Dissertation an der Fakultät für Mathematik der LMU München

Titel: Trees, Games and Reflections

Autor: Bernhard König

November 2001

Erster Berichterstatter: Prof. H.-D. Donder
Zweiter Berichterstatter: Prof. W. Buchholz

Tag der mündlichen Prüfung: 28.Februar 2002

TREES, GAMES AND REFLECTIONS

BERNHARD KÖNIG

Contents

The author was supported by the DFG (Deutsche Forschungsgemeinschaft).

Introduction

It is written somewhere in the scriptures that the appearance of every tree is completely specified by one seed alone. This in mind, a study of the growth of trees is probably futile. We nevertheless embark on this issue.

We actually try to understand the deeper structure of tree constructions in general. The crucial moments in the construction of trees are the limit levels. The crucial question will turn out to be: how many branches are extended at limit stages? This is where all the information is hidden, like in a seed. We try to find out how each limit level, a locally determined object, affects the shape of the global tree. In other words, we watch the seed as it springs and check if there is determinism in this process or not. It turns out that this is related to questions of reflection and large cardinals. If there is a lot of reflection in our universe, local properties of the tree will carry over to the global tree. If there is no such reflection, for example in the constructible universe, local properties might not affect the global tree at all.

In Part 2, we will define a class of minimal trees that answers our above question of how many branches to extend in the most economical way: as few branches as possible are extended. It will show up pretty soon that this strategy makes the tree trivial and very thin, even in the sense that there are no stationary antichains. The pivotal Lemmas 2.20 and 2.21 will characterize these 'trivial' trees in terms of elementary substructures. As an application, we are going to find that every non-trivial ω_1-tree has a stationary antichain in PFA-models. This completes the statement above and essentially says that whenever we try to construct a non-trivial ω_1-tree by conventional means, it will have a stationary antichain.

Although trivial trees can not be Aronszajn, there is a relativized notion of a trivial tree that allows this and we are going to call those trees *coherent*. Part 3 will provide a new construction of an ω_2-Aronszajn-tree of this kind from a weak version of a square-sequence. This question has been left open so far by the method of ϱ-functions, the most powerful tool in square-constructions.

Part 4 will eventually deal with already mentioned reflection principles. We ask if it is consistent that every ω_2-tree whose initial segments are trivial is trivial in a global sense. Call this statement *reflection of trivial coherence* or simply *reflection of coherence*. A positive answer towards its consistency will be given. This statement has a close relationship with the reflection of stationary sets (see Corollary 4.8), but the proofs indicate that it entails some more strength than simply stationary reflection as we still need the help of some forcing axiom in Theorem 4.19. But we have not been able to give a formal proof in this regard, distinguishing the reflection of coherence from the reflection of stationary sets.

Motivated by this phenomenon, we introduce another principle of *Game Reflection*. This time we reflect the notion of a winning strategy. It can be seen that this new principle implies the reflection of coherence and also the Continuum Hypothesis. Thus, these last two statements turn out to be consistent with each other, completing the results of the previous part. The Game Reflection Principle is still a quite remarkable strengthening over the stationary set reflection though. We show that it is equivalent to the fact that there is an ω-closed Forcing-extension that admits a non-trivial embedding of the universe into some transitive structure. This last assertion is known to be true in the Levy Collapse of a large cardinal.

Acknowledgments:

The thesis was produced under supervision of Hans-Dieter Donder and Stevo Todorčević. Either of them supplied me with instructive advice and indispensable help. Had I missed any of this, my thesis would not have been possible.

I would like to thank the DFG for their generous support during the three years of my thesis work. Additionally, I received support from the EU to cover conference travel expenses.

I enjoyed extraordinary hospitality at the following institutes: the Department of Mathematics at the University of Toronto, Equipe de Logique Mathematique at Paris VII, the Mittag-Leffler institute in Djursholm and Institut für formale Logik in Vienna.

Finally, I want to thank Matthew Foreman and Boban Veličković for giving me the opportunity for very fruitful discussions.

Part 1. Preliminaries

This part is devoted to the presentation of the basic notions and propositions used in the paper. All the other definitions will show up at the point where we need them. It will merely consist of a list of definitions and elementary Lemmas found in every introductory book on set theory such as for example [11] or [16]. We actually refer the reader to one of these two books in case that the author missed to define a couple of notions within the pages of Part 1. The educated reader, however, might want to skip these paragraphs if they happen to be too boring for him.

We start with very elementary notation: $f : A \xrightarrow{\text{1-1}} B$ means that f is a one-to-one mapping from A to B. $f : A \xrightarrow{\sim} B$ means that f is a one-to-one embedding of A into B that preserves structure. The reader could be confused by the fact that, in this case, f is not necessarily onto. If it is onto, we explicitly call f an *isomorphism* and denote the isomorphism relation by $\cong$.

Our shortcut of $f''A$ for the image of A under f is very natural though, just as well as the notations ${}^{\delta}\gamma$ for the set of all functions from δ into γ and ${}^{<\delta}\gamma$ for the set of all functions from ordinals smaller than δ into γ. We use the symbol $[A]^{\lambda}$ for the set of all subsets of A with cardinality λ, $[A]^{<\lambda}$ is defined analogously. $\mathfrak{P}(A)$ denotes the power set of A, the set of all subsets.

We call a set $C \subseteq \kappa$ *club* if it is closed and unbounded in ω_2. We define the notion of club for subsets of $[X]^{\kappa}$: $\mathcal{C}$ is *closed and unbounded in* $[X]^{\kappa}$ if

(i) for all $a \in [X]^{\kappa}$ there is a $b \in \mathcal{C}$ such that $a \subseteq b$,
(ii) whenever $\langle a_\xi : \xi < \kappa \rangle \subseteq \mathcal{C}$ is an increasing sequence, then $\bigcup_{\xi<\kappa} a_\xi \in \mathcal{C}$.

The above definition is common. Note that if $\kappa > \omega$, our notion of club no longer coincides with the property of being closed under an algebra of functions.

We confuse the notions of being club and containing a club. Furthermore, we need a more general version of clubs: a set of ordinals is called λ-*closed* if it is closed under sequences of order-type λ. A λ-closed and unbounded set is λ-*club* . The same generalization carries over to clubs in $[X]^{\kappa}$.

The set of all limit points of a given set of ordinals D is denoted by D'. The string $\mathrm{lh}(s)$ is a notation for the length of a sequence s. We write $M \prec N$ to say that M is an elementary submodel of N and $M \prec_{\omega_1} N$ to say that M is an elementary submodel of N and

$M \cap \omega_1 = N \cap \omega_1$. The chain of models $\langle M_\xi : \xi < \lambda \rangle$ is an $\in$-*chain* if $M_\xi \in M_{\xi+1}$ for all $\xi < \lambda$. $Sk(X)$ is the Skolem closure of the set X when it should be clear from the context which superstructure we are working in, usually some H_θ. Furthermore, we denote ideals by letters like $\mathcal{I}$ or $\mathcal{J}$, where $\mathcal{I}^+$ is the collection of all positive sets with respect to the ideal $\mathcal{I}$. NS_λ is the ideal of non-stationary subsets of λ; the subscript will be dropped if it is clear from the context.

ZFC^- is ZFC minus Power set and we use an abbreviation in the context of elementary embeddings: $j : M \longrightarrow N$ means that j is a non-trivial elementary embedding from M into N such that M and N are transitive. The *critical point* of such an embedding, i.e. the first ordinal moved by j, is denoted by $\mathrm{cp}(j)$. The reader has already noticed that we write symbols like κ and λ for cardinals and $\alpha, \beta, \gamma, \zeta, \xi, \ldots$ for ordinals.

All trees are considered to be *normal* , for our purposes this means that they are trees of functions closed under initial segments with the property that every point splits and has successors of arbitrary height.

We use the following tree terminology: for any element t of a tree, let $t \restriction \alpha$ be the predecessor of t on the αth level. If t is element of a successor level, let *immpred*(t) be the immediate predecessor of t. We write $x \perp y$ as a shorthand for x is incomparable with y. $x \wedge y$ is a symbol for the maximal $z \in T$ such that $z \leq_T x$ and $z \leq_T y$, the *infimum* of x and y. $\mathfrak{B}_T$ denotes the set of all cofinal branches T.

κ-*trees* are trees of height κ with the property that every level has size less than κ. λ-*closed* trees are trees that are closed under sequences of length less than λ.

The meanings of κ-*Aronszajn-tree* and κ-*Kurepa-tree* are the following: both kinds of trees have height κ and levels of size less than κ, but Aronszajn-trees lack branches, whereas Kurepa-trees have at least κ^+-many branches. κ-Aronszajn-trees are *special* if we can associate to them a regressive function $f : T \longrightarrow T$ such that the preimage of every point is a union of less than κ many antichains. Note that if κ is a successor cardinal, this is equivalent to the classical notion of speciality, saying that we can decompose the tree into less than κ many antichains. In Definition 2.31 we introduce a more general notion of *special* trees that applies to trees with branches as well.

Some words on Forcing: $p \leq q$ means that p is a stronger condition than q. Names are denoted with dots on top (e.g. $\dot{\tau}$) but our notation shall not be too strict in this. If $\mathbb{P}$ is a partial order, G a $\mathbb{P}$-generic filter and M a model of a large enough fragment of set theory, we let

$$M[G] = \{\dot{\tau}[G] : \dot{\tau} \text{ is a } \mathbb{P}\text{-name in } M\},$$

where $\dot{\tau}[G]$ is the G-interpretation of the name $\dot{\tau}$. On the other hand, $V^{\mathbb{P}}$ is sometimes written for $V[G]$, but no confusion should arise.

If λ is regular, we define the posets $Col(\lambda, \kappa)$ and $Coll(\lambda, < \kappa)$ to be the $(< \lambda)$-closed Levy Collapses of κ to λ and of everything less than κ to λ respectively:

$$Col(\lambda, \kappa) = \{p : \alpha \longrightarrow \kappa \,|\, \alpha < \lambda\}, \tag{1.1}$$

$$Coll(\lambda, < \kappa) = \{p \,|\, \mathrm{dom}(p) \subseteq \kappa \times \lambda, p(\delta, \alpha) < \delta \text{ and } p \text{ has cardinality less than } \lambda\}. \tag{1.2}$$

In both of these cases, the ordering is reverse inclusion.

We assume knowledge of the following axioms: $\Box(\kappa)$ is the statement that there be a sequence C_α $(\alpha < \kappa)$ such that

1 C_α is club in α,
2 if α is a limit point of C_β then $C_\alpha = C_\beta \cap \alpha$,
3 there is no club $C \subseteq \kappa$ with $C_\alpha = C \cap \alpha$ for every limit point α of C.

If $\kappa = \lambda^+$, we might require a stronger condition than 3:

*3 the order-type of C_α is at most λ for all $\alpha < \kappa$.

If this occurs, we call C_α $(\alpha < \kappa)$ a $\Box_\lambda$-sequence. Note that if λ is a regular cardinal, the non-existence of a $\Box_\lambda$-sequence is equiconsistent with a Mahlo cardinal, while the non-existence of a $\Box(\lambda^+)$-sequence is equiconsistent with a weakly compact.

The notions of *proper* and *semiproper* can be looked up in [1] and [18]. This is also the place where the axioms PFA, PFA$^+$ and SPFA can be found and familiarity with them acquired.

If $\gamma \leq \omega_2$, then SR_γ denotes the following reflection of stationary sets: for every regular θ and every sequence $\mathcal{E}_\alpha$ $(\alpha < \gamma)$ of stationary subsets of $[\mathrm{H}_\theta]^{\aleph_0}$ there is an $\in$-chain $\langle M_\xi : \xi < \omega_1 \rangle$ such that $\{\xi < \omega_1 : M_\xi \in \mathcal{E}_\alpha\}$ is stationary in ω_1 for every $\alpha \in (\bigcup_{\xi<\omega_1} M_\xi) \cap \omega_2$.

Rado's conjecture is known as the following statement: a family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies if and only if every subfamily of size $\aleph_1$ has this property.

Finally, we give three important Lemmas that are frequently used in the course of this work. We might sometimes even apply them without mentioning:

1.1 Lemma (Stretching stationary sets). *Suppose that α is an ordinal of uncountable cofinality λ. If $f : \lambda \longrightarrow \alpha$ enumerates a club set in α, then A is stationary in α if and only if $f^{-1}(A)$ is stationary in λ.* $\square$

1.2 Lemma (Pressing Down Lemma). *Let κ regular. If $f : S \longrightarrow \kappa$ is a regressive function on a stationary set $S \subseteq \kappa$ then there exist a stationary set $S_0 \subseteq S$ and $\gamma_0 < \kappa$ such that $f(\alpha) = \gamma_0$ for all $\alpha \in S_0$.*

Proof. We prefer to give the non-classical proof: take an elementary submodel $N \prec \mathrm{H}_\theta$ for some large enough regular θ such that N contains f as an element and $N \cap \kappa = \delta$ is an ordinal in S. Now $\gamma_0 = f(\delta) \in N$ and therefore the set $S_0 = f^{-1}(\gamma_0)$ is in N and contains δ. Thus, S_0 hits every club in N and by elementarity S_0 is stationary in κ. $\square$

The Extension Lemma was first proved by Silver:

1.3 Lemma (Extension Lemma). *Let $j : M \longrightarrow N$ and assume that G is $\mathbb{P}$-generic over M, H is $j(\mathbb{P})$-generic over N for a poset $\mathbb{P}$. If $j''G \subseteq H$ then there is a unique extension $j^* : M[G] \longrightarrow N[H]$ of j such that $j^*(G) = H$.*

Proof. For each $\mathbb{P}$-name $\dot{\tau}$, simply let $j^*(\dot{\tau}[G]) = j(\dot{\tau})[H]$. $\square$

Part 2. **A class of minimal trees**

a. Introducing Coherence

In parts 2 and 4 of the thesis we investigate the structure of the rational numbers $(\mathbb{Q},<)$. This object never received much attention from set theorists, reason for this being that the set of rationals is the same in every model of set theory. But we shall see in the following pages that statements about this seemingly trivial structure can have a complex set-theoretical nature with lots of independence results. These independencies, of course, only show up if we give the whole scenario a slight set-theoretical twist. In Part 4, we are going to relate it with principles of reflection, principles that mirror-image properties of large cardinals or non-trivial elementary embeddings. But we shall also need a generalization of the rational numbers to higher cardinalities. Before we do this in Definition 2.2, let us mention a general procedure to transform linear orderings into trees, called *atomization*[1]. The result of an atomization will be a *partition tree* of the linear ordering. On the other hand, every tree can be linearly ordered by lining up its elements lexicographically. This procedure is used in particular for the well-known one-to-one resemblance between Aronszajn-trees and Aronszajn-lines as well as Suslin-trees and Suslin-lines. If we apply this method to our object of study, the rationals, we will know where to go from here:

2.1 Note. *There is a partition tree of* $(\mathbb{Q},<)$ *of the form*

$$\mathbb{Q}^{\text{fin}}_{\leq\omega} = \{f \in {}^{\leq\omega}2 : \operatorname{supp}(f) \text{ is finite}\}.$$

Proof. By $\aleph_0$-categoricity of dense linear oderings, $(\mathbb{Q},<)$ can be identified with the set of all functions from ω to 2 with finite support ordered lexicographically. Define the partition tree T as follows: for $\vec{r} \in {}^n2$ $(n<\omega)$, let

$$I_{\vec{r}} = \{\vec{r}^\frown\vec{s} : \vec{s} \in {}^{\omega}2 \text{ with finite support}\}$$

and set $T_n = \{I_{\vec{r}} : \vec{r} \in {}^n2\}$. The set of intersections T_ω will consist of singletons only, so our partition tree has height $\omega+1$. But T is isomorphic to $\mathbb{Q}^{\text{fin}}_{\leq\omega}$ via the operation $I_{\vec{r}} \longmapsto \vec{r}$. □

We just motivated the next Definition:

2.2 Definition. We define a class of trees in the following fashion: if κ is regular and δ an ordinal, we let

$$\mathbb{Q}^{\kappa}_{<\delta} = \{f \in {}^{<\delta}2 : |\operatorname{supp}(f)| < \kappa\}.$$

[1]The notions of *atomization* and *partition tree* of a linear ordering are found e.g. in [21].

The special cases of $\kappa = \omega$ and $\kappa = \omega_1$ are also referred to by $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$ and $\mathbb{Q}^{\mathrm{ctbl}}_{<\delta}$ respectively. Our interest is fiercely located in the case of $\kappa = \omega$. This justifies the following way of speaking: we will call normal subtrees of $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$ *trivially coherent* or *trivial.*

The restriction to binary trees in the previous Definition 2.2 is just for notational simplicity and not essential as long as the splitting is reasonable. Nevertheless, we will stick to this restriction on binary splitting in the future.

2.3 Note. *$\mathbb{Q}^{\mathrm{fin}}_{\leq\delta}$ is a transitive subtree of any normal tree of height $\delta+1$.*

Proof. Let T a normal tree of height $\delta + 1$. Using normality, we can find a mapping $b : T_{<\delta} \longrightarrow T_\delta$ such that $x <_T b(x)$ and

$$x \leq_T y <_T b(x) \text{ implies } b(x) = b(y).$$

Let $B_0 = b''T_{<\delta}$ and define an even finer sublevel $B_1 \subseteq B_0$:

$$B_1 = \{f \in B_0 : \forall \text{ limit } \gamma < \delta\ \exists x \in T_{<\gamma}\ f \restriction \gamma <_T b(x)\}.$$

Let U be the downwards closure of B_1 and define the isomorphism $\pi : U \overset{\sim}{\longrightarrow} \mathbb{Q}^{\mathrm{fin}}_{\leq\delta}$ inductively:

$$\text{if } x \in U_{\alpha+1}, \text{ let } \pi(x)(\alpha) = 0 \text{ iff } x <_T b(x \restriction \alpha).$$

π maps into $\mathbb{Q}^{\mathrm{fin}}_{\leq\delta}$ by the definition of B_1 and it can be seen to be onto. $\square$

Note 2.3, in some sense, distinguishes the role played by trivially coherent trees. Of course, no other trees have this property, i.e. embeddability into everything else. In other words, $\mathbb{Q}^{\mathrm{fin}}_{\leq\delta}$ is a 'minimal model' within the class of trees.

2.4 Definition. Let T a tree of height δ. If $f : T \restriction C \longrightarrow T$ is regressive with $\mathrm{Lim}(\delta) \subseteq C$ and $x \in T \restriction C$, define a regressive *trace* tr^f_x from x to the root of the tree as follows:

$$\begin{aligned} \mathrm{tr}^f_x(0) &= x \\ \mathrm{tr}^f_x(n+1) &= \begin{cases} f(\mathrm{tr}^f_x(n)) & \text{if } \mathrm{tr}^f_x(n) \in T \restriction C, \\ \mathrm{immpred}(\mathrm{tr}^f_x(n)) & \text{if } \mathrm{tr}^f_x(n) \notin T \restriction C. \end{cases} \end{aligned}$$

2.5 Lemma. *The following are equivalent for any tree T of height δ:*

(1) *T is trivially coherent.*
(2) *$T = \bigcup_{n<\omega} S_n$, where each S_n has the properties:*
   - *there are no triangles, i.e. there are no $x, y, z \in S_n$ such that $z \leq_T x, y$ and $x \perp y$,*
   - *if $x \in S_n \cap T \restriction \mathrm{Lim}(\delta)$, then a final segment of the predecessors of x is in S_n.*

(3) *There is a regressive* $f : T \restriction \mathrm{Lim}(\delta) \longrightarrow T$ *such that*

$$(\perp)_f \qquad x_0 \perp x_1 \longrightarrow \exists i\ f(x_i) >_T x_0 \wedge x_1.$$

Proof. (1) $\Longrightarrow$ (2) can be achieved by letting

$$x \in S_n \quad \text{iff} \quad |\mathrm{supp}(x)| = n.$$

(2) $\Longrightarrow$ (3): just let $f(x) = \min\{y <_T x : [y, x) \subseteq S_n\}$ for an n.

For (3) $\Longrightarrow$ (1) fix a function f as in (3) and let $x \in T$. Now call $y \in T_{\mathrm{succ}}$ a *zero point* if there is $x \in T, n < \omega$ such that

$$\mathrm{tr}^f_x(n+1) <_T y <_T \mathrm{tr}^f_x(n).$$

Note that by $(\perp)_f$, every successor node contains at most one zero point, and this means that we can find an isomorphism π defined on T, with the property that $\pi(y)(\mathrm{dom}(y) - 1) = 0$ for all zero points y. But then, as is easily established, $\pi'' T \subseteq \mathbb{Q}^{\mathrm{fin}}_{<\delta}$. □

These reformulations of trivial coherence set the stage for the following lemma, revealing a crucial property of trivially coherent trees. We need the following definitions:

2.6 Definition. Let T a tree of height δ and $\mathrm{cf}(\delta) > \omega$. A subset $S \subseteq T$ is called *stationary*, if $ht'' S$ is stationary in δ.

A subset $S \subseteq T$ *projects 1-1*, if there is $\gamma < \delta$ so that the projection mapping $\mathrm{pr}_\gamma : S \longrightarrow T$ defined by $s \longmapsto s \restriction \gamma$ is 1-1.

An antichain $A \subseteq T$ is *non-trivial* , if A is stationary and no stationary $A_0 \subseteq A$ projects 1-1.

2.7 Note. *If T is a κ-tree and κ regular, then an antichain in T is stationary if and only if it is non-trivial.* □

2.8 Lemma. *If* $\mathrm{cf}(\delta) > \omega$, $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$ *has no non-trivial antichains.*

Proof. We assume towards a contradiction that A is a non-trivial antichain in $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$. Let $E = \mathrm{ht}'' A$ and $t_\nu = A \cap T_\nu$ ($\nu \in E$). Define a regressive mapping $h : E \longrightarrow \delta$ by setting

$$h(\nu) = \max\{\alpha < \nu : t_\nu(\alpha) \neq 0\} + 1.$$

The Pressing Down Lemma for singular ordinals (see e.g. [7, p.36]) will provide an ordinal $\xi < \delta$ and a stationary $E_0 \subseteq E$ such that $h'' E_0 \subseteq \xi$, i.e. the support of t_ν for $\nu \in E_0$ is below the ordinal ξ. Using the non-triviality of the antichain A, we have that $\{t_\nu : \nu \in E_0\}$ does not project 1-1, particularly not at the point ξ. As a consequence, there are $\nu, \nu' \in E_0$ such that $t_\nu \restriction \xi = t_{\nu'} \restriction \xi$. But this makes t_ν and $t_{\nu'}$ comparable. A contradiction. □

The just achieved lemma is actually the 'raison d'etre' for the notion of non-trivial antichains.

Note 2.7 yields:

2.9 Corollary. *If κ is regular, $\mathbb{Q}^{\text{fin}}_{<\kappa}$ has no stationary antichains.* $\square$

Theorems 2.34 and 4.13 will show that Lemma 2.8 is optimal in a much deeper sense.

b. Characterizing Coherence

This section is circling around Lemmas 2.20 and 2.21. Roughly speaking, they say that a tree T is trivially coherent if and only if every point $t \in T$ is definable from finitely many points below t. This is a much more intrinsic property than our original definition of trivially coherent trees.

In the following, let T be a tree. We will fix a regular cardinal θ much larger than the height of T. From now on, all elementary substructures $N \prec \mathrm{H}_\theta$ will be assumed countable (unless otherwise mentioned), with a fixed well-ordering $<_w$ of H_θ attached to them and, moreover, they contain T.

2.10 Definition. A chain $K \subseteq T \cap N$ is *captured* by an elementary submodel N, if there is a chain $L \in N$ such that $K \subseteq L$. A chain $K \subseteq T \cap N$ is *cofinally captured* by N, if there is a cofinal chain $L \in N$ such that $K \subseteq L$. In these cases, we may also say that K is *captured* or *cofinally captured* by L respectively.

Unfortunately, we need some additional notions.

2.11 Definition. Assume that T is a tree of height δ. We will call a chain K through T an *M-chain* if $K \subseteq M$. An M-chain K will be called *cofinal* (*in M*) if $\mathrm{dom}(\bigcup K) = \sup(M \cap \delta)$. If K is not cofinal, it is said to be *bounded* (*in M*).

$M \prec \mathrm{H}_\theta$ is called *locally T-simple* , if all converging bounded M-chains are captured by M. M is called *locally T-uniform* , if all converging bounded M-chains are cofinally captured by M.

$M \prec \mathrm{H}_\theta$ is called *T-simple*, if all converging cofinal M-chains are captured by M. Note that such a cofinal M-chain will always be cofinally captured.

$M \prec \mathrm{H}_\theta$ is said to be *strongly T-simple*, if it is both locally T-simple and T-simple. M is said to be *strongly T-uniform*, if it is both locally T-uniform and T-simple.

If a model $M \prec \mathrm{H}_\theta$ is not T-simple, we call it *T-complicated.* In this case, we will always let t_M be the $<_w$-minimal element $t \in T$ that witnesses complicatedness, i.e. t is the limit of an uncaptured and cofinal chain.

2.12 Note.

- $\mathrm{ht}(t_M) = \sup(M \cap \delta)$.
- *If M is T-complicated, t_M is definable in* $(\mathrm{H}_\theta, \in, <_w)$.

$\square$

2.13 Definition.

$$\begin{aligned}
\mathcal{L}_T &= \{M \prec \mathrm{H}_\theta : M \text{ is locally } T\text{-simple}\},\\
\mathcal{L}_T^- &= \{M \prec \mathrm{H}_\theta : M \text{ is locally } T\text{-uniform}\},\\
\mathcal{S}_T &= \{M \prec \mathrm{H}_\theta : M \text{ is } T\text{-simple}\},\\
\mathcal{S}_T^- &= \{M \prec \mathrm{H}_\theta : M \text{ is strongly } T\text{-simple}\} = \mathcal{L}_T \cap \mathcal{S}_T,\\
\mathcal{U}_T &= \{M \prec \mathrm{H}_\theta : M \text{ is strongly } T\text{-uniform}\} = \mathcal{L}_T^- \cap \mathcal{S}_T,\\
\mathcal{C}_T &= \{M \prec \mathrm{H}_\theta : M \text{ is } T\text{-complicated}\} = [\mathrm{H}_\theta]^{\aleph_0} \setminus \mathcal{S}_T.
\end{aligned}$$

2.14 Note.

- $\mathcal{L}_T^- \subseteq \mathcal{L}_T$.
- $\mathcal{U}_T \subseteq \mathcal{S}_T^- \subseteq \mathcal{S}_T$.

$\square$

2.15 Lemma. *Let T a tree of height δ with at most $|\delta|$-many cofinal branches and assume that $M \cap \delta = N \cap \delta$ for elementary M, N.*

Then M is T-complicated if and only if N is T-complicated. Moreover, a chain $K \subseteq T$ is captured by the model M if and only if it is captured by N.

Proof. Working in H_θ, choose enumerations of T and the set of all cofinal branches $\mathfrak{B}_T$:

$$\varphi_0 : \delta \xrightarrow{\text{1-1}} T,$$

$$\varphi_1 : \delta \xrightarrow{\text{1-1}} \mathfrak{B}_T.$$

Now, if M and N are as stated above, then

$$T \cap M = \varphi_0''(M \cap \delta) = \varphi_0''(N \cap \delta) = T \cap N.$$

Similarly, $\mathfrak{B}_T \cap M = \mathfrak{B}_T \cap N$ will hold. This makes it easy to check that T-complicatedness is preserved. $\square$

2.16 Definition. If X is a subset of a tree T, we define

$$\overline{X} = X \cup \{t \in T : \text{ there is a chain } K \subseteq X \text{ converging to } t\}.$$

2.17 Definition. Let T a tree of height δ. The game $\mathbb{G}_T^{\mathrm{coh}}$ is played as follows:

I	x_0	x_1	x_2	x_3	$\dots$
II	b_0	b_1	b_2	b_3	$\dots$

where player I plays points $x_n \in T$ $(n < \omega)$, while player II answers with branches b_n through T. By a branch we mean a downward closed chain that is not necessarily cofinal. We will see in Lemmas 2.20 and 2.21 what the distinction between cofinal and arbitrary branches amounts to.

II wins this play of the game, if $\overline{\{x_n : n < \omega\}} \subseteq \bigcup_{n<\omega} b_n$.

2.18 Note. *If σ is any winning strategy for II in $\mathbb{G}_T^{\mathrm{coh}}$ and x a limit point in T, then $\exists x_0 <_T \dots <_T x_n <_T x : x \in \sigma(x_0, \dots, x_n)$.*

Proof. Pick $N \prec \mathrm{H}_\theta$ containing σ, x and T. Let $\gamma = ht(x) \in N$ and $\gamma_N = \sup(N \cap \gamma)$. Now choose a sequence γ_n $(n < \omega)$ through N, cofinal in γ_N. Because σ is winning for II, there is $m < \omega$ such that

$$x \restriction \gamma_N \in b_N = \sigma(x \restriction \gamma_0, \dots, x \restriction \gamma_m).$$

By elementarity, the branch b_N hits x. This finishes the proof. □

There is a slight and insignificant variation of $\mathbb{G}_T^{\mathrm{coh}}$ that can relieve some technical difficulties in the proofs of the important Lemmas 2.20 and 2.21:

2.19 Definition. The game $\mathbb{G}_T^{\mathrm{coh}*}$ is the same as the game $\mathbb{G}_T^{\mathrm{coh}}$ with the additional rules that:

(i) $x_n <_T x_{n+1}$ for all $n < \omega$, and
(ii) $x_n \in b_n$.

2.20 Lemma. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*
(2) *$\mathcal{S}_T^-$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*
(3) *II has a winning strategy in the game $\mathbb{G}_T^{\mathrm{coh}}$.*
(4) *II has a winning strategy in the game $\mathbb{G}_T^{\mathrm{coh}*}$.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(1) $\Longrightarrow$ (2): pick $N \prec \mathrm{H}_\theta$ with $T \in N$. If $K \subseteq N \cap T$ is any converging chain in T, its support is finite so there is $\alpha \in N$ such that $K = (K \restriction \alpha)^\frown \vec{0}$. Define in N the branch $L : \gamma \longrightarrow 2$,

$$L(\xi) = \begin{cases} K(\xi) & \text{if } \xi < \alpha, \\ 0 & \text{if } \alpha \le \xi < \gamma, \end{cases}$$

where γ is the maximal ordinal such that L is branch through T. This L captures K and witnesses that N is strongly T-simple.

(2) $\Longrightarrow$ (3): in the first place, fix a reasonable pairing function

$$\langle \quad , \quad \rangle : \omega \times \omega \longrightarrow \omega.$$

Whenever $\vec{x} \in {}^{<\omega}T$, we will choose enumerations $S_{\vec{x}}$ of all the branches in the respective Skolem closures, i.e.

$$S_{\vec{x}} : \omega \longrightarrow \{b \in Sk(\vec{x}) : b \text{ is a branch through } T\},$$

where the Skolem closures $Sk(\vec{x})$ are strongly T-simple. If $v_0, \ldots, v_{m-1}$ is the start of a play, and $\langle k, l\rangle = m$, then we respond

$$\sigma(v_0, \ldots, v_{m-1}) = S_{\vec{v}\restriction k}(l).$$

Note that this is well-defined, since $m \geq k$.

2.20.1 Claim. *σ is winning for player II in the game $\mathbb{G}_T^{\mathrm{coh}}$.*

Proof. We assume that $K = \{v_{i(n)} : n < \omega\}$ converges and conclude the following:

$$K \subseteq Sk(v_n)_{n<\omega} = \bigcup_{n<\omega} Sk(v_0, \ldots, v_{n-1}).$$

But $Sk(v_n)_{n<\omega}$ is strongly T-simple since $\mathcal{S}_T^-$ is club, so it contains a branch $b \supseteq K$. By definition of the strategy σ, this branch will finally be played. □

(3) $\Longrightarrow$ (4): let σ be a winning strategy for II in the game $\mathbb{G}_T^{\mathrm{coh}}$. We define a winning strategy τ II in the game $\mathbb{G}_T^{\mathrm{coh}*}$: whenever the sequence $x_0 <_T x_1 <_T \ldots <_T x_n$ is a finite chain through T, let

$$\tau(x_0, x_1, \ldots, x_n) = \begin{cases} \sigma(x_0, x_1, \ldots, x_n) & \text{if } x_n \in \sigma(x_0, x_1, \ldots, x_n), \\ \text{any } b \text{ with } x_n \in b & \text{else.} \end{cases}$$

We may assume that $x_0 = \text{root}$, so that $x_n \in \tau(x_0, x_1, \ldots, x_n)$ for all sequences $\langle x_0, x_1, \ldots, x_n\rangle$ and τ is actually a strategy for the game $\mathbb{G}_T^{\mathrm{coh}*}$.

2.20.2 Claim. *τ is winning for player II in the game $\mathbb{G}_T^{\mathrm{coh}*}$.*

Proof. If there is a play:

I	x_0	x_1	x_2	x_3	$\ldots$
II	b_0	b_1	b_2	b_3	$\ldots$

according to τ and $x = \lim_{n<\omega} x_n$, set

$$B_0 = \bigcup_{n<\omega} b_n = \bigcup_{n<\omega} \tau(x_0, \ldots, x_n) \text{ and } B = \bigcup_{n<\omega} \sigma(x_0, \ldots, x_n).$$

But if $b \in B \setminus B_0$, then there is an integer $m < \omega$ such that $x_m \notin b$, where $b = \sigma(x_0, \ldots, x_m)$. Hence, $x \notin b$. This argument shows that τ is winning for player II. □

(4) $\Longrightarrow$ (1): let τ be a winning strategy for II in the game $\mathbb{G}_T^{\mathrm{coh}*}$. Our aim is to construct an embedding $\pi : T \stackrel{\sim}{\longrightarrow} \mathbb{Q}^{\mathrm{fin}}_{<\delta}$. So let $t \in T$ and inductively define t_n and α_n in the following way:

$$\alpha_0 = \mathrm{ht}(\tau(\mathrm{root}) \wedge t) \text{ and } t_0 = t \restriction (\alpha_0 + 1),$$
$$\alpha_n = \mathrm{ht}(\tau(\mathrm{root}, t_0, \ldots, t_{n-1}) \wedge t) \text{ and } t_n = t \restriction (\alpha_n + 1).$$

Clearly, $\alpha_n < \alpha_{n+1}$ as long as the process continues, i.e. as long as $\alpha_n < \mathrm{ht}(t)$. But actually, it will break down at some point:

2.20.3 Claim. *There is $k < \omega$ such that $\alpha_k = \mathrm{ht}(t)$.*

Proof. Assume not. Then there is a play

I	t_0	t_1	t_2	t_3	$\ldots$
II	b_0	b_1	b_2	b_3	$\ldots$

according to τ. But $t_n \notin b_{n-1} = \tau(t_0, \ldots, t_{n-1})$ by definition of t_n. So $x = \lim_{n<\omega} t_n \notin b_m$ for all $m < \omega$. This contradicts the fact that τ is winning. $\square$

We are in a position to define the embedding π: for $\xi < \mathrm{ht}(t)$ let

$$\pi(t)(\xi) = \begin{cases} 1 & \text{if } \xi = \alpha_n \text{ for some } n < k, \\ 0 & \text{else.} \end{cases}$$

Obviously, $\pi(t) \in \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.

2.20.4 Claim. *π is one-to-one and preserves the tree relation.*

Proof. It's easy to see that two incomparable elements differ in their π-values for the first time at the splitting node. This is enough to prove that π is one-to-one and preserves the tree relation. $\square$

$\square$

Rewriting the above proof of Lemma 2.20 yields:

2.21 Lemma. *The following are equivalent for any tree T of height δ:*

(1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.
(2) *$\mathcal{U}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*
(3) *II wins the game $\mathbb{G}_T^{\mathrm{coh}}$ by playing cofinal branches.*
(4) *II wins the game $\mathbb{G}_T^{\mathrm{coh}*}$ by playing cofinal branches.*

Proof. (1) $\Longrightarrow$ (2): pick $N \prec \mathrm{H}_\theta$ with $T \in N$. If $K \subseteq N \cap T$ is any converging chain in $\mathbb{Q}^{\mathrm{fin}}_{<\delta}$, its support is finite so there is $\alpha \in N$ such that $K = (K \restriction \alpha)^\frown \vec{0}$. Define in N the cofinal branch $L : \delta \longrightarrow 2$,

$$L(\xi) = \begin{cases} K(\xi) & \text{if } \xi < \alpha, \\ 0 & \text{if } \alpha \leq \xi < \delta. \end{cases}$$

L cofinally captures the chain K and so N is strongly T-uniform.

(2) $\Longrightarrow$ (3): now we choose enumerations $S_{\vec{x}} : \omega \longrightarrow Sk(\vec{x}) \cap \mathfrak{B}_T$, where $\vec{x} \in {}^{<\omega}T$ and the Skolem closures $Sk(\vec{x})$ are strongly T-uniform. Here, $\mathfrak{B}_T$ is the set of δ-branches through T. If $v_0, \ldots, v_{m-1}$ is the start of a play, and $\langle k, l\rangle = m$, then we respond

$$\sigma(v_0, \ldots, v_{m-1}) = S_{\vec{v}\restriction k}(l).$$

This is once more well-defined, since $m \geq k$.

2.21.1 Claim. *σ is winning for player II in the game $\mathbb{G}_T^{\mathrm{coh}}$ and plays δ-branches only.*

Proof. The strategy obviously plays δ-branches only. We repeat the proof of Claim 2.20.1 to show that σ is winning. $\square$

(3) $\Longrightarrow$ (4): is very much like (3) $\Longrightarrow$ (4) of Lemma 2.20.

(4) $\Longrightarrow$ (1): let τ be a winning strategy for II in the game $\mathbb{G}_T^{\mathrm{coh}*}$ that uses δ-branches only. Our aim is to construct an isomorphism $\pi : T \xrightarrow{\sim} \mathbb{Q}^{\mathrm{fin}}_{<\delta}$. So let $t \in T$ and inductively define t_n and α_n like in the proof of Lemma 2.20:

$$\alpha_0 = \mathrm{ht}(\tau(\mathrm{root}) \wedge t) \text{ and } t_0 = t \restriction (\alpha_0 + 1),$$

$$\alpha_n = \mathrm{ht}(\tau(\mathrm{root}, t_0, \ldots, t_{n-1}) \wedge t) \text{ and } t_n = t \restriction (\alpha_n + 1).$$

Again, $\alpha_n < \alpha_{n+1}$ as long as the process continues, and we have the following Claim:

2.21.2 Claim. *There is $k < \omega$ such that $\alpha_k = \mathrm{ht}(t)$.* $\square$

Define the embedding π: for $\xi < \mathrm{ht}(t)$ let

$$\pi(t)(\xi) = \begin{cases} 1 & \text{if } \xi = \alpha_n \text{ for some } n < k, \\ 0 & \text{else.} \end{cases}$$

2.21.3 Claim. *π is one-to-one, onto and preserves the tree relation.*

Proof. It was pointed out before that π is one-to-one and preserves the tree relation. We show that π is onto:

if $q \in \mathbb{Q}^{\mathrm{fin}}_{\gamma}$ $(\gamma < \delta)$ has support $\{\gamma_0, \ldots, \gamma_m\}$, define an ascending sequence in T, where $x_{n+1} \in T_{(\gamma_n+1)}$ $(n < m)$:

$$x_0 = \mathrm{root}$$

$$x_{n+1}(\xi) = \begin{cases} \tau(x_0, \ldots, x_n)(\xi) & \text{if } \xi < \gamma_n, \\ 1 - \tau(x_0, \ldots, x_n)(\xi) & \text{if } \xi = \gamma_n. \end{cases}$$

Note that this construction is possible only if τ provides us with cofinal branches. We go on to check that $\tau(x_0, \ldots, x_{m+1}) \restriction (\gamma)$ is the π-preimage of q. $\square$

$\square$

With this machinery developed, we make an interesting observation: since countable trees are really countable dense linear orderings, seen from the operations on page 8, it follows that every countable dense linear ordering is in fact isomorphic to $(\mathbb{Q}, <)$ and we did not use Cantor's back-and-forth method to prove this. In another respect, we are able to view Lemmas 2.20 and 2.21 as an advance in Kurepa's classification of countable trees in [17]. Kurepa actually used Cantor's technique to construct isomorphisms between any two countable normal trees of the same height.

We can't continue before mentioning the fact that the above two Lemmas are just a glance of a much more general fact. We can use the crucial notion of T-simpleness for elementary substructures of larger cardinality:

2.22 Definition. If an elementary substructure $N \prec \mathrm{H}_\theta$ has cardinality κ and $\kappa \subseteq N$ where κ is regular, we call it *strongly T-simple* if N captures every converging chain $K \subseteq N \cap T$ that has a κ-cofinal limit point.

The corresponding notions of *T-simple* and *T-uniform* are defined in analogy to Definition 2.11.

We might also play versions of the game $\mathbb{G}_T^{\mathrm{coh}}$ with length κ instead of length ω.

2.23 Definition. Let T a tree of height δ. The game $\mathbb{G}_T^{\kappa-\mathrm{coh}}$ is played as follows:

I	x_0		x_1		$\dots$		x_α		$x_{\alpha+1}$		$\dots$	
II		b_0		b_1		$\dots$		b_α		$b_{\alpha+1}$		$\dots$

where player I plays points $x_\alpha \in T$ $(\alpha < \kappa)$, while player II answers with δ-branches b_α through T.

II wins this play of the game, if $\overline{\{x_\alpha : \alpha < \kappa\}}^\kappa \subseteq \bigcup_{\alpha<\kappa} b_\alpha$, where

$\overline{X}^\kappa = X \cup \{t \in T :$ there is a κ-cofinal chain $K \subseteq X$ converging to $t.\}$.

Then we end up with the following general facts, mimics of Lemmas 2.20 and 2.21 respectively:

2.24 Lemma. *If κ is regular, the following are equivalent for any tree T of height δ:*

(1) *T is embeddable into $\mathbb{Q}^\kappa_{<\delta}$.*
(2) *$\mathcal{S}_T^-$ is club in $[\mathrm{H}_\theta]^\kappa$.*
(3) *II has a winning strategy in the game $\mathbb{G}_T^{\kappa-\mathrm{coh}}$.*

□

2.25 Lemma. *If κ is regular, the following are equivalent for κ-closed trees of height δ:*

(1) $T \cong \mathbb{Q}^{\kappa}_{<\delta}$.
(2) $\mathcal{U}_T$ *is club in* $[\mathrm{H}_\theta]^\kappa$.
(3) *II wins the game* $\mathbb{G}_T^{\kappa-\mathrm{coh}}$ *by playing cofinal branches.*

□

c. Substructure arguments

We are trying to apply what we just proved in a series of Corollaries. First, what is the structural difference between $\mathbb{Q}^{\mathrm{fin}}$ and its normal subtrees, the trivially coherent trees? The answer is given by the next Lemma.

2.26 Lemma. *The following are equivalent for any tree T of height δ:*

(1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\delta}$.
(2) (a) *T is trivially coherent,*
(b) *T is closed under chains of uncountable cofinality and*
(c) *for all $x \in T$ there is a cofinal branch b through x.*

Proof. (1) $\Longrightarrow$ (2): is immediate.

(2) $\Longrightarrow$ (1): we apply Lemma 2.21 and show that $\mathcal{U}_T$ is club. We actually show that $\mathcal{S}_T^- = \mathcal{U}_T$ in this case. So pick any $N \in \mathcal{S}_T^-$ and let $K \subseteq N \cap T$ be captured by $L \in N$, where $\lim(L)$ has minimal height. If $\lim(K)$ has height γ, then $\lim(L)$ has height $\gamma^* = \min(N \cap \mathrm{Ord} \setminus \gamma)$. Let us distinguish two cases:

(i): $\mathrm{cf}(\gamma^*) = \omega$. Now $\gamma = \gamma^*$ and therefore $\lim(K) \in N$. By (c), we can choose a cofinal branch $b \in N$ that goes through $\lim(K)$. This b cofinally captures the chain K.

(ii): $\mathrm{cf}(\gamma^*) > \omega$. If this happens, we know by (b) that L is converging in T. So let $t \in T$ be its limit point. Again by (c), there is a cofinal branch $b \in N$ that contains t and hence, cofinally captures K. □

Continuing in our applications, we see that Lemma 2.20 allows us to strengthen characterization (3) of Lemma 2.5 in case $\delta = \omega_1$:

2.27 Corollary. *The following are equivalent for any tree of height ω_1:*

(1) *T is trivially coherent.*
(2) *There is a regressive $f : T \restriction \mathrm{Lim}(\omega_1) \longrightarrow T$ such that the preimage of every point is a chain.*

Proof. In view of Lemma 2.5, we only need to show (2) $\Longrightarrow$ (1). For this, we choose a regressive f witnessing (2) and show that $\mathcal{S}_T^-$ is club: let $N \prec \mathrm{H}_\theta$ contain f as an element and take any N-chain $K \subseteq N \cap T$. If we set $t = \lim(K)$, we know that $f(t)$ has a countable height below the ordinal $N \cap \omega_1$, so $f(t) \in N$. But the f-preimage of $f(t)$ is a chain containing t, so its downward closure L is definable in N and will contain K. We proved that N is strongly T-simple. □

We present another game closely tied to the notion of coherence. Remember the convention that all trees are binary.

2.28 Definition. The game $\mathbb{G}_{\mathrm{i}}(T)$ is played as follows:

$$\begin{array}{c|cccccccccc} \mathrm{I} & x_0 & & x_1 & & x_2 & & x_3 & & \dots & \\ \hline \mathrm{II} & & i_0 & & i_1 & & i_2 & & i_3 & & \dots \end{array}$$

where $x_n \in T$, $i_n \in \{0,1\}$, $x_n{}^\frown i_n \leq_T x_{n+1}$ for all $n < \omega$ and

$$\text{II wins iff } x_n\ (n < \omega) \text{ does not converge in } T.$$

We will give a Lemma that is going to be of some importance later on in Part 5.c. It has to be compared with Davis' characterization of either countable or perfect sets of reals in terms of games (see [5]). His game is a special instance of our game in case when the considered tree has height ω:

2.29 Lemma. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*

(2) *II has a winning strategy in the game $\mathbb{G}_{\mathrm{i}}(T)$.*

Proof. (1) $\Longrightarrow$ (2): let $\sigma(x_0, \dots, x_n) = 1$. σ wins $\mathbb{G}_{\mathrm{i}}(T)$ for player II.

(2) $\Longrightarrow$ (1): we show that $\mathcal{S}_T^-$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$, where θ is big enough. In particular, every $N \prec \mathrm{H}_\theta$, containing a winning strategy σ, is strongly T-simple. Fix such an N and a chain $K \subseteq N$ converging in T. By going to a subchain, if necessary, we may assume that K has order-type ω.

Say that $\vec{x} = \langle x_0, \dots, x_n \rangle \in N$ is *nice*, if $\{x_l\}_{l \leq n} \subseteq K$ and furthermore $x_{l+1} \geq_T x_l{}^\frown\sigma(x_0, \dots, x_l)$ for all $l < n$.

2.29.1 Claim. *There is a maximal nice sequence $\vec{x}$.*

Proof. Assume that there isn't. In this case, there is a play

$$\begin{array}{c|cccccccccc} \mathrm{I} & x_0 & & x_1 & & x_2 & & x_3 & & \dots & \\ \hline \mathrm{II} & & i_0 & & i_1 & & i_2 & & i_3 & & \dots \end{array}$$

such that $i_l = \sigma(x_0, \dots, x_l)$ $(l < \omega)$ and $\{x_l\}_{l<\omega}$ is a cofinal subchain of K. But σ is winning, so K cannot converge, a contradiction. □

Choose such a sequence $\vec{x} = \langle x_0, \dots, x_n \rangle$, guaranteed by Claim 2.29.1. Define the following branch b within N: b is maximal with the property that

$$b \text{ extends } \bigcup_{l \leq n} x_l \text{ and for } \alpha \geq \mathrm{ht}(x_n), b(\alpha) = 1 - \sigma(\vec{x}{}^\frown b \restriction \alpha).$$

The last claim reveals that N is strongly T-simple.

2.29.2 Claim. *$K \subseteq b$.*

Proof. If not, let α_0 be the splitting point, i.e. the smallest point α such that $K(\alpha) \neq b(\alpha)$. Note that $\alpha_0 \in N$ and

$$j = \sigma(\vec{x}^\frown(b \restriction \alpha_0)) = 1 - b(\alpha_0).$$

Hence, $\vec{x}^\frown(b \restriction \alpha_0)^\frown j$ is nice, as it is inside of K and j is played according to σ. But this contradicts maximality of $\vec{x}$. $\square$

$\square$

Similar methods as in Lemma 2.20 yield the last result of this section:

2.30 Lemma. *The following are equivalent for any tree T:*

(1) *$\mathcal{S}_T^-$ is non-stationary in $[\mathrm{H}_\theta]^{\aleph_0}$.*

(2) *I has a winning strategy in the game $\mathbb{G}_T^{\mathrm{coh}}$.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(1) $\Longrightarrow$ (2): Again, choose a reasonable pairing function

$$\langle \quad , \quad \rangle : \omega \times \omega \longrightarrow \omega.$$

We define a winning strategy τ for player I: if the play begins with the moves

I	x_0		x_1		$\dots$	x_n			
II		b_0		b_1	$\dots$		b_n		

we let $N_k = Sk(b_i)_{i \leq k}$ and assume without restriction that N_k is not strongly T-simple for all $k \leq n$. Let $\langle k, l \rangle = n$ and set $\tau(b_0, \dots, b_n)$ to be the lth element of $N_k \cap T$ according to some enumeration of order-type ω.

2.30.1 Claim. *τ is winning for player I*

Proof. If there is a play according to τ, in which II plays $\{b_n\}_{n<\omega}$, then I plays all the points in $N_\omega \cap T$, where $N_\omega = \bigcup_{n<\omega} N_n$. N_ω is not strongly T-simple, so there is $K \subseteq N_\omega$ converging, but not captured. It follows that $x = \lim(K) \notin \bigcup_{n<\omega} b_n$, because if x were an element of b_n for some $n < \omega$, $N_\omega \supseteq \{b_n\}_{n<\omega}$ would capture K. $\square$

(2) $\Longrightarrow$ (1): we show that if an elementary $M \prec \mathrm{H}_\theta$ contains a winning strategy τ for player I, then it is not strongly T-simple. We enumerate the set of all branches in the substructure M, i.e. we let $\{b \in M : b \text{ is a branch through } T\} = \{b_n\}_{n<\omega}$. Consider the following play according to τ:

I	x_0		x_1		x_2		x_3		$\dots$	
II		b_0		b_1		b_2		b_3		$\dots$

Now $\{x_n\}_{n<\omega} \subseteq M$, since $\tau \in M$. But τ is winning, so there is a converging subsequence $x_{i(n)}$ $(n < \omega)$ which is not captured by $\{b_n\}_{n<\omega}$. This implies that M is not strongly T-simple. □

d. A dichotomy for trees

Here we give an independence result using the Proper Forcing Axiom. In fact, we introduce a dichotomy for trees that strengthens Suslin's Hypothesis but still follows from PFA.

2.31 Definition. A tree T of height κ^+ is called *special*, if there is a function $f : T \longrightarrow \kappa$ such that if $s \leq t, u$ and $f(s) = f(t) = f(u)$, then t and u are comparable.

Note that if the tree T has no cofinal branches, Definition 2.31 agrees with the notion of special for Aronszajn-trees as introduced on page 5.

The following has to be compared with Corollary 2.9.

2.32 Lemma. *For any special ω_1-tree T:*
either (1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\omega_1}$,
or (2) T has a stationary antichain.

Proof. Let $f : T \longrightarrow \omega$ specialize T, i.e. if there are $s \leq t, u$ with $f(s) = f(t) = f(u)$, then t and u are comparable.

Let T an ω_1-tree that is not isomorphic to $\mathbb{Q}^{\mathrm{fin}}_{<\omega_1}$. We know that there are three possibilities: either (a),(b) or (c) of Lemma 2.26 (2) is false. (b) is obviously void and if (c) were false, T would have a special Aronszajn-subtree that can easily be seen to contain a stationary antichain. So we may restrict ourselves to the case that T is not trivially coherent and by Lemma 2.20 we conclude that $\mathcal{S}^-_T$ does not contain a club. Since $T \cap N$ is transitive for any countable elementary N, we know that even $\mathcal{S}_T = \mathcal{S}^-_T$ does not contain a club, in other words $\mathcal{C}_T$ is stationary. We may assume that $f, T \in N$ for every $N \in \mathcal{C}_T$. As a consequence of the stationarity of $\mathcal{C}_T$, the projection to countable ordinals, i.e. $E = \{N \cap \omega_1 : N \in \mathcal{C}_T\}$, is stationary in ω_1. For every $\xi \in E$, pick one $N_\xi \in \mathcal{C}_T$ such that $\xi = N_\xi \cap \omega_1$ and a witness t_ξ for the T-complicatedness of N_ξ.

2.32.1 Claim. *$f(t) \neq f(t_\xi)$ for all $t <_T t_\xi$ $(\xi \in E)$.*

Proof. Assume that $f(t) = f(t_\xi)$ for some $t <_T t_\xi$ $(\xi \in E)$. Working in N_ξ, we define

$$b = \{u \in T : t \leq_T u \text{ and } f(t) = f(u)\}.$$

Note that this is possible, since t has countable height and is thus an element of N_ξ. Of course, every two elements of b will be comparable by the properties of f. So $b \in N_\xi$ and $t_\xi \in b$. This means that t_ξ is captured by $b \in N_\xi$, a contradiction. $\square$

Pressing Down via the specializing function f, we receive a stationary $E_0 \subseteq E$ such that $f(t_\xi)$ is constant for all $\xi \in E_0$. But with claim 2.32.1 accomplished, we know that $\{t_\xi : \xi \in E_0\}$ is in fact a stationary antichain in T. $\square$

Note that Lemma 2.32 has the following generalization:

2.33 Lemma. *If κ is regular, then for any special κ^+-tree T:*

either (1) T is embeddable into $\mathbb{Q}^{\kappa}_{<\kappa^+}$,
or (2) T has a stationary antichain on the κ-cofinals.

Proof. Copy the proof of Lemma 2.32 and apply Lemma 2.24 instead of Lemma 2.20. $\square$

We eventually point out an easy applications of this for ω_1-trees: PFA trivializes our coherence-business in a certain way.

2.34 Theorem. PFA *proves the following* ω_1**-tree dichotomy**.
For every ω_1-tree T:

either (1) $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\omega_1}$,
or (2) T has a stationary antichain.

Proof. Note that PFA implies that every tree of height ω_1 is special (see [1, p.951] for this) and we are done by Lemma 2.32. $\square$

2.35 Corollary. *The ω_1-tree dichotomy is consistent with* CH.

Proof. All we needed for the proof of Theorem 2.34 was the PFA-consequence that every tree of height ω_1 is special. Shelah [18, p.394] shows that this statement is consistent with CH. $\square$

Part 3. Coherent Aronszajn-trees

a. Non-trivial coherent sequences

We saw in the previous two sections that the most moderate way to construct a normal tree is to take a direct limit at every limit stage of the construction. Actually, we generated a full tree by a single function in allowing finite changes of values of the function, or equivalently we looked at all the mappings with finite support. If we expect to construct an Aronszajn-tree of that kind, we certainly have to relativize this: let us define a *coherent* sequence of functions to be a sequence $f_\alpha : \alpha \longrightarrow \alpha$ $(\alpha < \kappa)$ such that $f_\alpha =^* f_\beta \restriction \alpha$ for all $\alpha < \beta < \kappa$, where $=^*$ denotes equality modulo finite. The tree

$$T(f_\alpha : \alpha < \kappa) = \{f : \alpha \to \alpha \mid f =^* f_\alpha, \alpha < \kappa\}$$

will be the *coherent tree induced by* f_α $(\alpha < \kappa)$. Generally, a tree T of height κ is called *coherent*, if $T \subseteq T(f_\alpha : \alpha < \kappa)$ for a coherent sequence f_α $(\alpha < \kappa)$.

Note that these trees can be Aronszajn and we will actually construct some of them later. But if we really want them to be without cofinal branches, we have to consider *non-trivial* coherent sequences, i.e. sequences for which there is no $f : \kappa \longrightarrow \kappa$ such that $f_\alpha =^* f \restriction \alpha$ for all $\alpha < \kappa$. Otherwise, the whole sequence is generated by one function and only trees with branches will result. Indeed, non-triviality of the sequence is equivalent to the fact that $T(f_\alpha : \alpha < \kappa)$ is Aronszajn because of the following Lemma:

3.1 Lemma. *Let κ regular. If $T(f_\alpha : \alpha < \kappa)$ has a cofinal branch then every stationary $B \subseteq T(f_\alpha : \alpha < \kappa)$ contains a stationary chain $B_0 \subseteq B$.*

Proof. This is actually just a reproof of Lemma 2.8 in the case where κ is regular to get a slightly stronger conclusion. Assume that B is stationary in the tree $T(f_\alpha : \alpha < \kappa)$ and set $E = \{\mathrm{ht}(t) : t \in B\}$. We assume without restriction that B contains at most one point of every level of the tree, so let $t_\xi \in T_\xi$ be this point for all $\xi \in E$. Since $T(f_\alpha : \alpha < \kappa)$ has a cofinal branch, we can choose a function $f : \kappa \longrightarrow \kappa$ such that $f_\alpha =^* f \restriction \alpha$ for all $\alpha < \kappa$. Define a regressive mapping $h : E \longrightarrow \kappa$ by letting

$$h(\xi) = \max\{\gamma < \xi : t_\xi(\gamma) \neq f(\gamma)\} + 1.$$

Note that h is regressive for all limit ordinals $\xi \in E$ because the tree $T(f_\alpha : \alpha < \kappa)$ is coherent and f is a branch through it. By an application of the Pressing Down Lemma, there is a stationary $E_0 \subseteq E$ and

$\gamma_0 < \kappa$ such that $h(\xi) = \gamma_0$ for all $\xi \in E_0$. By a cardinality argument, there is another stationary $E_1 \subseteq E_0$ with $t_\zeta \restriction \gamma_0 = t_\xi \restriction \gamma_0$ for all $\zeta < \xi$ in E_1. Now, $B_0 = \{t_\xi : \xi \in E_1\}$ is the desired chain. $\square$

b. An axiomatic approach to coherent Aronszajn-trees

For the purpose of Section 3.b, we assume that all coherent Aronszajn-trees are *uniform*, i.e. they are closed under finite changes. In the language of Section 2.a, this is the same as saying that these trees are induced by some coherent sequence of functions $f_\alpha : \alpha \longrightarrow \alpha$ ($\alpha < \kappa$).

The next definition originates from [19], where the following notation is used:

$$T^{t_0} = \{t \in T : t \text{ is comparable with } t_0\}.$$

3.2 Definition. A κ-tree T is called *strongly homogeneous* if there is a family $\{h_{t_0,t_1} : t_0, t_1 \in T_\alpha, \alpha < \kappa\}$ of automorphisms with the following properties:

(1) $h_{t_0,t_1}(t_0) = t_1$ and h_{t_0,t_1} is the identity on $T \setminus (T^{t_0} \cup T^{t_1})$.
(2) (commutativity) $h_{s_0,s_2}(t_0) = h_{s_1,s_2}(h_{s_0,s_1}(t_0))$ holds whenever $s_0, s_1, s_2 \in T_\alpha$ and $s_0 \leq t_0$.
(3) (uniformity) If $s_0, s_1 \in T_\alpha$ with $s_0 \leq t_0$ and $s_1 \leq h_{s_0,s_1}(t_0) = t_1$ then $h_{t_0,t_1} = h_{s_0,s_1} \restriction T^{t_0}$.
(4) (transitivity) If α is a limit ordinal and $t_0, t_1 \in T_\alpha$, then there are $s_0, s_1 \in T_{<\alpha}$ such that $h_{s_0,s_1}(t_0) = t_1$.

We are going to show in the next Theorem that conditions (1)-(4) of Definition 3.2 are a precise characterization of uniform coherent trees. This means, (1)-(4) provide a structural description of these kinds of trees, very much in contrast to the original definition of coherent trees that depended upon the existence of an isomorphism onto some tree of functions $T(f_\alpha : \alpha < \kappa)$.

3.3 Note. *Every uniform coherent tree is strongly homogeneous.*

Proof. For the proof of this, it is actually important to assume that our tree is uniform. So we can assume that the coherent tree T is induced by some sequence $(f_\alpha : \alpha \to \alpha \,|\, \alpha < \kappa)$. Let $\alpha \leq \beta$, $s \in T_\beta$, $t_0, t_1 \in T_\alpha$ and define

$$h_{t_0,t_1}(s) = t_1 * s$$

where $(t_1 * s) \restriction \alpha = t_1$ and $(t_1 * s)(\gamma) = s(\gamma)$ for $\alpha \leq \gamma < \beta$. Then $(T, \subseteq)$ will be strongly homogeneous via $\{h_{t_0,t_1} : t_0, t_1 \text{ in a } T_\alpha\}$. $\square$

3.4 Theorem. *Every strongly homogeneous tree is isomorphic to a uniform coherent tree.*

Proof. From now on let $(T, \leq)$ a strongly homogeneous κ-tree via the family $h_{t_0,t_1} : t_0, t_1 \in T_\gamma (\gamma < \kappa)$ of automorphisms. For simplicity we assume that T is ω-splitting. We also assume without loss of generality that T has a root. Define $\pi : T \longrightarrow \{f : \alpha \to \omega \,|\, \alpha < \kappa\}$ inductively:

$\alpha = 0 : T_0 = \{\text{root}\}$, $\pi(\text{root}) = \emptyset$.

$lim(\alpha)$: for a $t \in T_\alpha$ set $\pi(t) = \bigcup \pi''\{s \in T : s < t\}$.

$\beta = \alpha + 1$: choose an $x \in T_\alpha$, let $\pi(x) = f : \alpha \to \omega$ and well-order the immediate successors of x by the enumeration $\{x_n : n < \omega\}$.

Define $\pi(x_n) = f \cup \{(\alpha, n)\}$ and for any other $s \in T_\beta$ set $r = s \upharpoonright \alpha$ and if $h_{r,x}(s) = x_m$, define $\pi(s) = \pi(r) \cup \{(\alpha, m)\}$.

3.4.1 Claim. $\pi(t)(\alpha) = \pi(h_{s_0,s_1}(t))(\alpha)$ *holds whenever* $\xi \leq \alpha < \beta$, $s_0, s_1 \in T_\xi$ *and* $t \in T_\beta$.

Proof. We may assume that $s_0 \leq t$. Now define $t' = h_{s_0,s_1}(t)$. We choose a 'master point' $x \in T_\alpha$ and a 'master enumeration' x_n $(n < \omega)$ of the immediate successors of x.

Let $\pi(t)(\alpha) = \pi(t \upharpoonright (\alpha + 1))(\alpha) = m$. Since $h_{s_0,s_1}(t \upharpoonright \alpha) = t' \upharpoonright \alpha$, an application of uniformity will yield

$$h_{t\upharpoonright\alpha, t'\upharpoonright\alpha}(t \upharpoonright (\alpha + 1)) = t' \upharpoonright (\alpha + 1).$$

By commutativity we can deduce

$$h_{x, t'\upharpoonright\alpha}(h_{t\upharpoonright\alpha, x}(t \upharpoonright (\alpha + 1))) = t' \upharpoonright (\alpha + 1),$$

so $h_{x,t'\upharpoonright\alpha}(x_m) = t' \upharpoonright (\alpha + 1)$. But this last equation finally means that $m = \pi(t' \upharpoonright (\alpha + 1))(\alpha) = \pi(t')(\alpha)$ and so we proved Claim 3.4.1. $\square$

3.4.2 Claim. *If* $t_0, t_1 \in T_\delta$ *then the set*

$$\{\alpha < \delta : \pi(t_0)(\alpha) \neq \pi(t_1)(\alpha)\}$$

is finite.

Proof. By induction on δ. There is nothing to show for successor steps, so let δ be limit: by transitivity choose $s_0, s_1 \in T_\eta$, $\eta < \delta$ such that $h_{s_0,s_1}(t_0) = t_1$. Claim 3.4.1 establishes the following equation:

$$\{\alpha < \delta : \pi(t_0)(\alpha) \neq \pi(t_1)(\alpha)\} = \{\alpha < \eta : \pi(s_0)(\alpha) \neq \pi(s_1)(\alpha)\}.$$

But this last set is finite by induction hypothesis. $\square$

3.4.3 Claim. *$\pi : T \longrightarrow \pi''T$ is an isomorphism.*

Proof. π is clearly order-preserving. To show that π is $1-1$, assume that $\pi(s_0) = \pi(s_1)$. We proceed by induction on β, so we can assume without restriction that $immpred(s_0) = immpred(s_1) = s$ (else use induction hypothesis). Then $h_{s,t}(s_0) = h_{s,t}(s_1)$ holds for any $t \in T_{\mathrm{ht}(s)}$ by the definition of π. But $h_{s,t}$ is an automorphism, so $s_0 = s_1$. $\square$

All that's left to show is the following Claim:

3.4.4 Claim. *$\pi''T$ is uniform coherent.*

Proof. It suffices to show the following: whenever $t \in T_\alpha, f = \pi(t)$ and $f =^* f' : \alpha \to \omega$ then $f' \in \pi''T$. But this is actually clear by checking the definition of π. $\square$

$\square$

Note that Theorem 3.4 can be further generalized if we consider trees that are coherent modulo λ (i.e. modulo sets of cardinality less than λ) where λ is regular instead of trees coherent modulo finite. The corresponding notion of *strong λ-homogeneity* has to be modified in the following way: change (transitivity) to (transitivity)$_\lambda$

(transitivity)$_\lambda$: for all ordinals δ of cofinality at least λ and $t_0, t_1 \in T_\delta$, there exist $s_0, s_1 \in T_{<\delta}$ such that $h_{s_0,s_1}(t_0) = t_1$.

Then a generalized theorem holds for these modified notions:

3.5 Theorem. *A λ-closed tree is strongly λ-homogeneous iff it is isomorphic to a uniform tree coherent modulo λ.*

Proof. A straightforward copy of the proof of Theorem 3.4. $\square$

c. Coherent Aronszajn-trees of larger height

Our aim is to construct coherent Aronszajn-trees of arbitrary height. In fact, the existence of a coherent ω_2-Suslin-tree has already been shown consistent by Veličković in [27] using a very strong combinatorial guessing principle that holds true in the constructible universe (called square with built-in-diamond). It is well-known and remarked in [19] that coherent ω_1-Suslin-trees are either constructed by $\diamondsuit$ or forced by a Cohen-real. Note also that there are various ZFC-constructions for coherent ω_1-Aronszajn-trees of the form

$$f_\nu : \nu \xrightarrow{\text{1-1}} \omega \ (\nu < \omega_1)$$

(see e.g.[16, p.70] or [22]). The following argument indicates that these constructions are not so easily generalized to ω_2.

3.6 Theorem. *There is no sequence* $(f_\nu : \nu < \omega_2)$ *with*

(i) $f_\nu : \nu \xrightarrow{1\text{-}1} \omega_1$ *and*
(ii) $f_\nu =^* f_\mu \restriction \nu$ *for* $\nu < \mu$.

Proof. Assume that $(f_\nu : \nu < \omega_2)$ satisfies (i) and (ii) and for every $\alpha < \omega_1, \nu < \omega_2$ let

$$F_\alpha(\nu) = \{\gamma < \nu : f_\nu(\gamma) < \alpha\}. \tag{3.1}$$

Set $\tau_\alpha(\nu) = \mathrm{otp}(F_\alpha(\nu)) < \omega_1$.

It's clear that for $\nu < \mu$, $F_\alpha(\nu) \subseteq^* F_\alpha(\mu)$ holds and thus

$$\nu \leq \mu \longrightarrow \tau_\alpha(\nu) < \tau_\alpha(\mu) + \omega. \tag{3.2}$$

The following is also true:

3.6.1 Claim.

$$\nu + \omega_1 \leq \mu \longrightarrow \exists \delta_{\nu,\mu} < \omega_1 \; \forall \alpha \geq \delta_{\nu,\mu} : \tau_\alpha(\nu) + \omega < \tau_\alpha(\mu).$$

Proof. To prove this, choose $\delta < \omega_1$ such that $\mathrm{otp}(\delta \cap f_\mu\text{"}[\nu, \mu)) \geq \omega + 1$. But then for all $\alpha \geq \delta$:

$$\begin{aligned} \tau_\alpha(\nu) + \omega &= \mathrm{otp}(\{\gamma < \nu : f_\mu(\gamma) < \alpha\}) + \omega \\ &< \mathrm{otp}(\{\gamma < \mu : f_\mu(\gamma) < \alpha\}) = \tau_\alpha(\mu). \end{aligned} \tag{3.3}$$

This proves the Claim. □

Now choose $\delta < \omega_1$, such that $\delta = \delta_{\lambda, \lambda+\omega_1}$ for stationarily many ω_1-cofinal λ's. Then for $\lambda < \lambda'$ such:

$$\tau_\delta(\lambda + \omega_1) < \tau_\delta(\lambda') + \omega < \tau_\delta(\lambda' + \omega_1), \tag{3.4}$$

but this contradicts $\tau_\alpha(\nu) < \omega_1$ for all $\alpha < \omega_1, \nu < \omega_2$. □

Another limitation is given by the next observation:

3.7 Note. *There are no coherent* ω_2*-Aronszajn-trees in the Levy-Collapse of a weakly compact cardinal to* ω_2*. Hence,* CH *does not imply the existence of a coherent* ω_2*-Aronszajn-tree.*

Proof. By [24], every ω_2-Aronszajn-tree in this model contains the complete binary tree ${}^{<\omega_1}2$. This violates coherence in a strong fashion. □

Remember that the folkloristic κ^+-Aronszajn-tree already follows from $\kappa^{<\kappa} = \kappa$. In this particular case, we need more than just cardinal arithmetic, but as we shall see, there is a construction of a coherent ω_2-Aronszajn-tree from the minimal possible assumption left, i.e. a $\square(\omega_2)$-sequence.

3.8 Theorem. *If $\Box(\kappa)$ holds then there is a coherent κ-Aronszajn-tree.*

Proof. Let C_ν ($\nu < \kappa$) be a $\Box(\kappa)$-sequence. Of course, we can choose a square sequence with $1 = \min(C_\nu)$ for all $\nu < \kappa$.

Inductively construct functions $f_\nu : \nu \longrightarrow \nu$ ($\nu \in \mathrm{Lim}\ \kappa$) with the following properties:

(i) $f_\lambda =^* f_\nu \restriction \lambda$ for all $\lambda < \nu < \kappa$,
(ii) all f_ν's are *almost one-to-one*, i.e. if $f_\nu(\alpha) = f_\nu(\beta) \neq 0$ with $\alpha < \beta < \nu$, then $\alpha = \beta$.

We will try to maintain two more induction hypotheses:

(iii) $C_\nu = \{\nu_\delta : \delta < \mathrm{otp}(C_\nu)\}$, where the ν_δ's are defined inductively:

$$\begin{aligned} \nu_0 &= 1 \\ \nu_{\delta+1} &= \text{the } \alpha > \nu_\delta \text{ with } f_\nu(\alpha) = \nu_\delta \\ \nu_\lambda &= \sup_{\delta<\lambda} \nu_\delta \text{ (as long as } \nu_\lambda < \nu). \end{aligned}$$

(iv) $\lambda \in C'_\nu \longleftrightarrow f_\lambda = f_\nu \restriction \lambda$.

Note that the right-to-left direction of (iv) follows from (iii). We now distinguish between the following cases:

Case A: $\sup C'_\nu = \nu$.

This is simple, just let $f_\nu = \bigcup_{\lambda \in C'_\nu} f_\lambda$. (i)-(iv) will be maintained.

Case B: $\lambda = \sup C'_\nu < \nu$ (i.e. cf $\nu = \omega$).

Case Ba: $\nu = \delta + \omega$ for a limit δ.

Let $C_\nu = C_\lambda \cup \{\lambda_n\}_{n<\omega}$, where $\lambda = \lambda_0 < \lambda_1 < \ldots < \nu$. We define f_ν by extending f_λ in the following way:

$$f_\nu(\alpha) = \begin{cases} f_\lambda(\alpha) & \text{if } \alpha < \lambda, \\ \lambda_n & \text{if } \alpha = \lambda_{n+1}, \\ f_\delta(\alpha) & \text{if } \alpha \in [\lambda, \delta) \setminus \{\lambda_n\}_{1 \leq n < \omega} \text{ and} \\ & \quad f_\delta(\alpha) \notin \mathrm{rng}(f_\lambda) \cup \{\lambda_n\}_{n<\omega}, \\ 0 & \text{else.} \end{cases}$$

Now $f_\lambda =^* f_\delta \restriction \lambda$ leads to $f_\delta =^* f_\nu \restriction \delta$, because the second case of the definition of f_ν is violated only finitely many times whenever $\alpha \in [\lambda, \delta)$ (except for the trivial case $f_\delta(\alpha) = f_\nu(\alpha) = 0$). (ii),(iii) and (iv) are immediately seen to hold by construction.

Case Bb: $\nu \neq \delta + \omega$ for any limit δ (i.e. ν is a limit of limits).

Again, let $C_\nu = C_\lambda \cup \{\lambda_n\}_{n<\omega}$ with $\lambda = \lambda_0 < \lambda_1 < \ldots < \nu$. We want to define f_ν to have the following properties:

(a) $f_\nu : \nu \longrightarrow \nu$ is almost one-to-one
(b) $f_\nu \restriction \lambda = f_\lambda$

(c) $f_\nu =^* f_{\delta_n}$ for all $n < \omega$
(d) $f_\nu(\lambda_{n+1}) = \lambda_n$.

Herefore, choose an increasing sequence δ_n $(n < \omega)$ of limit ordinals with $\delta_0 = \lambda$, $\sup_n \delta_n = \nu$ and an $a_0 \subseteq [\lambda, \nu)$ cofinal in ν with order-type ω such that $g : (\nu - a_0) \longrightarrow \nu$ defined as

$$g = \bigcup_{n<\omega} f_{\delta_n} \restriction ([\delta_{n-1}, \delta_n) - a_0)$$

is an almost one-to-one function. Now we define $a_1 = \{\lambda_n\}_{n<\omega}$ and $a_2 = \{\alpha < \nu : g(\alpha) \in a_1\}$. Set $A = a_0 \cup a_1 \cup a_2 \subseteq [\lambda, \nu)$ to formulate the following claim.

3.8.1 Claim. *A is cofinal in ν and has order-type ω.*

Proof. Since the claim is surely true for $a_0 \cup a_1$, it suffices to show that a_2 is either finite or ν-cofinal with order-type ω. Without any restriction, $a_0 \cap a_2 = \emptyset$ can be assumed for this. So suppose that a_2 is infinite. If a_2 was bounded in ν or had order-type bigger than ω, then in either case there would be $\beta < \nu$ and an infinite subset $\hat{a}_1 \subseteq a_1$ such that $g''(a_2 \cap \beta) = \hat{a}_1$. If this is so, pick $m < \omega$ such that $\delta_m > \beta$. By definition of g, $g \restriction \beta =^* f_{\delta_m} \restriction (\beta - a_0)$ and therefore ${f_{\delta_m}}''(a_2 \cap \beta) =^* \hat{a}_1$. Since $\hat{a}_1$ is always cofinal in ν, this contradicts the fact that $f_{\delta_m} : \delta_m \longrightarrow \delta_m$. $\square$

But Claim 3.8.1 is sufficient to change g on points in A in order to make (d) true and still maintain property (c). To see this, note that $\hat{g} = g \restriction (\nu - A)$ is such that

$$a_1 \cap \mathrm{dom}(\hat{g}) = a_1 \cap \mathrm{rng}(\hat{g}) = \emptyset,$$

so we may extend $\hat{g}$ to fulfill (a) and (d).

Finally define $f_\nu : \nu \longrightarrow \nu$ by letting

$$f_\nu(\alpha) = \begin{cases} g(\alpha) & \text{if} \quad \alpha \in (\nu - A), \\ \lambda_n & \text{if} \quad \alpha = \lambda_{n+1}, \\ 0 & \text{else.} \end{cases}$$

Now f_ν satisfies (a)-(d) and thus (i)-(iv). This finishes the construction of $(f_\nu : \nu \in \mathrm{Lim}\,\kappa)$.

3.8.2 Claim. *$(f_\nu : \nu < \kappa)$ induces a coherent κ-Aronszajn-tree.*

Proof. Note property (iv) of the construction:

$$\lambda \in C'_\nu \longleftrightarrow f_\lambda = f_\nu \restriction \lambda.$$

Since C_ν $(\nu < \kappa)$ is a $\square(\kappa)$-sequence, the following statement is a consequence of (iv):

$$\text{there is no unbounded } D \subseteq \operatorname{Lim} \kappa \text{ such that for all } \lambda < \nu \in D\text{: } f_\lambda \subseteq f_\nu. \tag{3.5}$$

Otherwise $C = \bigcup_{\lambda \in D} C_\lambda$ trivializes C_ν $(\nu < \kappa)$ and hence refutes the fact that this is a $\square(\kappa)$-sequence. But Lemma 3.1 applied to the set $B = \{f_\nu : \nu < \kappa\}$ now gives the statement of Claim 3.8.2. □

□

We can deduce by Lemma 3.1 that coherent Aronszajn-trees do not contain Aronszajn-trees of smaller height. This thought awards us with:

3.9 Corollary. *The following are equiconsistent under ZFC:*

(1) *there is a weakly compact cardinal,*
(2) *every ω_2-Aronszajn-tree contains an ω_1-Aronszajn-subtree.*

□

3.10 Corollary. *If $\square(\kappa)$ holds then there is a coherent κ-Aronszajn-tree of the form $f_\alpha : \alpha \longrightarrow 2$ $(\alpha < \kappa)$.*

Proof. We start with the coherent Aronszajn-tree induced by the sequence $f_\alpha : \alpha \longrightarrow \alpha$ $(\alpha < \kappa)$ constructed in Theorem 3.8. Now simply define a new sequence $g_\alpha : \alpha \longrightarrow 2$ $(\alpha < \kappa)$ by the formula

$$g_\alpha(\langle \zeta, \xi \rangle) = 1 \quad \text{iff} \quad f_\alpha(\zeta) = \xi,$$

where $\langle\ ,\ \rangle$ is any pairing function. We can look at the g_α's as being a code for the graph of f_α. It's easy to check that g_α $(\alpha < \kappa)$ is a coherent and non-trivial sequence with these properties inherited from the old sequence f_α $(\alpha < \kappa)$. □

We finally investigate the impact speciality has on coherent trees. It turns out that we can have coherent Aronszajn-trees both special and non-special.

3.11 Definition. If C_ν $(\nu < \kappa)$ is a $\square(\kappa)$-sequence, we define the associated tree $(\kappa, <^2)$ by letting

$$\alpha <^2 \beta \text{ if and only if } \alpha \in C'_\beta.$$

In this context, C_ν $(\nu < \kappa)$ is called *special* if the associated tree $(\kappa, <^2)$ is special.

The following Lemma can be found in [3, p.65], but we include the proof of it for convenience.

3.12 Lemma. *$\square_\lambda$ is equivalent to the existence of a special $\square(\lambda^+)$-sequence.*

Proof. For the left-to-right implication, we assume that C_α $(\alpha < \lambda^+)$ is a $\square_\lambda$-sequence. Our task is to specialize the tree $(\lambda^+, <^2)$, so we define a function $f : \lambda^+ \longrightarrow \lambda$ by letting $f(\alpha) = \mathrm{otp}(C_\alpha)$. Obviously, if $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$ then $\alpha \not<^2 \beta$.

Now we concentrate on the other direction, i.e. right-to-left: let $f : \lambda^+ \longrightarrow \lambda$ witness the speciality of the tree $(\lambda^+, <^2)$ with respect to the $\square(\lambda^+)$-sequence C_ν $(\nu < \lambda^+)$. Construct a continuous sequence $\beta_\nu(\xi)$ $(\xi \leq \theta_\nu)$ for every limit ordinal $\nu < \lambda^+$:

$$\begin{aligned} \beta_\nu(0) &= \text{the minimal limit point of } C_\nu, \\ \beta_\nu(\xi+1) &= \text{the first limit point } \beta > \beta_\nu(\xi) \text{ in } C_\nu \\ & \text{ with minimal possible } f\text{-value.} \end{aligned}$$

It is important to note that f really increases as ξ increases because f is a specializing function. When this process stops at the point $\beta_\nu(\theta_\nu)$, there are only finitely many points left in the set $C_\nu \cap (\beta_\nu(\theta_\nu), \nu)$, so we can define

$$\overline{C}_\nu = \{\beta_\nu(\xi) : \xi \leq \theta_\nu\} \cup (C_\nu \cap (\beta_\nu(\theta_\nu), \nu)).$$

Now the order-type of $\overline{C}_\nu$ is $\leq \lambda$ and $(\overline{C}_\nu : \nu < \lambda^+)$ is a $\square_\lambda$-sequence by the uniform definition of the sequence $\beta_\nu(\xi)$ $(\xi \leq \theta_\nu)$. $\square$

3.13 Theorem. *In the construction of Theorem 3.8, the following are equivalent:*

(1) *$T(f_\nu : \nu < \kappa)$ is special,*
(2) *C_ν $(\nu < \kappa)$ is special.*

Proof. For (1) $\Longrightarrow$ (2), choose a function g that specializes the constructed tree $T(f_\nu : \nu < \kappa)$. Now simply define a function $h : \kappa \longrightarrow \kappa$ by letting

$$h(\nu) = g(f_\nu).$$

Note that this function is regressive on reasonably closed ordinals. By property (iv), i.e. $\lambda \in C'_\nu \longleftrightarrow f_\lambda = f_\nu \restriction \lambda$, we know that h specializes the tree $(\kappa, <^2)$.

In (2) $\Longrightarrow$ (1), let the regressive $g : \kappa \longrightarrow \kappa$ specialize the tree $(\kappa, <^2)$. We use the normal Gödel pairing to induce a bijection

$$\varphi : [\kappa \times \kappa]^{<\omega} \longleftrightarrow \kappa.$$

Without restriction, $\varphi : [\nu \times \nu]^{<\omega} \longleftrightarrow \nu$ holds everywhere. We claim that the following function specializes the tree $T(f_\nu : \nu < \kappa)$:

$$h(f) = \langle g(\nu), \varphi(f) \rangle,$$

where $\nu = \mathrm{ht}(f)$ and $\varphi(f)$ codes the finitely many values of f differing from f_ν.

In order to see that h is specializing, we choose elements f, f' of T such that $\nu = \mathrm{ht}(f) < \mathrm{ht}(f') = \nu'$ and moreover we assume that $h(f) = h(f') = \langle \varepsilon_0, \varepsilon_1 \rangle$. In this case ε_1 codes a finite subset of ν and $\varepsilon_0 = g(\nu) = g(\nu')$ holds. But if f and f' would be compatible, then $f <_T f'$ would hold and since we coded properly the differences from the trunk $\langle f_\nu : \nu < \kappa \rangle$, $f_\nu <_T f_{\nu'}$ would result. By (iv), we conclude that ν is a limit point of $C_{\nu'}$ and obtain a contradiction to the fact that g specializes the sequence C_ν ($\nu < \kappa$). This finishes the proof. $\square$

3.14 Corollary.

(a) *If $\square_\lambda$ holds then there is a special coherent λ^+-Aronszajn-tree.*

(b) *If $\lambda \geq \omega_2$ and $\square(\lambda)$ holds then there is a non-special coherent λ-Aronszajn-tree.*

Proof. (a) is provided by Lemma 3.12.

For (b) we note that Todorčević has shown a way to construct a non-special $\square(\lambda)$-sequence from any given $\square(\lambda)$-sequence in [23]. $\square$

Part 4. Reflecting Coherence

a. Coherence might not reflect

4.1 Definition. A tree T is *locally coherent*, if $T_{<\gamma}$ is trivially coherent for every $\gamma < \mathrm{ht}(T)$.

4.2 Lemma. *The following are equivalent for any tree T:*

(1) *T is locally coherent.*
(2) *$\mathcal{L}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(1) $\Longrightarrow$ (2): take $N \prec \mathrm{H}_\theta$ that knows of our tree T. N will be locally T-simple, since N contains embeddings $\psi_\gamma : T_{<\gamma} \xrightarrow{\sim} \mathbb{Q}^{\mathrm{fin}}_{<\gamma}$ for all $\gamma \in N \cap \delta$.

(2) $\Longrightarrow$ (1): if $\gamma < \delta$, pick $N \in \mathcal{L}_T$ that contains the ordinal γ. By bounded T-simplicity, N is strongly $T_{<\gamma}$-simple, so $T_{<\gamma}$ is coherent using Lemma 2.20. $\square$

4.3 Corollary. *ω_1-trees are locally coherent.* $\square$

4.4 Lemma. *The following are equivalent for any tree T:*

(1) *T is trivially coherent.*
(2) *T is locally coherent and $\mathcal{S}_T$ is club in $[\mathrm{H}_\theta]^{\aleph_0}$.*

Proof. By Lemmas 2.20 and 4.2. $\square$

We want to exemplify that coherence does not necessarily reflect at the cardinality ω_2, i.e. there are locally coherent ω_2-trees that are not coherent. Remember that we already constructed a locally coherent tree that is not trivially coherent in Theorem 3.8. Now we are going to give trees of this sort that are not even coherent. The first interesting such example can be constructed by utilizing the next definition which is taken from [21, §4], where we let

$$S^\omega_\kappa = \{\xi < \kappa : \mathrm{cf}(\xi) = \omega\}.$$

4.5 Definition. Fix a converging sequence $\langle \eta_\xi(n) : n < \omega \rangle$ for every $\xi \in S^\omega_\kappa$. If $A \subseteq S^\omega_\kappa$, define

$$T(A) = \{f \in {}^{<\kappa}2 : \mathrm{supp}(f) = F \cup \{\eta_\xi(n) : n < \omega\}$$
$$\text{for some } \xi \in A \cup \{0\} \text{ and finite } F \subseteq \kappa \setminus \xi\}.$$

4.6 Note. *For $\xi \in S^\omega_\kappa$, define $t_\xi \in {}^\xi 2$ by* $\mathrm{supp}(t_\xi) = \{\eta_\xi(n) : n < \omega\}$. *Then $\{t_\xi : \xi \in A\}$ is an antichain in $T(A)$.* $\square$

4.7 Theorem. *Let κ a regular cardinal. If $E \subseteq S^\omega_\kappa$ is stationary non-reflecting, then $T(E)$ is locally coherent non-coherent.*

Proof. $T(E)$ is non-coherent by Corollary 2.9 and Note 4.6, because $T(E)$ has a cofinal branch and the set $\{t_\xi : \xi \in E\}$ is a stationary antichain.

In order to prove local coherence, pick $N \prec \mathrm{H}_\theta$ such that $E \in N$:

4.7.1 Claim. *$N \in \mathcal{L}_{T(E)}$, therefore $\mathcal{L}_{T(E)}$ is club.*

Proof. Take a bounded chain $K \subseteq N$ converging in $T(E)$ and let $\lim(K) = x \in T_\gamma$. We may assume that $x = t_\gamma$, so that $\gamma \in E$. Now define

$$\gamma^* = \min((N \cap \mathrm{Ord}) \setminus \gamma) < \kappa.$$

Case 1: $\mathrm{cf}(\gamma^*) = \omega$. Thus, $\gamma = \gamma^* \in N$, so t_γ is definable in N. But this means that K is captured by N.

Case 2: $\mathrm{cf}(\gamma^*) > \omega$. In N, there is a club $C \subseteq \gamma$ disjoint from E. Hence, $\gamma \notin E$, a contradiction. $\square$

This means we are done by Lemma 4.2. $\square$

4.8 Corollary. *If every locally coherent ω_2-tree is trivially coherent, then every stationary subset of $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ reflects.* $\square$

We remark that a similar construction works for $\square(\kappa)$-sequences:

4.9 Theorem. *Let κ a regular cardinal. If $\square(\kappa)$ holds then there is a locally coherent non-coherent κ-tree.*

Proof. Fix a $\square(\kappa)$-sequence $\vec{C} = \langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$. We define the tree $T(\vec{C})$ as follows:

$$T(\vec{C}) = \{f \in {}^{<\kappa}2 : \mathrm{supp}(f) = F \cup C_\alpha \text{ for some}$$
$$\text{limit ordinal } \alpha < \kappa \text{ and finite } F \subseteq \kappa \setminus \alpha\}.$$

First we note that $T(\vec{C})$ is really a normal tree when ordered by inclusion: if $f \in T(\vec{C})$ and $\gamma < \mathrm{dom}(f)$ we know that $f \restriction \gamma \in T(\vec{C})$ by coherence of $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa)\rangle$.

4.9.1 Claim. *$T(\vec{C})$ is locally coherent.*

Proof. Let $N \prec \mathrm{H}_\theta$ with $\vec{C} \in N$. If $K \subseteq N \cap T$ is a convergent chain, set $f = \lim(K)$ and $\gamma = \mathrm{ht}(f)$. In this context, we may assume that $\mathrm{supp}(f)$ is unbounded in γ, otherwise a capturing branch is easily defined within the substructure N. So the definition of $T(\vec{C})$ leaves only one remaining choice, i.e. $\mathrm{supp}(f) = C_\gamma$. In this case, define

$$\gamma^* = \min((N \cap \mathrm{Ord}) \setminus \gamma) < \kappa.$$

By elementarity, $\gamma \in C'_{\gamma^*}$, so we conclude that $C_\gamma = C_{\gamma^*} \cap \gamma$ by coherence. Let us define $g : \gamma^* \longrightarrow 2$ by $\mathrm{supp}(g) = C_{\gamma^*}$. Then $g \in N$ and $f \subseteq g$. This proves the Claim. □

Now we assume that $T(\vec{C})$ is trivially coherent. A familiar Pressing Down argument would yield an uncountable chain $\langle f_\alpha : \alpha \in B \rangle$, where $\mathrm{supp}(f_\alpha) = C_\alpha$. But then B' is a club trivializing the $\square(\kappa)$-sequence $\langle C_\alpha : \alpha \in \mathrm{Lim}(\kappa) \rangle$. So $T(\vec{C})$ is not trivially coherent and since the zero-sequence is a cofinal branch, it is non-coherent. □

b. An SPFA-result

The following arguments show that the constructions of Theorem 4.7 and 4.9 are not possible in ZFC alone. The statement that every locally coherent ω_2-tree is trivially coherent is actually consistent.

4.10 Definition. First note that a *finite continuous $\in$-chain* of submodels is a finite $\in$-chain that can be extended to a continuous $\in$-chain of length ω_1.

Again, let us fix a tree T and assume that every substructure referred to will contain this tree without further saying. Define the two posets $\mathcal{P}_T$ and $\mathcal{Q}_T$:

$$\mathcal{P}_T = \{p : p \text{ is a finite continuous } \in\text{-chain of models } N \prec \mathrm{H}_\theta.\}$$

$$\mathcal{Q}_T = \{p : p \text{ is a finite continuous } \in\text{-chain of models } N \prec \mathrm{H}_\theta \text{ such that either } \begin{array}{l} \bullet\ N \text{ is } T\text{-complicated, or} \\ \bullet\ \text{every } M \succ_{\omega_1} N \text{ is } T\text{-simple.}\} \end{array}$$

Both posets are ordered in the same way: let $q \leq p$ iff $q \supseteq p$ and

$$M \in q \setminus p, N \in p, M \in N \text{ implies } t_M \not\leq t_N, \text{ whenever } N, M \in \mathcal{C}_T.$$

4.11 Lemma. *For any tree T:*

(a) *$\mathcal{P}_T$ is proper.*
(b) *$\mathcal{Q}_T$ is semiproper.*

Proof. Fix $\delta = \mathrm{ht}(T)$.

(a): Pick an elementary $N \prec \mathrm{H}_\lambda$ for a large enough regular λ and a condition $p \in \mathcal{P}_T \cap N$. The following extension of p will be generic:

$$q = p \cup \{(\gamma, N \cap \mathrm{H}_\theta)\}, \text{ where } \gamma = N \cap \omega_1.$$

4.11.1 Claim. *q is $(N, \mathcal{P}_T)$-generic.*

Proof. Choose $\mathcal{D} \in N$ dense open and extend $q \geq r \in \mathcal{D}$. Now, let $n = |r \setminus N|$, $p_0 = r \cap N$ and $r \setminus N = \{N_1, \ldots, N_n\}$. For notational simplicity, we assume that all $N_1, \ldots, N_n$ are T-complicated. Working in N and using elementarity, we construct a tree $(S, \leq_S)$ of height $n+1$ with the following properties:

(a) elements of S are either $\mathrm{root}_S = p_0$ or tuples $s = (\alpha_s, K_s)$, where α_s is a countable ordinal and K_s a T-complicated substructure,
(b) if $s = (\alpha_s, K_s) \in S$ and $\{(\alpha_x, K_x)\}_{x \in \mathcal{I}_s}$ is the set of immediate successors of s, then $\{K_x\}_{x \in \mathcal{I}_s}$ is unbounded in $[\mathrm{H}_\theta]^{\aleph_0}$,
(c) every branch through S is a condition in $\mathcal{D}$.

Such a tree exists in N, since $r \in \mathcal{D}$ is a condition in the universe that guarantees the existence of arbitrary large copies of N_i in the dense open set $\mathcal{D}$ for every $1 \leq i \leq n$. Still in N, construct a path $B = \{p_0, s_1, \dots, s_n\}$ through S inductively: if s_l $(l < n)$ has been fixed, consider the set $X_l = \{(\alpha_x, K_x)\}_{x \in \mathcal{J}_l}$ of all immediate successors of s_l. Now apply the Dilworth-decomposition theorem (see [6]) to the partial order $\mathbb{A} = \{t_{K_x} : x \in \mathcal{J}_l\}$ with the inherited tree ordering: if there is an antichain of size $n+1$ in $\mathbb{A}$, choose $s_{l+1} = (\alpha_{l+1}, K_{l+1}) \in X_l$ such that $t_{K_{l+1}} \not\leq t_{N_i}$ for all $1 \leq i \leq n$. If there is no such antichain, represent

$$\mathbb{A} = \bigcup \{\mathcal{C}_0, \dots, \mathcal{C}_{n-1}\}$$

as a union of n-many chains $\mathcal{C}_i$. Let $\varepsilon < \delta$ be a bound for all the chains $\mathcal{C}_i$ not cofinal in T. Now by unboundedness of $\{K_x\}_{x \in \mathcal{J}_l}$, there is $x \in \mathcal{J}_l$ with $K = K_x \supseteq \{\mathcal{C}_0, \dots, \mathcal{C}_{n-1}\}$ and $\sup(K \cap \delta) > \varepsilon$. By this choice of K, we know that there is $i < n$ such that $b = \bigcup \mathcal{C}_i$ is a branch of length δ and $t_K \in \mathcal{C}_i$. Since $\mathcal{C}_i \in K$, we eventually get that t_K is captured by $b \in K$. This contradicts the T-complicatedness of K.

So we can construct a path $B = \{p_0, s_1, \dots, s_n\}$ through S such that $t_{K_j} \not\leq t_{N_i}$ for all $1 \leq j \leq n$ and $1 \leq i \leq n$. We are thus done by the fact that $B \cup r \in \mathcal{P}_T$ and $B \in \mathcal{D} \cap N$. □

(b): Pick an elementary $N \prec \mathrm{H}_\lambda$ for a large enough regular λ and a condition $p \in \mathcal{Q}_T \cap N$. If every $M \succ_{\omega_1} N \cap \mathrm{H}_\theta$ is T-simple, set $N^* = N$. Otherwise choose $M \succ_{\omega_1} N \cap \mathrm{H}_\theta$ for which t_M is defined and let

$$N^* = Sk_{\mathrm{H}_\lambda}(N \cup (M \cap \delta)).$$

4.11.2 Claim.

(i) *$N^* \cap \delta = M \cap \delta$*
(ii) *$N^* \succ_{\omega_1} N$*
(iii) *t_M witnesses T-complicatedness of $N^* \cap \mathrm{H}_\theta$, i.e. t_M is not captured by $N^* \cap \mathrm{H}_\theta$.*

Proof. See [3, p.18]. □

Now define the condition

$$q = p \cup \{(\gamma, N^* \cap \mathrm{H}_\theta)\}, \text{ where } \gamma = N^* \cap \omega_1.$$

The proof of the following Claim exactly matches the proof of Claim 4.11.1:

4.11.3 Claim. *q is $(N^*, \mathcal{Q}_T)$-generic, hence $(N, \mathcal{Q}_T)$-semigeneric.* □

□

4.12 Corollary. *If $\mathcal{C}_T$ is club then $\mathcal{Q}_T$ is equal to $\mathcal{P}_T$ and thence proper.*

□

4.13 Theorem. *Under* SPFA*, if* $\mathrm{cf}(\delta) \geq \omega_1$ *and $\mathcal{C}_T$ is stationary for a tree of height δ, then there is an ω_1-cofinal ordinal $\gamma \leq \delta$ such that $T_{<\gamma}$ contains a non-trivial antichain. Moreover, $\gamma = \delta$ if* $\mathrm{cf}(\delta) = \omega_1$.

Proof. In order to apply the Semiproper Forcing Axiom to the poset $\mathcal{Q}_T$, we fix an enumeration $\phi_N : \omega \longrightarrow N$ of every relevant submodel N and write $N(n)$ for $\phi_N(n)$. Now define the dense sets B_α, D^n_γ for all $\alpha < \omega_1, n < \omega$ and $\gamma \in \mathrm{Lim}(\omega_1)$:

$$B_\alpha = \{p \in \mathcal{Q}_T : \alpha \in \mathrm{dom}(p)\}, \tag{4.1}$$

$$D^n_\gamma = \{p \in \mathcal{Q}_T : \gamma \in \mathrm{dom}(p) \text{ and there is } \beta \in \gamma \cap \mathrm{dom}(p) \text{ such that } p(\gamma)(n) \in p(\beta)\}. \tag{4.2}$$

Note that D^n_γ is dense because conditions in $\mathcal{Q}_T$ are finite continuous $\in$-chains.

If we choose a generic filter for all B_α, D^n_γ defined above, we get a continuous $\in$-chain $\langle N_\xi : \xi < \omega_1\rangle$.

4.13.1 Claim. *$E = \{\xi < \omega_1 : N_\xi$ is T-complicated$\}$ is stationary.*

Proof. Assume that E is nonstationary. Then for every $M \prec \mathrm{H}_\theta$ with $\langle N_\xi : \xi < \omega_1\rangle \in M$, there is a club $C \subseteq \omega_1$ in M such that $C \cap E = \emptyset$. Hence

$$\gamma = M \cap \omega_1 = \sup(M \cap C) \in C,$$

i.e. N_γ is T-simple. Now we see that $N_\gamma \prec_{\omega_1} M$ holds, because M knows of the sequence $\langle N_\xi : \xi < \omega_1\rangle$ and hence $N_\gamma = \bigcup\{N_\xi\}_{\xi<\gamma} \subseteq M$. Of course $\gamma \subseteq N_\gamma$, so $N_\gamma \cap \omega_1 = M \cap \omega_1$ is true as well. By the definition of $\mathcal{Q}_T$ we get that M is T-simple. Thus, there is a club $\mathcal{D} = \{M \prec \mathrm{H}_\theta : \langle N_\xi : \xi < \omega_1\rangle \in M\} \subseteq [\mathrm{H}_\theta]^{\aleph_0}$ inside of $\mathcal{S}_T$, in contradiction to the assumption that $\mathcal{C}_T$ is stationary in $[\mathrm{H}_\theta]^{\aleph_0}$. □

4.13.2 Claim. *There is a stationary $E' \subseteq E$ such that $t_{N_\eta} \not\leq t_{N_\xi}$ for all $\eta < \xi \in E'$.*

Proof. By checking the definition of $\leq_{\mathcal{Q}_T}$ carefully, we learn that the set $\{\alpha \in \xi \cap E : t_{N_\alpha} < t_{N_\xi}\}$ is finite for all $\xi \in E$. This is because $t_{N_\alpha} < t_{N_\xi}$ only happens if $N_\alpha \in p$, where p forces that N_ξ is in the generic sequence. So let $\beta_\xi < \xi$ be a code for this finite set. By

Pressing Down, we receive a stationary $\bar{E}$ and β such that $\beta_\xi = \beta$ for all $\xi \in \bar{E}$. Now it can easily be seen that the statement of the claim holds for $E' = \bar{E} \setminus (\beta + 1)$. □

We set $\gamma_\xi = \sup(N_\xi \cap \delta)$ and $\gamma = \sup_{\xi<\omega_1} \gamma_\xi$. This is the time to note that $\gamma = \delta$ if $\mathrm{cf}(\delta) = \omega_1$ and $\gamma < \delta$ otherwise.

4.13.3 Claim. *$\{t_{N_\xi} : \xi \in E'\}$ is a non-trivial antichain in $T_{<\gamma}$.*

Proof. Note that $A = \{t_{N_\xi} : \xi \in E'\}$ is a stationary antichain in $T_{<\gamma}$ by Claim 4.13.2. It remains to show that no stationary subset $A_0 \subseteq A$ projects 1-1 into some level $\alpha < \gamma$, so fix $A_0 \subseteq A$ stationary and $\alpha < \gamma$. Now enumerate

$$N_{\omega_1} = \bigcup_{\xi<\omega_1} N_\xi = \{x_\alpha\}_{\alpha<\omega_1}$$

and assume without further restriction that $N_\xi = \{x_\alpha\}_{\alpha<\xi}$ for all ordinals $\xi \in E'$. Via this enumeration, we are able to define a regressive mapping on the stationary set $\{\xi \in E' : t_{N_\xi} \in A_0\} \setminus (\alpha + 1)$:

$$h(\xi) = \text{some } t <_T t_{N_\xi} \text{ in } M_\xi \text{ above the level } \alpha.$$

Pressing Down, we get a point $t_0 \in T$ above the level α and stationarily many points t_{N_ξ} with $t_0 <_T t_{N_\xi}$. This shows that A_0 does not project 1-1 into the level $\mathrm{ht}(t_0)$, so it does not project 1-1 into the level α. □

□

4.14 Corollary. *Under* SPFA*, if* $\mathrm{cf}(\delta) \geq \omega_2$ *then every locally coherent tree of height δ is trivially coherent.*

Proof. Assume T is not trivially coherent. By Lemma 4.4, we are able to assume that $\mathcal{C}_T$ is stationary in $[\mathrm{H}_\theta]^{\aleph_0}$. Theorem 4.13 states that there is $\gamma < \delta$ such that $T_{<\gamma}$ contains a non-trivial antichain. But this contradicts local coherence by Lemma 2.8. □

Now we see why there is no proper poset that does the job of $\mathcal{Q}_T$: Corollaries 4.14 and 4.8 show that we need to assume some amount of stationary reflection to reach the conclusion of Theorem 4.13. But it is proved in [2] that PFA does not reflect stationary sets.

The proof of Theorem 4.13 has a number of other corollaries. The first one is achieved by using Corollary 4.12:

4.15 Theorem. *Under* PFA*, if* $\mathrm{cf}(\delta) \geq \omega_1$ *and $\mathcal{C}_T$ is club for a tree of height δ, then there is an ω_1-cofinal ordinal $\gamma \leq \delta$ such that $T_{<\gamma}$ contains a non-trivial antichain.* □

Another observation is that SPFA can be replaced by the forcing version of PFA, i.e. PFA^+.

4.16 Theorem. *Under* PFA$^+$*, if* $\mathrm{cf}(\delta) \geq \omega_1$ *and* $\mathcal{C}_T$ *is stationary for a tree of height* δ*, then there is an* ω_1*-cofinal ordinal* $\gamma \leq \delta$ *such that* $T_{<\gamma}$ *contains a non-trivial antichain. Moreover,* $\gamma = \delta$ *if* $\mathrm{cf}(\delta) = \omega_1$*.*

Proof. Forcing with the proper poset $\mathcal{P}_T$, we get a generic sequence of models $\langle M_\xi : \xi < \omega_1\rangle$ in $V^{\mathcal{P}_T}$.

4.16.1 Claim.

$$V^{\mathcal{P}_T} \models \dot{E} = \{\xi < \omega_1 : M_\xi \textit{ is } T\textit{-complicated}\} \textit{ is stationary.}$$

Proof. Assume that $p \Vdash \dot{C}$ is club. In the extension, pick a T-complicated $N \prec \mathrm{H}_\theta{}^{V^{\mathcal{P}_T}}$ such that $p, \dot{C} \in N$. This can be done by stationarity of $\mathcal{C}_T$ in $V^{\mathcal{P}_T}$. Now let

$$\gamma = N \cap \omega_1 \quad \text{and} \quad q = p \cup \{(\gamma, N)\}.$$

Since $q \Vdash \gamma = N \cap \omega_1 = N \cap \dot{C} \in \dot{C}$, we finally have

$$q \Vdash \gamma \in \dot{E} \cap \dot{C}.$$

This proves the Claim. $\square$

With 4.16.1 accomplished and using PFA$^+$, we can choose a filter $G = \langle N_\xi : \xi < \omega_1\rangle$, generic for B_α, D^n_γ ($\alpha < \omega_1, n < \omega, \gamma \in \mathrm{Lim}(\omega_1)$) (as defined in (4.1),(4.2)) and with the additional property that the computed set $\dot{E}(G)$ is stationary in ω_1. Now continue exactly like in the proof of Theorem 4.13, i.e. repeat the proofs of Claims 4.13.2 and 4.13.3. $\square$

Like in the case of Corollary 4.14, we get:

4.17 Corollary. *Under* PFA$^+$*, if* $\mathrm{cf}(\delta) \geq \omega_2$ *then every locally coherent tree of height* δ *is trivially coherent.* $\square$

With an argument similar to the previous techniques, we can show that SR_2 determines the game $\mathbb{G}^{\mathrm{coh}}_T$:

4.18 Lemma. *Assume that* SR_2 *holds and* $\mathrm{cf}(\delta) \geq \omega_2$*. If* T *is a locally coherent tree of height* δ *with* $\mathcal{S}_T$ *stationary, then* T *is trivially coherent.*

Proof. Assume that $\mathcal{C}_T$ is stationary. Then SR_2 will provide a continuous $\in$-chain $\langle M_\xi : \xi \leq \omega_1\rangle$ of elementary submodels with the property that $E = \{\xi < \omega_1 : M_\xi \text{ is } T\text{-complicated}\}$ is stationary, co-stationary.

Let $\gamma_\xi = \sup(M_\xi \cap \delta)$ and $\gamma = \sup_{\xi<\omega_1} \gamma_\xi$. We will note that $\gamma < \delta$ by the large cofinality of δ.

4.18.1 Claim. $\{t_{M_\xi} : \xi \in E\}$ *contains a chain of order-type* ω_1.

Proof. By local coherence, we can pick an embedding $\pi : T_{<\gamma} \xrightarrow{\sim} \mathbb{Q}^{\text{fin}}_{<\gamma}$ and define $\varepsilon_\xi = \{\alpha < \gamma_\xi : \pi(t_{M_\xi}) \neq 0\}$. Like in preceding arguments, we Press Down with respect to a fixed enumeration of M_{ω_1}. For this let

$$g(\xi) = \text{ some } x \in M_\xi \cap T \text{ such that } x <_T t_{M_\xi} \text{ and } \operatorname{ht}(x) > \varepsilon_\xi.$$

The Pressing Down Lemma gives a stationary $E_0 \subseteq E$ and $x \in T$ such that $t_{M_\xi} = x^\frown\vec{0}$ for all $\xi \in E_0$. Thus, $\langle t_{M_\xi} : \xi \in E_0\rangle$ is an uncountable chain. □

But this contradicts the following Claim which will therefore finish the proof:

4.18.2 Claim. $\{\xi \in E : t_{M_\xi} \in b\}$ *is countable for all branches* $b \subseteq T$.

Proof. Assume that b is a branch through T hitting uncountably many t_{M_ξ}'s. Let us first check that the set

$$C = \{\xi < \omega_1 : b \cap M_\xi \text{ is cofinal in } M_\xi\}$$

is closed and unbounded. It is obviously closed and unboundedness holds since $E \subseteq C$. For all ξ in the stationary set $C \setminus E$, define

$$h(\xi) = \text{ some } L \in M_\xi \text{ such that } b \restriction \gamma_\xi \subseteq L.$$

Note that this is possible as M_ξ is T-simple and $\xi \in C$. Pressing Down again, there is $\zeta < \omega_1$ and a δ-branch $L \in M_\zeta$ such that $b \restriction \gamma \subseteq L$. Using the assumption about b, we pick $\eta > \zeta$ such that M_η is T-complicated and $t_{M_\eta} \in b$. But now $L \in M_\zeta \subseteq M_\eta$ contradicts the fact that t_{M_η} witnesses the T-complicatedness of M_η. □

□

Theorem 4.15 and Lemma 4.18 lead us to the following strengthening of both Corollary 4.14 and 4.17:

4.19 Theorem. *Under* PFA + SR_2, *if* $\operatorname{cf}(\delta) \geq \omega_2$ *then every locally coherent tree of height* δ *is trivially coherent.* □

Finally we note that if κ is regular, the statement above, i.e. the reflection of trivial coherence, is really a reflection of the tree $\mathbb{Q}^{\text{fin}}_{<\kappa}$ itself.

4.20 Theorem. *Let* κ *regular. Under* PFA + SR_2, *if* T *is a* κ*-tree such that*

$$T_{<\gamma} \cong \mathbb{Q}^{\text{fin}}_{<\gamma} \quad \textit{for all } \gamma < \kappa,$$

then $T \cong \mathbb{Q}^{\text{fin}}_{<\kappa}$.

Proof. By Theorem 4.19, T is trivially coherent. We use Lemma 2.26 to show that $T \cong \mathbb{Q}^{\mathrm{fin}}_{<\kappa}$. First note that by the assumption of T being locally isomorphic to $\mathbb{Q}^{\mathrm{fin}}_{<\kappa}$, every chain of uncountable cofinality converges in T. On the other hand, T does not contain any κ-Aronszajn-trees by Lemma 3.1, so (b) and (c) of Lemma 2.26 are true and we are done. $\square$

Part 5. Reformulating generic compactness

a. Reflecting games

We continue by approaching the problem, the reflection of coherence, from a different direction. This is done by introducing a new principle of stationary set reflection that holds true in the Levy Collapse of a weakly compact cardinal. In the following, it is a relevant concern of ours to point out that this principle of *Game Reflection* is no longer true in any of the other known collapses that make the continuum large. This will be shown in Theorem 5.9.

In what is going to follow, θ stands for an arbitrary regular cardinal.

5.1 Definition. A substructure $M \prec \mathrm{H}_\theta$ of size $\aleph_1$ is called *internally approachable* , if it is the limit of an increasing continuous $\in$-chain of countable elementary substructures, i.e. there is $\langle M_\xi : \xi < \omega_1\rangle$ such that $M = \bigcup_{\xi<\omega_1} M_\xi$.

5.2 Lemma. *Let $M \prec \mathrm{H}_\theta$ be of size $\aleph_1$. The following are equivalent under* CH*:*

(1) *M is internally approachable,*
(2) *${}^\omega M \subseteq M$.*

Proof. (1) $\Longrightarrow$ (2): let M be the union of the increasing continuous $\in$-chain $\langle M_\xi : \xi < \omega_1\rangle$. Now if $A \in [M]^{\aleph_0}$, there is $\zeta < \omega_1$ such that $A \subseteq M_\zeta$. But we are given that $\mathfrak{P}(M_\zeta) \in M_{\zeta+1} \subseteq M$ and $\mathfrak{P}(M_\zeta)$ has cardinality $\aleph_1$ by CH. Thus, $A \in \mathfrak{P}(M_\zeta) \subseteq M$ holds and we are done.

(2) $\Longrightarrow$ (1): let $\{x_\alpha\}_{\alpha<\omega_1}$ enumerate M. Build an increasing continuous $\in$-chain by letting

$$M_{\xi+1} = Sk(M_\xi, x_\xi).$$

The union of this chain will end up being exactly M since M is closed under countable sequences. $\Box$

5.3 Definition. If $\mathcal{A} \subseteq {}^{<\omega_1}\theta$, the game $\mathbb{G}(\mathcal{A})$ has length ω_1 and is played as follows:

$$\begin{array}{c|cccccccccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \dots & & \alpha_\xi & & \alpha_{\xi+1} & & \dots & \\ \hline \mathrm{II} & & \beta_0 & & \beta_1 & & \dots & & \beta_\xi & & \beta_{\xi+1} & & \dots \end{array}$$

both players I and II play ordinals below θ and

$$\text{II wins iff } \langle \alpha_\xi{}^\frown\beta_\xi : \xi < \omega_1\rangle \in [\mathcal{A}],$$

where $[\mathcal{A}] = \{f \in {}^{\omega_1}\theta : f \restriction \xi \in \mathcal{A} \text{ for all } \xi < \omega_1\}$.

For $B \subseteq \mathrm{H}_{(2^\theta)^+}$, define the game $\mathbb{G}^B(\mathcal{A})$ by letting the winning conditions be the same as in $\mathbb{G}(\mathcal{A})$ but imposing the restriction on both players to play ordinals in $B \cap \theta$.

5.4 Definition. The *Game Reflection Principle* or GRP is the following statement:

> Let $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$. If there is an ω_1-club $C \subseteq \omega_2$ such that II has a winning strategy in $\mathbb{G}^\alpha(\mathcal{A})$ for every $\alpha \in C$, then II has a winning strategy in $\mathbb{G}(\mathcal{A})$.

The *global Game Reflection Principle* or GRP^+ is the following statement:

> Let θ be regular and $\mathcal{A} \subseteq {}^{<\omega_1}\theta$. If there is an ω_1-club $\mathcal{C} \subseteq [\theta]^{\aleph_1}$ such that II has a winning strategy in $\mathbb{G}^B(\mathcal{A})$ for every $B \in \mathcal{C}$, then II has a winning strategy in $\mathbb{G}(\mathcal{A})$.

5.5 Note. *Let* IA *be the set of all internally approachable substructures of* $\mathrm{H}_{(2^\theta)^+}$ *of size* $\aleph_1$.

- IA *is* ω_1*-club in* $[\mathrm{H}_{(2^\theta)^+}]^{\aleph_1}$.
- $\mathrm{IA} \restriction \theta = \{M \cap \theta : M \in \mathrm{IA}\}$ *is* ω_1*-club in* $[\theta]^{\aleph_1}$.
- $\mathrm{IA} \restriction \omega_2 = \{M \cap \omega_2 : M \in \mathrm{IA}\}$ *is* ω_1*-club in* ω_2.

□

We use this note crucially: it suffices to show that II wins $\mathbb{G}^M(\mathcal{A})$ for all $M \in \mathrm{IA}$ to apply the Game Reflection Principle.

5.6 Remark.

(a) Notice that both formulations of the Game Reflection Principle are completely false when we replace the notion of an ordinary strategy with the notion of a point-strategy.
(b) We are going to show in Theorem 5.9 that GRP implies the Continuum Hypothesis. We can hence assume by Lemma 5.2 that all internally approachable substructures we consider are closed under countable sequences.

Let us note that there is a canonical ideal associated with GRP:

5.7 Definition. The *Game Reflection Ideal* $\mathcal{J}_{\mathrm{GR}}$ is defined as follows: for any $X \subseteq \omega_2$, let

$X \in \mathcal{J}_{\mathrm{GR}}$ iff there is $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$ such that II has no winning strategy for $\mathbb{G}(\mathcal{A})$, but II has a winning strategy for $\mathbb{G}^\alpha(\mathcal{A})$ whenever $\alpha \in X$.

Note that $\mathcal{J}_{\mathrm{GR}}$ is non-trivial if and only if GRP holds.

5.8 Lemma. $\mathcal{J}_{\mathrm{GR}}$ *is normal.*

Proof. Assume that X_ν ($\nu < \omega_2$) is a sequence in $\mathcal{J}_{\mathrm{GR}}$ witnessed by the games $\mathcal{A}_\nu$ ($\nu < \omega_2$). Build a game $\mathcal{B} \subseteq {}^{<\omega_1}\omega_2$ by letting player II choose the index ν for the game $\mathcal{A}_\nu$ that he wants to play and then resume with player I's first move. II will then win the game $\mathbb{G}(\mathcal{B})$ iff he wins the upcoming $\mathcal{A}_\nu$-game. Note that II does not have a winning strategy for $\mathcal{B}$ since he has none for the games $\mathcal{A}_\nu$ ($\nu < \omega_2$). But now let $\alpha \in \nabla_{\nu<\omega_2} X_\nu$. In this case, there is $\nu^* < \alpha$ such that $\alpha \in X_{\nu^*}$. We claim that player II has a winning strategy in the game $\mathbb{G}^\alpha(\mathcal{B})$ if he initiates it by selecting ν^*: since $\mathcal{A}_{\nu^*}$ witnesses that X_{ν^*} is a member of the ideal and $\alpha \in X_{\nu^*}$, we conclude that there is a winning strategy for II in the game $\mathbb{G}^\alpha(\mathcal{A}_{\nu^*})$. Therefore, II wins $\mathbb{G}^\alpha(\mathcal{B})$, and $\mathcal{B}$ witnesses that $\nabla_{\nu<\omega_2} X_\nu \in \mathcal{J}_{\mathrm{GR}}$. □

We will now fulfill the promise given at the beginning of this section. Theorem 5.9 shows that the Π^1_1-reflection of GRP implies CH. This contrasts the well-known fact that the tree property for ω_2, which is just as well a Π^1_1-reflection, implies $\neg$CH.

5.9 Theorem. *The Game Reflection Principle implies the Continuum Hypothesis.*

Proof. Assume that $2^{\aleph_0} \geq \aleph_2$. With the Axiom of Choice we can construct a *Bernstein set*, i.e. a set $B \subseteq \mathbb{R}$ of size continuum that does not contain a perfect subset (see e.g. [14, p.48]). So in particular, we get a set $A \subseteq \mathbb{R}$ of size $\aleph_2$ that does not contain a perfect subset. Enumerate $A = \{r_\beta : \beta < \omega_2\}$ and define the game $\mathbb{G}_{\mathrm{CH}}$ as follows: a typical play of this game is

$$\begin{array}{c|cccccccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \alpha_2 & & \alpha_3 & & \dots & \\ \hline \mathrm{II} & & i_0 & & i_1 & & i_2 & & i_3 & & \dots \end{array}$$

where $\alpha_n < \omega_2$ and $i_n \in \{0,1\}$ ($n < \omega$). We say that II wins the game if $\langle i_n : n < \omega\rangle = r_\beta \in A$ and $\beta > \sup_{n<\omega} \alpha_n$. Note that player II has a winning strategy in the game $\mathbb{G}^M_{\mathrm{CH}}$ for every internally approachable $M \prec \mathrm{H}_{(2^{\aleph_2})^+}$ since he can simply play the real $r_{(M\cap\omega_2)}$. By the Game Reflection Principle, II has a winning strategy σ for the game $\mathbb{G}_{\mathrm{CH}}$. From this we deduce a contradiction: let us identify our winning strategy with a function $\sigma : {}^{<\omega}\omega_2 \longrightarrow {}^{<\omega}2$ such that $\mathrm{lh}(\vec{\alpha}) = \mathrm{lh}(\sigma(\vec{\alpha}))$. For every $s \in {}^{<\omega}2$, choose a sequence of ordinals $\vec{\alpha}_s$ such that

(i) $\vec{\alpha}_r \subseteq \vec{\alpha}_s$ whenever $r \subseteq s$ are members of ${}^{<\omega}2$,
(ii) $\mathrm{lh}(r) = \mathrm{lh}(s) \longrightarrow \mathrm{lh}(\vec{\alpha}_r) = \mathrm{lh}(\vec{\alpha}_s)$,
(iii) $\sigma(\vec{\alpha}_{s^\frown 0}) \neq \sigma(\vec{\alpha}_{s^\frown 1})$ holds for every $s \in {}^{<\omega}2$.

Note that the length of $\vec{\alpha}_s$ typically differs from the length of s. Such a sequence exists because σ is a winning strategy for player II. If there were no such splitting, player I could predict the final outcome of II's choices and beat this very real. But now we claim:

5.9.1 Claim. $S = \{\sigma(\vec{\alpha}_s) \restriction n : s \in {}^{<\omega}2, n \leq \mathrm{lh}(\vec{\alpha}_s)\}$ *is a perfect tree.*

Proof. By condition (iii) in the construction of the sequences $\vec{\alpha}_s$, we know that for every element of S, there are two incomparable ones above. This is enough to prove our Claim. □

But $[S]$ is a subset of A, since for every branch through S there is a play associated to it and moreover it is played according to II's winning strategy σ. So A contains a perfect subset by Claim 5.9.1. This finishes the proof, because $A \supseteq X$ was chosen to avoid at least one point in every perfect set. □

The consistency of the Game Reflection Principle will be established in Corollary 5.18. We could go for this right now, but prefer to give an equivalent formulation first in Section 5.b and make the proofs more transparent.

Our next result points further down the road: no winning strategies are added in ω-closed forcing extensions. See the importance of it in the proofs of Theorems 5.14 and 5.17.

5.10 Lemma. *Let $\mathcal{A} \subseteq {}^{<\omega_1}\theta$ for some regular θ and $\mathcal{A} \in V$. If $\mathbb{P}$ is ω-closed and $\dot{\sigma}$ a $\mathbb{P}$-name for a winning strategy in the game $\mathbb{G}(\mathcal{A})$, then there is a winning strategy τ for $\mathbb{G}(\mathcal{A})$ in V.*

Proof. For simplicity we assume that $\dot{\sigma}$ is a name for a winning strategy of player II. We construct a winning strategy τ for player II in the ground model, using ω-closedness of $\mathbb{P}$.

Define τ as follows: first choose a condition $p_\emptyset \Vdash \dot{\sigma}$ is winning. Now for all countable sequences of ordinals $s = \langle \alpha_\xi : \xi < \gamma \rangle$, find conditions $p_s \in \mathbb{P}$ such that $p_s \Vdash \dot{\sigma}(\alpha_\xi : \xi < \gamma) = \beta_s$ for some $\beta_s \in V$ and

$$s \subseteq s' \longrightarrow p_s \geq p_{s'}.$$

Finally, let $\tau(\alpha_\xi : \xi < \gamma) = \beta_s$. We show that τ is winning for player II: for if there is a play

$$\begin{array}{c|cccccc} \mathrm{I} & \alpha_0 & \alpha_1 & \dots & \alpha_\xi & \alpha_{\xi+1} & \dots \\ \hline \mathrm{II} & \beta_0 & \beta_1 & \dots & \beta_\xi & \beta_{\xi+1} & \dots \end{array}$$

according to τ, we claim that II wins this play. Assume otherwise, then there is $\gamma < \omega_1$ and a sequence $s = \langle \alpha_\xi : \xi < \gamma \rangle$ such that $\langle \alpha_\xi{}^\frown \beta_\xi : \xi < \gamma \rangle \notin \mathcal{A}$. Remember that we constructed our tree of conditions in a way such that $p_s \Vdash$ 'II wins the play α_ξ, β_ξ $(\xi < \gamma)$', since p_s extends $p_\emptyset$. We choose any $\mathbb{P}$-generic filter H in V that contains the condition p_s. This will make the following true:

$$V[H] \models \langle \alpha_\xi{}^\frown \beta_\xi : \xi < \gamma \rangle \in \mathcal{A}. \tag{5.1}$$

But $\mathcal{A}$ is in the ground model, so we conclude

$$V \models \langle \alpha_\xi {}^\frown \beta_\xi : \xi < \gamma \rangle \in \mathcal{A}. \tag{5.2}$$

We have reached a contradiction. This proves that τ is a winning strategy for II in V. □

In the near future, we are sometimes going to play with arbitrary objects instead of ordinals. This is no restriction though, because only the cardinality of the underlying set matters: just fix any enumeration, make sure that it appears in all referred to structures and define the payoff-set relative to this enumeration. Note that we have to pay attention to this only in the case of the weaker GRP, where the underlying set is supposed to have cardinality at most $\aleph_2$.

b. Generic large cardinals

Now we shall prove a main result of this part: we can characterize the new principle of Game Reflection in terms of generic embeddings. The project of axiomatizing mathematics with the help of generic large cardinals has recently been pursued by Cummings and Foreman (see [4] and [9]). These mathematical universes, that we are going to live in while proving Theorems 5.12,5.14 and 5.17, have not been as carefully axiomatized as the respective forcing and reflection axioms that go along with ¬CH. Examples of known axioms in this well-studied 'other world' are SRP, SPFA or Woodin's (*) (see e.g. [29]).

5.11 Definition. We redefine the hierarchy of large cardinals modulo forcing extensions. The following properties can be true for smaller cardinals as well. Let κ a cardinal and Γ a class of posets.

κ is *generically weak compact by* Γ, if whenever the transitive structure $M \models \mathrm{ZFC}^-$ is of size κ with $\kappa \in M$, then there is $\mathbb{P} \in \Gamma$ such that the generic extension $V^{\mathbb{P}}$ supports $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \kappa$.

κ is *generically supercompact by* Γ, if for every regular λ there is $\mathbb{P} \in \Gamma$ such that $V^{\mathbb{P}}$ supports $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$ and $j''\lambda \in M$.

It is usually the case that generic large cardinals have the same consistency strength as their classical counterparts. A famous exception to this rule has been the notion of *generically almost huge* though (see [15] and [8]). But it might not entail such logical strength if a cardinal κ is generically weak compact, supercompact, etc. by the class of all posets. From now on, we will restrict Γ to the class of all ω-closed posets. This turns out to have considerable impact on the combinatorics of the cardinal κ (see Section 5.c).

We have the well-known fact:

5.12 Theorem. *Let* $\mathbb{P} = Coll(\omega_1, <\kappa)$.

(a) *If* κ *is weakly compact then*

$V^{\mathbb{P}} \models \omega_2$ *is generically weak compact by* ω*-closed forcing.*

(b) *If* κ *is supercompact then*

$V^{\mathbb{P}} \models \omega_2$ *is generically supercompact by* ω*-closed forcing.*

Proof. **(a):** Assume that G is $\mathbb{P}$-generic. If in $V^{\mathbb{P}}$, $M^* \models \mathrm{ZFC}^-$ is of size κ and contains κ as an element, choose $M \models \mathrm{ZFC}^-$ of size κ in the ground model such that $M^* \subseteq M[G]$. This can be accomplished by taking the Skolem Hull of a big enough set of names. We may assume without restriction that $M^* = M[G]$. Now since κ is weakly compact, there is $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \kappa$. Note that

$$j(\mathbb{P}) = \mathbb{P}_{j(\kappa)} = Coll(\omega_1, < j(\kappa)),$$

so we can identify $j(\mathbb{P}) \cong \mathbb{P} * \mathbb{S}$, where $\mathbb{S} = Coll(\omega_1, [\kappa, j(\kappa)))$. Let H be $\mathbb{S}$-generic over $V[G]$. Using Silver's argument (see Lemma 1.3), we can extend j to

$$j^* : M[G] \longrightarrow N[G * H].$$

This j^* exists in $V^{j(\mathbb{P})} = V^{\mathbb{P}*\mathbb{S}}$, but $\mathbb{S}$ is ω-closed, so we are done.

(b): This is an easy variation of (a). Fix $j : V \longrightarrow M$ in the ground model such that $\mathrm{cp}(j) = \kappa$ and $j''\lambda \in M$ for some λ. Just like before, we can identify $j(\mathbb{P})$ with $\mathbb{P} * \mathbb{S}$, where $\mathbb{S} = Coll(\omega_1, [\kappa, j(\kappa)))$. If G is $\mathbb{P}$-generic over V and H is $\mathbb{S}$-generic over $V[G]$, Silver's Lemma will apply again to provide us with

$$j^* : V[G] \longrightarrow M[G * H].$$

Of course, $j''\lambda \in M[G * H]$ and $\mathbb{S}$ is once more ω-closed. This finishes the proof. $\square$

5.13 Definition. Let $\mathcal{F} \subseteq {}^X\lambda$, for λ an ordinal and X an arbitrary set. Define the *filter-game* $\mathfrak{G}(\mathcal{F})$:

$$\begin{array}{c|cccccccccccc} \mathrm{I} & f_0 & & f_1 & & \dots & & f_\xi & & f_{\xi+1} & & \dots & \\ \hline \mathrm{II} & & \alpha_0 & & \alpha_1 & & \dots & & \alpha_\xi & & \alpha_{\xi+1} & & \dots \end{array}$$

where $f_\xi \in \mathcal{F}$, $\alpha_\xi < \lambda$ ($\xi < \omega_1$). II wins if the set

$$\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$$

contains at least 2 elements for every $\gamma < \omega_1$.

5.14 Theorem. *The following are equivalent:*

(1) GRP
(2) *II has a winning strategy in the game $\mathfrak{G}(\mathcal{F})$ for every $\mathcal{F} \subseteq {}^{\omega_2}\omega_1$ of size $\aleph_2$.*
(3) *ω_2 is generically weak compact by ω-closed forcing.*

Proof. (1) $\Longrightarrow$ (2): let $\mathcal{F}$ be any collection of functions from ω_2 to ω_1 of size $\aleph_2$ and take an internally approachable $K \prec \mathrm{H}_{(2^{\aleph_2})^+}$. We claim that player II wins the game $\mathfrak{G}^K(\mathcal{F})$: in the ξth move, he chooses $\alpha_\xi = f_\xi(\delta)$, the image of the point $\delta = K \cap \omega_2$. Since K is closed under countable sequences by Remark 5.6(b), the set $\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$ is in K and contains δ for all $\gamma < \omega_1$. So this intersection is stationary in ω_2 by elementarity. By GRP, II has a winning strategy for the game $\mathfrak{G}(\mathcal{F})$.

(2) $\Longrightarrow$ (3): let $M \models \mathrm{ZFC}^-$ be of size $\aleph_2$ with $\omega_2 \in M$ and set $\mathcal{F} = M \cap {}^{\omega_2}\omega_1$. We fix a winning strategy σ for player II in the game $\mathfrak{G}(\mathcal{F})$. Now look at the game $\mathfrak{G}(\mathcal{F})$ in $V^{Col(\omega_1,\omega_2)}$: let I play an enumeration $\{f_\xi : \xi < \omega_1\}$ of $\mathcal{F}$ of order-type ω_1. The game proceeds:

I	f_0	f_1	$\dots$	f_ξ	$f_{\xi+1}$	$\dots$
II	α_0	α_1	$\dots$	α_ξ	$\alpha_{\xi+1}$	$\dots$

where the α_ξ's are played according to σ. Define the M-filter $\mathcal{U}$ by letting $\mathcal{U} = \{f_\xi^{-1}(\alpha_\xi) : \xi < \omega_1\}$. Since σ wins $\mathfrak{G}(\mathcal{F})$, we are given that $\mathcal{U}$ is a complete ultrafilter with respect to M, i.e. $\mathcal{U}$ is closed under ω_1-sequences in M and for every set $A \subseteq \omega_2$ in M, either A or its complement is in $\mathcal{U}$. For non-triviality of $\mathcal{U}$ we use the fact that $\bigcap_{\xi<\gamma} f_\xi^{-1}(\alpha_\xi)$ contains at least 2 elements whenever $\gamma < \omega_1$. Hence, we can build the generic ultrapower ${}^{\omega_2}M/\mathcal{U}$ in M itself and get an embedding[2]

$$j : M \longrightarrow N$$

where N is the transitive collapse of ${}^{\omega_2}M/\mathcal{U}$. Note that this does not depend on the ultrafilter $\mathcal{U}$ being in M. The fact that $j \restriction \omega_2 = \mathrm{id} \restriction \omega_2$ follows from M-completeness of $\mathcal{U}$. Of course, $j(\omega_2) > \omega_2$ by non-triviality.

(3) $\Longrightarrow$ (1): Let $\mathcal{A} \subseteq {}^{<\omega_1}\omega_2$ and assume that there is an ω_1-club $C \subseteq \omega_2$ such that II has a winning strategy in $\mathbb{G}^\alpha(\mathcal{A})$ for all $\alpha \in C$. Let $M = Sk(\omega_2, \mathcal{A}, C, \alpha)_{\alpha<\omega_2}$ in $\mathrm{H}_{(2^{\aleph_2})^+}$. This M is a model of ZFC^-, contains ω_2 and has size $\aleph_2$. Thus, we find an elementary $j : M \longrightarrow N$ with $\mathrm{cp}(j) = \omega_2$ in some ω-closed extension $V^{\mathbb{P}}$. Let us work in this extension. We have:

[2]Arguments related to the equivalence between complete ultrafilters and elementary embddings are very well understood. For details, the reader is referred to the standard work [12, §5].

5.14.1 Claim. $\omega_2 \in j(C)$

Proof. Note that the following is true:

$$j''C = j(C) \cap \omega_2. \tag{5.3}$$

But $j(C)$ is closed under sequences of length less than $j(\omega_2)$, so we are done. □

Using elementarity, the claim gives that II has a winning strategy for the game $\mathbb{G}^{\omega_2}(j(\mathcal{A}))$. Now we need the equation:

$$j''\mathcal{A} = j(\mathcal{A}) \cap {}^{<\omega_1}\omega_2. \tag{5.4}$$

(5.4) yields that II has a winning strategy for the game $\mathbb{G}^{\omega_2}(j''\mathcal{A})$ and since j is the identity on ω_2, a winning strategy for $\mathbb{G}^{\omega_2}(\mathcal{A}) = \mathbb{G}(\mathcal{A})$ follows. It is still necessary to pull this winning strategy back into the ground model, but this can be done by Lemma 5.10. □

Let us define *generically measurable by* Γ analogous to Definition 5.11 and say that κ has this property if there is $\mathbb{P} \in \Gamma$ such that $V^{\mathbb{P}}$ supports $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \kappa$. It is possible to drop the restriction on the size of the algebra $\mathcal{F}$ and modify the previous proof. We would be given the following fact:

5.15 Corollary. *The following are equivalent:*

(1) *II has a winning strategy in the game $\mathfrak{G}({}^{\omega_2}\omega_1)$.*
(2) *II has a winning strategy in the game $\mathfrak{G}(\{f \in {}^{\omega_2}\omega_2 : f(\alpha) < \alpha\})$.*
(3) *ω_2 is generically measurable by ω-closed forcing.*

Proof. (2) $\Longrightarrow$ (1) is trivial and (1) $\Longrightarrow$ (3) follows the corresponding lines of the proof of Theorem 5.14. (3) $\Longrightarrow$ (2) is similar to the proof of Theorem 5.14 too, so we only sketch it: we describe the winning strategy for player II in the game $\mathfrak{G}(\{f \in {}^{\omega_2}\omega_2 : f(\alpha) < \alpha\})$. If in the ξth move player I plays the regressive $f_\xi : \omega_2 \longrightarrow \omega_2$, look at the regressive function

$$j(f) : j(\omega_2) \longrightarrow j(\omega_2)$$

and answer with $\alpha_\xi = j(f)(\omega_2)$. By elementarity arguments, II is going to win this play. □

A very powerful way of constructing a generic embedding is to force with $\mathcal{I}^+$, the collection of all positive sets with respect to some precipitous ideal $\mathcal{I}$. Laver proved consistent that there be a normal ideal on ω_2 that is *σ-dense*, i.e. $\mathcal{I}^+$ contains a σ-closed dense set. This means, of course, that $\mathcal{I}^+$ is an ω-closed poset, so we have the following:

5.16 Corollary. *If there is a normal σ-dense ideal on ω_2 then ω_2 is generically measurable by ω-closed forcing and hence* GRP *holds.* □

5.17 Theorem. *The following are equivalent:*

(1) GRP^+

(2) *For every regular* λ*, II has a winning strategy in the game* $\mathfrak{G}(\mathcal{F}_\lambda)$*, where*

$$\mathcal{F}_\lambda = \{f : [\lambda]^{\omega_1} \longrightarrow \lambda \mid f(A) \in A \text{ for every } A \text{ in } [\lambda]^{\omega_1}\}.$$

(3) ω_2 *is generically supercompact by* ω*-closed forcing.*

Proof. (1) $\Longrightarrow$ (2): let λ be regular. Again, all we need is a winning strategy for player II in the game $\mathfrak{G}^K(\mathcal{F}_\lambda)$ whenever $K \prec \mathrm{H}_{(2^\lambda)^+}$ is internally approachable and $\mathcal{F}_\lambda$ as in (2). Our strategy is similar: we answer the regressive function $f_\xi : [\lambda]^{\omega_1} \longrightarrow \lambda$ with $\alpha_\xi = f_\xi(K \cap \lambda)$. Such an answer is possible since $\alpha_\xi \in K$ by regressivity of f_ξ. Note that every subset of $[\lambda]^{\omega_1}$ in K is unbounded if it contains the element $K \cap \lambda$. This makes our strategy winning just like in the proof of Theorem 5.14.

(2) $\Longrightarrow$ (3): we continue in the same fashion as before. By collapsing $|\mathcal{F}_\lambda|$ to ω_1 with the ω-closed poset $Col(\omega_1, |\mathcal{F}_\lambda|)$, we fix an enumeration $\{f_\xi : \xi < \omega_1\}$ of $\mathcal{F}_\lambda$ of order-type ω_1 in the generic extension. We let player I play all functions in this enumeration and take into account II's replies α_ξ that make him win. Define the filter $\mathcal{U} = \{f_\xi^{-1}(\alpha_\xi) : \xi < \omega_1\}$, this time on the underlying set $[\lambda]^{\omega_1}$. Now $\mathcal{U}$ is a V-complete ultrafilter that is moreover normal. Let us build the generic ultrapower ${}^{[\lambda]^{\omega_1}}V/\mathcal{U}$ in V, yielding an elementary embedding[3]

$$j : V \longrightarrow M$$

where M is the transitive collapse of ${}^{[\lambda]^{\omega_1}}V/\mathcal{U}$. We check the properties of j: once again, $j \restriction \omega_2 = \mathrm{id} \restriction \omega_2$ holds by V-completeness of $\mathcal{U}$. It is an easily reviewed fact of ultrapower-embeddings with normal ultrafilters that $j''\lambda = [\mathrm{id}]_{\mathcal{U}}$, so $j''\lambda \in M$ is immediate. Finally, notice that $\mathrm{otp}([\mathrm{id}]_{\mathcal{U}}) < j(\omega_2)$ since this inequality holds really everywhere. But obviously,

$$\lambda = \mathrm{otp}(j''\lambda) = \mathrm{otp}([\mathrm{id}]_{\mathcal{U}}) < j(\omega_2).$$

(3) $\Longrightarrow$ (1): Let $\mathcal{A} \subseteq {}^{<\omega_1}\theta$ and assume that there is an ω_1-club $\mathcal{C} \subseteq [\theta]^{\aleph_1}$ such that II has a winning strategy in $\mathbb{G}^B(\mathcal{A})$ for all $B \in \mathcal{C}$. Choose $\lambda > \theta^{\aleph_1}$ and an ω-closed partial order $\mathbb{P}$ such that in $V^{\mathbb{P}}$ there is $j : V \longrightarrow M$ with $\mathrm{cp}(j) = \omega_2$, $j(\omega_2) > \lambda$ and $j''\lambda \in M$. Set $B = j''\theta \in M$. From now on we work in $V^{\mathbb{P}}$.

5.17.1 Claim. *The following two equations can be established:*

$$j''\mathcal{A} = j(\mathcal{A}) \cap {}^{<\omega_1}B. \tag{5.5}$$

[3]Basic facts about supercompact embeddings can be looked up in [12, §22].

$$j''\mathcal{C} = j(\mathcal{C}) \cap [B]^{\aleph_1}. \tag{5.6}$$

Proof. Consider (5.5): if $y \in j(\mathcal{A})$, then y is of the form $j(x)$ if and only if $y \in {}^{<\omega_1}B$. But $x \in \mathcal{A}$ if and only if $y = j(x) \in j(\mathcal{A})$. □

5.17.2 Claim. $B \in j(\mathcal{C})$

Proof. By elementarity we know that $j(\mathcal{C})$ is closed under sequences of length less than $j(\omega_2)$. From (5.6) we deduce that $j(\mathcal{C})$ is unbounded in $[B]^{\aleph_1}$, so B is the union of a directed system in $j(\mathcal{C})$ and $B \in j(\mathcal{C})$ holds. □

By Claim 5.17.2 and elementarity, II wins the game $\mathbb{G}^B(j(\mathcal{A}))$. By (5.5) we have that II wins $\mathbb{G}^B(j''\mathcal{A})$. But $j : \theta \longrightarrow B$ is one-to-one, so II wins $\mathbb{G}^\theta(\mathcal{A}) = \mathbb{G}(\mathcal{A})$. All these arguments take place in $V^{\mathbb{P}}$, so this winning strategy might only live in the generic extension, but we are done by an application of Lemma 5.10. □

5.18 Corollary. *Let $\mathbb{P} = Coll(\omega_1, < \kappa)$.*

(a) *If κ is weakly compact then $V^{\mathbb{P}} \models$ GRP.*

(b) *If κ is supercompact then $V^{\mathbb{P}} \models$ GRP$^+$.*

Proof. By Theorems 5.12, 5.14 and 5.17. □

c. Applications of GRP

This section is devoted to applications of either GRP or GRP$^+$ respectively. We are fully aware of the fact that the implications of Propositions 5.19, 5.20, 5.22 and 5.27 are already known for quite some time if we take into account that both of these principles above are reformulations of generic compactness. We still give their proofs in the language of games.

First we want to point out that in view of the game-representation of clubs in $[\omega_2]^{\aleph_0}$, we can easily see that GRP implies the diagonal reflection of ω_2-many stationary subsets of $[\omega_2]^{\aleph_0}$ simultaneously:

5.19 Proposition. *Under* GRP*, for every sequence $\mathcal{E}_\gamma$ $(\gamma < \omega_2)$ of stationary subsets of $[\omega_2]^{\aleph_0}$ there is an internally approachable*

$$M = \bigcup \langle M_\xi : \xi < \omega_1 \rangle$$

such that $\{\xi < \omega_1 : M_\xi \cap \omega_2 \in \mathcal{E}_\alpha\}$ is stationary in ω_1 for every $\alpha \in M \cap \omega_2$.

Proof. We play the following game:

I	γ, α_0	α_1	α_2	α_3	$\ldots$
II	β_0	β_1	β_2	β_3	$\ldots$

where $\gamma < \omega_2$ and α_n, β_n $(n < \omega)$ are ordinals below ω_2. We let player II win this game if

$$\{\alpha_n : n < \omega\} \cup \{\beta_n : n < \omega\} \in \mathcal{E}_\alpha.$$

Note that player I has no winning strategy in this game, since all $\mathcal{E}_\alpha$'s are stationary. So there is an internally approachable M for which he has no winning strategy. This finishes the proof. □

5.20 Proposition. GRP$^+$ *implies* SR$_{\omega_2}$.

Proof. Remember that we defined the principle SR$_\gamma$ on page 6. The proof is the same as the proof of Proposition 5.19, except that we play in some regular cardinal θ. □

We are going to define a number of games that will be very useful for further applications of the Game Reflection Principle.

5.21 Definition. If $(\mathbb{P}, \leq)$ is a partial ordering, define the *cut-and-choose game* $\mathcal{G}_\kappa(\mathbb{P})$ in the following way: the plays of $\mathcal{G}_\kappa(\mathbb{P})$ look like this

Empty	p, A_0	A_1	A_2	A_3	$\ldots$
Nonempty	p_0	p_1	p_2	p_3	$\ldots$

where $p \in \mathbb{P}$, $p_n \in A_n$ $(n < \omega)$, A_0 is a maximal antichain below p and A_{n+1} a maximal antichain below p_n for $n < \omega$. Furthermore, all maximal antichains A_n $(n < \omega)$ are of size $\leq \kappa$. Nonempty wins if there is $q \in \mathbb{P}$ such that $q \leq p_n$ for every $n < \omega$.

The game $\mathcal{G}_\infty(\mathbb{P})$ is the same game as $\mathcal{G}_\kappa(\mathbb{P})$, except that there is no restriction on the sizes of the antichains A_n $(n < \omega)$.

The *Banach-Mazur game* $\mathcal{G}(\mathbb{P})$ is defined in the following way:

Empty	p_0	p_2	p_4	p_6	$\ldots$
Nonempty	p_1	p_3	p_5	p_7	$\ldots$

where $p_n \in \mathbb{P}$ and $p_{n+1} \leq p_n$ $(n < \omega)$. Nonempty wins if there is a $q \in \mathbb{P}$ such that $q \leq p_n$ for every $n < \omega$.

It is proved in [28, p.732] that Nonempty has a winning strategy for $\mathcal{G}_\infty(\mathbb{P})$ if and only if he has a winning strategy for $\mathcal{G}(\mathbb{P})$.

We often want to play $\mathcal{G}$-games in Boolean algebras $\mathbb{B}$. To avoid trivialities, we abuse notation and write $\mathcal{G}_\kappa(\mathbb{B})$ for $\mathcal{G}_\kappa(\mathbb{B}^+)$ in this case.

The first use of Definition 5.21 results in a well-known Theorem of Gregory.

5.22 Proposition. GRP *implies that every ω_2-Suslin-tree is essentially σ-closed.*

Proof. Assume that T is ω_2-Suslin. By results in [28], it is enough to show that Nonempty has a winning strategy in the game $\mathcal{G}_\infty(T)$.

5.22.1 Claim. *If $M \prec \mathrm{H}_{(2^{\aleph_2})^+}$ is of size $\aleph_1$ then Nonempty wins the game $\mathcal{G}^M_\infty(T)$.*

Proof. Let $\delta = M \cap \omega_2$. To create his strategy, Nonempty picks any point $x \in T_\delta$. Note that by Suslinity, this point x is *generic* for M, i.e. the branch of all predecessors of x hits every maximal antichain in the model. Now it's easy for Nonempty to refine Empty's partitions along this branch and still remain in the tree. $\square$

This verifies a local winning strategy for Nonempty and we are done by an application of the Game Reflection Principle. $\square$

It has already been remarked that the reflection of locally coherent trees is one of the applications of the Game Reflection Principle. We show this in the following Theorem that will result in two more corollaries.

5.23 Theorem. GRP *implies that every locally coherent ω_2-tree is trivially coherent.*

Proof. We assume that T is locally coherent and $T = \omega_2$. Note that there is a payoff set $\mathcal{A}_T \subseteq {}^\omega T$ representing the rules and winning conditions of $\mathbb{G}_\mathrm{i}(T)$ (introduced on page 19), i.e. $\mathbb{G}_\mathrm{i}(T)$ is a game on ω_2 as defined in 5.3.

By Lemma 2.29, player II has a winning strategy in $\mathbb{G}_\mathrm{i}(T \cap M)$ for all internally approachable models $M \prec \mathrm{H}_{(2^{\aleph_2})^+}$. But the game $\mathbb{G}_\mathrm{i}(T \cap M)$ is equivalent to $\mathbb{G}^M_\mathrm{i}(T)$ and we conclude by GRP that II has a winning strategy in the game $\mathbb{G}_\mathrm{i}(T)$. Again by Lemma 2.29, T is trivially coherent. $\square$

5.24 Corollary. *The following are equiconsistent under ZFC:*

(1) *there is a weakly compact cardinal,*
(2) *every locally coherent ω_2-tree is trivially coherent.*

Proof. (1) $\Longrightarrow$ (2) is Corollary 5.18 and Theorem 5.23.

(2) $\Longrightarrow$ (1) is achieved by Theorem 4.9, constructing a locally coherent non-coherent ω_2-tree from a $\square(\omega_2)$-sequence. $\square$

5.25 Corollary. *The existence of an ω_2-morass does not imply the existence of a locally coherent ω_2-Kurepa-tree.*

Proof. By Corollary 5.18 and Theorem 5.23, there is no locally coherent ω_2-Kurepa-tree in the Levy collapse of a weakly compact to ω_2, but ω_3 may not be inaccessible in L. So there is an ω_2-morass in this model. $\square$

We can repeat the above argument and prove a global version of Theorem 5.23 with the help of GRP^+:

5.26 Theorem. *Under* GRP^+, *if* $\mathrm{cf}(\delta) \geq \omega_2$ *then every locally coherent tree of height δ is trivially coherent.* $\square$

A relative of Theorem 5.26 has actually been around for a long time now. This relative is known as Rado's conjecture and it is implied by GRP^+ as well.

5.27 Proposition. GRP^+ *implies Rado's conjecture.*

Proof. Todorčević proved in [25] that Rado's conjecture is equivalent to the following statement: a tree T is the union of countably many antichains if and only if every subtree of size $\aleph_1$ is the union of countably many antichains. We finish the proof of this proposition by noting that there is a canonical game to characterize the notion of being the union of countably many antichains: player I plays points in the tree and player II answers with rationals. After ω_1-many steps, player II wins if the play determines a strictly increasing partial map from the tree into the rationals. $\square$

We are going to see in the next theorem that GRP^+ has a strong influence on the ideal of non-stationary subsets of ω_2.

5.28 Theorem. *Under* GRP^+, *Nonempty wins* $\mathcal{G}_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$.

Proof. We investigate the game $\mathcal{G}^M_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$ for an internally approachable M: it is helpful for Nonempty to consider the 'universal' club set

$$C_M = \omega_2 \setminus \bigcup(\mathrm{NS} \cap M).$$

5.28.1 Claim. *If $E \subseteq \omega_2$ is in M, then $E \in \mathrm{NS}^+$ iff $E \cap C_M \neq \emptyset$.* $\square$

Now Empty starts by playing a positive set E and an NS-partition P_0 of E of size $\aleph_1$. It is Nonempty's strategy to pick $\gamma \in E \cap C_M$ and fix it for the rest of the game.

5.28.2 Claim. *There is $E_0 \in P_0$ such that $\gamma \in E_0$.*

Proof. Assume that there isn't. In this case $\gamma \in E \setminus \bigcup P_0$. But then by Claim 5.28.1, $E \setminus \bigcup P_0$ is stationary and disjoint from any member of the partition P_0. Thus, P_0 is not maximal, a contradiction. □

Let Nonempty play E_0 as an answer to P_0. Note that this is possible since $P_0 \subseteq M$ holds by the restricted size of P_0. Now Empty plays a partition P_1 of E_0 of size $\aleph_1$. Nonempty repeats proving Claim 5.28.2 for P_1 and plays an E_1 as a response and so on. At the end of this game, $\gamma \in \bigcap_{i<\omega} E_i$ will be true. Since M is closed under countable sequences, we know that $\bigcap_{i<\omega} E_i$ is a member of M and so it is positive, again by Claim 5.28.1. We have established a winning strategy for Nonempty in the game $\mathcal{G}^M_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$.

From GRP^+ we can deduce that Nonempty has a winning strategy for the game $\mathcal{G}_{\aleph_1}(\mathfrak{P}(\omega_2)/\mathrm{NS})$. □

The just proved statement is an echo of Corollary 5.15. But there are two important differences between the games $\mathfrak{G}(\mathcal{F})$ and $\mathcal{G}_\kappa(\mathbb{P})$. On the one hand, Empty's freedom of choosing the set of ω-cofinal points at the start of the game complicates things somewhat. On the other hand, the later considered game has length ω only, so we can not hope to recover full generic measurability by ω-closed forcing from the conclusion of Theorem 5.28. Nevertheless, it implies *strong Chang's conjecture* heavy-handedly: for every countable substructure N, we can find an ordinal $\gamma \notin N$, $\omega_1 < \gamma < \omega_2$ such that $f(\gamma) \in N$ for all Skolem functions $f : \omega_2 \longrightarrow \omega_1 \in N$.[4] To have a further estimate on its strength, we might also quote a result of Silver and Solovay, reproduced in [13, p.249], that still provides an inner model with a measurable cardinal (see also [28]). Their proof is actually a precursor of our Theorem 5.14 and shows that ω_2 is generically measurable by $Col(\omega, 2^{\omega_2})$.

We want to add that such a winning strategy is really an optimal result within the realm of cut-and-choose games for this particular algebra, since player Empty can show up with a winning strategy for the more liberal game $\mathcal{G}_{\aleph_2}(\mathfrak{P}(\omega_2)/\mathrm{NS})$:

5.29 Proposition. *Empty wins $\mathcal{G}_{\aleph_2}(\mathfrak{P}(\omega_2)/\mathrm{NS})$.*

Proof. In the first move, Empty chooses to play on the ω-cofinals, i.e. he picks the positive set $\{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$. He goes on to fix increasing sequences α_n $(n < \omega)$ for every ω-cofinal α. In the nth move,

[4] We get such a γ by letting player I play all Skolem functions in N and pick a big enough ordinal in the outcome of this play.

Empty plays the partition $f_n : \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\} \longrightarrow \omega_2$ defined by $f_n(\alpha) = \alpha_n$. This function f_n is actually a *regressive partition*. Note that every regressive partition contains an NS-partition by an easy application of the Pressing Down Lemma. Now assume that Nonempty plays the ordinal β_n in his nth move indicating his choice, i.e. the preimage $f_n^{-1}(\beta_n)$. At the end of the day, the outcome will be

$$\begin{aligned} \bigcap_{n<\omega} f_n^{-1}(\beta_n) &= \{\alpha < \omega_2 : f_n(\alpha) = \beta_n \text{ for all } n < \omega\} = \\ &= \{\alpha < \omega_2 : \alpha_n = \beta_n \text{ for all } n < \omega\}. \end{aligned}$$

But this set contains at most one element and we finished the proof. □

Theorem 5.28 and Proposition 5.29 show that there can be a huge difference between cut-and-choose games with partitions $f : \omega_2 \longrightarrow \omega_1$ on the one hand as opposed to regressive partitions $f : \omega_2 \longrightarrow \omega_2$. Notice that the exact opposite was true in Corollary 5.15.

Appendix

Page reference to mathematical symbols:

Index

References

[1] James Baumgartner. Applications of the Proper Forcing Axiom. In K. Kunen and J.E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 913–959. North-Holland, 1984.

[2] Robert E. Beaudoin. The Proper Forcing Axiom and Stationary Set Reflection. *Pacific Journal of Mathematics*, 149:13–24, 1991.

[3] Mohamed Bekkali. *Topics in set theory*, volume 1476 of *Lecture notes in Mathematics*. Springer-Verlag, 1992.

[4] James Cummings. Large cardinal properties of small cardinals. In *Set theory (Curacao, 1995; Barcelona, 1996)*, pages 23–39. Kluwer Academic Publishers, 1998.

[5] Morton Davis. Infinite games of perfect information. In *Advances in game theory*, pages 85–101. Princeton Univ. Press, 1964.

[6] Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51:161–166, 1950.

[7] P. Erdős, A. Hajnal, A. Máté, and R. Rado. *Combinatorial Set Theory: Partition Relations for Cardinals*. North-Holland, 1984.

[8] M. Foreman, Magidor M., and Shelah S. Martin's maximum, saturated ideals, and nonregular ultrafilters I. *Annals of Mathematics*, 127:1–47, 1988.

[9] Matthew Foreman. Generic large cardinals: new axioms for mathematics? In *Proceedings of the International Congress of Mathematicians, Berlin*, volume II, pages 11–21, 1998.

[10] F. Galvin, T. Jech, and M. Magidor. An ideal game. *Journal of Symbolic Logic*, 43:284–292, 1978.

[11] Thomas Jech. *Set Theory*. Perspectives in Mathematical Logic. Springer-Verlag, 1997.

[12] Akihiro Kanamori. *The Higher Infinite*. Perspectives in Mathematical Logic. Springer-Verlag, 1997.

[13] Akihiro Kanamori and Menachem Magidor. The evolution of large cardinal axioms in set theory. In *Higher set theory*, volume 669 of *Lecture notes in Mathematics*, pages 99–275. Springer-Verlag, 1978.

[14] Alexander S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.

[15] Kenneth Kunen. Saturated ideals. *Journal of Symbolic Logic*, 43:65–76, 1978.

[16] Kenneth Kunen. *An introduction to independence proofs*. North-Holland, 1980.

[17] Djuro Kurepa. Ensembles ordonnés et ramifiés. *Publ. Math. Univ. Belgrade*, 4:1–138, 1935.

[18] Saharon Shelah. *Proper and Improper Forcing*. Perspectives in Mathematical Logic. Springer-Verlag, 1998.

[19] Saharon Shelah and Jindřich Zapletal. Canonical models for $\aleph_1$-combinatorics. *Annals of Pure and Applied Logic*, 98:217–259, 1999.

[20] Stevo Todorčević. A note on the Proper Forcing Axiom. *Contemporary Mathematics*, 31:209–218, 1984.

[21] Stevo Todorčević. Trees and linearly ordered sets. In K. Kunen and J.E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 235–293. North-Holland, 1984.

[22] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Mathematica*, 159:261–294, 1987.

[23] Stevo Todorčević. Special square sequences. *Proceedings of the American Mathematical Society*, 105:199–205, 1989.

[24] Stevo Todorčević. Trees, subtrees and order types. *Annals of Mathematical Logic*, 20:233–268, 1981.

[25] Stevo Todorčević. On a conjecture of R. Rado. *Journal of the London Mathematical Society*, 27:1–8, 1983.

[26] Stevo Todorčević. Conjectures of Rado and Chang and Cardinal Arithmetic. In Sauer N. et al., editors, *Finite and Infinite Combinatorics*, pages 385–398. Kluwer Academic Publishers, 1993.

[27] Boban Veličković. Jensen's $\square$ principles and the Novák number of partially ordered sets. *Journal of Symbolic Logic*, 51:47–58, 1986.

[28] Boban Veličković. Playful Boolean Algebras. *Transactions of the American Mathematical Society*, 296:727–740, 1986.

[29] W. Hugh Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*. Walter de Gruyter & Co., Berlin, 1999.

Lebenslauf

- Bernhard Andreas König, geb. am 23.04.1974 in München
- Eltern: Helmut König und Andrea König, geb. Vollmeyer
- Schulbildung:

1980-1984	Grundschule in Freising (Neustift)
1984-1993	Josef-Hofmiller-Gymnasium in Freising

- Studium:

1993-1998	Student der Logik und Wissenschaftstheorie an der LMU München
1998-2002	Doktorand im Fach Mathematik

E-mail address: B.Koenig@lrz.uni-muenchen.de