}{}_{d} = M_{1}{}^{b}{}_{b'} M_{2}{}^{a}{}_{a'} R^{a'b'}{}_{cd} \,, \tag{1.26}$$

[1]During the transition from groups to quantum groups the arrows of mappings have to be reversed.

which completes the algebraic structure of the q-Lorentz group.[2]

To summarize, let us give a compact and rigorous definition of the q-Lorentz group [21, 22]. First we need to define the so-called coquasitriangular map $R : SL_q(2) \otimes SL_q(2) \to \mathbb{C}$ on the generators by

$$R(M^a{}_c, M^b{}_d) := q^{-\frac{1}{2}} R^{ab}{}_{cd}\,, \tag{1.27}$$

which can be shown to extend to all of $SL_q(2)$ by linearity in both arguments and by demanding

$$R(fg, h) := R(f, h_{(1)})R(g, h_{(2)})\,, \qquad R(f, gh) := R(f_{(1)}, h)R(f_{(2)}, g)\,. \tag{1.28}$$

The factor $q^{-\frac{1}{2}}$ has been introduced for convenience. The map R has a unique convolution inverse, that is, a map $R^{-1} : SL_q(2) \otimes SL_q(2) \to \mathbb{C}$ with

$$R(a_{(1)}, b_{(1)})R^{-1}(a_{(2)}, b_{(2)}) = R^{-1}(a_{(1)}, b_{(1)})R(a_{(2)}, b_{(2)}) = \varepsilon(a)\varepsilon(b)\,, \tag{1.29}$$

simply defined by

$$R^{-1}(M^a{}_c, M^b{}_d) := q^{\frac{1}{2}} (R^{-1})^{ab}{}_{cd}\,. \tag{1.30}$$

Using R, the commutation relations of $SL_q(2)$ can be written as

$$R(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = b_{(1)}a_{(1)}R(a_{(2)}, b_{(2)})\,. \tag{1.31}$$

Definition 1. *Let R denote the coquasitriangular map of $SL_q(2)$ and R^{-1} its convolution inverse. The vector space $SL_q(2) \otimes SL_q(2)$ with tensor product coalgebra structure, $\Delta(g \otimes h) = (g_{(1)} \otimes h_{(1)}) \otimes (g_{(2)} \otimes h_{(2)})$, $\varepsilon(g \otimes h) = \varepsilon(g)\varepsilon(h)$, with multiplication*

$$(g \otimes h)(g' \otimes h') = g g'_{(2)} \otimes h_{(2)} h'\, R^{-1}(h_{(1)}, g'_{(1)}) R(h_{(3)}, g'_{(3)}) \tag{1.32}$$

antipode $S(g \otimes h) = (1 \otimes S(h))(S(g) \otimes 1)$, and $*$-structure

$$(g \otimes h)^{*_{SL_q(2,\mathbb{C})}} = h^{*_{SU_q(2)}} \otimes g^{*_{SU_q(2)}} \tag{1.33}$$

is the q-Lorentz group $SL_q(2, \mathbb{C})$.

1.3 The q-Lorentz Algebra as Dual of the q-Lorentz Group

For a symmetry of a quantum mechanical system the mathematical object with a direct physical interpretation is the enveloping algebra of the symmetry group's Lie algebra rather than the group itself. The Hilbert space representations of its generators are the observables of the conserved quantities corresponding to the symmetry. Consequently, rather than in the quantum group itself we are interested in its dual, the quantum enveloping algebra.

[2] If we drop the requirement of a subsymmetry of rotations, we can construct an alternative q-Lorentz group with two *commuting* copies of $SL_q(2)$. It turns out to be unphysical, however, insofar as the according q-Poincaré algebra has no mass Casimir.

1.3.1 $\mathcal{U}_q(\mathrm{su}_2)$ as dual of $SU_q(2)$

We will call two Hopf-$*$ algebras A and H dual to each other if there is a dual pairing [46] between them:

Definition 2. *Let A and H be Hopf-$*$ algebras. A non-degenerate bilinear map*

$$\langle\,\cdot\,,\,\cdot\,\rangle : A \times H \longrightarrow \mathbb{C}\,, \quad (a,h) \longmapsto \langle a,h\rangle \tag{1.34}$$

is called a dual pairing of A and H if it satisfies

$$\begin{aligned}
(i):&\quad \langle \Delta(a), g\otimes h\rangle = \langle a, gh\rangle\,, \quad \langle a\otimes b, \Delta(h)\rangle = \langle ab, h\rangle\\
(ii):&\quad \langle a, 1\rangle = \varepsilon(a)\,, \quad \langle 1, h\rangle = \varepsilon(h)\\
(iii):&\quad \langle a^*, h\rangle = \overline{\langle a, (Sh)^*\rangle}\,.
\end{aligned} \tag{1.35}$$

Remark that for property (i) we have to extend the dual pairing on tensor products by

$$\langle a\otimes b, g\otimes h\rangle := \langle a, g\rangle\langle b,h\rangle\,. \tag{1.36}$$

From the properties of the dual pairing it follows that

$$\langle S(a), h\rangle = \langle a, S(h)\rangle\,. \tag{1.37}$$

The following algebra is dual to $SU_q(2)$

Definition 3. *The algebra generated by E, F, K, and K^{-1} with commutation relations $KK^{-1} = K^{-1}K = 1$ and*

$$KE = q^2 EK\,, \qquad KF = q^{-2} FK\,, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,, \tag{1.38}$$

Hopf structure

$$\begin{aligned}
\Delta(E) &= E\otimes K + 1\otimes E\,, & S(E) &= -EK^{-1}\,, & \varepsilon(E) &= 0\\
\Delta(F) &= F\otimes 1 + K^{-1}\otimes F\,, & S(F) &= -KF\,, & \varepsilon(F) &= 0\\
\Delta(K) &= K\otimes K\,, & S(K) &= K^{-1}\,, & \varepsilon(K) &= 1
\end{aligned} \tag{1.39}$$

and $*$-structure

$$E^* = FK\,, \qquad F^* = K^{-1}E\,, \qquad K^* = K \tag{1.40}$$

is called $\mathcal{U}_q(\mathrm{su}_2)$, the q-deformation of the enveloping algebra $\mathcal{U}(\mathrm{su}_2)$ [48, 49].

The dual pairing of $\mathcal{U}_q(\mathrm{su}_2)$ and $SU_q(2)$ is defined on the generators by

$$\langle E, M^a{}_b\rangle := \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad \langle F, M^a{}_b\rangle := \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \langle K, M^a{}_b\rangle := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}. \tag{1.41}$$

There is a universal $\mathcal{R}$-matrix (Sec. A.3) for $\mathcal{U}_q(\mathrm{sl}_2)$ defined by the formal power series

$$\mathcal{R} = q^{(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(q)(E^n \otimes F^n) \tag{1.42}$$

where $R_n(q) := q^{n(n-1)/2}(q - q^{-1})^n([n]!)^{-1}$, and $K = q^H$ [12]. It is dual to the coquasitriangular map R of $SU_q(2)$ in the sense that

$$\langle \mathcal{R}, g \otimes h\rangle = R(g, h) \tag{1.43}$$

for all $g, h \in SU_q(2)$. This duality is the reason why we have introduced the factor $q^{-\frac{1}{2}}$ in the definition (1.27) of the coquasitriangular map R. We will sometimes write in a Sweedler like notation $\mathcal{R} = \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}$, where the subscripts stand for an index which is summed over.

1.3.2 Computing the Dual of the q-Lorentz Group

The map of the dual pairing of $\mathcal{U}_q(\mathrm{sl}_2)$ and $SL_q(2)$ naturally extends to a pairing of the tensor product spaces $\mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ and $SL_q(2, \mathbb{C}) \cong SL_q(2) \otimes SL_q(2)$ by

$$\langle a \otimes b, g \otimes h\rangle := \langle a, g\rangle\langle b, h\rangle \tag{1.44}$$

for all $a, b \in \mathcal{U}_q(\mathrm{sl}_2)$ and $g, h \in SL_q(2)$. By construction, this pairing is non-degenerate. We now want to define a Hopf algebra structure on $\mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ which turns this into a pairing of Hopf algebras. Firstly, the multiplication must satisfy

$$\begin{aligned} \langle (a \otimes b)(a' \otimes b'), g \otimes h\rangle &\overset{!}{=} \langle (a \otimes b) \otimes (a' \otimes b'), \Delta(g \otimes h)\rangle \\ &= \langle a \otimes a', \Delta(g)\rangle\langle b \otimes b', \Delta(h)\rangle = \langle aa', g\rangle\langle bb', h\rangle \\ &= \langle aa' \otimes bb', g \otimes h\rangle\,. \end{aligned} \tag{1.45}$$

Hence, the multiplication on the vector space $\mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ must be defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$, which means that as an algebra the dual of the q-Lorentz group is just the tensor algebra of two copies of $\mathcal{U}_q(\mathrm{sl}_2)$. Secondly, we want to define a coproduct that is consistent with the pairing.

$$\begin{aligned}
&\langle \Delta(a\otimes b),(g\otimes h)\otimes(g'\otimes h')\rangle \stackrel{!}{=} \langle a\otimes b,(g\otimes h)(g'\otimes h')\rangle\\
&\quad= \langle a\otimes b, gg'_{(2)}\otimes h_{(2)}h'\rangle\, R^{-1}(h_{(1)},g'_{(1)})R(h_{(3)},g'_{(3)})\\
&\quad= \langle \Delta(a), g\otimes g'_{(2)}\rangle\langle\Delta(b),h_{(2)}\otimes h'\rangle\langle\mathcal{R}^{-1},h_{(1)}\otimes g'_{(1)}\rangle\langle\mathcal{R},h_{(3)}\otimes g'_{(3)}\rangle\\
&\quad= \langle a_{(1)},g\rangle\langle\mathcal{R}^{-1}_{[1]},h_{(1)}\rangle\langle b_{(1)},h_{(2)}\rangle\langle\mathcal{R}_{[1]},h_{(3)}\rangle\\
&\qquad\times\langle\mathcal{R}^{-1}_{[2]},g'_{(1)}\rangle\langle a_{(2)},g'_{(2)}\rangle\langle\mathcal{R}_{[2]},g'_{(3)}\rangle\langle b_{(2)},h'\rangle\\
&\quad= \langle a_{(1)},g\rangle\langle\mathcal{R}^{-1}_{[1]}b_{(1)}\mathcal{R}_{[1]},h\rangle\langle\mathcal{R}^{-1}_{[2]}a_{(2)}\mathcal{R}_{[2]},g'\rangle\langle b_{(2)},h'\rangle\\
&\quad= \langle(a_{(1)}\otimes\mathcal{R}^{-1}_{[1]}b_{(1)}\mathcal{R}_{[1]})\otimes(\mathcal{R}^{-1}_{[2]}a_{(2)}\mathcal{R}_{[2]}\otimes b_{(2)}),(g\otimes h)\otimes(g'\otimes h')\rangle
\end{aligned}\tag{1.46}$$

From the last line we read off the coproduct

$$\Delta(a\otimes b)=\mathcal{R}^{-1}_{23}\Delta^{\otimes 2}(a\otimes b)\mathcal{R}_{23}\,,\tag{1.47}$$

where $\mathcal{R}_{23}=1\otimes\mathcal{R}\otimes 1$ and $\Delta^{\otimes 2}(a\otimes b)=(a_{(1)}\otimes b_{(1)})\otimes(a_{(2)}\otimes b_{(2)})$. This tells us, that the coproduct of the q-Lorentz algebra is the tensor coproduct of $\mathcal{U}_q(\mathrm{sl}_2)\otimes\mathcal{U}_q(\mathrm{sl}_2)$ with the two inner tensor factors twisted by the universal $\mathcal{R}$-matrix.

Thirdly, the same reasoning for the antipode

$$\begin{aligned}
&\langle S(a\otimes b),g\otimes h\rangle \stackrel{!}{=} \langle a\otimes b,S(g\otimes h)\rangle = \langle a\otimes b,(1\otimes Sh)(Sg\otimes 1)\rangle\\
&\quad= \langle a\otimes b,(Sg)_{(2)}\otimes(Sh)_{(2)}\rangle\, R^{-1}\big((Sh)_{(1)},(Sg)_{(1)}\big)R\big((Sh)_{(3)},(Sg)_{(3)}\big)\\
&\quad= \langle a,S(g_{(2)})\rangle\langle b,S(h_{(2)})\rangle\, R^{-1}\big(h_{(3)},g_{(3)}\big)R\big(h_{(1)},g_{(1)}\big)\\
&\quad= \langle\mathcal{R}_{[2]},g_{(1)}\rangle\langle S(a),g_{(2)}\rangle\langle\mathcal{R}^{-1}_{[2]},g_{(3)}\rangle\langle\mathcal{R}_{[1]},h_{(1)}\rangle\langle S(b),h_{(2)}\rangle\langle\mathcal{R}^{-1}_{[1]},h_{(3)}\rangle\\
&\quad= \langle\mathcal{R}_{[2]}S(a)\mathcal{R}^{-1}_{[2]},g\rangle\langle\mathcal{R}_{[1]}S(b)\mathcal{R}^{-1}_{[1]},h\rangle\\
&\quad= \langle\mathcal{R}_{[2]}S(a)\mathcal{R}^{-1}_{[2]}\otimes\mathcal{R}_{[1]}S(b)\mathcal{R}^{-1}_{[1]},g\otimes h\rangle
\end{aligned}\tag{1.48}$$

leads to

$$S(a\otimes b)=\mathcal{R}_{21}(Sa\otimes Sb)\mathcal{R}^{-1}_{21}\,,\tag{1.49}$$

where $\mathcal{R}_{21}=\mathcal{R}_{[2]}\otimes\mathcal{R}_{[1]}$. The antipode is the tensor antipode twisted by the transposed universal R-matrix.

The counit $\varepsilon(a\otimes b)=\varepsilon(a)\varepsilon(b)$ follows directly from the definition of the pairing. Finally, we need to calculate the star structure.

$$\begin{aligned}
&\langle(a\otimes b)^*,g\otimes h\rangle \stackrel{!}{=} \overline{\langle a\otimes b,S((g\otimes h)^*)\rangle} = \overline{\langle S(a\otimes b),h^*\otimes g^*\rangle}\\
&\quad= \overline{\langle\mathcal{R}_{[2]}S(a)\mathcal{R}^{-1}_{[2]},h^*\rangle\langle\mathcal{R}_{[1]}S(b)\mathcal{R}^{-1}_{[1]},g^*\rangle}\\
&\quad= \langle\mathcal{R}_{[1]}a^*\mathcal{R}^{-1}_{[1]},h\rangle\langle\mathcal{R}_{[2]}b^*\mathcal{R}^{-1}_{[2]},g\rangle
\end{aligned}\tag{1.50}$$

Here we have used that $\mathcal{R}$ is real, $\mathcal{R}^{*\otimes *}=\mathcal{R}_{21}$. Thus, we find

$$(a\otimes b)^*=\mathcal{R}_{21}(b^*\otimes a^*)\mathcal{R}^{-1}_{21}\,,\tag{1.51}$$

which completes the structure of the q-Lorentz algebra.

To summarize, we have the following

Proposition 1. *The tensor product algebra* $\mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ *with the Hopf-$*$-structure*

$$\begin{aligned} \Delta(a \otimes b) = \mathcal{R}_{23}^{-1} \Delta^{\otimes 2}(a \otimes b) \mathcal{R}_{23}\,, \quad S(a \otimes b) = \mathcal{R}_{21}(Sa \otimes Sb)\mathcal{R}_{21}^{-1} \\ \varepsilon(a \otimes b) = \varepsilon(a)\varepsilon(b)\,, \quad (a \otimes b)^* = \mathcal{R}_{21}(b^* \otimes a^*)\mathcal{R}_{21}^{-1} \end{aligned} \tag{1.52}$$

is the Hopf-$*$-dual of the q-Lorentz group $SL_q(2,\mathbb{C})$. Therefore, we will call it the q-Lorentz algebra $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ *[42].*

There are two universal $\mathcal{R}$-matrices of the q-Lorentz algebra, which are composed of the $\mathcal{R}$-matrix of $\mathcal{U}_q(\mathrm{sl}_2)$ according to

$$\mathcal{R}_{\mathrm{I}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}\,, \qquad \mathcal{R}_{\mathrm{II}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{13}\mathcal{R}_{24}\mathcal{R}_{23}\,. \tag{1.53}$$

$\mathcal{R}_{\mathrm{I}}$ is anti-real while $\mathcal{R}_{\mathrm{II}}$ is real.

Chapter 2

Structure of the q-Lorentz Algebra

2.1 Representation Theory of the q-Lorentz Algebra

2.1.1 The Clebsch-Gordan Series of $\mathcal{U}_q(\mathrm{sl}_2)$

Let us review some facts about the representation theory of $\mathcal{U}_q(\mathrm{sl}_2)$ and $\mathcal{U}_q(\mathrm{su}_2)$ [50]. For any $j \in \frac{1}{2}\mathbb{N}_0$ there is an irreducible representation on the $(2j+1)$-dimensional Hilbert space D^j with orthonormal basis $\{|j,m\rangle \,|\, m = -j, -j+1, \dots j\}$ and representation map $\rho^j : \mathcal{U}_q(\mathrm{sl}_2) \to \mathrm{Aut}(D^j)$ given by[1]

$$\begin{aligned}
\rho^j(E)|j,m\rangle &= q^{(m+1)}\sqrt{[j+m+1][j-m]}\,|j,m+1\rangle \\
\rho^j(F)|j,m\rangle &= q^{-m}\sqrt{[j+m][j-m+1]}\,|j,m-1\rangle \\
\rho^j(K)|j,m\rangle &= q^{2m}|j,m\rangle\,.
\end{aligned} \tag{2.1}$$

For the real form $\mathcal{U}_q(\mathrm{su}_2)$ these are even $*$-representations. D^0 is called the scalar representation, $D^{\frac{1}{2}}$ the fundamental or spinor representation, and D^1 the vector representation.

Recall that the coproduct of a Hopf algebra enables us to construct the tensor product of two representations: Let D^j and $D^{j'}$ be representations of $\mathcal{U}_q(\mathrm{sl}_2)$ as defined above, with representation maps ρ^j and $\rho^{j'}$. Then there is a representation on the tensor product space $D^j \otimes D^{j'}$ with representation map $(\rho^j \otimes \rho^{j'}) \circ \Delta$. We denote this tensor product of representations also by $D^j \otimes D^{j'}$.

In general, the tensor product of two irreducible representation is no longer irreducible. In fact, in complete analogy to the classical case we have an isomorphism of representations

$$D^j \otimes D^{j'} \cong D^{|j-j'|} \oplus D^{|j-j'|+1} \oplus \dots \oplus D^{j+j'} \tag{2.2}$$

decomposing the tensor product into the Clebsch-Gordan series. This isomorphism, viewed as a transformation of basis

$$|j,m\rangle = \sum_{j_1,j_2,m_1,m_2} \mathrm{C}_q(j_1,j_2,j\,|\,m_1,m_2,m)\,|j_1,m_1\rangle \otimes |j_2,m_2\rangle \tag{2.3}$$

[1] There is a second series of irreducible representations with negative eigenvalues of K, which we will not take into account, since they have no undeformed limit.

defines the q-Clebsch-Gordan coefficients, which can be calculated in a closed form (Sec. A.1.1). The two most important cases are the construction of a scalar and the construction of a vector out of two vector representations, where the right hand side of Eq. (2.3) may be viewed as the scalar and the vector product of two 3-vectors.

2.1.2 Clebsch-Gordan Coefficients of the q-Lorentz Algebra

As an algebra the q-Lorentz algebra is the tensor product of two $\mathcal{U}_q(\mathrm{sl}_2)$. Hence, every finite irreducible representation is composed of two irreducible representations D^{j_1} and D^{j_2} of $\mathcal{U}_q(\mathrm{sl}_2)$, that is, the vector space $D^{(j_1,j_2)} := D^{j_1} \otimes D^{j_2}$ with the representation map $\rho^{(j_1,j_2)} := \rho^{j_1} \otimes \rho^{j_2}$. Viewing the decomposition of the q-Lorentz algebra into two $\mathcal{U}_q(\mathrm{sl}_2)$ as chiral decomposition, we call D^{j_1} the left handed and D^{j_2} the right handed part of the representation. $D^{(j_1,j_2)}$ is not a $*$-representation, since the $*$-operation of the q-Lorentz algebra is not the tensor product of the $*$'s of $\mathcal{U}_q(\mathrm{sl}_2)$. Therefore, all finite irreducible representations are non-unitary. This is a sign of the non-compactness of the q-Lorentz algebra on a representation theoretic level.

Next, we consider the tensor product of two representations. Again, its vector space is just the tensor product $D^{(j_1,j_2)} \otimes D^{(j'_1,j'_2)}$. The representation map is again $\rho = (\rho^{(j_1,j_2)} \otimes \rho^{(j'_1,j'_2)}) \circ \Delta$, where Δ is now the coproduct of the q-Lorentz algebra as defined in Eq. (1.52). The coproduct is calculated by, firstly, taking the $\mathcal{U}_q(\mathrm{sl}_2)$ coproduct of the two $\mathcal{U}_q(\mathrm{sl}_2)$ tensor factors, then interchanging the 2. and 3. tensor factor, and, finally, conjugating with the universal $\mathcal{R}$-matrix in the 2. and 3. position of the 4-fold tensor product. Algebraically, the last step is a complicated inner automorphism, since $\mathcal{R}$ exists only as an infinite formal power series. However, when we apply the representation maps, $\mathcal{R}$ becomes a finite $(j_2 j'_1) \times (j_2 j'_1)$ matrix $R = (R^{ab}{}_{cd})$

$$\rho^{(j_1,j_2)} \otimes \rho^{(j'_1,j'_2)}(\mathcal{R}_{23}) = 1 \otimes \big((\rho^{j_2} \otimes \rho^{j'_1})(\mathcal{R})\big) \otimes 1 =: 1 \otimes R \otimes 1\,, \tag{2.4}$$

and the inner automorphism becomes a simple basis transformation.

Putting things together, we see how to reduce the product of two q-Lorentz representations. Up to a change of basis we reduce the tensor product of the 1. with the 3. and the 2. with the 4. of the $\mathcal{U}_q(\mathrm{sl}_2)$-subrepresentations, each by means of the Clebsch-Gordan series of $\mathcal{U}_q(\mathrm{sl}_2)$.

$$D^{(j_1,j_2)} \otimes D^{(j'_1,j'_2)} \cong \bigoplus_{\substack{|j_1-j'_1|\le k_1\le j_1+j'_1\\ |j_2-j'_2|\le k_2\le j_2+j'_2}} D^{(k_1,k_2)} \tag{2.5}$$

Written out for the important case of the product of two vector representations, this formula becomes

$$D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})} \cong D^{(0,0)} \oplus D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1,1)}\,, \tag{2.6}$$

which corresponds to the decomposition of a 4×4 matrix viewed as a second rank Lorentz tensor into the scalar trace part, a left and a right chiral 3-vector, and the traceless symmetric part (Sec. A.1.3).

So far, the representation theory is in complete accordance with the undeformed case. New is the appearance of an R-matrix, which matters as soon as we want to write down the above isomorphisms explicitly. The matrix representing isomorphism 2.5 is the product of two Clebsch-Gordan coefficients contracted with the R-matrix. Musing for a while about the right positions of the indices, we find

$$\begin{aligned} |(k_1,k_2),(n_1,n_2)\rangle = \sum \mathrm{C}_q(j_1,j_1',k_1\,|\,m_1,b,n_1)\mathrm{C}_q(j_2,j_2',k_2\,|\,a,m_2',n_2) \\ \times (R^{-1})^{m_2 m_1'}{}_{ab}\,|(j_1,j_2),(m_1,m_2)\rangle \otimes |(j_1',j_2'),(m_1',m_2')\rangle\,, \end{aligned} \tag{2.7}$$

where we sum over repeated indices, and where the labeling of the free indices is the same as in Eq. (2.5). This defines the Clebsch-Gordan coefficients of the q-Lorentz algebra

$$\begin{aligned} \left[\begin{array}{ccc|ccc} j_1 & j_1' & k_1 & m_1 & m_1' & n_1 \\ j_2 & j_2' & k_2 & m_2 & m_2' & n_2 \end{array}\right]_q := \\ \sum_{a,b} \mathrm{C}_q(j_1,j_1',k_1\,|\,m_1,b,n_1)\mathrm{C}_q(j_2,j_2',k_2\,|\,a,m_2',n_2)(R^{-1})^{m_2 m_1'}{}_{ab}\,. \end{aligned} \tag{2.8}$$

2.2 Tensor Operators

2.2.1 Tensor Operators in Hopf Algebras

Recall that there is a left and right action of any Hopf algebra H on itself given by

$$\mathrm{ad_L}(g) \triangleright h := g_{(1)} h S(g_{(2)})\,, \qquad h \triangleleft \mathrm{ad_R}(g) := S(g_{(1)}) h g_{(2)} \tag{2.9}$$

for $g, h \in H$, called the left and right Hopf adjoint action, respectively. In general, this action will be highly reducible. In fact, if a linearly independent set $\{A_\mu \in H\}$ of operators generates an invariant subspace D of the left Hopf adjoint action, this induces a matrix representation map ρ of H by

$$\mathrm{ad_L}(h) \triangleright A_\mu = A_{\mu'} \rho(h)^{\mu'}{}_\mu\,, \tag{2.10}$$

turning D into a representation. The set of operators $\{A_\mu\}$ with this property is called a left D-tensor operator of H, indicated by a lower index. It will be called irreducible if D is irreducible. If in addition H is equipped with a $*$-operation, we can demand that D is a $*$-representation.

There are other useful types of tensor operators. If a set of operators $\{A^\mu\}$ transforms as

$$(\mathrm{ad_L} h) \triangleright A^\mu = \rho(Sh)^\mu{}_{\mu'} A^{\mu'}\,, \tag{2.11}$$

we will call it a left upper or congredient tensor operator, denoted by an upper index. Its transformation is congredient in the sense that

$$\begin{aligned}(\mathrm{ad_L}h) \triangleright (A_\mu B^\mu) &= [(\mathrm{ad_L}h_{(1)}) \triangleright A_\mu][(\mathrm{ad_L}h_{(2)}) \triangleright B^\mu] \\ &= A_{\mu'}\rho(h_{(1)})^{\mu'}{}_\mu\, \rho\big(S(h_{(1)})\big)^\mu{}_{\mu''} B^{\mu''} = \varepsilon(h)\, A_\mu B^\mu\,, \end{aligned} \tag{2.12}$$

that is, $A_\mu B^\mu$ is a scalar operator. If $g^{\mu\nu}$ is a metric for the representation under consideration and A_μ and B_ν are left tensor operator then $g^{\mu\nu}A_\mu B_\nu$ is a scalar. This is true for the q-spinor metric ε^{ab}, the metric g^{AB} of vector representations of $\mathcal{U}_q(\mathrm{su}_2)$ and the q-Minkowski metric $\eta^{\mu\nu}$, as defined in Eqs. (1.10), (2.23), and (3.16), respectively. We conclude, that the convention for the position of tensor operator indices is consistent with raising and lowering indices as usual, $A^\mu = g^{\mu\mu'}A_{\mu'}$. Moreover, we conclude that

$$g^{\mu\mu'}g_{\nu'\nu}\rho(h)^{\nu'}{}_{\mu'} = \rho(Sh)^\mu{}_\nu\,. \tag{2.13}$$

If we deal with a Hopf-$*$-algebra and ρ is a $*$-representation, we can apply $*$ to Eq. (2.10) and get

$$\begin{aligned}(*\circ S)(h_{(2)})(A_\mu)^*(h_{(1)})^* &= \big[((Sh)^*)_{(1)}\big](A_\mu)^* S\big[((Sh)^*)_{(2)}\big] = (\mathrm{ad_L}(Sh)^*) \triangleright (A_\mu)^* \\ &= (A_{\mu'})^*\overline{\rho(h)^{\mu'}{}_\mu} = (A_{\mu'})^*\rho(h^*)^\mu{}_{\mu'} = (A_{\mu'})^*\rho(S[(Sh)^*])^\mu{}_{\mu'}\,, \end{aligned} \tag{2.14}$$

from which we deduce

$$(\mathrm{ad_L}(Sh)^*) \triangleright (A_\mu)^* = \rho(S[(Sh)^*])^\mu{}_{\mu'}\,(A_{\mu'})^*\,. \tag{2.15}$$

Comparing this with Eq. (2.11), we conclude that $(A_\mu)^*$ is a congredient left tensor operator.

Let us now consider tensor operators $A^{\tilde\mu}$ with respect to the right Hopf-adjoint action

$$A^{\tilde\mu} \triangleleft (\mathrm{ad_R}h) = S(h_{(1)})A^{\tilde\mu}h_{(2)} = \rho(h)^\mu{}_{\mu'}\,A^{\tilde\mu'}\,, \tag{2.16}$$

which we call *right* upper tensor operators, distinguished from left upper tensor operators by putting a tilde over their indices.

Let A^μ be a left upper tensor operator and let there be an extension of the antipode of H on A^μ, for example, A^μ might be an element of H. Then we can apply S to Eq. (2.11) and obtain

$$\begin{aligned}S(S(h_{(2)}))S(A^\mu)S(h_{(1)}) &= S((Sh)_{(1)}))S(A^\mu)(Sh)_{(2)}) \\ &= S(A^\mu) \triangleleft (\mathrm{ad_R}Sh) = \rho(Sh)^\mu{}_{\mu'}\,S(A^{\mu'})\,. \end{aligned} \tag{2.17}$$

Thus, $S(A^\mu)$ is a right upper tensor operator.

Note, that within a Lie algebra we would have $S(A^\mu) = -A^\mu$. Hence, in a Lie algebra a right tensor operator is the same as a left tensor operator. This is why in the undeformed case we need not distinguish between indices with and without a tilde.

Finally, we define a right lower tensor operator $A_{\tilde{\mu}}$ to transform as $S(A_\mu)$, that is,

$$A_{\tilde{\mu}} \triangleleft (\mathrm{ad_R} h) = A_{\tilde{\mu}'} \rho(S^{-1}h)^{\mu'}{}_{\mu} \,. \tag{2.18}$$

One can check that we can raise and lower indices as usual, $A_{\tilde{\mu}} = g_{\mu\nu} A^{\tilde{\nu}}$, and that $A^{\tilde{\mu}} B_{\tilde{\mu}}$ is a scalar operator. Note that being a left or a right scalar is the same thing: A scalar is an operator that commutes with H.

2.2.2 Tensor Operators of $\mathcal{U}_q(\mathrm{su}_2)$

Most tensor operators of $\mathcal{U}_q(\mathrm{su}_2)$ that we will consider are D^0-tensor operators, which will be called $\mathcal{U}_q(\mathrm{sl}_2)$-scalars, and D^1-tensor operators, called $\mathcal{U}_q(\mathrm{sl}_2)$-vectors. One big advantage of grouping several operators to a $\mathcal{U}_q(\mathrm{sl}_2)$-tensor operator lies in the q-Wigner-Eckart theorem:

Theorem 1. *Let A_μ be a left D^λ-tensor operator of $\mathcal{U}_q(\mathrm{su}_2)$ and let there be a representation of $\mathcal{U}_q(\mathrm{su}_2)$ with irreducible subrepresentations D^j and $D^{j'}$ with bases $\{|j,m\rangle\}$ and $\{|j',m'\rangle\}$. Then there exists a number $\langle j'\|A\|j\rangle$ such that*

$$\langle j',m'|A_\mu|j,m\rangle = \mathrm{C}_q(\lambda,j,j' \,|\, \mu,m,m')\langle j'\|A\|j\rangle \tag{2.19}$$

for all m, m'. This number is called the reduced matrix element of the tensor operator A_μ [51].

If we have degeneracy of the $|j,m\rangle$ basis, the reduced matrix elements will depend on additional quantum numbers but not on m. Whenever a q-Clebsch-Gordan coefficient $\mathrm{C}_q(\lambda,j,j' \,|\, \mu,m,m')$ vanishes for all m, m', the reduced matrix element is not defined uniquely. In that case we set $\langle j'\|A\|j\rangle := 0$ for convenience.

Looking at the definition (2.9), we see that $\mathrm{ad_L}(g) \triangleright (hh') = (\mathrm{ad_L}(g_{(1)}) \triangleright h)(\mathrm{ad_L}(g_{(2)}) \triangleright h')$. Hence, the product of a D- and a D'-tensor operator is a $D \otimes D'$-tensor operator. Just as for the representations of $\mathcal{U}_q(\mathrm{sl}_2)$ we have a Clebsch-Gordan decomposition of the product of tensor operators:

Proposition 2. *Let A_α be a D^a-tensor operator and B_β a D^b-tensor operator of $\mathcal{U}_q(\mathrm{sl}_2)$. Then*

$$C_\gamma := \sum_{\alpha,\beta} \mathrm{C}_q(a,b,c \,|\, \alpha,\beta,\gamma) A_\alpha B_\beta \tag{2.20}$$

is a D^c-tensor operator of $\mathcal{U}_q(\mathrm{sl}_2)$.

If we now take the matrix elements of a tensor operator C_γ constructed in this way, we find with the aid of the q-Wigner-Eckart theorem relations between the reduced matrix elements

$$\langle j''\|C\|j\rangle = \sum_{j'} \mathrm{R}_q(a,b,j\,|\,c,j',j'')\langle j''\|A\|j'\rangle\langle j'\|B\|j\rangle\,. \tag{2.21}$$

Here R_q denote the q-Racah coefficients defined by the expression

$$\begin{aligned}\mathrm{R}_q(a,b,j\,|\,c,j',j'') &:= \mathrm{C}_q(c,j,j''\,|\,\gamma,m,m'')^{-1}\\ &\times \sum_{\alpha,\beta,m'} \mathrm{C}_q(a,b,c\,|\,\alpha,\beta,\gamma)\mathrm{C}_q(a,j,j''\,|\,\alpha,m',m'')\mathrm{C}_q(b,j,j'\,|\,\beta,m,m')\,,\end{aligned} \tag{2.22}$$

which can be proven not to depend on m, m'' as the arguments of R_q indicate. Values of the q-Racah coefficients are given in Sec. A.1.1.

The two cases of Eq. (2.20) that we encounter most frequently are the construction of a scalar and the construction of a vector operator out of two vector operators. This suggests the definition

$$g^{AB} := -\sqrt{[3]}\mathrm{C}_q(1,1,0\,|\,A,B,0)\,,\quad \varepsilon^{AB}{}_C = -\sqrt{\frac{[4]}{[2]}}\mathrm{C}_q(1,1,1\,|\,A,B,C)\,, \tag{2.23}$$

where the capital Roman indices run through $\{-1,0,1\} = \{-,3,+\}$. Values are given in Sec. A.1.2. Proposition 2 tells us that we can define a scalar and a vector product of two vector operators X_A and Y_A by

$$\vec{X}\cdot\vec{Y} := X_A Y_B\, g^{AB}\,, \qquad (\vec{X}\times\vec{Y})_C := \mathrm{i}X_A Y_B\,\varepsilon^{AB}{}_C\,, \tag{2.24}$$

where the imaginary unit is needed for the right undeformed limit.[2] By definition, the scalar product is a scalar and the vector product is a vector operator in the sense of Eq. (2.10).

2.2.3 The Vector Form of $\mathcal{U}_q(\mathrm{su}_2)$

For a set of operators to be interpreted as q-angular momentum, it will have to generate the symmetry of rotations on the one hand, but on the other hand it will itself have to transform like a vector under rotations. In other words, this set must be a vector operator generating $\mathcal{U}_q(\mathrm{su}_2)$. In the EFK-form of $\mathcal{U}_q(\mathrm{sl}_2)$ it is not obvious, what this vector operator could be.

We begin our search for such a vector operator by giving the explicit conditions for A_μ to be a irreducible D^j-tensor operator of $\mathcal{U}_q(\mathrm{sl}_2)$: Inserting Eqs. (2.1) in

[2] See Sec. A.1.2, in particular the remark above Eq. (A.16).

Eq. (2.10) we get

$$\begin{aligned} EA_\mu - A_\mu E &= q^{(\mu+1)}\sqrt{[j+\mu+1][j-\mu]}\ A_{\mu+1}K \\ FA_\mu - q^{-2\mu}A_\mu F &= q^{-\mu}\sqrt{[j+\mu][j-\mu+1]}\ A_{\mu-1} \\ KA_\mu &= q^{2\mu}A_\mu K\,. \end{aligned} \tag{2.25}$$

To find a vector operator in $\mathcal{U}_q(\mathrm{su}_2)$ satisfying these conditions we first look for a highest weight vector J_+ and let $\mathcal{U}_q(\mathrm{sl}_2)$ act on it by the left Hopf-adjoint action, giving us the subrepresentation generated by J_+. The condition $\mathrm{ad}_\mathrm{L}(E) \triangleright J_+ = 0$ for J_+ to be a highest weight vector is equivalent to $[E, J_+] = 0$. Thus, J_+ must be in the centralizer of E, a very restrictive condition most obviously satisfied by E itself. The results of the Hopf-adjoint action of the ladder operators E and F on E are

$$\begin{gathered} \mathrm{ad}_\mathrm{L}(F)\triangleright E = K^{-1}(KFE - EKF)\,, \quad \mathrm{ad}_\mathrm{L}(F^2)\triangleright E = -[2]KF \\ ,\quad \mathrm{ad}_\mathrm{L}(F^3)\triangleright E = 0\,, \quad \mathrm{ad}_\mathrm{L}(EF)\triangleright E = [2]E \\ \mathrm{ad}_\mathrm{L}(EF^2)\triangleright E = [2]K^{-1}(KFE - EKF)\,, \end{gathered} \tag{2.26}$$

which shows that we can indeed interpret E as a highest weight vector of a vector representation. Comparing the Hopf-adjoint action with the vector representation as given in Eqs. (2.1), one finds that

$$\begin{aligned} J_- &:= q[2]^{-\frac{1}{2}}KF \\ J_3 &:= -q[2]^{-1}K^{-1}(KFE - EKF) = [2]^{-1}(q^{-1}EF - qFE) \\ J_+ &:= -[2]^{-\frac{1}{2}}E \end{aligned} \tag{2.27}$$

form a vector operator.[3]

How can we describe the subalgebra of $\mathcal{U}_q(\mathrm{su}_2)$ generated by J_A? After some calculations we find that the commutation relations of the J's do not close. Since the commutation relations (2.10) are given by the adjoint action of the set of generators on itself, this is due to the fact that coproduct and antipode of the J's cannot be expressed by J's again. We can help ourselves out by introducing the additional generator

$$W := K - \lambda J_3 = K - \lambda[2]^{-1}(q^{-1}EF - qFE)\,, \tag{2.28}$$

so the commutation relations can be written as

$$J_A J_B \varepsilon^{AB}{}_C = WJ_C\,, \qquad J_A W = WJ_A\,, \qquad W^2 - \lambda^2 J_A J_B g^{AB} = 1\,, \tag{2.29}$$

where the last equation expresses that W and the J's are not algebraically independent. The $*$-structure reads on the generators

$$J_+^* = -qJ_-\,, \qquad J_3^* = J_3\,, \qquad W^* = W\,, \tag{2.30}$$

[3]Elsewhere [52], the vector generators have been defined as $L^A = -q^{-3}J_A$.

that is, $(J_A)^* = J^A$. We will call the subalgebra of $\mathcal{U}_q(\mathrm{sl}_2)$ generated by J_A, W with relations (2.29) and $*$-structure (2.30) the vectorial form of $\mathcal{U}_q(\mathrm{su}_2)$. Note that the vectorial form of $\mathcal{U}_q(\mathrm{sl}_2)$ is a proper subalgebra of $\mathcal{U}_q(\mathrm{sl}_2)$ since it does not contain K^{-1}. We do need K^{-1} to write down the Hopf structure: the coproduct

$$\begin{aligned}\Delta(J_\pm) &= J_\pm \otimes K + 1 \otimes J_\pm \\ \Delta(J_3) &= J_3 \otimes K + K^{-1} \otimes J_3 + \lambda(qK^{-1}J_+ \otimes J_- + q^{-1}K^{-1}J_- \otimes J_+) \\ \Delta(W) &= W \otimes K - \lambda K^{-1} \otimes J_3 - \lambda^2(qK^{-1}J_+ \otimes J_- + q^{-1}K^{-1}J_- \otimes J_+)\,,\end{aligned} \tag{2.31}$$

the antipode

$$\begin{aligned}S(J_\pm) &= -J_\pm K^{-1} \\ S(J_3) &= J_3 - \lambda^{-1}(K - K^{-1}) \\ S(W) &= W\,,\end{aligned} \tag{2.32}$$

and the counit $\varepsilon(J_A) = 0$, $\varepsilon(W) = 1$.

2.3 The q-Lorentz Algebra as Quantum Double

2.3.1 Rotations and the $SU_q(2)^{\mathrm{op}}$ Algebra of Boosts

In Sec. 1.2.2 the commutation relations of the q-Lorentz group have been chosen to preserve an $SU_q(2)$ substructure, physically interpreted as rotations. That is, the multiplication of the two copies of $SL_q(2)$ is a Hopf-$*$-homomorphism projecting the q-Lorentz group onto $SU_q(2)$. On the quantum algebra level, the dual of multiplication is comultiplication. Hence, the mapping

$$i : \mathcal{U}_q(\mathrm{su}_2) \xrightarrow{\Delta} \mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2) = \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C})) \tag{2.33}$$

ought to define a $\mathcal{U}_q(\mathrm{su}_2)$ Hopf-$*$-subalgebra of the q-Lorentz algebra.

Given the properties of the coproduct, it is obvious that i is an algebra homomorphism. It is less clear, whether i preserves the Hopf structure and the $*$-structure of $\mathcal{U}_q(\mathrm{su}_2)$. For the coproducts we find

$$\begin{aligned}(\Delta_{\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))} \circ i)(h) &= \mathcal{R}_{23}^{-1}(h_{(1)} \otimes h_{(3)} \otimes h_{(2)} \otimes h_{(4)})\mathcal{R}_{23} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \otimes h_{(4)} \\ &= \big((i \otimes i) \circ \Delta_{\mathcal{U}_q(\mathrm{sl}_2)}\big)(h)\,,\end{aligned} \tag{2.34}$$

which shows that i is a coalgebra map. In the same manner we find that i preserves the counit (trivial), the antipode

$$\begin{aligned}(S_{\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))} \circ i)(h) &= \mathcal{R}_{21}\big(S(h_{(1)}) \otimes S(h_{(2)})\big)\mathcal{R}_{21}^{-1} = \mathcal{R}_{21}\big((Sh)_{(2)} \otimes (Sh)_{(1)}\big)\mathcal{R}_{21}^{-1} \\ &= (Sh)_{(1)} \otimes (Sh)_{(2)} = (i \circ S_{\mathcal{U}_q(\mathrm{sl}_2)})(h)\,,\end{aligned} \tag{2.35}$$

and the $*$-structure

$$\begin{aligned}(i(h))^* &= \mathcal{R}_{21}\big((h_{(2)})^* \otimes (h_{(1)})^*\big)\mathcal{R}_{21}^{-1} = \mathcal{R}_{21}\big((h^*)_{(2)} \otimes (h^*)_{(1)}\big)\mathcal{R}_{21}^{-1} \\ &= (h^*)_{(1)} \otimes (h^*)_{(2)} = i(h^*)\,.\end{aligned} \tag{2.36}$$

We conclude that $i(\mathcal{U}_q(\mathrm{su}_2))$ is indeed a $\mathcal{U}_q(\mathrm{su}_2)$ Hopf-$*$ subalgebra of the q-Lorentz algebra.[4] Since in the undeformed case the embedding of the rotations in the Lorentz algebra is given by the coproduct, too, $i(\mathcal{U}_q(\mathrm{su}_2))$ has the right undeformed limit. This strongly suggests to interpret $i(\mathcal{U}_q(\mathrm{su}_2))$ as the quantum subsymmetry of physical rotations.

There is another Hopf-$*$ subalgebra of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. Let $\langle\ ,\ \rangle$ denote the dual pairing of $\mathcal{U}_q(\mathrm{sl}_2)$ and $SL_q(2)$ as defined in Sec. 1.3.1. We define a map $j : SU_q(2) \to \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ by

$$j(h) := \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, h_3 \rangle\,, \tag{2.37}$$

where the subscripts denote the position in the tensor product, $h_3 := 1 \otimes 1 \otimes h$, and where the dual pairing acts only on the third tensor factor. Let us show some properties of this map. we have

$$\begin{aligned} j(gh) &= \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, g_3 h_3 \rangle = \langle \Delta_3(\mathcal{R}_{31}^{-1}\mathcal{R}_{23}), g_3 h_4 \rangle = \langle \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}, g_3 h_4 \rangle \\ &= \langle \mathcal{R}_{41}^{-1}\mathcal{R}_{24}\mathcal{R}_{31}^{-1}\mathcal{R}_{23}, g_3 h_4 \rangle = j(h) j(g)\,, \end{aligned} \tag{2.38}$$

telling us that j is an algebra anti-homomorphism. Next we consider the coproduct

$$\begin{aligned} (\Delta_{\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))} \circ j)(h) &= \langle \Delta \otimes \mathrm{id}\ (\mathcal{R}_{31}^{-1}\mathcal{R}_{23}), h_3 \rangle = \langle \mathcal{R}_{23}^{-1} R_{51}^{-1} \mathcal{R}_{53}^{-1}\mathcal{R}_{25}\mathcal{R}_{45}\mathcal{R}_{23}, h_5 \rangle \\ &= \langle \mathcal{R}_{51}^{-1}\mathcal{R}_{25}\mathcal{R}_{53}^{-1}\mathcal{R}_{45}, h_5 \rangle = \big((j \otimes j) \circ \Delta_{SU_q(2)}\big)(h)\,, \end{aligned} \tag{2.39}$$

so j is a coalgebra homomorphism, too. The calculation for the counit is trivial.

So far we can say that j is a bialgebra homomorphism from $SU_q(2)^{\mathrm{op}}$ to the q-Lorentz algebra. $SU_q(2)^{\mathrm{op}}$ becomes a Hopf algebra, when we equip it with a antipode and $*$-structure according to

$$S^{\mathrm{op}} := S^{-1}\,, \qquad\qquad *^{\mathrm{op}} := * \circ S^2\,, \tag{2.40}$$

where S is the usual antipode of $SU_q(2)$. Let us check now if j preserves this Hopf structure as well. We begin with the antipode

$$\begin{aligned} (j \circ S^{\mathrm{op}})(h) &= \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, S^{-1}(h_3) \rangle = \langle \mathcal{R}_{23}^{-1}\mathcal{R}_{31}, h_3 \rangle = \langle \mathcal{R}_{21}\mathcal{R}_{31}\mathcal{R}_{23}^{-1}\mathcal{R}_{21}^{-1}, h_3 \rangle \\ &= \langle \mathcal{R}_{21}[(S \otimes S \otimes \mathrm{id})(\mathcal{R}_{31}^{-1}\mathcal{R}_{23})]\mathcal{R}_{21}^{-1}, h_3 \rangle = (S_{\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))} \circ j)(h)\,, \end{aligned} \tag{2.41}$$

[4]It is the appearance of the $\mathcal{R}$-matrices in the Hopf structure of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$, which ensures the compliance of the embedding i with the Hopf structures. This is why $\mathcal{U}_q(\mathrm{so}_4)$ does not possess a Hopf subalgebra of rotations.

which is indeed preserved. Finally, we have the $*$-structure

$$
\begin{aligned}
j\big(h^{*^{\mathrm{op}}}\big) &= \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, (S^2 h_3)^* \rangle = \overline{\langle [\mathrm{id}\otimes\mathrm{id}\otimes(S^2\circ *\circ S)](\mathcal{R}_{31}^{-1}\mathcal{R}_{23}), h_3\rangle} \\
&= \overline{\langle [\mathrm{id}\otimes\mathrm{id}\otimes(*\circ S^{-1})](\mathcal{R}_{31}^{-1}\mathcal{R}_{23}), h_3\rangle} = \overline{\langle (*\otimes *\otimes\mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{32}^{-1}), h_3\rangle} \\
&= \overline{\langle (*\otimes *\otimes\mathrm{id})(\mathcal{R}_{12}^{-1}\mathcal{R}_{32}^{-1}\mathcal{R}_{13}\mathcal{R}_{12}), h_3\rangle} \\
&= \overline{\langle \mathcal{R}_{21}[(*\otimes *\otimes\mathrm{id})(\mathcal{R}_{32}^{-1}\mathcal{R}_{13})]\mathcal{R}_{21}^{-1}, h_3\rangle} = \mathcal{R}_{21}\,\langle \mathcal{R}_{32}^{-1}\,\mathcal{R}_{13}, h_3\rangle^{(*\otimes *)}\,\mathcal{R}_{21}^{-1} \\
&= \big(j(h)\big)^* .
\end{aligned} \tag{2.42}
$$

We conclude that j is a Hopf-$*$ algebra homomorphism from $SU_q(2)^{\mathrm{op}}$ to the q-Lorentz algebra. Hence, $j(SU_q(2)^{\mathrm{op}})$ is indeed a Hopf-$*$ subalgebra of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. We will call it the subalgebra of the boosts.

2.3.2 *L*-Matrices and the Explicit Form of the Boost Algebra

To calculate the explicit form of the algebra of boosts we introduce the computational tool of L-Matrices [16]. Let ρ^j be the representation map of the D^j-representation of $\mathcal{U}_q(\mathrm{sl}_2)$. We define matrices of generators by applying ρ^j to one tensor factor of the universal $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{R}_{[1]}\otimes\mathcal{R}_{[2]}$,

$$
(L^j_+)^a{}_b := \mathcal{R}_{[1]}\rho^j(\mathcal{R}_{[2]})^a{}_b\,, \qquad (L^j_-)^a{}_b := \rho^j(\mathcal{R}_{[1]}^{-1})^a{}_b\,\mathcal{R}_{[2]}^{-1}\,. \tag{2.43}
$$

Here, we need the L-matrices for $j=\frac{1}{2}$, where we get

$$
L_+^{\frac{1}{2}} = \begin{pmatrix} K^{-\frac{1}{2}} & q^{-\frac{1}{2}}\lambda K^{-\frac{1}{2}}E \\ 0 & K^{\frac{1}{2}} \end{pmatrix}, \qquad L_-^{\frac{1}{2}} = \begin{pmatrix} K^{\frac{1}{2}} & 0 \\ -q^{\frac{1}{2}}\lambda F K^{\frac{1}{2}} & K^{-\frac{1}{2}} \end{pmatrix} \tag{2.44}
$$

with respect to the basis $\{-,+\}$. The appearance of the square roots of K comes from the fact that $\mathcal{R}$ only exists as formal power series.

We can derive some properties of the L-matrices from the properties of $\mathcal{R}$: Applying $\mathrm{id}\otimes\rho^j\otimes\rho^j$ to the quantum Yang-Baxter equation $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$ we obtain

$$
(L^j_+)^a{}_{c'}(L^j_+)^d{}_{d'}R^{c'd'}{}_{cd} = R^{ab}{}_{a'b'}(L^j_+)^{b'}{}_d(L^j_+)^{a'}{}_c \tag{2.45}
$$

and in an analogous manner

$$
\begin{aligned}
(L^j_-)^a{}_{c'}(L^j_-)^d{}_{d'}R^{c'd'}{}_{cd} &= R^{ab}{}_{a'b'}(L^j_-)^{b'}{}_d(L^j_-)^{a'}{}_c \\
(L^j_-)^a{}_{c'}(L^j_+)^d{}_{d'}R^{c'd'}{}_{cd} &= R^{ab}{}_{a'b'}(L^j_+)^{b'}{}_d(L^j_-)^{a'}{}_c\,.
\end{aligned} \tag{2.46}
$$

From the coproduct properties $(\Delta\otimes\mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$, $(\mathrm{id}\otimes\Delta)(\mathcal{R}^{-1}) = \mathcal{R}_{12}^{-1}\mathcal{R}_{13}^{-1}$ and from $(\varepsilon\otimes\mathrm{id})(\mathcal{R}) = 1 = (\mathrm{id}\otimes\varepsilon)(\mathcal{R}^{-1})$ it follows that

$$
\Delta\big((L^j_\pm)^a{}_c\big) = (L^j_\pm)^a{}_b\otimes(L^j_\pm)^b{}_c\,, \qquad \varepsilon\big((L^j_\pm)^a{}_b\big) = \delta^a_b\,. \tag{2.47}
$$

Finally, we apply $\mathrm{id} \otimes \rho^j \otimes \mathrm{id}$ to the form $\mathcal{R}_{13}^{-1}\mathcal{R}_{23}^{-1}\mathcal{R}_{12}\mathcal{R}_{13} = \mathcal{R}_{12}\mathcal{R}_{23}^{-1}$ of the Yang-Baxter equation in order to get

$$\mathcal{R}^{-1}\,(L_+^j)^b{}_c \otimes (L_-^j)^a{}_b\,\mathcal{R} = (L_+^j)^a{}_b \otimes (L_-^j)^b{}_c\,. \tag{2.48}$$

Now, we can compute the explicit form of the boosts. Observing that the dual pairing of $SU_q(2)$ and $\mathcal{U}_q(\mathrm{su}_2)$ (Sec. 1.3.1) can be expressed on the matrix $M^a{}_b$ of generators of $SU_q(2)$ by $\langle h, M^a{}_b\rangle = \rho^{\frac{1}{2}}(h)^a{}_b$, we get for the boost generators

$$\begin{aligned} B^a{}_c &:= j(M^a{}_c) = \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, 1 \otimes 1 \otimes M^a{}_c\rangle = (\mathcal{R}_{[2]}^{-1} \otimes \mathcal{R}_{[1']})\rho^{\frac{1}{2}}(\mathcal{R}_{[1]}^{-1}\mathcal{R}_{[2']})^a{}_c \\ &= \mathcal{R}_{[2]}^{-1}\rho^{\frac{1}{2}}(\mathcal{R}_{[1]}^{-1})^a{}_b \otimes \mathcal{R}_{[1']}\rho^{\frac{1}{2}}(\mathcal{R}_{[2']})^b{}_c = \left(L_-^{\frac{1}{2}}\right)^a{}_b \otimes \left(L_+^{\frac{1}{2}}\right)^b{}_c\,, \end{aligned} \tag{2.49}$$

explicitly,

$$B^a{}_b = \begin{pmatrix} K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & q^{-\frac{1}{2}}\lambda K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \\ -q^{\frac{1}{2}}\lambda F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & K^{-\frac{1}{2}} \otimes K^{\frac{1}{2}} - \lambda^2 F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.50}$$

The commutation relations are

$$\begin{gathered} ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd \\ bc = cb, \quad da - ad = (q - q^{-1})bc, \quad da - qbc = 1\,. \end{gathered} \tag{2.51}$$

The coproduct, $\Delta(B^a{}_c) = B^a{}_b \otimes B^b{}_c$, is the same as for $SU_q(2)$ just as the counit, $\varepsilon(B^a{}_b) = \delta^a_b$. For the antipode we had $S^{\mathrm{op}} = S^{-1}$ and for the $*$-structure $*^{\mathrm{op}} := * \circ S^2$. Since $(M^a{}_b)^* = S(M^b{}_a)$, it follows that $(M^a{}_b)^{*^{\mathrm{op}}} = S^{\mathrm{op}}(M^b{}_a)$ in $SU_q(2)^{\mathrm{op}}$ and, consequently, the unitarity condition $(B^a{}_b)^* = S(B^b{}_a)$ holds in $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ as well. Written out this is

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix}. \tag{2.52}$$

If we want to verify that the $B^a{}_b$ are the generators of a $SU_q(2)^{\mathrm{op}}$ subalgebra using the definition of the q-Lorentz algebra only, we find that this is extremely tedious.

2.3.3 Commutation Relations between Boosts and Rotations

Now, we have to figure out the commutation relations between rotations and boost, embedded into $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ by the maps i and j, respectively. For $l \in \mathcal{U}_q(\mathrm{su}_2)$

and $h \in SU_q(2)^{\mathrm{op}}$ the embedding is

$$
\begin{aligned}
j(h)i(l) &= \langle \mathcal{R}_{31}^{-1}\mathcal{R}_{23}, h_3\rangle (l_{(1)} \otimes l_{(2)}) \\
&= \mathcal{R}_{[2]}^{-1} l_{(1)} \otimes \mathcal{R}_{[1']} l_{(2)} \,\langle \mathcal{R}_{[1]}^{-1}\mathcal{R}_{[2']} l_{(3)} S(l_{(4)}), h\rangle \\
&= \mathcal{R}_{[2]}^{-1} l_{(1)} \otimes l_{(3)} \mathcal{R}_{[1']} \,\langle \mathcal{R}_{[1]}^{-1} l_{(2)} \mathcal{R}_{[2']} S(l_{(4)}), h\rangle \\
&= l_{(2)} \mathcal{R}_{[2]}^{-1} \otimes l_{(3)} \mathcal{R}_{[1']} \,\langle l_{(1)} \mathcal{R}_{[1]}^{-1} \mathcal{R}_{[2']} S(l_{(4)}), h\rangle \\
&= l_{(2)} \mathcal{R}_{[2]}^{-1} \otimes l_{(3)} \mathcal{R}_{[1']} \,\langle l_{(1)}, h_{(1)}\rangle \langle \mathcal{R}_{[1]}^{-1} \mathcal{R}_{[2']}, h_{(2)}\rangle \langle S(l_{(4)}), h_{(3)}\rangle \\
&= \langle l_{(1)}, h_{(1)}\rangle \, i(l_{(2)}) j(h_{(2)}) \,\langle S(l_{(3)}), h_{(3)}\rangle \,.
\end{aligned} \tag{2.53}
$$

The commutation relations which can be read off this equation are precisely the ones of the quantum double [12, 13]. For the generators they write out

$$
\begin{aligned}
B^a{}_b E &= E B^a{}_{a'} \rho^{\frac{1}{2}} (K^{-1})^{a'}{}_b + K \rho^{\frac{1}{2}} (E)^a{}_{a'} B^{a'}{}_{b'} \rho^{\frac{1}{2}} (K^{-1})^{b'}{}_b - B^a{}_{a'} \rho^{\frac{1}{2}} (E K^{-1})^{a'}{}_b \\
B^a{}_b F &= F \rho^{\frac{1}{2}} (K^{-1})^{a'}{}_{a'} B^{a'}{}_b - K^{-1} \rho^{\frac{1}{2}} (K^{-1})^a{}_{a'} B^{a'}{}_{b'} \rho^{\frac{1}{2}} (K F)^{b'}{}_b + \rho^{\frac{1}{2}} (F)^a{}_{a'} B^{a'}{}_b \\
B^a{}_b K &= \rho^{\frac{1}{2}} (K)^a{}_{a'} B^{a'}{}_{b'} \rho^{\frac{1}{2}} (K^{-1})^{b'}{}_b \,.
\end{aligned} \tag{2.54}
$$

Explicitly, this gives us

$$
\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix} E = \begin{pmatrix} qEa - q^{\frac{3}{2}} b & q^{-1} E b \\ qEc + q^{\frac{3}{2}} K a - q^{\frac{3}{2}} d & q^{-1} E d + q^{-\frac{1}{2}} K b \end{pmatrix} \\
&\begin{pmatrix} a & b \\ c & d \end{pmatrix} F = \begin{pmatrix} qFa + q^{-\frac{1}{2}} c & qFb - q^{-\frac{1}{2}} K^{-1} a + q^{-\frac{1}{2}} d \\ q^{-1} F c & q^{-1} F d - q^{-\frac{5}{2}} K^{-1} c \end{pmatrix} \\
&\begin{pmatrix} a & b \\ c & d \end{pmatrix} K = K \begin{pmatrix} a & q^{-2} b \\ q^2 c & d \end{pmatrix} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} K^{-1} = K^{-1} \begin{pmatrix} a & q^2 b \\ q^{-2} c & d \end{pmatrix} .
\end{aligned} \tag{2.55}
$$

We summarize:

Definition 4. *The Hopf-$*$ algebra generated by $SU_q(2)^{\mathrm{op}}$ and $\mathcal{U}_q(\mathrm{su}_2)$ with cross commutation relations*

$$
hl = \langle l_{(1)}, h_{(1)}\rangle \, l_{(2)} h_{(2)} \,\langle S(l_{(3)}), h_{(3)}\rangle \tag{2.56}
$$

or, equivalently,

$$
lh = \langle S(l_{(1)}), h_{(1)}\rangle \, h_{(2)} l_{(2)} \,\langle l_{(3)}, h_{(3)}\rangle \tag{2.57}
$$

for $h \in SU_q(2)^{\mathrm{op}}$ and $l \in \mathcal{U}_q(\mathrm{su}_2)$, is the quantum double form of the q-Lorentz algebra [21].

Finally, if we want to invert the embedding $i \otimes j : \mathcal{U}_q(\mathrm{su}_2) \otimes SU_q(2)^{\mathrm{op}} \to \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ we find

$$
\begin{aligned}
E \otimes 1 &= q K^{-\frac{1}{2}} (Ea - q^{\frac{3}{2}} \lambda^{-1} b) \,, & 1 \otimes E &= q^{\frac{1}{2}} \lambda^{-1} a^{-1} b \\
F \otimes 1 &= -q^{-\frac{1}{2}} \lambda^{-1} c a^{-1} \,, & 1 \otimes F &= q K^{\frac{1}{2}} (Fa + q^{-\frac{1}{2}} \lambda^{-1} c) \\
K \otimes 1 &= K^{\frac{1}{2}} a \,, & 1 \otimes K &= K^{\frac{1}{2}} a^{-1}
\end{aligned} \tag{2.58}
$$

For these expressions to make sense we had to add the generator a^{-1} to $SU_q(2)^{\mathrm{op}}$ and $K^{\pm\frac{1}{2}}$ to $\mathcal{U}_q(\mathrm{su}_2)$. From the viewpoint of representation theory this modification seems to be insignificant.

2.4 The Vectorial Form of the q-Lorentz Algebra

2.4.1 Tensor Operators of the q-Lorentz Algebra

The definition of tensor operators in Eq. (2.10) has been general. We just have to work it out for the q-Lorentz algebra. We begin by calculating for $g \otimes h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$

$$\begin{aligned}(g \otimes h)_{(1)} \otimes S\big((g \otimes h)_{(2)}\big) &= (g_{(1)} \otimes \mathcal{R}^{-1}_{[1]} h_{(1)} \mathcal{R}_{[1']}) \otimes S(\mathcal{R}^{-1}_{[2]} g_{(2)} \mathcal{R}_{[2']} \otimes h_{(2)}) \\ &= (g_{(1)} \otimes \mathcal{R}^{-1}_{[1]} h_{(1)} \mathcal{R}_{[1']}) \otimes (\mathcal{R}_{[2'']} S(\mathcal{R}^{-1}_{[2]} g_{(2)} \mathcal{R}_{[2']}) \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1'']} S(h_{(2)}) \mathcal{R}^{-1}_{[1''']}) \\ &= (g_{(1)} \otimes \mathcal{R}_{[1]} h_{(1)} \mathcal{R}_{[1']}) \otimes (\mathcal{R}_{[2'']} S(\mathcal{R}_{[2']}) S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1'']} S(h_{(2)}) \mathcal{R}^{-1}_{[1''']})\,,\end{aligned} \tag{2.59}$$

where in the last step we have used that $(\mathrm{id} \otimes S)(\mathcal{R}^{-1}) = \mathcal{R}$. Hence, for $T_{\mu\nu} = \sum_n A^n_{\mu\nu} \otimes B^n_{\mu\nu}$ (no summation of μ and ν) to be a $D^{(i,j)}$-tensor operator of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$

$$\begin{aligned}& T_{\mu'\nu'} \rho^i(g)^{\mu'}{}_{\mu} \rho^j(h)^{\nu'}{}_{\nu} = \mathrm{ad}_{\mathrm{L}}(g \otimes h) \triangleright (T_{\mu\nu}) \\ &= \sum_n g_{(1)} A^n_{\mu\nu} \mathcal{R}_{[2'']} S(\mathcal{R}_{[2']}) S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1]} h_{(1)} \mathcal{R}_{[1']} B^n_{\mu\nu} \mathcal{R}_{[1'']} S(h_{(2)}) \mathcal{R}^{-1}_{[1''']}\end{aligned} \tag{2.60}$$

must hold for all $g \otimes h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$.

Some tensor operators of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ can be derived from tensor operators of $\mathcal{U}_q(\mathrm{su}_2)$: If A_μ is a D^j-tensor operator of $\mathcal{U}_q(\mathrm{su}_2)$ then $A_\mu \otimes 1$ is a $D^{(j,0)}$-tensor operator. We check this by inserting $T_{\mu\nu} = A_\mu \otimes 1$ in the last equation:

$$\begin{aligned}&\mathrm{ad}_{\mathrm{L}}(g \otimes h) \triangleright (A_\mu \otimes 1) \\ &\quad = g_{(1)} A_\mu S(\mathcal{R}_{[2']} \mathcal{R}^{-1}_{[2'']}) S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1]} h_{(1)} (\mathcal{R}_{[1']} \mathcal{R}^{-1}_{[1'']}) S(h_{(2)}) \mathcal{R}^{-1}_{[1''']} \\ &\quad = g_{(1)} A_\mu S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1]} h_{(1)} S(h_{(2)}) \mathcal{R}^{-1}_{[1''']} = g_{(1)} A_\mu S(g_{(2)}) \otimes \varepsilon(h) \\ &\quad = (A_{\mu'} \otimes 1)\, \rho^j(g)^{\mu'}{}_{\mu}\, \rho^0(h)\,.\end{aligned} \tag{2.61}$$

In the same manner we verify that $\mathcal{R}_{21}(1 \otimes A_\mu)\mathcal{R}^{-1}_{21}$ is a $D^{(0,j)}$-tensor operator:

$$\begin{aligned}&\mathrm{ad}_{\mathrm{L}}(g \otimes h) \triangleright \mathcal{R}_{21}(1 \otimes A_\mu)\mathcal{R}^{-1}_{21} \\ &\quad = g_{(1)} \mathcal{R}_{[2'']} S(\mathcal{R}_{[2']}) S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1]} h_{(1)} \mathcal{R}_{[1']} \mathcal{R}_{[1'']} A_\mu S(h_{(2)}) \mathcal{R}^{-1}_{[1''']} \\ &\quad = g_{(1)} S(g_{(2)}) \mathcal{R}_{[2]} \mathcal{R}^{-1}_{[2''']} \otimes \mathcal{R}_{[1]} h_{(1)} A_\mu S(h_{(2)}) \mathcal{R}^{-1}_{[1''']} \\ &\quad = \mathcal{R}_{21}(1 \otimes h_{(1)} A_\mu S(h_{(2)}))\mathcal{R}^{-1}_{21} \varepsilon(g) \\ &\quad = \mathcal{R}_{21}(1 \otimes A_{\mu'})\mathcal{R}^{-1}_{21}\, \varepsilon(g) \rho^j(h)^{\mu'}{}_{\mu}\,.\end{aligned} \tag{2.62}$$

2.4.2 The Vectorial Generators

Now, it is obvious how we can define vectorial generators of the q-Lorentz algebra. Let J_A be the vector generator of $\mathcal{U}_q(\mathrm{su}_2)$ as defined in Eqs. (2.27). We define[5]

$$S_A := J_A \otimes 1\,, \qquad\qquad R_A := \mathcal{R}_{21}(1 \otimes J_A)\mathcal{R}_{21}^{-1}\,. \tag{2.63}$$

From the last section it is obvious that S_A is a $D^{(1,0)}$-tensor and R_A is a $D^{(0,1)}$-tensor operator, that is, a left and right chiral vector operator, respectively. Moreover, both R_A and S_A are vector operators with respect to rotations since $D^{(1,0)}$ and $D^{(0,1)}$ induce a D^1 vector representation of the $\mathcal{U}_q(\mathrm{su}_2)$ subalgebra.

We can raise the indices with the 3-metric of $\mathcal{U}_q(\mathrm{su}_2)$ introduced in Eq. (2.23), $S^A = g^{AB}S_B$, giving us a congredient vector operator,

$$\begin{aligned}\mathrm{ad_L}(g \otimes h) \triangleright S^A &= \mathrm{ad_L}(g \otimes h) \triangleright (J_{A'} \otimes 1)g^{AA'} \\ &= (J^B \otimes 1)g^{AA'}g_{B'B}\,\rho^j(g)^{B'}{}_{A'}\,\varepsilon(h) \\ &= S^B \rho^j(Sg)^A{}_B\,\varepsilon(Sh)\,,\end{aligned} \tag{2.64}$$

and the same for R_A. By looking at the definition of the $*$-structure of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ we immediately see that

$$(R_A)^* = S^A\,. \tag{2.65}$$

For the commutation relations of the algebra generated by R_A and S_A to close we yet have to embed the Casimir operator W of the vectorial form of $\mathcal{U}_q(\mathrm{su}_2)$, as defined in Eq. (2.28), in the q-Lorentz algebra, that is[6]

$$V := W \otimes 1\,, \qquad\qquad U := \mathcal{R}_{21}(1 \otimes W)\mathcal{R}_{21}^{-1} = 1 \otimes W\,. \tag{2.66}$$

By construction the commutation relations of the R's and U among each other are the same as for the L's and W as given in Eqs. (2.29). The same holds for the S's and V since these generators are embedded by an inner automorphism. To calculate the commutation relations of R_A with S_B we first note that commuting $\mathcal{R}_{21}$ with $1 \otimes J_A$ shows us that

$$R_A = \mathcal{R}_{[2]} \otimes J_{A'}\rho^1(\mathcal{R}_{[1]})^{A'}{}_A\,. \tag{2.67}$$

Then we commute this expression with S_A

$$\begin{aligned}R_A S_B &= \mathcal{R}_{[2]}J_B \otimes J_{A'}\rho^1(\mathcal{R}_{[1]})^{A'}{}_A \\ &= J_{B'}\rho^1(\mathcal{R}_{[2]})^{B'}{}_B\mathcal{R}_{[2']} \otimes J_{A'}\rho^1(\mathcal{R}_{[1']}\mathcal{R}_{[1]})^{A'}{}_A \\ &= S_{B'}R_{A'}\rho^1(\mathcal{R}_{[1]})^{A'}{}_A\rho^1(\mathcal{R}_{[2]})^{B'}{}_B\,.\end{aligned} \tag{2.68}$$

[5]The operators R and S defined here correspond to the operators $q^2[2]R$ and $-q^2[2]S$ of [44].
[6]The operator V defined here corresponds to U' in [44].

The representation of the universal $\mathcal{R}$-matrix appearing on the last line is proportional to the R-matrix of $SO_q(3)$, defined in Eq. (A.56). The RS-commutation relations can now be written as

$$R_A S_B = q^2 S_{B'} R_{A'}\, R_{\mathrm{so}_3}^{A'B'}{}_{AB}\,, \tag{2.69}$$

where R_{so_3} is given explicitly in Eq. (A.58). We summarize

Definition 5. *The algebra generated by R_A, U, S_A, V, where A runs through $\{-,+,3\}$, with relations*

$$R_A R_B \varepsilon^{AB}{}_C = U R_C\,, \qquad R_A U = U R_A\,, \qquad U^2 - \lambda^2 g^{AB} R_A R_B = 1 \tag{2.70a}$$

$$S_A S_B \varepsilon^{AB}{}_C = V S_C\,, \qquad S_A V = V S_A\,, \qquad V^2 - \lambda^2 g^{AB} S_A S_B = 1 \tag{2.70b}$$

$$R_C S_D = q^2 S_C R_D - q^{-1}\lambda\, g_{CD}(g^{AB} S_A R_B) + \varepsilon_C{}^X{}_D \varepsilon^{AB}{}_X\, S_A R_B \tag{2.70c}$$

$$R^A V = V R^A\,, \qquad UV = VU\,, \qquad S^A U = U S^A \tag{2.70d}$$

and $*$-structure

$$R_A^* = g^{AB} S_B\,, \qquad\qquad U^* = V \tag{2.70e}$$

is called the vectorial or RS-form of the q-Lorentz algebra [44].

2.4.3 Relations with the other Generators

Let us first express the vectorial generators R_A and S_A by the original generators of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. For S_A and V the case is simple. We merely have to look up the expressions for J_A and W in Eqs. (2.27) and (2.28). For completeness we write them down once more

$$\begin{aligned} S_- &:= q[2]^{-\frac{1}{2}} KF \otimes 1 \\ S_3 &:= [2]^{-1}(q^{-1}EF - qFE) \otimes 1 \\ S_+ &:= -[2]^{-\frac{1}{2}} E \otimes 1 \\ V &:= [K - \lambda[2]^{-1}(q^{-1}EF - qFE)] \otimes 1\,. \end{aligned} \tag{2.71}$$

For R_A one might at first sight expect formal power series, but as we have shown in the preceding section

$$R_A = \mathcal{R}_{[2]} \otimes L_{A'}\, \rho^1(\mathcal{R}_{[1]})^{A'}{}_A = S\big[(L^1_-)^{A'}{}_A\big] \otimes J_{A'}\,. \tag{2.72}$$

We only have to sum over the L^1_--matrix of $\mathcal{U}_q(\mathrm{su}_2)$, which has been computed in Eq. (A.62) where we get

$$S\big[(L^1_-)^A{}_B\big] = \begin{pmatrix} K^{-1} & 0 & 0 \\ \lambda[2]^{\frac{1}{2}} F & 1 & 0 \\ q^2\lambda^2 K F^2 & q\lambda[2]^{\frac{1}{2}} KF & K \end{pmatrix} \tag{2.73}$$

with respect to the $\{-1,0,1\} = \{-,3,+\}$ basis, so the expressions for the R's become

$$\begin{aligned}
R_- &= q[2]^{-\frac{1}{2}} K^{-1} \otimes KF + \lambda [2]^{-\frac{1}{2}} F \otimes (q^{-1}EF - qFE) \\
&\quad - q^2\lambda^2 [2]^{-\frac{1}{2}} KF^2 \otimes E \\
R_3 &= 1 \otimes [2]^{-1}(q^{-1}EF - qFE) - q\lambda KF \otimes E \\
R_+ &= -[2]^{-\frac{1}{2}} K \otimes E \\
U &= 1 \otimes [K - \lambda[2]^{-1}(q^{-1}EF - qFE)]\,.
\end{aligned} \tag{2.74}$$

Next, let us express R_A and S_A by the generators of the quantum double form of the q-Lorentz algebra. For S_A we find

$$\begin{aligned}
S_- &= -q^{-\frac{1}{2}}\lambda^{-1}[2]^{-\frac{1}{2}} K^{\frac{1}{2}} c \\
S_3 &= q^{-\frac{3}{2}}\lambda^{-1}[2]^{-1} K^{-\frac{1}{2}}(qcE - Ec) \\
&= \lambda^{-1}[2]^{-1} K^{-\frac{1}{2}}(q^{-\frac{1}{2}}\lambda Ec + qKa - qd) \\
S_+ &= q[2]^{-\frac{1}{2}}\, K^{-\frac{1}{2}}(q^{\frac{3}{2}}\lambda^{-1} b - Ea) \\
V &= [2]^{-1} K^{-\frac{1}{2}}(q^{-1}Ka - q^{-\frac{1}{2}}\lambda Ec + qd)\,.
\end{aligned} \tag{2.75}$$

To compute the corresponding expressions for R_A we remember that $S^*_- = -q^{-1}R_+$, $S^*_3 = R_3$, and $S^*_+ = -qR_-$. With the $*$-structure of rotations and boosts as given in Eqs. (2.30) and (2.52) this yields

$$\begin{aligned}
R_- &= [2]^{-\frac{1}{2}}\, K^{-\frac{1}{2}}(Fd + q^{-\frac{5}{2}}\lambda^{-1} c) \\
R_3 &= q^{-\frac{1}{2}}\lambda^{-1}[2]^{-1} K^{\frac{1}{2}}(bF - q^3 Fb) \\
&= \lambda^{-1}[2]^{-1} K^{\frac{1}{2}}(-q^{\frac{3}{2}}\lambda Fb - q^{-1}K^{-1}a + q^{-1}d) \\
R_+ &= -q^{\frac{1}{2}}\lambda^{-1}[2]^{-\frac{1}{2}} K^{\frac{1}{2}} b \\
U &= [2]^{-1} K^{\frac{1}{2}}(q^{\frac{3}{2}}\lambda Fb + q^{-1}K^{-1}a + qd)\,.
\end{aligned} \tag{2.76}$$

We also want to express the generators of boosts and rotations within the RS-algebra. For the vectorial generators of the rotations we find [31, 52]

$$\begin{aligned}
J_C &= VR_C + US_C + q\lambda R_A S_B\, \varepsilon^{AB}{}_C \\
W &= UV + q^2\lambda^2 g^{AB}\, R_A S_B\,.
\end{aligned} \tag{2.77}$$

While this yields an expression of $K = W + \lambda J_3$, K^{-1} is not a member of the RS-algebra proper. We must add $K^{-\frac{1}{2}}$ by hand to the RS-algebra to write down expressions for the boosts

$$a = K^{-\frac{1}{2}}(V + \lambda S_3)\,, \qquad b = -q^{-\frac{1}{2}}\lambda[2]^{\frac{1}{2}} K^{-\frac{1}{2}} R_+ \tag{2.78a}$$

$$c = -q^{\frac{1}{2}}\lambda[2]^{\frac{1}{2}} K^{-\frac{1}{2}} S_-\,, \qquad d = K^{-\frac{1}{2}}(U + \lambda R_3)\,. \tag{2.78b}$$

Chapter 3

Algebraic Structure of the q-Poincaré Algebra

3.1 The q-Poincaré Algebra

3.1.1 Construction of the q-Minkowski-Space Algebra

As in the undeformed case, we want to construct the coordinate functions of Minkowski space to form a matrix $X_{a\dot{b}}$ with a lower undotted and dotted index. For the cotransformations to be compliant with the $*$-structure the $*$ has to act on $X_{a\dot{b}}$ as on a product $\phi_a \psi_{\dot{b}}$, that is, $(X_{a\dot{b}})^* := X_{b\dot{a}}$. For our purposes it is more convenient to work with the index structure ${X_a}^{\dot{b}}$,

$$ {X_a}^{\dot{b}} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad\qquad ({X_a}^{\dot{b}})^* = \begin{pmatrix} -qD & B \\ C & -q^{-1}A \end{pmatrix}, \tag{3.1} $$

with respect to the $\{-,+\}$ basis. With this index structure the cotransformation is[1]

$$ \rho_{\mathrm{R}}({X_a}^{\dot{b}}) = {X_{a'}}^{\dot{b}'} \otimes ({M^{a'}}_a \otimes {M^{b'}}_b) \,. \tag{3.2} $$

Upon dualizing, this right coaction of the q-Lorentz group becomes a left $D^{(\frac{1}{2},\frac{1}{2})}$ action of the q-Lorentz algebra.

We want to construct the space algebra out of the algebra $\mathbb{C}\langle {X_a}^{\dot{b}}\rangle$ freely generated by the generators ${X_a}^{\dot{b}}$ divided by some relations. The generators have the dimension of a length, so we need homogeneous relations,[2] which for the correct undeformed limit have to be of second order. We demand the resulting quotient algebra to be a q-Lorentz module algebra.

This last requirement means that the quadratic terms that will be set zero must be the basis of a q-Lorentz submodule. For only if we divide the free module $\mathbb{C}\langle {X_a}^{\dot{b}}\rangle$ by an ideal generated by a submodule, the quotient will be a module again. The vector space generated by ${X_a}^{\dot{b}}{X_c}^{\dot{d}}$ naturally forms a $D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})}$ representation of the q-Lorentz algebra. By the Clebsch-Gordan-Series (2.6) this representation has the same four subrepresentations as in the undeformed case. To obtain the correct undeformed limit where the space-functions commute, it is

[1] Recall, that we think of the dot as belonging to X rather than to the index itself.

[2] For inhomogeneous relations we would need to introduce an additional dimensional parameter.

the submodules $D^{(1,0)}$ and $D^{(0,1)}$ that have to be set zero. The bases of those two submodules as computed in Eqs. (A.23) and (A.24) yield the relations

$$\begin{aligned}
0 &= qBA - q^{-1}AB \\
0 &= DA - AD + \lambda BB + BC - q^{-2}CB \\
0 &= DC - CD + \lambda DB \\
0 &= CA - AC + \lambda BA \\
0 &= DA - AD + \lambda BB + CB - q^{-2}BC \\
0 &= qDB - q^{-1}BD\,,
\end{aligned} \tag{3.3}$$

which can be written more compactly as

$$\begin{gathered}
AB = q^2 BA\,, \qquad BD = q^2 DB\,, \qquad BC = CB \\
AC - CA = \lambda BA\,, \qquad CD - DC = \lambda DB \\
AD - DA = \lambda B(B + q^{-1}C)\,.
\end{gathered} \tag{3.4}$$

Now we can give the definition of the q-Minkowski-Space Algebra.

Definition 6. *The $*$-algebra generated by $\{A, B, C, D\}$ with $*$-structure as in Eq.* (3.1) *and commutation relations* (3.4) *is called the q-Minkowski-Space algebra $\mathcal{M}_q$ [20].*

The basis vector of the $D^{(0,0)}$ submodule yields a q-Lorentz scalar, that corresponds to the invariant quadratic length, X^2, of Minkowski space. Up to normalization we get from Eq. (A.25)

$$X^2 := [2]^{-1}(qDA + q^{-1}AD - q^{-1}BC - q^{-1}CB - q^{-1}\lambda BB)\,, \tag{3.5}$$

which can be simplified with the commutation relations (3.4) to

$$X^2 = DA - q^{-2}BC\,. \tag{3.6}$$

It turns out that this expression commutes with all generators of $\mathcal{M}_q$. Hence, it can be viewed as the length Casimir of q-Minkowski space or, within a momentum representation, as mass Casimir of the q-Poincaré algebra.

3.1.2 4-Vectors and the *q*-Pauli Matrices

We have constructed the q-Lorentz algebra to possess a $\mathcal{U}_q(\mathrm{su}_2)$ Hopf-$*$ subalgebra, viewed as the algebra of rotations. Hence, we are able to write $X_a{}^{\dot{b}}$ in a manifest 4-vector form, that is, split up its 4 degrees of freedom with respect to rotations into a scalar and a 3-vector.

The $D^{(\frac{1}{2},\frac{1}{2})}$ representation induces a representation on the subalgebra of rotations. To compute the representation map ρ of the latter we have to embed $\mathcal{U}_q(\mathrm{su}_2)$ with $i = \Delta$ and then apply the representation map $\rho^{(\frac{1}{2},\frac{1}{2})}$ yielding

$$\rho = (\rho^{\frac{1}{2}} \otimes \rho^{\frac{1}{2}}) \circ \Delta\,. \tag{3.7}$$

In other words, this induced representation is simply the tensor representation $D^{\frac{1}{2}} \otimes D^{\frac{1}{2}}$ which reduces according to the Clebsch-Gordan series

$$D^{\frac{1}{2}} \otimes D^{\frac{1}{2}} \cong D^0 \oplus D^3 \tag{3.8}$$

to the direct sum of a scalar and a vector representation. Explicitly, this reduction of $X_a{}^{\dot{b}}$ into a 4-vector is expressed by the q-Clebsch-Gordan coefficients,

$$X_0 = q^{-1}[2]^{-\frac{1}{2}} \mathrm{C}_q(\tfrac{1}{2}, \tfrac{1}{2}, 0 \,|\, a, b, 0) X_a{}^{\dot{b}}, \quad X_C = [2]^{-\frac{1}{2}} \mathrm{C}_q(\tfrac{1}{2}, \tfrac{1}{2}, 1 \,|\, a, b, C) X_a{}^{\dot{b}}, \tag{3.9}$$

where C runs through $(-1, 0, 1) = (-, 3, +)$ and we sum over repeated indices. The factor $[2]^{-\frac{1}{2}}$ has been introduced to ensure the right undeformed limit, the factor of q^{-1} in the definition of X_0 is traditional [52]. Written out, we get

$$\begin{aligned} X_0 &= q^{-1}[2]^{-1}(q^{\frac{1}{2}} C - q^{-\frac{1}{2}} B) \\ X_- &= [2]^{-\frac{1}{2}} A \\ X_+ &= [2]^{-\frac{1}{2}} D \\ X_3 &= [2]^{-1}(q^{-\frac{1}{2}} C + q^{\frac{1}{2}} B) \,. \end{aligned} \tag{3.10}$$

The back transformation is

$$A = [2]^{\frac{1}{2}} X_- \,, \qquad B = q^{\frac{1}{2}}(X_3 - X_0) \tag{3.11a}$$

$$C = q^{-\frac{1}{2}} X_3 + q^{\frac{3}{2}} X_0 \,, \qquad D = [2]^{\frac{1}{2}} X_+ \,. \tag{3.11b}$$

Expressed in terms of the 4-vector generators, the commutation relations (3.4) become

$$\begin{gathered} X_- X_0 = X_0 X_- \,, \quad X_+ X_0 = X_0 X_+ \,, \quad X_3 X_0 = X_0 X_3 \\ q^{-1} X_- X_3 - q X_3 X_- = -\lambda X_- X_0 \,, \quad q^{-1} X_3 X_+ - q X_+ X_3 = -\lambda X_+ X_0 \\ X_- X_+ - X_+ X_- - \lambda X_3 X_3 = -\lambda X_3 X_0 \end{gathered} \tag{3.12}$$

Using the q-deformed ε-tensor (2.23) this can be written more compactly as

$$X_0 X_A = X_A X_0 \,, \qquad X_A X_B \, \varepsilon^{AB}{}_C = -\lambda X_0 X_C \,. \tag{3.13}$$

For the $*$-structure we get

$$X_0^* = X_0 \,, \qquad (X_A)^* = X^A \,, \tag{3.14}$$

for the scalar product (3.5)

$$X^2 = X_0^2 + q^{-1} X_- X_+ + q X_+ X_- - X_3^2 = X_0^2 - X_A X_B g^{AB} \,. \tag{3.15}$$

From this, we can read off the 4-metric, $X^2 = X_\mu X_\nu \eta^{\mu\nu}$, with

$$\eta^{00} = 1 \,, \qquad \eta^{AB} = -g^{AB} \tag{3.16}$$

and zero otherwise. We also could have computed the metric directly from the formulas of the Clebsch-Gordan coefficients.

If we write the back transformation (3.11) as

$$X_a{}^{\dot{b}} = \sum_\mu X_\mu (\sigma_\mu)_a{}^{\dot{b}}\,, \tag{3.17}$$

this defines the q-Pauli matrices

$$(\sigma_0)_a{}^{\dot{b}} = q[2]^{\frac{1}{2}} \mathrm{C}_q(\tfrac{1}{2},\tfrac{1}{2},0\,|\,a,b,0)\,,\quad (\sigma_C)_a{}^{\dot{b}} = [2]^{\frac{1}{2}} \mathrm{C}_q(\tfrac{1}{2},\tfrac{1}{2},1\,|\,a,b,C)\,. \tag{3.18}$$

For the usual index structure we have to lower the dotted index.

$$(\sigma_\mu)_{a\dot{b}} = (\sigma_\mu)_a{}^{\dot{b}'} \varepsilon_{b'b} \tag{3.19}$$

The q-Pauli matrices with lower undotted and dotted indices are

$$\sigma_0 = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}, \quad \sigma_- = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \sigma_+ = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ -q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix} \tag{3.20}$$

with respect to the basis $\{-,+\}$. If we compare the q-Pauli matrices with the spin-$\frac{1}{2}$ representation of J_A we find

$$\rho^{\frac{1}{2}}(J_A) = [2]^{-1}\sigma_A\,. \tag{3.21}$$

This tells us that if we raise (and lower) the vector index of σ_A as usual by $\sigma^A := g^{AA'}\sigma_{A'}$ we get $(\sigma_A)^\dagger = \sigma^A$, that is,

$$\overline{(\sigma_A)_{b\dot{a}}} = (\sigma^A)^{a\dot{b}}\,. \tag{3.22}$$

From Eq. (2.29) we deduce

$$\sigma_A\,\sigma_B\,\varepsilon^{AB}{}_C = [4][2]^{-1}\,\sigma_C\,. \tag{3.23}$$

Further relations which are not representations of relations within the algebra of rotations can be found by explicit calculations

$$\sigma^A\sigma^B = \varepsilon^{BA}{}_C\,\sigma^C + g^{BA}\,, \qquad \sigma_A\sigma_B = \sigma_C\,\varepsilon_A{}^C{}_B + g_{AB}\,. \tag{3.24}$$

The basis transformation from the matrix generators $X_a{}^{\dot{b}}$ to the 4-vector generators X_μ defines a matrix representation Λ of the q-Lorentz algebra by

$$(g\otimes h) \triangleright X_\mu = X_{\mu'}\,\Lambda(g\otimes h)^{\mu'}{}_\mu\,. \tag{3.25}$$

Using the formulas for the basis transformation Eqs. (3.9) and (3.17) we get

$$\begin{aligned}(g\otimes h)\triangleright X_0 &= (g\otimes h)\triangleright\big(q^{-2}[2]^{-1}(\sigma_0)_a{}^{\dot b}X_a{}^{\dot b}\big)\\ &= q^{-2}[2]^{-1}(\sigma_0)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b X_{a'}{}^{\dot b'}\\ &= q^{-2}[2]^{-1}(\sigma_0)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b(\sigma_\mu)_{a'}{}^{\dot b'}X_\mu\\ &= X_\mu\,\Lambda(g\otimes h)^\mu{}_0\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}(g\otimes h)\triangleright X_A &= (g\otimes h)\triangleright\big([2]^{-1}(\sigma_A)_a{}^{\dot b}X_a{}^{\dot b}\big)\\ &= [2]^{-1}(\sigma_A)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b X_{a'}{}^{\dot b'}\\ &= [2]^{-1}(\sigma_A)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b(\sigma_\mu)_{a'}{}^{\dot b'}X_\mu\\ &= X_\mu\,\Lambda(g\otimes h)^\mu{}_A\,,\end{aligned}\tag{3.27}$$

for any $(g\otimes h)\in\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. From this we can read off explicit formulas for Λ in terms of the $D^{\frac12}$-representation of $\mathcal{U}_q(\mathrm{su}_2)$ and the q-Pauli matrices

$$\begin{aligned}\Lambda(g\otimes h)^\mu{}_0 &= q^{-2}[2]^{-1}(\sigma_0)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b(\sigma_\mu)_{a'}{}^{\dot b'}\\ \Lambda(g\otimes h)^\mu{}_A &= [2]^{-1}(\sigma_A)_a{}^{\dot b}\rho^{\frac12}(g)^{a'}{}_a\,\rho^{\frac12}(h)^{b'}{}_b(\sigma_\mu)_{a'}{}^{\dot b'}\,.\end{aligned}\tag{3.28}$$

The matrices representing the generators of rotations and boosts have been calculated explicitly in Eqs. (A.50) and (A.51).

3.1.3 Commutation Relations of the q-Poincaré Algebra

In order to construct the q-Poincaré algebra we have to view $\mathcal{M}_q$ as the algebra of translations, so we write P_μ instead of X_μ. By construction $\mathcal{M}_q$ is a left $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$-module $*$-algebra. Denoting the action of $h\in\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ on $p\in\mathcal{M}_q$ by $h\triangleright p$ this means

$$h\triangleright pp' = (h_{(1)}\triangleright p)(h_{(2)}\triangleright p')\,,\qquad (h\triangleright p)^* = (Sh)^*\triangleright p^*\,.\tag{3.29}$$

As in the undeformed case, $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ and $\mathcal{M}_q$ can then be joined together in a semidirect product:

Definition 7. *The $*$-algebra of the Hopf semidirect product $\mathcal{M}_q\rtimes\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$, that is, the vector space $\mathcal{M}_q\otimes\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ with multiplication*

$$(p\otimes h)(p'\otimes h') := p(h_{(1)}\triangleright p')\otimes h_{(2)}h'\tag{3.30}$$

and $*$-structure $(p\otimes h)^* = (1\otimes h^*)(p^*\otimes 1)$, is $\mathcal{P}_q$, the q-Poincaré algebra.

By construction we have

$$(\mathrm{ad_L} h) \triangleright P_\mu = h_{(1)} P_\mu S(h_{(2)}) = h \triangleright P_\mu = P_{\mu'} \Lambda(h)^{\mu'}{}_\mu \tag{3.31}$$

for all $h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$, that is, P_μ a 4-vector operator.

We want to calculate the commutation relations between q-Lorentz generators and momenta explicitly. By construction of the 4-vectors the zero component P_0 commutes with all rotations. According to Eq. (2.25) we get for the 3-vector part

$$\begin{aligned} EP_A &= P_A E + q^{(A+1)} \sqrt{[A+2][1-A]}\; P_{A+1} K \\ FP_A &= q^{-2A} P_A F + q^{-A} \sqrt{[1+A][2-A]}\; P_{A-1} \\ KP_A &= q^{2A} P_A K\,, \end{aligned} \tag{3.32}$$

where A runs through $\{-1, 0, 1\} = \{-, 3, +\}$. In terms of the vectorial generators this becomes

$$\begin{aligned} J_A P_B &= P_A J_B - \varepsilon_A{}^C{}_B \varepsilon^{DE}{}_C\, P_D J_E + \varepsilon_A{}^C{}_B\, P_C W \\ WP_A &= (\lambda^2 + 1) P_A W - \lambda^2 \varepsilon^{BC}{}_A\, P_B J_C\,. \end{aligned} \tag{3.33}$$

For the commutation relations between momenta and boosts we use Eq. (A.48) to write in an obvious matrix notation

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} P_0 &= \begin{pmatrix} [2]^{-1}\left(\frac{[4]}{[2]} P_0 + q^{-1}\lambda P_3\right) & q^{-\frac{1}{2}}\lambda [2]^{-\frac{1}{2}} P_+ \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}} P_- & [2]^{-1}\left(\frac{[4]}{[2]} P_0 - q\lambda P_3\right) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} P_- &= \begin{pmatrix} P_- & q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}(P_3 - P_0) \\ 0 & P_- \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} P_+ &= \begin{pmatrix} P_+ & 0 \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}(P_3 - P_0) & P_+ \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} P_3 &= \begin{pmatrix} [2]^{-1}(2P_3 + q\lambda P_0) & q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}} P_+ \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}} P_- & [2]^{-1}(2P_3 - q^{-1}\lambda P_0) \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{3.34}$$

The commutation relations between momenta and the vectorial RS-generators as defined in Eq. (2.63) are more complicated but involve only 3-vectors and scalars with respect to rotations.

$$R_C P_0 = [4][2]^{-2} P_0 R_C + \lambda[2]^{-1} \varepsilon^{AB}{}_C \, P_A R_B - q[2]^{-1} P_C \, U \tag{3.35a}$$
$$S_C P_0 = [4][2]^{-2} P_0 S_C + \lambda[2]^{-1} \varepsilon^{AB}{}_C \, P_A S_B + q^{-1}[2]^{-1} P_C \, U \tag{3.35b}$$

$$\begin{aligned} R_C P_D = {} & q P_C R_D - \lambda[2]^{-1} \varepsilon_C{}^B{}_D \, P_0 R_B - q^{-1}\lambda[2]^{-1} g_{CD} \, (g^{AB} P_A R_B) \\ & - 2[2]^{-1} \varepsilon_C{}^X{}_D \varepsilon^{AB}{}_X \, P_A R_B - q^{-1}[2]^{-1} g_{CD} \, P_0 U + [2]^{-1} \varepsilon_C{}^A{}_D \, P_A U \end{aligned} \tag{3.35c}$$
$$\begin{aligned} S_C P_D = {} & q P_C S_D - \lambda[2]^{-1} \varepsilon_C{}^B{}_D \, P_0 S_B + q\lambda[2]^{-1} g_{CD} \, (g^{AB} P_A S_B) \\ & - 2[2]^{-1} \varepsilon_C{}^X{}_D \varepsilon^{AB}{}_X \, P_A S_B + q[2]^{-1} g_{CD} \, P_0 V + [2]^{-1} \varepsilon_C{}^A{}_D \, P_A V \end{aligned} \tag{3.35d}$$

$$U P_0 = [4][2]^{-2} P_0 U - q^{-1}\lambda^2[2]^{-1}(g^{AB} P_A R_B) \tag{3.35e}$$
$$V P_0 = [4][2]^{-2} P^0 V + q\lambda^2[2]^{-1}(g^{AB} P_A S_B) \tag{3.35f}$$

$$U P_C = [4][2]^{-2} P_A U - q\lambda^2[2]^{-1} P_0 R_A - \lambda^2[2]^{-1} \, \varepsilon^{AB}{}_C \, P_A R_B \tag{3.35g}$$
$$V P_C = [4][2]^{-2} P_A V + q^{-1}\lambda^2[2]^{-1} P_0 S_A - \lambda^2[2]^{-1} \, \varepsilon^{AB}{}_C \, P_A S_B \tag{3.35h}$$

Finally, we want to indicate how one can boost 4-vector operators. Let V_0 be some element of the q-Poincaré algebra. If we assume that V_0 is the zero component of a left 4-vector operator the action of the boosts on V_0 must be the same as on P_0, so according to Eq. (A.48) we must define the other components by

$$\begin{aligned} V_- &:= \mathrm{ad_L}(-q^{-\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}} \, c) \triangleright V_0 \\ V_+ &:= \mathrm{ad_L}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}} \, b) \triangleright V_0 \\ V_3 &:= \mathrm{ad_L}(\lambda^{-1}\,(d-a)) \triangleright V_0 \,. \end{aligned} \tag{3.36}$$

We will make use of this method of computing 4-vectors in Sec. 5.2.2 in order to compute the γ-matrices. In case we know the zero component $V_{\tilde{0}}$ of a *right* 4-vector the other components must be defined by

$$\begin{aligned} V_{\tilde{-}} &:= V_{\tilde{0}} \triangleleft \mathrm{ad_R}(-q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}} \, b) \\ V_{\tilde{+}} &:= V_{\tilde{0}} \triangleleft \mathrm{ad_R}(q^{-\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}} \, c) \\ V_{\tilde{3}} &:= V_{\tilde{0}} \triangleleft \mathrm{ad_R}(\lambda^{-1}\,(d-a)) \,. \end{aligned} \tag{3.37}$$

3.2 The *q*-Pauli-Lubanski Vector and the Spin Casimir

3.2.1 The *q*-Euclidean Algebra

Rotations and translations generate a $*$-subalgebra of the q-Poincaré algebra, the q-Euclidean subalgebra $\mathcal{E}_q$. Since rotations form a $\mathcal{U}_q(\mathrm{su}_2)$ Hopf subalgebra of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ this q-Euclidean subalgebra is a semidirect product

$$\mathcal{E}_q = \mathcal{M}_q \rtimes \mathcal{U}_q(\mathrm{su}_2)\,. \tag{3.38}$$

By comparing Eq. (3.13) with Eq. (2.29) we note that $\mathcal{M}_q$ and $\mathcal{U}_q(\mathrm{su}_2)$ are very similar as algebras. One could identify the generators by a map $\xi : \mathcal{M}_q \to \mathcal{U}_q(\mathrm{su}_2)$ with $\xi(P_A) = \alpha J_A$, $\xi(P_0) = \beta W$, for some numbers α, β. More precisely, ξ is a homomorphism of algebras as long as $\alpha/\beta = -\lambda$.

We cannot invert ξ, though, since there is no relation like

$$W^2 - \lambda^2 J_A J_B g^{AB} = 1 \tag{3.39}$$

in $\mathcal{M}_q$. However, for the case of constant positive mass, $P_\mu P^\mu = m^2$, we find

$$\xi(P_\mu P^\mu) = \beta^2 W^2 - \alpha^2 J_A J_B g^{AB} = m^2 \,. \tag{3.40}$$

We conclude that the image of the constant mass relation in $\mathcal{M}_q$ holds in $\mathcal{U}_q(\mathrm{su}_2)$ if $\alpha = -m\lambda$ and $\beta = m$. This is consistent with the requirement $\alpha/\beta = -\lambda$. We conclude that $\mathcal{M}_q/\langle P_\mu P^\mu = m^2\rangle$ is isomorphic to the vectorial form of $\mathcal{U}_q(\mathrm{su}_2)$. Setting aside the lack of K^{-1} in the vectorial $\mathcal{U}_q(\mathrm{su}_2)$ we thus have an isomorphism

$$\mathcal{E}_q/\langle P_\mu P^\mu = m^2\rangle \xrightarrow{\phi} \mathcal{U}_q(\mathrm{su}_2) \rtimes \mathcal{U}_q(\mathrm{su}_2) \,, \tag{3.41}$$

where the action of the semidirect product on the right hand side is the left Hopf adjoint action of $\mathcal{U}_q(\mathrm{su}_2)$ on itself. The isomorphism is given by $\xi \rtimes 1$ on the momenta and $1 \rtimes \mathrm{id}$ on the rotations,

$$\phi(P_A) = -m\lambda J_A \rtimes 1 \,, \qquad \phi(P_0) = mW \rtimes 1 \tag{3.42a}$$

$$\phi(J_A) = 1 \rtimes J_A \,, \qquad \phi(W) = 1 \rtimes W \,. \tag{3.42b}$$

Introducing

$$J_0 := -\lambda^{-1} W \,, \tag{3.43}$$

we can write ϕ more compactly as

$$\phi(P_\mu) = -m\lambda J_\mu \rtimes 1 \,, \qquad \phi(J_\mu) = 1 \rtimes J_\mu \,, \tag{3.44}$$

where μ runs through $\{0, -, +, 3\}$. Note, however, that J_μ is no 4-vector operator. The introduction of J_0 merely allows for a more compact notation. For example, Eq. (3.39) can be written as $\lambda^2 J_\mu J^\mu = 1$. Furthermore, it is convenient to give the pre-image of K a name

$$\pi := m\phi^{-1}(K) = (P_0 - P_3) \,. \tag{3.45}$$

For the semidirect product of a Hopf algebra H with itself by the left Hopf adjoint action we have the following isomorphism of algebras

$$\psi : \; H \rtimes H \longrightarrow H \otimes H \,, \qquad \psi(g \rtimes h) = g h_{(1)} \otimes h_{(2)} \,. \tag{3.46}$$

First, we prove that ψ is a homomorphism

$$\begin{aligned}\psi[(g \rtimes h)(g' \rtimes h')] &= \psi[g(h_{(1)} \triangleright g') \rtimes h_{(2)}h'] = gh_{(1)}g'S(h_{(2)})h_{(3)}h'_{(1)} \otimes h_{(4)}h'_{(2)} \\ &= gh_{(1)}g'h'_{(1)} \otimes h_{(2)}h'_{(2)} = (gh_{(1)} \otimes h_{(2)})(g'h'_{(1)} \otimes h'_{(2)}) \\ &= \psi(g \rtimes h)\, \psi(g' \rtimes h') \,. \end{aligned} \tag{3.47}$$

The invertibility can be shown directly, by defining

$$\psi^{-1}(g \otimes h) := gS(h_{(1)}) \rtimes h_{(2)} \,, \tag{3.48}$$

and checking that

$$\begin{aligned}(\psi \circ \psi^{-1})(g \otimes h) &= \psi[gS(h_{(1)}) \rtimes h_{(2)}] = gS(h_{(1)})h_{(2)} \otimes h_{(3)} = g \otimes h \\ (\psi^{-1} \circ \psi)(g \rtimes h) &= \psi[gh_{(1)} \otimes h_{(2)}] = gh_{(1)}S(h_{(2)}) \rtimes h_{(3)} = g \rtimes h \,.\end{aligned} \tag{3.49}$$

Thus, Eq. (3.46) tells us, that we have the sequence of Isomorphisms

$$\mathcal{E}_q/\langle P_\mu P^\mu = m^2 \rangle \xrightarrow{\phi} \mathcal{U}_q(\mathrm{su}_2) \rtimes \mathcal{U}_q(\mathrm{su}_2) \xrightarrow{\psi} \mathcal{U}_q(\mathrm{su}_2) \otimes \mathcal{U}_q(\mathrm{su}_2) \,. \tag{3.50}$$

Through these isomorphisms we get a full understanding of the structure of the q-Euclidean algebra.

One particularly interesting fact is that there is a whole $\mathcal{U}_q(\mathrm{su}_2)$ subalgebra of $\mathcal{E}_q$ which commutes with the momenta $\mathcal{M}_q$. This subalgebra is embedded by the map

$$i : \mathcal{U}_q(\mathrm{su}_2) \longrightarrow \mathcal{E}_q \,, \qquad i = \phi^{-1} \circ \psi^{-1} \circ (1 \otimes \mathrm{id}) \,, \tag{3.51}$$

which computes to

$$\begin{aligned} i(J_\pm) &= J_\pm + \lambda^{-1} P_\pm \pi^{-1} K \\ i(J_3) &= m\lambda^{-1}\pi^{-1}K - m^{-1}(\lambda^{-1}P_0 W + g^{AB}P_A J_B) \\ i(W) &= m^{-1}(P_0 W + \lambda g^{AB} P_A J_B) \\ i(K) &= m\, \pi^{-1} K \,. \end{aligned} \tag{3.52}$$

Observe that the images of J_A do not exist in $\mathcal{E}_q$ proper, since they all involve the inverse of $\pi = P_0 - P_3$, which is not an element of $\mathcal{M}_q$.

3.2.2 The Center of the q-Euclidean Algebra

We wonder where precisely the condition $P_\mu P^\mu = m^2$ has entered into our considerations. Which of the results do still hold if the mass shell condition is relaxed?

Towards this end we list the commutation relations between rotations and translations

$$\begin{array}{lll} [J_-, P_-] = 0 & [J_-, P_3] = q^{-1} P_- K & [J_-, P_+] = P_3 K \\ [J_+, P_-] = -P_3 K & [J_+, P_3] = -q P_+ K & [J_+, P_+] = 0 \\ [K, P_-] = -q^{-1}\lambda P_- K & [K, P_3] = 0 & [K, P_+] = q\lambda P_+ K \end{array} \tag{3.53}$$

Let us check what relations still hold within $i(\mathcal{U}_q(\mathrm{su}_2))$. We compute for example

$$\begin{aligned} i(K)i(J_+) &= m\pi^{-1}K(J_+ + \lambda^{-1}P_+\pi^{-1}K) \\ &= (q^2 J_+ + qP_+\pi^{-1}K)m\pi^{-1}K + m\lambda^{-1}P_+(\pi^{-1}K)^2 \\ &= (q^2 J_+ + (q+\lambda^{-1})P_+\pi^{-1}K)m\pi^{-1}K \\ &= q^2 i(J_+)i(K)\,, \end{aligned} \tag{3.54}$$

telling us that the relation $KJ_+ = q^2 J_+ K$ is preserved under i. Similarly, we find that the image of $KJ_- = q^{-2}J_-K$ still holds in $i(\mathcal{U}_q(\mathrm{su}_2))$. Hence, we did not use the mass shell condition for these two relations. However, for the relation $\lambda[2](qJ_+J_- - q^{-1}J_-J_+) = 1 - K^2$ we find

$$i[\lambda[2](qJ_+J_- - q^{-1}J_-J_+)] = 1 - \frac{P_\mu P^\mu}{m^2}\, i(K)^2\,, \tag{3.55}$$

such that this relation holds in $\mathcal{E}_q$ precisely if the mass shell condition $P_\mu P^\mu = m^2$ holds. We conclude that without the mass shell condition i is no longer a homomorphism of algebras.

Now, we check if $i(\mathcal{U}_q(\mathrm{su}_2))$ still commutes with all translations. Setting aside the problem that π^{-1} does not exist in $\mathcal{E}_q$ proper, we compute for example

$$\begin{aligned} P_+ i(J_-) &= P_+(J_- + \lambda^{-1}P_-\pi^{-1}K) = J_-P_+ + (\lambda^{-1}P_+P_- - P_3\pi)\pi^{-1}K \\ &= J_-P_+ + \lambda^{-1}(P_-P_+)\pi^{-1}K = i(J_-)P_+\,. \end{aligned} \tag{3.56}$$

In the same manner we find, that all of $i(\mathcal{U}_q(\mathrm{su}_2))$ commutes with all translations. This holds in particular for $i(W)$ which is furthermore a scalar with respect to rotations, since it is made up of the scalars P_0, W, and $\vec{P}\cdot\vec{J}$. In conclusion we have[3]

Proposition 3. *The center of the q-Euclidean algebra $\mathcal{E}_q$ is generated by $P_\mu P^\mu$, P_0 and*

$$Z := m\, i(W) = P_0 W + \lambda g^{AB} P_A J_B = -\lambda P_\mu J^\mu\,. \tag{3.58}$$

3.2.3 The Pauli-Lubanski Vector in the q-Deformed Setting

In the undeformed case one considers the Pauli-Lubanski (pseudo) vector

$$W_\mu^{q=1} := -\frac{1}{2}\,\varepsilon_{\mu\nu\sigma\tau} V^{\nu\sigma} P^\tau\,, \tag{3.59}$$

where $V^{\nu\sigma}$ is the matrix of Lorentz generators. Its usefulness is due to the following two properties:

[3]Using the Casimir operators of $\mathcal{E}_q$, the orbital angular momentum relation of [31] can be equivalently written as

$$\lambda(\vec{P}\cdot\vec{J}) = P_0(1-W) \qquad \Leftrightarrow \qquad Z = P_0\,. \tag{3.57}$$

(i) W_μ is a 4-vector operator of the Poincaré algebra.

(ii) Each component W_μ commutes with all translations P^τ.

If we demand further that W_μ be linear in the Lorentz generators and the translations, conditions (i) and (ii) determine the Pauli-Lubanski vector up to a constant factor. From (i) and (ii) we deduce that $W_\mu W^\mu$ is a Casimir operator. Physically, this Casimir operator turns out to correspond to spin.

In the q-deformed case we are tempted to define W_μ analogously by Eq. (3.59) with the q-deformed versions of the epsilon tensor, the matrix of Lorentz generators, and the translations. By construction, this would be a 4-vector operator. However, it turns out that with this naive approach property (ii) will not hold. Therefore, we will try to find a way to construct W_μ such that (ii) holds, as well.

Let us start with the zero component W_0. It has to commute with all translations to satisfy (ii) and with all rotations since the zero component of a 4-vector is a scalar with respect to rotations. Thus, it has to commute with all of the q-Euclidean algebra $\mathcal{E}_q$. If we assume that as in the undeformed case W_0 is itself a member of $\mathcal{E}_q$, we conclude that W_0 has to be an element of the center of the q-Euclidean algebra, which we computed in the preceding section. Since the momenta carry dimensions W_0 has to be linear in the momenta. Hence W_0 must be a linear combination of P_0 and Z. The additional requirement that W_0 has to have the right undeformed limit determines

$$W_0 := \lambda^{-1}(Z - P_0) = \lambda^{-1}(W - 1)P_0 + g^{AB} J_A P_B \tag{3.60}$$

up to an overall factor that tends to one as $q \to 1$.

Now that we have a good candidate for the zero component of the q-Pauli-Lubanski vector we have to see if it can be boosted to a 4-vector. First we have to ask what type of vector operator we would expect it to be. Recall from Sec. 2.2.1 that we have to distinguish between left and right tensor operators. A short calculation shows that for any translation $p \in \mathcal{M}_q$ and any Lorentz transformation $h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$

$$\begin{aligned}(W_0 \triangleleft \mathrm{ad_R} h)\, p &= S(h_{(1)}) W_0 h_{(2)} p = S(h_{(1)}) W_0 (h_{(2)} \triangleright p) h_{(3)} \\ &= S(h_{(1)})(h_{(2)} \triangleright p) W_0 h_{(3)} = (S(h_{(1)})_{(1)} h_{(2)} \triangleright p) S(h_{(1)})_{(2)} W_0 h_{(3)} \\ &= (S(h_{(2)}) h_{(3)} \triangleright p) S(h_{(1)}) W_0 h_{(4)} \\ &= p\, (W_0 \triangleleft \mathrm{ad_R} h)\end{aligned} \tag{3.61}$$

Hence, a right boosted W_0 commutes with all translations. This is not be the case for $\mathrm{ad_L} \triangleright W_0$. Hence, the q-Pauli-Lubanski vector will satisfy property (ii) only if it is a right vector operator $W_{\tilde{\mu}}$.

3.2.4 Boosting the q-Pauli-Lubanski Vector

If $W_0 = W_{\tilde{0}}$ as defined in (3.60) really is a left 4-vector operator, which is not necessarily so, then the other components are given, uniquely, by Eqs. (3.37). We will now determine $W_{\tilde{\mu}}$ and rigorously show that it is a right 4-vector operator. Up to a constant factors $W_{\tilde{0}}$ is the sum of two parts, Z and P_0, which we will treat separately.

Boosting $\boldsymbol{Z}$ The explicit calculations of the right adjoint action of the boosts on Z by Eqs. (3.37) turn out to be very lengthy. It is more efficient to start with a more abstract consideration.

We observe that for all boosts $h \in SU_q(2)^{\mathrm{op}}$ we have

$$\langle (J_\mu)_{(1)}, h\rangle (J_\mu)_{(2)} = J_{\mu'} \Lambda(h)^{\mu'}{}_{\mu}\,, \tag{3.62}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the dual pairing of $\mathcal{U}_q(\mathrm{su}_2)$ and $SU_q(2)$. We exemplify this result for J_+,

$$\begin{aligned}
\langle (J_+)_{(1)}, B^a{}_b\rangle (J_+)_{(2)} &= \langle J_+, B^a{}_b\rangle K + \langle 1, B^a{}_b\rangle J_+ \\
&= \lambda[2]^{-1}(\sigma_+)_{ab}\,(J_3 - J_0) + \delta^a_b\, J_+ \\
&= \begin{pmatrix} J_+ & 0 \\ -q^{\frac{1}{2}}\lambda[2]^{-1/2}(J_3 - J_0) & J_+ \end{pmatrix} \\
&= J_{\mu'}\Lambda(h)^{\mu'}{}_{+}\,.
\end{aligned} \tag{3.63}$$

Applying the map ϕ^{-1} as defined in Eq. (3.44) to Eq. (3.62) we get

$$\mathrm{ad_L} h \triangleright \phi^{-1}(l) = \langle l_{(1)}, h\rangle \phi^{-1}(l_{(2)})\,. \tag{3.64}$$

for all $l \in \mathcal{U}_q(\mathrm{su}_2)$ and $h \in SU_q(2)^{\mathrm{op}}$. For example, for $l = \phi(P_+) = -m\lambda J_+$ the left adjoint action of the boost generators on P_+ can be written as

$$\mathrm{ad_L} B^a{}_b \triangleright P_+ = \delta^a_b\, P_+ - \lambda\langle J_+, B^a{}_b\rangle\pi = \delta^a_b\, P_+ + \lambda[2]^{-1}\,(\sigma_+)_{ab}\,(P_3 - P_0)\,, \tag{3.65}$$

which is the same as in Eq. (A.48).

Let i be the map that has been defined in Eq. (3.51). We try to commute $i(l)$, $l \in \mathcal{U}_q(\mathrm{su}_2)$, with a boost $h \in SU_q(2)^{\mathrm{op}}$ using Eq. (3.64):

$$\begin{aligned}
i(l)\,h &= \phi^{-1}[S(l_{(1)})]l_{(2)}h = \phi^{-1}[S(l_{(1)})]h_{(2)}l_{(3)}\langle S(l_{(2)}), h_{(1)}\rangle\langle l_{(4)}, h_{(3)}\rangle \\
&= h_{(3)}\{\mathrm{ad_L} S^{-1}(h_{(2)}) \triangleright \phi^{-1}[S(l_{(1)})]\}l_{(3)}\langle S(l_{(2)}), h_{(1)}\rangle\langle l_{(4)}, h_{(4)}\rangle \\
&= h_{(3)}\langle S(l_{(2)}), S^{-1}(h_{(2)})\rangle\phi^{-1}[S(l_{(1)})]l_{(4)}\langle S(l_{(3)}), h_{(1)}\rangle\langle l_{(5)}, h_{(4)}\rangle \\
&= h_{(1)}\phi^{-1}[S(l_{(1)})]l_{(2)}\langle l_{(3)}, h_{(2)}\rangle
\end{aligned} \tag{3.66}$$

This leads to a remarkably simple formula for the right adjoint action of a boost on $i(l)$

$$i(l) \triangleleft \mathrm{ad_R} h = i(l_{(1)}\langle l_{(2)}, h\rangle)\,. \tag{3.67}$$

For $l = S(J_\mu)$ this formula becomes

$$\begin{aligned} i[S(J_\mu)] \triangleleft \mathrm{ad_R} h &= i[S(J_\mu)_{(1)}\langle S(J_\mu)_{(2)}, h\rangle] = i[S((J_\mu)_{(2)}\langle (J_\mu)_{(1)}, S^{-1}h\rangle)] \\ &= i[S(J_{\mu'})\Lambda(S^{-1}h)^{\mu'}{}_\mu]\,, \end{aligned} \tag{3.68}$$

which tells us that $i(S(J_\mu))$ transforms under boosts as a right lower 4-vector operator.

It remains to check whether $i(S(J_\mu))$ transforms as right 4-vector under rotations. We observe that ϕ^{-1} maps the 3-vector J_μ to the 3-vector P_μ and the $\mathcal{U}_q(\mathrm{su}_2)$-scalar J_0 to the scalar P_0. Hence, for $a, b \in \mathcal{U}_q(\mathrm{su}_2)$ we have

$$\mathrm{ad_L} b \triangleright \phi^{-1}(a) = \phi^{-1}(\mathrm{ad_L} b \triangleright a)\,. \tag{3.69}$$

Now we are prepared to tackle the right action of a rotation on $i(Sa)$

$$\begin{aligned} i(Sa) \triangleleft \mathrm{ad_R} b &= S(b_{(1)})\phi^{-1}[S((Sa)_{(1)})](Sa)_{(2)}b_{(2)} \\ &= \{S(b_{(1)})_{(1)} \triangleright \phi^{-1}[S((Sa)_{(1)})]\}\{S(b_{(1)})_{(2)}(Sa)_{(2)}b_{(2)}\} \\ &= \phi^{-1}[(S(b_{(2)})_{(1)}S((Sa)_{(1)})S(S(b_{(2)})_{(2)})]S(b_{(1)})(Sa)_{(2)}b_{(3)} \\ &= \phi^{-1}[S(S(b_{(2)})(Sa)_{(1)}b_{(3)})]S(b_{(1)})(Sa)_{(2)}b_{(4)} \\ &= \phi^{-1}[S(S(b_{(1)})_{(1)}(Sa)_{(1)}b_{(2)(1)})]S(b_{(1)})_{(2)}(Sa)_{(2)}b_{(2)(2)} \\ &= i(Sa \triangleleft \mathrm{ad_R} b) = i(S(\mathrm{ad_L} S^{-1}b \triangleright a))\,. \end{aligned} \tag{3.70}$$

This shows that since $S(J_\mu)$ transforms as a right lower 4-vector under rotations, so does $i(S(J_\mu))$. In conclusion we have

Proposition 4. *The set of operators*

$$Z_{\tilde\mu} := -m\lambda\, i(S(J_\mu)) \tag{3.71}$$

is a right lower 4-vector operator of the q-Lorentz algebra. Since furthermore $Z_{\tilde 0} = Z$, $Z_{\tilde\mu}$ is the unique right lower 4-vector operator with zero component Z.

All that remains to do is to compute $Z_{\tilde\mu}$ explicitly:

$$\begin{aligned} Z_{\tilde 0} &= m\, i(W) = P_0 W + \lambda g^{AB} P_A J_B \\ Z_{\tilde\pm} &= P_\pm + \lambda J_\pm K^{-1}\pi \\ Z_{\tilde 3} &= m\, i(W - K^{-1}) = P_0 W + \lambda g^{AB} P_A J_B - K^{-1}\pi\,. \end{aligned} \tag{3.72}$$

Observe that these expressions do not contain π^{-1}, hence, they are proper members of $\mathcal{E}_q$, that is,

$$\begin{aligned} Z_{\tilde 0} &= W P_0 - q\lambda J_+ P_- - q^{-1}\lambda J_- P_+ + \lambda J_3 P_3 \\ Z_{\tilde\pm} &= P_\pm + \lambda J_\pm K^{-1}(P_0 - P_3) \\ Z_{\tilde 3} &= (W - K^{-1})P_0 - q\lambda J_+ P_- - q^{-1}\lambda J_- P_+ + (\lambda J_3 + K^{-1})P_3\,. \end{aligned} \tag{3.73}$$

Finally, we recall that the square of $Z_{\tilde\mu}$ must be a Casimir operator. After lengthy calculations we find

$$Z^{\tilde\mu} Z_{\tilde\mu} = P_\mu P^\mu\,. \tag{3.74}$$

We conclude that squaring $Z_{\tilde\mu}$ alone does not yield a new Casimir operator.

Boosting P_0 The next step in our calculation of the q-Pauli-Lubanski vector is to find a right 4-vector operator with P_0 as zero component. With a universal $\mathcal{R}$-matrix of the q-Lorentz algebra, we can generically turn a left 4-vector operator into a right 4-vector operator. Defining for the left 4-vector operator P_μ

$$j(P_\mu) := S^2(\mathcal{R}_{[1]})(\mathcal{R}_{[2]} \triangleright P_\mu) = S^2\big(\mathcal{R}_{[1]}\Lambda(\mathcal{R}_{[2]})^{\mu'}{}_\mu\big)P_{\mu'} \tag{3.75}$$

we check that for any q-Lorentz transformation $h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ we have

$$\begin{aligned} j(P_\mu) \triangleleft \mathrm{ad}_\mathrm{R} h &= S(h_{(1)})S^2(\mathcal{R}_{[1]})(\mathcal{R}_{[2]} \triangleright P_\mu)h_{(2)} \\ &= S(h_{(1)})S^2(\mathcal{R}_{[1]})h_{(3)}\big(S^{-1}(h_{(2)})\mathcal{R}_{[2]} \triangleright P_\mu\big) \\ &= S\big(S(\mathcal{R}_{[1]})h_{(1)}\big)h_{(3)}\big(S^{-1}(S(\mathcal{R}_{[2]})h_{(2)}) \triangleright P_\mu\big) \\ &= S\big(h_{(2)}S(\mathcal{R}_{[1]})\big)h_{(3)}\big(S^{-1}(h_{(1)}S(\mathcal{R}_{[2]})) \triangleright P_\mu\big) \\ &= S^2(\mathcal{R}_{[1]})S(h_{(2)})h_{(3)}\big(\mathcal{R}_{[2]}S^{-1}(h_{(1)}) \triangleright P_\mu\big) \\ &= S^2(\mathcal{R}_{[1]})(\mathcal{R}_{[2]}S^{-1}h \triangleright P_\mu) \\ &= j(P_{\mu'})\Lambda(S^{-1}h)^{\mu'}{}_\mu\,, \end{aligned} \tag{3.76}$$

thus, $j(P_\mu)$ is indeed a right 4-vector operator. Recall from Sec. 2.3.2, that the object

$$(L_+^\Lambda)^\mu{}_\nu := \mathcal{R}_{[1]}\Lambda(\mathcal{R}_{[2]})^\mu{}_\nu \tag{3.77}$$

that appears in the definition of $j(P_\mu)$ is an L-matrix. Furthermore, we recall from Eq. (1.53) that there are two universal $\mathcal{R}$-matrices of the q-Lorentz algebra, which are composed of the $\mathcal{R}$-matrix of $\mathcal{U}_q(\mathrm{sl}_2)$ according to

$$\mathcal{R}_\mathrm{I} = \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}\,, \qquad \mathcal{R}_\mathrm{II} = \mathcal{R}_{41}^{-1}\mathcal{R}_{13}\mathcal{R}_{24}\mathcal{R}_{23}\,. \tag{3.78}$$

We will now compute the L-Matrix for $\mathcal{R}_\mathrm{I}$. We have for the $(\frac{1}{2},\frac{1}{2})$-form of the vector representation

$$\begin{aligned} \big(L_{\mathrm{I}+}^{(\frac{1}{2},\frac{1}{2})}\big)^{ab}{}_{cd} &= \big(\mathrm{id} \otimes \mathrm{id} \otimes \rho^{\frac{1}{2}} \otimes \rho^{\frac{1}{2}}\big)(\mathcal{R}_\mathrm{I})^{ab}{}_{cd} \\ &= \big(L_-^{\frac{1}{2}}\big)^b{}_{b'}\big(L_-^{\frac{1}{2}}\big)^a{}_{a'} \otimes \big(L_+^{\frac{1}{2}}\big)^{b'}{}_d\big(L_+^{\frac{1}{2}}\big)^{a'}{}_c \\ &= B^b{}_d B^a{}_c\,, \end{aligned} \tag{3.79}$$

where $B^a{}_b \in SU_q(2)^\mathrm{op}$ is the matrix of boosts. For the 4-vector form of this L-matrix we then find

$$(L_{\mathrm{I}+}^\Lambda)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a^2 & b^2 & q^{\frac{1}{2}}[2]^{\frac{1}{2}}ab \\ 0 & c^2 & d^2 & q^{\frac{1}{2}}[2]^{\frac{1}{2}}cd \\ 0 & q^{\frac{1}{2}}[2]^{\frac{1}{2}}ac & q^{\frac{1}{2}}[2]^{\frac{1}{2}}bd & (1+[2]bc) \end{pmatrix} \tag{3.80}$$

with respect to the basis $\{0,-,+,3\}$. This matrix of generators becomes more familiar if we write it in block diagonal form

$$(L^{\Lambda}_{\mathrm{I}+})^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & t^{A}{}_{B} \end{pmatrix} , \tag{3.81}$$

so we can see that $t^{A}{}_{B}$, $A, B \in \{-1, 0, 1\}$ is the 3-dimensional corepresentation matrix of $SU_q(2)^{\mathrm{op}}$ [53, 54]. From the block diagonal form we deduce that

$$j(P_0) = P_0 \,, \tag{3.82}$$

so we get

Proposition 5. *The set of operators*

$$j(P_\mu) := S^2\big[(L^{\Lambda}_{\mathrm{I}+})^{\mu'}{}_{\mu}\big] P_{\mu'} \tag{3.83}$$

is a right lower 4-vector operator of the q-Lorentz algebra. Since furthermore $j(P_0) = P_0$*,* $j(P_\mu)$ *is the unique right lower 4-vector operator with zero component* P_0*.*

With Eq. (3.83) we find the explicit expressions

$$\begin{aligned}
j(P_0) &= P_0 \\
j(P_-) &= a^2\, P_- + q^{-4}c^2\, P_+ + q^{-\frac{3}{2}}[2]^{\frac{1}{2}} ac\, P_3 \\
j(P_+) &= q^4 b^2\, P_- + d^2\, P_+ + q^{\frac{5}{2}}[2]^{\frac{1}{2}} bd\, P_3 \\
j(P_3) &= q^{\frac{5}{2}}[2]^{\frac{1}{2}} ab\, P_- + q^{-\frac{3}{2}}[2]^{\frac{1}{2}} cd\, P_+ + (1+[2]bc)\, P_3 \,.
\end{aligned} \tag{3.84}$$

Finally, we want to calculate the square of $j(P_\mu)$ which must be a Casimir operator. First, we note that since P_0 commutes with all momenta and $j(P_\mu)$ is the right boosted P_0, the reasoning of Eq. (3.61) applies, that is, all momenta P_μ commute with $j(P_\nu)$,

$$P_\mu\, j(P_\nu) = j(P_\nu)\, P_\mu \,. \tag{3.85}$$

Moreover, we have

$$\begin{aligned}
(L^{\Lambda}_{\mathrm{I}+})^{\mu}{}_{\nu}(L^{\Lambda}_{\mathrm{I}+})^{\sigma}{}_{\tau}\, \eta^{\tau\nu} &= R_{[1]}R_{[1']}\Lambda(R_{[2]})^{\mu}{}_{\nu}\Lambda(R_{[2']})^{\sigma}{}_{\tau}\, \eta^{\tau\nu} \\
&= R_{[1]}R_{[1']}\Lambda(R_{[2]})^{\mu}{}_{\nu}\Lambda(R_{[2']})^{\sigma''}{}_{\tau}\, \eta^{\tau\nu}\eta_{\sigma'\sigma''}\eta^{\sigma\sigma'} \\
&= R_{[1]}R_{[1']}\Lambda(R_{[2]})^{\mu}{}_{\nu}\Lambda(S^{-1}R_{[2']})^{\nu}{}_{\sigma'}\, \eta^{\sigma\sigma'} \\
&= R_{[1]}R^{-1}_{[1']}\Lambda(R_{[2]}R^{-1}_{[2']})^{\mu}{}_{\sigma'}\, \eta^{\sigma\sigma'} \\
&= \eta^{\sigma\mu} \,,
\end{aligned} \tag{3.86}$$

where we have used Eq. (2.13). With the last two equations we can compute the square of $j(P_\mu)$ quite easily

$$\begin{aligned} j(P^\mu)j(P_\mu) &= S^2\big((L^\Lambda_{\mathrm{I}+})^{\mu'}{}_\mu\big)\,P_{\mu'}\,S^2\big((L^\Lambda_{\mathrm{I}+})^{\nu'}{}_\nu\big)\,P_{\nu'}\,\eta^{\nu\mu} \\ &= \big[S^2\big((L^\Lambda_{\mathrm{I}+})^{\mu'}{}_\mu\big)S^2\big((L^\Lambda_{\mathrm{I}+})^{\nu'}{}_\nu\big)\eta^{\nu\mu}\big]\,P_{\nu'}P_{\mu'} \\ &= \eta^{\nu\mu}\,P_\nu P_\mu\,. \end{aligned} \tag{3.87}$$

Again, the square of one half of the q-Pauli-Lubanski vector alone yields only the mass Casimir.

The q-Pauli-Lubanski Vector We come to the following conclusion:

Proposition 6. *The set of operators*

$$W_{\tilde\mu} := \lambda^{-1}[Z_{\tilde\mu} - j(P_\mu)] = -m\,i(S(J_\mu)) - \lambda^{-1}j(P_\mu) \tag{3.88}$$

has the following properties:

(i) It is a right lower 4*-vector operator.*

(ii) Each component W_μ commutes with all translations P_τ.

Furthermore, it is the unique right lower 4-vector operator with zero component $W_0 = \lambda^{-1}(Z - P_0)$. We will therefore call it the q-Pauli-Lubanski vector.

Explicitly, the q-Pauli-Lubanski vector is

$$\begin{aligned} W_{\tilde 0} &= \lambda^{-1}(W-1)P_0 - qJ_+P_- - q^{-1}J_-P_+ + J_3P_3 \\ W_{\tilde -} &= \lambda^{-1}[\lambda J_-K^{-1}P_0 + (1-a^2)P_- - q^{-4}c^2P_+ - (\lambda J_-K^{-1} + q^{-\frac{3}{2}}[2]^{\frac{1}{2}}ac)P_3] \\ W_{\tilde +} &= \lambda^{-1}[\lambda J_+K^{-1}P_0 - q^4b^2P_- + (1-d^2)P_+ - (\lambda J_+K^{-1} + q^{\frac{5}{2}}[2]^{\frac{1}{2}}bd)P_3] \\ W_{\tilde 3} &= \lambda^{-1}(W-K^{-1})P_0 - (qJ_+ + q^{\frac{5}{2}}\lambda^{-1}[2]^{\frac{1}{2}}ab)P_- - (q^{-1}J_- + q^{-\frac{3}{2}}\lambda^{-1}[2]^{\frac{1}{2}}cd)P_+ \\ &\quad + (J_3 + \lambda^{-1}K^{-1} - \lambda^{-1}(1+[2]bc))P_3\,. \end{aligned} \tag{3.89}$$

3.3 The Little Algebras

3.3.1 Little Algebras in the q-Deformed Setting

In classical relativistic mechanics the state of a free particle is completely determined by its 4-momentum. In quantum mechanics particles can have an additional degree of freedom called spin. What is spin?

Let us assume we have a free relativistic particle described by an irreducible representation of the Poincaré algebra. We pick all states with a given momentum,

$$\mathcal{H}_p := \{|\psi\rangle \in \mathcal{H} : P_\mu|\psi\rangle = p_\mu|\psi\rangle\}\,, \tag{3.90}$$

where $\mathcal{H}$ is the Hilbert space of the particle and $p = (p_\mu)$ is the 4-vector of momentum eigenvalues. If the state of the particle is *not* uniquely determined by the eigenvalues of the momentum, then the eigenspace $\mathcal{H}_p$ will be degenerate. In that case we need, besides the momentum eigenvalues, an additional quantity to label the basis of our Hilbert space uniquely. This additional degree of freedom is spin. The spin symmetry is then the set of Lorentz transformations that leaves the momentum eigenvalues invariant and, hence, acts on the spin degrees of freedom only,

$$\mathcal{K}'_p := \{h \in \mathcal{L} : P_\mu h|\psi\rangle = p_\mu h|\psi\rangle \text{ for all } |\psi\rangle \in \mathcal{H}_p\}\,, \tag{3.91}$$

where $\mathcal{L}$ is the enveloping Lorentz algebra. In mathematical terms, $\mathcal{K}'_p$ is the stabilizer of $\mathcal{H}_p$. Clearly, $\mathcal{K}'_p$ is an algebra, called the little algebra.

A priori, there are a lot of different little algebras for each representation and each vector p of momentum eigenvalues. In the undeformed case it turns out that for the physically relevant representations (real mass) there are (up to isomorphism) only two little algebras, depending on the mass being either positive or zero [1]. For positive mass we get the algebra of rotations, $\mathcal{U}(\mathrm{su}_2)$, for zero mass an algebra that is isomorphic to the algebra of rotations and translations of the 2-dimensional plane denoted by $\mathcal{U}(\mathrm{iso}_2)$. The proof that $\mathcal{K}'_p$ does not depend on the particular representation but on the mass, does not generalize to the q-deformed case: If we defined for representations of the q-Poincaré algebra the little algebra as in Eq. (3.91), it could well happen that $\mathcal{K}'_p$ for a spin-$\frac{1}{2}$ particle is not the same as for spin-1. We will therefore define the q-little algebras differently.

In the undeformed case there is an alternative but equivalent definition of the little algebras. $\mathcal{K}'_p$ is the algebra generated by the components of the q-Pauli-Lubanski vector as defined in Eq. (3.59) with the momentum generators replaced by their eigenvalues. Let us formalize this to see why this definition works and how it is generalized to the q-deformed case.

Let $\mathcal{T}$ be the algebra of translations, $\mathcal{L}$ the Lorentz algebra, both joined in a semidirect product to form the Poincaré algebra $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$. Let χ_p be the map that maps the momentum generators to the eigenvalues, $\chi_p(P_\mu) = p_\mu$. Being the restriction of a representation, χ_p must extend to a one dimensional $*$-representation $\chi_p : \mathcal{T} \to \mathbb{C}$, a non-trivial condition only in the q-deformed case. Noting that every element of $\mathcal{P}$ can be written as a sum of products of Lorentz transformations and translations, $\sum_i l_i t_i$, we extend χ_p to a linear map $\tilde{\chi}_p : \mathcal{P} \to \mathcal{L}$ by

$$\tilde{\chi}_p\big(\sum l_i t_i\big) := \sum l_i \chi_p(t_i). \tag{3.92}$$

The little algebra can now be alternatively defined as the unital algebra generated by the images of the q-Pauli-Lubanski vector under $\tilde{\chi}_p$,

$$\mathcal{K}_p := \mathbb{C}\langle \tilde{\chi}_p(W_\mu)\rangle\,. \tag{3.93}$$

Why is this a reasonable definition? By construction the action of every element of $\mathcal{P}$ on $\mathcal{H}_p$ is the same as of its image under $\tilde{\chi}_p$. For any $|\psi\rangle \in \mathcal{H}_p$ this means

$$P_\mu \tilde{\chi}_p(W_\nu)|\psi\rangle = \tilde{\chi}_p(P_\mu W_\nu)|\psi\rangle = \tilde{\chi}_p(W_\nu P_\mu)|\psi\rangle = p_\mu \tilde{\chi}_p(W_\nu)|\psi\rangle\,, \tag{3.94}$$

which shows that $\mathcal{K}_p \subset \mathcal{K}'_p$. It still could happen, that $\mathcal{K}_p$ is strictly smaller than $\mathcal{K}'_p$. In the undeformed case there are theorems telling us [55, 56] that this cannot happen, so we really have $\mathcal{K}_p = \mathcal{K}'_p$. For the q-deformed case no such theorem is known [57]. However, if there were more generators in the stabilizer of some momentum eigenspace they would have to vanish for $q \to 1$. In this sense Eq. (3.93) with the q-deformed Pauli-Lubanski vector can be considered to define the q-deformed little algebras.

3.3.2 Computation of the q-Little Algebras

To begin the explicit calculation of the q-deformed little algebras, we need to figure out if there are eigenstates of q-momentum at all. That is, we want to determine the one-dimensional $*$-representations of $\mathcal{M}_q$, that is the homomorphisms of $*$-Algebras $\chi : \mathcal{M}_q \mapsto \mathbb{C}$. Let us again denote the eigenvalues of the generators by lower case letters $p_\mu := \chi(P_\mu)$. For χ to be a $*$-map we must have p_0, p_3 real and $p_+^* = -qp_-$. To find the conditions for χ to be a homomorphism of algebras, we apply χ to the relations (3.12) of $\mathcal{M}_q$, yielding

$$p_A(p_0 - p_3) = 0\,. \tag{3.95}$$

There are two cases. The first is $p_0 \neq p_3$, which immediately leads to $p_A = 0$, and $p_0 = \pm m$. The second case is $p_0 = p_3$, leading to $m^2 = -|p_-|^2 - |p_+|^2$. Hence, if the mass m is to be real, we must have $p_\pm = 0$.

To summarize, for real mass m we have a massive and a massless type of momentum eigenstates with eigenvalues given by

$$(p_0, p_-, p_+, p_3) = \begin{cases} (\pm m, 0, 0, 0) & m > 0 \\ (k, 0, 0, k) & m = 0,\ k \in \mathbb{R} \end{cases} \tag{3.96}$$

Now, we need to move the momentum generators in the expressions of the q-Pauli-Lubanski vector to the right and replace them with these eigenvalues.

The Massive Case In Eqs. (3.89) the momenta have already been moved to the right, so we can simply replace them with $(P_0, P_-, P_+, P_3) \to (m, 0, 0, 0)$. We get

$$\begin{aligned} \tilde{\chi}_p(W_{\tilde{0}}) &= \lambda^{-1}(W - 1)m \\ \tilde{\chi}_p(W_{\tilde{-}}) &= J_- K^{-1} m \\ \tilde{\chi}_p(W_{\tilde{+}}) &= J_+ K^{-1} m \\ \tilde{\chi}_p(W_{\tilde{3}}) &= \lambda^{-1}(W - K^{-1})m\,, \end{aligned} \tag{3.97}$$

so the set of generators of the little algebra is essentially $\{W, K^{-1}, J_\pm K^{-1}\}$. Since K^{-1} stabilizes the momentum eigenspace, so does its inverse K. Hence, it is safe to add K to the little algebra which would exist, anyway, as operator within a representation. We thus get

$$\mathcal{K}_m := \mathcal{K}_{(m,0,0,0)} = \mathcal{U}_q(\mathrm{su}_2)\,, \tag{3.98}$$

completely analogous to the undeformed case.

The Massless Case The massless case is more complicated. Replacing in Eqs. (3.89) the momentum generators with $(P_0, P_-, P_+, P_3) \to (k, 0, 0, k)$ we get

$$\begin{aligned}
\tilde{\chi}_p(W_{\tilde{0}}) &= \lambda^{-1}(K-1)k \\
\tilde{\chi}_p(W_{\tilde{-}}) &= -\lambda^{-1} q^{-\frac{3}{2}} [2]^{\frac{1}{2}} ac\, k \\
\tilde{\chi}_p(W_{\tilde{+}}) &= -\lambda^{-1} q^{\frac{5}{2}} [2]^{\frac{1}{2}} bd\, k \\
\tilde{\chi}_p(W_{\tilde{3}}) &= \lambda^{-1}\big(K - (1 + [2]bc)\big)k\,.
\end{aligned} \tag{3.99}$$

The set of generators of the little algebra is essentially $\{K, ac, bd, bc\}$. The commutation relations of these generators can be written more conveniently in terms of K and

$$N_- := q^{\frac{1}{2}}[2]^{\frac{1}{2}} ac\,, \qquad N_+ := q^{\frac{1}{2}}[2]^{\frac{1}{2}} bd\,, \qquad N_3 := 1 + [2]bc\,, \tag{3.100}$$

or equivalently $N_A = t^3{}_A$, for $t^A{}_B$ as defined by Eqs. (3.80) and (3.81). The commutation relations are

$$N_B N_A\, \varepsilon^{AB}{}_C = -\lambda N_C\,, \qquad N_A N_B\, g^{BA} = 1\,, \qquad K N_A = q^{-2A} N_A K\,, \tag{3.101}$$

and the conjugation properties

$$N_A^* = N_B\, g^{BA}\,, \qquad\qquad K^* = K\,. \tag{3.102}$$

In words: The N_A generate the opposite algebra of a unit quantum sphere, $\mathcal{S}^{\mathrm{op}}_{q\infty}$ [58]. K, the generator of $\mathcal{U}_q(\mathrm{u}_1)$, acts on N_A as on a right 3-vector operator. In total we have

$$\mathcal{K}_0 := \mathcal{K}_{(k,0,0,k)} = \mathcal{U}_q(\mathrm{u}_1) \ltimes \mathcal{S}^{\mathrm{op}}_{q\infty}\,. \tag{3.103}$$

As opposed to the massive case, $\mathcal{K}_0$ is no Hopf algebra. However, since $N_A = t^3{}_A$ and $\Delta(t^A{}_C) = t^A{}_B \otimes t^B{}_C$, we have

$$\Delta(N_B) = N_A \otimes t^A{}_B\,, \tag{3.104}$$

hence, $\mathcal{K}_0$ is a left coideal.

The only irreducible $*$-representations of $\mathcal{K}_0$ are one-dimensional. They depend on a real parameter α and are defined on the single basis vector $|\alpha\rangle$ by

$$K|\alpha\rangle = \alpha|\alpha\rangle\,, \qquad N_\pm|\alpha\rangle = 0\,, \qquad N_3|\alpha\rangle = |\alpha\rangle\,. \tag{3.105}$$

Unlike for the undeformed case, no infinite-dimensional irreducible representation exists.

Chapter 4

Massive Spin Representations

4.1 Representations in an Angular Momentum Basis

4.1.1 The Complete Set of Commuting Observables

We want to construct a massive irreducible representations in a basis that can be given a physical interpretation. Massive irreducible means that within the representation we have

$$P_\mu P^\mu = m^2 \tag{4.1}$$

for some real positive constant m, $P_\mu P^\mu$ being the mass Casimir operator. We have shown in Sec. 3.3.2 that there are rest states, that is, momentum eigenstates, $P_\mu|\psi_0\rangle = p_\mu|\psi_0\rangle$, with $(p_\mu) = (p_0, p_-, p_+, p_3) = (m, 0, 0, 0)$. On these rest states the q-Pauli-Lubanski vector acts as

$$W_{\tilde{0}}|\psi_0\rangle = m\lambda^{-1}(W-1)|\psi_0\rangle\,, \qquad W_{\tilde{A}}|\psi_0\rangle = -mS(J_A)|\psi_0\rangle\,, \tag{4.2}$$

from which it follows that

$$W^{\tilde{\mu}}W_{\tilde{\mu}}|\psi_0\rangle = 2m^2\lambda^{-2}(1-W)|\psi_0\rangle\,. \tag{4.3}$$

The spin Casimir $W^{\tilde{\mu}}W_{\tilde{\mu}}$ must be constant, thus, the angular momentum must be constant within the rest frame. According to Eq. (A.33) the possible values are

$$W|\psi_0\rangle = [2]^{-1}\big(q^{(2s+1)} + q^{-(2s+1)}\big)|\psi_0\rangle\,, \tag{4.4}$$

where $s \in \frac{1}{2}\mathbb{N}_0$ is a half integer. For the spin Casimir this means

$$W^{\tilde{\mu}}W_{\tilde{\mu}}|\psi_0\rangle = -2[2]^{-1}m^2[s+1][s]|\psi_0\rangle\,. \tag{4.5}$$

In accordance with the undeformed case we will call s the spin of the representation. The space of all rest states is stabilized by the algebra generated by the little algebra for the massive case, $\mathcal{U}_q(\mathrm{su}_2)$, and the momenta, that is, by the q-Euclidean algebra $\mathcal{E}_q$. The observables that are most commonly diagonalized are all elements of $\mathcal{E}_q$: energy P_0, momentum $\vec{P}$, angular momentum $\vec{J}$, helicity $\vec{J}\cdot\vec{P}$.

We opt for an angular momentum basis, where we diagonalize J_3 and $\vec{J}^2 = \vec{J}\cdot\vec{J}$. If we add the Casimir operators of $\mathcal{E}_q$, P_0 and Z as defined in Eq. (3.58), we get a complete set of commuting observables.[1] Instead of J_3 and $\vec{J}^2$ it is more practical to work with K and W, whose possible eigenvalues can be looked up in Sec. A.2.1. From Sec. 3.2.1 we know that P_0 and Z are Casimir operators of a $\mathcal{U}_q(\mathrm{su}_2)$ algebra, so we know their possible eigenvalues, as well. Labeling the states of the yet to be constructed representation by their possible eigenvalues we get including the Casimirs

$$\begin{aligned}
K|j,m,n,k\rangle &= q^{2m}|m,j,n,k\rangle && (4.6a)\\
W|j,m,n,k\rangle &= [2]^{-1}\big(q^{(2j+1)}+q^{-(2j+1)}\big)|j,m,n,k\rangle && (4.6b)\\
P_0|j,m,n,k\rangle &= m[2]^{-1}\big(q^{(2n+1)}+q^{-(2n+1)}\big)|j,m,n,k\rangle && (4.6c)\\
Z|j,m,n,k\rangle &= m[2]^{-1}\big(q^{(2k+1)}+q^{-(2k+1)}\big)|j,m,n,k\rangle && (4.6d)\\
P_\mu P^\mu|j,m,n,k\rangle &= m^2|j,m,n,k\rangle && (4.6e)\\
W^{\tilde{\mu}}W_{\tilde{\mu}}|j,m,n,k\rangle &= -2[2]^{-1}m^2[s+1][s]|j,m,n,k\rangle\,. && (4.6f)
\end{aligned}$$

The eigenvalues of W, P_0, and Z are all of the same form, $\xi(j)$, $m\xi(n)$, and $m\xi(k)$, where

$$\xi(j) := [2]^{-1}\big(q^{(2j+1)}+q^{-(2j+1)}\big)\,. \tag{4.7}$$

For the operators with a more obvious undeformed limit J_3, $\vec{J}^2$, and $\vec{J}\cdot\vec{P}$ we get

$$\begin{aligned}
J_3|j,m,n,k\rangle &= \big(q^m[m]-\lambda[2]^{-2}[2j+2][2j]\big)|m,j,n,k\rangle\\
\vec{J}^2|j,m,n,k\rangle &= [2]^{-2}[2j+2][2j]|m,j,n,k\rangle\\
(\vec{J}\cdot\vec{P})|j,m,n,k\rangle &= \lambda[2]^{-2}\big([n+j+k+2][n+j-k]\\
&\qquad + [n-j+k][n-j-k]\big)|m,j,n,k\rangle\,,
\end{aligned} \tag{4.8}$$

which shows why it is more efficient to work with K, W, and Z instead.

One further advantage of using an angular momentum basis is, that the q-Wigner-Eckart theorem of Page 26 applies. The problem of finding the matrix elements of 3-vector or scalar operators with respect to rotations is reduced to finding the reduced matrix elements. For 3-vector operators such as P_A, J_A, R_A, and S_A we get

$$\langle j',m',n',k'|P_A|j,m,n,k\rangle = \mathrm{C}_q(1,j,j'\,|\,A,m,m')\langle j',n',k'\|\vec{P}\|j,n,k\rangle\,, \tag{4.9}$$

while for scalars with respect to rotations such as Z, W, U, and V we get

$$\langle j',m',n',k'|Z|j,m,n,k\rangle = \delta_{mm'}\delta_{jj'}\langle j',n',k'\|Z\|j,n,k\rangle\,. \tag{4.10}$$

The values of the q-Clebsch-Gordan coefficients that we will need are given in Sec. A.1.1. Useful relations for the reduced matrix elements can be derived from Eq. (2.21), which has been done explicitly in Eqs. (A.19).

[1] The authors of [34,35] failed to add Z or $\vec{J}\cdot\vec{P}$ to their set of commuting observables (cf. [35], p. 67). This is the reason why they only found spin zero representations.

4.1.2 Representations of the q-Euclidean Algebra

If we keep n and k constant, we fix the eigenvalues of the Casimirs operators P_0 and Z of the q-Euclidean algebra $\mathcal{E}_q$. For constant n, k we must thus get an irreducible representation of $\mathcal{E}_q$. This irreducible representation of $\mathcal{E}_q$ on the mass shell is by isomorphism (3.50) simply the product $D^n \otimes D^k$ of two representations of $\mathcal{U}_q(\mathrm{su}_2)$. We describe them briefly in terms of reduced matrix elements.

The reduced matrix element of J_A can be read off Eq. (A.32),

$$\langle j \| \vec{J} \| j \rangle = -[2]^{-1} \sqrt{[2j+2][2j]} \,. \tag{4.11}$$

Due to the Clebsch-Gordan series (2.2) j takes on the values $\{|k-n|, |k-n|+1, \dots, k+n\}$. Taking the matrix elements of Eq. (4.6d) we find

$$\langle j,n,k \| \vec{P} \| j,n,k \rangle = m\lambda \, \frac{[k+n+j+2][j-k+n] - [k+n-j][j+k-n]}{[2]\sqrt{[2j+2][2j]}} \,. \tag{4.12}$$

If we take the diagonal matrix elements of the relation $P_A P_B \, \varepsilon^{AB}{}_C = -\lambda P_0 P_C$ and of Eq. (4.6f) we get, using Eqs. (A.19), two equations for the reduced matrix elements from which we can eliminate the $\langle j \| \vec{P} \| j-1 \rangle \langle j-1 \| \vec{P} \| j \rangle$ term

$$\begin{aligned} &[2]\sqrt{[2j+3][2j+1]} \, \langle j \| \vec{P} \| j+1 \rangle \langle j+1 \| \vec{P} \| j \rangle = \\ &\qquad [2j] \langle j \| \vec{P} \| j \rangle^2 + \lambda E \sqrt{[2j+2][2j]} \langle j \| \vec{P} \| j \rangle - [2j+2](P^0 P^0 - m^2) \,. \end{aligned} \tag{4.13}$$

Upon inserting Eq. (4.12),

$$\begin{aligned} &\langle j \| \vec{P} \| j+1 \rangle \langle j+1 \| \vec{P} \| j \rangle = \\ &\qquad - m^2 \lambda^2 \frac{[k+n+j+2][k+n-j][k-n+j+1][n-k+j+1]}{[2][2j+2]\sqrt{[2j+3][2j+1]}} \,, \end{aligned} \tag{4.14}$$

and using Eq. (A.19e) we finally get

$$\begin{aligned} &\langle j+1,n,k \| \vec{P} \| j,n,k \rangle = \\ &\qquad m\lambda \frac{\sqrt{[k+n+j+2][k+n-j][k-n+j+1][n-k+j+1]}}{\sqrt{[2][2j+3][2j+2]}} \end{aligned} \tag{4.15a}$$

$$\begin{aligned} &\langle j-1,n,k \| \vec{P} \| j,n,k \rangle = \\ &\qquad - m\lambda \frac{\sqrt{[k+n+j+1][k+n-j+1][k-n+j][n-k+j]}}{\sqrt{[2][2j][2j-1]}} \,. \end{aligned} \tag{4.15b}$$

4.1.3 Possible Transitions of Energy and Helicity

Next, we will determine the possible transitions of the quantum numbers n and k under the action of the non-Euclidean generators. To find restrictions on the possible transitions we consider Eq. (3.35e) and the contraction of Eq. (3.35a) with $g^{DC}P_D$ from the left

$$[2]^2UP^0 = [4]P^0U - q^{-1}\lambda^2[2](\vec{P}\cdot\vec{R}) \tag{4.16a}$$

$$[2](\vec{P}\cdot\vec{R})P_0 = 2P^0(\vec{P}\cdot\vec{R}) - q(\vec{P}\cdot\vec{P})U\,. \tag{4.16b}$$

Taking the matrix elements of these equations yields a system of linear equations

$$0 = m\big([4]\xi(n') - [2]^2\xi(n)\big)\langle U\rangle \qquad -q^{-1}\lambda^2[2]\langle\vec{P}\cdot\vec{R}\rangle \tag{4.17a}$$

$$0 = \qquad m^2q(1-\xi(n')^2)\langle U\rangle + m\big(2\xi(n') - [2]\xi(n)\big)\langle\vec{P}\cdot\vec{R}\rangle\,, \tag{4.17b}$$

where we have used the abbreviation $\langle U\rangle := \langle j,m,n',k'|U|j,m,n,k\rangle$ and analogously for $\langle\vec{P}\cdot\vec{R}\rangle$. For a nontrivial solution to exist, the determinant of the coefficient matrix must vanish,

$$\begin{aligned}0 &\overset{!}{=} m^2[2]^2(\xi(n')^2 - [2]\xi(n')\xi(n') + \xi(n)^2) + m^2\lambda^2\\ &= m^2\lambda^4\,\left[n+\tfrac{1}{2}-n'\right]\left[n-\tfrac{1}{2}-n'\right]\left[n+n'+\tfrac{1}{2}\right]\left[n+n'+\tfrac{3}{2}\right]\,,\end{aligned} \tag{4.18}$$

which is, since $n \geq 0$, precisely the case for $n' = n \pm \frac{1}{2}$.

To obtain conditions on the transitions of k we contract Eqs. (3.35g) and (3.35a) with $g^{DC}J_D$ from the left

$$[2]^2U(\vec{J}\cdot\vec{P}) = [4](\vec{L}\cdot\vec{P})U - q\lambda^2[2]P_0(\vec{L}\cdot\vec{R}) + \mathrm{i}\lambda^2[2]\,\vec{J}\cdot(\vec{P}\times\vec{R}) \tag{4.19a}$$

$$[2](\vec{J}\cdot\vec{R})P_0 = -q[2](\vec{J}\cdot\vec{P})U + [4]P_0(\vec{J}\cdot\vec{R}) - \mathrm{i}\lambda[2]\,\vec{J}\cdot(\vec{P}\times\vec{R})\,. \tag{4.19b}$$

Contracting Eq. (3.35c) with J^BJ^A from the right and eliminating the $\vec{P}\cdot\vec{R}$ term using Eq. (4.16a) yields

$$\begin{aligned}\lambda(\vec{J}\cdot\vec{R})(\vec{J}\cdot\vec{P}) &= q^2\{q\lambda[2]\,(\vec{J}\cdot\vec{P}) - \lambda^2WP_0\}(\vec{J}\cdot\vec{P}) + (\vec{J}\cdot\vec{J})UP_0\\ &\quad - \{q(\vec{J}\cdot\vec{J})P_0 - \lambda W(\vec{J}\cdot\vec{P})\}U - 2\mathrm{i}q^{-1}[2]^{-1}\lambda W\,\vec{J}\cdot(\vec{P}\times\vec{R})\,.\end{aligned} \tag{4.19c}$$

Eliminating the $\vec{J}\cdot(\vec{P}\times\vec{R})$ term from the last three equations we obtain

$$\lambda^2\{(\vec{J}\cdot\vec{R})Z - qZ(\vec{J}\cdot\vec{R})\} = q(P_0 - WZ)U - U(P_0 - WZ) \tag{4.20a}$$

$$\lambda^2\{(\vec{J}\cdot\vec{R})P_0 - q^{-1}P_0(\vec{J}\cdot\vec{R})\} = q^{-1}(Z - WP^0)U - U(Z - WP^0)\,. \tag{4.20b}$$

Again we take the matrix elements of these two equations

$$\begin{aligned}0 &= \{[\xi(n) - \xi(j)\xi(k)] - q[\xi(n') - \xi(j)\xi(k')]\}\langle U\rangle + \lambda^2\{\xi(k) - q\xi(k')\}\langle\vec{J}\cdot\vec{R}\rangle\\ 0 &= \{q[\xi(k) - \xi(j)\xi(n)] - [\xi(k') - \xi(j)\xi(n')]\}\langle U\rangle + \lambda^2\{q\xi(n) - \xi(n')\}\langle\vec{J}\cdot\vec{R}\rangle\,.\end{aligned} \tag{4.21}$$

Provided Eq. (4.18) holds, the determinant condition for a nontrivial solution is

$$\begin{aligned}0 &= [2]^2\{[\xi(k')^2 - [2]\xi(k')\xi(k') + \xi(k)^2] - [\xi(n')^2 - [2]\xi(n')\xi(n') + \xi(n)^2]\} \\ &= [2]^2[\xi(k')^2 - [2]\xi(k')\xi(k') + \xi(k)^2] + \lambda^2 \\ &= \lambda^4 \left[k + \tfrac{1}{2} - k'\right]\left[k - \tfrac{1}{2} - k'\right]\left[k + k' + \tfrac{1}{2}\right]\left[k + k' + \tfrac{3}{2}\right] ,\end{aligned} \tag{4.22}$$

which is fulfilled precisely for $k' = k \pm \frac{1}{2}$. We conclude that the possible transitions of the quantum numbers n and k are $n \to n \pm \frac{1}{2}$ and $k \to k \pm \frac{1}{2}$.

4.1.4 Dependence on Total Angular Momentum

Eq. (4.21) establishes a correspondence between the reduced matrix elements of $\vec{J} \cdot \vec{R}$ and U. With Eq. (4.11) we get for $j > 0$

$$\begin{aligned}\langle j, n', k' \| \vec{R} \| j, n, k\rangle &= \left(\frac{\xi(n) - q\xi(n')}{\xi(k) - q\xi(k')} - \xi(j)\right)\frac{[2]\langle j, n', k' \| U \| j, n, k\rangle}{\lambda^2\sqrt{[2j+2][2j]}} \\ &=: A_1(n', k', n, k, j)\langle j, n', k' \| U \| j, n, k\rangle .\end{aligned} \tag{4.23}$$

The reduced matrix elements of Eq. (3.35a) between $\langle j+1, n', k'\|$ and $\|j, n, k\rangle$, $\langle j-1, n', k'\|$ and $\|j, n, k\rangle$ yield

$$\langle j+1, n', k' \| \vec{R} \| j, n, k\rangle = A_2(n', k', n, k, j)\langle j, n', k' \| U \| j, n, k\rangle \tag{4.24a}$$

$$\langle j-1, n', k' \| \vec{R} \| j, n, k\rangle = A_3(n', k', n, k, j)\langle j, n', k' \| U \| j, n, k\rangle , \tag{4.24b}$$

where

$$A_2 := \frac{(\lambda\sqrt{\frac{[2j]}{[2j+2]}}\,A_1 - q)\langle j+1, n', k' \| \vec{P} \| j, n', k'\rangle}{m\big([2]\xi(n) - \frac{[4]}{[2]}\xi(n')\big) + \lambda\sqrt{\frac{[2j+4]}{[2j+2]}}\langle j+1, n', k' \| \vec{P} \| j+1, n', k'\rangle} \tag{4.25a}$$

$$A_3 := \frac{-(\lambda\sqrt{\frac{[2j+2]}{[2j]}}\,A_1 + q)\langle j-1, n', k' \| \vec{P} \| j, n', k'\rangle}{m\big([2]\xi(n) - m\frac{[4]}{[2]}\xi(n')\big) - \lambda\sqrt{\frac{[2j-4]}{[2j]}}\langle j-1, n', k' \| \vec{P} \| j-1, n', k'\rangle} . \tag{4.25b}$$

This again can be used to calculate the reduced matrix elements of Eq. (3.35g) between $\langle j+1, n', k'\|$ and $\|j, n, k\rangle$

$$\langle j+1, n', k' \| U \| j+1, n, k\rangle = A_4(n', k', n, k, j)\langle j, n', k' \| U \| j, n, k\rangle , \tag{4.26}$$

where

$$\begin{aligned}A_4 := \Bigg\{&\left(\tfrac{[4]}{[2]} - \lambda^2\sqrt{\tfrac{[2j]}{[2j+2]}}\,A_1\right)\langle j+1, n', k' \| \vec{P} \| j, n', k'\rangle - \lambda^2 A_2\Big(mq\xi(n') \\ &- \sqrt{\tfrac{[2j+4]}{[2j+2]}}\,\langle j+1, n', k' \| \vec{P} \| j+1, n', k'\rangle\Big)\Bigg\}[2]^{-1}\langle j+1, n, k \| \vec{P} \| j, n, k\rangle^{-1} .\end{aligned} \tag{4.27}$$

The calculation of the auxiliary functions A_1, A_2, A_3, and A_4 is elementary but lengthy.[2] The results can be written most compactly introducing the functions $u(n', k', n, k)$ and $v(n', k', n, k)$ by

$$\begin{aligned} u(n+\Delta n, k+\Delta k, n, k) &:= \Delta n\,(2n+1) + \Delta k\,(2k+1) \\ v(n+\Delta n, k+\Delta k, n, k) &:= \Delta n\,(2n+1) - \Delta k\,(2k+1)\,, \end{aligned} \tag{4.28}$$

for $\Delta n, \Delta k = \pm\frac{1}{2}$, that is,

$$\begin{aligned} n' &= n - \tfrac{1}{2}\,, & k' &= k - \tfrac{1}{2} & \Rightarrow \quad u &= -n-k-1\,, & v &= -n+k \\ n' &= n - \tfrac{1}{2}\,, & k' &= k + \tfrac{1}{2} & \Rightarrow \quad u &= -n+k\,, & v &= -n-k-1 \\ n' &= n + \tfrac{1}{2}\,, & k' &= k - \tfrac{1}{2} & \Rightarrow \quad u &= n-k\,, & v &= n+k+1 \\ n' &= n + \tfrac{1}{2}\,, & k' &= k + \tfrac{1}{2} & \Rightarrow \quad u &= n+k+1\,, & v &= n-k \end{aligned} \tag{4.29}$$

Using u we can write A_4 as

$$A_4(n', k', n, k, j) = \frac{\sqrt{[j+u+2][j-u+1]}}{\sqrt{[j+u+1][j-u]}} = \frac{A_5(n', k', n, k, j+1)}{A_5(n', k', n, k, j)}\,, \tag{4.30}$$

where

$$A_5(n', k', n, k, j) := \sqrt{[j+u+1][j-u]}\,. \tag{4.31}$$

Defining

$$\langle n', k' \| U \| n, k \rangle := \frac{\langle j, n', k' \| U \| j, n, k \rangle}{A_5(n', k', n, k, j)}\,, \tag{4.32}$$

Eq. (4.26) tells us by induction that $\langle n', k' \| U \| n, k \rangle$ does not depend on j. With Eqs. (4.23) and (4.24) we conclude that the j-dependence of all reduced matrix elements can be absorbed in reduction coefficients according to

$$\begin{aligned} \langle j', n', k' \| U \| j, n, k \rangle &= B^0_q(j', n', k' \,|\, j, n, k) \langle n', k' \| U \| n, k \rangle \\ \langle j', n', k' \| \vec{R} \| j, n, k \rangle &= B^1_q(j', n', k' \,|\, j, n, k) \langle n', k' \| U \| n, k \rangle\,, \end{aligned} \tag{4.33}$$

if we define the coefficients as

$$B^0_q(j', n', k' \,|\, j, n, k) := \begin{cases} A_5(n', k', n, k, j)\,, & j' = j > 0 \\ 0\,, & \text{else} \end{cases} \tag{4.34a}$$

$$B^1_q(j', n', k' \,|\, j, n, k) := \begin{cases} A_3(n', k', n, k, j) A_5(n', k', n, k, j)\,, & j' = j-1 \\ A_1(n', k', n, k, j) A_5(n', k', n, k, j)\,, & j' = j > 0 \\ A_2(n', k', n, k, j) A_5(n', k', n, k, j)\,, & j' = j+1 \\ 0\,, & \text{else}\,. \end{cases} \tag{4.34b}$$

[2] The calculation of the auxiliary functions has been done by computer algebra [59].

Explicitly, the formulas for the B-coefficients are

$$
\begin{aligned}
B^0_q(j', n', k' \,|\, j, n, k) &= \delta_{jj'}\sqrt{[j+u+1][j-u]} \\
B^1_q(j-1, n', k' \,|\, j, n, k) &= -\frac{q^{-j}\sqrt{[2][j+v][j-v][j-u][j-u-1]}}{\lambda\sqrt{[2j][2j-1]}} \\
B^1_q(j, n', k' \,|\, j, n, k) &= -(q^{(j+1)}[j-v] - q^{-(j+1)}[j+v])\frac{\sqrt{[j+u+1][j-u]}}{\lambda\sqrt{[2j+2][2j]}} \\
B^1_q(j+1, n', k' \,|\, j, n, k) &= -\frac{q^{j+1}\sqrt{[2][j+v+1][j-v+1][j+u+2][j+u+1]}}{\lambda\sqrt{[2j+3][2j+2]}}\,,
\end{aligned}
\tag{4.35}
$$

which can be written more compactly as

$$
B^\alpha_q(j', n', k' \,|\, j, n, k) =
$$

$$
(-\lambda)^{-\alpha}\,\mathrm{C}_q(\alpha, j', j \,|\, 0, v, v) \times \begin{cases} q^{-j}\sqrt{[j'-u+1][j'-u]}\,, & j' = j-1 \\ \sqrt{[j'+u+1][j'-u]}\,, & j' = j \\ -q^{j+1}\sqrt{[j'+u+1][j'+u]}\,, & j' = j+1\,. \end{cases} \tag{4.36}
$$

4.1.5 Dependence on the other Quantum Numbers

Using the B-coefficients, equations in the reduced matrix elements of R, U can be reduced further to equations in the double reduced matrix elements $\langle n', k' \| U \| n, k\rangle$ as defined in Eq. (4.32). We start by taking the matrix elements of the RR-relations (2.70a), $U^2 - \lambda^2\,(\vec{R}\cdot\vec{R}) = 1$ and $R^A U - U R^A = 0$ between $\langle j, n, k\|$ and $\|j, n, k\rangle$. We obtain

$$
\sum_{n',k'} A_6(n', k', n, k, j)\langle n, k\|U\|n', k'\rangle\langle n', k'\|U\|n, k\rangle = 1 \tag{4.37a}
$$

$$
\sum_{n',k'} A_7(n', k', n, k, j)\langle n, k\|U\|n', k'\rangle\langle n', k'\|U\|n, k\rangle = 0\,, \tag{4.37b}
$$

where the summation indices run through $n' = n \pm \frac{1}{2}$, $k' = k \pm \frac{1}{2}$ and

$$
\begin{aligned}
A_6(n', k', n, k, j) &:= B^0_q(j, n, k \,|\, j, n', k')B^0_q(j, n', k' \,|\, j, n, k) \\
&\quad - \lambda^2 \sum_{j'=j-1}^{j+1} (-1)^{j'-j}\sqrt{\tfrac{[2j'+1]}{[2j+1]}}\, B^1_q(j, n, k \,|\, j', n', k')B^1_q(j', n', k' \,|\, j, n, k)
\end{aligned}
\tag{4.38a}
$$

$$
\begin{aligned}
A_7(n', k', n, k, j) &:= -\frac{\lambda}{[2]}\sqrt{[2j+2][2j]}\Big\{B^1_q(j, n, k \,|\, j, n', k')B^0_q(j, n', k' \,|\, j, n, k) \\
&\quad - B^0_q(j, n, k \,|\, j, n', k')B^1_q(j, n', k' \,|\, j, n, k)\Big\}\,.
\end{aligned}
\tag{4.38b}
$$

The values of these coefficients are

$$\begin{aligned} A_6(n',k',n,k,j) &= 4\Delta k\,\Delta n\,[2][2k'+1][2n'+1] \\ A_7(n',k',n,k,j) &= [2v][j+u+1][j-u] = \lambda^{-2}[2][2v]\big(\xi(j)-\xi(u)\big)\,. \end{aligned} \tag{4.39}$$

Eq. (4.37b) must hold for all values of j, which turns out to lead to two independent equations. Thus, Eqs. (4.37) form a system of three independent equations in four unknowns of the type $\langle n,k\|U\|n',k'\rangle\langle n',k'\|U\|n,k\rangle$. Eliminating two unknowns in each equation we can interpret them as recursion relations

$$\rho(\mu,\nu) = \rho(\mu,\nu-1) + [2\nu+2] \tag{4.40a}$$

$$\omega(\mu,\nu) = \omega(\mu+1,\nu) + [2\mu] \tag{4.40b}$$

$$\omega(\mu+1,\nu) = -\rho(\mu,\nu) + [\nu+\mu+2][\nu-\mu+1] \tag{4.40c}$$

where we use the abbreviations $\mu := k-n$, $\nu := k+n$ and

$$\rho(\mu,\nu) := [2]^2[2k+2][2k+1][2n+2][2n+1] \times \langle n,k\|U\|n+\tfrac{1}{2},k+\tfrac{1}{2}\rangle\langle n+\tfrac{1}{2},k+\tfrac{1}{2}\|U\|n,k\rangle \tag{4.41a}$$

$$\omega(\mu,\nu) := [2]^2[2k+1][2k][2n+2][2n+1] \times \langle n,k\|U\|n+\tfrac{1}{2},k-\tfrac{1}{2}\rangle\langle n+\tfrac{1}{2},k-\tfrac{1}{2}\|U\|n,k\rangle\,. \tag{4.41b}$$

In order to determine the initial conditions, we recall Eq. (4.4) which tells us that $n=0$ implies $k=s$. Hence, matrix elements involving states with $n=0$ and $k\neq s$ have to vanish, in particular

$$\rho(s,s-1) = 0\,. \tag{4.42}$$

The solution of recursion relation (4.40a) with this initial value is

$$\rho(s,\nu) = \sum_{\nu'=s}^{\nu}[2\nu'+2] = [\nu+s+2][\nu-s+1] \tag{4.43}$$

where we used $\sum_{i'=a}^{b}[2i'+c] = [a+b+c][b-a+1]$. Inserting this result in Eq. (4.40c) yields $\omega(s+1,\nu)=0$. The solution of Eq. (4.40b) with this initial value is

$$\omega(\mu,\nu) = \sum_{\mu'=\mu}^{s}[2\mu'] = [\mu+s][s-\mu+1]\,. \tag{4.44}$$

Inserting this again in Eq. (4.40c) results in

$$\begin{aligned} \omega(\mu,\nu) &= \omega(\mu) = [\mu+s][s-\mu+1] \\ \rho(\mu,\nu) &= \rho(\nu) = [\nu+s+2][\nu-s+1] \end{aligned} \tag{4.45}$$

for $-s \leq \mu \leq s+1$ and $s-1 \leq \nu$. At the border of this half-closed strip in $\mu\nu$-space ρ and ω vanish, so there are no transitions to the outside. For an irreducible representation we must not have two disconnected regions, hence, ρ and ω must vanish outside this strip. The allowed quantum numbers form a strip in nk-space given by

$$|\mu| = |k-n| \leq s\,, \qquad\qquad \nu = n+k \geq s\,. \tag{4.46}$$

To derive from Eq. (4.45) formulas for the matrix elements we need to take the RS-relations (2.70c) into account. We begin with the matrix elements of $UV = VU$ between $\langle j, n+\frac{1}{2}, k+\frac{1}{2}\|$ and $\|j, n-\frac{1}{2}, k-\frac{1}{2}\rangle$ using the conjugation $U^* = V$ to obtain

$$\begin{aligned}\langle n+\tfrac{1}{2}, k+\tfrac{1}{2}\|U\|n,k\rangle\overline{\langle n-\tfrac{1}{2}, k-\tfrac{1}{2}\|U\|n,k\rangle} = \\ \overline{\langle n,k\|U\|n+\tfrac{1}{2}, k+\tfrac{1}{2}\rangle}\langle n,k\|U\|n-\tfrac{1}{2}, k-\tfrac{1}{2}\rangle\,,\end{aligned} \tag{4.47}$$

which can be written as

$$\frac{\overline{\langle \mu, \nu-1\|U\|\mu,\nu\rangle}}{\langle \mu,\nu\|U\|\mu,\nu-1\rangle} = \frac{\overline{\langle \mu,\nu\|U\|\mu,\nu+1\rangle}}{\langle \mu,\nu+1\|U\|\mu,\nu\rangle}\,. \tag{4.48}$$

with $\mu := k-n$, $\nu := k+n$ as above. Reading this as recursion relation, it follows that

$$\overline{\langle \mu,\nu-1\|U\|\mu,\nu\rangle} = \alpha_\mu\langle \mu,\nu\|U\|\mu,\nu-1\rangle\,, \tag{4.49a}$$

where the yet to be determined number α_μ may depend on μ but not on ν. Taking the matrix elements of $UU' = U'U$ between $\langle j, n+\frac{1}{2}, k+\frac{1}{2}\|$ and $\|j, n-\frac{1}{2}, k-\frac{1}{2}\rangle$, it follows analogously that

$$\overline{\langle \mu,\nu\|U\|\mu-1,\nu\rangle} = \beta_\nu\langle \mu-1,\nu\|U\|\mu,\nu\rangle\,, \tag{4.49b}$$

with β_ν independent of μ.

Next, we take the diagonal matrix elements of $W = UV + q^2\lambda^2(\vec{R}\cdot\vec{S})$ as in Eq. (2.77) using the conjugation relations (A.19e) to obtain

$$\sum_{n',k'} A_8(n',k',n,k,j)|\langle n,k\|U\|n',k'\rangle|^2 = \xi(j)\,, \tag{4.50}$$

where

$$A_8(n',k',n,k,j) := |B_q^0(j,n,k\,|\,j,n',k')|^2 + q^2\lambda^2\sum_{j'=j-1}^{j+1}|B_q^1(j,n,k\,|\,j',n',k')|^2\,. \tag{4.51}$$

Eq. (4.50) must hold for all possible values of j, thus yielding two independent equations from which we can derive

$$[2]^{-2}[\mu+\nu+1]^{-1} = q^{-2\mu}[\nu-\mu]|\langle\mu,\nu\|U\|\mu,\nu-1\rangle|^2 + q^{-2(\nu+1)}[\nu-\mu+2]|\langle\mu,\nu\|U\|\mu-1,\nu\rangle|^2 . \quad (4.52)$$

Relations (4.46) tell us that the first term on the right hand side vanishes for $\nu = s$ while the second vanishes for $\mu = -s$, that is,

$$\begin{aligned} |\langle s,\nu\|U\|s,\nu-1\rangle|^2 &= \frac{q^{-2s}}{[2]^2[\nu-s+1][\nu+s]} \\ |\langle\mu,s\|U\|\mu-1,s\rangle|^2 &= \frac{q^{2(s+1)}}{[2]^2[\mu+s+1][s-\mu+2]} . \end{aligned} \quad (4.53)$$

If we compare this with $\rho(-s,\nu-1)$ and $\omega(\mu,s)$ as computed in Eqs. (4.45), we find

$$\alpha_\mu = q^{2s} , \qquad \beta_\nu = q^{2(s+1)} . \quad (4.54)$$

With this result Eqs. (4.45) can be written as formulas for the squares of matrix elements. For example,

$$\begin{aligned} &[\mu+s][s-\mu+1] = \omega(\mu,\nu) \\ &\quad = [2]^2[2k+1][2k][2n+2][2n+1]\langle n,k\|U\|n+\tfrac{1}{2},k-\tfrac{1}{2}\rangle\langle n+\tfrac{1}{2},k-\tfrac{1}{2}\|U\|n,k\rangle \\ &\quad = q^{2(s+1)}[2]^2[2k+1][2k][2n+2][2n+1]\ |\langle n+\tfrac{1}{2},k-\tfrac{1}{2}\|U\|n,k\rangle|^2 . \end{aligned} \quad (4.55)$$

This is an equation for the absolute value of the double reduced matrix elements. In fact, none of the commutation relations of the q-Poincaré algebra gives us a condition on the phase of the reduced matrix elements, that is, the phase can be chosen arbitrarily. We choose it, such that

$$\langle n+\tfrac{1}{2},k-\tfrac{1}{2}\|U\|n,k\rangle = \frac{q^{-2(s+1)}\sqrt{[s+k-n][s-k+n+1]}}{[2]\sqrt{[2k+1][2k][2n+2][2n+1]}} . \quad (4.56)$$

Analogously, we determine the other matrix elements. The end result is

$$\langle n',k'\|U\|n,k\rangle = \frac{q^{2(n-n')s+(n'-k'-n+k)}\sqrt{[s+u+1][s-u]}}{[2]\sqrt{[k'+k+\frac{3}{2}][k'+k+\frac{1}{2}][n'+n+\frac{3}{2}][n'+n+\frac{1}{2}]}} . \quad (4.57)$$

Summary We summarize the results for the reduced matrix elements. As before, the abbreviations u and v as defined in Eq. (4.28) are being used. The relation between the reduced and the ordinary matrix elements is given by Eqs. (4.9) and (4.10).

$$\langle j', n', k' \| \vec{J} \| j, n, k \rangle = -[2]^{-1} \delta_{jj'} \delta_{nn'} \delta_{kk'} \sqrt{[2j+2][2j]} \tag{4.58a}$$

$$\langle j-1, n', k' \| \vec{P} \| j, n, k \rangle = -m\lambda \delta_{nn'} \delta_{kk'} \times \frac{\sqrt{[k+n+j+1][k+n-j+1][k-n+j][n-k+j]}}{\sqrt{[2][2j][2j-1]}} \tag{4.58b}$$

$$\langle j, n', k' \| \vec{P} \| j, n, k \rangle = m\lambda \delta_{nn'} \delta_{kk'} \times \frac{[k+n+j+2][j-k+n] - [k+n-j][j+k-n]}{[2]\sqrt{[2j+2][2j]}} \tag{4.58c}$$

$$\langle j+1, n, k \| \vec{P} \| j, n, k \rangle = m\lambda \delta_{nn'} \delta_{kk'} \times \frac{\sqrt{[k+n+j+2][k+n-j][k-n+j+1][n-k+j+1]}}{\sqrt{[2][2j+3][2j+2]}} \tag{4.58d}$$

$$\langle j', n', k' \| U \| j, n, k \rangle = \delta_{jj'}\, q^{2(n-n')s+(n'-k'-n+k)} \times \frac{\sqrt{[j+u+1][j-u][s+u+1][s-u]}}{[2]\sqrt{[k'+k+\frac{3}{2}][k'+k+\frac{1}{2}][n'+n+\frac{3}{2}][n'+n+\frac{1}{2}]}}\,. \tag{4.58e}$$

$$\langle j', n', k' \| \vec{R} \| j, n, k \rangle = \frac{q^{2(n-n')s+(n'-k'-n+k)}\sqrt{[s+u+1][s-u]}}{\lambda[2]\sqrt{[k'+k+\frac{3}{2}][k'+k+\frac{1}{2}][n'+n+\frac{3}{2}][n'+n+\frac{1}{2}]}} \times \mathrm{C}_q(1, j', j \,|\, 0, v, v) \times \begin{cases} -q^{-j}\sqrt{[j'-u+1][j'-u]}\,, & j' = j-1 \\ -\sqrt{[j'+u+1][j'-u]}\,, & j' = j \\ q^{j+1}\sqrt{[j'+u+1][j'+u]}\,, & j' = j+1\,. \end{cases} \tag{4.58f}$$

$$\langle j', n', k' \| V \| j, n, k \rangle = \delta_{jj'}\, q^{2(n'-n)s+(n-k-n'+k')} \times \frac{\sqrt{[j+u+1][j-u][s+u+1][s-u]}}{[2]\sqrt{[k'+k+\frac{3}{2}][k'+k+\frac{1}{2}][n'+n+\frac{3}{2}][n'+n+\frac{1}{2}]}}\,. \tag{4.58g}$$

$$\langle j', n', k' \| \vec{S} \| j, n, k \rangle = \frac{q^{2(n'-n)s+(n-k-n'+k')}\sqrt{[s+u+1][s-u]}}{\lambda[2]\sqrt{[k'+k+\frac{3}{2}][k'+k+\frac{1}{2}][n'+n+\frac{3}{2}][n'+n+\frac{1}{2}]}} \times \mathrm{C}_q(1, j', j \,|\, 0, -v, -v) \times \begin{cases} -q^{j}\sqrt{[j'-u+1][j'-u]}\,, & j' = j-1 \\ -\sqrt{[j'+u+1][j'-u]}\,, & j' = j \\ q^{-(j+1)}\sqrt{[j'+u+1][j'+u]}\,, & j' = j+1 \end{cases} \tag{4.58h}$$

4.2 Representations by Induction

We want to describe briefly how representations of the q-Poincaré algebra can be constructed using the method of induced representations.

4.2.1 The Method of Induced Representations of Algebras

Let us assume that we do have an irreducible representation of the undeformed Poincaré algebra $\mathcal{P}$ on a Hilbert space $\mathcal{H}$,

$$\sigma : \mathcal{P} \otimes \mathcal{H} \longrightarrow \mathcal{H} \,. \tag{4.59}$$

Let the situation be as in Sec. 3.3.1, where we denoted by $\mathcal{H}_p$ a momentum eigenspace and by $\mathcal{K}_p$ its stabilizer (little algebra). By definition, the restriction of σ to $\mathcal{H}_p$ defines representations on translations $\mathcal{T}$ and the little algebra $\mathcal{K}_p$ by

$$\begin{aligned} \chi_p : \mathcal{T} &\longrightarrow \mathbb{R} \,, \quad \text{where} \quad \sigma(t \otimes |\psi_p\rangle) = \chi_p(t)|\psi_p\rangle \\ \rho : \mathcal{K}_p \otimes \mathcal{H}_p &\longrightarrow \mathcal{H}_p \,, \quad \rho(k \otimes |\psi_p\rangle) = \sigma(k \otimes |\psi_p\rangle) \end{aligned} \tag{4.60}$$

for all $|\psi_p\rangle \in \mathcal{H}_p$. Together, χ_p and ρ define a representation of $\mathcal{T} \rtimes \mathcal{K}_p$ on $\mathcal{H}_p$. Let us assume for a moment that we did not know about σ but were given only χ_p and ρ. There is a generic method to extend a representation of an subalgebra to a representation of the whole algebra.

Definition 8. *Let $\mathcal{A}$ be an algebra, $\mathcal{S}$ a subalgebra and V a left $\mathcal{S}$-module. Then the tensor product of $\mathcal{A}$ and V over $\mathcal{S}$, $\mathcal{A} \otimes_{\mathcal{S}} V$ becomes a left $\mathcal{A}$-module by left multiplication. It is called the module (or representation) induced by V.*

Explicitly, $\mathcal{A} \otimes_{\mathcal{S}} V$ is the vector space $\mathcal{A} \otimes V$ (ordinary tensor product over the complex numbers), divided by the relations

$$as \otimes v = a \otimes sv \,, \quad \textit{for all} \quad a \in \mathcal{A} \,, s \in \mathcal{S} \,, v \in V \,, \tag{4.61}$$

with the left $\mathcal{A}$-action defined by

$$a'(a \otimes v) = a'a \otimes v \tag{4.62}$$

and linear extension.

For given χ_p, $\mathcal{H}_p$, ρ, and $\mathcal{K}_p$, the induced representation acts on the tensor product

$$\mathcal{P} \otimes_{\mathcal{T} \rtimes \mathcal{K}_p} \mathcal{H}_p = (\mathcal{T} \rtimes \mathcal{L}) \otimes_{\mathcal{T} \rtimes \mathcal{K}_p} \mathcal{H}_p \cong \mathcal{L} \otimes_{\mathcal{K}_p} \mathcal{H}_p. \tag{4.63}$$

While this construction may look somewhat abstract, its great practical value lies in the following

Theorem 2. *Let $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$ be the Poincaré algebra, χ_p a one dimensional representation of $\mathcal{T}$, $\mathcal{K}_p = \{k \in \mathcal{L} \,|\, \chi_p([k,t]) = 0 \quad \text{for all} \quad t \in \mathcal{T}\}$ the according little algebra, and ρ an* irreducible *representation of $\mathcal{K}_p$ on the finite vector space $\mathcal{H}_p$. With the action defined by χ_p and ρ the space $\mathcal{H}_p$ becomes a left $\mathcal{T} \rtimes \mathcal{K}_p$-module. Then the induced representation $\mathcal{P} \otimes_{\mathcal{T} \rtimes \mathcal{K}_p} \mathcal{H}_p$ is irreducible. Furthermore, all irreducible representations of $\mathcal{P}$ are of this form [55, 56].*

This means that all we have to do in order to construct the irreducible representations of $\mathcal{P}$ is

1. determine the little algebras,
2. construct the irreducible representations of the little algebras,
3. induce these representations.

Using the Lie group version of this method, Wigner [1] was the first to construct all irreducible representations of the Poincaré group (see also [60]). Theorem 2 cannot be generalized to Hopf semidirect products but in very special cases [57, 61, 62]. The method of induced representations, however, works for any algebra.

4.2.2 Induced Representations of the q-Poincaré Algebra

We will deal only with the massive case, $p = (p_\mu) = (m, 0, 0, 0) = \chi_p(P_\mu)$, where we have $\mathcal{K}_p = \mathcal{U}_q(\mathrm{su}_2)$, as calculated in Sec. 3.3.2. Let D^j be an irreducible $\mathcal{U}_q(\mathrm{su}_2)$-module. Recall (p. 42) the definition of the q-Poincaré algebra $\mathcal{P}_q = \mathcal{M}_{q_q} \rtimes \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. In the quantum double form (Sec. 2.3) the q-Lorentz algebra is $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C})) \cong SU_q(2)^{\mathrm{op}} \otimes \mathcal{U}_q(\mathrm{su}_2)$ as vector space. We conclude that the induced representation of D^j acts on the vector space

$$\begin{aligned}
\mathcal{P}_q \otimes_{\mathcal{M}_q \rtimes \mathcal{U}_q(\mathrm{su}_2)} D^j &= [\mathcal{M}_q \rtimes \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))] \otimes_{\mathcal{M}_q \rtimes \mathcal{U}_q(\mathrm{su}_2)} D^j \\
&\cong \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C})) \otimes_{\mathcal{U}_q(\mathrm{su}_2)} D^j \\
&\cong (SU_q(2)^{\mathrm{op}} \otimes \mathcal{U}_q(\mathrm{su}_2)) \otimes_{\mathcal{U}_q(\mathrm{su}_2)} D^j \\
&\cong SU_q(2)^{\mathrm{op}} \otimes D^j .
\end{aligned} \tag{4.64}$$

Let e_m be a basis of D^j. The action of some boost $h' \in SU_q(2)^{\mathrm{op}}$ on $h \otimes e_m \in SU_q(2)^{\mathrm{op}} \otimes D^j$ is simply given by left multiplication

$$h'(h \otimes e_m) = h'h \otimes e_m \,. \tag{4.65}$$

For the action of a rotation $l \in \mathcal{U}_q(\mathrm{su}_2)$ we have to commute lh using Eq. (2.57) and let l act on e_m

$$l(h \otimes e_m) = \langle S(l_{(1)}), h_{(1)} \rangle \langle l_{(3)}, h_{(3)} \rangle \, (h_{(2)} \otimes e_{m'} \, \rho^j(l_{(2)})^{m'}{}_m) \,. \tag{4.66}$$

Finally, for the action of $P_\mu \in \mathcal{M}_q$ we must use Eq. (3.30),

$$P_\mu(h \otimes \psi) = p_{\mu'} \Lambda(S^{-1} h_{(1)})^{\mu'}{}_\mu \, (h_{(2)} \otimes e_m) \,, \tag{4.67}$$

where $p_\mu = \chi_p(P_\mu)$ are the momentum eigenvalues.

We can equip this representation with a scalar product using the Haar measure of $SU_q(2)$ ([15], see also [63], pp. 111-117). An orthogonal basis is provided by the Peter-Weyl theorem ([63], pp. 106-111).

Chapter 5

Free Wave Equations

5.1 General Wave Equations

5.1.1 Wave Equations by Representation Theory

On the way from free theories to theories with interaction we need to leave the mass shell. The space of on-shell states is clearly too small as to allow for interactions where energy and momentum can be transfered from one sort of particle onto another. Moreover, we need a way to describe several particle types and their coupling in one common formalism.

These issues are resolved by introducing Lorentz spinor wave functions, that is, tensor products of the algebra of functions on spacetime with a finite vector space containing the spin degrees of freedom, the whole space carrying a tensor representation of the Lorentz symmetry. The additional mathematical structure we need to describe coupling is provided by the multiplication within the algebra of space functions. This structure is equally present in the undeformed as in the deformed case.

Using such Lorentz spinors has some consequences that have to be dealt with:

(a) The Lorentz spinor representations cannot be irreducible. Otherwise they would have to be on shell and the spinor degrees of freedom would have to carry a representation of the little algebra.

(b) The Lorentz spinor representations cannot be unitary since the spin degrees of freedom carry a finite representation of the non-compact Lorentz algebra.

The solution to these problems are:

(a) We consider only an irreducible subrepresentation to be the space of physical states. This subrepresentation is described as kernel of a linear operator $\mathbb{A}$, that is, we demand all physical states ψ to satisfy the wave equation $\mathbb{A}\psi = 0$.

(b) We introduce a non-degenerate but indefinite pseudo scalar product, such that the spinor representation becomes a $*$-representation with respect to the corresponding pseudo adjoint. This amounts to introducing a new conjugation j on states and operators.

For $\ker \mathbb{A}$ to be a subrepresentation, the operator must satisfy

$$\mathbb{A}\psi = 0 \quad \Rightarrow \quad \mathbb{A}h\psi = 0 \tag{5.1}$$

for all q-Poincaré transformations h. Depending on the particle type under consideration we might include charge and parity transformations. $\mathbb{A}$ is not unique since the wave equations for $\mathbb{A}$ and $\mathbb{A}'$ must be considered equivalent as long as their solutions are the same, $\ker(\mathbb{A}) = \ker(\mathbb{A}')$.

Ideally, the operator $\mathbb{A}$ is a projector, $\mathbb{A} = \mathbb{P}$, with $\mathbb{P}^2 = \mathbb{P}$, $\mathbb{P}^* = \mathbb{P}$. Condition (5.1) is then equivalent to

$$[\mathbb{P}, h] = 0 \tag{5.2}$$

for all q-Poincaré transformations h. Whether the wave equation is written with a projection is a matter of convenience. The Dirac equation is commonly written with such a projection which is determined uniquely (up to complement) by condition (5.2). For the Maxwell equations a projection can be found but yields a second order differential equation. For this reason, the Maxwell equations are commonly described by a more general operator $\mathbb{A}$, which leads to a first order equation. So far, all considerations pertain equally to the deformed as to the undeformed case.

5.1.2 q-Lorentz Spinors

We define a general, single particle q-Lorentz spinor wave function as element of the tensor product $\mathcal{S} \otimes \mathcal{M}_q$ of a finite vector space $\mathcal{S}$ holding the spin degrees of freedom and the space of q-Minkowski space functions $\mathcal{M}_q$ (Sec. 3.1.1).

Let $\{e_k\}$ be a basis of $\mathcal{S}$ transforming under a q-Lorentz transformation $h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ as $h \triangleright e_j = e_i\, \rho(h)^i{}_j$, where $\rho : \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C})) \to \mathrm{End}(\mathcal{S})$ is the representation map. Any spinor ψ can be written as

$$\psi = e_j \otimes \psi^j\,, \tag{5.3}$$

where j is summed over and the ψ^j are elements of $\mathcal{M}_q$. The total action of $h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ on a spinor is

$$h\psi = (h_{(1)} \triangleright e_j) \otimes (h_{(2)} \triangleright \psi^j) = e_i \otimes \rho(h_{(1)})^i{}_j (h_{(2)} \triangleright \psi^j)\,. \tag{5.4}$$

This tells us that, if we want to work directly with the $\mathcal{M}_q$-valued components ψ^j, the action of h is

$$h\psi^i = \rho(h_{(1)})^i{}_j (h_{(2)} \triangleright \psi^j)\,. \tag{5.5}$$

Do not confuse the total action $h\psi^i$ with the action of h on each component of ψ^i denoted by $h \triangleright \psi^i$. The transformation of ψ^i can easily be generalized to the case where $\mathcal{S}$ carries a tensor representation of two finite representations, that is, we have spinors with two or more indices

$$h\psi^{ij} = \rho(h_{(1)})^i{}_{i'}\rho'(h_{(2)})^j{}_{j'}(h_{(3)} \triangleright \psi^{i'j'}) , \tag{5.6}$$

where ρ and ρ' are the representation maps of the first and second index, respectively.

Furthermore, we can derive spinors by the action of tensor operators: Let T^i be a upper left ρ-tensor operator and $\psi = e_j \otimes \psi^j$ a ρ'-spinor field. Any operator T^i can be written as $T^i = \sum_k A^i_k \otimes B^i_k \in \mathrm{End}(\mathcal{S}) \otimes \mathrm{End}(\mathcal{M}_q)$ such that the action of T^i becomes

$$T^i\psi = e_j \otimes \sum_k \rho(A^i_k)^j{}_{j'} B^i_k \triangleright \psi^{j'} = e_j \otimes (T^i\psi^j) =: e_j \otimes \phi^{ij} . \tag{5.7}$$

How does this new array of wave functions $\phi^{ij} = T^i\psi^j$ transform under q-Lorentz transformations? Letting act h from the left, we find

$$h\phi^{ij} = \rho(h_{(1)})^j{}_{j'}(h_{(2)} \triangleright \phi^{ij'}) , \tag{5.8}$$

that is, h acts only on the index that came from the wave functions ψ^j. However, if we transform ϕ^{ij} by transforming ψ^j inside, we find

$$\begin{aligned} T^i(h\psi^j) &= (T^i h)\psi^j = h_{(2)}[\mathrm{ad_L} S^{-1}(h_{(1)}) \triangleright T^i]\psi^j \\ &= \rho(h_{(1)})^i{}_{i'} h_{(2)} T^{i'}\psi^j = \rho(h_{(1)})^i{}_{i'} h_{(2)}\phi^{i'j} \\ &= \rho(h_{(1)})^i{}_{i'}\rho'(h_{(2)})^j{}_{j'}(h_{(3)} \triangleright \phi^{i'j'}) . \end{aligned} \tag{5.9}$$

In other words, if ψ^j is transformed $\phi^{ij} = T^i\psi^j$ will transform as a $\rho \otimes \rho'$-spinor. Note, that for the last calculation the order in the tensor product $\mathcal{S} \otimes \mathcal{M}_q$ is essential. This reasoning would not have worked out as nicely if we had constructed the spinor space as $\mathcal{M}_q \otimes \mathcal{S}$. Chief examples of this construction are the gauge term $P^\mu\phi$ of the vector potential A^μ, or the derivatives of the vector potential $P^\mu A^\nu$ which are used to construct the electromagnetic field strength tensor $F^{\mu\nu}$.

We have not said yet how the momenta P^μ act on q-Lorentz spinors. One might be tempted to assume that they act on the wave function part only, that is, as $1 \otimes P^\mu$ on the tensor product. However, this is not possible, as in general $1 \otimes P^\mu$ is no 4-vector operator and thus cannot represent 4-momentum. We can turn $1 \otimes P_\mu$ into a vector operator, though, by twisting $1 \otimes P_\mu$ with an $\mathcal{R}$-matrix of the q-Lorentz algebra,

$$P^\mu := \mathcal{R}^{-1}(1 \otimes P^\mu)\mathcal{R} = (L^\Lambda_+)^\mu{}_{\mu'} \otimes P^{\mu'} , \tag{5.10}$$

with the L-matrix for the 4-vector representation as defined in Eq. (2.43). Of the two universal $\mathcal{R}$-matrices of the q-Lorentz algebra we opt for $\mathcal{R}_{\mathrm{I}}$, because only then the twisting is compatible with the $*$-structure. The momenta act on a ρ-spinor as

$$P^{\mu}\psi^{i} = \rho\big((L^{\Lambda}_{\mathrm{I}+})^{\mu}{}_{\mu'}\big)^{i}{}_{j}\,(P^{\mu'} \triangleright \psi^{j})\,, \tag{5.11}$$

where the L-matrix has been calculated in Eq. (3.80). The action of P^{μ} on each component of ψ^{j} can be viewed as derivation within the algebra of q-Minkowski space functions $\mathcal{M}_q$. The q-derivation operators are

$$\partial^{\mu} := 1 \otimes \mathrm{i}P^{\mu}\,. \tag{5.12}$$

Now we can interpret an operator linear in the momenta as q-differential operator. If $C_{\mu} = C_{\mu} \otimes 1$ are operators that act on the spinor indices only,

$$\mathrm{i}\,C_{\mu}P^{\mu} = C_{\mu}\,\rho\big((L^{\Lambda}_{\mathrm{I}+})^{\mu}{}_{\mu'}\big)\partial^{\mu'} = \tilde{C}_{\mu'}\partial^{\mu'}\,, \tag{5.13}$$

where

$$\tilde{C}_{\mu'} := C_{\mu}\,\rho\big((L^{\Lambda}_{\mathrm{I}+})^{\mu}{}_{\mu'}\big) \tag{5.14}$$

such that $\tilde{C}_{\mu'}$ still acts on the spinor index only, while $\partial^{\mu'}$ acts componentwise, so the two operators commute $[\tilde{C}_{\mu}, \partial^{\nu}] = 0$. It remains to calculate the transformation $C_{\mu} \to \tilde{C}_{\mu}$ for particular representations ρ. Finally, we remark that for the mass Casimir we have $P_{\mu}P^{\mu} = \mathcal{R}^{-1}(1 \otimes P_{\mu}P^{\mu})\mathcal{R} = 1 \otimes P_{\mu}P^{\mu}$, hence, $P_{\mu}P^{\mu} = -\partial_{\mu}\partial^{\mu}$. This means, that mass irreducibility for a spinor is the same as mass irreducibility for each component of the spinor.

5.1.3 Conjugate Spinors

One of the effects of using Lorentz spinors is that the underlying representations can no longer be unitary, since there are no unitary finite representations of the non-compact Lorentz algebra. However, we can introduce non-degenerate but indefinite bilinear forms playing the role of the scalar product. With respect to these pseudo scalar products the spinors carry $*$-representations, that is, the $*$-operation on the algebra side is the same as the pseudo adjoint on the operator side.

The problem of non-unitarity arises from the finiteness of the spin part, $\mathcal{S}$, within the space of spinor wave functions $\mathcal{S}\otimes\mathcal{M}_q$, so we can assume that the wave function part $\mathcal{M}_q$ does carry a $*$-representation. It is then sufficient to redefine the scalar product on $\mathcal{S}$ only. Consider a $D^{(j,0)}$-representation of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ with orthonormal basis $\{e_m\}$ and the canonical scalar product $\langle e_m|e_n\rangle = \delta_{mn}$. We want to define a pseudo scalar product by

$$(e_m|e_n) := A_{mn} \quad \text{such that} \quad (e_m|(g\otimes h)\triangleright e_n) = ((g\otimes h)^{*}\triangleright e_m|e_n) \tag{5.15}$$

for any $g \otimes h \in \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. For a pseudo scalar product we must suppose A_{mn} to be a non-degenerate, hermitian, but not necessarily positive definite matrix. Inserting the definition of the pseudo scalar product, the pseudo-unitarity condition (5.15) reads

$$\begin{aligned}(e_m|(g\otimes h)\triangleright e_n) &= (e_m|\,e_{n'}\rho^j(g)^{n'}{}_n\,\varepsilon(h)) = A_{mn'}\rho^j(g)^{n'}{}_n\,\varepsilon(h)\\ &\stackrel{!}{=}((g\otimes h)^*\triangleright e_m|\,e_n) = (e_{m'}\varepsilon(g^*)\rho^j(h^*)^{m'}{}_m|\,e_n)\\ &= A_{m'n}\,\overline{\varepsilon(g^*)\rho^j(h^*)^{m'}{}_m} = A_{m'n}\,\varepsilon(g)\rho^j(h)^m{}_{m'}\,, \end{aligned}\tag{5.16}$$

where we have used the definition (1.52) of $(g \otimes h)^*$ observing that $\varepsilon(\mathcal{R}_{[1]})\mathcal{R}_{[2]} = 1$. Traditionally, the scalar product is not described by a matrix A_{mn} but by introducing a conjugate spinor basis $\{\bar{e}_m\}$ demanding

$$(e_m|e_n) = \langle \bar{e}_m|e_n\rangle \quad \Rightarrow \quad \bar{e}_m = e_{m'}A_{m'm}\,. \tag{5.17}$$

Using (5.16) the conjugate basis turns out to transform as

$$\begin{aligned}(g\otimes h)\triangleright \bar{e}_n &= e_{m'}\rho^j(g)^{m'}{}_m\varepsilon(h)\,A_{mn} = e_{m'}A_{m'n'}\varepsilon(g)\rho^j(h)^{n'}{}_n\\ &= \bar{e}_{n'}\varepsilon(g)\rho^j(h)^{n'}{}_n\,,\end{aligned}\tag{5.18}$$

that is, $\bar{e}_m$ ought to transform according to a $D^{(0,j)}$-representation. $D^{(j,0)}$ and $D^{(0,j)}$ being inequivalent representations, the conjugate basis $\bar{e}_m$ cannot be expressed as a linear combination of the original basis vectors e_m. In order to allow for a conjugate spinor basis we must consider a representation that contains both, $D^{(j,0)}$ and $D^{(0,j)}$, and thus at least their direct sum $D^{(j,0)} \oplus D^{(0,j)}$ as subrepresentation.

So far it seems that everything is almost trivially analogous to the undeformed case. It is not. If we consider irreducible representations of mixed chirality, $D^{(i,j)}$, we find that the appearance of the $\mathcal{R}$-matrix in $(g \otimes h)^*$ makes it impossible to define conjugate spinors. It only works for $D^{(j,0)}$, because $\rho^0 = \varepsilon$ and $\varepsilon(\mathcal{R}_{[1]})\mathcal{R}_{[2]} = 1$. Fortunately, we do have conjugate spinors for the most interesting cases: Dirac spinors ($D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$) and the Maxwell tensor ($D^{(1,0)} \oplus D^{(0,1)}$). For these cases everything is analogous to the undeformed case.

Let us consider a $D^{(j,0)} \oplus D^{(0,j)}$ representation with basis $\{e^{\mathrm{L}}_m\}$ for the left chiral subrepresentation $D^{(j,0)}$ and the basis $\{e^{\mathrm{R}}_m\}$ for $D^{(0,j)}$. We define the conjugate basis by $\overline{e^{\mathrm{L}}_m} := e^{\mathrm{R}}_m$ and $\overline{e^{\mathrm{R}}_m} = e^{\mathrm{L}}_m$. Let us call $\mathcal{P}$ the parity operator that exchanges the left and right chiral part. Its matrix representation in the basis $\{e^{\mathrm{L}}_m, e^{\mathrm{R}}_m\}$ is

$$\mathcal{P}_{mn} = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\,, \tag{5.19}$$

where 1 is the $(2j + 1)$-dimensional unit matrix. This is the matrix that represents our new pseudo scalar product as a bilinear form. The pseudo Hermitian conjugate of some operator A can now be written as

$$j(A) := \mathcal{P}A^\dagger\mathcal{P}\,, \tag{5.20}$$

which is an involution because $\mathcal{P} = \mathcal{P}^\dagger$ and an algebra anti-homomorphism because $\mathcal{P} = \mathcal{P}^{-1}$.

We apply this result to the whole space of spinor wave functions $\mathcal{S} \otimes \mathcal{M}_q$. Let us assume that the scalar product of two wave functions $f, g \in \mathcal{M}_q$ can be written (at least formally) as some sort of integral $\langle f|g\rangle = \int f^* g$. The pseudo scalar product of two $D^{(j,0)} \oplus D^{(0,j)}$ spinors ψ, ϕ becomes

$$\begin{aligned}(\psi|\phi) &= (e_m \otimes \psi^m | e_n \otimes \phi^n) = (e_m|e_n)\langle\psi^m|\phi^n\rangle \\ &= \textstyle\int (\psi^m)^* \mathcal{P}_{mn} \phi^n = \int \bar{\psi}^n \phi^n \,, \end{aligned} \tag{5.21}$$

with the conjugate spinor wave function defined as

$$\bar{\psi}^n := (\psi^m)^* \mathcal{P}_{mn} \,. \tag{5.22}$$

To summarize, we have convinced ourselves that in the case of $D^{(j,0)} \oplus D^{(0,j)}$ representations the conjugation of spinors, spinor wave functions and operators works exactly as in the undeformed case.

5.2 The *q*-Dirac Equation

5.2.1 The *q*-Dirac Equation in the Rest Frame

In this section we consider q-Dirac spinors $\psi = e_j \otimes \psi^j$ with the spin part transforming according to a $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation. We hope that we can write the projection onto an irreducible component of the space of q-Dirac spinors as expression which involves momenta only to first order terms, corresponding to a first order differential equation. The general expression for such a q-Dirac equation would be

$$\mathbb{P}\psi := \frac{1}{2m}(m + \gamma_\mu P^\mu)\psi = 0 \,, \tag{5.23}$$

with γ_μ being some operators acting on ψ^j. We can already say that γ_μ must be a left 4-vector operator. If it were not, $\gamma_\mu P^\mu$ would not be scalar and, hence, would not commute with the q-Lorentz transformations as required in Eq. (5.2).

We consider here a massive q-Dirac spinor representation, so there is a rest frame (Sec. 3.3.2), that is, a set of states ψ^j, which the momenta act upon as $P^0\psi^j = m\psi^j$, $P^A\psi^j = 0$. We start the search for a projector $\mathbb{P}$ that reduces the q-Dirac representation by computing how it has to act on the rest frame, where we have

$$\mathbb{P}_0 = \tfrac{1}{2}(1 + \gamma_0) \,, \tag{5.24}$$

the zero indicating that this is a projector within the rest frame only. We assume that we can realize the operator γ_0 as 4×4-matrix that acts on the spin degrees of freedom only. This is not unreasonable, for if γ_μ is a set of matrices that form a 4-vector operator in the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation then $\gamma_\mu \otimes 1$ will also be a 4-vector operator in the representation of spinor wave functions. So, let us assume we can write $\mathbb{P}_0 = \mathbb{P}_0 \otimes 1$ in block form as

$$\mathbb{P}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{5.25}$$

where A, B, C, D are 2×2-matrices.

Recall that $\mathbb{P}_0$ must satisfy condition (5.2). This tells us in particular that $\mathbb{P}_0$ must commute with rotations, the symmetry of the rest frame. A rotation l is represented by

$$\rho(l) = \begin{pmatrix} \rho^{\frac{1}{2}}(l) & 0 \\ 0 & \rho^{\frac{1}{2}}(l) \end{pmatrix} . \tag{5.26}$$

Since the $\rho^{\frac{1}{2}}$-representations of the rotations generate all 2×2-matrices (the q-Pauli matrices are a basis), $\mathbb{P}_0$ will only commute with all rotations if A, B, C, D are numbers, that is, complex multiples of the unit matrix.

Furthermore, $\mathbb{P}_0$ has to be a projector, $\mathbb{P}_0^2 = \mathbb{P}_0$, $\mathbb{P}_0^\dagger = \mathbb{P}_0$, and, as in the undeformed case, we require it to commute with the parity operator, $[\mathbb{P}_0, \mathcal{P}] = 0$. Altogether these conditions fix $\mathbb{P}_0$ and hence γ_0 uniquely to be

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.27}$$

the same as in the undeformed case.

5.2.2 The q-Gamma Matrices and the q-Clifford Algebra

If γ_0 is to be a 4-vector operator, we have to define the other gamma matrices as in Eq. (3.36) by

$$\begin{aligned} \gamma_- &= \mathrm{ad_L}(-q^{-\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\, c) \triangleright \gamma_0 \\ \gamma_+ &= \mathrm{ad_L}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\, b) \triangleright \gamma_0 \\ \gamma_3 &= \mathrm{ad_L}(\lambda^{-1}\,(d-a)) \triangleright \gamma_0 \, , \end{aligned} \tag{5.28}$$

where the adjoint action is understood with respect to the q-Dirac representation. To compute this, explicitly, we have to calculate the representations of the boosts first.

$$\rho(a) = \begin{pmatrix} \rho^{\frac{1}{2}}(K^{\frac{1}{2}}) & 0 \\ 0 & \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}) \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-\frac{1}{2}}\lambda\rho^{\frac{1}{2}}(K^{-\frac{1}{2}}E) \end{pmatrix} \tag{5.29a}$$

$$\rho(c) = \begin{pmatrix} -q^{\frac{1}{2}}\lambda\rho^{\frac{1}{2}}(FK^{\frac{1}{2}}) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho(d) = \begin{pmatrix} \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}) & 0 \\ 0 & \rho^{\frac{1}{2}}(K^{\frac{1}{2}}) \end{pmatrix} \tag{5.29b}$$

To demonstrate the simplicity of the technique of boosting, let us demonstrate it with an example.

$$\begin{aligned}\gamma_+ &= \mathrm{ad}_{\mathrm{L}}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\, b) \triangleright \gamma_0 = q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}[\rho(b)\gamma_0\rho(a) - q\rho(a)\gamma_0\rho(b)] \\ &= [2]^{\frac{1}{2}}\left[\begin{pmatrix} 0 & 0 \\ \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}EK^{\frac{1}{2}}) & 0\end{pmatrix} - q\begin{pmatrix} 0 & \rho^{\frac{1}{2}}(E) \\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix} 0 & q\,\sigma_+ \\ -q^{-1}\sigma_+ & 0\end{pmatrix} \end{aligned} \tag{5.30}$$

Here, σ_+ is the q-Pauli matrix (Sec. 3.1.2). After doing the other calculations we get

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \gamma_A = \begin{pmatrix} 0 & q\,\sigma_A \\ -q^{-1}\sigma_A & 0\end{pmatrix}, \tag{5.31}$$

where A runs as usual through $\{-,+,3\}$.

This result can be easily generalized to higher spin massive particles. All we have to do for a massive $D^{(j,0)} \oplus D^{(0,j)}$-spinor is to replace $\rho^{\frac{1}{2}}$ with ρ^j in the above calculations. The result are higher dimensional γ-matrices

$$\gamma_0^{(j)} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \gamma_A^{(j)} = [2]\begin{pmatrix} 0 & q\,\rho^j(J_A) \\ -q^{-1}\rho^j(J_A) & 0\end{pmatrix}. \tag{5.32}$$

Now we want to write the q-Dirac equation as q-differential equation. Towards this end we need to calculate $\tilde{\gamma}_\mu$ by formula (5.14). Using Eqs (A.68) and (A.70) we get for the q-Pauli matrices

$$\sigma_A\,\rho^{(\frac{1}{2},0)}\big((L^{\Lambda}_{\mathrm{I}+})^A{}_B\big) = q^2\tilde{\sigma}_B\,, \qquad \sigma_A\,\rho^{(0,\frac{1}{2})}\big((L^{\Lambda}_{\mathrm{I}+})^A{}_B\big) = q^{-2}\tilde{\sigma}_B\,, \tag{5.33}$$

where

$$\tilde{\sigma}_- = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & q^{\frac{1}{2}} \\ 0 & 0\end{pmatrix}, \quad \tilde{\sigma}_+ = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ -q^{-\frac{1}{2}} & 0\end{pmatrix}, \quad \tilde{\sigma}_3 = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q\end{pmatrix} \tag{5.34}$$

with respect to the $\{-,+\}$ basis. We can write this more compactly as

$$\tilde{\sigma}_A = -[2]\,\rho^{\frac{1}{2}}(SJ_A)\,. \tag{5.35}$$

In this sense the transformed q-Pauli matrices, $\tilde{\sigma}_A$, can be viewed as antipodes of the original ones. For the transformed q-gamma matrices we obtain

$$\tilde{\gamma}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \qquad \tilde{\gamma}_A = \begin{pmatrix} 0 & q^{-1}\,\tilde{\sigma}_A \\ -q\tilde{\sigma}_A & 0\end{pmatrix}, \tag{5.36}$$

so the q-Dirac equation written as q-differential equation becomes

$$(m - \mathrm{i}\tilde{\gamma}_\mu\partial^\mu)\psi = 0\,. \tag{5.37}$$

What commutation relations do the gamma matrices satisfy? Using Eqs. (5.36) and (A.39) we find after some lengthy calculations

$$\tilde{\gamma}_c\tilde{\gamma}_d = \eta_{dc} + \tilde{\gamma}_a\tilde{\gamma}_b\mathbb{P}_{\mathrm{A}}{}^{ba}{}_{dc}\,, \tag{5.38}$$

where $\mathbb{P}_{\mathrm{A}}$ is the antisymmetric projector defined in Eq. (A.29). This is the q-deformation of the Clifford algebra. Using the relations of the q-Clifford algebra it can be shown that the square of q-Dirac operator is indeed the mass Casimir,

$$(\tilde{\gamma}_\mu\partial^\mu)^2 = \partial_\mu\partial^\mu = -P_\mu P^\mu\,. \tag{5.39}$$

As in the undeformed case we conclude that a solution ψ to the q-Dirac equation satisfies automatically the mass shell condition $P_\mu P^\mu\psi = m^2\psi$, and that the operator $\mathbb{P} = \frac{1}{2m}(m + \gamma_\mu P^\mu)$ really is a projector. The q-Clifford relations (5.38) can be written in equivalent but more familiar forms as

$$\tilde{\gamma}_a\tilde{\gamma}_b\mathbb{P}_{\mathrm{S}}{}^{ba}{}_{dc} = \eta_{dc}\,, \qquad \text{or} \qquad \tilde{\gamma}_c\tilde{\gamma}_d + \tilde{\gamma}_a\tilde{\gamma}_b R_{II}{}^{ab}{}_{dc} = q[2]\eta_{cd}\,, \tag{5.40}$$

with the symmetrizer (A.29) and the R-matrix (A.66).

One could have started directly from these relations trying to find matrices that satisfy them [36]. This approach has a number of disadvantages: a) It is computationally much more cumbersome than boosting γ_0. b) The result is not unique, that is, we would get many solutions to the q-Clifford algebra not knowing which representations they belong to. c) Having determined a solution $\tilde{\gamma}_\mu$, the covariance of the q-Dirac equation remains unclear as $\tilde{\gamma}_\mu$ cannot be a 4-vector operator.

5.2.3 The Zero Mass Limit and the q-Weyl Equations

The zero mass limit of the q-Dirac equations, $(m + \gamma_\mu P^\mu)\psi = 0$, is formally

$$\gamma_\mu P^\mu\psi = 0\,, \tag{5.41}$$

where γ_μ is defined as in Eq. (5.32). The operator $\mathbb{A} := \gamma_\mu P^\mu$ is no longer a projection. For $m \to 0$ the wave equation decouples into two independent equations for a left handed $D^{(\frac{1}{2},0)}$-spinor ψ_{L} and a right handed $D^{(0,\frac{1}{2})}$-spinor ψ_{R},

$$\sigma_A P^A\psi_{\mathrm{L}} = q^{-1}P^0\psi_{\mathrm{L}}\,, \qquad \sigma_A P^A\psi_{\mathrm{R}} = -qP^0\psi_{\mathrm{R}}\,, \tag{5.42}$$

the q-Weyl equations for massless left and right handed spin-$\frac{1}{2}$ particles. Written as q-differential equation they become

$$\tilde{\sigma}_A\partial^A\psi_{\mathrm{L}} = -q\partial^0\psi_{\mathrm{L}}\,, \qquad \tilde{\sigma}_A\partial^A\psi_{\mathrm{R}} = q^{-1}\partial^0\psi_{\mathrm{R}}\,. \tag{5.43}$$

The operator $\mathbb{A}$ inherits property (5.1) from the massive q-Dirac projector $\mathbb{P}$, so $\mathbb{A}\psi = 0$ is a viable wave equation. Let us see what it looks like in the momentum eigenspace $\mathcal{H}_p$ for the momentum eigenvalues $p = (p_0, p_-, p_+, p_3) = (k, 0, 0, k)$ for some real parameter k (Sec. 3.3.2). On this subspace $\mathbb{A}$ acts as

$$\mathbb{A}|_{\mathcal{H}_p} = k \begin{pmatrix} 0 & 1 - q\sigma_3 \\ 1 + q^{-1}\sigma_3 & 0 \end{pmatrix} = k[2] \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \end{pmatrix} . \tag{5.44}$$

The kernel of this operator is 2-dimensional leaving us with two states corresponding to helicity $\pm\frac{1}{2}$.

If we generalize these considerations to higher spin Dirac type spinors, we find that the corresponding operator $\mathbb{A}$ has a zero kernel, $\ker \mathbb{A} = 0$, which can be easily verified in the momentum eigenspace $\mathcal{H}_p$. In other words: the wave equation for massive $D^{(j,0)} \oplus D^{(0,j)}$ spinor wave functions leads for $m \to 0$ to a wave equation that has no solutions. This applies in particular to q-Maxwell spinors. Therefore, we need a different approach to find the q-Maxwell equations.

5.3 The *q*-Maxwell Equations

5.3.1 The *q*-Maxwell Equations in the Momentum Eigenspaces

In this section we consider massless spinors ψ^j with the spinor index transforming according to a $D^{(1,0)} \oplus D^{(0,1)}$ representation. According to the Clebsch-Gordan series (2.6) this type of spinor is equivalent to considering an antisymmetric tensor $F^{\mu\nu}$ with two 4-vector indices. These are the types of spinor wave functions commonly used to describe the electromagnetic field, a massless field of spin-1.

We start our calculations in the massless momentum eigenspace $\mathcal{H}_p$ with momentum eigenvalues $\mathrm{p} = (p_0, p_-, p_+, p_3) = (k, 0, 0, k)$ for some real parameter k. In Sec. 3.3.2 we have shown this eigenspace to be invariant under the little algebra $\mathcal{K}_0$, whose generators K, and N_A have been defined in Eq. (3.100). Within $\mathcal{H}_p$ the little algebra acts only on the spinor index. The $D^{(1,0)} \oplus D^{(0,1)}$ matrix representation of the generators are given by

$$N_- = -q[2] \begin{pmatrix} \rho^1(J_-) & 0 \\ 0 & 0 \end{pmatrix}, \quad N_+ = -q^{-1}[2] \begin{pmatrix} 0 & 0 \\ 0 & \rho^1(J_+) \end{pmatrix}, \quad N_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$K = \begin{pmatrix} \rho^1(K) & 0 \\ 0 & \rho^1(K) \end{pmatrix}, \tag{5.45}$$

where ρ^1 is the vector representation map of $\mathcal{U}_q(\mathrm{su}_2)$.

We seek a projector $\mathbb{P} = \mathbb{P} \otimes 1$ that projects onto an irreducible subrepresentation of the little algebra. We write it in block form as

$$\mathbb{P} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.46}$$

where A, B, C, D are 3×3-matrices. We must have $\mathbb{P}_0^\dagger = \mathbb{P}_0$, so A and D must be Hermitian matrices and $C = B^\dagger$. Recall from Eq. (3.105) that within an irreducible representation of $\mathcal{K}_0$ we have $N_\pm = 0$. Therefore, we must have

$$N_\pm \mathbb{P} = 0 \tag{5.47}$$

within the $D^{(1,0)} \oplus D^{(0,1)}$ spinor representation. This leads to the conditions

$$\rho^1(J_-)\, A = 0\,, \qquad \rho^1(J_+)\, D = 0\,, \qquad \rho^1(J_-)\, B = 0\,, \qquad \rho^1(J_+)\, B^\dagger = 0\,. \tag{5.48}$$

To satisfy these conditions A, B, and D must be of the form

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix}, \tag{5.49}$$

for α, δ real and β complex. Furthermore, $\mathbb{P}$ must project on an eigenvector of K. From this it follows that $\beta = 0$ and either $\alpha = 1$, $\delta = 0$ or $\alpha = 0$, $\delta = 1$. To summarize, there are two possible projectors

$$\mathbb{P}_{\mathrm{L}} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad \mathbb{P}_{\mathrm{R}} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{5.50}$$

projecting each on a irreducible one-dimensional representation of the little algebra $\mathcal{K}_0$. The image of $\mathbb{P}_{\mathrm{L}}$ is part of the left handed $D^{(1,0)}$ component while $\mathbb{P}_{\mathrm{R}}$ projects to the right handed $D^{(0,1)}$ component of the spinor. Physically, this corresponds to left and right handed circular waves. We want to allow for parity transformations exchanging the left and right handed parts, so we need both parts

$$\mathbb{P} = \mathbb{P}_{\mathrm{L}} + \mathbb{P}_{\mathrm{R}}\,. \tag{5.51}$$

With the parity transformation included, the two dimensional space which $\mathbb{P}$ projects onto is irreducible.

5.3.2 Computing the q-Maxwell Equation

We would like to find the q-Maxwell equation in the form of a first order differential equation

$$\mathbb{A}\psi = C_\mu P^\mu\, \psi = 0\,, \tag{5.52}$$

hoping that the operators C_μ can be chosen to act on the spinor index only, $C_\mu = C_\mu \otimes 1$. This wave equation has to fulfill condition (5.1): The q-Lorentz transform of a solution must again be a solution. For this, it would be sufficient but not necessary, if $\mathbb{A}$ were a scalar operator, as it has been the case for the q-Dirac equation and its zero mass limit, the q-Weyl equations.

Recall from the last section, that as long as we do not include parity transformations, we must have two independent equations for the right and the left handed part of the spinor, ψ_{L} carrying a $D^{(1,0)}$ representation and ψ_{R} carrying a $D^{(0,1)}$ representation

$$\mathbb{A}_{\mathrm{L}}\psi_{\mathrm{L}} = 0\,, \qquad \mathbb{A}_{\mathrm{R}}\psi_{\mathrm{R}} = 0\,. \tag{5.53}$$

Let us try to choose $\mathbb{A}_{\mathrm{L}} = C^{\mathrm{L}}_{\mu}P^{\mu}$ and $\mathbb{A}_{\mathrm{R}} = C^{\mathrm{R}}_{\mu}P^{\mu}$, so they commute with rotations. For this to be possible C^{L}_0, C^{R}_0 must be scalars with respect to rotations while C^{L}_A, C^{R}_A must transform as 3-vectors. The only scalar operators within the D^1-representation of rotations are multiples of the unit matrix, while every 3-vector operator is proportional to $\rho^1(J_A)$. Hence, up to an overall constant factor our wave equations can be written as

$$\big(P^0 + \alpha_{\mathrm{L}}\,\rho^1(J_A)P^A\big)\psi_{\mathrm{L}} = 0\,, \qquad \big(P^0 + \alpha_{\mathrm{R}}\,\rho^1(J_A)P^A\big)\psi_{\mathrm{R}} = 0\,, \tag{5.54}$$

where α_{L}, α_{R} are constants. To determine these constants, we consider the wave equations in the momentum eigenspace, where they take the form

$$\big(1 + \alpha_{\mathrm{L}}\,\rho^1(J_3)\big)\psi_{\mathrm{L}} = 0\,, \qquad \big(1 + \alpha_{\mathrm{R}}\,\rho^1(J_3)\big)\psi_{\mathrm{R}} = 0\,. \tag{5.55}$$

The space of solutions of each of these equations must equal the image of the projectors $\mathbb{P}_{\mathrm{L}}$ and $\mathbb{P}_{\mathrm{R}}$, respectively. This requirement fixes the constants to $\alpha_{\mathrm{L}} = q^{-1}$ and $\alpha_{\mathrm{R}} = -q$.

Although this determines our candidate for the q-Maxwell equation, condition (5.1) has yet to be checked for the boosts. Let $\psi_0 \in \mathcal{H}_p$ be an element of the momentum eigenspace, $P_\mu\psi_0 = p_\mu\psi_0$, with $p_\mu = (p_0, p_-, p_+, p_3) = (k, 0, 0, k)$. Using the commutation relations between boosts and momentum generators we find

$$P_\mu(a\psi_0) = q^{-1}p_\mu(a\psi_0)\,, \qquad P_\mu(b\psi_0) = q^{-1}p_\mu(b\psi_0) \tag{5.56a}$$

$$P_\mu(c\psi_0) = qp_\mu(c\psi_0)\,, \qquad P_\mu(d\psi_0) = qp_\mu(d\psi_0)\,. \tag{5.56b}$$

By induction it follows, that for any monomial in the boosts, $h = a^i b^j c^k d^l$, we have $P_\mu(h\psi_0) = q^{k+l-i-j}\,p_\mu(h\psi_0)$. Thus, for any such $\psi := h\psi_0$, the wave equation (5.52) takes the form

$$(C_0 - C_3)\psi = 0\,. \tag{5.57}$$

Looking separately at the left and right handed part of $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}$ this equation writes out

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & q^{-2} & 0 \\ 0 & 0 & q^{-1}[2] \end{pmatrix} \begin{pmatrix} \psi^-_{\mathrm{L}} \\ \psi^3_{\mathrm{L}} \\ \psi^+_{\mathrm{L}} \end{pmatrix} = 0\,, \qquad \begin{pmatrix} q[2] & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^-_{\mathrm{R}} \\ \psi^3_{\mathrm{R}} \\ \psi^+_{\mathrm{R}} \end{pmatrix} = 0\,, \tag{5.58}$$

which is equivalent to

$$\psi_{\mathrm{L}}^{3} = \psi_{\mathrm{L}}^{+} = 0\,, \qquad\qquad \psi_{\mathrm{R}}^{-} = \psi_{\mathrm{R}}^{3} = 0\,. \tag{5.59}$$

If we now have a solution of Eq. (5.57), that is, a spinor ψ whose only non-vanishing components are ψ_{L}^{-} and ψ_{R}^{+}, could it happen that by boosting it gets other non-vanishing components, thus turning a solution into a non-solution? The answer to this question is no. We exemplify this, applying formula (5.5) for the action of the boost generator c on a left handed spinor,

$$\begin{aligned}
c\,\psi_{\mathrm{L}}^{A} &= \rho^{(1,0)}(c_{(1)})^{A}{}_{A'}\big(c_{(2)} \triangleright \psi_{\mathrm{L}}^{A'}\big) \\
&= \rho^{(1,0)}(c)^{A}{}_{A'}\big(a \triangleright \psi_{\mathrm{L}}^{A'}\big) + \rho^{(1,0)}(d)^{A}{}_{A'}\big(c \triangleright \psi_{\mathrm{L}}^{A'}\big) \\
&= -q^{\frac{1}{2}}\lambda\rho^{1}(FK^{\frac{1}{2}})^{A}{}_{A'}\big(a \triangleright \psi_{\mathrm{L}}^{A'}\big) + \rho^{1}(K^{-\frac{1}{2}})^{A}{}_{A'}\big(c \triangleright \psi_{\mathrm{L}}^{A'}\big) \\
&= -q^{\frac{1}{2}}\lambda[2]^{\frac{1}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \triangleright \psi_{\mathrm{L}}^{-} \\ a \triangleright \psi_{\mathrm{L}}^{3} \\ a \triangleright \psi_{\mathrm{L}}^{+} \end{pmatrix} + \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} \begin{pmatrix} c \triangleright \psi_{\mathrm{L}}^{-} \\ c \triangleright \psi_{\mathrm{L}}^{3} \\ c \triangleright \psi_{\mathrm{L}}^{+} \end{pmatrix} \\
&= \begin{pmatrix} -q^{\frac{1}{2}}\lambda[2]^{\frac{1}{2}} a \triangleright \psi_{\mathrm{L}}^{3} + q\, c \triangleright \psi_{\mathrm{L}}^{-} \\ -q^{\frac{1}{2}}\lambda[2]^{\frac{1}{2}} a \triangleright \psi_{\mathrm{L}}^{+} + c \triangleright \psi_{\mathrm{L}}^{3} \\ q^{-1} c \triangleright \psi_{\mathrm{L}}^{+} \end{pmatrix},
\end{aligned} \tag{5.60}$$

which clearly shows that, if ψ_{L}^{3} and ψ_{L}^{+} vanish, so do $c\psi_{\mathrm{L}}^{3}$ and $c\psi_{\mathrm{L}}^{+}$. Similar calculations can be done for the other boost generators and right handed spinors.

By induction we conclude, that if $\psi_0 \in \mathcal{H}_p$ is a solution of Eq. (5.57) and $h = a^i b^j c^k d^l$ is a monomial in the boosts, $h = a^i b^j c^k d^l$, the spinor $\psi = h\psi_0$ will be a solution, as well. The algebra of all boosts, $SU_q(2)^{\mathrm{op}}$, is generated as linear space by monomials, thus, $h\psi_0$ is a solution for any boost $h \in SU_q(2)^{\mathrm{op}}$. Since furthermore every q-Lorentz transformation can be written as a sum of products of rotations and boost, $h\psi_0$ is a solution for *any* q-Lorentz transformation h. We assume that the space of solutions, $\ker \mathbb{A}$, is an irreducible representation. This means in particular that the q-Lorentz algebra acts transitively on $\ker \mathbb{A}$, so any solution can be written as $h\psi_0$. Hence, the wave equations

$$\rho^{1}(J_A)P^{A}\psi_{\mathrm{L}} = -qP_0\psi_{\mathrm{L}}\,, \qquad \rho^{1}(J_A)P^{A}\psi_{\mathrm{R}} = q^{-1}P_0\psi_{\mathrm{R}} \tag{5.61}$$

do indeed satisfy property (5.1).

We want to write these equations, $C_\mu P^\mu \psi = 0$ as q-differential equations $\tilde{C}_\mu \partial^\mu \psi = 0$, where $\tilde{C}_\mu$ is defined in Eq. (5.14). After lengthy calculations using Eqs. (A.42), (A.68), and (A.70) we get for the left and right handed part separately

$$\begin{aligned}
\rho^{1}(J_{A'})^{B}{}_{C'}\,\rho^{(1,0)}\big((L_{+}^{\Lambda})^{A'}{}_{A}\big)^{C'}{}_{C} &= -q^{2}\varepsilon_{C}{}^{B}{}_{A} \\
\rho^{1}(J_{A'})^{B}{}_{C'}\,\rho^{(0,1)}\big((L_{+}^{\Lambda})^{A'}{}_{A}\big)^{C'}{}_{C} &= -q^{-2}\varepsilon_{C}{}^{B}{}_{A}\,,
\end{aligned} \tag{5.62}$$

so the wave equations can be written as

$$\vec{\partial} \times \vec{\psi}_{\mathrm{L}} = \mathrm{i}q^{-1}\partial_0\vec{\psi}_{\mathrm{L}}\,, \qquad \vec{\partial} \times \vec{\psi}_{\mathrm{R}} = -\mathrm{i}q\,\partial_0\vec{\psi}_{\mathrm{R}}\,, \tag{5.63}$$

where $\vec{\psi}_{\mathrm{R}} = (\psi^A_{\mathrm{R}})$, $\vec{\psi}_{\mathrm{L}} = (\psi^A_{\mathrm{L}})$, and where the cross product is defined in Eq. (2.24). A spinor $\vec{\psi}_{\mathrm{L}}$ which is a solution to this equation must yet satisfy the mass zero condition. Using the identities (A.18) for the cross product, the commutation relations of the derivations $\vec{\partial} \times \vec{\partial} = -\mathrm{i}\lambda\partial_0\vec{\partial}$, and the wave equation (5.63), we rewrite the mass zero condition as

$$\begin{aligned}
0 &= \partial_\mu\partial^\mu\vec{\psi}_{\mathrm{L}} = (\partial_0^2 - \vec{\partial}\cdot\vec{\partial})\vec{\psi}_{\mathrm{L}} \\
&= \partial_0^2\vec{\psi}_{\mathrm{L}} - (\vec{\partial}\times\vec{\partial})\times\vec{\psi}_{\mathrm{L}} + \vec{\partial}\times(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= \partial_0^2\vec{\psi}_{\mathrm{L}} + \mathrm{i}\lambda\partial_0(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) + \vec{\partial}\times(\mathrm{i}q^{-1}\partial_0\vec{\psi}_{\mathrm{L}}) - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= \partial_0^2\vec{\psi}_{\mathrm{L}} - q^{-1}\lambda\partial_0^2\vec{\psi}_{\mathrm{L}} - q^{-2}\partial_0^2\vec{\psi}_{\mathrm{L}} - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= -\vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})\,.
\end{aligned} \tag{5.64}$$

Contracting the wave equation with $\vec{\partial}$

$$\vec{\partial}\cdot(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) = (\vec{\partial}\times\vec{\partial})\cdot\vec{\psi}_{\mathrm{L}} = -\mathrm{i}\lambda\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = \mathrm{i}q^{-1}\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})\,, \tag{5.65}$$

we see that $\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = 0$ if $\vec{\psi}_{\mathrm{L}}$ is to satisfy the wave equation. Together with Eq. (5.64) this means that the mass zero condition is equivalent to $\partial_\mu(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = 0$, that is, $\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}$ must be a constant number. In a momentum eigenspace we have $\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = k(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})$, so this constant number must be zero. The same reasoning applies for the right handed spinor $\vec{\psi}_{\mathrm{R}}$.

We conclude that the wave equations (5.63) together with the mass zero condition $\partial_\mu\partial^\mu\psi = 0$ are equivalent to

$$\vec{\partial}\times\vec{\psi}_{\mathrm{L}} = \mathrm{i}q^{-1}\partial_0\vec{\psi}_{\mathrm{L}}\,, \qquad \vec{\partial}\cdot\vec{\psi}_{\mathrm{L}} = 0 \tag{5.66}$$

$$\vec{\partial}\times\vec{\psi}_{\mathrm{R}} = -\mathrm{i}q\,\partial_0\vec{\psi}_{\mathrm{R}}\,, \qquad \vec{\partial}\cdot\vec{\psi}_{\mathrm{R}} = 0\,, \tag{5.67}$$

which we will call the q-Maxwell equations.

5.3.3 The *q*-Electromagnetic Field

Finally, we write the q-Maxwell equations in a more familiar form, that is, in terms of the q-deformed electric and magnetic fields. In the undeformed case the electric and magnetic fields can — up to constant factors — be characterized within the $D^{(1,0)} \oplus D^{(0,1)}$ representation as eigenstates of the parity operator. The electric field should transform like a polar vector $\mathcal{P}\vec{E} = -\vec{E}$, while the magnetic field must be an axial vector $\mathcal{P}\vec{B} = \vec{B}$. Recall, that the parity operator

$\mathcal{P}$ acts on q-spinors by exchanging the left and the right handed parts $\mathcal{P}\psi_{\mathrm{L}} = \psi_{\mathrm{R}}$, $\mathcal{P}\psi_{\mathrm{R}} = \psi_{\mathrm{L}}$. This fixes the fields

$$\vec{E} = \mathrm{i}(\vec{\psi}_{\mathrm{R}} - \vec{\psi}_{\mathrm{L}})\,, \qquad \vec{B} = \vec{\psi}_{\mathrm{R}} + \vec{\psi}_{\mathrm{L}} \tag{5.68}$$

up to constant factors which have been chosen to give the right undeformed limit. Spinor conjugation of the fields is now the same as ordinary conjugation $\bar{E}^A = (E^A)^*$, $\bar{B}^A = (B^A)^*$. In terms of these fields, the q-Maxwell equations (5.66) take the form

$$\vec{\partial} \times \vec{E} = \tfrac{1}{2}[2]\,\partial_0 \vec{B} - \tfrac{1}{2}\mathrm{i}\lambda\,\partial_0 \vec{E}\,, \qquad \vec{\partial}\cdot\vec{E} = 0 \tag{5.69}$$

$$\vec{\partial} \times \vec{B} = -\tfrac{1}{2}[2]\,\partial_0 \vec{E} - \tfrac{1}{2}\mathrm{i}\lambda\,\partial_0 \vec{B}\,, \qquad \vec{\partial}\cdot\vec{B} = 0\,. \tag{5.70}$$

We would also like to express the q-Maxwell equations in terms of a field strength tensor $F^{\mu\nu}$. According to the Clebsch-Gordan series the left and right chiral 3-vectors ψ_{L} and ψ_{R} can be each identified with a 4-vector matrix. If we replace in $\psi_{\mathrm{L}} = e_C \otimes \psi_{\mathrm{L}}^C$ the spinor basis e_C with $E^{1,0}_{C,0}$ from formula (A.26),

$$\begin{aligned}\psi_{\mathrm{L}} = e_C \otimes \psi_{\mathrm{L}}^C &= (E_A \otimes E_B\,\varepsilon^{AB}{}_C + qE_0 \otimes E_C - q^{-1}E_C \otimes E_0) \otimes \psi_{\mathrm{L}}^C \\ &= (E_\mu \otimes E_\nu) \otimes F_{\mathrm{L}}^{\mu\nu}\,,\end{aligned} \tag{5.71}$$

this defines the matrix

$$F_{\mathrm{L}}^{\mu\nu} := \begin{pmatrix} F_{\mathrm{L}}^{00} & F_{\mathrm{L}}^{0N} \\ F_{\mathrm{L}}^{M0} & F_{\mathrm{L}}^{MN} \end{pmatrix} = \begin{pmatrix} 0 & q\psi_{\mathrm{L}}^N \\ -q^{-1}\psi_{\mathrm{L}}^M & \varepsilon^{MN}{}_C\,\psi_{\mathrm{L}}^C \end{pmatrix}, \tag{5.72}$$

where M, N run through $\{-,+,3\}$. In the same manner we obtain for the right handed part

$$F_{\mathrm{R}}^{\mu\nu} := \begin{pmatrix} 0 & -q^{-1}\psi_{\mathrm{R}}^N \\ q\psi_{\mathrm{R}}^M & \varepsilon^{MN}{}_C\,\psi_{\mathrm{R}}^C \end{pmatrix}. \tag{5.73}$$

In terms of these matrices the q-Maxwell equations (5.66) take the form $\partial_\nu F_{\mathrm{L}}^{\mu\nu} = 0$ and $\partial_\nu F_{\mathrm{R}}^{\mu\nu} = 0$. This suggests to introduce the field strength tensor and its dual

$$F^{\mu\nu} := \mathrm{i}(F_{\mathrm{L}}^{\mu\nu} + F_{\mathrm{R}}^{\mu\nu})\,, \qquad \tilde{F}^{\mu\nu} := \mathrm{i}(F_{\mathrm{L}}^{\mu\nu} - F_{\mathrm{R}}^{\mu\nu})\,, \tag{5.74}$$

where the factor i is needed for the right undeformed limit. In terms of the electric and the magnetic field, we have

$$\begin{aligned} F^{\mu\nu} &:= \begin{pmatrix} 0 & -\tfrac{1}{2}[2]E^N + \tfrac{1}{2}\mathrm{i}\lambda B^N \\ \tfrac{1}{2}[2]E^M + \tfrac{1}{2}\mathrm{i}\lambda B^M & \mathrm{i}\varepsilon^{MN}{}_C\,B^C \end{pmatrix} \\ \tilde{F}^{\mu\nu} &:= \begin{pmatrix} 0 & \tfrac{1}{2}[2]\mathrm{i}B^N - \tfrac{1}{2}\lambda E^N \\ -\tfrac{1}{2}[2]\mathrm{i}B^M - \tfrac{1}{2}\lambda E^M & -\varepsilon^{MN}{}_C\,E^C \end{pmatrix}. \end{aligned} \tag{5.75}$$

The q-Maxwell equations become

$$\partial_\nu F^{\mu\nu} = 0\,, \qquad \partial_\nu \tilde{F}^{\mu\nu} = 0\,, \tag{5.76}$$

in complete analogy to the undeformed case.

Appendix A

Useful Formulas

A.1 Clebsch-Gordan Coefficients

A.1.1 Clebsch-Gordan and Racah Coefficients for $\mathcal{U}_q(\mathrm{su}_2)$

We first list some formulas which enable us to calculate some q-Clebsch-Gordan coefficients explicitly [63, 64]:

$$
\begin{aligned}
\mathrm{C}_q(0,j,j'\,|\,0,m,m') &= \delta_{mm'}\delta_{jj'} \\
\mathrm{C}_q(j,\tfrac{1}{2},j+\tfrac{1}{2}\,|\,m,\pm\tfrac{1}{2},m\pm\tfrac{1}{2}) &= q^{\pm(j\mp m)/2}([j\pm m+\tfrac{1}{2}][2j+1]^{-1})^{\frac{1}{2}} \\
\mathrm{C}_q(j,\tfrac{1}{2},j-\tfrac{1}{2}\,|\,m,\pm\tfrac{1}{2},m\pm\tfrac{1}{2}) &= \mp q^{\mp(j\pm m+1)/2}([j\mp m][2j+1]^{-1})^{\frac{1}{2}}
\end{aligned}
\tag{A.1}
$$

For $\mathrm{C}_q(1,j,j+\Delta j\,|\,\Delta m,m,m+\Delta m)$ we have the formulas

Δj	$\Delta m=-1$	$\Delta m=0$	$\Delta m=+1$
-1	$q^{m-j-1}\sqrt{\frac{[j+m][j+m-1]}{[2j+1][2j]}}$	$-q^{m}\sqrt{\frac{[2][j+m][j-m]}{[2j+1][2j]}}$	$q^{m+j+1}\sqrt{\frac{[j-m][j-m-1]}{[2j+1][2j]}}$
0	$-q^{m-1}\sqrt{\frac{[2][j+m][j-m+1]}{[2j+2][2j]}}$	$q^{m}\,\frac{q^{(j+1)}[j-m]-q^{-(j+1)}[j+m]}{\sqrt{[2j+2][2j]}}$	$q^{m+1}\sqrt{\frac{[2][j+m+1][j-m]}{[2j+2][2j]}}$
$+1$	$q^{m+j}\sqrt{\frac{[j-m+2][j-m+1]}{[2j+2][2j+1]}}$	$q^{m}\sqrt{\frac{[2][j+m+1][j-m+1]}{[2j+2][2j+1]}}$	$q^{m-j}\sqrt{\frac{[j+m+2][j+m+1]}{[2j+2][2j+1]}}$

The q-Clebsch-Gordan coefficients obey the symmety

$$
\mathrm{C}_q(n,j,j'\,|\,\nu,m,m') = (-1)^{j'-j}(-q)^{\nu}\sqrt{\frac{[2j'+1]}{[2j+1]}}\mathrm{C}_q(n,j',j\,|\,-\nu,m',m)\,. \tag{A.2}
$$

For the q-Racah coefficients we have

$$
-\sqrt{[3]}\,\mathrm{R}_q(1,1,j\,|\,0,j',j') = (-1)^{j'+j}\sqrt{\frac{[2j'+1]}{[2j+1]}}\,. \tag{A.3}
$$

For $-\sqrt{\frac{[4]}{[2]}}\,\mathrm{R}_q(1,1,j\,|\,1,j',j'')$ there are the formulas

	$j''=j-1$	$j''=j$	$j''=j+1$
$j'=j-1$	$\sqrt{\frac{[2j-2]}{[2j]}}$	$\sqrt{\frac{[2j+2][2j-1]}{[2j+1][2j]}}$	0
$j'=j$	$-\sqrt{\frac{[2j+2]}{[2j]}}$	$\frac{[2j]-[2j+2]}{\sqrt{[2j+2][2j]}}$	$\sqrt{\frac{[2j]}{[2j+2]}}$
$j'=j+1$	0	$-\sqrt{\frac{[2j+3][2j]}{[2j+2][2j+1]}}$	$-\sqrt{\frac{[2j+4]}{[2j+2]}}$

A.1.2 Metric and Epsilon Tensor

We define the 3-metric and the epsilon tensor as[1]

$$g^{AB} := -\sqrt{[3]}\mathrm{C}_q(1,1,0\,|\,A,B,0)\,, \quad \varepsilon^{AB}{}_C = -\sqrt{\frac{[4]}{[2]}}\mathrm{C}_q(1,1,1\,|\,A,B,C)\,, \qquad \text{(A.4)}$$

where the capital roman indices run through $\{-1,0,1\}=\{-,3,+\}$. The positions of the indices are chosen such that the basis vectors are written with lower indices. From this definition it is clear that the projectors on the subspaces on the right hand side of the Clebsch-Gordan series

$$D^1 \otimes D^1 \cong D^0 \oplus D^1 \oplus D^3, \qquad \text{(A.5)}$$

that we denote by $\mathbb{P}_0$, $\mathbb{P}_1$, $\mathbb{P}_3$, can be written as

$$\begin{aligned} \mathbb{P}_0^{AB}{}_{CD} &= [3]^{-1}\, g^{AB} g_{CD} \\ \mathbb{P}_1^{AB}{}_{CD} &= [2][4]^{-1}\, \varepsilon^{ABX}\varepsilon_{DCX} \\ \mathbb{P}_3^{AB}{}_{CD} &= \delta^A_C\delta^B_D - \mathbb{P}_0^{AB}{}_{CD} - \mathbb{P}_1^{AB}{}_{CD}\,, \end{aligned} \qquad \text{(A.6)}$$

where the projectors act on lower indices, $\mathbb{P}\triangleright E_C E_D := E_A E_B \mathbb{P}^{AB}{}_{CD}$. The nonzero values of the metric are

$$g^{-+} = -q^{-1}\,, \qquad g^{+-} = -q\,, \qquad g^{33} = 1\,. \qquad \text{(A.7)}$$

By definition g_{AB} is the inverse of g^{AB}

$$g_{AB}g^{BC} = \delta^C_A = g^{CB}g_{BA}\,, \qquad \text{(A.8)}$$

implying $g_{AB} = g^{AB}$. This means that we can not raise and lower the indices of the metric as usual. Instead, we get

$$g_{AA'}g_{BB'}g^{A'B'} = g_{BA}\,. \qquad \text{(A.9)}$$

[1] Metric and epsilon tensor, g_{AB} and $\varepsilon^{AB}{}_C$, as defined here correspond to g^{AB} and $q\varepsilon_{BA}{}^C$ in [52].

The nonzero values of the epsilon tensor are

$$\varepsilon^{-3}{}_{-} = q^{-1} \qquad \varepsilon^{3-}{}_{-} = -q \tag{A.10a}$$
$$\varepsilon^{-+}{}_{3} = 1 \qquad \varepsilon^{+-}{}_{3} = -1 \qquad \varepsilon^{33}{}_{3} = -\lambda \tag{A.10b}$$
$$\varepsilon^{3+}{}_{+} = q^{-1} \qquad \varepsilon^{+3}{}_{+} = -q\,. \tag{A.10c}$$

Lowering the first index as usual by $\varepsilon_A{}^B{}_C := g_{AA'}\,\varepsilon^{A'B}{}_C$ we get

$$\varepsilon_{+}{}^{3}{}_{-} = -1 \qquad \varepsilon_{3}{}^{-}{}_{-} = -q \tag{A.11a}$$
$$\varepsilon_{+}{}^{+}{}_{3} = -q \qquad \varepsilon_{-}{}^{-}{}_{3} = q^{-1} \qquad \varepsilon_{3}{}^{3}{}_{3} = -\lambda \tag{A.11b}$$
$$\varepsilon_{3}{}^{+}{}_{+} = q^{-1} \qquad \varepsilon_{-}{}^{3}{}_{+} = 1\,. \tag{A.11c}$$

Lowering the second index

$$\varepsilon^{-}{}_{3-} = q^{-1} \qquad \varepsilon^{3}{}_{+-} = q^{2} \tag{A.12a}$$
$$\varepsilon^{-}{}_{-3} = -q^{-1} \qquad \varepsilon^{+}{}_{+3} = q \qquad \varepsilon^{3}{}_{33} = -\lambda \tag{A.12b}$$
$$\varepsilon^{3}{}_{-+} = -q^{-2} \qquad \varepsilon^{+}{}_{3+} = -q\,. \tag{A.12c}$$

With all indices down $\varepsilon_{ABC} := g_{AA'}\varepsilon^{A'}{}_{BC}$

$$\varepsilon_{+3-} = -1 \qquad \varepsilon_{3+-} = q^{2} \tag{A.13a}$$
$$\varepsilon_{+-3} = 1 \qquad \varepsilon_{-+3} = -1 \qquad \varepsilon_{333} = -\lambda \tag{A.13b}$$
$$\varepsilon_{3-+} = -q^{-2} \qquad \varepsilon_{-3+} = 1\,. \tag{A.13c}$$

Various contractions of ε-tensor and metric yield useful identities

$$\begin{gathered}
\varepsilon^{AB'C} g_{B'B} = \varepsilon^{CA}{}_{B}\,, \quad \varepsilon^{AB}{}_{C'} g^{C'C} = \varepsilon^{BCA} \\
\varepsilon^{A'B'C} g_{A'A} g_{B'B} = \varepsilon_{B}{}^{C}{}_{A}\,, \quad \varepsilon_{A'BC'} g^{A'A} g^{C'C} = \varepsilon^{CA}{}_{B} \\
g_{AB}\varepsilon^{ABC} = 0\,, \quad g_{CA}\varepsilon^{ABC} = 0\,, \quad \varepsilon_{ABC}\, g^{BA} = 0\,, \quad \varepsilon_{ABC}\, g^{AC} = 0 \\
\varepsilon^{AXB}\varepsilon_{CXD} = \varepsilon^{BA}{}_{X}\varepsilon_{C}{}^{X}{}_{D} = \varepsilon^{BAX}\varepsilon_{DCX} \\
\varepsilon_{A}{}^{B}{}_{C}\,\varepsilon^{AC}{}_{D} = [4][2]^{-1}\delta^{B}_{D}\,, \quad \varepsilon_{ABC}\,\varepsilon^{BAD} = \varepsilon_{BCA}\,\varepsilon^{ADB} = [4][2]^{-1}\delta^{D}_{C} \\
\varepsilon^{AB}{}_{X}\varepsilon^{XC}{}_{D} + g^{AB}\delta^{C}_{D} = \varepsilon^{AX}{}_{D}\varepsilon^{BC}{}_{X} + \delta^{A}_{D} g^{BC}\,.
\end{gathered} \tag{A.14}$$

There are relations between ε-tensors with the same index in an upper and a lower position

$$\varepsilon_{ABC} = \varepsilon^{ACB}\,, \quad \varepsilon_{A}{}^{B}{}_{C} = \varepsilon^{AC}{}_{B}\,, \quad \varepsilon^{A}{}_{BC} = \varepsilon_{A}{}^{CB}\,. \tag{A.15}$$

With the metric and the epsilon tensor we can define a scalar and a vector product. Note, that if we defined real coordinates by $X_1 := \mathrm{i}(X_+ - X_+^*)$, $X_2 := X_+ + X_+^*$ we would get, e.g., $\varepsilon^{123} = -q\mathrm{i}$. In the limit $q \to 1$ our epsilon tensor will tend to $-\mathrm{i}$ times the usual undeformed epsilon tensor. We therefore define for 3-vector operators X_A and Y_B

$$\vec{X} \cdot \vec{Y} := g^{AB} X_A Y_B \,, \qquad (\vec{X} \times \vec{Y})_C := \mathrm{i}\, X_A Y_B \varepsilon^{AB}{}_C \,, \tag{A.16}$$

where we use arrows to indicate the 3-vector operators. Raising and lowering the indices we get

$$\vec{X} \cdot \vec{Y} := g_{BA}\, X^A Y^B \,, \qquad (\vec{X} \times \vec{Y})^C := \mathrm{i}\, X^A Y^B \varepsilon_B{}^C{}_A \,. \tag{A.17}$$

With this notation some of the identities (A.14) take on a very intuitive form

$$\begin{aligned} \vec{X} \cdot (\vec{Y} \times \vec{Z}) &= (\vec{X} \times \vec{Y}) \cdot \vec{Z} \\ (\vec{X} \times \vec{Y}) \times \vec{Z} - (\vec{X} \cdot \vec{Y})\vec{Z} &= \vec{X} \times (\vec{Y} \times \vec{Z}) - \vec{X}(\vec{Y} \cdot \vec{Z}) \,, \end{aligned} \tag{A.18}$$

from which more relations can be deduced very easily. Finally, we apply Eq. (2.21) to the scalar and the vector product

$$\langle j \| \vec{X} \cdot \vec{Y} \| j \rangle = \sum_{j'} (-1)^{j'+j} \sqrt{\frac{[2j'+1]}{[2j+1]}}\, \langle j \| \vec{X} \| j' \rangle \langle j' \| \vec{Y} \| j \rangle \tag{A.19a}$$

$$\begin{aligned} \langle j-1 \| \vec{X} \times \vec{Y} \| j \rangle = {} & \mathrm{i} \sqrt{\frac{[2j-2]}{[2j]}}\, \langle j-1 \| \vec{X} \| j-1 \rangle \langle j-1 \| \vec{Y} \| j \rangle \\ & -\mathrm{i} \sqrt{\frac{[2j+2]}{[2j]}}\, \langle j-1 \| \vec{X} \| j \rangle \langle j \| \vec{Y} \| j \rangle \end{aligned} \tag{A.19b}$$

$$\begin{aligned} \langle j \| \vec{X} \times \vec{Y} \| j \rangle = {} & \mathrm{i} \sqrt{\frac{[2j+2][2j-1]}{[2j+1][2j]}}\, \langle j \| \vec{X} \| j-1 \rangle \langle j-1 \| \vec{Y} \| j \rangle \\ & +\mathrm{i} \frac{[2j]-[2j+2]}{\sqrt{[2j+2][2j]}}\, \langle j \| \vec{X} \| j \rangle \langle j \| \vec{Y} \| j \rangle \\ & -\mathrm{i} \sqrt{\frac{[2j+3][2j]}{[2j+2][2j+1]}}\, \langle j \| \vec{X} \| j+1 \rangle \langle j+1 \| \vec{Y} \| j \rangle \end{aligned} \tag{A.19c}$$

$$\begin{aligned} \langle j+1 \| \vec{X} \times \vec{Y} \| j \rangle = {} & \mathrm{i} \sqrt{\frac{[2j]}{[2j+2]}}\, \langle j+1 \| \vec{X} \| j \rangle \langle j \| \vec{Y} \| j \rangle \\ & -\mathrm{i} \sqrt{\frac{[2j+4]}{[2j+2]}}\, \langle j+1 \| \vec{X} \| j+1 \rangle \langle j \| \vec{Y} \| j \rangle \,. \end{aligned} \tag{A.19d}$$

If furthermore there is a $*$-structure $X_A^* = Y^A$, this implies for the reduced matrix elements of a $*$-representation

$$\langle j' \| \vec{X} \| j \rangle = (-1)^{j'-j} \sqrt{\frac{[2j+1]}{[2j'+1]}}\, \overline{\langle j \| \vec{Y} \| j' \rangle} \,. \tag{A.19e}$$

A.1.3 Clebsch-Gordan Coefficients for the q-Lorentz Algebra

The Clebsch-Gordan Coefficients for the q-Lorentz algebra can be read off the formula for the basis vectors of the irreducible subrepresentations

$$\begin{aligned}|(k_1,k_2),(n_1,n_2)\rangle &= \sum \mathrm{C}_q(j_1,j_1',k_1\,|\,m_1,b,n_1)\mathrm{C}_q(j_2,j_2',k_2\,|\,a,m_2',n_2)\\ &\quad\times (R^{-1})^{m_2 m_1'}_{ab}\,|(j_1,j_2),(m_1,m_2)\rangle\otimes|(j_1',j_2'),(m_1',m_2')\rangle\,, \end{aligned}\tag{A.20}$$

As the R-matrix is in general not unitary, these basis vectors have yet to be normalized. We are in particular interested in the q-Clebsch-Gordan coefficients for the decomposition of a tensor product of two vector representations

$$D^{(\frac{1}{2},\frac{1}{2})}\otimes D^{(\frac{1}{2},\frac{1}{2})}\cong D^{(0,0)}\oplus D^{(1,0)}\oplus D^{(0,1)}\oplus D^{(1,1)}\,. \tag{A.21}$$

For a more compact notation we write

$$E^{j_1 j_2}_{m_1 m_2} := |(j_1,j_2),(m_1,m_2)\rangle\,, \qquad E_{ab} := |(\tfrac{1}{2},\tfrac{1}{2}),(a,b)\rangle\,, \tag{A.22}$$

where a, b run through $\{-\frac{1}{2},\frac{1}{2}\}=\{-,+\}$. We get for the *unnormalized* basis vectors of the $D^{(1,0)}$ subrepresentation

$$\begin{aligned}E^{1,0}_{-1,0} &= qE_{-+}\otimes E_{--} - q^{-1}E_{--}\otimes E_{-+}\\ E^{1,0}_{0,0} &= E_{++}\otimes E_{--} - E_{--}\otimes E_{++} + \lambda E_{-+}\otimes E_{-+}\\ &\quad + E_{-+}\otimes E_{+-} - q^{-2}E_{+-}\otimes E_{-+}\\ E^{1,0}_{+1,0} &= E_{++}\otimes E_{+-} - E_{+-}\otimes E_{++} + \lambda E_{++}\otimes E_{-+}\,,\end{aligned}\tag{A.23}$$

for the $D^{(0,1)}$ subrepresentation

$$\begin{aligned}E^{0,1}_{0,-1} &= E_{+-}\otimes E_{-+} - E_{--}\otimes E_{+-} + \lambda E_{-+}\otimes E_{--}\\ E^{0,1}_{0,0} &= E_{++}\otimes E_{--} - E_{--}\otimes E_{++} + \lambda E_{-+}\otimes E_{-+}\\ &\quad + E_{+-}\otimes E_{-+} - q^{-2}E_{-+}\otimes E_{+-}\\ E^{0,1}_{0,+1} &= qE_{++}\otimes E_{-+} - q^{-1}E_{-+}\otimes E_{++}\,,\end{aligned}\tag{A.24}$$

and for the $D^{(0,0)}$ subrepresentation

$$\begin{aligned}E^{0,0}_{0,0} &= qE_{++}\otimes E_{--} + q^{-1}E_{--}\otimes E_{++} - q^{-1}E_{-+}\otimes E_{+-}\\ &\quad - q^{-1}E_{+-}\otimes E_{+-} - q^{-1}\lambda E_{-+}\otimes E_{-+}\,.\end{aligned}\tag{A.25}$$

Expressed in terms of a 4-vector basis we find bases for the $D^{(1,0)}$, $D^{(0,1)}$, and $D^{(0,0)}$ subrepresentations

$$\begin{aligned}E^{1,0}_{C,0} &= E_A\otimes E_B\,\varepsilon^{AB}{}_C + qE_0\otimes E_C - q^{-1}E_C\otimes E_0\\ E^{0,1}_{0,C} &= E_A\otimes E_B\,\varepsilon^{AB}{}_C + qE_C\otimes E_0 - q^{-1}E_0\otimes E_C\\ E^{0,0}_{0,0} &= E_\mu\otimes E_\nu\,\eta^{\mu\nu}\,,\end{aligned}\tag{A.26}$$

which are neither orthogonal nor normalized. The last equation defines up to a constant factor the 4-vector metric $\eta^{\mu\nu}$ whose non-zero values are

$$\eta^{00} = 1\,, \qquad \eta^{-+} = q^{-1}\,, \qquad \eta^{+-} = q\,, \qquad \eta^{33} = -1\,, \tag{A.27}$$

which means in particular that $\eta^{AB} = -g^{AB}$. Let us denote the projectors on the subrepresentations of the $D^{(\frac{1}{2},\frac{1}{2})}$ representation in an obvious notation[2] by

$$1 = \mathbb{P}_{(0,0)} + \mathbb{P}_{(1,0)} + \mathbb{P}_{(0,1)} + \mathbb{P}_{(1,1)}\,. \tag{A.28}$$

The projectors on the symmetric and antisymmetric part are denoted by

$$\mathbb{P}_{\mathrm{S}} := \mathbb{P}_{(0,0)} + \mathbb{P}_{(1,1)}\,, \qquad \mathbb{P}_{\mathrm{A}} := \mathbb{P}_{(1,0)} + \mathbb{P}_{(0,1)}\,. \tag{A.29}$$

These projectors can be determined from the bases of the corresponding spaces, which we just computed. Note however that $D^{(1,0)}$ and $D^{(0,1)}$ are not mutually orthogonal, so we have to project on $D^{(1,0)}$ along $D^{(0,1)}$ and vice versa. We obtain for the trace part

$$(\mathbb{P}_{(0,0)})^{ab}{}_{cd} = [2]^{-2}\,\eta^{ab}\eta_{cd}\,, \tag{A.30}$$

for the left chiral and right chiral part:

$[2]^2(\mathbb{P}_{(1,0)})^{ab}{}_{cd} =$

	$C0$	$0D$	CD
$A0$	δ^A_C	$-q^{-2}\delta^A_D$	$-q^{-1}\varepsilon_C{}^A{}_D$
$0B$	$-q^2\delta^B_C$	δ^B_D	$q\varepsilon_C{}^B{}_D$
AB	$-q\varepsilon^{AB}{}_C$	$q^{-1}\varepsilon^{AB}{}_D$	$\varepsilon^{AB}{}_X\varepsilon_C{}^X{}_D$

$[2]^2(\mathbb{P}_{(0,1)})^{ab}{}_{cd} =$

	$C0$	$0D$	CD
$A0$	δ^A_C	$-q^2\delta^A_D$	$q\varepsilon_C{}^A{}_D$
$0B$	$-q^{-2}\delta^B_C$	δ^B_D	$-q^{-1}\varepsilon_C{}^B{}_D$
AB	$q^{-1}\varepsilon^{AB}{}_C$	$-q\varepsilon^{AB}{}_D$	$\varepsilon^{AB}{}_X\varepsilon_C{}^X{}_D$

For the anti-symmetrizer this yields

$[2]^2(\mathbb{P}_{\mathrm{A}})^{ab}{}_{cd} =$

	$C0$	$0D$	CD
$A0$	$2\delta^A_C$	$-[4][2]^{-1}\delta^A_D$	$\lambda\varepsilon_C{}^A{}_D$
$0B$	$-[4][2]^{-1}\delta^B_C$	$2\delta^B_D$	$\lambda\varepsilon_C{}^B{}_D$
AB	$-\lambda\varepsilon^{AB}{}_C$	$-\lambda\varepsilon^{AB}{}_D$	$2\varepsilon^{AB}{}_X\varepsilon_C{}^X{}_D$

The traceless symmetric part is given by Eq. (A.28).

[2] Elsewhere [52] the same projectors have been denoted by P_T, P_+, P_-, P_S, in that order.

A.2 Representations

A.2.1 Representations of $\mathcal{U}_q(\mathrm{su}_2)$

The action of the generators within the D^j representation of $\mathcal{U}_q(\mathrm{su}_2)$ is given by

$$\begin{aligned} E|j,m\rangle &= q^{(m+1)}\sqrt{[j+m+1][j-m]}\,|j,m+1\rangle \\ F|j,m\rangle &= q^{-m}\sqrt{[j+m][j-m+1]}\,|j,m-1\rangle \\ K|j,m\rangle &= q^{2m}|j,m\rangle\,. \end{aligned} \tag{A.31}$$

For for the vectorial generators this means

$$J_A|j,m\rangle = -[2]^{-1}\sqrt{[2j+2][2j]}\,\mathrm{C}_q(1,j,j\,|\,A,m,m+A)\,|j,m+A\rangle\,. \tag{A.32}$$

The value of the Casimir W within such a representation is given by

$$\rho^j(W) = [2]^{-1}\big(q^{(2j+1)} + q^{-(2j+1)}\big)\,. \tag{A.33}$$

For $j=\frac{1}{2}$ the generators are represented by

$$E := \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} & 0 \end{pmatrix}, \qquad F := \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \qquad K := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}, \tag{A.34}$$

with respect to the $\{-,+\}$ basis. The representation of the vector generators J_A is proportional to the q-Pauli matrices

$$\sigma_A = [2]\,\rho^{\frac{1}{2}}(J_A)\,, \qquad \tilde\sigma_A = -[2]\,\rho^{\frac{1}{2}}(SJ_A)\,, \tag{A.35}$$

where

$$\sigma_- = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \sigma_+ = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ -q^{\frac{1}{2}} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix} \tag{A.36}$$

$$\tilde\sigma_- = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & q^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \tilde\sigma_+ = [2]^{\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \tilde\sigma_3 = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix}, \tag{A.37}$$

with respect to the $\{-\frac{1}{2},\frac{1}{2}\} = \{-,+\}$ basis. The q-Pauli matrices satisfy the relations

$$\sigma_A\,\sigma_B\,\varepsilon^{AB}{}_C = [4][2]^{-1}\,\sigma_C\,, \qquad \sigma_A\sigma_B = g_{AB} + \sigma_C\,\varepsilon_A{}^C{}_B \tag{A.38}$$

$$\tilde\sigma_A\,\tilde\sigma_B\,\varepsilon^{BA}{}_C = -[4][2]^{-1}\,\tilde\sigma_C\,, \qquad \tilde\sigma_A\tilde\sigma_B = g_{BA} - \tilde\sigma_C\,\varepsilon_B{}^C{}_A, \tag{A.39}$$

For the $j=1$ vector representations we get

$$E := [2]^{\frac{1}{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad F := [2]^{\frac{1}{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & q^{-1} \\ 0 & 0 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix} \tag{A.40}$$

$$J_- := \begin{pmatrix} 0 & q^{-1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_+ := \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -q & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} -q & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & q^{-1} \end{pmatrix} \tag{A.41}$$

with respect to the $\{-1, 0, 1\} = \{-, 3, +\}$ basis. The matrix representations of the vector generator is proportional to the epsilon tensor,

$$\rho^1(J_A)^B{}_C = \varepsilon_A{}^B{}_C\,. \tag{A.42}$$

A.2.2 Representations of the q-Lorentz Algebra

The representation maps for the $D^{(j_1,j_2)}$ representations of the q-Lorentz algebra, $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$, are composed of the representation maps of $\mathcal{U}_q(\mathrm{su}_2)$ according to $\rho^{(j_1,j_2)} := \rho^{j_1} \otimes \rho^{j_2}$. Particularly simple are the chiral representations $D^{(j,0)}$ and $D^{(0,j)}$. For any rotations $l \in \mathcal{U}_q(\mathrm{su}_2)$ and for the boosts as defined in Eq. (A.63) we get

$$\begin{gathered} \rho^{(j,0)}(l) = \rho^j(l) = \rho^{(0,j)}(l) \\ \rho^{(j,0)}(B^a{}_b) = \rho^j\big((L_-^{\frac{1}{2}})^a{}_b\big)\,, \qquad \rho^{(0,j)}(B^a{}_b) = \rho^j\big((L_+^{\frac{1}{2}})^a{}_b\big)\,. \end{gathered} \tag{A.43}$$

If we denote the basis of a $D^{(\frac{1}{2},\frac{1}{2})}$ representation as in Eq. (3.1) by

$$E_{ab} = \begin{pmatrix} E_{--} & E_{-+} \\ E_{+-} & E_{++} \end{pmatrix} =: \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{A.44}$$

we get for the action

$$\begin{aligned} E \otimes 1 \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix}, & 1 \otimes E \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} B & 0 \\ D & 0 \end{pmatrix} \\ F \otimes 1 \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}, & 1 \otimes F \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & A \\ 0 & C \end{pmatrix} \\ K \otimes 1 \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} q^{-1}A & q^{-1}B \\ qC & qD \end{pmatrix}, & 1 \otimes K \triangleright \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} q^{-1}A & qB \\ q^{-1}C & qD \end{pmatrix}. \end{aligned} \tag{A.45}$$

For the boost generators (A.63) this means in particular

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright A = \begin{pmatrix} A & q^{-1}\lambda B \\ 0 & A \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright B = \begin{pmatrix} q^{-1}B & 0 \\ 0 & qB \end{pmatrix} \tag{A.46a}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright C = \begin{pmatrix} qC & \lambda D \\ -q\lambda A & q^{-1}C - \lambda^2 B \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright D = \begin{pmatrix} D & 0 \\ -\lambda B & D \end{pmatrix} \tag{A.46b}$$

In terms of the 4-vector basis E_μ of $D^{(\frac{1}{2},\frac{1}{2})}$, defined as in Eq. (3.10) by

$$\begin{aligned} E_0 &= q^{-1}[2]^{-1}(q^{\frac{1}{2}}C - q^{-\frac{1}{2}}B) \\ E_- &= [2]^{-\frac{1}{2}}A \\ E_+ &= [2]^{-\frac{1}{2}}D \\ E_3 &= [2]^{-1}(q^{-\frac{1}{2}}C + q^{\frac{1}{2}}B)\,. \end{aligned} \tag{A.47}$$

the action becomes

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright E_0 &= \begin{pmatrix} [2]^{-1}\left(\frac{[4]}{[2]}E_0 + q^{-1}\lambda E_3\right) & q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}E_+ \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}E_- & [2]^{-1}\left(\frac{[4]}{[2]}E_0 - q\lambda E_3\right) \end{pmatrix} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright E_- &= \begin{pmatrix} E_- & q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}(E_3 - E_0) \\ 0 & E_- \end{pmatrix} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright E_+ &= \begin{pmatrix} E_+ & 0 \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}(E_3 - E_0) & E_+ \end{pmatrix} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright E_3 &= \begin{pmatrix} [2]^{-1}(2E_3 + q\lambda E_0) & q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}E_+ \\ -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}}E_- & [2]^{-1}(2E_3 - q^{-1}\lambda E_0) \end{pmatrix}
\end{aligned} \tag{A.48}$$

Now we can calculate the 4-vector matrix representation Λ defined by

$$h \triangleright E_\mu = E_{\mu'}\,\Lambda(h)^{\mu'}{}_\mu \tag{A.49}$$

for all q-Lorentz transformations h. For the rotations $l \in \mathcal{U}_q(\mathrm{su}_2)$ we get by construction of the 4-vector basis

$$\Lambda(l) = \begin{pmatrix} \rho^0(l) & 0 \\ 0 & \rho^1(l) \end{pmatrix} . \tag{A.50}$$

For the boost we calculate

$$\begin{aligned}
\Lambda(a) &= \begin{pmatrix} [4][2]^{-2} & 0 & 0 & q\lambda[2]^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1}\lambda[2]^{-1} & 0 & 0 & 2[2]^{-1} \end{pmatrix}, \quad \Lambda(b) = q^{-\frac{1}{2}}\lambda[2]^{-\frac{1}{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\Lambda(c) &= -q^{\frac{1}{2}}\lambda[2]^{-\frac{1}{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Lambda(d) = \begin{pmatrix} [4][2]^{-2} & 0 & 0 & -q^{-1}\lambda[2]^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q\lambda[2]^{-1} & 0 & 0 & 2[2]^{-1} \end{pmatrix},
\end{aligned} \tag{A.51}$$

with respect to the $\{0, -, +, 3\}$ basis.

A.3 $\mathcal{R}$-matrices

For a Hopf algebra H a universal $\mathcal{R}$-matrix is an invertible element $\mathcal{R} \in H \otimes H$, which we will also write in a Sweedler like notation as $\mathcal{R} := \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}$, with

$$\begin{gathered}
(\tau \circ \Delta)(h) = \mathcal{R}\,\Delta(h)\,\mathcal{R}^{-1} \\
(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}\,, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}\,,
\end{gathered} \tag{A.52}$$

where the indices indicate the position of the tensor factors, $\mathcal{R}_{13} := \mathcal{R}_{[1]} \otimes 1 \otimes \mathcal{R}_{[2]}$ etc. If there is a $*$-structure on H the $\mathcal{R}$-matrix is said to be real if $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}$ and anti-real if $\mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}$. There are some useful properties of $\mathcal{R}$ that can be deduced from Eqs. (A.52):

$$\begin{aligned} &\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,, \quad (\varepsilon \otimes \mathrm{id})(\mathcal{R}) = 1 \quad (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1 \\ &(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}\,, \quad (\mathrm{id} \otimes S)(\mathcal{R}^{-1}) = \mathcal{R}\,, \quad (S \otimes S)(\mathcal{R}) = \mathcal{R}\,. \end{aligned} \tag{A.53}$$

A.3.1 The $\mathcal{R}$-Matrix of $\mathcal{U}_q(\mathrm{su}_2)$

There is a universal $\mathcal{R}$-matrix for $\mathcal{U}_q(\mathrm{su}_2)$,

$$\mathcal{R} = q^{(H\otimes H)/2} \sum_{n=0}^{\infty} R_n(q)(E^n \otimes F^n) \tag{A.54}$$

which is not an element $\mathcal{U}_q(\mathrm{su}_2) \otimes \mathcal{U}_q(\mathrm{su}_2)$ proper, since it is described as an infinite power series. For our purposes this does not raise serious problems. This $\mathcal{R}$-matrix is real. For representations ρ^j, $\rho^{j'}$ of $\mathcal{U}_q(\mathrm{su}_2)$ we can define R-matrices and a variant, the $\hat{R}$-matrices, by

$$R^{(j,j')} := (\rho^j \otimes \rho^{j'})(\mathcal{R})\,, \qquad (\hat{R}^{(j,j')})^{ab}{}_{cd} := (R^{(j,j')})^{ba}{}_{cd}\,. \tag{A.55}$$

Traditionally, the R-matrices are normalized differently. We will use

$$R_{\mathrm{su}_2} := q^{\frac{1}{2}} R^{(\frac{1}{2},\frac{1}{2})}\,, \qquad R_{\mathrm{so}_3} := q^{-2} R^{(1,1)}\,. \tag{A.56}$$

Explicitly, we get

$$(R_{\mathrm{su}_2})^{ab}{}_{cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{A.57}$$

with respect to the basis $\{--, -+, +-, ++\}$, and

$$\begin{aligned} (R_{\mathrm{so}_3})^{AB}{}_{CD} &= \delta^B_C \delta^A_D - q^{-3}\lambda\, g^{BA} g_{CD} - q^{-2} \varepsilon^{BAX} \varepsilon_{DCX} \\ (R^{-1}_{\mathrm{so}_3})^{AB}{}_{CD} &= \delta^A_D \delta^B_C - q^{3}\lambda\, g^{AB} g_{DC} - q^{2} \varepsilon^{ABX} \varepsilon_{CDX}\,. \end{aligned} \tag{A.58}$$

This means that we have a projector decomposition

$$\hat{R}_{\mathrm{so}_3} = 1 - q^{-3}\lambda[3]^{-1}\mathbb{P}_0 - q^{-2}[4][2]^{-1}\mathbb{P}_1 = -q^{-6}\mathbb{P}_0 - q^{-4}\mathbb{P}_1 + \mathbb{P}_3\,. \tag{A.59}$$

Applying a representation to one half of the $\mathcal{R}$-matrix only leads to the definition of the L-matrices

$$(L^j_+)^a{}_b := \mathcal{R}_{[1]}\rho^j(\mathcal{R}_{[2]})^a{}_b\,, \qquad (L^j_-)^a{}_b := \rho^j(\mathcal{R}^{-1}_{[1]})^a{}_b\,\mathcal{R}^{-1}_{[2]}\,. \tag{A.60}$$

We calculate the L-matrices for $j=\frac{1}{2}$ and $j=1$, explicitly.

$$L_+^{\frac{1}{2}} = \begin{pmatrix} K^{-\frac{1}{2}} & q^{-\frac{1}{2}}\lambda K^{-\frac{1}{2}}E \\ 0 & K^{\frac{1}{2}} \end{pmatrix}, \qquad L_-^{\frac{1}{2}} = \begin{pmatrix} K^{\frac{1}{2}} & 0 \\ -q^{\frac{1}{2}}\lambda F K^{\frac{1}{2}} & K^{-\frac{1}{2}} \end{pmatrix} \tag{A.61}$$

$$L_+^{1} = \begin{pmatrix} K^{-1} & \lambda[2]^{\frac{1}{2}}K^{-1}E & \lambda^2 K^{-1}E^2 \\ 0 & 1 & q^{-1}\lambda[2]^{\frac{1}{2}}E \\ 0 & 0 & K \end{pmatrix}, \quad L_-^{1} = \begin{pmatrix} K & 0 & 0 \\ -\lambda[2]^{\frac{1}{2}}FK & 1 & 0 \\ \lambda^2F^2K & -q\lambda[2]^{\frac{1}{2}}F & K^{-1} \end{pmatrix} \tag{A.62}$$

These results are being used in Eq. (2.50) to calculate the boost generators defined as

$$B^a{}_c := \left(L_-^{\frac{1}{2}}\right)^a{}_b \otimes \left(L_+^{\frac{1}{2}}\right)^b{}_c =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.63}$$

which yields

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & q^{-\frac{1}{2}}\lambda K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \\ -q^{\frac{1}{2}}\lambda F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & K^{-\frac{1}{2}} \otimes K^{\frac{1}{2}} - \lambda^2 F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \end{pmatrix}. \tag{A.64}$$

A.3.2 The $\mathcal{R}$-Matrices of the q-Lorentz Algebra

There are two universal $\mathcal{R}$-matrices of the q-Lorentz algebra, which are composed of the $\mathcal{R}$-matrix of $\mathcal{U}_q(\mathrm{sl}_2)$ according to

$$\mathcal{R}_{\mathrm{I}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}\,, \qquad \mathcal{R}_{\mathrm{II}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{13}\mathcal{R}_{24}\mathcal{R}_{23}\,. \tag{A.65}$$

$\mathcal{R}_{\mathrm{I}}$ is anti-real while $\mathcal{R}_{\mathrm{II}}$ is real. Their vector representations are normalized as

$$R_{\mathrm{I}} := (\Lambda\otimes\Lambda)(\mathcal{R}_{\mathrm{I}})\,, \qquad R_{\mathrm{II}} := q(\Lambda\otimes\Lambda)(\mathcal{R}_{\mathrm{II}})\,, \tag{A.66}$$

where Λ is the 4-vector representation map of the q-Lorentz algebra. These matrices can be decomposed into projectors

$$\begin{aligned} \hat{R}_{\mathrm{I}} &= \mathbb{P}_{(0,0)} - q^2\mathbb{P}_{(1,0)} - q^{-2}\mathbb{P}_{(0,1)} + \mathbb{P}_{(1,1)} \\ \hat{R}_{\mathrm{II}} &= q^2\mathbb{P}_{(0,0)} - \mathbb{P}_{(1,0)} - \mathbb{P}_{(0,1)} + q^{-2}\mathbb{P}_{(1,1)}\,. \end{aligned} \tag{A.67}$$

The L_+-matrix of $\mathcal{R}_{\mathrm{I}}$ has a simple form:

$$\left(L_{\mathrm{I}+}^{\Lambda}\right)^a{}_b := \mathcal{R}_{\mathrm{I}[1]}\,\Lambda(\mathcal{R}_{\mathrm{I}[2]})^a{}_b = \begin{pmatrix} 1 & 0 \\ 0 & t^A{}_B \end{pmatrix}, \tag{A.68}$$

where $t^A{}_B$ is the vector corepresentation matrix of $SU_q(2)^{\mathrm{op}}$,

$$t = \begin{pmatrix} a^2 & q^{\frac{1}{2}}[2]^{\frac{1}{2}}ab & b^2 \\ q^{\frac{1}{2}}[2]^{\frac{1}{2}}ac & (1+[2]bc) & q^{\frac{1}{2}}[2]^{\frac{1}{2}}bd \\ c^2 & q^{\frac{1}{2}}[2]^{\frac{1}{2}}cd & d^2 \end{pmatrix} \tag{A.69}$$

with respect to the basis $\{-1,0,1\}=\{-,3,+\}$. For chiral representations we get

$$\rho^{(j,0)}(t^A{}_B) = \rho^j\left((L_-^1)^A{}_B\right), \qquad \rho^{(0,j)}(t^A{}_B) = \rho^j\left((L_+^1)^A{}_B\right). \tag{A.70}$$

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