An Infinitesimal Approach to Stochastic Analysis on Abstract Wiener Spaces

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Introduction

The concept of abstract Wiener spaces, introduced by L. Gross in [14], arises from the problem of finding a measure on an infinite dimensional Hilbert space $\mathbb{H}$. The Gaussian measure on the cylinder sets of $\mathbb{H}$ fails to be σ-additive; it is categorically impossible to find measures on $\mathbb{H}$ which are translation invariant and positive on nonempty open sets. But if one considers an appropriate weaker norm on $\mathbb{H}$ and denotes the Banach space completion of $\mathbb{H}$ with respect to this new norm by $\mathbb{B}$, then there exists a Gaussian measure P on $\mathbb{B}$. The triple $(\mathbb{B}, \mathbb{H}, P)$ is called an abstract Wiener space. This terminology is justified since the classical Wiener space of continuous functions on the unit interval can be regarded as an abstract Wiener space. In fact, any separable Banach space appears as an abstract Wiener space.

The Malliavin calculus is an infinite-dimensional differential calculus and was introduced by P. Malliavin in [21]. The derivative is a linear but unbounded operator defined on a closed subspace of the space of square integrable functions $f : \mathbb{B} \to \mathbb{R}$. It takes values in the space of square Bochner integrable functions $g : \mathbb{B} \to \mathbb{H}$. The integral is defined as the adjoint operator of the derivative. Originally conceived to investigate regularity properties of the law of solutions of stochastic differential equations, the Malliavin calculus evolved into an area of study in its own right. Recently it has also been applied to the theory of finance (cf. [13]).

In contrast to the classical Wiener space, in the case of abstract Wiener spaces there is no natural notion of time. Naturally, this leads to difficulties when defining a stochastic integral. In a new approach (cf. [32]), Üstünel and Zakai solve this problem by working with a resolution of the identity on $\mathbb{H}$. This provides a notion of time and adaptedness, enabling them to define the stochastic integral of certain adapted $\mathbb{H}$-valued random variables. The integrators are $\mathbb{B}$-valued random variables called abstract Wiener processes. Both, the integrands and the integrator live on an arbitrary probability space.

It is a well known result from model theory that there exist polysaturated models of mathematics. On the one hand these models have the same formal properties as the standard model. On the other hand all sets in saturated models are essentially compact: for every cardinal number κ there is a polysaturated model in which each set is κ-compact. As a consequence we obtain the existence of numbers $\alpha \neq 0$ in the extended models which are smaller than any real number in the standard model. These numbers are called infinitesimals. Certainly we have to pay a price for obtaining such properties: the sets in the saturated models, called the internal sets, are no more unrestricted closed under subset formation. But sets which can be defined in terms of other internal sets are internal themselves. Using polysaturated models, we can replace objects of the standard model by internal objects of considerably lower complexity. For example we can work with the set $\{1, ..., H\}$ instead of the continuous time interval $[0, 1]$. Here H is a hyperfinite natural number, i.e. an internal natural number greater than every standard natural number. The passage to the extended model can also be reversed: it is often possible to identify internal sets with standard sets via the so-called standard part map. For example, we can define the Lebesgue integral as the standard part of a hyperfinite summation. Another example is H. Osswald's (see [24] and [25]) method to introduce a notion of time in abstract Wiener spaces: instead of $\mathbb{B}$, one works with the space of $\mathbb{B}$-valued functions on the unit interval. This allows for the definition of a Brownian motion as the standard part of an internal random walk, and to define a stochastic integral as the standard part of an internal Riemann-Stieltjes-integral.

By means of polysaturated models we can define a Loeb probability space which allows us to treat the Malliavin calculus on abstract Wiener spaces and the Üstünel-Zakai-integral simultaneously. First, we replace $\mathbb{B}$ by a hyperfinite dimensional subspace $\mathbb{F}$ of ${}^*\mathbb{H}$ which contains $\mathbb{H}$. We then equip $\mathbb{F}$ with the Loeb measure induced by the internal Gaussian measure, a method due to Lindstrøm (cf. [18]). In [8], Cutland and Ng establish an infinitesimal approach to the Malliavin calculus for the classical Wiener space. In this approach the basic operators have natural descriptions as classical differential operators on internal Euclidean spaces. Using the hyperfinite space $\mathbb{F}$, we generalize this saturated model approach of the Malliavin calculus to abstract Wiener spaces. In this setup, a resolution of the identity is a family of orthogonal projections on $\mathbb{H}$, indexed by the unit interval. By working with $\mathbb{F}$ instead of $\mathbb{H}$, we can express the resolution by an internal family of projections on ${}^*\mathbb{H}$, indexed by a hyperfinite set. Furthermore, we can describe the internal projections in terms of an internal orthonormal basis of $\mathbb{F}$. Using a saturation argument, we manage to establish a linear dependence between the index set of the internal family of projections and the index set of the orthonormal basis of $\mathbb{F}$. This paves the way for our further proceeding: we define a canonical internal Itô integral, whose standard part turns out to be the Üstünel-Zakai-integral.

One of the most interesting theorems of this thesis is the Clark Ocone formula for abstract Wiener spaces, because it establishes a connection between the Malliavin derivative and the stochastic integral. This kind of fundamental theorem of calculus states, roughly speaking, that each Malliavin differentiable function equals the stochastic integral of its derivative. This theorem is well known for the classical Wiener space but new in this general form. The orthogonal projection pr from the space of square Bochner integrable functions $\psi : \mathbb{F} \to \mathbb{H}$ onto the space of adapted functions plays a major role in the Clark Ocone formula. The crucial point is that our saturated model approach makes it possible to define the assignment $\psi \mapsto pr\psi$ constructively, i.e. given a generic ψ we have the information how $pr\,\psi$ explicitely looks like. Endowed with this information, the proof of the Clark Ocone formula is a simple hyperfinite computation.

This shows once again that polysaturated models are not only an isolated field of research. By the transfer to internal objects many operations can be expressed constructively. This leads to new results in various mathematical disciplines, such as functional analysis and stochastic analysis. See for example the treatment of stochastic differential equations by Hoover and Perkins in [15].

In the appendix, which is independent of the rest of the thesis, we consider the so called Lévy transformation of Brownian motion L. This transformation operates on the space of continuous functions on the unit interval. It is one of the most famous open problems in stochastic analysis whether it is ergodic. (See Question 1 in Chapter VI of [28].) We construct an internal transformation τ on a hyperfinite dimensional space and show that L is ergodic if and only if τ is ergodic. This allows us to see the open question concerning the ergodicity of L from a different point of view, since τ is given explicitely, whereas L is defined by a stochastic integral.

1 Some Definitions and Notations

Let us start with some definitions and notations. For sets A and B we write $A := B$ if A equals B by definition. If A and B are formulae, we write $A :\Leftrightarrow B$ if A is equivalent to B by definition. Let $\mathbb{R}$ denote the set of the real numbers. Set further $\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$, $\mathbb{N} := \{1, 2, ...\}$ and $\mathbb{N}_0 := \{0, 1, 2, ...\}$. For $n \in \mathbb{N}$ and any set I we define

$$I^n_{\neq} := \{(i_1, ..., i_n) \in I^n \mid k \neq l \Rightarrow i_k \neq i_l\}$$

and if I is a set of real numbers we define

$$I^n_{<} := \{(i_1, ..., i_n) \in I^n \mid k < l \Rightarrow i_k < i_l\}.$$

Therefore $I^1_{\neq} = I^1_{<} = I$. For any topological space X let $\mathfrak{b}_X$ denote the Borel σ-algebra on X and for $B \subset X$ let $\overline{B}$ denote the closure of B in X. For any subset A of a real vector space X let $\operatorname{span}(A)$ denote the smallest subspace of X containing A. For subsets A and B of a set X let

$$A \bigtriangleup B := \{x \in A \mid x \notin B\} \cup \{x \in B \mid x \notin A\}$$

denote the **symmetric difference** of A and B. For any Banach space $\mathbb{B}$ we denote by $\mathbb{B}'$ the **topological dual** of $\mathbb{B}$, i.e. the space of all continuous linear functions $\varphi : \mathbb{B} \to \mathbb{R}$.

Given any Hilbert space $\mathbb{H}$, we denote the scalar product by $<\cdot\,,\cdot>_{\mathbb{H}}$ and the norm by $\|\cdot\|_{\mathbb{H}}$. Sometimes we omit the index $\mathbb{H}$. Fix $x, y \in \mathbb{H}$ and $A, B \subset \mathbb{H}$. The vectors x and y are said to be **orthogonal** if $< x, y >= 0$. In this case we write $x \perp y$. We further set

$$\begin{aligned} x\perp A \;&:\Leftrightarrow\; x\perp z \text{ for all } z \in A\,, \\ A\perp B \;&:\Leftrightarrow\; z\perp B \text{ for all } z \in A \text{ and} \\ A^{\perp} &:= \{z \in \mathbb{H} \mid z\perp A\}. \end{aligned}$$

For a finite dimensional subspace A of $\mathbb{H}$ let $\dim(A)$ denote the dimension of A. By the Riesz representation theorem, we can identify $\mathbb{H}$ with $\mathbb{H}'$. If $\mathbb{H}$ is an $\mathcal{L}^2$-space, we sometimes write $<\cdot\,,\cdot>_2$ instead of $<\cdot\,,\cdot>_{\mathbb{H}}$ and $\|\cdot\|_2$ instead of $\|\cdot\|_{\mathbb{H}}$. For a closed subspace A of $\mathbb{H}$, denote by $pr^{\mathbb{H}}_A$ the orthogonal projection onto A. For $x \in \mathbb{H}$ and a finite subset A of $\mathbb{H}$ we set

$$x_A := pr^{\mathbb{H}}_{\operatorname{span}(A)} x := pr^{\mathbb{H}}_{\operatorname{span}(A)}(x).$$

For $h \in \mathbb{H}$ and $m \in \mathbb{N}$ let

$$\begin{aligned} h^{\odot m} \ : \ \ \mathbb{H}^m &\to \mathbb{R}\,, \\ (k_1, ..., k_m) &\mapsto \prod_{i=1}^{m} < k_i, h > \ . \end{aligned}$$

Then $h^{\odot m}$ is a symmetric multilinear form on $\mathbb{H}^m$. Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of closed subspaces of $\mathbb{H}$ such that $A_n \perp A_m$ if $n \neq m$. Set

$$\oplus_{n=1}^{\infty} A_n := \left\{ \sum_{n=1}^{\infty} x_n \mid x_n \in A_n \right\} .$$

Then $\oplus_{n=1}^{\infty} A_n$ itself is a closed subspace of $\mathbb{H}$.

Fix a probability space $(\Lambda, \mathcal{C}, \mu)$ and let $\mathcal{S} \subset \mathcal{C}$ be a sub-σ-algebra. If $f : \Lambda \to \mathbb{R}$ is $\mathcal{C}$-measurable, we write $f : (\Lambda, \mathcal{C}) \to \mathbb{R}$ or simply $f \in \mathcal{C}$. If $(\Lambda', \mathcal{C}', \mu')$ is another probability space and $g : \Lambda \to \Lambda'$ is a measurable function, we write $g : (\Lambda, \mathcal{C}) \to (\Lambda', \mathcal{C}')$. We denote by $\mathcal{C} \otimes \mathcal{C}'$ the product σ-algebra of $\mathcal{C}$ and $\mathcal{C}'$ and by $\mu \otimes \mu'$ the product measure of μ and μ'. Set

$$\mathcal{N}_\mu := \{A \in \mathcal{C} \mid \mu(A) = 0\}.$$

Denote by $\mathcal{S} \vee \mathcal{N}_\mu$ the smallest σ-algebra that contains $\mathcal{S}$ and $\mathcal{N}_\mu$. Then a subset A of Λ is in $\mathcal{S} \vee \mathcal{N}_\mu$ if and only if there is a set $B \in \mathcal{S}$ such that $A \triangle B \in \mathcal{N}_\mu$. Set $\mathbb{E}_\mu F := \mathbb{E} F := \int_\Lambda F d\mu$. If $\mathcal{V}$ is a set of functions $f : (\Lambda, \mathcal{C}) \to \mathbb{R}$ we denote by ${}^\sigma\mathcal{V}$ the smallest σ-algebra such that all $f \in \mathcal{V}$ are ${}^\sigma\mathcal{V}$-measurable. If $P = P(\omega)$ is a property which depends on $\omega \in \Lambda$ we say that P holds μ-**almost surely** if $\{\omega \in \Lambda \mid P(\omega) \text{ fails}\} \in \mathcal{N}_\mu$. In this case we write P holds μ-a.s. A function $f : \Lambda \to \mathbb{R}$ is measurable with respect to $\mathcal{S} \vee \mathcal{N}_\mu$ if and only if there is a function $g \in \mathcal{S}$ such that $f = g$ μ-a.s.

Let

$$\mathcal{L}^2(\mu) := \mathcal{L}^2_{\mathcal{C}}(\mu) := \mathcal{L}^2(\Lambda, \mathcal{C}, \mu)$$

denote the Hilbert space of square μ-integrable functions $f : \Lambda \to \mathbb{R}$. For an $f \in \mathcal{L}^2_{\mathcal{C}}(\mu)$ denote by $\mathbb{E}^{\mathcal{S}} f$ the conditional expectation of f with respect to $\mathcal{S}$.

We are working with the standard model of mathematics and with a polysaturated model. The interplay between these universes is established by an elementary embedding $*$ from the standard model into the extended model. In [1] and [20] you can find introductions to these concepts. Now we are specifying the compactness property we have mentioned in the introduction. Fix a standard set I and a set A in the extended model. Assume further that for each $i \in I$ there is an internal subset A_i of A such that the family $(A_i)_{i\in I}$ has the **finite intersection property**, which means that the intersection of each finite subfamily is nonempty. Then we obtain $\cap_{i\in I} A_i \neq \emptyset$. This property is called **polysaturation**. For $\alpha, \beta \in {}^*\mathbb{R}$ we write $\alpha \approx \beta$ if $|\alpha - \beta| < 1/n$ for all $n \in \mathbb{N}$. A number $\alpha \in {}^*\mathbb{R}$ is called **limited** if there is an $n \in \mathbb{N}$ such that $|\alpha| < n$, otherwise α is called **unlimited**. Note that an $\alpha \in {}^*\mathbb{R}$ is limited if and only if there is an $a \in \mathbb{R}$ such that $a \approx \alpha$. In this case a is uniquely determined. Set

$$\mathrm{Lim} := \{\alpha \in {}^*\mathbb{R} \mid \alpha \text{ is limited } \}.$$

For any $\alpha \in {}^*\mathbb{R}$ the **standard part** ${}^\circ\alpha$ of α is defined by

$${}^\circ\alpha = \begin{cases} a & \text{if } \alpha \text{ is limited and } a \in \mathbb{R} \text{ with } \alpha \approx a\,, \\ \infty & \text{if } \alpha \text{ is not limited and } \alpha > 0\,, \\ -\infty & \text{if } \alpha \text{ is not limited and } \alpha < 0\,. \end{cases}$$

Let $(\Omega, \mathcal{B}, \Gamma)$ be an internal probability space. Set

$$\mathcal{N} := \{M \subset \Omega \mid \forall \varepsilon \in \mathbb{R}_0^+ \, \exists N \in \mathcal{B} \text{ with } M \subset N \text{ and } \Gamma(N) < \varepsilon\}$$

and

$$L_\Gamma(\mathcal{B}) := \{\, B \subset \Omega \mid \exists A \in \mathcal{B} \text{ with } B \triangle A \in \mathcal{N} \,\}\ .$$

For $B \in L_\Gamma(\mathcal{B})$ and $A \in \mathcal{B}$ with $B \triangle A \in \mathcal{N}$ set $\widehat{\Gamma}(B) := {}^\circ\Gamma(A)$. Then $(\Omega, L_\Gamma(\mathcal{B}), \widehat{\Gamma})$ is a probability space in the standard sense and $\mathcal{N}$ equals the set of all $\widehat{\mu}$-nullsets $\mathcal{N}_{\widehat{\mu}}$. $(\Omega, L_\Gamma(\mathcal{B}), \widehat{\Gamma})$ is called the **Loeb space** of $(\Omega, \mathcal{B}, \Gamma)$, see again [1] or [20] for details. For an internal function $F : (\Omega, \mathcal{B}) \to \mathbb{R}$ the implications

$$\int |F|\, d\Gamma \in \mathrm{Lim} \ \Rightarrow\ |F| \text{ is limited } \widehat{\Gamma}\text{-a.s.}$$

and

$$\int |F|\, d\Gamma \approx 0 \ \Rightarrow\ |F| \approx 0 \ \widehat{\Gamma}\text{-a.s.}$$

are valid. An internal function $F : \Omega \to {}^*\mathbb{R}$ is called a **lifting** of a function $f : \Omega \to \mathbb{R}$ if $F \approx f$ $\widehat{\Gamma}$-a.s. It is well known that a function $f : \Omega \to \mathbb{R}$ is $L_\Gamma(\mathcal{B})$-measurable if and only if there is a $\mathcal{B}$-measurable lifting F of f. The notion of S-integrability provides an important connection between integration on Loeb spaces and internal integrals. Fix an internal function $F : (\Omega, \mathcal{B}) \to {}^*\mathbb{R}$. Then F is called S_Γ**-integrable** if

$$\int_{\{|F| \geq N\}} |F|\, d\Gamma \approx 0$$

for each unlimited $N \in {}^*\mathbb{N}$. This property is equivalent to the conditions

- $\int_\Omega |F|\, d\Gamma \in \mathrm{Lim}$ and
- $\int_\Omega |F|\, d\Gamma \approx 0$ for each $N \in \mathcal{B}$ with $\Gamma(N) \approx 0$.

For $p \in [1, \infty[$ denote by $SL^p(\Gamma)$ the set of all $G \in \mathcal{B}$ such that $|G|^p$ is S_Γ-integrable. By Hölder's inequality we obtain that $F \in SL^p(\Gamma)$ if $\mathbb{E}\,|F|^{2p}$ is limited. We say that F is s**quare S_Γ-integrable** if $F \in SL^2(\Gamma)$. This is the case if for each $n \in \mathbb{N}$ there is a $G \in SL^2(\Gamma)$ such that $\|G - F\|_2 < 1/n$. The term S-integrable is justified by the fact that a function $f : (\Omega, L_\Gamma(\mathcal{B})) \to \mathbb{R}$ is $\widehat{\Gamma}$-integrable if and only if there is an S-integrable lifting G of f. In this case

$$\int f d\widehat{\Gamma} \approx \int G d\Gamma.$$

If $F \in \mathcal{B}$ is S-integrable, the implications from above become equivalences:

$$\int |F|\, d\Gamma \in \mathrm{Lim} \;\Leftrightarrow\; |F| \text{ is limited } \widehat{\Gamma}\text{-a.s.}$$

and

$$\int |F|\, d\Gamma \approx 0 \;\Leftrightarrow\; |F| \approx 0 \;\widehat{\Gamma}\text{-a.s.}$$

We further mention that all of the assertions about Loeb spaces and S-integrability we have made remain true if $\Gamma(\Omega) \in \mathrm{Lim}$ but Γ is no longer necessarily a probability measure. The proof of the next lemma is a typical example for the use of polysaturation arguments.

1.1 Lemma

Let $(X, \|\cdot\|)$ be an internal normed space. Fix a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X$ and suppose that there is a strictly monotone increasing map $g : \mathbb{N} \to \mathbb{N}$ such that the implication

$$l, k \geq g(n) \Rightarrow \|x_k - x_l\| \leq \tfrac{1}{n}$$

is valid for all $l, k, n \in \mathbb{N}$. Then there is an $x \in X$ such that $\|x - x_k\| \leq \frac{2}{n}$ for all $k, n \in \mathbb{N}$ with $k \geq g(n)$.

Proof. A first saturation argument shows that, since the sequence

$$\mathcal{G}_n := \{F : {}^*\mathbb{N} \to X \mid \forall 1 \leq k \leq n\, (F(k) = x_k)\} \;\; , n \in \mathbb{N}$$

has the finite intersection property, there is an internal sequence $(y_n)_{n\in{}^*\mathbb{N}}$ in X such that $y_n = x_n$ for all $n \in \mathbb{N}$. Note further that ${}^*g : {}^*\mathbb{N} \to {}^*\mathbb{N}$ is an internal strictly monotone increasing function. Now

$$\mathcal{F}_m := \left\{ x \in X \mid \left\| y_{*g(m)} - x \right\| \leq \frac{1}{m} \right\}, \ m \in \mathbb{N}$$

has also the finite intersection property. For $x \in \cap_{m \in \mathbb{N}} \mathcal{F}_m$ we obtain

$$\|x - x_k\| \leq \left\| x - x_{g(n)} \right\| + \left\| x_{g(n)} - x_k \right\| \leq 2 \cdot \tfrac{1}{n}$$

for $n, k \in \mathbb{N}$ and $k \geq g(n)$. $\square$

Note that for a set A in the standard model the inclusion $\{^*a \mid a \in A\} \subset {}^*A$ is strict if and only if A is not finite. Therefore there is a great difference between A and $^*A$. But if we are only interested in A as an object and do not care about the elements of A, we sometimes identify A with $^*A$. For example, we do not distinguish between x and $^*x$ if x is a real number or an element of a Banach space.

For $x, y \in {}^*\mathbb{H}$ we write $x \approx y$ if $\|x - y\| \approx 0$. A function $f : (\Lambda, \mathcal{C}) \to (\mathbb{H}, \mathfrak{b}_{\mathbb{H}})$ is called **square Bochner integrable** if $\|f\|$ is in $\mathcal{L}^2(\mu)$. Note that for $\mathbb{H} = \mathbb{R}$ square Bochner integrable is the same as square integrable. Denote by $\mathcal{L}^2_{\mathcal{C}}(\mu, \mathbb{H})$ the space of square Bochner integrable functions. Sometimes we write $\mathcal{L}^2(\mu, \mathbb{H})$ instead of $\mathcal{L}^2_{\mathcal{C}}(\mu, \mathbb{H})$. If we identify $f, g \in \mathcal{L}^2_{\mathcal{C}}(\mu, \mathbb{H})$ if $f = g$ μ-a.s. then $\mathcal{L}^2_{\mathcal{C}}(\mu, \mathbb{H})$ becomes a Hilbert space with respect to the scalar product

$$< f, g >_{\mathcal{L}^2_{\mathcal{C}}(\mu, \mathbb{H})} \ := \mathbb{E}_\mu < f, g >_{\mathbb{H}} \ .$$

Now suppose that $(\Lambda, \mathcal{C}, \mu) = (\Omega, L_\Gamma(\mathcal{B}), \widehat{\Gamma})$. Fix an internal function $F : (\Omega, \mathcal{B}) \to ({}^*\mathbb{H}, \mathfrak{b}_{{}^*\mathbb{H}})$ and any mapping $f : \Omega \to \mathbb{H}$. Then F is called a **lifting** of f if $f \approx F$ $\widehat{\Gamma}$-a.s. We will make use of the following lifting theorem.

1.2 Proposition

(See [2], Theorem 6 and [26], Theorem 8.9.1.) *A mapping $f : \Omega \to \mathbb{H}$ is measurable (with respect to the σ-algebras $L_\Gamma(\mathcal{B})$ and $\mathfrak{b}_{\mathbb{H}}$) if and only if there exists a lifting $F : (\Omega, \mathcal{B}) \to ({}^*\mathbb{H}, \mathfrak{b}_{{}^*\mathbb{H}})$ of f. Furthermore, f is square Bochner integrable if and only if f has a lifting F such that $\|F\| \in SL^2(\mu)$.*

Set $T := \{1, ..., H\}$ for an unlimited $H \in {}^*\mathbb{N}$. Let $(\mathcal{B}_k)_{k \in T}$ be an internal filtration in $\mathcal{B}$. Using a method of Keisler (see [16]) we construct a filtration in the standard sense. Define for $t \in [0, 1]$

$$\mathfrak{c}_t := \left\{ B \in L_\Gamma(\mathcal{B}) \mid \exists k \in T \, \exists A \in \mathcal{B}_k \text{ with } \tfrac{k}{H} \approx t \text{ and } A \triangle B \in \mathcal{N} \right\}.$$

1.3 Proposition

(A) (See [16]. See also [20], Theorem 5.2.10 for a detailed proof.) *The system* $(\mathfrak{c}_t)$ *is an increasing family of* σ*-algebras such that* $\mathcal{N} \subset \mathfrak{c}_0$*. Furthermore, the filtration* $(\mathfrak{c}_t)$ *is right continuous, i.e. we have*

$$\mathfrak{c}_s = \cap \{\mathfrak{c}_t \mid t \in]s,1]\}$$

for each $s \in [0,1[$.

(B) Fix $k \in T$ *and* $t \in [0,1]$ *with* $\frac{k}{H} \approx t$*. Then for each* $F \in \mathcal{B}_k$ *the standard part* ${}^\circ F$ *is* $\mathfrak{c}_t$*-measurable.*

(C) Fix $t \in [0,1]$ *and suppose that* $f \in \mathcal{L}^2(\Omega, \mathfrak{c}_t, \widehat{\Gamma})$*. Then there is an* $F \in SL^2(\Gamma)$ *and a* $k \in T$ *with* $\frac{k}{H} \approx t$ *such that* $F \in \mathcal{B}_k$ *and* F *is a lifting of* f*.*

Proof. Assertion (B) follows from the fact that $L_\Gamma(\mathcal{B}_k) \subset \mathfrak{c}_t$. It remains to prove (C). We can assume that $t < 1$. Fix any SL^2-lifting G of f. It suffices to show that for each $n \in \mathbb{N}$ the set

$$\mathcal{K}_n := \left\{(F,k) \mid k \in T,\ F \in \mathcal{B}_k,\ \left|t - \tfrac{k}{H}\right| < \tfrac{1}{n} \text{ and } \|F - G\|_2 < \tfrac{1}{n}\right\}$$

is nonempty. Because this implies that the decreasing system $(\mathcal{K}_m)$ has the finite intersection property, and any pair (F,k) in the intersection of all sets $\mathcal{K}_m$ will satisfy the conditions of assertion (C). So let $n \in \mathbb{N}$. Fix a $k \in T$ such that $t < {}^\circ\frac{k}{H} < t + \frac{1}{n}$. Then $\mathfrak{c}_t \subset L_\Gamma(\mathcal{B}_k) \vee \mathcal{N}$. Without loss of generality we may assume that $f \in L_\Gamma(\mathcal{B}_k)$. Thus there is a $\mathcal{B}_k$-measurable SL^2-lifting F of f. We obtain that $(F,k) \in \mathcal{K}_n$. $\square$

2 Abstract Wiener Spaces

The Lebesgue measure on $\mathbb{R}^n$ is uniquely determined (up to some constant) by the following conditions: it is translation invariant, it assigns finite values to bounded Borel sets and it assigns positive numbers to non-empty open sets. It is easy to see that there cannot be a measure with these properties on the Borel σ-algebra $\mathfrak{b}_{\mathbb{H}}$ of an infinite dimensional Hilbert space $\mathbb{H}$, even if we replace translation invariant by rotation invariant. On the other hand we have to face the fact that Gaussian measure on the cylinder sets of $\mathbb{H}$ is not σ-additive (see below). A possible solution to the problem of finding a kind of natural measure on $\mathfrak{b}_{\mathbb{H}}$ lies in the concept of abstract Wiener spaces, introduced by L. Gross in [14]. The crucial point lies in the introduction of a weaker norm on $\mathbb{H}$. The theory of L. Gross, as well as the infinitesimal approach to abstract Wiener spaces by Lindstrøm (see [19]) are sketched in this section. We further recommend the introduction to abstract Wiener spaces presented by Kuo in [17]. We start from an arbitrary infinite dimensional and separable Hilbert space $\mathbb{H}$. Set

$$\mathcal{E} := \{E \subset \mathbb{H} \mid E \text{ is a finite dimensional subspace of } \mathbb{H}\}\,.$$

Fix $E \in \mathcal{E}$ with orthonormal basis $(a_1, ..., a_n)$. The Gaussian probability measure μ_E on $\mathfrak{b}_E$ is given by

$$\mu_E(A) := \sqrt{2\pi}^{-n} \int_{\{(\alpha_1,...,\alpha_n)\in\mathbb{R}^n \mid \sum_{i=1}^n \alpha_i a_i \in A\}} e^{-\frac{1}{2}(x_1^2+...+x_n^2)}\, d\, x_1...x_n\,.$$

2.1 Lemma

(See [26], Lemma 2.2.3, for a detailed proof.) *The probability measure μ_E does not depend on the choice of the orthonormal basis* $(a_1, ..., a_n)$.

The **cylinder sets in** $\mathbb{H}$ are the sets

$$Z := (\varphi_1, ..., \varphi_n)^{-1}(A)$$

with $n \in \mathbb{N}$, $\varphi_i : \mathbb{H} \to \mathbb{R}$ linear and continuous and $A \in \mathfrak{b}_{\mathbb{R}^n}$. Denote by $\mathcal{Z}_{\mathbb{H}}$ the system of the cylinder sets in $\mathbb{H}$. It is easy to verify that $\mathcal{Z}_{\mathbb{H}}$ is an algebra. A different characterization of the cylinder sets allows to define a measure on $\mathcal{Z}_{\mathbb{H}}$.

2.2 Proposition

(See Definition 4.2 and Proposition 4.1 in [17].) *A set* $Z \in \mathfrak{b}_{\mathbb{H}}$ *is a cylinder set in* $\mathbb{H}$ *if and only if there are sets* $E \in \mathcal{E}$ *and* $B \in \mathfrak{b}_E$ *such that* $Z = B + E^{\perp}$*. In this case set* $\mu_{\mathbb{H}}(Z) := \mu_E(B)$*. Then* $\mu_{\mathbb{H}}$ *is well defined and finitely-additive, but not* σ*-additive on* $\mathcal{Z}_{\mathbb{H}}$*.*

A norm $|\cdot|$ on $\mathbb{H}$ is called **measurable** if for each $\varepsilon > 0$ there is an $E_\varepsilon \in \mathcal{E}$ such that for each $E \in \mathcal{E}$

$$E \perp E_\varepsilon \;\Rightarrow\; \mu_E(\{x \in E \mid |x| > \varepsilon\}) < \varepsilon \;.$$

For example, each norm on $\mathbb{H}$ given by a Hilbert-Schmidt operator is measurable.

2.3 Lemma

(See [17], Lemma 4.2.) *If* $|\cdot|$ *is a measurable norm on* $\mathbb{H}$ *then there is a* $c \in \mathbb{R}^+$ *such that* $|h| \leq c \cdot \|h\|$ *for all* $h \in \mathbb{H}$*.*

We mention that $\mathfrak{b}_{\mathbb{H}}$ is always understood with respect to the Hilbert space norm $\|\cdot\|$. Fix a measurable norm $|\cdot|$. Let $\mathbb{B}$ denote the completion of $(\mathbb{H}, |\cdot|)$. Let $\mathbb{B}'$ denote the topological dual of $\mathbb{B}$. The cylinder sets in $\mathbb{B}$ are the sets

$$Z := (\varphi_1, ..., \varphi_n)^{-1}(A)$$

with $n \in \mathbb{N}$, $\varphi_i \in \mathbb{B}'$ and $A \in \mathfrak{b}_{\mathbb{R}^n}$. Denote by $\mathcal{Z}_{\mathbb{B}}$ the system of the **cylinder sets in** $\mathbb{B}$. Note that if B is a cylinder set in $\mathbb{B}$ then $B \cap \mathbb{H}$ is a cylinder set in $\mathbb{H}$. Now we can define a finitely-additive measure P on $\mathcal{Z}_{\mathbb{B}}$ by setting

$$P(B) := \mu_{\mathbb{H}}(B \cap \mathbb{H}) \;.$$

This construction leads to a well behaved probability measure on $\mathfrak{b}_{\mathbb{B}}$.

2.4 Proposition

(See [17], Theorem 4.1 and Theorem 4.2.) *The system* $\mathcal{Z}_{\mathbb{B}}$ *generates the* σ-algebra $\mathfrak{b}_{\mathbb{B}}$*. Furthermore, there is a* σ*-additive extension of* P *to* $\mathfrak{b}_{\mathbb{B}}$*.*

This extension is uniquely determined and should also be denoted by P. The triple $(\mathbb{B}, \mathbb{H}, P)$ is called an **abstract Wiener space**.

Now we want to define a canonical map

$$\delta_{\mathbb{H}} : \mathbb{H} \to \mathcal{L}^2(P).$$

First we need some technical prerequisites.

2.5 Proposition

(A) For $\varphi \in \mathbb{B}'$ *the restriction* $\varphi^{\restriction \mathbb{H}}$ *of* φ *to* $\mathbb{H}$ *is continuous with respect to* $\|\cdot\|$.

Therefore and since $\mathbb{H}$ *is dense in* $\mathbb{B}$ *we can consider* $\mathbb{B}'$ *as a subspace of* $\mathbb{H}$.

(B) The space $\mathbb{B}'$ *is dense in* $(\mathbb{H}, \|\cdot\|)$.

(C) Each $\varphi \in \mathbb{B}'$ *is normal distributed with mean* 0 *and variance* $\|\varphi\|^2$.

Proof. (A) This is a consequence of Lemma 2.3.

(B) Fix $h \in \mathbb{H}$ with $h \perp \mathbb{B}'$. Then $h \in \mathbb{B}$ and $\varphi(h) = 0$ for each $\varphi \in \mathbb{B}'$. This implies $h = 0$.

(C) We can assume that $\|\varphi\| = 1$. Set $h := \varphi^{\restriction \mathbb{H}}$ and $E := \operatorname{span}\{h\}$. For any $c \in \mathbb{R}$ we obtain

$$P(\{\varphi \leq c\}) = \mu_{\mathbb{H}}(\{\varphi \leq c\} \cap \mathbb{H}) =$$
$$\mu_{\mathbb{H}}(\{\alpha \cdot h \mid \alpha \in \mathbb{R} \text{ and } \varphi(\alpha \cdot h) \leq c\} + E^{\perp}) =$$
$$\mu_E(\{\alpha \cdot h \mid \alpha \leq c\}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{1}{2}x^2} dx. \qquad \square$$

By Proposition 2.5, the map

$$(\mathbb{B}', \|\cdot\|) \to \mathcal{L}^2(\mathbb{B}, \mathfrak{b}_{\mathbb{B}}, P)\,,\ \varphi \mapsto \varphi$$

is linear and norm preserving, therefore it possesses a uniquely determined linear and norm preserving extension $\delta_{\mathbb{H}} : (\mathbb{H}, \|\cdot\|) \to \mathcal{L}^2(\mathbb{B}, \mathfrak{b}_{\mathbb{B}}, P)$. The isometry $\delta_{\mathbb{H}}$ is called the **divergence operator**.

By saturation, there is an internal set $\mathbb{F} \in {}^*\mathcal{E}$ such that $\mathbb{H} \subset \mathbb{F}$. For the moment, fix any $\mathbb{F}$ with this property. In the next section we will specify $\mathbb{F}$. Define $\mu := \mu_{\mathbb{F}}$ and

$$Ns(\mathbb{F}) := \{x \in \mathbb{F} \mid \exists y \in \mathbb{B} \text{ with } |x - y| \approx 0\}\,.$$

Note that if for an $x \in \mathbb{F}$ there is a $y \in \mathbb{B}$ such that $|x - y| \approx 0$ then this y is uniquely determined. Therefore we can define the **standard part map** on $\mathbb{F}$ by

$$St : \mathbb{F} \to \mathbb{B}$$
$$x \mapsto \begin{cases} y & \text{if } x \in Ns(\mathbb{F}) \text{ and if y} \in \mathbb{B} \text{ with } |x - y| \approx 0 \\ 0 & \text{if } x \notin Ns(\mathbb{F}) \end{cases} \quad .$$

2.6 Proposition

(See Lemma 9 and Theorem 10 in [19].) *We have* $\widehat{\mu}(Ns(\mathbb{F})) = 1$. *Furthermore, the mapping* $St : (\mathbb{F}, L_\mu(\mathfrak{b}_\mathbb{F})) \to (\mathbb{B}, \mathfrak{b}_\mathbb{B})$ *is measurable and* $\widehat{\mu} \circ St^{-1} = P$.

In fact this construction gives rise to an alternative proof of $P : \mathcal{Z}_\mathbb{B} \to [0, 1]$ having a σ-additive extension.

3 Resolutions of the Identity

In [32], Üstünel and Zakai introduce a stochastic integral for Hilbert space valued random variables. In order to obtain a notion of time, they are working with a resolution of the identity on $\mathbb{H}$. Using saturation, we manage to obtain a hyperfinite dimensional subspace $\mathbb{F}$ of ${}^*\mathbb{H}$ which is related very closely to the resolution of the identity and which contains $\mathbb{H}$. This will pave the way for a canonical infinitesimal approach to this new stochastic integral.

Note that a linear and continuous operator $T : \mathbb{H} \to \mathbb{H}$ is the orthogonal projection on a closed subspace of $\mathbb{H}$ if and only if $T^2 = T$ and $< Tx, y > = < x, Ty >$ for all $x, y \in \mathbb{H}$. (See [9], Theorem 4.7.1.) A sequence $(\pi_t)_{t\in[0,1]}$ of mappings $\pi_t : \mathbb{H} \to \mathbb{H}$ is called a **resolution of the identity** if the following properties are fulfilled.

- Each π_t is an orthogonal projection.
- For $s < t$ we have $\pi_s\mathbb{H} \subsetneqq \pi_t\mathbb{H}$.
- For each $h \in \mathbb{H}$ the map $[0, 1] \ni t \mapsto \pi_t h \in \mathbb{H}$ is continuous.
- We have $\pi_0 = 0$ and π_1 is the identity on $\mathbb{H}$.

We fix a resolution of the identity (π_t). Set

$$\widehat{\prod} : {}^*[0,1] \times {}^*\mathbb{H} \to {}^*\mathbb{H}$$
$$(s, x) \mapsto ({}^*\pi)(s, x) \;.$$

If $s \in {}^*[0, 1]$ we often write $\widehat{\prod}_s$ for $\widehat{\prod}(s, \cdot)$. Note that for each $t \in [0, 1]$ we have

$${}^*(\pi_t\mathbb{H}) = \widehat{\prod}_t {}^*\mathbb{H} \;.$$

Our purpose is to prove the existence of a hyperfinite dimensional subspace of ${}^*\mathbb{H}$ which fits together very well with (π_t). Set for $h \in \mathbb{H}$ and $m \in \mathbb{N}$

$$\mathcal{U}_{h,m} := \{\mathbb{F} \in {}^*\mathcal{E} \mid h \in \mathbb{F},\ m \text{ divides } \dim(\mathbb{F}) \text{ and}$$
$$\forall 1 \le k \le m \ \dim(\widehat{\prod}_{\frac{k}{m}} {}^*\mathbb{H} \cap \mathbb{F}) = \tfrac{k}{m} \dim(\mathbb{F})\,\}.$$

3.1 Proposition

The sequence $(\mathcal{U}_{h,m})_{h\in\mathbb{H},m\in\mathbb{N}}$ has the finite intersection property.

Proof. Fix $l \in \mathbb{N} \setminus \{1\}$, $h_1, ..., h_l \in \mathbb{H}$ and $m_1, ..., m_l \in \mathbb{N}$. Set $n := m_1 \cdot ... \cdot m_l \cdot l$. For $K \in \{1, ..., n\}$ set

$$A_K := \widehat{\prod}_{\frac{K}{n}} {}^*\mathbb{H} \cap \left(\widehat{\prod}_{\frac{K-1}{n}} {}^*\mathbb{H}\right)^{\perp} .$$

Then each A_K is an internal closed subspace of ${}^*\mathbb{H}$. By the properties of the resolution of the identity, each A_K is infinite dimensional. Note further that

$$\widehat{\prod}_{\frac{\mu}{n}} {}^*\mathbb{H} = \oplus_{K=1}^{\mu} A_K \text{ for } 1 \leq \mu \leq n .$$

For $1 \leq K \leq n$ chose an internal orthonormal system $c_K^1, ..., c_K^n$ in A_K such that

$$pr_{A_K}^{{}^*\mathbb{H}} h_1, ..., pr_{A_K}^{{}^*\mathbb{H}} h_l \in \operatorname{span}(\{c_K^1, ..., c_K^n\}). \tag{1}$$

We show that

$$\mathbb{F} := \operatorname{span}\left(\{c_K^i \mid i, K \in \{1, ..., n\}\}\right) \in \bigcap_{\eta=1}^{l} \mathcal{U}_{h_\eta, m_\eta} ,$$

i.e. we show that the following holds,

- (A) $h_1, ..., h_l \in \mathbb{F}$,
- (B) n divides $\dim(\mathbb{F})$ and
- (C) $\forall 1 \leq \eta \leq l \, \forall 1 \leq k \leq m_\eta \; \dim(\widehat{\prod}_{\frac{k}{m_\eta}} {}^*\mathbb{H} \cap \mathbb{F}) = \frac{k}{m_\eta} \dim(\mathbb{F})$.

Property (A) follows from (1) and (B) holds since $\dim(\mathbb{F}) = n^2$. To prove (C), fix $1 \leq \eta \leq l$ and $1 \leq k \leq m_\eta$. Set $w := \frac{n}{m_\eta}$. Since

$$\widehat{\prod}_{\frac{k \cdot w}{n}} {}^*\mathbb{H} \cap \mathbb{F} = \left(\oplus_{K=1}^{k \cdot w} A_K\right) \cap \mathbb{F} = \operatorname{span}\left(\{c_K^i \mid 1 \leq i \leq n, 1 \leq K \leq k \cdot w\}\right) ,$$

we obtain

$$\dim(\widehat{\prod}_{\frac{k}{m_\eta}} {}^*\mathbb{H} \cap \mathbb{F}) = \dim(\widehat{\prod}_{\frac{k \cdot w}{n}} {}^*\mathbb{H} \cap \mathbb{F}) = k \cdot w \cdot n = \frac{k}{m_\eta} \cdot n^2 = \frac{k}{m_\eta} \cdot \dim(\mathbb{F}). \qquad \square$$

By saturation, there is an $\mathbb{F} \in \bigcap_{m \in \mathbb{N}, h \in \mathbb{H}} \mathcal{U}_{h,m}$. This implies that $\mathbb{H} \subset \mathbb{F}$. Set $\omega := \dim(\mathbb{F})$ and $I := \{1, ..., \omega\}$. Again by saturation, there is an unlimited $H \in {}^*\mathbb{N}$ such that $\frac{\omega}{H} \in {}^*\mathbb{N} \setminus \mathbb{N}$ and such that for $1 \leq k \leq H$ we have

$$\dim(\widehat{\prod}_{\frac{k}{H}} {}^*\mathbb{H} \cap \mathbb{F}) = \frac{k}{H} \cdot \omega . \tag{2}$$

Set $T := \{1, ..., H\}$ and define

$$\sigma : T \cup \{0\} \ni k \mapsto \tfrac{k}{H} \cdot \omega \in I \cup \{0\}.$$

Now we set $\prod_k := \widehat{\prod}_{\frac{k}{H}}$ for $k \in T \cup \{0\}$. We will often write $\prod_k x$ instead of $\prod_k(x)$. Note that (2) shows the close relationship between $\mathbb{F}$ and π: the internal resolution $(\prod_k)_{1\leq k\leq H}$ cuts $\mathbb{F}$ into slices of dimension $\frac{\omega}{H}$. We fix an internal orthonormal basis $(b_i)_{i\in I}$ of $\mathbb{F}$ such that for each $k \in T$

$$\prod_k {}^*\mathbb{H} \cap \mathbb{F} = \operatorname{span}\left(\{b_1, ..., b_{\sigma(k)}\}\right).$$

For any $x \in \mathbb{F}$ set $pr_x : \mathbb{F} \ni y \mapsto < x, y >$. Note the difference between pr_x and $pr^{\mathbb{F}}_{\{x\}}$. For $i \in I$ set $pr_i := pr_{b_i}$. Finally, for $J \subset I$ and $x \in \mathbb{F}$ set $x_J := pr^{\mathbb{F}}_{\{b_i|i\in J\}} x$. By Lemma 2.1, $\mu := \mu_{\mathbb{F}}$ is a well defined internal probability measure on $\mathfrak{b}_{\mathbb{F}}$. We state some basic properties of the mappings pr_i.

3.2 Proposition

For any $x \in \mathbb{F}$ the map pr_x is normal distributed with mean 0 and variance $\|x\|^2$. Therefore if $\|x\|$ is limited, then $pr_x \in SL^p(\mu)$ for every $p \in [1, \infty[$. Furthermore, $(pr_i)_{i\in I}$ is a sequence of independent random variables. Finally, for $k \in T$ and $1 \leq i \leq \sigma(k)$ we have $b_i = \prod_k b_i$.

Proof. We can assume that $\|x\| = 1$. Chose $a_2, ..., a_\omega \in \mathbb{F}$ such that $(x, a_2, ..., a_\omega)$ is an orthonormal basis of $\mathbb{F}$. Then we obtain for $c \in {}^*\mathbb{R}$

$$\mu(\{pr_x \leq c\}) =$$

$$\frac{1}{\sqrt{2\pi}^{\omega}} \int_{\{(\alpha_1,...,\alpha_\omega)\in{}^*\mathbb{R}^\omega|\alpha_1 x+\sum_{i=2}^{\omega}\alpha_i a_i\in\{pr_x\leq c\}\}} e^{-\frac{1}{2}(y_1^2+...+y_\omega^2)} dy_1...y_\omega =$$

$$\frac{1}{\sqrt{2\pi}^{\omega}} \int_{\{(\alpha_1,...,\alpha_\omega)\in{}^*\mathbb{R}^\omega|\alpha_1\leq c\}\}} e^{-\frac{1}{2}(y_1^2+...+y_\omega^2)} dy_1...y_\omega =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{1}{2}y^2} dy\,.$$

Hence pr_x is normal distributed with mean 0 and variance 1. If $\|x\|$ is limited, then for each $p \in [1, \infty[$ the expectation of $|pr_x|^{2p}$ is limited, which implies that $pr_x \in SL^p(\mu)$.

To show that (pr_i) is a sequence of independent variables fix $c_1, ..., c_\omega \in {}^*\mathbb{R}$ and note that

$$\mu(\{x \in \mathbb{F} \mid \forall 1 \leq i \leq \omega : pr_i(x) \leq c_i\}) =$$

$$\frac{1}{\sqrt{2\pi}^{\omega}} \int_{\{(\alpha_1,...,\alpha_\omega)\in {}^*\mathbb{R}^\omega \mid \forall i\in I:\, \alpha_i\le c_i\}} e^{-\frac{1}{2}(y_1^2+...+y_\omega^2)} dy_1...y_\omega =$$

$$\prod_{i=1}^{\omega} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c_i} e^{-\frac{1}{2}y^2} dy = \prod_{i=1}^{\omega} \mu(\{x \in \mathbb{F} \mid pr_i(x) \le c_i\}).$$

Now fix $k \in T$ and $i \in I$ with $i \le \sigma(k)$. Note that $\prod_k {}^*\mathbb{H} \cap \mathbb{F} = \operatorname{span}(\{b_1, ..., b_{\sigma(k)}\})$, thus there is an $h \in {}^*\mathbb{H}$ with $b_i = \prod_k h$. Therefore $\prod_k b_i = \prod_k \prod_k h = \prod_k h = b_i$. $\square$

An important fact, namely that if $h \in \mathbb{H}$ and $k \in T$, then $\prod_k h$ is almost the same as the projection of h onto $\operatorname{span}(\{b_i \mid 1 \le i \le \sigma(k)\})$, is a consequence of the following proposition. Recall that for $x, y \in {}^*\mathbb{H}$ the formula $x \approx y$ means $\|x - y\| \approx 0$.

3.3 Proposition

Fix $t \in [0,1]$, $k \in T$ and $i \in I$ such that $t \approx \frac{k}{H} \approx \frac{i}{\omega}$. Then, for any $h \in \mathbb{H}$, we obtain

$$\pi_t h \approx \prod_k h \approx h_{\{1,...,i\}}. \tag{3}$$

Therefore, $< h, b_i >^2 \approx 0$ and $\sum_{j\in I} < h, b_j >^4 \approx 0$.

Proof. The first assertion, $\pi_t h \approx \prod_k h$, follows from the continuity of the resolution of the identity. If an $l \in T$ has the property that $\frac{l}{H} \in [0,1]$, then $\prod_l h \in \mathbb{F}$, in this case we obtain

$$\prod_l h = \sum_{i=1}^{\omega} < \prod_l h, b_i > \cdot\, b_i = \sum_{i=1}^{\sigma(l)} < \prod_l h, b_i > \cdot\, b_i = h_{\{1,...,\sigma(l)\}}.$$

Now fix an $\varepsilon \in \mathbb{R}_0^+$ and a $\delta \in \mathbb{R}_0^+$ such that

$$\forall l, m \in T \left(\frac{|m-l|}{H} < \delta \Rightarrow \|\prod_m h - \prod_l h\| < \frac{\varepsilon}{2} \right).$$

We show that $\left\|\prod_k h - h_{\{1,...,i\}}\right\| < \varepsilon$. Therefore we chose l_1 and l_2 in $T \cup \{0\}$ such that

- $\sigma(l_1) \le i \le \sigma(l_2)$,
- $\frac{l_1}{H} \in \mathbb{R}$, $\frac{l_2}{H} \in \mathbb{R}$,
- $l_1 \le k \le l_2$ and

- $\frac{l_2-l_1}{H} < \delta$.

We obtain

$$\left\|\prod_k h - h_{\{1,\dots,i\}}\right\| \leq \left\|\prod_k h - \prod_{l_2} h\right\| + \left\|\prod_{l_2} h - h_{\{1,\dots,i\}}\right\| <$$

$$\frac{\varepsilon}{2} + \sqrt{\sum_{j=i+1}^{\sigma(l_2)} < h, b_j >^2} \leq \frac{\varepsilon}{2} + \sqrt{\sum_{j=\sigma(l_1)+1}^{\sigma(l_2)} < h, b_j >^2} =$$

$$\frac{\varepsilon}{2} + \left\|\prod_{l_2} h - \prod_{l_1} h\right\| < \varepsilon.$$

This proves (3). Now (3) implies that for each $j \in I$ we have $< h, b_j >^2 \approx 0$, therefore $\max_{j\in I} < h, b_j >^2 \approx 0$. Finally we obtain

$$\sum_{i\in I} < h, b_i >^4 \leq \sum_{i\in I} < h, b_i >^2 \cdot \max_{j\in I} < h, b_j >^2 =$$

$$\|h\|^2 \cdot \max_{j\in I} < h, b_j >^2 \approx 0. \qquad \square$$

Define

$$\mathcal{H}_1 := \{x \in \mathbb{F} \mid \exists h \in \mathbb{H}\,(x \approx h)\}.$$

One of the most important tools in the theory of saturated models is the concept of S-continuity. An internal function $F : I \to {}^*\mathbb{R}$ is called **S-continuous** if $F(i) \in \mathrm{Lim}$ for all $i \in I$ and if the implication

$$\forall i, j \in I \left(\tfrac{j-i}{\omega} \approx 0 \Rightarrow F(j) - F(i) \approx 0\right)$$

is valid. The S-continuity of an internal function $G : T \to {}^*\mathbb{R}$ is defined analogously. The next proposition is an immediate consequence of Proposition 3.3.

3.4 Proposition

For each $F \in \mathcal{H}_1$ the function

$$I \to {}^*\mathbb{R},\ i \mapsto \sum_{j=1}^{i} < F, b_j >^2$$

is S-continuous.

Set

$$\mathcal{W} := {}^{\sigma}\{ {}^{\circ}pr_x \mid x \in \mathcal{H}_1\} \vee \mathcal{N}_{\widehat{\mu}} \ .$$

Now we define two internal filtrations in $\mathfrak{b}_{\mathbb{F}}$. For $i \in I$ set

$$\mathcal{B}_i := {}^{\sigma}\{pr_j \mid 1 \leq j \leq i\}.$$

Define $\mathcal{B}_0 := \{\emptyset, \mathbb{F}\}$. For $k \in T \cup \{0\}$ set

$$\mathcal{C}_k := \mathcal{B}_{\sigma(k)} \ .$$

Let $(\mathfrak{c}_t)_{t\in[0,1]}$ be the standard part of $(\mathcal{C}_k)_{k\in T}$, see Section 1. For $t \in [0,1]$ set

$$\mathcal{W}_t := \mathfrak{c}_t \cap \mathcal{W} \ .$$

Now we have two internal filtered probability spaces and a filtered probability space in the standard sense, namely $(\mathbb{F}, \mathfrak{b}_{\mathbb{F}}, (\mathcal{B}_i)_{i\in I}, \mu)$, $(\mathbb{F}, \mathfrak{b}_{\mathbb{F}}, (\mathcal{C}_k)_{k\in T}, \mu)$ and $(\mathbb{F}, \mathcal{W}, (\mathcal{W}_t)_{t\in[0,1]}, \widehat{\mu})$.

4 The Chaos Decomposition Theorem

For $n \in \mathbb{N}_0$ the n^{th} **Hermite polynomial** is given by

$$H_n(x) := \frac{(-1)^n}{n!} e^{\frac{1}{2}x^2} \cdot \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2} .$$

If f, g are random variables which are normal distributed with mean 0 and variance 1 then the variables $I_n(f)$ and $I_m(g)$ are orthogonal in the respective $\mathcal{L}^2$-space, if $n \neq m$ and if any linear combination of f and g is also normal distributed. (See Lemma 1.1.1 in [23].) This property gives rise to an orthogonal decomposition of $\mathcal{L}^2(P)$. Set $\mathcal{K}_0 := \mathbb{R}$ and

$$\mathcal{K}_n := \overline{\operatorname{span}\{H_n(\delta_{\mathbb{H}} h) \mid h \in \mathbb{H} \text{ with } \|h\| = 1\}},\ n \in \mathbb{N}.$$

Then we have, see Theorem 1.1.1 in [23],

$$\mathcal{L}^2(\mathbb{B}, \mathfrak{b}_{\mathbb{B}}, P) = \oplus_{n=0}^{\infty} \mathcal{K}_n \ .$$

Thus for each $\varphi \in \mathcal{L}^2(P)$ there is a sequence $(\varphi_n)_{n \in \mathbb{N}_0}$ such that $\varphi_k \in \mathcal{K}_k$ and

$$\varphi = \sum_{n=0}^{\infty} \varphi_n \text{ in } \mathcal{L}^2(\mathbb{B}, \mathfrak{b}_{\mathbb{B}}, P).$$

Fix $m \in \mathbb{N}$. In the case of the classical Wiener space $C[0,1]$ each φ_m can be written as a multiple stochastic integral. But if we consider abstract Wiener spaces, we cannot regard φ_m as a multiple stochastic integral, since in this situation there is no natural notion of a stochastic integral. Using saturated models we obtain a new approach to the chaos decomposition of $\mathcal{L}^2(P)$. Instead of $\mathcal{L}^2(P)$ we will work with the space $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. (In Section 13 we will prove that the two spaces can be identified.) Generalizing the methods in [8] and [20] to abstract Wiener spaces, we obtain a chaos decomposition of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ where the components are standard parts of well behaved internal linear combinations of products $pr_{i_1} \cdot ... \cdot pr_{i_m}$. Thus the components of the decomposition are explicitely given.

A proof of the following well known facts about the Hermite polynomials can be found in [20] (Lemma 6.4.8).

- We have $(n+1)H_{n+1} + H_{n-1} = H_n \cdot id$ for all $n \in \mathbb{N}$.
- For each $n \in \mathbb{N}_0$, H_n is a polynomial of degree n and each polynomial of degree n is a linear combination of $H_0, ..., H_n$.

We need the following proposition about dense subspaces of $\mathcal{L}^2$-spaces.

4.1 Proposition

Fix any probability space $(\Omega, \mathcal{B}, \Gamma)$. *Let* $\mathcal{V}$ *be a vector space of functions* $f \in \mathcal{B}$ *such that* $e^f \in \mathcal{L}^2(\Omega, \mathcal{B}, \Gamma)$. *Set* $\mathcal{A} := {}^\sigma\mathcal{V} \vee \mathcal{N}_\Gamma$. *Then*

$$M := \operatorname{span}\left(\{f^n \mid f \in \mathcal{V},\, n \in \mathbb{N}_0\}\right)$$

is dense in $\mathcal{L}^2(\Omega, \mathcal{A}, \Gamma)$.

Proof. Fix any $\varphi \in \mathcal{L}^2(\Omega, \mathcal{A}, \Gamma)$ with $\varphi \perp M$. We have to show that $\varphi = 0$. Without loss of generality we can assume that $\varphi \in {}^\sigma\mathcal{V}$. Define two measures P^+ and P^- by setting

$$P^\pm(A) = \int_A \varphi^\pm d\Gamma$$

for $A \in {}^\sigma\mathcal{V}$. We show that for every $f \in \mathcal{V}$

$$\int_\Omega e^f dP^+ = \int_\Omega e^f dP^- < \infty\ .$$

Then, see [20], Lemma 6.4.12, the measures P^+ and P^- coincide and therefore $\varphi = 0$ Γ-a.s. Set

$$f_m := \sum\nolimits_{n=0}^m \frac{f^n}{n!}$$

and note that $f_m \to e^f$ Γ-a.s. Since $|f_m| \leq e^{|f|}$, the Lebesgue theorem implies that $f_m \to e^f$ in $\mathcal{L}^2(\Gamma)$. But then $\varphi \perp M$ and the continuity of the scalar product yields

$$\int_\Omega e^f dP^+ - \int_\Omega e^f dP^- = < e^f, \varphi >_2 = \lim_{m\to\infty} < f_m, \varphi >_2 = 0\ . \qquad \square$$

Recall the definition of $\mathcal{H}_1$ in Section 3. We identify $x \in \mathcal{H}_1$ with the map pr_x. Now let $n \geq 2$ and set

$$\widetilde{\mathcal{H}}_n := \{F : \mathbb{F}^n \to {}^*\mathbb{R}\ | F \text{ is an internal symmetric multilinear form}\}\,.$$

Fix a function G and a sequence of functions $(G_m)_{m\in\mathbb{N}}$ with $G, G_m \in \widetilde{\mathcal{H}}_n$. We call G an S^n**-limit** of (G_m) if for each $k \in \mathbb{N}$ there is an $m_0 \in \mathbb{N}$ such that

$$\sum_{i_1<\ldots<i_n} (G_m - G(b_{i_1}, .., b_{i_n}))^2 < \frac{1}{k}$$

for each $m \geq m_0$. Let $\mathcal{H}_n$ be the smallest $\mathbb{R}$-linear subspace of $\widetilde{\mathcal{H}}_n$ which is closed under S^n-limits and which contains the functions $G^{\odot n}$ for $G \in \mathcal{H}_1$. For $n \in \mathbb{N}$ and $G \in \mathcal{H}_n$ define

$$I_n(G) := \sum_{i_1 < \ldots < i_n} G(b_{i_1}, ..., b_{i_n}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_n} \ .$$

If $n = 1$, we write $I(G) := I_1(G)$. Note that $I(G) = pr_G$. Define further $\mathcal{H}_0 := \mathrm{Lim}$ and

$$\begin{aligned} I_0(F) : \mathbb{F} &\to {}^*\mathbb{R}\,, \\ x &\mapsto F \end{aligned}$$

for $F \in \mathcal{H}_0$. We state some basic properties of the functions $I_n(G)$.

4.2 Proposition

(A) For $n \in \mathbb{N}$ and $G \in \mathcal{H}_n$ we have

$$\sum_{i_1 < \ldots < i_n} G(b_{i_1}, ..., b_{i_n})^2 \in \mathrm{Lim} \quad \textit{and} \quad \sum_{i_1 < \ldots < i_n} G(b_{i_1}, ..., b_{i_n})^4 \approx 0\,.$$

(B) Fix $n, m \in \mathbb{N}$, $F \in \mathcal{H}_n$ and $G \in \mathcal{H}_m$. Then

$$< I_n(F), I_m(G) >_2 = \begin{cases} \sum_{i_1 < \ldots < i_n} F(b_{i_1}, ..., b_{i_n}) \cdot G(b_{i_1}, ..., b_{i_n}) & \textit{if } n = m \\ 0 & \textit{otherwise} \end{cases} .$$

(C) For $n \in \mathbb{N}$ and $F \in \mathcal{H}_n$ we have $I_n(F_n) \in SL^2(\mu)$.

Proof. (A) For $n = 1$, the assertion follows from Proposition 3.3. Now let $n > 1$ and set

$$\begin{aligned} \mathcal{G}_n^1 &:= \{G \in \mathcal{H}_n \mid \sum_{i_1 < \ldots < i_n} G(b_{i_1}, ..., b_{i_n})^2 \in \mathrm{Lim}\}, \\ \mathcal{G}_n^2 &:= \{G \in \mathcal{H}_n \mid \sum_{i_1 < \ldots < i_n} G(b_{i_1}, ..., b_{i_n})^4 \approx 0\}\,. \end{aligned}$$

Then the spaces $\mathcal{G}_n^i$ are closed under S^n-limits and contain the functions $G^{\odot n}$ for $G \in \mathcal{H}_1$. Thus $\mathcal{H}_n = \mathcal{G}_n^i$ for $i = 1, 2$.

(B) The assertion follows from the fact that (pr_i) is a sequence of independent random variables such that $\mathbb{E}pr_i = 0$.

(C) We show that $\mathbb{E}\left(I_n(F)\right)^4 \in \mathrm{Lim}$.

$$\mathbb{E}\left(I_n(F)\right)^4 \;=\; \mathbb{E}\sum_{i_1<\ldots<i_n} F(b_{i_1},...,b_{i_n})^4\cdot pr_{i_1}^4\cdot\ldots\cdot pr_{i_n}^4 +$$

$$\mathbb{E}\sum_{\substack{i_1<\ldots<i_n\\ j_1<\ldots<j_n\\ \{i_1,\ldots,i_n\}\neq\{j_1,\ldots,j_n\}}} F(b_{i_1},...,b_{i_n})^2\cdot F(b_{j_1},...,b_{j_n})^2\cdot pr_{i_1}^2\cdot\ldots\cdot pr_{i_n}^2\cdot pr_{j_1}^2\cdot\ldots\cdot pr_{j_n}^2 \;\leq$$

$$\sum_{i_1<\ldots<i_n} F(b_{i_1},...,b_{i_n})^4\cdot 3^n + \sum_{\substack{i_1<\ldots<i_n\\ j_1<\ldots<j_n\\ \{i_1,\ldots,i_n\}\neq\{j_1,\ldots,j_n\}}} F(b_{i_1},...,b_{i_n})^2\cdot F(b_{j_1},...,b_{j_n})^2\cdot 3^n \;\leq$$

$$3^n \;+\; 3^n\left(\sum_{i_1<\ldots<i_n} F(b_{i_1},...,b_{i_n})^2\right)^2 \;\in\; \mathrm{Lim} \qquad \square$$

The following proposition is of great importance, since it establishes the connection between the Hermite polynomials and the iterated Itô integrals. On the other hand, the proof is long and it consists of uninteresting technical details, so the reader can omit it in the first reading.

4.3 Proposition

(A) For $G\in\mathcal{H}_1$ and $n\geq 2$ we have $\widehat{\mu}$-a.s.

$$\sum_{i_1,\ldots,i_{n-1}\in I_{\neq}^{n-1}}\;\sum_{s\in\{i_1,\ldots,i_{n-1}\}} G(b_{i_1})\cdot\ldots\cdot G(b_{i_{n-1}})\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_{n-1}}\cdot G(b_s)^2\cdot pr_s^2 \;\approx\; 0\ .$$

(B) For $G\in\mathcal{H}_1$, $n\geq 2$ and $1\leq k\leq n$ we have $\widehat{\mu}$-a.s.

$$\sum_{i_1,\ldots,i_n\in I_{\neq}^n} G(b_{i_1})\cdot\ldots\cdot G(b_{i_k})^2\cdot\ldots\cdot G(b_{i_n})\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_k}^2\cdot\ldots\cdot pr_{i_n} \approx$$

$$\sum_{i_1,\ldots,i_{n-1}\in I_{\neq}^{n-1}}\;\sum_{s\in I} G(b_{i_1})\cdot\ldots\cdot G(b_{i_{n-1}})\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_{n-1}}\cdot G(b_s)^2\cdot pr_s^2\ .$$

(C) For $n\in\mathbb{N}$ and $G\in\mathcal{H}_1$ with $\sum_{i\in I} G(b_i)^2\approx 1$ we have $\widehat{\mu}$-a.s.

$$(n+1)I_{n+1}(G^{\odot n+1}) \;\approx\; I_n(G^{\odot n})I\left(G\right) - I_{n-1}(G^{\odot n-1})\ ,$$

where $I_0(G^{\odot 0}) := 1$.

(D) *Fix $n \in \mathbb{N}$ and $G \in \mathcal{H}_1$ with $\sum_{i\in I} G(i)^2 \approx 1$. Then*

$$H_n(I(G)) \approx I_n(G^{\odot n}) \ \ \widehat{\mu}\text{-a.s.}$$

Proof. (A) Assume that $n = 2$. Then

$$\mathbb{E}\left(\sum_{i\in I} G(b_i)^3 \cdot pr_i^3\right)^2 = \mathbb{E}\sum_{i\in I} G(b_i)^6 \cdot pr_i^6 = 15 \cdot \sum_{i\in I} G(b_i)^6 \approx 0.$$

Now assume that $n > 2$. Fix $k \in \{1, ..., n-1\}$. We obtain

$$\mathbb{E}\left(\sum_{i_1,...,i_{n-1}\in I_{\neq}^{n-1}} G(b_{i_1}) \cdot ... \cdot G(b_{i_k})^3 \cdot ... \cdot G(b_{i_{n-1}}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_k}^3 \cdot ... \cdot pr_{i_{n-1}}\right)^2 =$$

$$\mathbb{E} \sum_{i_1,...,i_{n-1}\in I_{\neq}^{n-1}} G(b_{i_1})^2 \cdot ... \cdot G(b_{i_k})^6 \cdot ... \cdot G(b_{i_{n-1}})^2 \cdot pr_{i_1}^2 \cdot ... \cdot pr_{i_k}^6 \cdot ... \cdot pr_{i_{n-1}}^2 =$$

$$15 \cdot \sum_{i_1,...,i_{n-1}\in I_{\neq}^{n-1}} G(b_{i_1})^2 \cdot ... \cdot G(b_{i_k})^6 \cdot ... \cdot G(b_{i_{n-1}})^2 \leq$$

$$15 \cdot \sum_{i\in I} G(b_i)^6 \cdot \left(\sum_{i\in I} G(b_i)^2\right)^{n-2} \approx 0\ .$$

(B) We have $\widehat{\mu}$-a.s.

$$\sum_{i_1,...,i_{n-1}\in I_{\neq}^{n-1}} \sum_{s\in I} G(b_{i_1}) \cdot ... \cdot G(b_{i_{n-1}}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} \cdot G(b_s)^2 \cdot pr_s^2 -$$

$$\sum_{i_1,...,i_n\in I_{\neq}^{n}} G(b_{i_1}) \cdot ... \cdot G(b_{i_k})^2 \cdot ... \cdot G(b_{i_n}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_k}^2 \cdot ... \cdot pr_{i_n} \ =$$

$$\sum_{i_1,...,i_{n-1}\in I_{\neq}^{n-1}} \sum_{s\in\{i_1,...,i_{n-1}\}} G(b_{i_1}) \cdot ... \cdot G(b_{i_{n-1}}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} \cdot G(b_s)^2 \cdot pr_s^2 \ \approx \ 0\,,$$

because of (A).

(C) First suppose that $n = 1$. We first show that

$$\sum_{i\in I} G\left(b_i\right)^2 pr_i^2 \ \approx \ 1 \ \widehat{\mu}\text{-a.s.} \tag{4}$$

Note that

$$\mathbb{E}\left(\sum_{i\in I} G\left(b_i\right)^2 pr_i^2 - \sum_{i\in I} G(b_i)^2\right)^2 = \mathbb{E}\left(\sum_{i\in I} G\left(b_i\right)^2\left(pr_i^2-1\right)\right)^2 =$$

$$\mathbb{E}\sum_{i\in I} G\left(b_i\right)^4\left(pr_i^4+1-2pr_i^2\right) = \sum_{i\in I} G\left(b_i\right)^4 \cdot 2 \approx 0 \ .$$

Thus we get $\widehat{\mu}$-a.s.

$$I\left(G\right)^2 = \left(\sum_{i\in I} G\left(b_i\right)\cdot pr_i\right)^2 =$$

$$2\sum_{i<j} G\left(b_i\right)\cdot G(b_j)\cdot pr_i\cdot pr_j + \sum_{i\in I} G\left(b_i\right)^2\cdot pr_i^2 \approx 2\, I_2(G^{\odot 2}) + 1 \ .$$

Now assume that $n>1$. We get $\widehat{\mu}$-a.s.

$$I_n(G^{\odot n})I\left(G\right) =$$

$$\frac{1}{n!}\sum_{i_1,\dots,i_n\in I^n_{\neq}}\sum_{s\in I} G(b_{i_1})\cdot \dots \cdot G(b_{i_n})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_n}\cdot G(b_s)\cdot pr_s =$$

$$\frac{1}{n!}\sum_{i_1,\dots,i_n\in I^n_{\neq}}\sum_{s\in I\setminus\{i_1,\dots,i_n\}} G(b_{i_1})\cdot \dots \cdot G(b_{i_n})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_n}\cdot G(b_s)\cdot pr_s +$$

$$\frac{1}{n!}\sum_{i_1,\dots,i_n\in I^n_{\neq}}\sum_{s\in \{i_1,\dots,i_n\}} G(b_{i_1})\cdot \dots \cdot G(b_{i_n})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_n}\cdot G(b_s)\cdot pr_s =$$

$$\frac{1}{n!}\sum_{i_1,\dots,i_{n+1}\in I^{n+1}_{\neq}} G(b_{i_1})\cdot \dots \cdot G(b_{i_{n+1}})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_{n+1}} +$$

$$\frac{1}{n!}\sum_{k=1}^{n}\sum_{i_1,\dots,i_n\in I^n_{\neq}} G(b_{i_1})\cdot \dots \cdot G(b_{i_k})^2\cdot\dots\cdot G(b_{i_n})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_k}^2\cdot\dots\cdot pr_{i_n} \overset{(i)}{=}$$

$$\frac{n+1!}{n!}\sum_{i_1<\dots<i_{n+1}} G(b_{i_1})\cdot \dots \cdot G(b_{i_{n+1}})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_{n+1}} +$$

$$\frac{n}{n!}\sum_{i_1,\dots,i_{n-1}\in I^{n-1}_{\neq}}\sum_{s\in I} G(b_{i_1})\cdot \dots \cdot G(b_{i_{n-1}})\cdot pr_{i_1}\cdot\dots\cdot pr_{i_{n-1}}\cdot G(b_s)^2\cdot pr_s^2 \overset{(ii)}{\approx}$$

$$(n+1)I_{n+1}(G^{\odot n+1}) +$$

$$\frac{n}{n!} \sum_{i_1,...,i_{n-1} \in I_{\neq}^{n-1}} G(b_{i_1}) \cdot ... \cdot G(b_{i_{n-1}}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} =$$

$$(n+1) I_{n+1}(G^{\odot n+1}) \; + \; I_{n-1}(G^{\odot n-1}) \; ,$$

where (i) follows from (B) and (ii) follows from (4).

(D) For $n = 1$ there is nothing to prove. Now fix $n \in \mathbb{N}$ and suppose that the assertion is already proved for $1 \leq k \leq n$. From assertion (C) and from the induction hypothesis it follows that $\widehat{\mu}$-a.s.

$$H_{n+1}(I(G)) \; = \; \frac{1}{n+1} \left(H_n(I\,(G)) I\,(G) - H_{n-1}\,(I\,(G)) \right) \; \approx$$

$$\frac{1}{n+1} \left(I_n(G^{\odot n}) I\,(G) - I_{n-1}(G^{\odot n-1}) \right) \; \approx \; I_{n+1}(G^{\odot n+1}) \, . \qquad \square$$

Now we can prove the chaos decomposition theorem for abstract Wiener spaces.

4.4 Proposition

If $n \in \mathbb{N}_0$ *and* $G \in \mathcal{H}_n$ *then* ${}^\circ I_n(G) \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$*. Furthermore, for* $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ *there is a sequence* $(F_n)_{n \in \mathbb{N}_0}$ *of functions* $F_n \in \mathcal{H}_n$ *such that*

$$\varphi \; = \; \sum_{n=0}^{\infty} {}^\circ I_n(F_n) \quad \text{in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}) \, .$$

If $(G_n)_{n \in \mathbb{N}}$ *is another sequence with this property, then* $F_0 \approx G_0$ *and*

$$\sum_{i_1 < ... < i_n} \left(F_n - G_n(b_{i_1}, ..., b_{i_n}) \right)^2 \approx 0$$

for each $n \in \mathbb{N}$*.*

Proof. For $n \geq 2$ the set $\mathcal{G}_n := \{F \in \mathcal{H}_n \mid {}^\circ I_n(F) \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})\}$ is a linear space over $\mathbb{R}$ containing functions of the type $G^{\odot n}$ for $G \in \mathcal{H}_1$. The space $\mathcal{G}_n$ is also closed under S^n-limits. Therefore $\mathcal{G}_n = \mathcal{H}_n$. This proves the first assertion of the proposition. We now show, using similar arguments as in the proof of Lemma 1.1, that for $n \in \mathbb{N}$ the space

$$\{ {}^\circ I_n(G_n) \mid G_n \in \mathcal{H}_n \}$$

is closed in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Let (G^k) be a sequence in $\mathcal{H}_n$ such that $({}^\circ I_n(G^k))$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Then there exists a strictly monotone increasing function $g : \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$ and for $k_1, k_2 \geq g(k)$ we have

$$\left\| {}^\circ I_n(G^{k_1}) - {}^\circ I_n(G^{k_2}) \right\|_2^2 < \frac{1}{2 \cdot k}.$$

By part (B) of Proposition 4.2, this implies that

$$\sum_{i_1 < \ldots < i_n} \left(G^{k_1}(b_{i_1}, ..., b_{i_n}) - G^{k_2}(b_{i_1}, ..., b_{i_n}) \right)^2 < \frac{1}{k}$$

for $k \in \mathbb{N}$ and $k_1, k_2 \geq g(k)$. Now extend $(G^k)_{k \in \mathbb{N}}$ to an internal sequence $(G^k)_{k \in {}^*\mathbb{N}}$ in $\widetilde{\mathcal{H}}_n$ and verify that

$$\mathcal{F}_k := \left\{ G \in \widetilde{\mathcal{H}}_n \mid \sum_{i_1 < \ldots < i_n} \left(G(b_{i_1}, ..., b_{i_n}) - G^{{}^*g(k)}(b_{i_1}, ..., b_{i_n}) \right)^2 < \frac{1}{k} \right\}$$

has the finite intersection property. Fix a $G \in \cap_{k \in \mathbb{N}} \mathcal{F}_k$. Then $G \in \mathcal{H}_n$ and ${}^\circ I_n(G^k) \to {}^\circ I_n(G)$ in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Thus $\{ {}^\circ I_n(G_n) \mid G_n \in \mathcal{H}_n \}$ is a closed subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Since ${}^\circ I_n(G_n) \perp {}^\circ I_m(G_m)$ for $n \neq m$, this implies that

$$M := \{ \sum_{n=0}^{\infty} {}^\circ I_n(G_n) \mid G_n \in \mathcal{H}_n \}$$

is also a closed linear subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Because of part (D) of Proposition 4.3, ${}^\circ I(G)^n \in M$ for all $n \in \mathbb{N}$ and $G \in \mathcal{H}_1$. But this implies $M = \mathcal{L}^2(\mathbb{F}, \mathcal{W}, \widehat{\mu})$ because of Proposition 4.1. $\square$

In this situation we call the functions F_n the **kernels** of φ.

5 A Decomposition of Hilbert Space Valued Functions

Recall that $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ denotes the Hilbert space of square Bochner integrable functions $f : (\mathbb{F}, \mathcal{W}) \to \mathbb{H}$. Since the domain of the Skorohod integral is a subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ and since we want to define the Skorohod integral via a chaos decomposition of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$, we will present such a decomposition in this section, in a similar way as in Section 4. A function $f \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ is called **simple** if there is a $g \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ and an $h \in \mathbb{H}$ such that $f(x) = g(x) \cdot h$ for all $x \in \mathbb{F}$.

5.1 Lemma

The space of linear combinations of simple functions is dense in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$.

Proof. We show that $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ is null if it is orthogonal to every simple function. Fix an orthonormal basis (e_n) of $\mathbb{H}$ and an $n \in \mathbb{N}$. We have to show that $\langle \varphi, e_n \rangle_{\mathbb{H}} = 0$ in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. But this follows from the fact that for each $g \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ we have

$$\mathbb{E} \langle \varphi, e_n \rangle_{\mathbb{H}} \cdot g = \mathbb{E} \langle \varphi, g \cdot e_n \rangle_{\mathbb{H}} = \langle \varphi, g \cdot e_n \rangle_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})} = 0 \,. \qquad \square$$

Let $SL^2(\mu, \mathbb{F})$ denote the space of all internal functions

$$F : (\mathbb{F}, \mathfrak{b}_{\mathbb{F}}) \to (\mathbb{F}, \mathfrak{b}_{\mathbb{F}})$$

for which $\|F\|$ is in $SL^2(\mu)$. A function $F \in SL^2(\mu, \mathbb{F})$ is called **nearstandard** if there is an $f \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ such that $f \approx F$ $\widehat{\mu}$-a.s. Then we say that f is the **standard part** of F and that F is an $SL^2(\mu, \mathbb{F})$**-lifting** of f. In this case f is uniquely determined and we set ${}^\circ F := f$. Therefore ${}^\circ F \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ for any nearstandard function F. Note that if internal functions F, G are $SL^2(\mu, \mathbb{F})$-liftings of standard functions $f, g \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ then

$$< f, g >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})} \approx \; \mathbb{E}_\mu < F, G >_{\mathbb{F}} \quad \text{and} \quad \|f\|_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})} = \sqrt{\mathbb{E}\, \|F\|^2_{\mathbb{F}}} \,.$$

5.2 Lemma

Each $f \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ has an $SL^2(\mu, \mathbb{F})$-lifting.

Proof. Because of Proposition 1.2 there is an internal function

$$G : (\mathbb{F}, \mathfrak{b}_{\mathbb{F}}) \to ({}^*\mathbb{H}, \mathfrak{b}_{{}^*\mathbb{H}})$$

such that $\|G\| \in SL^2(\mu)$ and $G \approx f$ $\widehat{\mu}$-a.s. Set $F := pr_{\mathbb{F}}^{*\mathbb{H}} G$. Since $\|F\| \leq \|G\|$, the function F is in $SL^2(\mu, \mathbb{F})$. We obtain

$$F(x) = pr_{\mathbb{F}}^{*\mathbb{H}} G(x) \approx pr_{\mathbb{F}}^{*\mathbb{H}} f(x) = f(x)$$

for $\widehat{\mu}$-almost all $x \in \mathbb{F}$. $\square$

Set $\mathcal{H}_{0,1} := \mathcal{H}_1$. Fix $n \in \mathbb{N}$. Let $\widetilde{\mathcal{H}}_{n,1}$ be the internal space of all multilinear forms $F : \mathbb{F}^{n+1} \to {}^*\mathbb{R}$ that are symmetric in the first n variables. A function $F \in \widetilde{\mathcal{H}}_{n,1}$ is called an $S^{n,1}$**-limit** of a sequence $(F_m)_{m\in\mathbb{N}}$ of functions F_m in $\widetilde{\mathcal{H}}_{n,1}$ if for every $k \in \mathbb{N}$ there is an $m_0 \in \mathbb{N}$ such that

$$\sum_{i\in I} \sum_{i_1<\ldots<i_n} \left(F - F_m(b_{i_1}, ..., b_{i_n}, b_i)\right)^2 < \frac{1}{k}$$

for each $m \geq m_0$. Now let $\mathcal{H}_{n,1}$ be the smallest linear space over $\mathbb{R}$ that is closed under $S^{n,1}$-limits and that contains the functions $F \odot G$ for $F \in \mathcal{H}_n$ and $G \in \mathcal{H}_1$, where

$$\begin{aligned} F \odot G \quad : \quad & \mathbb{F}^{n+1} \to {}^*\mathbb{R}\,, \\ & (h_1, ..., h_{n+1}) \mapsto F(h_1, ..., h_n) \cdot G\left(h_{n+1}\right) . \end{aligned}$$

Set for $F \in \mathcal{H}_{n,1}$

$$\begin{aligned} & I_{n,1}(F) : \mathbb{F} \to \mathbb{F}'\,, \\ & \qquad x \mapsto \left(y \mapsto \sum_{i_1<\ldots<i_n} F(b_{i_1}, ..., b_{i_n}, y) \cdot pr_{i_1}(x) \cdot ... \cdot pr_{i_n}(x) \right) . \end{aligned}$$

Since $\mathbb{F} = \mathbb{F}'$ we can regard $I_{n,1}(F)$ as an $\mathbb{F}$-valued function. For $F \in \mathcal{H}_{0,1}$ we define

$$\begin{aligned} I_{0,1}(F) : \mathbb{F} &\to \mathbb{F}\,, \\ x &\mapsto F\,. \end{aligned}$$

Now we sum up some properties of such functions.

5.3 Proposition

Fix $n, m \in \mathbb{N}_0$, $F \in \mathcal{H}_{n,1}$ and $G \in \mathcal{H}_{m,1}$.

(A) *We have*

$$\mathbb{E} < I_{n,1}(F), I_{m,1}(G) >_{\mathbb{F}} =$$

$$\begin{cases} \sum_{i\in I}\sum_{i_1<...<i_n} F(b_{i_1},...,b_{i_n},b_i)\cdot G(b_{i_1},...,b_{i_n},b_i) & \textit{if } n=m\,, \\ 0 & \textit{otherwise}\,. \end{cases}$$

(B) *The function $I_{n,1}(F_n)$ is in $SL^2(\mu,\mathbb{F})$.*

(C) *The function $I_{n,1}(F)$ is nearstandard.*

Proof. Assertion (A) follows from a straightforward calculation. For the proof of the assertions (B) and (C) we can assume that $n \geq 1$. Set

$$\mathcal{G}_{n,1} := \{F \in \mathcal{H}_{n,1} \mid I_{n,1}(F) \text{ is in } SL^2(\mu,\mathbb{F}) \text{ and nearstandard }\}\ .$$

The set $\mathcal{G}_{n,1}$ is a linear space over $\mathbb{R}$, because for $F, L \in \mathcal{G}_{n,1}$ and $a, b \in \mathbb{R}$

$$I_{n,1}(aF + bL) = a \cdot I_{n,1}(F) + b \cdot I_{n,1}(L)$$

and the set of nearstandard functions in $SL^2(\mu,\mathbb{F})$ is closed under linear combinations. Now fix $F \in \mathcal{H}_n$ and $L \in \mathcal{H}_1$. Since

$$I_{n,1}(F \odot L)(x) = I_n(F)(x) \cdot L$$

for all $x \in \mathbb{F}$, we obtain that $I_{n,1}(F \odot L) \in SL^2(\mu,\mathbb{F})$. There is an $h \in \mathbb{H}$ with $h \approx L$. We obtain

$$I_{n,1}(F \odot L)(x) \approx {}^\circ I_n(F)(x) \cdot h$$

for $\widehat{\mu}$-almost all $x \in \mathbb{F}$, which implies that $I_{n,1}(F \odot L)$ is nearstandard. Now fix an $F \in \mathcal{H}_{n,1}$ and a sequence (F_k) in $\mathcal{G}_{n,1}$ which converges to F in the $S^{n,1}$-sense. We have to show that F belongs to $\mathcal{G}_{n,1}$. Since for $k \in \mathbb{N}$

$$\mathbb{E}\left(\|I_{n,1}(F)\| - \|I_{n,1}(F_k)\|\right)^2 \leq$$

$$\mathbb{E}\,\|I_{n,1}(F) - I_{n,1}(F_k)\|^2 =$$

$$\sum_{i\in I}\sum_{i_1<...<i_n}\left(F - F_k(b_{i_1},...,b_{i_n},b_i)\right)^2\ ,$$

the function $\|I_{n,1}(F)\|$ is in $SL^2(\mu)$, thus $I_{n,1}(F) \in SL^2(\mu,\mathbb{F})$. A similar argument shows that ${}^\circ I_{n,1}(F_k)$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$ and therefore converges to an $f \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$. Let Φ be an $SL^2(\mu,\mathbb{F})$-lifting of f. For any $k \in \mathbb{N}$ we obtain

$$\sqrt{\mathbb{E}\,\|I_{n,1}(F) - \Phi\|^2} \leq$$

$$\sqrt{\mathbb{E}\left\|I_{n,1}(F)-I_{n,1}(F_k)\right\|^2}+\sqrt{\mathbb{E}\left\|I_{n,1}(F_k)-\Phi\right\|^2}\approx$$

$$\sqrt{\sum_{i\in I}\sum_{i_1<\ldots<i_n}(F-F_k(b_{i_1},...,b_{i_n},b_i))^2}+\|{}^\circ I_{n,1}(F_k)-f\|_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})}\,.$$

This implies that $I_{n,1}(F)$ is nearstandard. Thus $F\in\mathcal{G}_{n,1}$ and the proof is finished. $\square$

Now we are ready to prove a chaos decomposition theorem for Hilbert space valued functions.

5.4 Proposition

Fix $\varphi\in\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$. Then there is a sequence $(F_n)_{n\in\mathbb{N}_0}$ of functions $F_n\in\mathcal{H}_{n,1}$ such that

$$\varphi=\sum_{n=0}^{\infty}{}^\circ I_{n,1}(F_n)\quad\text{in }\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})\,.$$

If $(G_n)_{n\in\mathbb{N}_0}$ is another sequence with this property, then

$$\sum_{i\in I}\sum_{i_1<\ldots<i_n}(F_n-G_n(b_{i_1},...,b_{i_n},b_i))^2\approx 0$$

for each $n\in\mathbb{N}_0$.

Proof. A similar saturation argument as in the proof of Proposition 4.4 shows that

$$M:=\{\sum_{n=0}^{\infty}{}^\circ I_{n,1}(F_n)\mid F_n\in\mathcal{H}_{n,1}\}$$

is a closed subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$. Because of Lemma 5.1 and Proposition 4.4 it suffices to show that for any $f\in\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ with

$$f=\sum_{n=0}^{\infty}{}^\circ I_n(G_n)\,,\ G_n\in\mathcal{H}_n\,,$$

and for any $h\in\mathbb{H}$ the operator $f\cdot h$ is in M. First note that for $n\in\mathbb{N}_0$ we have

$$I_{n,1}(G_n\odot h)=I_n(G_n)\cdot h\,,$$

therefore the series $\sum_{n=0}^{\infty}{}^\circ I_{n,1}(G_n\odot h)$ converges in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$. It suffices to show that

$$f\cdot h=\sum_{n=0}^{\infty}{}^\circ I_{n,1}(G_n\odot h)\ \text{ in }\ \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})\,.$$

This follows from the fact that for $g \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ and $k \in \mathbb{H}$ we have

$$< \sum_{n=0}^{\infty} {}^{\circ}I_{n,1}(G_n \odot h), g \cdot k >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} = \sum_{n=0}^{\infty} < {}^{\circ}I_{n,1}(G_n \odot h), g \cdot k >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} =$$

$$\sum_{n=0}^{\infty} < {}^{\circ}I_n(G_n) \cdot h, g \cdot k >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} = \sum_{n=0}^{\infty} < {}^{\circ}I_n(G_n), g >_2 \cdot < h, k >_{\mathbb{H}} =$$

$$< \sum_{n=0}^{\infty} {}^{\circ}I_n(G_n), g >_2 \cdot < h, k >_{\mathbb{H}} =< f \cdot h, g \cdot k >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} . \qquad \square$$

In this situation we call the functions F_n the **kernels** of φ. We mention that ${}^{\circ}I_{0,1}(F)$ equals the Bochner integral of φ. But since we do not use this fact we do without a proof.

6 Adapted Hilbert Space Valued Functions

In this section we define a closed subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$, namely the space of the adapted functions. This notion of adaptedness is introduced in [31] and based on the resolution of the identity π. Furthermore we show that each adapted function has an $SL^2(\mu, \mathbb{F})$-lifting which is adapted in an internal sense. This will allow us later to define the stochastic integral as the standard part of an internal stochastic integral.

A function $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ is called **adapted** if for each $t \in [0,1]$ and each $h \in \mathbb{H}$ the mapping $< \varphi, \pi_t h >$ is $\mathfrak{c}_t$-measurable, which implies that it is even $\mathcal{W}_t$-measurable. Denote by $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ the space of all adapted functions. Thus $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ is a closed subspace of $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$. A function $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ is called **simple adapted** if

$$f \;=\; g \cdot (\pi_t - \pi_s) h$$

for $s < t$ in $[0,1]$, $g \in \mathcal{L}^2(\mathbb{F}, \mathcal{W}_s, \widehat{\mu})$ and $h \in \mathbb{H}$.

6.1 Lemma

(See [31], Lemma 2.2.) *The space of linear combinations of simple adapted functions is dense in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$.*

Proof. We show that $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ is zero if it is orthogonal to every simple adapted function. First observe that under this circumstances for any $h \in \mathbb{H}$ the process $m := (< f, \pi_t h >)_{t \in [0,1]}$ is a continuous $(\mathfrak{c}_t)$-martingale with $m_0 = 0$. We will see that m is of finite variation, which implies that it is zero. (See Proposition 1.2 in Chapter IV of [28].) For every partition $0 = t_0 < ... < t_n = 1$ of $[0,1]$ and every $x \in \mathbb{F}$ we have

$$\sum_{i=1}^{n} \left| < f(x), \pi_{t_i} - \pi_{t_{i-1}} h > \right| \;=\; \sum_{i=1}^{n} \left| < \pi_{t_i} - \pi_{t_{i-1}} f(x), \pi_{t_i} - \pi_{t_{i-1}} h > \right| \leq$$

$$\sum_{i=1}^{n} \left\| \pi_{t_i} - \pi_{t_{i-1}} f(x) \right\| \cdot \left\| \pi_{t_i} - \pi_{t_{i-1}} h \right\| \leq \| f(x) \| \cdot \| h \| \quad .$$

This implies that for every $x \in \mathbb{F}$ $m(x, \cdot)$ is of finite variation. $\square$

An element F of $SL^2(\mu, \mathbb{F})$ is called **adapted** if

$$< F, b_i > \in \mathcal{B}_{i-1} \text{ for all } i \in I. \tag{5}$$

The next proposition claims that the functions in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ are exactly the standard parts of the adapted nearstandard functions in $SL^2(\mu, \mathbb{F})$.

6.2 Proposition

If $F \in SL^2(\mu, \mathbb{F})$ is adapted and nearstandard, then $^\circ F$ is in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. On the other hand, every $g \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ has an adapted $SL^2(\mu, \mathbb{F})$-lifting G.

Proof. Fix $h \in \mathbb{H}, t \in [0,1]$ and $k \in T$ with $\frac{k}{H} \approx t$. We have to show that $< {}^\circ F, \pi_t h >$ is $\mathfrak{c}_t$-measurable. Proposition 3.3 implies that

$$< {}^\circ F, \pi_t h > \approx < F, h_{\{1,\ldots,\sigma(k)\}} > = \sum_{i=1}^{\sigma(k)} < h, b_i > \cdot < F, b_i > \qquad (6)$$

$\widehat{\mu}$-a.s. Since F is adapted, the right hand side of (6) is $\mathcal{C}_k$-measurable. Now part (B) of Proposition 1.3 implies that the left hand side of (6) is $\mathfrak{c}_t$-measurable. It remains to show that the $\mathbb{R}$-linear space

$$M := \left\{ g \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H}) \mid g \text{ has an adapted } SL(\mu, \mathbb{F})\text{-lifting} \right\}$$

is a closed subspace of $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ which contains each simple adapted function $f \cdot (\pi_t - \pi_s)h$. Because of part (C) of Proposition 1.3, there is a $k \in \{2, ..., H\}$ with $\frac{k}{H} \approx s$ and a $\mathcal{C}_k$-measurable $F \in SL^2(\mu)$ such that F is a lifting of f. If $l \in T$ with $\frac{l}{H} \approx t$ then $F \cdot (\prod_l - \prod_k)h$ is an adapted $SL^2(\mu, \mathbb{F})$-lifting of $f \cdot (\pi_t - \pi_s)h$. Now fix a sequence (f_n) in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ such that each f_n has an adapted $SL^2(\mu, \mathbb{F})$-lifting F_n. Fix further an $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ such that $f_n \to f$. Because of Lemma 5.2 there is an $SL^2(\mu, \mathbb{F})$-lifting F of f, but F is not necessarily adapted. Extend $(F_n)_{n \in \mathbb{N}}$ to an internal sequence $(F_n)_{n \in {}^*\mathbb{N}}$ of mappings such that each $F_n : (\mathbb{F}, \mathfrak{b}_{\mathbb{F}}) \to (\mathbb{F}, \mathfrak{b}_{\mathbb{F}})$ has the property (5). Then there is an unlimited $N \in {}^*\mathbb{N}$ such that

$$\mathbb{E} \left\| F_N - F \right\|^2_{\mathbb{F}} \approx 0.$$

Therefore, F_N is an adapted $SL^2(\mu, \mathbb{F})$-lifting of f. $\square$

7 The Orthogonal Projection from $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ onto $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$

Now we are ready to express the image of the orthogonal projection of a $\psi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ onto $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ in terms of the kernels of ψ. This result, which is interesting for its own sake, allows for example a straightforward approach to the Clark Ocone formula in Section 12.

Up to now we have used the term adapted for two kinds of functions, namely for certain $f \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ and for certain internal functions $F : \mathbb{F} \to \mathbb{F}$. Fix an $m \in \mathbb{N}$. In addition, we call an element G of $\mathcal{H}_{m,1}$ **adapted** if for $i_1, ..., i_m, i \in I$ the implication

$$G(b_{i_1}, ..., b_{i_m}, b_i) \neq 0 \;\Rightarrow\; i_1, ..., i_m < i$$

is valid. The use of the term adapted in this situation is justified by the following fact.

7.1 Lemma

If $G \in \mathcal{H}_{m,1}$ is adapted then $I_{m,1}(G)$ is adapted, which implies that ${}^\circ I_{m,1}(G)$ is also adapted.

Proof. For an adapted $G \in \mathcal{H}_{m,1}$ we have

$$< I_{m,1}(G), b_i > = \sum_{i_1 < ... < i_m < i} G(b_{i_1}, ..., b_{i_m}, b_i) \cdot pr_{i_1} \cdot ... \cdot pr_{i_m} \in \mathcal{B}_{i-1}$$

for each $i \in I$. Thus $I_{m,1}(G)$ is adapted. Because of Proposition 6.2, this implies that ${}^\circ I_{m,1}(G) \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. $\square$

For $G \in \mathcal{H}_{m,1}$ define $G^< : \mathbb{F}^{n+1} \to {}^*\mathbb{R}$ by

$$G^<(x_1, ..., x_m, x) :=$$

$$\sum_{i_1, ..., i_m < i} < x_1, b_{i_1} > \cdot ... \cdot < x_m, b_{i_m} > \cdot < x, b_i > \cdot G(b_{i_1}, ..., b_{i_m}, b_i)\,.$$

Note that $G^<$ is an element of $\widetilde{\mathcal{H}}_{m,1}$ which is adapted and closely related to G. In order to be able to build the standard part of $I_{m,1}(G^<)$ we must show that $G^<$ is even in $\mathcal{H}_{m,1}$.

7.2 Proposition

If $F \in \mathcal{H}_{m,1}$ then $F^< \in \mathcal{H}_{m,1}$.

Proof. Set

$$\mathcal{G}_{m,1} := \{F \in \mathcal{H}_{m,1} \mid F^< \in \mathcal{H}_{m,1}\}$$

and verify that $\mathcal{G}_{m,1}$ is an $\mathbb{R}$-linear subspace of $\mathcal{H}_{m,1}$ which is closed under $S^{m,1}$-limits. Therefore it suffices to show that $(G \odot L)^<$ is in $\mathcal{H}_{m,1}$ for $G \in \mathcal{H}_m$ and $L \in \mathcal{H}_1$. Fix $L \in \mathcal{H}_1$ and note that

$$\mathcal{G}_m := \{G \in \mathcal{H}_m \mid (G \odot L)^< \in \mathcal{H}_{m,1}\}$$

is an $\mathbb{R}$-linear subspace of $\mathcal{H}_m$ which is closed under S^m-limits. It remains to show that for fixed $G \in \mathcal{H}_1$ we have

$$\left(G^{\odot m} \odot L\right)^< \in \mathcal{H}_{m,1} \,. \tag{7}$$

The formula (7) is valid if we can find a sequence $(N_k)_{k\in\mathbb{N}}$ with $N_k \in \mathcal{H}_{m,1}$ such that $(G^{\odot m} \odot L)^<$ is an $S^{m,1}$-limit of (N_k). Fix $k \in \mathbb{N}$ and $1 \le n \le k$ and set

$$I_n^k := \{(n-1)\,\tfrac{\omega}{k} + 1, ..., n \cdot \tfrac{\omega}{k}\} \,.$$

Furthermore set $i_n^k := \max I_n^k$. By Proposition 3.3, for any interval $J \subset I$ and for any $x \in \mathcal{H}_1$ we have $x_J \in \mathcal{H}_1$. Therefore the function N_k, defined by

$$N_k := \sum_{1 \le n < k} G^{\odot m}_{\{1,...,i_n^k\}} \odot L_{\{i_n^k+1,...,i_{n+1}^k\}}$$

is an element of $\mathcal{H}_{m,1}$. Note that N_k is adapted and for each set of indices $i_1 < ... < i_m < i$ we have

$$N_k(b_{i_1}, ..., b_{i_m}, b_i) = G^{\odot m} \odot L(b_{i_1}, ..., b_{i_m}, b_i)$$

if there is an $n \in \{1, ..., k-1\}$ such that $i_m \in \{1, ..., i_n^k\}$ and $i \in I_{n+1}^k$. If this is not the case we have

$$N_k(b_{i_1}, ..., b_{i_m}, b_i) = 0 \,.$$

Let $\varepsilon > 0$. Since

$$\sum_{i_1 < ... < i_m} G^{\odot m}(b_{i_1}, ..., b_{i_m})^2 \in \mathrm{Lim}$$

and because of Proposition 3.4 there is a $k_0 \in \mathbb{N}$ such that

$$\sum_{i_1<\ldots<i_m} \left(G^{\odot m}(b_{i_1}, \ldots, b_{i_m})^2 \cdot \sum_{\substack{i\in I \\ |i-i_m|<\frac{\omega}{k}}} L\left(b_i\right)^2 \right) < \varepsilon$$

for $k \geq k_0$. Thus we obtain for $k \geq k_0$

$$\sum_{\substack{i_1<\ldots<i_m \\ i\in I}} \left(\left(G^{\odot m} \odot L\right)^< - N_k(b_{i_1}, \ldots, b_{i_m}, b_i) \right)^2 =$$

$$\sum_{i_1<\ldots<i_m<i} \left(\left(G^{\odot m} \odot L\right) - N_k(b_{i_1}, \ldots, b_{i_m}, b_i) \right)^2 =$$

$$\sum_{\substack{i_1<\ldots<i_m<i \\ i-i_m<\frac{\omega}{k}}} \left(\left(G^{\odot m} \odot L\right) - N_k(b_{i_1}, \ldots, b_{i_m}, b_i) \right)^2 \leq$$

$$\sum_{\substack{i_1<\ldots<i_m<i \\ i-i_m<\frac{\omega}{k}}} \left(G^{\odot m} \odot L\right)^2 (b_{i_1}, \ldots, b_{i_m}, b_i) \leq$$

$$\sum_{i_1<\ldots<i_m} \left(G^{\odot m}(b_{i_1}, \ldots, b_{i_m})^2 \cdot \sum_{\substack{i\in I \\ |i-i_m|<\frac{\omega}{k}}} L\left(b_i\right)^2 \right) < \varepsilon.$$

Thus $(G^{\odot m} \odot L)^<$ is an $S^{m,1}$-limit of (N_k) and therefore the proof is finished.

$\square$

Denote by $pr_{\mathcal{A}}^{\mathcal{W}}\psi$ the image of the orthogonal projection of a $\psi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ onto $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. How does $pr_{\mathcal{A}}^{\mathcal{W}}\psi$ look like? For $\psi = {}^\circ I_{m,1}(G)$ a natural candidate for $pr_{\mathcal{A}}^{\mathcal{W}}\psi$ is ${}^\circ I_{m,1}(G^<)$. The next proposition guarantees that this is indeed the case, therefore we can express the orthogonal projection from $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ onto $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ by the assignment $G \mapsto G^<$. For $F \in \mathcal{H}_{0,1}$ set $F^< := F$.

7.3 Proposition

Fix a $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ *with chaos decomposition*

$$\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n) \;\; in \;\; \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H}),$$

where $F_n \in \mathcal{H}_{n,1}$ *for* $n \in \mathbb{N}_0$*. Then*

$$pr_{\mathcal{A}}^{\mathcal{W}}\varphi = \sum_{n=0}^{\infty} {}^{\circ}I_{n,1}(F_n^{<}) \;\; in \;\; \mathcal{L}_{\mathcal{A}}^2(\widehat{\mu}, \mathbb{H})\,. \tag{8}$$

Proof. First note that the right hand side of (8) converges, since

$$\|{}^{\circ}I_{n,1}(F_n^{<})\|_{\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu},\mathbb{H})} \leq \|{}^{\circ}I_{n,1}(F_n)\|_{\mathcal{L}_{\mathcal{W}}^2(\widehat{\mu},\mathbb{H})}\,.$$

The operator $pr_{\mathcal{A}}^{\mathcal{W}}$ is linear and continuous, therefore it suffices to show that

$$pr_{\mathcal{A}}^{\mathcal{W}}\,{}^{\circ}I_{m,1}(F_m) = {}^{\circ}I_{m,1}(F_m^{<}) \tag{9}$$

in $\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu}, \mathbb{H})$ for a fixed $m \in \mathbb{N}$. Since both functions in (9) are adapted and since the linear combinations of simple adapted functions are dense in $\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu}, \mathbb{H})$, see Lemma 6.1, it suffices to show that

$$< pr_{\mathcal{A}}^{\mathcal{W}}\,{}^{\circ}I_{m,1}(F_m), g\cdot(\pi_t - \pi_s)\,h >_{\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu},\mathbb{H})} =$$

$$< {}^{\circ}I_{m,1}(F_m^{<}), g\cdot(\pi_t - \pi_s)\,h >_{\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu},\mathbb{H})}$$

for $s < t$ in $[0,1]$, $g \in \mathcal{L}^2(\mathbb{F}, \mathcal{W}_s, \widehat{\mu})$ and $h \in \mathbb{H}$. According to part (C) of Proposition 1.3 there is a $k \in T$ with $\frac{k}{H} \approx s$ and a $\mathcal{C}_k$-measurable $G \in SL^2(\mu)$ which is a lifting of g. We further fix an $l \in T$ with $\frac{l}{H} \approx t$ and obtain

$$< pr_{\mathcal{A}}^{\mathcal{W}}\,{}^{\circ}I_{m,1}(F_m), g\cdot(\pi_t - \pi_s)\,h >_{\mathcal{L}_{\mathcal{A}}^2(\widehat{\mu},\mathbb{H})} =$$

$$< {}^{\circ}I_{m,1}(F_m), g\cdot(\pi_t - \pi_s)\,h >_{\mathcal{L}_{\mathcal{W}}^2(\widehat{\mu},\mathbb{H})} \approx$$

$$\mathbb{E}_{\mu} < I_{m,1}(F_m), G\cdot h_{\{\sigma(k)+1,\dots,\sigma(l)\}} >_{\mathbb{F}} =$$

$$\sum_{i_1<\dots<i_m} F_m(b_{i_1},\dots,b_{i_m}, h_{\{\sigma(k)+1,\dots,\sigma(l)\}})\cdot \mathbb{E}_{\mu} G\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_m} =$$

$$\sum_{i_1<\dots<i_m\leq\sigma(k)} F_m(b_{i_1},\dots,b_{i_m}, h_{\{\sigma(k)+1,\dots,\sigma(l)\}})\cdot \mathbb{E}_{\mu} G\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_m} =$$

$$\sum_{i_1<\dots<i_m\leq\sigma(k)} F_m^{<}(b_{i_1},\dots,b_{i_m}, h_{\{\sigma(k)+1,\dots,\sigma(l)\}})\cdot \mathbb{E}_{\mu} G\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_m} =$$

$$\sum_{i_1<\dots<i_m} F_m^{<}(b_{i_1},\dots,b_{i_m}, h_{\{\sigma(k)+1,\dots,\sigma(l)\}})\cdot \mathbb{E}_{\mu} G\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_m} =$$

$$\mathbb{E}_{\mu} < I_{m,1}(F_m^{<}), G\cdot h_{\{\sigma(k)+1,\dots,\sigma(l)\}} >_{\mathbb{F}} \approx$$

$$< {}^{\circ}I_{m,1}(F_m^<), g \cdot (\pi_t - \pi_s)\, h >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} =$$
$$< {}^{\circ}I_{m,1}(F_m^<), g \cdot (\pi_t - \pi_s)\, h >_{\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H})},$$

where we have used that

$$(\pi_t - \pi_s)\, h \approx h_{\{\sigma(k)+1,\ldots,\sigma(l)\}}$$

and that

$$\mathbb{E}_\mu\, G \cdot pr_{i_1} \cdot \ldots \cdot pr_{i_m} = 0$$

for $\sigma(k) < i_m$. $\square$

After our exhaustive survey of the spaces $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$, $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$ and $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H})$ it is quite easy to introduce the stochastic integral, the Skorohod integral and the Malliavin derivative. This will be the topics of the next three chapters.

8 The Stochastic Integral

In this section we show that for an adapted $F \in SL^2(\mu, \mathbb{F})$ the function

$$\mathbb{F} \ni x \mapsto < F(x), x >_{\mathbb{F}}$$

is in $SL^2(\mu)$ and that

$$\| < F(\,\cdot\,),\,\cdot\, >_{\mathbb{F}} \|_2 = \sqrt{\mathbb{E}\, \|F\|_{\mathbb{F}}^2}\,.$$

This allows us to define the stochastic integral of an $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ by

$$\int f d\pi := {}^\circ < F(\,\cdot\,),\,\cdot\, >,$$

where $F \in SL^2(\mu, \mathbb{F})$ is an adapted lifting of f. We further show that $\int f d\pi$ is $\mathcal{W}$-measurable. In Section 13 we will see that this definition of the stochastic integral coincides with the integral introduced in [32].

8.1 Proposition

Fix an adapted $F \in SL^2(\mu, \mathbb{F})$. Then for each $j \in I$ the function

$$\mathbb{F} \ni x \mapsto \sum_{i=1}^{j} < F(x), b_i > \cdot\, pr_i(x)$$

is square S_μ-integrable.

Proof. Let ν_I be the internal counting probability measure on the internal powerset ${}^*\mathcal{P}(I)$ of I. Define for $x \in \mathbb{F}$ and $j \in I$

$$M(x, j) := \sum_{i=1}^{j} < F(x), b_i > \cdot\, pr_i(x)\ .$$

Then M is an internal $(\mathcal{B}_i)_{i \in I}$-martingale. Due to a result of Lindstrøm (cf. [18]) and Hoover and Perkins (cf. [15]) (see also [26], Theorem 8.14.1 for a detailed proof) it is sufficient to show that the function

$$\sum_{i \in I} < F, b_i >^2 \cdot\, pr_i^2$$

is in $SL^1(\mu)$. Set

$$\begin{aligned} \widetilde{F} &: \mathbb{F} \times I \ni (x, i) \mapsto < F(x), b_i > \quad \text{and} \\ \widetilde{pr} &: \mathbb{F} \times I \ni (x, i) \mapsto pr_i(x)\ . \end{aligned}$$

Since
$$\mathbb{E}_{\mu\otimes\nu_I}\widetilde{F}^2\omega = \mathbb{E}\sum_{i\in I} < F, b_i >^2 = \mathbb{E}\,\|F\|_{\mathbb{F}}^2 \in \mathrm{Lim}\ ,$$
the measure
$$\widetilde{\mu} : \mathfrak{b}_{\mathbb{F}} \otimes {}^*\mathcal{P}(I) \ni A \mapsto \int_A \widetilde{F}^2\omega\, d\,\mu \otimes \nu_I$$
takes only limited values. And since
$$\mathbb{E}_{\widetilde{\mu}}\,\widetilde{pr}^4 = \int_{\mathbb{F}\times I} \widetilde{pr}^4 \cdot \widetilde{F}^2\omega d\mu \otimes \nu_I = 3\cdot\mathbb{E}\,\|F\|_{\mathbb{F}}^2 \in \mathrm{Lim},$$
we obtain that $\widetilde{pr}^2 \in SL^1(\widetilde{\mu})$. Now fix $A \in \mathfrak{b}_{\mathbb{F}}$ with $\mu(A) \approx 0$. Observe that the S_μ-integrability of $\|F\|_{\mathbb{F}}^2$ implies that $\widetilde{\mu}(A\times I) \approx 0$. We obtain
$$\int_A \sum_{i\in I} < F, b_i >^2\ \cdot pr_i^2 d\mu = \int_{A\times I} \widetilde{pr}^2 d\widetilde{\mu} \approx 0. \qquad \square$$

For an adapted function $F \in SL^2(\mu, \mathbb{F})$ the function $\int F \bigtriangleup \prod$, given by
$$\begin{aligned} \textstyle\int F \bigtriangleup \prod : \mathbb{F} &\to {}^*\mathbb{R}\,, \\ x &\mapsto < F\,(x)\,, x >_{\mathbb{F}} \end{aligned}$$
is called **the internal stochastic integral of** F. Note that
$$\textstyle\int F \bigtriangleup \prod = \sum_{i\in I} < F, b_i > \ \cdot pr_i\,.$$

Because of Proposition 8.1 the function $\int F \bigtriangleup \prod$ is square S_μ-integrable. A straightforward calculation shows that
$$< \textstyle\int F \bigtriangleup \prod, \int G \bigtriangleup \prod >_2\ =\ \mathbb{E} < F, G >_{\mathbb{F}} \tag{10}$$
for adapted functions $F, G \in SL^2(\mu, \mathbb{F})$. This gives rise to the following definition. Take $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ and fix an adapted $SL^2(\mu, \mathbb{F})$-lifting F of f. Set
$$\textstyle\int f d\pi :=\ {}^\circ\!\int F \bigtriangleup \prod.$$
Because of (10) the mapping $\int f d\pi$ is well defined, because of Proposition 8.1 it is in $\mathcal{L}^2(\mathbb{F}, L_\mu(\mathfrak{b}_{\mathbb{F}}), \widehat{\mu})$. We call this map **the stochastic integral of** f. The stochastic integral fulfills the following continuity and measurability conditions.

8.2 Lemma

The map

$$\begin{aligned} \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H}) &\to \mathcal{L}^2(\mathbb{F}, L_\mu(\mathfrak{b}_\mathbb{F}), \widehat{\mu}) \\ f &\mapsto \int f d\pi \end{aligned}$$

is linear and norm preserving. Furthermore, $\int f d\pi \in \mathcal{W}$ *for all* $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$.

Proof. Fix an $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ and let F be an adapted $SL^2(\mu, \mathbb{F})$-lifting of f. Then we have

$$\mathbb{E}_{\widehat{\mu}} \left(\int f d\pi \right)^2 \approx \mathbb{E}_\mu \left(\int F \vartriangle \prod \right)^2 = \mathbb{E}_\mu \, \|F\|^2_\mathbb{F} \approx \|f\|^2_{\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})}$$

because of (10). Thus $f \mapsto \int f d\pi$ is norm preserving. Since for a simple adapted function g the map $\int g d\pi$ is $\mathcal{W}$-measurable and since the linear combinations of simple adapted functions are dense in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$, the integral $\int f d\pi$ is $\mathcal{W}$-measurable for all $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. $\square$

9 The Skorohod Integral

Now we define the Skorohod integral $\delta\psi$ of suitable (i.e. integrable) variables $\psi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$. For $\psi = {}^\circ I_{n,1}(F),\ F \in \mathcal{H}_{n,1}$ we set $\delta\psi = {}^\circ I_{n+1}(\widetilde{F})$, where $\widetilde{F}$ is a symmetrization of F. The crucial point is to show that $\widetilde{F}$ is in $\mathcal{H}_{n+1}$.
Fix $n \in \mathbb{N}$ and $F \in \mathcal{H}_{n,1}$. Set

$$\begin{aligned} \widetilde{F} \quad &: \quad \mathbb{F}^{n+1} \to {}^*\mathbb{R} \\ (x_1, ..., x_{n+1}) &\mapsto \sum_{k=1}^{n+1} F(x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1}, x_k)\,. \end{aligned}$$

Note that $\widetilde{F} \in \widetilde{\mathcal{H}}_{n+1}$. Note further that for $\alpha, \beta \in {}^*\mathbb{R}$ and $G \in \mathcal{H}_{n,1}$ we have

$$\widetilde{\alpha F + \beta G} = \alpha\widetilde{F} + \beta\widetilde{G}\,.$$

The next lemma implies that for a sequence (F_k) in $\widetilde{\mathcal{H}}_{n,1}$ which converges to an $F \in \widetilde{\mathcal{H}}_{n,1}$ in the $S^{n,1}$-sense the function $\widetilde{F}$ is an S^{n+1}-limit of $(\widetilde{F}_k)$.

9.1 Lemma

There is an $\alpha \in \mathbb{R}$ such that for each $G \in \widetilde{\mathcal{H}}_{n,1}$

$$\sum_{i_1<...<i_{n+1}} \widetilde{G}(b_{i_1}, ..., b_{i_{n+1}})^2 \le \alpha \cdot \sum_{i \in I} \sum_{i_1<...<i_n} G(b_{i_1}, ..., b_{i_n}, b_i)^2.$$

Proof. Set $\alpha := 3 \cdot n! \cdot (n+1)^2$. We obtain

$$\sum_{i_1<...<i_{n+1}} \widetilde{G}(b_{i_1}, ..., b_{i_{n+1}})^2 =$$

$$\sum_{i_1<...<i_{n+1}} \left(\sum_{k=1}^{n+1} G(b_{i_1}, ..., b_{i_{k-1}}, b_{i_{k+1}}, ..., b_{i_{n+1}}, b_{i_k}) \right)^2 =$$

$$\sum_{i_1<...<i_{n+1}} \sum_{k=1}^{n+1} G(b_{i_1}, ..., b_{i_{k-1}}, b_{i_{k+1}}, ..., b_{i_{n+1}}, b_{i_k})^2 +$$

$$2 \cdot \sum_{\substack{i_1<...<i_{n+1} \\ 1 \le k < l \le n+1}} G(b_{i_1}, ..., b_{i_{k-1}}, b_{i_{k+1}}, ..., b_{i_{n+1}}, b_{i_k}) \cdot G(b_{i_1}, ..., b_{i_{l-1}}, b_{i_{l+1}}, ..., b_{i_{n+1}}, b_{i_l}) \le$$

$$(n+1)\cdot \sum_{(i_1,\dots,i_{n+1})\in I_{\neq}^{n+1}} G(b_{i_1},\dots,b_{i_{n+1}})^2 +$$

$$2\cdot (n+1)^2 \cdot \sum_{(i_1,\dots,i_{n+1})\in I_{\neq}^{n+1}} G(b_{i_1},\dots,b_{i_{n+1}})^2 \leq$$

$$3\cdot (n+1)^2\cdot n!\cdot \sum_{i\in I}\sum_{i_1<\dots<i_n} G(b_{i_1},\dots,b_{i_n},b_i)^2. \qquad \square$$

Now we show that $\widetilde{F}$ is even in $\mathcal{H}_{n+1}$. This guarantees that ${}^{\circ}I_{n+1}(\widetilde{F})$ is in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$.

9.2 Lemma

The function $\widetilde{F}$ is in $\mathcal{H}_{n+1}$.

Proof. Set

$$\mathcal{G}_{n,1} := \{L\in \mathcal{H}_{n,1} \mid \widetilde{L} \text{ is in } \mathcal{H}_{n+1}\}$$

and note that this set is an $\mathbb{R}$-linear space which, by Lemma 9.1, is closed under $S^{n,1}$-limits. We show that $\mathcal{G}_{n,1}=\mathcal{H}_{n,1}$. To this end we must prove that for a fixed $G\in\mathcal{H}_1$ the set

$$\mathcal{G}_n := \{L\in\mathcal{H}_n \mid L\widetilde{\odot}G\in\mathcal{H}_{n+1}\}$$

equals $\mathcal{H}_n$. Again it is easy to see that $\mathcal{G}_n$ is an $\mathbb{R}$-linear space which is closed under S^n-limits. It remains to show that for each $N\in\mathcal{H}_1$ the function $N^{\odot n}\in\mathcal{G}_n$, i.e. that $N^{\odot n}\widetilde{\odot}G\in\mathcal{H}_{n+1}$. Set

$$\Phi := \sum_{i_1<\dots<i_{n+1}} N^{\odot n}\widetilde{\odot}G(b_{i_1},\dots,b_{i_{n+1}})\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_{n+1}}.$$

It is sufficient to show that $\Phi\in SL^2(\mu)$ and that ${}^{\circ}\Phi\in\mathcal{W}$. This follows from the fact that we have $\widehat{\mu}$-a.s.

$$\Phi = \sum_{\substack{i_1<\dots<i_n\\ i\in I\setminus\{i_1,\dots,i_n\}}} N^{\odot n}(b_{i_1},\dots,b_{i_n})\cdot G(b_i)\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_n}\cdot pr_i =$$

$$\sum_{\substack{i_1<\dots<i_n\\ i\in I}} N^{\odot n}(b_{i_1},\dots,b_{i_n})\cdot G(b_i)\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_n}\cdot pr_i -$$

$$\sum_{\substack{i_1<\dots<i_n\\ i\in\{i_1,\dots,i_n\}}} N^{\odot n}(b_{i_1},\dots,b_{i_n})\cdot G(b_i)\cdot pr_{i_1}\cdot\ldots\cdot pr_{i_n}\cdot pr_i =$$

$$I_n(N^{\odot n}) \cdot I(G) - \sum_{\substack{i_1<...<i_{n-1} \\ i \in I \setminus \{i_1,...,i_{n-1}\}}} N^{\odot n-1}(b_{i_1},...,b_{i_{n-1}}) \cdot N(b_i) \cdot G(b_i) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} \cdot pr_i^2 \approx$$

$$I_n(N^{\odot n}) \cdot I(G) - \sum_{\substack{i_1<...<i_{n-1} \\ i \in I \setminus \{i_1,...,i_{n-1}\}}} N^{\odot n-1}(b_{i_1},...,b_{i_{n-1}}) \cdot N(b_i) \cdot G(b_i) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} \cdot 1 \approx$$

$$I_n(N^{\odot n}) \cdot I(G) - \sum_{\substack{i_1<...<i_{n-1} \\ i \in I}} N^{\odot n-1}(b_{i_1},...,b_{i_{n-1}}) \cdot N(b_i) \cdot G(b_i) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n-1}} \cdot 1 \approx$$

$$I_n(N^{\odot n}) \cdot I(G) - I_{n-1}(N^{\odot n-1}) \cdot \sum_{i \in I} N(b_i) \cdot G(b_i)\,. \qquad \square$$

For $G \in \mathcal{H}_{0,1}$ set $\widetilde{G} := G$. Now define for $n \in \mathbb{N}_0$

$$\delta\, {}^\circ I_{n,1}(F) := {}^\circ I_{n+1}(\widetilde{F}).$$

This definition is possible because of Lemma 9.1. Now we set

$$\Delta := \{\sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n) \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H}) \mid \sum_{n=0}^{\infty} {}^\circ I_{n+1}(\widetilde{F}_n) \text{ converges in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})\}$$

and define the **Skorohod integral** by

$$\delta : \Delta \to \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$$
$$\sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n) \mapsto \sum_{n=0}^{\infty} {}^\circ I_{n+1}(\widetilde{F}_n)\ .$$

In Section 13 we will show that our definition of δ coincides with the usual definition of the Skorohod integral.

Now we will prove that the Skorohod integral is an extension of the stochastic integral, i.e. that each adapted $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$ is Skorohod integrable and that the Skorohod integral of φ coincides with its stochastic integral. This fact follows from a straightforward internal calculation.

9.3 Lemma

For each $n \in \mathbb{N}$ and for each adapted $F \in \mathcal{H}_{n,1}$ we have

$$I_{n+1}(\widetilde{F}) = \int I_{n,1}(F) \triangle \prod .$$

Proof. The statement holds even for $n = 0$, since

$$I_1(\widetilde{F}) = I(F) = \int I_{0,1}(F) \triangle \prod .$$

Now assume that $n \geq 1$. Since F is adapted,

$$I_{n+1}(\widetilde{F}) = \sum_{i_1<...<i_{n+1}} \sum_{k=1}^{n+1} F(b_{i_1}, ..., b_{i_{k-1}}, b_{i_{k+1}}, ..., b_{i_{n+1}}, b_{i_k}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n+1}} =$$

$$\sum_{i_1<...<i_{n+1}} F(b_{i_1}, ..., b_{i_{n+1}}) \cdot pr_{i_1} \cdot ... \cdot pr_{i_{n+1}} =$$

$$\sum_{i \in I} \sum_{i_1<...<i_n} F(b_{i_1}, ..., b_{i_n}, b_i) \cdot pr_{i_1} \cdot ... \cdot pr_{i_n} \cdot pr_i =$$

$$\sum_{i \in I} I_n(F(\,\cdot\,, b_i)) \cdot pr_i = \int I_{n,1}(F) \triangle \prod . \qquad \square$$

9.4 Lemma

We have $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H}) \subset \triangle$.

Proof. This follows from the fact that for each $n \in \mathbb{N}$ and for each adapted $F \in \mathcal{H}_{n,1}$ we have

$$\left\| {}^\circ I_{n+1}(\widetilde{F}) \right\|_2 = \| {}^\circ I_{n,1}(F) \|_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})} . \qquad \square$$

9.5 Proposition

Let $\varphi \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. Then $\delta\varphi = \int \varphi d\pi$.

Proof. Because of Proposition 7.3 there is a sequence $(F_n)_{n \in \mathbb{N}}$ of adapted functions $F_n \in \mathcal{H}_{n,1}$ such that

$$\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n) \text{ in } \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H}).$$

By the continuity of the stochastic integral, by the definition of the Skorohod integral and by Lemma 9.3 we obtain

$$\delta\varphi = \delta\sum_{n=0}^{\infty} {}^{\circ}I_{n,1}(F_n) = \sum_{n=0}^{\infty} {}^{\circ}I_{n+1}(\widetilde{F}_n) =$$

$$\sum_{n=0}^{\infty} {}^{\circ}\textstyle\int I_{n,1}(F_n) \bigtriangleup \prod = \displaystyle\sum_{n=0}^{\infty} \textstyle\int {}^{\circ}I_{n,1}(F_n) d\pi = \int \sum_{n=0}^{\infty} {}^{\circ}I_{n,1}(F_n) d\pi = \int \varphi d\pi \ .$$

$\square$

10 The Malliavin Derivative

Now we define the Malliavin derivative $D\varphi$ of suitable (i.e. differentiable) variables $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. For $F \in \mathcal{H}_m$, $m \geq 1$ we set $D\,{}^\circ I_m(F) := {}^\circ I_{m-1,1}(F^{\neq})$, where $F^{\neq}$ is a slight modification of F.

For $m \in \mathbb{N}$ and $F \in \mathcal{H}_m$ we define $F^{\neq} : \mathbb{F}^m \to {}^*\mathbb{R}$ by setting

$$F^{\neq}(x_1, ..., x_m) :=$$

$$\sum_{(i_1,...,i_m)\in I^m_{\neq}} < x_1, b_{i_1} > \cdot ... \cdot < x_m, b_{i_m} > \cdot F(b_{i_1}, ..., b_{i_m})\,.$$

(Note that $F^{\neq} = F$ for $m = 1$.) Since we have

$$\sum_{i_1<...<i_m} \left(F - F^{\neq}(b_{i_1}, ..., b_{i_m})\right)^2 = 0,$$

the function $F^{\neq}$ is in $\mathcal{H}_m$. Note further that for $\alpha, \beta \in {}^*\mathbb{R}$ and $G \in \mathcal{H}_n$ we have

$$(\alpha F + \beta G)^{\neq} = \alpha F^{\neq} + \beta G^{\neq}\,.$$

10.1 Lemma

If $F \in \mathcal{H}_m$ then $F^{\neq} \in \mathcal{H}_{m-1,1}$.

Proof. We can assume that $m \geq 2$. Set

$$\mathcal{G}_m := \left\{G \in \mathcal{H}_m \mid G^{\neq} \in \mathcal{H}_{m-1,1}\right\}.$$

The fact that, for $L \in \mathcal{H}_1$,

$$\sum_{\substack{i_1<...<i_{m-1}\\ i\in I}} \left((L^{\odot m})^{\neq} - L^{\odot m}\right)^2 (b_{i_1}, ..., b_{i_{m-1}}, b_i) \approx 0$$

implies that $L^{\odot m} \in \mathcal{G}_m$. Furthermore, $\mathcal{G}_m$ is an $\mathbb{R}$-linear space. Since for $G \in \mathcal{H}_m$ we have

$$\begin{aligned} \sum_{\substack{i_1<...<i_{m-1}\\ i\in I}} G^{\neq}(b_{i_1}, ..., b_{i_m-1}, b_i)^2 &= \\ \frac{1}{(m-1)!} \cdot \sum_{(i_1,...,i_m)\in I^m_{\neq}} G(b_{i_1}, ..., b_{i_m})^2 &= \\ m \cdot \sum_{i_1<...<i_m} G(b_{i_1}, ..., b_{i_m})^2\,, \end{aligned} \tag{11}$$

the space $\mathcal{G}_m$ is closed under S^m-limits. Therefore $\mathcal{G}_m$ equals $\mathcal{H}_m$. $\square$

Now set

$$\mathbb{D}^{1,2} := \left\{ \sum_{n=0}^{\infty} {}^\circ I_n(F_n) \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}) \mid \sum_{n=1}^{\infty} {}^\circ I_{n-1,1}(F_n^{\neq}) \text{ converges in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H}) \right\}$$

and define the Malliavin derivative by

$$\begin{aligned} D : \mathbb{D}^{1,2} &\to \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H}) \\ \sum_{n=0}^{\infty} {}^\circ I_n(F_n) &\mapsto \sum_{n=1}^{\infty} {}^\circ I_{n-1,1}(F_n^{\neq}) \,. \end{aligned}$$

By (11) the operator D is well defined. In Section 13 we will show that D coincides with the Malliavin derivative as it is defined in the literature.

11 Representation of Martingales

After introducing their new stochastic integral in [32], Üstünel and Zakai are showing that the well known result about the representation of martingales as stochastic integrals extends to the setup of abstract Wiener spaces. Using our infinitesimal approach, we obtain a new proof of this result.

We show that a function $m : \mathbb{F} \times [0,1] \to \mathbb{R}$ is a square integrable $(\mathcal{W}_t)$-martingale if and only if there is a $\varphi \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ such that

$$m_t = \mathbb{E}m_0 + \int \pi_t \varphi d\pi \quad \widehat{\mu}\text{-a.s.}$$

An internal martingale $M : \mathbb{F} \times T \to {}^*\mathbb{R}$ is called square S_μ-integrable if for each $k \in T$ the function M_k is in $SL^2(\mu)$. The process M is called S-continuous if for $\widehat{\mu}$-almost all $x \in \mathbb{F}$ the function

$$M(\,\cdot\,, x) : T \to {}^*\mathbb{R},\, t \mapsto M(t,x)$$

is S-continuous. The S-continuity of a process $N : \mathbb{F} \times I \to {}^*\mathbb{R}$ is defined analogously. The next proposition is about sufficient conditions for a process being a continuous $\mathcal{L}^2$-martingale. A detailed proof of this well known criterion can be found in [5] (Satz 1.5.3).

11.1 Proposition

Let $m : \mathbb{F} \times [0,1] \to \mathbb{R}$ be any mapping. Suppose that $M : \mathbb{F} \times T \to {}^*\mathbb{R}$ is an internal S-continuous and square S_μ-integrable $(\mathcal{C}_k)_{k \in T}$-martingale. Assume further that for each $k \in T$ and each $t \in [0,1]$ with $\frac{k}{H} \approx t$

$$M_k \approx m_t \qquad \widehat{\mu}\text{-a.s.}$$

Then m is a square $\widehat{\mu}$-integrable continuous $(\mathfrak{c}_t)_{t \in [0,1]}$-martingale.

In this context M is called a **lifting** of m. Now we fix an adapted nearstandard function $F \in SL^2(\mu, \mathbb{F})$.

11.2 Proposition

The internal process

$$M : \mathbb{F} \times I \ni (\,\cdot\,, j) \mapsto \sum_{i=1}^{j} < F, b_i > \cdot pr_i$$

is an S-continuous, square S_μ-integrable $(\mathcal{B}_i)_{i\in I}$-martingale.

Proof. By Proposition 8.1 the function M_j is square S_μ-integrable for each $j \in I$. Since F is adapted the process M is an internal $(\mathcal{B}_i)$-martingale. It remains to show that M is S-continuous. By a result of Hoover and Perkins (cf. [15]; see also [26], Theorem 8.15.1 for a detailed proof) it is sufficient to show that the process

$$[M] : \mathbb{F} \times I \ni (\,\cdot\,, j) \mapsto \sum_{i=1}^{j} < F, b_i >^2 \ \cdot pr_i^2$$

is S-continuous. Let E be the set which consists of all $x \in \mathbb{F}$ such that the following properties are fulfilled.

- (A) There is an $h \in \mathbb{H}$ such that $F(x) \approx h$,
- (B) $\sum_{i\in I} < x, b_i >^4 \ \cdot \ < F(x), b_i >^2 \in \mathrm{Lim}$,
- (C) $\sum_{i\in I} < F(x), b_i >^2 \in \mathrm{Lim}$ and
- (D) $\sum_{i\in I} < x, b_i >^2 \ \cdot \ < F(x), b_i >^2 \in \mathrm{Lim}$.

Note that $\widehat{\mu}(E) = 1$. Fix an $x \in E$. By (D), $[M](x, j) \in \mathrm{Lim}$ for each $j \in I$. Now fix $j_1 < j_2$ in I with $\frac{j_2 - j_1}{\omega} \approx 0$. We have to show that

$$\sum_{i=j_1+1}^{j_2} < F(x), b_i >^2 \ \cdot \ < x, b_i >^2 \approx \ 0\,.$$

Define an internal counting measure $\widetilde{\nu}$ on ${}^*\mathcal{P}(I)$ by setting

$$\widetilde{\nu}(\{i\}) := < F(x), b_i >^2\ .$$

Because of (C), $\widetilde{\nu}(I) \in \mathrm{Lim}$. Set $\kappa : I \ni i \mapsto < x, b_i >^2$. Because of (B), $\kappa \in SL^1(\widetilde{\nu})$. Because of (A) and because of Proposition 3.4

$$\widetilde{\nu}(\{j_1 + 1, ..., j_2\}) = \sum_{i=j_1+1}^{j_2} < F(x), b_i >^2 \approx 0.$$

Therefore,

$$\sum_{i=j_1+1}^{j_2} < F(x), b_i >^2 \ \cdot \ < x, b_i >^2 \ = \ \int_{j_1+1}^{j_2} \kappa d\widetilde{\nu} \approx 0\,. \qquad \square$$

For $k \in T$ we set

$$\textstyle\int_1^k F \bigtriangleup \prod := \displaystyle\sum_{i=1}^{\sigma(k)} \ < F, b_i > \ \cdot pr_i\,.$$

11.3 Proposition

The internal process

$$\left(\int_1^k F \bigtriangleup \prod\right)_{k\in T}$$

is an S-continuous, square S_μ-integrable $(\mathcal{C}_k)_{k\in T}$-martingale.

Proof. It remains to show the S-continuity of the process. Set

$$M : \mathbb{F} \times I \ni (\,\cdot\,, j) \mapsto \sum_{i=1}^{j} < F, b_i > \cdot pr_i \ .$$

Due to Proposition 11.2, M is S-continuous. Fix an $x \in \mathbb{F}$ such that

- $M(x,i) \in \mathrm{Lim}$ for each $i \in I$ and
- $M(x,i) \approx M(x,j)$ for $i,j \in I$ with $\frac{j-i}{\omega} \approx 0$.

Since for each $k \in T$ we have $\frac{\sigma(k)}{\omega} = \frac{k}{H}$ it follows that

- $\int_1^k F \bigtriangleup \prod(x) \in \mathrm{Lim}$ for each $k \in T$ and
- $\int_1^k F \bigtriangleup \prod(x) \approx \int_1^l F \bigtriangleup \prod(x)$ for $k,l \in T$ with $\frac{l-k}{H} \approx 0$. □

Now we can show that the stochastic integral $\int f d\pi$ corresponds to a martingale. Therefore we must equip the integral with a notion of time. Again this will happen with the resolution of the identity. Note that for $\varphi \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ and $t \in [0,1]$ the function $\pi_t f$ is in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$.

11.4 Proposition

For each $f \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$ the process $(\int \pi_t f d\pi)$ has a lifting in the sense of Proposition 11.1 and therefore is a continuous $\mathcal{L}^2$-martingale with respect to $(\mathfrak{c}_t)$. Furthermore, $(\int \pi_t f d\pi)$ is adapted to $(\mathcal{W}_t)$.

Proof. Let F be an adapted $SL^2(\mu, \mathbb{F})$-lifting of f. Then $\left(\int_1^k F \bigtriangleup \prod\right)_{k\in T}$ is S-continuous. We will show that $\left(\int_1^k F \bigtriangleup \prod\right)_{k\in T}$ is a lifting of $\left(\int \pi_t f d\pi\right)_{t\in[0,1]}$. Therefore fix $k \in T$ and $t \in [0,1]$ with $\frac{k}{H} \approx t$. We will show that $\int_1^k F \bigtriangleup \prod$ is a lifting of $\int \pi_t f d\pi$. Set $\widehat{F} := \sum_{i=1}^{\sigma(k)} < F, b_i > \cdot b_i$. Note that $\widehat{F}$ is adapted and in $SL^2(\mu, \mathbb{F})$ and that $\int_1^k F \triangle \prod = \int \widehat{F} \triangle \prod$. We show that $\widehat{F}$ is an $SL^2(\mu, \mathbb{F})$-lifting of $\pi_t f$. Take an $x \in \mathbb{F}$ such that $f(x) \approx F(x)$. Then Proposition 3.3 implies

$$\widehat{F}(x) = pr_{\{1,\dots,\sigma(k)\}} F(x) \approx pr_{\{1,\dots,\sigma(k)\}} f(x) \approx \pi_t f(x)\,.$$

The second assertion follows from Lemma 8.2. $\square$

Now we are going to show that every square integrable $(\mathcal{W}_t)$-martingale can be written as a stochastic integral. For this purpose we fix a process

$$m : \mathbb{F} \times [0,1] \to \mathbb{R}$$

which is a square integrable $(\mathcal{W}_t)$-martingale. We further fix a sequence $(F_n)_{n\in\mathbb{N}}$ with $F_n \in \mathcal{H}_n$ such that

$$m_1 = \mathbb{E} m_1 + \sum_{n=1}^{\infty} {}^\circ I_n(F_n) \text{ in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})\,.$$

11.5 Lemma

For $n \in \mathbb{N}$ we have $\widehat{\mu}$-a.s.

$$^\circ I_n(F_n) = \int {}^\circ I_{n-1,1}(F_n^<) d\pi$$

which, by Lemma 8.2, implies that

$$\|^\circ I_n(F_n)\|_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})} = \|^\circ I_{n-1,1}(F_n^<)\|_{\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H})}\,.$$

Proof. We obtain $\widehat{\mu}$-a.s.

$$^\circ I_n(F_n) \approx I_n(F_n) = \sum_{i_1<\dots<i_n} F_n(b_{i_1},\dots,b_{i_n}) \cdot pr_{i_1} \cdot \ldots \cdot pr_{i_n} =$$

$$\sum_{i_1<\dots<i_n} F_n^<(b_{i_1},\dots,b_{i_n}) \cdot pr_{i_1} \cdot \ldots \cdot pr_{i_n} =$$

$$\sum_{i\in I} \sum_{i_1<\dots<i_{n-1}} F_n^<(b_{i_1},\dots,b_{i_{n-1}},b_i) \cdot pr_{i_1} \cdot \ldots \cdot pr_{i_{n-1}} \cdot pr_i =$$

$$\sum_{i\in I} I_{n-1}(F_n^<(\,\cdot\,,b_i)) \cdot pr_i = \int I_{n-1,1}(F_n^<) \triangle \prod \approx \int {}^\circ I_{n-1,1}(F_n^<) d\pi\,. \qquad \square$$

Therefore the sequence $\left(\sum_{n=1}^m {}^\circ I_{n-1,1}(F_n^<)\right)_{m\in\mathbb{N}}$ converges in $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H})$. Denote its limit by ψ.

11.6 Proposition

The variable m_1 *has the representation* $m_1 = \mathbb{E}m_1 + \int \psi d\pi$ *in* $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$.

Proof. By the continuity of the stochastic integral we obtain $\widehat{\mu}$-a.s.

$$\mathbb{E}m_1 + \int \psi d\pi = \mathbb{E}m_1 + \int \sum_{n=1}^{\infty} {}^{\circ}I_{n-1,1}(F_n^{<}) d\pi =$$

$$\mathbb{E}m_1 + \sum_{n=1}^{\infty} \int {}^{\circ}I_{n-1,1}(F_n^{<}) d\pi = \mathbb{E}m_1 + \sum_{n=1}^{\infty} {}^{\circ}I_n(F_n) = m_1 \ . \qquad \square$$

11.7 Proposition

For each $t \in [0,1]$ *we have*

$$m_t = \mathbb{E}m_1 + \int \pi_t \psi d\pi \ \ in \ \ \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}).$$

Proof. This follows immediately from Proposition 11.4 and from Proposition 11.6. Just take expectations. $\square$

As a consequence we obtain that each square integrable $(\mathcal{W}_t)$-martingale is continuous. This corresponds to the well-known fact that a Brownian filtration does not admit a discontinuous martingale (cf. Theorem 3.4 in Chapter V of [28]).

12 The Clark Ocone Formula

The Clark Ocone formula provides a connection between the Malliavin derivative and the stochastic integral. It can be regarded as the stochastic version of the fundamental theorem of calculus. This formula is well known in the case of the classical Wiener space, see Proposition 1.3.5 in [23]. Our version for abstract Wiener spaces is, as far as we know, new and constitutes the main result of these notes.

12.1 Proposition

For $\varphi \in \mathbb{D}^{1,2}$ we have

$$\varphi = \mathbb{E}\varphi + \int pr_{\mathcal{A}}^{\mathcal{W}}\,(D\varphi)\, d\pi\,.$$

Proof. Since both, the projection operator $pr_{\mathcal{A}}^{\mathcal{W}}$ and the stochastic integral are linear and continuous operators and by the definition of the Malliavin derivative, it suffices to show that for a fixed $n \in \mathbb{N}$ and for any $F \in \mathcal{H}_n$ we have

$$^{\circ}I_n(F) = \int pr_{\mathcal{A}}^{\mathcal{W}}\,(D\,^{\circ}I_n(F))\, d\pi \ \text{ in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}).$$

From Proposition 7.3 and from

$$\sum_{\substack{i_1<\ldots<i_n\\ i\in I}} \left((F^{\neq})^{<} - F^{<}\right)^2 (b_{i_1},\ldots,b_{i_n},b_i) = 0$$

it follows that

$$pr_{\mathcal{A}}^{\mathcal{W}}\,(D\,^{\circ}I_n(F)) \ = \ ^{\circ}I_{n-1,1}(F^{<}) \ \text{ in } \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H}).$$

By Lemma 11.5 we obtain

$$\int pr_{\mathcal{A}}^{\mathcal{W}}\,(D\,^{\circ}I_n(F))\, d\pi = \int {}^{\circ}I_{n-1,1}(F^{<})d\pi = \ ^{\circ}I_n(F) \ \text{ in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})\,. \qquad \square$$

Proposition 11.6 yields that each $\varphi \in \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})$ has the representation

$$\varphi = \mathbb{E}\varphi + \int \psi d\pi$$

for a $\psi \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu},\mathbb{H})$. If $\varphi \in \mathbb{D}^{1,2}$ then Proposition 12.1 implies that ψ equals the projection of the derivative of φ onto the space of adapted functions. The common ground of these two propositions is the formula

$$^{\circ}I_n(F) = \int {}^{\circ}I_{n-1,1}(F^{<})d\pi\,.$$

13 Reference to Abstract Wiener Spaces

In this section we show that the operators D and δ are indeed the operators of the Malliavin calculus. Furthermore we will show that the integral $\int f d\pi$ is the stochastic integral of Üstünel and Zakai. Since this reference to the standard theory does not influence the correctness of the theory we have established up to now, some of the proofs are omitted.

In order to get a relation to the standard world of abstract Wiener spaces, we must identify the spaces $\mathcal{L}^2(P)$ and $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. We also must identify the spaces $\mathcal{L}^2(P, \mathbb{H})$ and $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$. This is possible because of the choice of the σ-algebra $\mathcal{W}$. Remember the definition of the divergence operator $\delta_{\mathbb{H}}$ in Section 2.

13.1 Proposition

The σ-algebra $\mathcal{W}$ is generated by the mapping St, i.e. $St^{-1}(\mathfrak{b}_{\mathbb{B}}) \vee \mathcal{N}_{\widehat{\mu}} = \mathcal{W}$. Furthermore, the mappings

$$\mathcal{L}^2(P) \to \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}), \;\; f \mapsto f \circ St$$

$$\mathcal{L}^2(P, \mathbb{H}) \to \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H}), \;\; g \mapsto g \circ St$$

are surjective isometries. Furthermore, we have

$$\delta_{\mathbb{H}}(h) \circ St = {}^{\circ}I(h) \tag{12}$$

for each $h \in \mathbb{H}$.

Proof. We only show (12). First assume that there is a $\varphi \in \mathbb{B}'$ such that $\varphi(y) = < h, y >$ for each $y \in \mathbb{H}$. Fix further an $x \in Ns(\mathbb{F})$. (See Section 2.) By the continuity of φ we obtain

$$\delta_{\mathbb{H}}(h) \circ \mathrm{St}(x) = \varphi(\mathrm{St}(x)) \approx {}^{*}\varphi(x) = < h, x > = I(h)(x) \approx {}^{\circ}I(h)(x)\,.$$

Since $\mathbb{B}'$ is dense in $\mathbb{H}$, the result is also true for an arbitrary $h \in \mathbb{H}$. $\square$

Now we sum up the introduction to the Malliavin derivative as it is presented in Appendix B of [33]. Let $\mathcal{S}(\mathbb{R}^n)$ denote the **functions of rapid decrease**. (See [27], page 133.) We call a function $f \in \mathcal{L}^2(P)$ **cylindrical** if

$$f = F(\delta_{\mathbb{H}}(h_1), ..., \delta_{\mathbb{H}}(h_n))$$

for an $F \in \mathcal{S}(\mathbb{R}^n)$ and for $h_1, ..., h_n \in \mathbb{H}$. We define the derivative $\widetilde{D}f$ of such a cylindrical function by

$$\widetilde{D}f := \sum_{i=1}^{n} \partial_i F(\delta_{\mathbb{H}}(h_1), ..., \delta_{\mathbb{H}}(h_n)) \cdot h_i,$$

where $\partial_i F$ denotes the i-th partial derivative of F. The Cameron-Martin theorem implies that $\widetilde{D}f_1 = \widetilde{D}f_2$ in $\mathcal{L}^2(P, \mathbb{H})$ if $f_1 = f_2$ in $\mathcal{L}^2(P)$. The operator $\widetilde{D}$ is **closable**: if a sequence (f_n) of cylindrical functions converges to zero in $\mathcal{L}^2(P)$ and if $(\widetilde{D}f_n)$ is a Cauchy sequence in $\mathcal{L}^2(P, \mathbb{H})$, then $(\widetilde{D}f_n)$ converges also to zero. Therefore we can extend the operator $\widetilde{D}$ to the space

$$\widetilde{\mathbb{D}^{1,2}} :=$$

$$\left\{ f \in \mathcal{L}^2(P) \mid \exists (f_n) \text{ such that } f_n \to f \text{ and such that } (\widetilde{D}f_n) \text{ is Cauchy} \right\}$$

by setting

$$\widetilde{D}f := \lim_{n \to \infty} \widetilde{D}f_n$$

if the sequence of cylindrical functions (f_n) converges to f and $(\widetilde{D}f_n)$ is a Cauchy sequence.

Now we show that the operator $\widetilde{D}$ corresponds to the operator D we have defined in Section 10. To this end we need the following result about the derivative D.

13.2 Proposition

(A) *Suppose that $h_1, ..., h_n \in \mathbb{H}$ and fix a function $F \in \mathcal{S}(\mathbb{R}^n)$. Then $F({}^\circ I(h_1), ..., {}^\circ I(h_n)) \in \mathbb{D}^{1,2}$ and*

$$DF({}^\circ I(h_1), ..., {}^\circ I(h_n)) = \sum_{i=1}^{n} \partial_i F({}^\circ I(h_1), ..., {}^\circ I(h_n)) \cdot h_i.$$

(B) *Let (φ_n) converge to φ in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$ and suppose that each $\varphi_n \in \mathbb{D}^{1,2}$. Assume further that $(D\varphi_n)$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}, \mathbb{H})$. Then $\varphi \in \mathbb{D}^{1,2}$ and $D\varphi_n \to D\varphi$.*

13.3 Proposition

Suppose that $f \in \widetilde{\mathbb{D}^{1,2}}$ and set $\varphi := f \circ St$. Then $\varphi \in \mathbb{D}^{1,2}$ and $D\varphi = \widetilde{D}f \circ St$.

Proof. First suppose that f is cylindrical, i.e. that

$$f = F(\delta_{\mathbb{H}}(h_1), ..., \delta_{\mathbb{H}}(h_n))$$

for an $F \in \mathcal{S}(\mathbb{R}^n)$ and for $h_1, ..., h_n \in \mathbb{H}$. Since $\delta_{\mathbb{H}}(h_i) \circ \mathrm{St} = {}^\circ I(h_i)$, we obtain $\varphi = F({}^\circ I(h_1), ..., {}^\circ I(h_n))$. Thus $\varphi \in \mathbb{D}^{1,2}$ and

$$D\varphi = \sum_{i=1}^{n} \partial_i F({}^\circ I(h_1), ..., {}^\circ I(h_n)) \cdot h_i =$$

$$\left(\sum_{i=1}^{n} \partial_i F(\delta_{\mathbb{H}}(h_1), ..., \delta_{\mathbb{H}}(h_n)) \cdot h_i \right) \circ \mathrm{St} = \widetilde{D} f \circ \mathrm{St}.$$

Now suppose that f is a genuine element of $\widetilde{\mathbb{D}^{1,2}}$. Fix a sequence (f_n) of cylindrical functions, converging to f such that $(\widetilde{D} f_n)$ is a Cauchy sequence. Set $\varphi_n = f_n \circ \mathrm{St}$ for $n \in \mathbb{N}$. Then $\varphi_n \to \varphi$ and $(D\varphi_n)$ is a Cauchy sequence. Thus $\varphi \in \mathbb{D}^{1,2}$ and

$$D\varphi = \lim_{n\to\infty} D\varphi_n = \lim_{n\to\infty} (\widetilde{D} f_n \circ \mathrm{St}) = (\lim_{n\to\infty} \widetilde{D} f_n) \circ \mathrm{St} = \widetilde{D} f \circ \mathrm{St}. \qquad \square$$

Now we sketch the definition of the Skorohod integral, again following the presentation in Appendix B of [33]. The subspace $\widetilde{\triangle}$ of $\mathcal{L}^2(P, \mathbb{H})$ is defined as followed: a function $g \in \mathcal{L}^2(P, \mathbb{H})$ is in $\widetilde{\triangle}$ if and only if there is a constant $c \in \mathbb{R}$ such that for each $f \in \widetilde{\mathbb{D}^{1,2}}$ the estimation

$$\left| < \widetilde{D} f, g >_{\mathcal{L}^2(P,\mathbb{H})} \right| \ \leq \ \|f\|_2 \cdot c$$

holds. For each such g the assignment

$$\widetilde{\mathbb{D}^{1,2}} \to \mathbb{R}, \ f \mapsto < \widetilde{D} f, g >_{\mathcal{L}^2(P,\mathbb{H})}$$

is a bounded operator, which, since $\widetilde{\mathbb{D}^{1,2}}$ is dense in $\mathcal{L}^2(P)$, implies that there is an element of $\mathcal{L}^2(P)$, denoted by $\delta_{\mathbb{H}} g$ such that

$$< \widetilde{D} f, g >_{\mathcal{L}^2(P,\mathbb{H})} = < f, \delta_{\mathbb{H}} g >_2$$

for all $f \in \widetilde{\mathbb{D}^{1,2}}$. This gives rise to a function

$$\delta_{\mathbb{H}} : \widetilde{\triangle} \to \mathcal{L}^2(P).$$

The operator $\delta_{\mathbb{H}}$ is called the Skorohod integral. Later we will see that for a $h \in \mathbb{H}$ and for

$$g : \mathbb{F} \to \mathbb{H}, \ x \mapsto h$$

we have $\delta_{\mathbb{H}} g = \delta_{\mathbb{H}}(h)$.

Now we show that $\delta_{\mathbb{H}}$ corresponds to the operator δ we have introduced in Section 9. We want to make use of the following fact.

13.4 Proposition

For $\varphi \in \mathbb{D}^{1,2}$ and $\psi \in \triangle$ we have $< \varphi, \delta\psi >_2 = < D\varphi, \psi >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})}$.

13.5 Proposition

Fix a $g \in \widetilde{\triangle}$ and set $\psi := g \circ St$. Then $\psi \in \triangle$ and $\delta\psi = \delta_{\mathbb{H}} g \circ St$.

Proof. Let ψ have the decomposition $\psi = \sum_{n=0}^{\infty} {}^{\circ}I_{n,1}(F_n)$ with $F_n \in \mathcal{H}_{n,1}$. By the definition of $\triangle$, we have to show that the series

$$\sum_{n=0}^{\infty} {}^{\circ}I_{n+1}(\widetilde{F}_n) \tag{13}$$

converges in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$. Since $g \in \widetilde{\triangle}$, there is a $c \in \mathbb{R}$ such that

$$\left| < \widetilde{D} f, g >_{\mathcal{L}^2(P,\mathbb{H})} \right| \ \leq \ c \cdot \|f\|_2$$

for all $f \in \widetilde{\mathbb{D}^{1,2}}$. This implies that

$$\left| < D\varphi, \psi >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} \right| \ \leq \ c \cdot \|\varphi\|_2$$

for all $\varphi \in \mathbb{D}^{1,2}$. Thus we obtain for each $m \in \mathbb{N}$

$$\left\| \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n) \right\|_2^2 = < \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n), \delta \sum_{n=0}^{m} {}^{\circ}I_{n,1}(F_n) >_2 =$$

$$< D \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n), \sum_{n=0}^{m} {}^{\circ}I_{n,1}(F_n) >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} =$$

$$< D \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n), \psi >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} \leq \ c \cdot \left\| \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n) \right\|_2 .$$

But this implies that

$$\left\| \sum_{n=0}^{m} {}^{\circ}I_{n+1}(\widetilde{F}_n) \right\|_2 \ \leq \ c,$$

and this estimation does not depend on m, therefore the series (13) converges. It remains to show that

$$\delta\psi = \delta_{\mathbb{H}} g \circ \mathrm{St}.$$

Since $\mathbb{D}^{1,2}$ is dense in $\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})$, it suffices to show that

$$< \delta\psi, \varphi >_2 = < \delta_{\mathbb{H}} g \circ \mathrm{St}, \varphi >_2$$

for each $\varphi \in \mathbb{D}^{1,2}$. Fix such a φ and let $f \in \widetilde{\mathbb{D}^{1,2}}$ with $f \circ \mathrm{St} = \varphi$. We obtain

$$< \delta\psi, \varphi >_2 = < \psi, D\varphi >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} = < g \circ \mathrm{St}, \widetilde{D} f \circ \mathrm{St} >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} =$$

$$< g, \widetilde{D} f >_{\mathcal{L}^2(P,\mathbb{H})} = < \delta_{\mathbb{H}} g, f >_2 = < \delta_{\mathbb{H}} g \circ \mathrm{St}, \varphi >_2 . \qquad \square$$

The next lemma states a relation between functions of the kind $\delta_{\mathbb{H}} g$ and functions of the kind $\delta_{\mathbb{H}}(h)$.

13.6 Lemma

Fix a $h \in \mathbb{H}$ and set $g : \mathbb{B} \to \mathbb{H}$, $x \mapsto h$. Then $g \in \widetilde{\triangle}$ and $\delta_{\mathbb{H}} g = \delta_{\mathbb{H}}(h)$ in $\mathcal{L}^2(P)$.

Proof. Since for $f \in \widetilde{\mathbb{D}^{1,2}}$ we have

$$< \widetilde{D} f, g >_{\mathcal{L}^2(P,\mathbb{H})} = < \widetilde{D} f \circ St, g \circ St >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} = < D(f \circ St), h >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu},\mathbb{H})} =$$

$$< f \circ St, \delta h >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})} = < f \circ St, {}^{\circ}I(h) >_{\mathcal{L}^2_{\mathcal{W}}(\widehat{\mu})} \leq \|f\|_{\mathcal{L}^2(P)} \, \|h\|_{\mathbb{H}} \, ,$$

we obtain $g \in \widetilde{\triangle}$. And since

$$\delta_{\mathbb{H}} g \circ \mathrm{St} \;=\; \delta(g \circ \mathrm{St}) \;=\; \delta({}^{\circ}I_{0,1}(h)) \;=\; {}^{\circ}I_1(h) \;=\; \delta_{\mathbb{H}}(h) \circ \mathrm{St} \;\text{ in } \mathcal{L}^2_{\mathcal{W}}(\widehat{\mu}),$$

we obtain the desired equality

$$\delta_{\mathbb{H}} g = \delta_{\mathbb{H}}(h) \;\text{ in } \mathcal{L}^2(P) . \qquad \square$$

Now we sketch the definition of the stochastic integral on abstract Wiener spaces due to Üstünel and Zakai (cf. [32]), which is based on a resolution of the identity $(\pi_t)_{t\in[0,1]}$. Both, the integrands and the integrator live on an arbitrary filtered probability space $(\Lambda, \mathcal{C}, (\mathcal{C}_t)_{t\in[0,1]}, \nu)$. We fix a function

$$b : (\Lambda, \mathcal{C}) \to (\mathbb{B}, \mathfrak{b}_{\mathbb{B}})$$

with $\upsilon \circ b^{-1} = P$. Let us define yet another **divergence operator** by setting

$$\begin{aligned} \delta_\nu : \mathbb{H} &\to \mathcal{L}^2(\nu) \, , \\ h &\mapsto \delta_{\mathbb{H}}(h) \circ b \, . \end{aligned}$$

Note that δ_ν inherits the characteristic properties of $\delta_{\mathbb{H}}$ such as continuity and linearity. The mapping b is called **abstract Wiener process** if for each $f \in \mathbb{B}'$ the process $(\delta_\nu(\pi_t f))$ is a martingale with respect to $(\mathcal{C}_t)$. More detailed we say that the triple $(b, \mathbb{H}, (\pi_t))$ is an abstract Wiener process on $(\Lambda, \mathcal{C}, (\mathcal{C}_t), \nu)$. Let the space $\mathcal{L}^2_{\mathcal{A}}(\nu, \mathbb{H})$ be defined in analogy to $\mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$. Again we call a function $\psi \in \mathcal{L}^2_{\mathcal{A}}(\nu, \mathbb{H})$ **simple adapted** if

$$\psi = \varphi \cdot (\pi_t - \pi_s)h$$

for $s < t$ in $[0,1]$, $\varphi \in \mathcal{L}^2(\Lambda, \mathcal{C}_s, \nu)$ and $h \in \mathbb{H}$. For such functions ψ we set

$$\delta'_\nu \psi := \varphi \cdot \delta_\nu((\pi_t - \pi_s)h). \tag{14}$$

By the properties of abstract Wiener processes, the function $\delta_\nu((\pi_t - \pi_s)h)$ is independent of $\mathcal{C}_s$. (See page 132 in [32].) This implies that the functions φ and $\mathcal{C}_s$ are independent, therefore $\delta'_\nu \psi$ is square ν-integrable. Note further that

$$\|\delta'_\nu \psi\|_2 = \|\psi\|_{\mathcal{L}^2_{\mathcal{A}}(\nu, \mathbb{H})}.$$

Therefore and since the linear combinations of adapted functions are dense in $\mathcal{L}^2(\nu, \mathbb{H})$, there exists a uniquely determined linear and norm preserving operator

$$\delta'_\nu : \mathcal{L}^2_{\mathcal{A}}(\nu, \mathbb{H}) \to \mathcal{L}^2(\nu)$$

that fulfills (14) for each simple adapted function ψ. This is the stochastic integral of Üstünel and Zakai. The next proposition makes it clear that what we have defined in Section 8 coincides with this integral.

13.7 Proposition

The triple $(St, \mathbb{H}, (\pi_t))$ is an abstract Wiener process on $(\mathbb{F}, \mathcal{W}, (\mathcal{W}_t), \widehat{\mu})$. Furthermore, we have

$$\delta'_{\widehat{\mu}} \psi = \int \psi d\pi \tag{15}$$

for each $\psi \in \mathcal{L}^2_{\mathcal{A}}(\widehat{\mu}, \mathbb{H})$.

Proof. By Proposition 2.6 and Proposition 13.1 the map St is measurable and measure preserving. We show that for each $h \in \mathbb{H}$ the process $(\delta_{\widehat{\mu}}(\pi_t h))$ is a $(\mathcal{W}_t)$-martingale. Therefore fix $h \in \mathbb{H}$ and $t \in [0,1]$ and set

$$\begin{aligned} g &: \mathbb{F} \to \mathbb{H},\ x \mapsto h\ , \\ g_t &: \mathbb{F} \to \mathbb{H},\ x \mapsto \pi_t h \quad \text{and} \\ f_t &: \mathbb{B} \to \mathbb{H},\ x \mapsto \pi_t h\,. \end{aligned}$$

We obtain

$$\delta_{\widehat{\mu}}(\pi_t h) = \delta_{\mathbb{H}}(\pi_t h) \circ St \overset{(i)}{=} (\delta_{\mathbb{H}} f_t) \circ St \overset{(ii)}{=} \delta(f_t \circ St) = \delta g_t \overset{(iii)}{=} \int \pi_t g d\pi \,,$$

where (i) follows from Lemma 13.6, (ii) follows from Proposition 13.5 and (iii) follows from Proposition 9.5. It remains to show (15). Since both integrals are linear and norm preserving, it is sufficient to show this equation for simple integrands ψ. Therefore fix

$$\psi = \varphi \cdot (\pi_t - \pi_s) h$$

with $s < t$ in $[0,1]$, $\varphi \in \mathcal{L}^2(\mathbb{F}, \mathcal{W}_s, \widehat{\mu})$ and $h \in \mathbb{H}$. Fix further indices k, l in T such that $\frac{k}{H} \approx s$ and $\frac{l}{H} \approx t$ and a $\mathcal{B}_k$-measurable SL^2-lifting Φ of φ. Then

$$\Psi := \Phi \cdot (\prod_l - \prod_k) h \, \in SL^2(\mu, \mathbb{F})$$

is an adapted lifting of ψ. We obtain $\widehat{\mu}$-a.s.

$$\int \psi d\pi \approx \int \Psi \bigtriangleup \prod = \sum_{i \in I} < \Psi, b_i > \cdot pr_i =$$

$$\Phi \cdot \sum_{i \in I} < (\prod_l - \prod_k) h, b_i > \cdot pr_i = \Phi \cdot I((\prod_l - \prod_k) h) \approx$$

$$\varphi \cdot {}^\circ I((\pi_t - \pi_s) h) = \varphi \cdot \delta_{\mathbb{H}}((\pi_t - \pi_s) h) \circ St = \varphi \cdot \delta_{\widehat{\mu}}((\pi_t - \pi_s) h) = \delta'_{\widehat{\mu}} \psi \,. \qquad \square$$

Given any stochastic basis $(\Lambda, \mathcal{C}, (\mathcal{C}_t)_{t \in [0,1]}, \nu)$ and any abstract Wiener process b we can carry out the Malliavin calculus on $\mathcal{L}^2_{\mathcal{C}}(\nu)$ and $\mathcal{L}^2_{\mathcal{C}}(\nu, \mathbb{H})$ if $\mathcal{C}$ is generated by b. This follows from the fact that each abstract Wiener process is measure preserving. In this situation the question concerning the validity of the Clark Ocone formula arises. We have seen that in the setup of this thesis the answer is 'yes'. It remains to investigate if this result depends on the choice of the stochastic basis.

14 Appendix: an Internal Representation of the Lévy Transformation of Brownian Motion

In this self-contained section we construct an internal transformation τ on a hyperfinite dimensional Euclidean space and show that τ is ergodic if and only if L is ergodic, where L denotes the Lévy transformation of Brownian motion. As, in opposition to L, the mapping τ is constructively given, this approach allows to see the open problem whether L is ergodic from a different angle.

Let $\mathfrak{b}_{C[0,1]}$ be the Borel σ-algebra on $C[0,1]$ and let W be Wiener-measure on $\mathfrak{b}_{C[0,1]}$. Furthermore let ψ be the canonical Brownian motion on the classical Wiener-space $(C[0,1], \mathfrak{b}_{C[0,1]}, W)$, i.e.

$$\psi : C[0,1] \times [0,1] \ni (x,t) \mapsto x(t) \ .$$

Define a measure-preserving transformation L by

$$C[0,1] \ni \omega \mapsto L(\omega) := \int_0^{\cdot} \operatorname{sgn}(\psi)\, d\,\psi\,(\omega) \in C[0,1]\,,$$

where the sgn-mapping is defined as usual by

$$\operatorname{sgn} : \mathbb{R} \ni x \mapsto \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} .$$

The map L is referred to as the **Lévy transformation of Brownian motion.** It is an open problem whether L is ergodic, i.e. whether for each $A \in \mathfrak{b}_{C[0,1]}$ the implication

$$A \text{ is } L\text{-invariant} \quad \Longrightarrow \quad W(A) \in \{0,1\}$$

is valid, where A is called L-**invariant** if $W(A \triangle L^{-1}(A)) = 0$. This question is mentioned for example in [3], [10], [22], [28], [30] and [34]. In [11], Dubins, Émery and Yor have introduced a condition in terms of time-changing a Brownian motion and show that this condition is equivalent to L being ergodic; in [12], Dubins and Smorodinsky have proven that a discrete analogue of L is ergodic.

There is an infinitesimal approach to the Malliavin calculus (on the classical Wiener space) due to Cutland, Ng (see [8]) and Osswald (see [20]) that replaces $C[0,1]$ by a hyperfinite dimensional Euclidean space. Our aim is to search for a transformation in this internal setting which corresponds to L.

We want the internal counterpart of the mapping sgn, denoted by Sgn, to take only values in $\{-1,1\}$ and therefore define

$$\mathrm{Sgn} : {}^*\mathbb{R} \ni x \mapsto \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Let $H \in {}^*\mathbb{N}$ be **unlimited**, i.e. $H > n$ for every $n \in \mathbb{N}$. Define $T := \{1, .., H\}$ and $\Omega := {}^*\mathbb{R}^H$. For $k \in T$ let

$$\pi_k : \Omega \ni X = (X_1, ..., X_H) \mapsto X_k$$

be the projection onto the k-th component. Let $\mathfrak{b}_\Omega$ be the internal Borel ${}^*\sigma$-algebra on Ω. There is exactly one internal probability measure such that the projections π_k are independent and normal distributed with mean 0 and variance $\frac{1}{H}$. Denote this measure by Γ. This construction of Cutland and Ng gives rise to an internal probability space $(\Omega, \mathfrak{b}_\Omega, \Gamma)$. Define $\tau : \Omega \to \Omega$ by

$$\tau(X)_t := \begin{cases} X_t & \text{if } t = 1 \\ \mathrm{Sgn}(\sum_{i=1}^{t-1} X_i) X_t & \text{if } t \in \{2, ..., H\} \end{cases}.$$

14.1 Lemma

The mapping $\tau : (\Omega, \mathfrak{b}_\Omega, \Gamma) \to (\Omega, \mathfrak{b}_\Omega, \Gamma)$ is bijective and measure preserving.

Proof. Fix X,$Y \in \Omega$ with $X \neq Y$. Set $k := \min\{l \in T \mid X_l \neq Y_l\}$. Obviously, $\tau(X)_k \neq \tau(Y)_k$ and therefore $\tau(X) \neq \tau(Y)$, thus τ is injective. Now let $Y \in \Omega$. Define $X_1 := Y_1,\ X_{n+1} := \mathrm{Sgn}(\sum_{i=1}^{n} X_i) Y_{n+1}$ and verify that $\tau(X) = Y$. Hence τ is onto. Since the projections π_t and the function Sgn are measurable, τ is measurable. Finally, prove that $(\pi_t \circ \tau)_{t \leq H}$ is a sequence of independent variables and that each $\pi_t \circ \tau$ is normal distributed with mean 0 and variance $\frac{1}{H}$, which implies that $\Gamma = \Gamma \circ \tau^{-1}$. $\square$

14.2 Lemma

The transformation $\tau : (\Omega, \mathfrak{b}_\Omega, \Gamma) \to (\Omega, \mathfrak{b}_\Omega, \Gamma)$ is not ergodic.

Proof. Ergodicity fails, since the set $\{\pi_1 < 0\}$ is τ-invariant but has measure 1/2. $\square$

Let $(\Omega, L_\Gamma(\mathfrak{b}_\Omega), \widehat{\Gamma})$ be the Loeb space of $(\Omega, \mathfrak{b}_\Omega, \Gamma)$. Set

$$\mathcal{N}_{\widehat{\Gamma}} := \left\{ A \in L_\Gamma(\mathfrak{b}_\Omega) \mid \widehat{\Gamma}(A) = 0 \right\}.$$

The next lemma is an immediate consequence of Lemma 14.1 and of the definition of Loeb spaces.

14.3 Lemma

The transformation $\tau : (\Omega, L_\Gamma(\mathfrak{b}_\Omega), \widehat{\Gamma}) \to (\Omega, L_\Gamma(\mathfrak{b}_\Omega), \widehat{\Gamma})$ *is bijective and measure-preserving.*

After introducing a sub-σ-field $\mathcal{W}$ of $L_\Gamma(\mathfrak{b}_\Omega)$ that corresponds to $\mathfrak{b}_{C[0,1]}$, we will show that L is ergodic if and only if

$$\tau : (\Omega, \mathcal{W}, \widehat{\Gamma}) \to (\Omega, \mathcal{W}, \widehat{\Gamma})$$

is ergodic. This approach allows a new view on the Lévy transformation, since τ is bijective and defined pointwise, whereas

$$L(x) = L(-x)$$

for W-almost all $x \in C[0,1]$.
We will use the following lemma about stochastic integrals.

14.4 Lemma

Fix two probability spaces $(\Lambda, \mathcal{C}, \mu)$, $(\Lambda', \mathcal{C}', \mu')$ *and a measure-preserving mapping*

$$\kappa : (\Lambda, \mathcal{C}, \mu) \to (\Lambda', \mathcal{C}', \mu').$$

Let $(\mathcal{C}_t)_{t\in[0,1]}$ *and* $(\mathcal{C}'_t)_{t\in[0,1]}$ *be filtrations in* $\mathcal{C}$, $\mathcal{C}'$ *respectively such that*

$$\mathcal{C}_t = \kappa^{-1}(\mathcal{C}'_t) \vee \mathcal{N}_\mu$$

for each $t \in [0,1]$. *Assume further that*

$$X : \Lambda' \times [0,1] \to \mathbb{R}$$

is a $(\mathcal{C}'_t)$*-Brownian motion and that*

$$Y : \Lambda' \times [0,1] \to \mathbb{R}$$

is in $\mathcal{L}^2(\Lambda' \times [0,1])$ *and* $(\mathcal{C}'_t)$*-predictable. Then the process* $(X_t \circ \kappa)$ *is a* $(\mathcal{C}_t)$*-Brownian motion, the process* $(Y_t \circ \kappa)$ *is* $(\mathcal{C}_t)$*-predictable and*

$$\left(\int Y_t \, d\, X_t\right) \circ \kappa \;=\; \int \,(Y_t \circ \kappa)\; d\; (X_t \circ \kappa)\,.$$

For $k \in T \cup \{0\}$ set $B_k := \sum_{l=1}^{k} \pi_l$. We define a sub-$\sigma$-algebra of $L_\Gamma(\mathcal{B})$ by

$$\mathcal{W} := {}^{\sigma}\{\,{}^{\circ}B_k \mid k \in T\} \vee \mathcal{N}_{\widehat{\Gamma}}\,,$$

i.e. $\mathcal{W}$ is the smallest σ-field on Ω that includes $\mathcal{N}_{\widehat{\Gamma}}$ and such that the functions ${}^{\circ}B_k, k \in T$ are measurable. Define further for $X \in \Omega$, $t \in [0,1]$ and $k \in T$ with $\frac{k}{H} \approx t$

$$b(X,t) := {}^{\circ}B_k(X) \quad \text{and set} \quad \kappa(X) := b(X,\;\cdot\;)\;.$$

A proof of the next statement can be found in [20].

14.5 Lemma

The process b *is* $\widehat{\Gamma}$*-almost surely well defined and a Brownian motion. Furthermore, the mapping*

$$\kappa : (\Omega, \mathcal{W}, \widehat{\Gamma}) \to (C[0,1], \mathfrak{b}_{C[0,1]}, W)$$

is measure-preserving and

$$\mathcal{W} = \kappa^{-1}(\mathfrak{b}_{C[0,1]}) \vee \mathcal{N}_{\widehat{\Gamma}}. \tag{16}$$

Let $(\mathcal{B}_t)_{t\in[0,1]}$ be the filtration in $\mathfrak{b}_{C[0,1]}$ which is generated by ψ and set for $t \in [0,1]$

$$\mathcal{W}_t := \kappa^{-1}(\mathcal{B}_t) \vee \mathcal{N}_{\widehat{\Gamma}}.$$

14.6 Lemma

For each $t \in [0,1]$ *and* $k \in T$ *with* $\frac{k}{H} \approx t$ *we have* $\widehat{\Gamma}$*-a.s.*

$$\int_0^t \operatorname{sgn}(b_s)\, d\, b_s \;\approx\; \sum_{l=1}^{k} \operatorname{Sgn}(B_{l-1}) \cdot \pi_l\,.$$

Proof. This follows immediately from the definition of the stochastic integral in [20], since

$$\operatorname{sgn}(b_t) \approx \operatorname{Sgn}(B_k)$$

for $\widehat{\Gamma}$-almost all $X \in \Omega$. $\square$

The main proposition of this section states the equivalence of τ and L concerning ergodicity.

14.7 Proposition

The transformation

$$L : (C[0,1], \mathfrak{b}_{C[0,1]}, W) \to (C[0,1], \mathfrak{b}_{C[0,1]}, W)$$

is ergodic if and only if the transformation

$$\tau : (\Omega, \mathcal{W}, \widehat{\Gamma}) \to (\Omega, \mathcal{W}, \widehat{\Gamma})$$

is ergodic.

Proof. The assertion follows from Equation (16) and from the fact that the diagram

$$\begin{array}{ccc} C[0,1] & \xrightarrow{L} & C[0,1] \\ \uparrow \kappa & & \uparrow \kappa \\ \Omega & \xrightarrow{\tau} & \Omega \end{array}$$

commutes $\widehat{\Gamma}$-a.s., which implies that $A \in \mathfrak{b}_{C[0,1]}$ is L-invariant if and only if $\kappa^{-1}(A)$ is τ-invariant. Note that the fact that the diagram commutes also implies that

$$\tau^{-1}(\mathcal{W}) \subset \mathcal{W}.$$

In order to prove that $\kappa \circ \tau = L \circ \kappa$ $\widehat{\Gamma}$-a.s. we fix a $t \in [0,1]$ and show that

$$\psi_t(L \circ \kappa) = \psi_t(\kappa \circ \tau) \;\; \widehat{\Gamma}\text{-a.s.}$$

Fix a $k \in T$ with $\frac{k}{H} \approx t$ and note that because of Lemma 14.4 and Lemma 14.6

$$\psi_t(L \circ \kappa)(X) = \int_0^t \operatorname{sgn}(\psi_s)\, d\,\psi_s(\kappa(X)) = \int_0^t \operatorname{sgn}(b_s)\, d\, b_s(X) \approx$$

$$\sum_{l=1}^{k} \operatorname{Sgn}(B_{l-1})(X) \cdot X_l = B_k(\tau(X)) \approx b_t(\tau(X)) = \psi_t(\kappa \circ \tau)(X)$$

for $\widehat{\Gamma}$-almost all $X \in \Omega$. $\square$

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