$\mathcal{K}$-essence: cosmology, causality and emergent geometry

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Contents

Zusammenfassung

Diese Arbeit beschäftigt sich mit verschiedenen Aspekten der K-*essence*-Theorien. Diese sind skalare Feldtheorien, die sich durch allgemeinkovariante und Lorentz-invariante Wirkungen mit nicht kanonischen kinetischen Termen auszeichnen. Es wird gezeigt, dass wegen der intrinsischen Nichlinearität diese Theorien unerwartete und ungewöhnliche physikalische Eigenschaften haben können. Jede nichttriviale dynamische Lösung der K-*essence*-Bewegungsgleichungen bricht die Lorentzsymmetrie spontan genauso wie in den kanonischen relativistischen Feldtheorien der Fall ist. Demzufolge breiten sich die Störungen um diese Lösung in einem *neuen Äther* aus. Im Gegensatz zu den üblichen relativistischen Feldtheorien propagieren sich diese Störungen in einer effektiven Metrik, welche sich von der gewöhnlichen Gravitationsmetrik unterscheidet. Diese *effektive Metrik* wird als die sogenannte *emergent* Raumzeit interpretiert. Es wird gezeigt, dass die Dynamik der Störungen durch eine Wirkung beschrieben werden kann, die im Bezug auf die Reparametrisierung dieser *emergent* Raumzeit allgmeinkovariant ist. Das interessanteste Artefakt dieser *emergent* Raumzeit ist das mögliche Vorhandensein der Überlichtgeschwindigkeiten für die Ausbreitung der Störungen. Es wird gezeigt, dass die kausalen Paradoxen trotz Ausbreitung mit der Überlichtgeschwindigkeit in diesen Theorien nicht auftreten, und in diesem Sinne sind solche Theorien nicht weniger unsicher als Allgemeine Relativitätstheorie. Die Ausbreitung der Störungen mit der Überlichtgeschwindigkeit hat interessante Folgen für Kosmologie und Physik der Schwarze Löcher. Insbesondere ist es möglich Inflationsmodelle mit verstärkter Produktion an Gravitationswellen zu konstruieren. Dies wiederum kann in der nahen Zukunft durch Beobachtung der Kosmischen Hintergrundstrahlung überprüft werden.

Außerdem wird gezeigt, dass es K-*essence*-Modelle existieren, die prinzipiell erlauben würden, Informationen aus der Region hinter dem Ereignishorizont eines Schwarzen Loches zu bekommen. Diese Informationen sind in den Störungen um eine solche Lösung implementiert, die stationäre Akretion der K-*essence* in das Schwarze Loch beschreibt.

Zusätzlich wird eine mögliche dynamische Verletzung der Null-Energie-Bedingung (NEB) in den K-*essence*-Modellen diskutiert. Die Verletzung der NEB durch die dynamische Dunkle Energie (DE) ist nicht ausgeschlossen und sogar ein wenig bevorzugt bei den Beobachtungen[1]. Außerdem spielt die Verletzung der NEB eine äußerst wichtige Rolle in den Pre-Big Bang-Szenarien. Es wird gezeigt, dass im Rahmen der K-*essence* eine dynamische Verletzung der NEB physikalisch nicht plausibel ist.

[1]Die Dunkle Energie, die die Null-Energie-Bedingung verletzt, wird manchmal als *Phantom* bezeichnet.

Abstract

In this work we consider different aspects of k-*essence* theories, which are scalar field theories described by the generally covariant and Lorentz invariant action with non-canonical kinetic terms. It is shown that, because of the intrinsic nonlinearity, these theories can have rather unexpected and unusual physical properties. As in the usual relativistic field theories, any nontrivial dynamical solution of the k-*essence* equation of motion spontaneously breaks the Lorentz invariance. Thus the perturbations around such solutions propagate in a *new aether*. In contrast to the usual relativistic field theories, these perturbations propagate in an *effective metric* which is different form the usual gravitational metric. This *effective metric* can be interpreted as the so-called *emergent spacetime*. In this thesis we show that the dynamics of the perturbations can be described by the action which is generally covariant with respect to the reparameterization of this *emergent spacetime*. The most interesting manifestation of this *emergent spacetime* is that perturbations can propagate faster than light. We show that despite the superluminal propagation the causal paradoxes do not arise in these theories, and in this respect these theories are not less safe than General Relativity. This superluminal propagation of perturbations has interesting consequences for cosmology and black hole physics. In particular, it is possible to construct models of inflation with an enhanced production of gravitational waves. This in turn can be verified in the nearest future by the observations of the B-mode polarization of the Cosmic Microwave Background Radiation (CMBR).

Moreover, we have shown that there exist k-*essence* models which in principle allow to obtain information from the region beyond the black hole horizon. This information is encoded in the perturbations around the solution describing the stationary accretion of the k-*essence* onto the black hole.

In addition, we discuss the possible dynamical violation of the Null Energy Condition (NEC) in the k-*essence* models. The violation of NEC by the dynamical Dark Energy (DE) is not excluded and even slightly preferred by observations[2]. Moreover, the violation of NEC plays a crucial role in the pre-Big-Bang scenarios. We have shown that in the framework of general k-*essence* the dynamical violation of NEC is physically implausible.

[2]The NEC-violating Dark Energy is sometimes called *Phantom*.

Chapter 1

Introduction and Discussion

During last years theories described by the action with non-standard kinetic terms, attracted a considerable interest. The first theory of this kind was introduced in 1934 by M. Born and L. Infeld [1] to avoid the infinite self-energy of the electron[1]. Further in 1939 and 1952 such nonlinear scalar field theories were studied by W. Heisenberg in connection to physics of cosmic rays [4] and meson production [5] respectively. The ideas of M. Born and L. Infeld were further developed by P. Dirac in [6] in 1962. The non-canonical kinetic terms are rather common for effective field theories arising from string theory and in particular in D-branes models e.g. [7, 8, 9, 10, 11]. In cosmology such theories were first studied in the context of k-*inflation* [12], and then the k-*essence* models were suggested as dynamical Dark Energy (DE) for solving the cosmic coincidence problem [13, 14], see also [15]. One can also try to describe dark matter using k-*essence* or tachyon fields [10, 16]. The *ghost condensation* scenario [17], *ghost inflation* [18] and *phantom* dark energy [19] can be thought of as the further developments of this ideas.

Throughout the text we will refer to general scalar field theories with non-quadratic kinetic terms as the k-*essence.*

An interesting difference between the relativistic field theories with canonical kinetic terms and k-*essence* is that nontrivial dynamical solutions of the k-*essence* equation of motion not only spontaneously break Lorentz invariance but also change the metric for the perturbations around these solutions. Thus the perturbations propagate not only in the *new aether* determined by the background solution but also in the so-called *emergent* or *analogue* curved spacetime [20] with the metric different from the gravitational one.

Recently spontaneous breaking of the Lorentz invariance and questions related to this issue, such as superluminal propagation of perturbations in nontrivial backgrounds, attracted renewed interest among physicists. One of the basic questions here is whether the theories allowing superluminal velocities possess internal inconsistencies and, in particular, inevitably lead to the causality paradoxes namely to the appearance of the Closed Causal Curves (CCCs). Concerning this issue there exist two contradicting points of view. Some authors (see, for instance classical textbook [21] and recent papers [22, 23, 24, 25, 26, 27,

[1] A similar way of limiting the curvature of spacetime was considered in [2, 3]

28, 29, 30, 31, 32]) argue that the subluminal propagation condition should a priori be imposed to make the theory physically acceptable. For example, in [21] on page 60 the authors introduce the *"Postulate of Local Causality"* which excludes the superluminal velocities from the very beginning. The requirement of subluminality is sometimes used to impose rather strong restrictions on the form of the admissible Lagrangians for the vector and higher spin fields [29, 30] and gravity modifications [27, 31, 32]. The effective field theories (EFT) allowing superluminal propagation were considered in [28], where it was argued that in such theories global causality and analyticity of the S-matrix may be easily violated. The main conclusion of [28] is not favorable for the theories with superluminal propagation. In particular the authors claim that the UV-completion of such theories must be very nontrivial if it exists at all (for a different attitude see [33, 34, 35]).

An open minded opinion concerning the superluminal propagation is expressed in [36], where one argues that the proper change of the chronological ordering of spacetime in non-linear field theory with superluminal propagation allows us to avoid causal paradoxes.

Recently, several cases were discussed in the literature where faster-than-light propagation arises in a rather natural way. In particular we would like to mention the noncommutative solitons [37, 38, 39], Einstein aether waves [40], "superluminal" photons in the Drummond-Hathrell effect [41, 42, 43, 44] and in the Scharnhorst effect [45, 46, 47, 48], see also [49, 50, 51] [2]. These last two phenomena are due to the vacuum polarization i.e. higher-order QED corrections. It was argued that this superluminal propagation leads to the causal paradoxes in the gedanken experiment involving either two black holes [52] or two pairs of Casimir plates [53] moving with the high relative velocities. To avoid the appearance of the closed causal curved in such experiments the authors of [53] invoked the *Chronology Protection Conjecture* [54] (see also [55, 56, 57]) and showed that the photons in the Scharnhorst effect causally propagate in effective metric different from the Minkowski one.

Note that the superluminal propagation cannot be the sole reason for the appearance of Closed Causal Curves. There are numerous examples of spacetimes in General Relativity, where the *"Postulate of Local Causality"* is satisfied and, nevertheless, the Closed Causal Curves are present, see e.g. Refs. [58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68] and for the recent review of the CCCs-time machines see [68]. Therefore an interesting question arises whether the superluminal propagation leads to additional problems related with causality compared to the situation in General Relativity.

In this thesis we will consider the k-*essence* fields [12, 13, 14, 69, 70, 71, 16, 72, 73, 74, 75] and repeat our arguments from [75] that contrary to the claim of [23, 22] the causality is not violated in generic k-*essence* models with superluminal propagation (this point of view was also advocated in [76, 77, 78, 79, 80]). In particular we show that in the *test field* approximation one can always describe the perturbations by the action which is generally covariant in the curved *emergent* spacetime. This action correspond to the action of the

[2] In this work under *superluminal* we always mean "faster than light in usual QED vacuum in unbounded empty space". To avoid confusion one could say that photons propagate faster than gravitons in the Scharnhorst effect. When this thesis was in the stage of preparation the superluminal wave-front velocity in these effects was putted under question [35, 34].

canonical Klein-Gordon scalar field[3] in the curved spacetime and is valid for perturbations around any solution of arbitrary k-*essence* in any gravitational background. Here it is important to note that for the perfect fluid similar results were derived in [81], for details see Appendix A. Further we show that also in the case of superluminal propagation the *emergent* spacetime is stably causal in most of the interesting cases and therefore does not allow any causal paradoxes. Thus in this sense, in spite of the presence of superluminal signals on nontrivial backgrounds, the k-*essence* theories are not less safe and legitimate than General Relativity.

The possibility of the superluminal propagation of perturbations has interesting and falsifiable consequences for inflationary cosmology see e.g. [82, 83, 84]. The main consequence of inflation is the generation of primordial cosmological perturbations [85] and the production of long wavelength gravitational waves (tensor perturbations) [86]. The predicted slightly red-tilted spectrum of the scalar perturbations is at present in excellent agreement with the measurements of the CMB fluctuations [87, 88]. The detection of a small deviation of the spectrum from flat together with the observation of primordial gravitational waves would make us completely confident in early-time cosmic acceleration. The detection of primordial gravitational waves is not easy, but they can be seen indirectly in the B-mode of the CMBR polarization (see, for example, [82]). In standard slow-roll inflationary scenarios [89] the amplitude of the tensor perturbations can, in principle, be large enough to be discovered. However, it is only on the border of detectability in future experiments.

There is no problem to modify the inflationary scenarios in a way to suppress the tensor component produced during inflation. In particular, in models such as new inflation [90, 91] and hybrid inflation [92], tensor perturbations are typically small [82]. Moreover, in the curvaton scenario [93, 94, 95, 96, 97, 98, 99] and k-*inflation* [12, 69], they can be suppressed completely. Finally in the current string theory inspired inflationary models the gravitational waves production is completely suppressed [100] by the factor 10^{-24} relatively to chaotic inflation. Thus gravitational waves from string inflation are far beyond the detectability in the future experiments. Therefore as it was mentioned in [100] a possible experimental discovery of tensor modes may present a challenge to string cosmology.

An interesting question is whether the gravitational waves can be significantly enhanced compared to our naive expectations. Recently it was argued that the contribution of tensor perturbations to the CMB anisotropy can be much greater than expected [101, 102]. However, it was found in [103] that in the models considered in [101, 102] one cannot avoid the production of too large scalar perturbations and therefore they are in contradiction with observations. It is also possible to produce blue-tilted spectrum of gravitational waves in a so-called super-inflation [104], which is, however, plagued by graceful exit problem [105].

Before our paper [70] there did not exist any inflationary model with graceful exit [4]where the B-mode of polarization would exceed the value predicted by simple chaotic inflation. In [70] we have shown that such models can be easily constructed even within

[3]generically with spacetime-dependent mass.

[4]see however [106].

the class of simple slow-roll inflationary scenarios, if we allow a nontrivial dependence of the Lagrangian on the kinetic term. These models resemble k-*inflation* with only difference that here inflation is due to the potential term in the Lagrangian.

Another natural question arising in regard to the superluminal propagation concerns the physics of Black Holes. Can one use the superluminal propagation in order to obtain information from the region beyond the horizon? In our papers [73, 75] we have shown that it is possible to construct a model which allows one to send signals from the interior of the black hole. These signals are small perturbations around the solution describing the accretion of the k-*essence* onto the black hole.
The inherent nonlinearity of k-*essence* could theoretically account for the dynamical violation of the Null Energy Condition (NEC). This would correspond to the super-acceleration of the universe filled with this k-*essence*. The recent observations in turn allow for (and, as it is also sometimes claimed even slightly prefer) the super-acceleration of our universe now. One of the greatest challenges in modern cosmology is understanding the nature of the observed late-time acceleration of the universe. The present acceleration expansion seems to be an experimental fact, now that data from supernovae type Ia [107, 108, 109], corroborated later by those from the cosmic microwave background [110] have been recently confirmed by the observations of the largest relaxed galaxy clusters [111]. Although the observations are in a good agreement with the simplest explanation given by a cosmological constant Λ of order $(10^{-3}\,\mathrm{eV})^4$, the mysterious origin of this tiny number which is about 120 orders smaller than the naive expectations, gives rise to the idea of a dynamical nature of this energy. Possible dynamical explanations of this phenomenon are given in various frameworks. One of them is known as quintessence (see e.g. [112] and other references from the reviews [113, 114]). In this framework the equation of state $p = w\varepsilon$ is such that $w \geq -1$. Another proposal is the phantom scalar fields (see e.g. [19, 115]) which possess the super-negative equation of state $w \leq -1$, usually due to the "wrong" sign before the kinetic term in the Lagrangian. Alternatively there is a more general possibility under the name k-*essence* [13, 14] which is an effective scalar field theory described by a Lagrangian with a nonlinear kinetic term. For this model the equation of state w is not constrained to be larger or smaller than -1. Allowing the dark energy to be dynamical provides an opportunity to study the so-called coincidence problem which asks why dark energy domination begins just at the epoch when sentient beings are able to observe it. The main advantage of k-*essence* is its ability to solve this problem in a generic way (for details see [13, 14]), whereas the first two models require a fine-tuning of parameters.

Without imposing the prior constraint $w \geq -1$, the observations seem to favor the dark energy with the present equation of state parameter $w < -1$ (see e.g. Ref. [111, 109, 116, 117]). Moreover, recently it was argued (see Ref. [118, 119] and other constraints on $w(z)$ obtained in Ref. [120, 121, 122, 123, 124, 125, 126]) that the dark energy with the equation of state parameter $w(z)$ rapidly evolving from the dust like $w \simeq 0$ at high redshift $z \sim 1$, to phantom like $-1.2 \lesssim w \lesssim -1$ at present $z \simeq 0$ provides the best fit for the supernovae Ia data and their combinations with other currently available CMBR and 2dFGRS data (see more also more recent 2dFGRS result [127] where the $w < -1$ is not preferred but still consistent with the data). For the recent discussion see reviews [128, 114].

In General Relativity, matter with $w < -1$ violates all classical energy conditions [21] see also [129]. Therefore for such models one cannot guarantee the stability of vacuum on the classical level. The instability can reveal itself at the quantum level as well. In fact, it was shown that the *phantom* scalar fields are quantum-mechanically unstable with respect to decay of the vacuum into gravitons and *phantom* particles with negative energy [130, 129, 131, 132, 21, 133]. Assuming that the phantom dark energy is an effective theory allows one to escape this problem through the appropriate fine-tuning of the Lorentz symmetry violating cutoff parameter. Here it is worth mentioning that quantum effects on a locally de Sitter background could lead to the effective parameter $w < -1$ (see Ref. [134, 135, 136]). If the dark energy could dynamically change its equation of state from a phantom-like one to that with $w \geq -1$, then this transition might prevent the undesirable particle production with less fine-tuning of the Lorentz violating cutoff mentioned above. Sometimes this dynamical transition through $w = -1$ is called *phantom divide crossing* [128] or *cosmological-constant boundary crossing* [137].

Another fundamental physical issue where this transition could play an important role is the cosmological singularity problem. If $w < -1$ in the expanding Friedmann universe, then the positive energy density of such phantom matter generally becomes infinite in finite time, overcoming all other forms of matter and hence leads to the late-time singularity called the "big rip" [115]. The transition under consideration could naturally prevent this late-time singularity. Here it is worthwhile to mention that for a certain potentials and initial conditions the phantom scalar fields can escape this singularity by evolving to a late time asymptotic which is the de Sitter solution with $w = -1$ [138, 117]. Moreover, it was argued that the quantum effects can prevent the developing of the "big-rip" singularity as well [139].

On the other hand, to avoid the big crunch singularity, which arises in various pre-big bang and cyclic scenarios (see e.g. [140, 141, 142]), one assumes that the universe can bounce instead of collapsing to the singularity. The existence of a non-singular bouncing solution in a flat (or open) Friedmann universe requires the violation of the NEC during the bounce [143]. If the energy density ε is constrained to be positive, then it follows that NEC violation or $w < -1$ is the necessary condition for the bounce. But the energy density of such phantom matter would rapidly decrease during the collapse and therefore only the transition from $w \geq -1$ to $w < -1$ just before the bounce could explain the non-singular bouncing without a fine-tuning in initial energy densities of phantom and other forms of matter present in the universe.

It is worth noting as well that for regimes where the equation of state of the k-*essence* field is greater than -1 it is possible to find a quintessence model which gives the same cosmological evolution but behaves differently with respect to cosmological perturbations [144]. Hence it is interesting whether this equivalence can be broken dynamically.

Note that the models with two fields dubbed *Quintom* [124, 145, 146] (see also [147]) and higher derivative models [148, 149, 150] can account for the phantom divide crossing as well but they seem to suffer from the quantum-mechanical instability mentioned above. There are more complicated models like vector fields [151, 152], gravity modifications (for consistent models see Refs. [153, 154, 155] and references therein), some braneworld models e.g. [156], which can account for the crossing without violating the stability. Moreover, there sting field theory inspired models e.g. [157, 158, 132] which are inherently nonlocal and can evolve through the *phantom divide* as well. Recently, it was claimed [159, 160, 161, 162] that it is possible to construct higher derivative effective field theories which violate NEC dynamically without introducing new dynamical degrees of freedom. The quantum stability of this models seems to be at least questionable because the fluctuations have negative energy and the instability discussed long ago in [21] in connection with *Creation* field [163, 147] should take place here as well. Hovewer, this issue requires additional consideration. Another type of instability, which is relevant for these models, and which was used in the arguments in [105] is associated with the negative square of the sound speed. As we discuss in the Section 3.1 this correspond to Riemannian signature of the *emergent* spacetime and to the elliptic equations of motion. It is well known that the Cauchy problem for elliptic equations is ill defined see e.g. [164]. This is the well posedness of the Cauchy problem that forces our spacetime to have the Lorentzian signature.

Thus without proper consideration of the higher derivatives the theories [159, 160, 161, 162] were physically meaningless. On the other hand the dynamical violation of the NEC is generically related to the change of signature in the *emergent* spacetime. Thus the hypersurfaces in the *emergent* spacetime where the signature changes correspond to the curvature singularities or degeneracy of the *emergent* metric. One could expect therefore that the quantum effects like vacuum polarization and particle production e.g.[165] may became significant to modify the imposed dynamics of the system. We think that this interesting issue requires further detailed investigation. It is important to note that there are analogy of the signature change in the Bose-Einstein condensates [166].

In this work we review our arguments from [105] and show that the phantom divide crossing is physically implausible in the framework of k-*essence*. Therefore in [105] we conclude that, if future observations reveal the *phantom divide crossing* behaviour of DE then it should contain at least additional degrees of freedom or higher derivatives (see also [147]). Thus the possible *phantom divide crossing* would be a "smoking gun" for the very unusual new physics.

The thesis is organized as follows. In the Chapter 2 we introduce general formalism for k-*essence.* In particular in the Section 2.1 we discuss general properties of the k-*essence* equation of motion and energy-momentum tensor. The next Section 2.2 is devoted to the basic equations describing the dynamics of the Friedmann universe filled with k-*essence.* Further in the Section 2.3 we very briefly present the basic facts from the theory of cosmological perturbations.

The next Chapter 3 deals with the causality and *emergent* geometry. This Chapter is based on our paper [75]. In the Section 3.1 we discuss perturbations around arbitrary k-*essence* solution and gravitational background in the *test field* approximation. Here we present the generally covariant action for perturbations and introduce the *emergent* geometry. In the Section 3.2 we consider the *emergent* geometry and generally covariant form of the action for cosmological perturbations. The derivation of the generalized Dirac-Born-Infeld action from the simplicity requirement of the *emergent* geometry can be found in the Section 3.3.

General aspects of causality and propagation of perturbations on a nontrivial background, determining the *new aether*, are discussed in the Section 3.4. In particular, here we prove that no causal paradoxes arise in the cases studied in our previous works [71, 70, 72, 73, 13, 14].

Section 3.5 is devoted to the Cauchy problem for k-*essence* equation of motion. We investigate under which restrictions on the initial conditions the Cauchy problem is well posed.

In the Section 3.6 we study the Cauchy problem for small perturbations in the *new aether* rest frame and in the fast moving spacecraft.

Section 3.7 is focused on the *Chronology Protection Conjecture,* which is used to avoid the CCCs in gedanken experiments considered in [28].

In the Section 3.8 we discuss the universal role of the gravitational metric. Namely, we show that for the physically justified k-*essence* theories the boundary of the smooth field configuration localized in Minkowski vacuum, can propagate only with the speed not exceeding the speed of light. In agreement with this result we derive that exact solitary waves in purely kinetic k-*essence* propagate in vacuum with the speed of light.

Our main conclusions concerning the causality and *emergent geometry* in k-*essence* models are summarized in the Section 3.9.

The next Chapter is based on our work [105] and contains strong arguments why the phantom divide crossing cannot be realized in the k-*essence* framework (without higher derivatives).

Firs in the Section 4.1 we determine the properties of a general Lagrangian, which are necessary for the smooth transition of the k-*essence* scalar field ϕ from the equation of state $w(\phi, \dot{\phi}) \geq -1$ to $w(\phi, \dot{\phi}) < -1$ or vice versa. The transition obviously happens if the system passes through the boundaries of the domains in the space $(\phi, \dot{\phi})$, defined by these inequalities. In most of this Chapter we assume that the k-*essence* dark energy dominates in the spatially flat Friedmann universe. The main question is whether trajectories connecting these domains on the phase space $(\phi, \dot{\phi})$ do exist and are stable with respect to cosmological perturbations. In the case of the phase curves which do not violate the usual stability condition, we study their asymptotic behavior in the neighborhood of the points where the transition could occur. To proceed with this analysis we linearize the equation of motion in the neighborhood of these points and then use the results of the qualitative theory of differential equations. From this analysis we infer that the solutions either change the sign of the square of the sound speed or have measure zero. In the next Section 4.2 we perform this investigation in the linear as well as beyond the linear approximation for the case of dark energy models described by Lagrangians linear in kinetic terms. For this class of Lagrangians we illustrate the outcome of our analysis by numerically obtained phase curves.

Further in the Section 4.3 we generalize the results to the cases of spatially not-flat Friedmann universes filled with a mixture or the dark energy and other forms of matter. Our conclusions concerning the dynamical violation of NEC violation the reader can find in the Section 4.4.

The Chapter 5 is based on our works [70, 71] and is devoted to the model of inflation where the production of the gravitational waves can be substantially enhanced with respect to the usual chaotic inflation [89].

In the first Section of this Chapter 5.1 we discuss general properties of inflation with potential domination and non-canonical kinetic term. Here we present the main idea of this work. The next Section 5.2 we present the a simple model based on the DBI-like k-*essence.*

We briefly discuss the result of this Chapter in the Section 5.3.

The last Chapter 6 of this thesis is focused on the possibility to obtain information from the region inside the Black Hole horizon. Here we consider the stationary spherically symmetric accretion of the DBI-like k-*essence* onto the Schwarzschild Black Hole. This section is based on our works [72, 75, 73]. In the Section 6.1 we present the model based on the DBI-like k-*essence* . Then we continue by studying the background solution in the Section 6.2. In the next Section 6.3 we consider small perturbations and fix the integration constants obtained for the background solution. Here we also present the formula for the *emergent* geometry and derive the redshift of the acoustic signals which can be sent by the spacecraft falling into the Black Hole together with the k-*essence* field. In the last Section 6.4 of this Chapter we briefly discuss our results.

All derivations of more technical nature we tried to put into Appendices. In Appendix A we consider the connection between k-*essence* and hydrodynamics. In Appendix B we derive characteristics of the equation of motion and discuss local causality. Appendix C is devoted to the derivation of the generally covariant action for perturbations. In Appendix D we show how the action derived in Appendix C is related to the action for cosmological perturbations from [69, 82]. The derivation of Green functions is given in Appendix E.

Chapter 2

General Framework

2.1 Equation of motion and Energy-Momentum Tensor for k-*essence*

In this thesis we consider the k-*essence* scalar field ϕ minimally coupled to the gravitational field $g_{\mu\nu}$. Thus the k-*essence* action is:

$$S_k[\phi,\, g_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}\left(X,\phi\right), \tag{2.1}$$

where

$$X \equiv \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi,$$

is the canonical kinetic term and by ∇_μ we always denote the covariant derivative associated with metric $g_{\mu\nu}$. We would like to stress that this action is explicitly generally covariant and in particular this action is Lorentz invariant in the Minkowski spacetime.

The total action $S_\Sigma[\phi, g_{\mu\nu}]$ describing dynamics of k-*essence* and the General Relativity is the sum of $S_k[\phi,\, g_{\mu\nu}]$ and the Einstein-Hilbert action:

$$S_\Sigma[\phi, g_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}M_{Pl}^2 R + \mathcal{L}\left(X,\phi\right)\right], \tag{2.2}$$

where R is the Ricci scalar and $M_{Pl} = (8\pi G_N)^{-1/2} = 1.72\times 10^{18}$ GeV is the reduced Planck mass. Throughout this thesis we mostly use the units where $\hbar = c = G_N = 1$, but in some parts of this work we set $M_{Pl} = 1$ or write the speed of light c and the Newton constant G_N explicitly. This change of units will be always indicated.

The variation of the action (2.1) with respect to $g_{\mu\nu}$ gives us the following energy-momentum tensor for the k-*essence* scalar field:

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_k}{\delta g^{\mu\nu}} = \mathcal{L}_{,X}\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{L}, \tag{2.3}$$

where $(...)_{,X}$ is the partial derivative with respect to X. It is well-known that for timelike field derivatives $\nabla_\nu\phi$ ($X > 0$ in our signature) one can employ the hydrodynamic approach to describe the system with the action (2.1). For details see Appendix A. To begin with one can introduce the effective "four-velocity" as follows:

$$u_\mu \equiv \sigma\frac{\nabla_\mu\phi}{\sqrt{2X}}, \tag{2.4}$$

where $\sigma = \mathrm{sign}\,(\partial_0\phi)$. Further by using (2.4), the energy-momentum tensor (2.3) can be rewritten in the perfect fluid form:

$$T_{\mu\nu} = (\varepsilon + p)\,u_\mu u_\nu - p g_{\mu\nu},$$

where the pressure p is given by the Lagrangian density, $p = \mathcal{L}(X,\phi)$, and the energy density is

$$\varepsilon\,(X,\phi) = 2X\mathcal{L}_{,X} - \mathcal{L}. \tag{2.5}$$

It should be stressed that the energy density ε and pressure p introduced in this way are scalars and correspond to T^0_0 and $-\frac{1}{3}T^i_i$ only in the *rest frame* where $u_i = 0$ and the scalar field is locally isotropic. For various cosmological applications it is convenient to introduce the equation of state parameter w

$$w \equiv \frac{p}{\varepsilon}. \tag{2.6}$$

Note that w defined in this way characterizes intrinsic properties of k-*essence* in the coordinate independent way. Sometimes we will interchange p and $\mathcal{L}$ and use the function $\varepsilon\,(X,\phi)$ also for $X \leq 0$ when this function does not correspond to the energy density. There are several conditions on the energy-momentum tensor which are usually supposed to be physically reasonable and which are substantial for such important results in General Relativity as *singularity theorems, vacuum conservation theorem* and *black hole area theorem* see Refs. [21, 167], for a recent discussion see Ref. [129] . One of these conditions is the Null Energy Condition (NEC): $T_{\mu\nu}n^\mu n^\nu \geq 0$, where n^μ is any null vector: $g_{\mu\nu}n^\mu n^\nu = 0$. This is one of the necessary conditions for the *black hole area theorem* and for some of the *singularity theorems.* In the theory under consideration the NEC is satisfied provided $\mathcal{L}_{,X} \geq 0$. For $X > 0$ we can use the hydrodynamical language where this inequality corresponds to $\varepsilon + p \geq 0$. The fields which violate NEC are sometimes called *ghosts* or *phantoms* [19]. A possible violation of Null Energy Condition by a k-*essence* model would imply the Hamiltonian unbounded from below and hence signifies an inherent instability of the system [130, 129, 131, 132]. As it follows from the action (2.1) the k-*essence* field always interacts with gravity, and thus through gravity with all other matter fields. Therefore, due to the gravitational interaction, the absolute value of the energy density for phantom-like k-*essence* can grow along with the energy density of all other normal fields. This unbounded growth corresponds to the unavoidable intrinsic instability of the system. Long ago in [21] it was argued in connection with the so called *Creation-* or *C-field* [163, 168, 169] that the difficulty associated with this instability is even worse from the quantum mechanical point of view. Even if the cross-section for the creation of pairs of normal quanta and *C-field* or *phantom* quanta were very small, the infinite phase space available for these positive and negative energy quanta would seem to result in an infinite number of these pairs being produced in a finite region of spacetime. For this and other possible instabilities in NEC-violating models see also [133, 170, 171].

Another condition which is important for our discussion is the Dominant Energy Condition (DEC): for every timelike t^{μ}, $T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$, and $T^{\mu\nu}t_{\nu}$ is a non-spacelike vector i.e. $g^{\mu\nu}T_{\mu\alpha}t^{\alpha}T_{\nu\beta}t^{\beta} \geq 0$. This may be interpreted as saying that to any observer the local energy density appears non-negative and the local energy flow vector is non-spacelike. In the hydrodynamical language this condition is equivalent to $\varepsilon \geq |p|$. This condition is used in the *vacuum conservation theorem* and forbids the *superluminal* propagation of energy.

These conditions hold for all today known forms of matter. However, there are a lot of self-consistent theories where at least the Dominant Energy Condition is violated. The simplest example of such theories is the negative cosmological constant. There are also speculations that there are might be fields for which mass renormalization could lead to pressure being greater than the energy density [172, 173, 174].

The equation of motion for the scalar field is obtained by variation of the action (2.1) with respect to ϕ,

$$-\frac{1}{\sqrt{-g}}\frac{\delta S_k}{\delta\phi} = \tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi + 2X\mathcal{L}_{,X\phi} - \mathcal{L}_{,\phi} = 0, \tag{2.7}$$

where the "effective" metric is given by

$$\tilde{G}^{\mu\nu}(\phi, \nabla\phi) \equiv \mathcal{L}_{,X}g^{\mu\nu} + \mathcal{L}_{,XX}\nabla^{\mu}\phi\nabla^{\nu}\phi. \tag{2.8}$$

This second order differential equation is hyperbolic (that is, $\tilde{G}^{\mu\nu}$ has the Lorentzian signature) and hence physically meaningful describes the time evolution of the system provided [26, 175, 80]

$$1 + 2X\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}} > 0. \tag{2.9}$$

When this condition holds everywhere the effective metric $\tilde{G}^{\mu\nu}$ determines the characteristics (cone of influence) for k-*essence*, see e.g. [175, 80, 176, 9, 177, 178, 25]. For the *nontrivial configurations* of the k-*essence* field $\partial_{\mu}\phi \neq 0$ the metric $\tilde{G}^{\mu\nu}$ is generally not conformally equivalent to $g^{\mu\nu}$; hence in this case the characteristics do not coincide with those ones for canonical scalar field the Lagrangian of which depends linearly on the kinetic term X. In turn, the characteristics determine the *local causal structure* of the space time in every point of the manifold. Hence, the *local causal structure* for the k-*essence* field is generically different from those one defined by the metric $g_{\mu\nu}$ (see Appendix B for details).

Finally by varying the total action with respect to $g_{\mu\nu}$ one arrives to the Einstein equations

$$-\frac{2}{\sqrt{-g}}\frac{\delta S_{\Sigma}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - 8\pi G_{N}T_{\mu\nu} = 0, \tag{2.10}$$

where $R_{\mu\nu}$ is the Ricci tensor. For the coupled system of equations of motion for the gravitational field Eq. (2.10) and k-*essence* Eq. (2.7) the Cauchy problem is well posed only if the initial conditions are set on the hypersurface which is spacelike with respect to both metrics: $g^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ (see P. 251 of Ref. [167] and Refs. [175, 179, 180] for details). We postpone the detailed discussion of this issue until Section 3.5.

2.2 k-*essence* in the Friedmann universe, background dynamics

In this section we consider the dynamics of the Friedmann universe filled with k-*essence.* In accordance with the standard cosmology (see e.g. [82]) we live in the universe which is spatially homogeneous and isotropic on large scales. Moreover from the modern precise observations of the Cosmic Microwave Background Radiation (CMBR) [110, 87, 88, 181] and other measurements we know that our universe was also spatially homogeneous and isotropic more than 13 billions years ago with the relative precision more than 10^{-4}. To honor Alexander Friedmann who has introduced this cosmological model for the first time [182, 183], we will refer to this model as *the Friedmann universe.* There is increasing evidence that the total energy density of the universe is equal to the critical value, and hence in the most part of the paper we will consider the flat Friedmann universe. Thus the background line element reads

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t)\,\mathrm{d}\mathbf{x}^2. \tag{2.11}$$

For this background metric the Einstein equations (2.10) can be written in the following form

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G_{_N}\left(\varepsilon + 3p\right), \tag{2.12}$$

$$H^2 = \frac{8\pi G_{_N}}{3}\varepsilon, \tag{2.13}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and a dot denotes derivative with respect to the physical time t. These equations also imply a continuity equation:

$$\dot{\varepsilon} = -3H(\varepsilon + p), \tag{2.14}$$

In general whenever $\dot{a} \neq 0$ any two of these three last equations imply the third one (by compatible initial conditions). Usually it is easier to work with the second and the third equations (these are the Friedmann equations). Note that, from Eq. (2.13), ε is constrained to be nonnegative.

Because of the homogeneity and isotropy of the background, we get $X = \dot{\phi}^2/2$ and the energy density is the Legendre transform of the Lagrangian

$$\varepsilon(\phi, \dot{\phi}) = \dot{\phi}p_{,\dot{\phi}} - p, \tag{2.15}$$

as if ε where the Hamiltonian in usual 1D classical mechanics of ϕ. Expressing H from the first Friedmann equation (2.13) we can rewrite Eq. (2.7) in the case of the homogeneous and isotropic flat background (2.11) and $\phi(t)$ as follows:

$$\ddot{\phi}\varepsilon_{,X} + 2\dot{\phi}p_{,X}\sqrt{6\pi G_N \varepsilon} + \varepsilon_{,\phi} = 0. \tag{2.16}$$

So far as $\dot{a}(t) \neq 0$ the whole information about the background dynamics of the gravity and scalar field contains in the equation written above. Using the hydrodynamical description from Appendix A, this equation of motion can be also easily obtained from the continuity equation (2.14).

2.3 Cosmological perturbations

In this section we follow the textbook [82] and briefly summarize some results from the cosmological perturbation theory which we will need in our forthcoming discussion. Notice that in this section we set $G_N = 1$. Let us consider the spatially flat Friedmann universe with small perturbations:

$$\mathrm{d}s^2 = (1+2\Psi)\,\mathrm{d}t^2 - a^2(t)\,\left[(1-2\Psi)\,\delta_{ik} + h_{ik}\right]\mathrm{d}x^i\mathrm{d}x^k, \tag{2.17}$$

where $\Psi = \Phi$ is the gravitational Newtonian potential characterizing scalar metric perturbations and h_{ik} is a traceless, transverse perturbations describing the gravitational waves. Here we have used the Longitudinal (conformal-Newtonian) gauge and have neglected the vector perturbations, because they decay as a^{-2}. In the linear approximation the decomposition of the cosmological perturbations into scalar, vector and tensor irreducible pieces allows to consider their dynamics separately. For details of the cosmological perturbation theory see [82, 184, 69]. Further it is useful to introduce the conformal time $\eta = \int dt/a\,(t)$ and denote $()' = \partial_\eta\,()$ and $\mathcal{H} \equiv a'/a$. Hereafter we suppose that the k-*essence* is the dominant source for the spacetime curvature. Let us assume that $\phi_0\,(t)$ is a background cosmological solution for the k-*essence* equation of motion (2.16) . The gauge invariant perturbations in the k-*essence* field $\delta\phi$ are connected with the scalar metric perturbations Ψ through the constraint (see Eq. (8.53) from [82])

$$\Psi' + \mathcal{H}\Psi = 4\pi a^2\,(\varepsilon + p)\left(\frac{\delta\phi}{\phi_0'}\right), \tag{2.18}$$

therefore Ψ and $\delta\phi$ cannot be considered as independent variables. However, it is possible [185, 69] to construct a canonical variable

$$\overline{\delta\phi} \equiv \delta\phi + \frac{\phi'}{\mathcal{H}}\Psi, \tag{2.19}$$

which can account for the self-consistent dynamics of the cosmological perturbations. This canonical variable is sometimes called "scalar perturbations on the spatially flat slicing". Rescaling this variable in accordance with

$$\upsilon \equiv \left(\sqrt{\varepsilon_{,X}}a\right)\overline{\delta\phi} \tag{2.20}$$

one arrives to the new variable υ. The advantage of this variable is that the whole dynamics of the scalar cosmological perturbations is described by the action

$$S_{\mathrm{cosm}} = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\eta\,\left[(\upsilon')^2 - c_s^2(\vec{\nabla}\upsilon)^2 - m_{\mathrm{cosm}}^2\upsilon^2\right] \tag{2.21}$$

where the effective mass is

$$m_{\mathrm{cosm}}^2 \equiv -\frac{z''}{z}, \tag{2.22}$$

auxiliary variable z is defined as

$$z \equiv \frac{\phi'}{\mathcal{H}}\sqrt{\varepsilon_{,X}}a, \tag{2.23}$$

and the sound speed is

$$c_s^2 \equiv \frac{p_{,X}}{\varepsilon_{,X}}. \tag{2.24}$$

For the general k-*essence* this result was obtained in [69]. Note that the formula (2.24) for the effective sound speed can be guessed from the effective hydrodynamical description of the system which is described in Appendix A. Note that the hyperbolicity requirement (2.9) reduces to the usual condition of the hydrodynamical stability with respect to the high frequency perturbations $c_s^2 > 0$. Thus the complicated constrained dynamics of the cosmological scalar perturbations is reduced to the intuitively much more clear dynamics of the canonical scalar field υ with the time dependent sound speed $c_s(\eta)$ and effective mass $m_{\mathrm{cosm}}(\eta)$. Another obvious advantage of the canonical variable υ is that it allows the quantization of the cosmological perturbations.

The gravitational waves are described by the action

$$S_h = \frac{1}{64\pi}\int \mathrm{d}\eta\mathrm{d}^3x a^2\left(\left(h_j^i\right)'\left(h_i^j\right)' - h_{j,k}^i h_i^{j,k}\right), \tag{2.25}$$

where the spatial indices are raised and lowered with the help of the unit tensor δ_{ik}.

Chapter 3

Causality and Emergent Geometry

3.1 Emergent Geometry

In this chapter we follow our paper [75]. In the this Section we study to the behavior of small perturbations π on a given arbitrary background $g_{\mu\nu}$ and $\phi_0(x)$. Thus we consider solutions $\phi(x) = \phi_0(x) + \pi(x)$ of (2.7) with the fixed $g_{\mu\nu}$ and neglect the metric perturbations $\delta g_{\mu\nu}$, induced by π. This corresponds to the test field approximation. It is convenient to rewrite the formula (2.24) for the sound speed and consider it as a function

$$c_s^2(X,\phi) \equiv \left(1 + 2X\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)^{-1}. \tag{3.1}$$

From the Section 2.3 it is clear that in the case $X > 0$ this function plays the role of "speed of sound" measured in the preferred reference frame, where the background is at rest and background field configuration $\phi_0(x)$ is locally isotropic.

The Leray's theorem (see P. 251 of Ref. [167] and Ref. [180]) states that the perturbations π on given background $\phi_0(x)$ propagate causally in metric $\tilde{G}^{\mu\nu}(\phi_0, \nabla\phi_0)$. In Appendix C we show that performing the conformal transformation

$$G^{\mu\nu} \equiv \frac{c_s}{\mathcal{L}_{,X}^2}\tilde{G}^{\mu\nu} \tag{3.2}$$

one can rewrite the equation of motion for the scalar field perturbations in the following form

$$\frac{1}{\sqrt{-G}}\partial_\mu\left(\sqrt{-G}G^{\mu\nu}\partial_\nu\pi\right) + M_{\mathrm{eff}}^2\pi = 0, \tag{3.3}$$

here we have denoted $\sqrt{-G} \equiv \sqrt{-\mathrm{det}G_{\mu\nu}^{-1}}$ and we have introduced the inverse metric $G_{\mu\nu}^{-1}$: $G_{\mu\lambda}^{-1}G^{\lambda\nu} = \delta_\mu^\nu$ along with the effective mass

$$M_{\mathrm{eff}}^2 \equiv \frac{c_s}{\mathcal{L}_{,X}^2}\left(2X\mathcal{L}_{,X\phi\phi} - \mathcal{L}_{,\phi\phi} + \frac{\partial\tilde{G}^{\mu\nu}}{\partial\phi}\nabla_\mu\nabla_\nu\phi_0\right). \tag{3.4}$$

Note that the metric $G^{\mu\nu}$ is conformally equivalent to $\tilde{G}^{\mu\nu}$ and hence describes the same causal structure as it must be. The equation for the perturbations has exactly the same form as equation for the massive Klein-Gordon field in the curved spacetime. Therefore the metric $G^{\mu\nu}$ describes the *emergent* or *analogue* spacetime where the perturbations live. In particular this means that the action for perturbations

$$S_\pi = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-G}\,\left[G^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - M_{\text{eff}}^2\pi^2\right], \tag{3.5}$$

and the equation of motion (3.3) are generally covariant in the geometry $G^{\mu\nu}$. Introducing the covariant derivatives D_μ associated with metric $G^{\mu\nu}$ ($D_\mu G^{\alpha\beta} = 0$), equation (3.3) becomes

$$G^{\mu\nu}D_\mu D_\nu\pi + M_{\text{eff}}^2\pi = 0. \tag{3.6}$$

Using the inverse to $G^{\mu\nu}$ matrix

$$G^{-1}_{\mu\nu} = \frac{\mathcal{L}_{,X}}{c_s}\left[g_{\mu\nu} - c_s^2\left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\nabla_\mu\phi_0\nabla_\nu\phi_0\right], \tag{3.7}$$

one can define the "emergent" interval

$$\mathrm{d}S^2 \equiv G^{-1}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{3.8}$$

which determines the influence cone for small perturbations of k-*essence* on a given background[1]. This influence cone is larger than those one determined by the metric $g_{\mu\nu}$, provided [26, 175, 80, 186, 9, 177, 178, 25]

$$\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}} < 0, \tag{3.9}$$

and the superluminal propagation of small perturbations becomes possible (see Appendix B). It is important to note that the Dominant Energy Condition precludes the spacelike energy flows. Whereas a stronger condition $\mathcal{L}_{,XX}/\mathcal{L}_{,X} \geq 0$ guarantees the absence of superluminal signals on a given background, see also [28]. At first glance it looks like the theory under consideration has emergent bimetric structure. However, this theory is inherently different from the bimetric theories of gravity [187, 188, 189] because the emergent metric refers only to the perturbations of k-*essence* and is due to the non-linearity of the theory, while in the bimetric gravity theories both metrics have fundamental origin and are on the same footing.

The derived above form of the action and of the equation of motion for perturbations is very useful. In particular, it simplifies the stability analysis of the background with respect to the perturbations of arbitrary wavelengths, while the hyperbolicity condition (2.9) guarantees this stability only with respect to the short-wavelength perturbations.

[1] Note that in order to avoid confusion we will be raising and lowering the indices of tensors by gravitational metric $g^{\mu\nu}$ ($g_{\mu\nu}$) throughout the paper.

It is important to mention that besides of the usual hyperbolicity condition (2.9) one has to require that $\mathcal{L}_{,X}$ is nowhere vanishes or becomes infinite. The points where $\mathcal{L}_{,X}$ vanishes or diverges, generally correspond to the singularities of the emergent geometry. It follows from equations (3.2) and (3.7) that these singularities are of the true nature and cannot be avoided by the change of the coordinate system. Therefore one can argue that before the singularities are formed the curvature of the emergent spacetime becomes large enough for efficient quantum production of the k-*essence* perturbations which will destroy the classical background and therefore $\mathcal{L}_{,X}$ cannot dynamically change its sign. Hence, if one assumes that at some moment of time the k-*essence* satisfies the null energy condition, that is, $\mathcal{L}_{,X} > 0$ everywhere in the space (or $\varepsilon + p > 0$ in hydrodynamical language; see Appendix A) then this condition can be violated only if one finds the way to pass through the singularity in the emergent geometry with taking into account the quantum production of the perturbations. This doubts the possibility of the smooth crossing of the equation of state $w = -1$ and puts under question recently suggested models of the bouncing universe [159, 160, 162, 161]. The statements above generalize the results obtained in [105] and rederived later in different ways in [137, 190, 191, 192, 193] in cosmological context. We will repeat the arguments from [105] in the Chapter 4. However, this issue can be much more involved because the models under consideration [159, 160, 162, 161] relay on a stabilization mechanism involving higher derivative terms in the Lagrangian. In this work we are not going to address the problems involving the higher derivative terms.

If the hyperbolicity condition (2.9) is satisfied, then at any given point of spacetime the metric $G^{-1}_{\mu\nu}$ can always be brought to the canonical Minkowski form $\mathrm{diag}\,(1, -1, -1, -1)$ by the appropriate coordinate transformation. However, the quadratic forms $g_{\mu\nu}$ and $G^{-1}_{\mu\nu}$ are not positively defined and therefore for a general background there exist no coordinate system where they are both simultaneously diagonal. In some cases both metrics can be nevertheless simultaneously diagonalized at a given point, so that, e.g. gravitational metric $g_{\mu\nu}$ is equal Minkowski metric and the induced metric $G^{-1}_{\mu\nu}$ is proportional to $\mathrm{diag}\,(c_s^2, -1, -1, -1)$, where c_s is the speed of sound (3.1). For instance, in isotropic homogeneous universe both metrics are always diagonal in the Friedmann coordinate frame.

3.2 Emergent geometry for cosmological perturbations

In deriving (3.5) and (3.6) we have assumed that the k-*essence* is sub-dominant component in producing the gravitational field and consequently have neglected the metric perturbations induced by the scalar field. In particular the formalism developed is applicable for accretion of a *test* scalar field onto black hole [73, 72]. For k-*essence* dark energy [13, 14] action (3.5) can be used only when k-*essence* is a small fraction of the total energy density of the universe, in particular, this action is applicable during the stage when the speed of sound of a successful k-*essence* has to be larger than the speed of light [23, 22, 79]. During k-*inflation* [12, 70, 194] or DBI inflation [195, 196] the geometry $g_{\mu\nu}$ is determined by the scalar field itself and therefore the induced scalar metric perturbations are of the same order of magnitude as the perturbations of the scalar field. As we discussed in the Section

2.3, in this case the action for cosmological perturbations was derived in [69]. We have shown in Appendix D that the correct action for perturbations in k-*inflation* (2.21) has, however, the same structure of the kinetic terms as (3.5) or, in other words, the canonical scalar field $\overline{\delta\phi} = \delta\phi + \Psi\phi_0'/\mathcal{H}$ plays the role of π and lives in the same emergent spacetime with geometry $G^{\mu\nu}$. In fact in Appendix D we have proven that the action (2.21) can be written in the form

$$S_{\text{cosm}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-G}\,\left[G^{\mu\nu}\partial_\mu\overline{\delta\phi}\partial_\nu\overline{\delta\phi} - M_{\text{cosm}}^2\overline{\delta\phi}^2\right], \tag{3.10}$$

where the effective mass is given in the generally covariant form

$$M_{\text{cosm}}^2 = -c_s^{-3}\varepsilon_{,X}^{-1}\left(\sqrt{\frac{\varepsilon}{X}}\Box_g\sqrt{\frac{X}{\varepsilon}} + \nabla_\mu \ln\left(\varepsilon_{,X}\right)\nabla^\mu \ln\sqrt{\frac{X}{\varepsilon}}\right). \tag{3.11}$$

Note that generically $M_{\text{cosm}}^2 \neq M_{\text{eff}}^2$, where M_{eff}^2 is given by (3.4). One can expect that this emergent geometry $G^{\mu\nu}$ has a much broader range of applicability and determines the causal structure for perturbations also in the case of other backgrounds, where one cannot neglect the induced metric perturbations.

We conclude this section with the following interesting observation. The effective metric (3.7) can be expressed through the energy momentum tensor (2.3) as

$$G_{\mu\nu}^{-1} = \alpha g_{\mu\nu} + \beta T_{\mu\nu} \tag{3.12}$$

where

$$\alpha = \frac{\mathcal{L}_{,X}}{c_s} - \mathcal{L}c_s\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}} \quad \text{and} \quad \beta = -c_s\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}.$$

As we have pointed out the cosmological perturbations propagate in $G_{\mu\nu}^{-1}$ even if the background field determines the dynamics of the universe. In this case the energy momentum tensor for the scalar filed satisfies the Einstein equations (2.10) [2]and eventually we can rewrite the effective metric in the following form

$$G_{\mu\nu}^{-1} = \left(\alpha - \frac{\beta}{2}R\right)g_{\mu\nu} + \beta R_{\mu\nu}. \tag{3.13}$$

This looks very similar to the "metric redefinition" $g_{\mu\nu} \leftrightarrow G_{\mu\nu}^{-1}$ in string theory where the quadratic in curvature terms in the effective action are fixed only up to "metric redefinition" (3.13) see e.g. [197, 198, 199, 200, 201, 202]. The "metric redefinition" does not change the light cone and hence the *local causality* only in the Ricci flat $R_{\mu\nu} = 0$ spacetimes. However, neither in the matter dominated universe nor during inflation the *local causals structures* determined by $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$ are equivalent.

[2]Here we used units $M_{Pl} = 1$.

3.3 Dirac-Born-Infeld Lagrangians

Here we derive the Dirac-Born-Infeld (DBI) Lagrangian by requiring that the structure of the emergent metric $G_{\mu\nu}$ should be as simple as it is possible for a theory which is nonlinear in kinetic term X. An arguably simple induced metric corresponds to the case when the additional term $c_s^2\left(\mathcal{L}_{,XX}/\mathcal{L}_{,X}\right)\nabla_\mu\phi_0\nabla_\nu\phi_0$ which changes the causal structure depends only quadratically on field derivatives thus

$$c_s^2\left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\equiv-\frac{1}{\alpha(\phi)}.$$

This condition can be brought to the form

$$\mathcal{L}_{,XX}\left(X,\phi\right)\left(2X\alpha\left(\phi\right)-1\right)+\mathcal{L}_{,X}\left(X,\phi\right)\alpha\left(\phi\right)=0. \tag{3.14}$$

Integrating this differential equation with respect to X one can find the class of Lagrangians $\mathcal{L}\left(\phi,X\right)$ corresponding to these simple emergent geometry:

$$\mathcal{L}\left(\phi,X\right)=\alpha(\phi)\sigma(\phi)\sqrt{1+\frac{2X}{\alpha(\phi)}}-V(\phi), \tag{3.15}$$

which is a generalized Born-Infeld Lagrangian [1], and is also often called Dirac-Born-Infeld Lagrangian because of the work [6]. It is interesting to note that the theories of this type were also studied in the papers of Heisenberg [4, 5] even before the work of Dirac [6]. We will use the today standard notation Dirac-Born-Infeld (DBI) Lagrangian for the theories of type (3.15). Thus the generalized DBI Lagrangian generically depends on three free functions $\sigma\left(\phi\right)$,$V\left(\phi\right)$ and $\alpha\left(\phi\right)$. The sound speed in this case is given by

$$c_s^2=1+\frac{2X}{\alpha(\phi)}. \tag{3.16}$$

From the hyperbolicity condition (2.9) we have $1+2X/\alpha>0$. Note that this condition corresponds to the condition of reality of the DBI Lagrangian (3.15). The sign of the function $\sigma\left(\phi\right)$ controls whether the theory respects the Null Energy Condition. Indeed if $\sigma(\phi)>0$ then $\mathcal{L}_{,X}=\sigma/\sqrt{1+2X/\alpha}>0$ and the theory does not violate the NEC. The sign of the function $\alpha\left(\phi\right)$ is responsible for the maximal sound speed present in the theory. If $\alpha(\phi)<0$ then $\mathcal{L}_{,XX}/\mathcal{L}_{,X}=-\alpha^{-1}\left(1+2X/\alpha\right)^{-1}>0$ and the model does not permit the *superluminal* propagation, see (3.9), whereas for $\alpha(\phi)>0$ there is reference frame where the sound speed always exceed the speed of light. The energy density can be calculated from Eq. (2.5)

$$\varepsilon\left(\phi,X\right)=-\alpha\left(\phi\right)\sigma\left(\phi\right)\left(1+\frac{2X}{\alpha\left(\phi\right)}\right)^{-1/2}+V\left(\phi\right)=-\frac{\alpha\left(\phi\right)\sigma\left(\phi\right)}{c_s}+V\left(\phi\right). \tag{3.17}$$

While the pressure can be written in terms of c_s as

$$p\left(\phi,X\right)=\alpha(\phi)\sigma(\phi)c_s-V(\phi). \tag{3.18}$$

In the limit of small nonlinearities $2X \ll \alpha(\phi)$ in kinetic term the Lagrangian is that of the nonlinear sigma model

$$\mathcal{L}(\phi, X) \simeq \sigma(\phi)X - [V(\phi) - \alpha(\phi)\sigma(\phi)] .$$

For the emergent geometry we have

$$G^{-1}_{\mu\nu} = \frac{\sigma}{c_s^2}\left[g_{\mu\nu} + \frac{\nabla_\mu\phi\nabla_\nu\phi}{\alpha}\right], \tag{3.19}$$

and

$$G^{\mu\nu} = \frac{c_s^2}{\sigma}\left[g^{\mu\nu} - \frac{\nabla^\mu\phi\nabla^\nu\phi}{\alpha c_s^2}\right]. \tag{3.20}$$

The DBI Lagrangians are rather common in the context of string theory see e.g. [203, 204, 205, 186, 9, 177, 178]. Moreover, these models were rather extensively studied in the cosmological context [206, 207, 208, 209, 210, 211, 212, 213, 196]. We will use this kind of models in the forthcoming Chapters 5 and 6.

3.4 Causality on nontrivial backgrounds

In this section we discuss the causality issue for superluminal propagation of perturbations on some nontrivial backgrounds, in particular, in Minkowski spacetime with the scalar field, in Friedmann universe and for black hole surrounded by the accreting scalar field.

First, we would like to recall a well-known paradox sometimes called "tachyonic anti-telephone" [214] arising in the presence of the superluminal hypothetical particles *tachyons* possessing unbounded velocity $c_{tachyon} > 1$. In this case we could send a message to our own past. Indeed, let us consider some observer, who is at rest at $x = 0$ with respect to the reference frame (x, t) and sends along OR a tachyon signal to an astronaut in the spacecraft R (see Fig. 3.1). In turn, after receiving this signal, the astronaut communicates back sending the tachyon signal, RP. As this signal propagates the astronaut proper time t' grows. However, if the speed of the spacecraft is larger than $1/c_{tachyon}$, then the signal RP propagates *backward in time* in the original rest frame of the observer. Thus, the observers can in principle send information from "their future" to "their past". It is clear that such situation is unacceptable from the physical point of view.

Now let us turn to the case of the Minkowski space-time filled with the scalar field, which allows the "superluminal" propagation of perturbations in its background. For simplicity we consider a homogeneous time dependant field $\phi_0(t)$. Its "velocity" $\partial_\mu\phi$ is directed along the timelike vector, $u^\mu = (1, 0, 0, 0)$. Why does the paradox above not arise here? This is because the superluminal propagation of the signals is possible only in the presence of nontrivial background of scalar field which serves as the *aether* for sonic perturbations. The *aether* selects the preferred reference frame and clearly the equation of motion for acoustic perturbations is not invariant under the Lorentz transformations unless $c_s = 1$. In the moving frame of the astronaut the equation for perturbations has more complicated

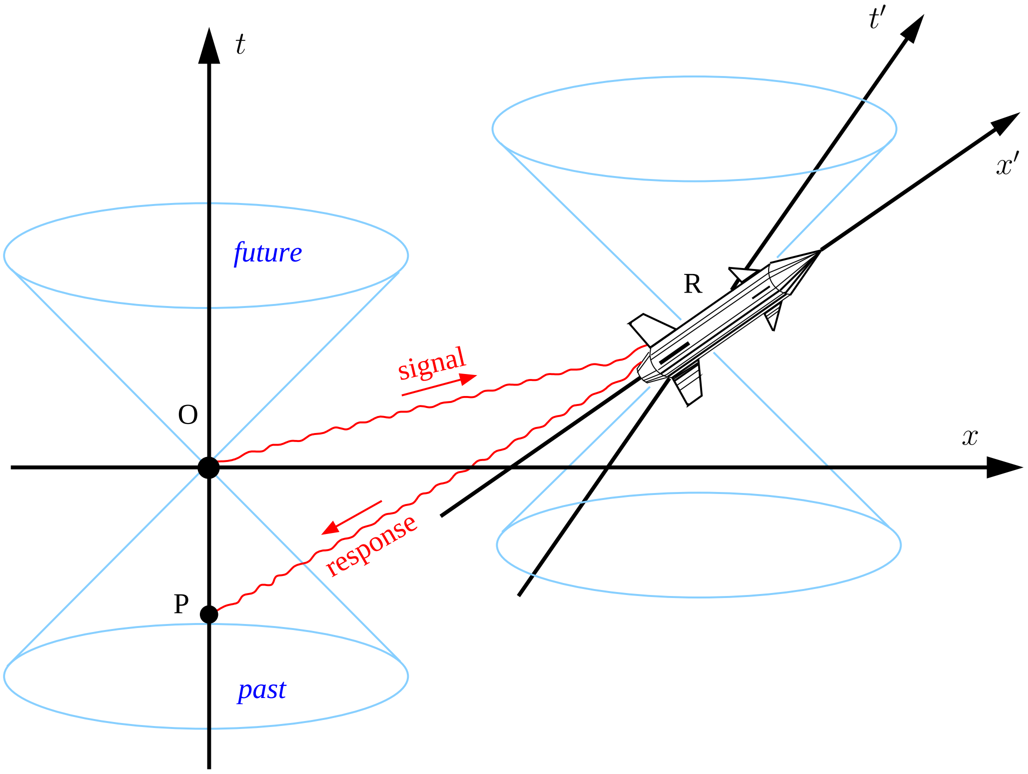

Figure 3.1: This figure represents the causal paradox constructed using *tachyons*. Someone living along the worldline $x = 0$ sends a tachyon signal to the astronaut in a fast moving spacecraft, OR. In the spacecraft frame (x', t'), the astronaut sends a tachyon signal back, RP. The signal RP propagates in the direction of growing t' as it is seen by the astronaut, however it travels *"back in time"* in the rest frame. Thus it is possible to send a message back in the own past.

form than in the rest frame and the analysis of its solutions is more involved. However, keeping in mind that k-*essence* signals propagate along the characteristics which are coordinate independent hypersurfaces in the spacetime we can study the propagation of sonic perturbations, caused by the astronaut, in the rest frame of the aether and easily find that the signal propagates always *forward in time* in this frame (see Fig. 3.2). Hence no closed causal curves can arise here.

We would like to make a remark concerning the notion of "future-" and "past" directed signals. It was argued in [23] that in order to have no CCCs for the k-*essence* during the "superluminal" stage, "...the observers travelling at high speeds with respect to the cosmological frame must send signals backwards in their time for some specific direction". One should remember, however, that the notion of past and future is determined by the past and future cones in the spacetime and has nothing to do with a particular choice of coordinates. Thus, the signals, which are future-directed in the rest-frame remain the future-directed also in a fast-moving spacecraft, in spite of the fact that this would corre-

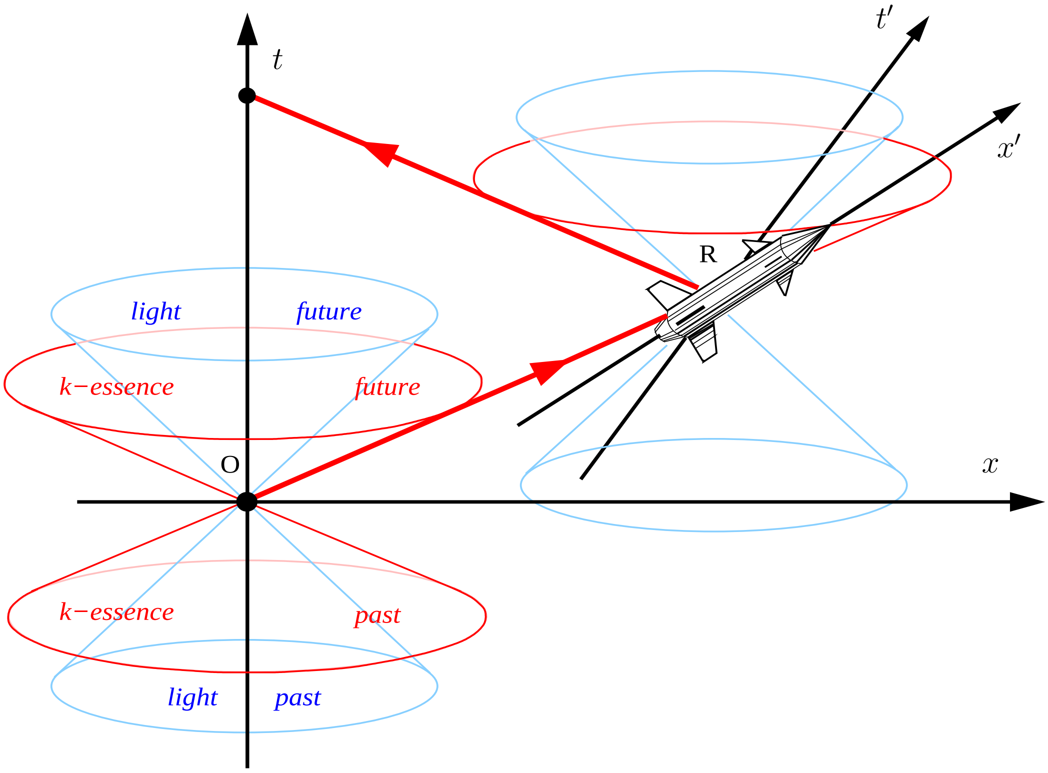

Figure 3.2: The causality paradox is avoided when superluminal signals propagate in the background which breaks the Lorentz symmetry (compare with Fig. 3.1). The observers cannot send a message to themselves in the past.

spond to the decreasing time coordinate t'. As we show in Section 3.6, the confusion arises because of a poor choice of coordinates, when decreasing t' correspond to future-directed signals and vice versa. The example shown in Fig. 3.4 illustrates this point: one can see that even without involving superluminal signals, an increasing coordinate time does not always imply the future direction.

Another potentially confusing issue is related to the question which particular velocity must be associated with the speed of signal propagation, namely, phase, group or front velocity. For example, in [23] an acausal paradox is designed using different superluminal group velocities for different wavenumbers. One should remember, however, that neither group nor phase velocities have any direct relation with the causal structure of the space-time. Indeed the characteristic surfaces of the partial differential equations describe the propagation of the wavefront. This front velocity coincides with the phase velocity only in the limit of the short wavelength perturbations. Generally the wavefront corresponds to the discontinuity of the second derivatives and therefore it moves "off-shell" (a more detailed discussion can be found in e.g. [33]). The group velocity can be less or even larger than the wavefront velocity. One can recall the simple examples of the canonical free scalar field theories: for normal scalar fields the mass squared, $m^2 > 0$, is positive and the phase velocity is larger than c while the group velocity is smaller than c; on the other hand for tachyons ($m^2 < 0$) the situation is opposite. Thus, if the group velocity were the speed of the signal transfer, one could easily build the time-machine similar to those described in [23] using canonical scalar field with negative mass squared, $m^2 < 0$. This, however, is impossible because the causal structure in both cases ($m^2 > 0$ and $m^2 < 0$) is governed by the same *light* cones. Finally we would like to mention that the faster-than-light group velocity has been already measured in the experiment [215, 216].

To prove the absence of the closed causal curves (CCC) in those known situations where the superluminal propagation is possible, we use the theorem from Ref. [167] (see p. 198): *A spacetime* $(\mathcal{M}, g_{\mu\nu})$ *is stably causal if and only if there exists a differentiable function* f *on* $\mathcal{M}$ *such that* $\nabla^{\mu} f$ *is a future directed timelike vector field.* Here $\mathcal{M}$ is a manifold and $g_{\mu\nu}$ is metric with Lorentzian signature. Note, that the notion of *stable causality* implies that the spacetime $(\mathcal{M},\, g_{\mu\nu})$ possesses no CCCs and thus no causal paradoxes can arise in this case. The theorem above has a kinematic origin and does not rely on the dynamical equations. In the case of the effective acoustic geometry the acoustic metric $G^{-1}_{\mu\nu}$ plays the role of $g_{\mu\nu}$ and the function f serves as the "*global time*" of the emergent spacetime $\left(\mathcal{M},\, G^{-1}_{\mu\nu}\right)$. For example, in the Minkowski spacetime filled with the scalar field "aether" one can take the Minkowski time t of the rest frame, where this field is homogeneous, as the global time function. Then we have

$$G^{\mu\nu}\partial_{\mu}t\partial_{\nu}t = \frac{c_s}{\mathcal{L}_{,X}} g^{00}\left(1 + 2X\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right) = \frac{g^{00}}{\mathcal{L}_{,X} c_s}. \tag{3.21}$$

Even for those cases when the speed of perturbations can exceed the speed of light, $c_s > 1$, this expression is positive, provided that $\mathcal{L}_{,X} > 0$, and the hyperbolicity condition (2.9) is satisfied. Thus $\partial_{\mu}t$ is timelike with respect to the effective metric $G^{-1}_{\mu\nu}$; hence the conditions of the theorem above are met and no CCCs can exist.

Now we consider the Minkowski spacetime with an arbitrary inhomogeneous background $\phi_0(x)$ and verify under which conditions one can find a global time t for both geometries $g_{\mu\nu}$ and $G^{-1}_{\mu\nu}$ and thus guarantee the absence of CCCs. Let us take the Minkowski t, $\eta^{\mu\nu}\partial_{\mu}t\partial_{\nu}t = 1$, and check whether this time can also be used as a global time for $G^{-1}_{\mu\nu}$. We have

$$G^{\mu\nu}\partial_{\mu}t\partial_{\nu}t = \frac{c_s}{\mathcal{L}_{,X}}\left[1 + \left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)(\partial_{\mu}t\nabla^{\mu}\phi_0)^2\right] = \frac{c_s}{\mathcal{L}_{,X}}\left[1 + \left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\dot{\phi}_0^2\right], \tag{3.22}$$

and assuming that $c_s > 0$, $\mathcal{L}_{,X} > 0$ we arrive to the conclusion that t is a global time for emergent spacetime provided

$$1 + \left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\left(\dot{\phi}_0\left(x^{\mu}\right)\right)^2 > 0, \tag{3.23}$$

holds everywhere on the manifold $\mathcal{M}$. This inequality is obviously always satisfied in the subluminal case. It can be rewritten in the following form

$$1 + c_s^2\left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\left(\vec{\nabla}\phi_0\left(x^{\mu}\right)\right)^2 > 0, \tag{3.24}$$

from where it is obvious that, if the spatial derivatives are sufficiently small then this condition can also be satisfied even if $c_s > 1$. Note that the breaking of the above condition for some background field configuration $\phi_0(x)$ does not automatically mean the appearance of the CCCs. This just tells us that the time coordinate t cannot be used as the global time coordinate. However it does not exclude the possibility that there exists another function serving as the global time. Only, if one can prove that such global time for both metrics does not exist at all, then there arise causal paradoxes.

In the case of the Friedmann universe with "superluminal" scalar field, one can choose the cosmological time t as the global time function and then we again arrive to (3.21), thus concluding that there exist no CCCs. In particular, the k-*essence* models, where the superluminal propagation is the generic property of the fluctuations during some stage of expansion of the universe [23, 79], *do not lead to causal paradoxes* contrary to the claim by [23, 22].

The absence of the closed causal curves in the Friedmann universe with k-*essence* can also be seen directly by calculating of the "effective" line element (3.8). Taking into account that the Friedmann metric is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2, \tag{3.25}$$

we find that the line element (3.8), corresponding to the effective acoustic metric, is

$$\mathrm{d}S^2 = G^{-1}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathcal{L}_{,X}}{c_s}\left(c_s^2\mathrm{d}t^2 - a^2(t)\mathrm{d}\mathbf{x}^2\right). \tag{3.26}$$

The theory under consideration is generally covariant. After making redefinitions,

$$\sqrt{\mathcal{L}_{,X}c_s}\mathrm{d}t \to \mathrm{d}t, \quad \text{and} \quad a^2(t)\mathcal{L}_{,X}/c_s \to a^2(t) \tag{3.27}$$

the line element (3.26) reduces to the interval for the Friedmann universe (3.25), where obviously no causality violation can occur. Thus we conclude that both the k-*essence* [13, 14, 15] and the "superluminal" inflation with large gravity waves [71, 70] are completely safe and legitimate on the side of causality.

When $X = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0$ is positive everywhere in the spacetime the background field itself can be used as the global time function. Indeed for general gravitational background $g_{\mu\nu}$ and $c_s > 0$, $\mathcal{L}_{,X} > 0$ we have

$$g^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0 > 0 \text{ and } G^{\mu\nu}\partial_\mu\phi_0\partial_\nu\phi_0 = \frac{2X}{\mathcal{L}_{,X}c_s} > 0,$$

and due to the fact that $X > 0$ the sign in front $\nabla^\mu\phi_0$ can be chosen so that the vector $\nabla^\mu\phi_0$ is always future directed on $\mathcal{M}$. Therefore $\phi_0(x)$ or ($-\phi_0(x)$ if necessary) can serve as a global time in both spacetimes $(\mathcal{M}, g_{\mu\nu})$ and $\left(\mathcal{M}, G^{-1}_{\mu\nu}\right)$, and no causal paradoxes arise.

In particular this is applicable for the accretion of the "superluminal" scalar field onto the Schwarzschild black hole [73, 72]. In this case sound horizon is located inside the

Schwarzschild radius and therefore the Schwarzschild time coordinate cannot be used as a global time function. However, $X > 0$ outside the acoustic horizon (see [73, 72] and the Chapter 6) and in accordance with the theorem and discussion above we can take ϕ as the global time coordinate and hence the acoustic spacetime is stably causal.

In all examples above we have considered the "superluminal" acoustic metric. Thus, if there exist no CCCs in $\left(\mathcal{M},\, G^{-1}_{\mu\nu}\right)$ then there are no CCCs with respect to metric $g_{\mu\nu}$ because acoustic cone is larger than the light cone. It may happen that in some cases it is not enough to prove that there no CCCs separately in $\left(\mathcal{M},\, G^{-1}_{\mu\nu}\right)$ and $(\mathcal{M},\, g_{\mu\nu})$ and one has to use the maximal cone or introduce an artificial cone [76, 77, 78] encompassing all cones arising in the problem. It is interesting to note that, if the k-*essence* realizes both "superluminal" and subluminal speed of sound in the different regions of the manifold, then there exist hypersurface where the k-*essence* metric is conformally equivalent to the $g_{\mu\nu}$ and one can smoothly glue the maximal cones together everywhere on $\mathcal{M}$. After that one can consider a new "artificial metric" $G^{\Sigma}_{\mu\nu}$ as determining the complete causal structure of the manifold.

We would like to point out that although the theorem on stable causality allowed us to prove that there is no causal paradoxes in those cases we considered above, it is no guaranteed that CCCs cannot arise for some other backgrounds. Indeed, in [28] the authors have found some configurations of fields possessing CCCs: one for the scalar field with non-canonical kinetic term and another for the "wrong"-signed Euler-Heisenberg system. In both cases the small perturbations propagate superluminally on rather non-trivial backgrounds. We will pursue this issue further in Section 3.7.

3.5 Which initial data are allowed for the well posed Cauchy problem?

Using the theorem on stable causality we have proven that the "superluminal" k-*essence* does not lead to any causal paradoxes for cosmological solutions and for accretion onto black hole. However, the consideration above is of a kinematic nature and it does not deal with the question how to pose the Cauchy problem for the background field ϕ_0 and it's perturbations π.

It was pointed out in [28] that in the reference frame of the spacecraft moving with respect to nontrivial background, where $c_s > 1$, with the speed $v = 1/c_s$ the Cauchy problem for small perturbations π is ill posed. This happens because the hypersurface of the constant proper time t' of the astronaut is a null-like with respect to the acoustic metric $G^{-1}_{\mu\nu}$. Hence $t' = const$ is tangential to the characteristic surface (or sonic cone see Appendix B) and cannot be used to formulate the Cauchy problem for perturbations which "live" in this acoustic metric. Intuitively this happens because the perturbations propagate instantaneously with respect to the hypersurface $t' = const$. Moreover, for $v > 1/c_s$, the sonic cone deeps below the surface $t' = const$ (see Figs. 3.3 and 3.2) and in the spacetimes of dimension $D > 2$ the Cauchy problem is ill posed as well because there always exist two directions along which the perturbations propagate "instantaneously" in time t' (red vectors in Fig. 3.3). This tell us that not every imaginable configuration of the background can be realized as the result of evolution of the system with the well formulated Cauchy problem and hence not every set of initial conditions for the scalar field is allowed.

In this Section we will find under which restrictions on the initial configuration of the scalar field the Cauchy problem for equation (2.7) is well-posed. For this purpose it is more convenient not to split the scalar field into background and perturbations and consider instead the total value of the field $\phi = \phi_0 + \pi$. The k-*essence* field interacts with gravity and therefore for consistency one has to consider the coupled system of equations for the gravitational metric $g_{\mu\nu}$ and the k-*essence* field ϕ. In this case the Cauchy problem is well posed only if the initial data are set up on a hypersurface Σ which is simultaneously spacelike in both metrics: $g_{\mu\nu}$ and $G^{-1}_{\mu\nu}$ (for details see P. 251 of Ref. [167] and Refs. [180], [175], [179]). We will work in the synchronous coordinate system, where the metric takes the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \gamma_{ik}\mathrm{d}x^i\mathrm{d}x^k, \tag{3.28}$$

and select the spacelike in $g_{\mu\nu}$ hypersurface Σ to be a constant time hypersurface $t = t_0$. The 1-form $\mathrm{d}t = \partial_\mu t \mathrm{d}x^\mu$ vanishes on any vector R^μ tangential to Σ: $R^\mu \partial_\mu t = 0$ (see Fig. 3.3). This 1-form is timelike with respect to the gravitational metric $g_{\mu\nu}$, that is $g^{\mu\nu}\partial_\mu t \partial_\nu t > 0$. In case when Lagrangian for k-*essence* depends at maximum on the first derivatives of scalar field the initial conditions which completely specify the unambiguous solution of the equations of motion are the initial field configuration $\phi(\mathbf{x})$ and and it's first time derivative $\dot{\phi}(\mathbf{x}) \equiv (g^{\mu\nu}\partial_\mu t \partial_\nu \phi)_\Sigma$. Given these initial conditions one can calculate the metric $G^{-1}_{\mu\nu}$ and consequently the influence cone at every point on Σ. First we have to require that for a given set of initial data the hyperbolicity condition (2.9) is not violated. This imposes the following restriction on the allowed initial values $\phi(\mathbf{x})$ and $\dot{\phi}(\mathbf{x})$:

$$c_s^{-2} = 1 + \left[\left(\dot{\phi}(\mathbf{x})\right)^2 - \left(\vec{\nabla}\phi(\mathbf{x})\right)^2\right]\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}} > 0, \tag{3.29}$$

where we have denoted $\left(\vec{\nabla}\phi(\mathbf{x})\right)^2 = \gamma^{ik}\partial_i\phi\partial_k\phi$. In addition we have to require that the hypersurface Σ is spacelike also with respect to emergent metric $G^{\mu\nu}$, that is, for every vector R^μ, tangential to Σ, we have $G^{-1}_{\mu\nu}R^\mu R^\nu < 0$, or

$$1 + c_s^2 \left(\vec{\nabla}\phi(\mathbf{x})\right)^2 \frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}} > 0. \tag{3.30}$$

If at some point on Σ the vector R^μ becomes null-like with respect to $G^{-1}_{\mu\nu}$, that is, $G^{-1}_{\mu\nu}R^\mu R^\nu = 0$, the signals propagate instantaneously (red propagation vectors from cone B on Fig. 3.3) and one cannot guarantee the continuous dependence on the initial data or even the existence and uniqueness of the solution, see e.g. [164]. Using (3.1) the last inequality can be rewritten as

$$c_s^2 \left(1 + \left(\dot{\phi}(\mathbf{x})\right)^2 \frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right) > 0. \tag{3.31}$$

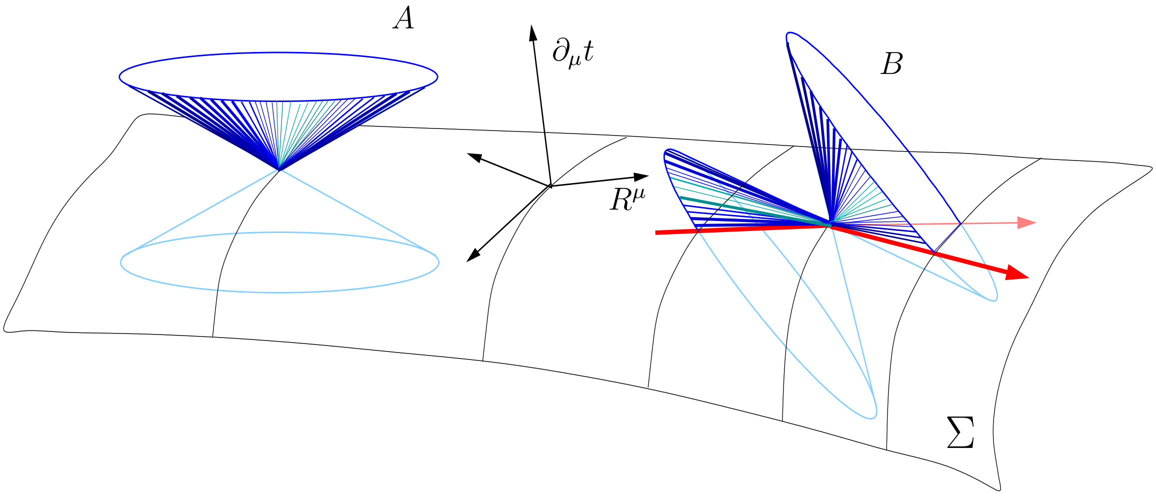

Figure 3.3: The Cauchy problem for the equation of motion of k-*essence* is set up on the hypersurface Σ: $t = t_0$. The vector R^μ is tangential to Σ: $R^\mu \partial_\mu t = 0$. For the hyperbolic equation of motion, the Cauchy problem is well posed provided that $\partial_\mu t$ is timelike with respect to $G^{\mu\nu}$ everywhere on Σ, or, equivalently, the hypersurface Σ is spacelike with respect to $G^{\mu\nu}$ (the cone A in the figure). The cone B represents an ill-posed Cauchy problem for the hyperbolic equation. In particular the red propagation vectors are tangent to Σ.

Therefore, given Lagrangian $\mathcal{L}(\phi, X)$ and hypersurface Σ one has to restrict the initial data $\left(\phi(\mathbf{x}), \dot{\phi}(\mathbf{x})\right)$ by inequalities (3.29) and (3.30) (or equivalently (3.31)), to have a well posed Cauchy problem. The condition (3.31) is always satisfied in the subluminal case for which $\mathcal{L}_{,XX}/\mathcal{L}_{,X} \geq 0$. In addition, we conclude that, if these conditions are satisfied everywhere on the manifold $\mathcal{M}$ and the selected synchronous frame is nonsingular in $\mathcal{M}$, then time t plays the role of global time and in accordance with the theorem about stable causality no causal paradoxes arise in this case.

As a concrete application of the conditions derived let us find which restrictions should satisfy the admissible initial conditions for the low energy effective field theory with Lagrangian $\mathcal{L}(X) \simeq X - X^2/\mu^4 + ...$, where μ is a cut off scale Ref. [28]. In this case (3.31) imply that not only $X \ll \mu^4$, but also $\left(\dot{\phi}(\mathbf{x})\right)^2 \ll \mu^4$ and $\left(\vec{\nabla}\phi(\mathbf{x})\right)^2 \ll \mu^4$. Note that these restrictions can be rewritten in the Lorentz invariant way: for example the first condition takes the form $\left(g^{\mu\nu}\partial_\mu t \partial_\nu \phi\right)^2 \ll \mu^4$.

Finally let us note that even well-posed Cauchy problem cannot guarantee the global existence of the unique solution for nonlinear system of the equations of motion: for example, the solution can develop caustics [206] or can become multi-valued [217, 218].

3.6 How to pose the initial conditions in a fast moving spacecraft?

In this section we resolve "paradoxes" which at first glance seems arising in the case of superluminal propagation of perturbations [28, 23, 22] when one tries to formulate the Cauchy problem in a fast moving spacecraft. To simplify the consideration we restrict ourselves by purely kinetic k-*essence*, for which $\mathcal{L}(\phi, X) = \mathcal{L}(X)$ and assume that for the background solution $X_0 = const > 0$ and $c_s > 1$. This is a reasonable approximation for more general backgrounds with $X_0 > 0$ on the scales much smaller than the curvature scale of the emergent geometry $G^{-1}_{\mu\nu}$. There is always the preferred reference frame (t, x^i) in which the background is isotropic and homogeneous. We refer to this frame as the *rest frame*. In the presence of an external source δJ equation (3.3) in this frame takes the following form

$$\partial_t^2 \pi - c_s^2 \triangle_x \pi = \xi \delta J, \tag{3.32}$$

where $\xi \equiv (c_s^2/\mathcal{L}_{,X})$, for details see Appendix C, equations (C.20) and (C.1). Note that in this case we have $\mathcal{L}_{,X} = const$, $c_s = const$ and the emergent interval becomes standard Minkowski

$$\mathrm{d}S^2 \equiv G^{-1}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\mathcal{L}_{,X}}{c_s}\left(c_s^2 \mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2\right) = \eta_{\mu\nu}\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu \tag{3.33}$$

after the rescalation of the coordinates:

$$\tilde{t} = \sqrt{\mathcal{L}_{,X} c_s} t \quad \text{and} \quad \tilde{\mathbf{x}} = \sqrt{\frac{\mathcal{L}_{,X}}{c_s}}\mathbf{x}. \tag{3.34}$$

Note that our theory is generally covariant therefore we can use whatever coordinates we want. Now let us consider a spacecraft moving in x-direction with velocity v through the k-*essence* background and denote the Lorentz boosted comoving spacecraft coordinates by (t', x'^i). As we have already mentioned above, if the velocity of the spacecraft is larger than c^2/c_s then the Cauchy problem for π cannot be well posed on the hypersurface $t' = const$ [3]. After Lorentz transformation to comoving spacecraft frame, equation (3.32) becomes

$$\left(1 - \frac{v^2}{c^2}\right)^{-1}\left[\left(1 - \frac{c_s^2 v^2}{c^4}\right)\partial^2_{t'}\pi - 2v\left(1 - \frac{c_s^2}{c^2}\right)\partial_{t'}\partial_{x'}\pi + \left(v^2 - c_s^2\right)\partial^2_{x'}\pi\right] - c_s^2 \partial_J \partial_J \pi = \xi\delta J, \tag{3.35}$$

where prime denotes comoving coordinates and index $J = 2, 3, ...$ stands for the spatial directions other than x' [note that in Ref. [28] the factor $(1 - v^2/c^2)^{-1}$ in front of squire brackets is missing]. For $v = c^2/c_s$ the second time derivative drops out of (3.35) and the necessary conditions for applicability of the Cauchy-Kowalewski theorem are not satisfied; hence the existence and uniqueness of the solution (3.35) are not guaranteed. For $v > c^2/c_s$ the necessary conditions of the Cauchy-Kowalewski theorem are met and the unique

[3] Throughout this section we explicitly write the speed of light c and without loss of generality we assume $v > 0$.

solution of (3.35) exists; however, this solution contains exponentially growing modes in the spatial directions, perpendicular to x'. Indeed, substituting

$$\pi \propto \exp\left(-i\omega' t' + ik_{x'}x' + ik_J x^J\right),$$

in (3.35) we find that in the boosted frame:

$$\omega'_{\pm} = \left(1 - \frac{v^2 c_s^2}{c^4}\right)^{-1} \left\{k_{x'} v \left(\frac{c_s^2}{c^2} - 1\right) \pm c_s \sqrt{\left(1 - \frac{v^2}{c^2}\right)\left[k_{x'}^2 \left(1 - \frac{v^2}{c^2}\right) - \left(\frac{v^2 c_s^2}{c^4} - 1\right) k_\perp^2\right]}\right\}. \tag{3.36}$$

where we have denoted $k_\perp = \{k_J\}$ and $k_\perp^2 = k_J k_J$. For $D = 2$, when $k_\perp = 0$, the frequencies ω are always real and no instability modes exist (note that $v < c$). However, if $D > 2$ and $v > c^2/c_s$ then for

$$k_\perp^2 > k_{x'}^2 \left(\frac{1 - v^2/c^2}{v^2 c_s^2/c^4 - 1}\right), \tag{3.37}$$

the general solution of (3.35) contains exponentially growing modes. Note that these are the high frequency modes and hence the instability would imply catastrophic consequences for the theory. At first glance, this looks like a paradox, because equation (3.32), which has no unstable solutions in the rest frame, acquired exponentially unstable solutions in the boosted frame. On the other hand, any solution of (3.32) after performing the Lorentz transformation with $v > c^2/c_s$ does not contain exponentially growing modes with $k_\perp^2$ satisfying (3.37). Indeed, given $(k_x, k_\perp)$ in the rest frame one can perform the Lorentz transformation and obtain:

$$\{\omega',\, k_{x'},\, k'_\perp\} = \left\{\frac{\omega + v k_x}{\sqrt{1 - v^2/c^2}},\, \frac{k_x + \omega v/c^2}{\sqrt{1 - v^2/c^2}},\, k_\perp\right\}, \tag{3.38}$$

were $\omega = \pm c_s \sqrt{k_x^2 + k_\perp^2}$. Expressing ω' via $k_{x'}$ and $k_\perp$ we again arrive to (3.36). However, it follows from (3.38) that if $v > c^2/c_s$ then the components of the Lorentz boosted wavevector satisfy the condition

$$k_\perp^2 \leq k_{x'}^2 \left(\frac{1 - v^2/c^2}{v^2 c_s^2/c^4 - 1}\right), \tag{3.39}$$

and hence unstable modes are not present. This raises the question whether the unstable modes which cannot be generated in the rest frame of k-*essence*, can nevertheless be exited by any physical device in the spacecraft. We will show below that such device does not exist. With this purpose we have to find first the Greens function in both frames.

Let us begin with two-dimensional spacetime. In this case the retarded Green's function for (3.32) in the rest frame (rf) is (see e.g. [164]):

$$G_R^{\mathrm{rf}}(t, x) = \frac{1}{2c_s}\theta\left(c_s t - |x|\right). \tag{3.40}$$

In the boosted Lorentz frame it becomes

$$G_R^{\mathrm{rf}}(t', x') = \frac{1}{2c_s}\theta\left(\frac{c_s\,(t' + vx'/c^2) - |x' + vt'|}{\sqrt{1 - v^2/c^2}}\right). \tag{3.41}$$

For $c_s v < c^2$, the Fourier transform of (3.41) is the retarded in t' Green's function:

$$G_R^{\mathrm{rf}}(t', k') = \frac{\theta(t')}{2ic_s k'}\left(e^{i\omega'_+ t'} - e^{i\omega'_- t'}\right), \tag{3.42}$$

whereas for $c_s v > c^2$ it is given by:

$$G_R^{\mathrm{rf}}(t', k') = -\frac{\theta(t')e^{i\omega'_+ t'} + \theta(-t')e^{i\omega'_- t'}}{2ic_s k'}. \tag{3.43}$$

This Green's function corresponds to the Feynman's boundary conditions in the boosted frame. Thus, in the fast moving spacecraft, the *retarded* Green's function (3.43), obtained as a result of Lorentz transformation from (3.40) looks like a mixture of the retarded [proportional to $\theta(t')$] and the advanced [proportional to $\theta(-t')$] Green's functions with respect to the spacecraft time t'. In fact, the situation is even more complicated. If from the very beginning we work in the comoving spacecraft frame (*sc*), then solving (3.35) we obtain the following expression for the retarded Green's function,

$$G_R^{\mathrm{sc}}(t', k') = \frac{\theta(t')}{2ik' c_s}\left(e^{i\omega'_+ t'} - e^{i\omega'_- t'}\right). \tag{3.44}$$

which coincides with equation (3.40), only if $c_s v < c^2$. However, for fast moving spacecraft, $c_s v > c^2$, formula (3.44) does not coincide with formula (3.43).

The situation is more interesting in the four dimensional spacetime. Similar to the 2d case, after we apply the Lorentz boost to the retarded (in the rest frame) Green's function (see e.g. [164])

$$G_R^{\mathrm{rf}}(t, x^i) = \frac{\theta\,(t)}{2c_s\pi}\delta\left(c_s^2 t^2 - |x|^2\right), \tag{3.45}$$

and calculate its Fourier transform (see Appendix E for the details) we find that for the slowly moving spacecraft, $vc_s < c^2$,

$$G_R^{\mathrm{rf}}(t', k') = \frac{\theta\,(t')}{2ic_s}\left(k_{x'}^2 + k_\perp^2 \frac{1 - c_s^2 v^2/c^4}{1 - v^2/c^2}\right)^{-1/2}\left(\mathrm{e}^{i\omega'_+ t'} - \mathrm{e}^{i\omega'_- t'}\right). \tag{3.46}$$

That is, the resulting Green's function is also retarded with respect to the spacecraft time t'. On the other hand, for the fast moving spacecraft, $vc_s > c^2$, we obtain:

$$G_R^{\mathrm{rf}}(t', k') = -\frac{1}{2ic_s}\left(k_{x'}^2 + k_\perp^2 \frac{1 - c_s^2 v^2/c^4}{1 - v^2/c^2}\right)^{-1/2}\left(\theta\,(t')\,\mathrm{e}^{i\omega'_+ t'} + \theta\,(-t')\,\mathrm{e}^{i\omega'_- t'}\right). \tag{3.47}$$

Similar to the 2d case formula (3.47) is the Feynman Green's function in the spacecraft frame. Note that formula (3.47) can be rewritten as:

$$G_R^{\mathrm{rf}}(t',k') = \frac{1}{2c_s}\left(k_{x'}^2 + k_\perp^2 \frac{1-c_s^2v^2/c^4}{1-v^2/c^2}\right)^{-1/2} \times \tag{3.48}$$
$$\times \exp\left(-ik_{x'}vt'\frac{1-c_s^2/c^2}{1-c_s^2v^2/c^4} - \frac{1-v^2/c^2}{c_s^2v^2/c^4-1}c_s\left|t'\right|\sqrt{k_\perp^2\frac{1-v^2/c^2}{c_s^2v^2/c^4-1} - k_{x'}^2}\right).$$

It is obvious from here that the modes with large $k_\perp$are exponentially suppressed and therefore very high frequency source δJ cannot excite perturbations with $k_\perp^2$ satisfying inequality (3.37).

In the spacecraft frame the retarded Green's function calculated directly for Fourier modes of (3.35) is:

$$G_R^{\mathrm{sc}}(t',k') = \frac{\theta\left(t'\right)}{2ic_s}\left(k_{x'}^2 + k_\perp^2\frac{1-c_s^2v^2/c^4}{1-v^2/c^2}\right)^{-1/2}\left(\mathrm{e}^{i\omega'_+t'} - \mathrm{e}^{i\omega'_-t'}\right).$$

It coincides with Green's function (3.46), obtained by applying the Lorentz transformation, only in the case of slow motion with $v < c^2/c_s$. However, the results drastically differ for the fast moving spacecraft - compare equations (3.46) and (3.47). The function $G_R^{\mathrm{sc}}(t',k')$ contains exponentially growing modes for sufficiently large $k_\perp$ and it's Fourier transform to coordinate space $G_R^{\mathrm{sc}}(t',x')$ does not exist. Physically this means that we have failed to find the Green's function, which describes the propagation of the signal which the source δJ in the fast moving spacecraft tries to send in the direction of growing t'. Instead, the response to any source in the spacecraft is always driven by (3.48) (or the Lorentz transformed Green's function in the rest frame (3.45)). Because we cannot send a signal in the direction of growing t' one cannot associate growing t' with the arrow of time contrary to the claims in [22].

Now we will discuss in more details how the problem of initial conditions for perturbations π must be correctly formulated in the fast moving spacecraft. The first question here whether the fast moving astronaut can create an arbitrary initial field configurations π and $\dot{\pi}$ at a given moment of his proper time $t'_1 = const$. This hypersurface is not *space-like* with respect to the metric $G_{\mu\nu}^{-1}$ and therefore as it follows from the consideration in the previous section the Cauchy problem is not well posed on it. Hence not all possible configurations are admissible on this hypersurface but only those which could be obtained as a result of evolution of some initial configuration chosen on the hypersurface which is simultaneously spacelike with respect to both metrics $g_{\mu\nu}$ and $G_{\mu\nu}^{-1}$. If the astronaut disturbs the background with some device (source function δJ) which he/she switches off at the moment of time t'_1, then the resulting configuration of the field on the hypersurface $t'_1 = const$ obtained using the correct Green's function (3.48) will always satisfy the conditions needed for unambiguous prediction of the field configuration everywhere in the spacetime irrespective of the source $\delta J(x)$. The presence of the advanced mode in this Green's function plays an important role in obtaining a consistent field configuration on $t'_1 = const$. Thus we see that not "everything" is in the hand of the astronaut: he has no "complete freedom" in the choice of the "initial" field configuration at time t_1'. Nonrecognition of this fact leads to the fictitious causal paradoxes discussed in the literature [23, 22].

For a slowly moving spacecraft, $v < c^2/c_s$, the retarded Green's function in the rest frame is transformed in the retarded Green's function in the spacecraft frame. Therefore we can obtain any a priori given field configuration on the hypersurface $t_1' = const$ by arranging the source function δJ in the corresponding way. Thus, the choice of the initial conditions for the perturbations at $t_1' = const$ is entirely in the hand of the astronaut. This is in complete agreement with our previous consideration because in the slowly moving spacecraft the hypersurface $t_1' = const$ is spacelike with respect to both metrics.

The appearance of the advance part in the correct Green's function for the fast moving spacecraft still looks a little bit strange because according to the clocks of the astronaut the head of the spacecraft can "feel" signals sent at the same moment of time by a device installed on the stern of the spacecraft. However, in this case the proper time of the astronaut is simply not a good coordinate for the time ordering of the events at different points of the space related by the k-*essence* superluminal signals. The causality is also preserved in this case but it is determined by the superluminal k-*essence* cone which is larger than the light cone and as we have already seen no causal paradoxes arise in this case. If the astronaut synchronizes his clocks using the superluminal sonic signals then the new time coordinate $\tilde{t}$ becomes a good coordinate for the time ordering of the causal events in different points of the space. The hypersurface $\tilde{t} = const$ being spacelike in both metrics can then be used as the initial hypersurface for the well posed Cauchy problem in the fast moving spacecraft, that is, any initial configuration of the field can be freely created by the astronaut on this hypersurface. In the "well synchronized' reference frame $\left(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z}\right)$ the equation of motion for perturbations (3.32) takes the same form as in the rest frame of the k-*essence* background:

$$\partial_{\tilde{t}}^2 \pi - c_s^2 \bigtriangleup \pi = \xi \delta J. \tag{3.49}$$

It follows from here that

$$\omega_{\pm} = \pm c_s \sqrt{k_{\tilde{x}}^2 + k_{\tilde{y}}^2 + k_{\tilde{z}}^2},$$

and hence no exponentially growing modes exist for any $k_{\tilde{x}}$, $k_{\tilde{y}}$ and $k_{\tilde{z}}$.

The causal Green's function in the spacecraft frame contains only the retarded with respect to the time $\tilde{t}$ part. For example, in four-dimensional spacetime it is given by

$$G_R^{\rm sc}(\tilde{t}, \tilde{x}^i) = \frac{\theta\left(\tilde{t}\right)}{2c_s\pi} \delta\left(c_s^2 \tilde{t}^2 - |\tilde{x}^i|^2\right). \tag{3.50}$$

This result can be obtained either by applying the Lorentz transformation with the invariant speed c_s to (3.40), or directly by solving equation (3.49). Thus, no paradoxes with Green's functions arise for the superluminal perturbations. The same conclusions are valid in 4d spacetime.

To make the consideration above even more transparent we conclude this section by considering analogous situation with *no* superluminal signals involved. Namely, we take

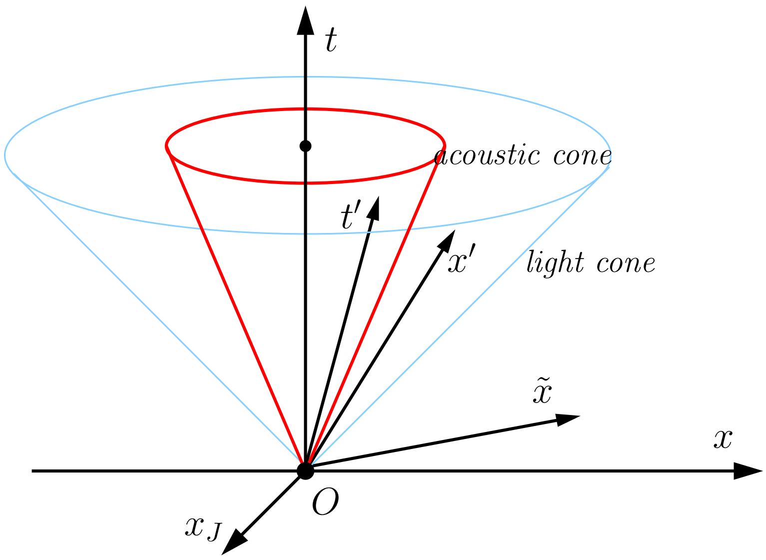

Figure 3.4: It is shown how one can create a would -be "paradox" similar to that discussed in this section, without involving any superluminal signals. The fluid is at rest and the perturbations propagate subluminally in the fluid, $c_s < c$. The reference frame (t', x') is connected to the rest frame by the Lorentz boost with the invariant speed c_s. If the boost speed v is such that $c_s/c < v/c_s < 1$, then the hypersurface of constant t' is inside the light cone and the Cauchy problem for the electromagnetic field is ill posed in this reference frame. Instead, one should use the "correct" frame $(\tilde{t}, \tilde{x})$, obtained by the Lorentz boost with the invariant fastest speed $c = 1$. In this frame the Cauchy problem is well-posed.

a fluid at rest with a *subluminal* speed of sound, $c_s < c$. Then we can make the Lorentz transformation using the invariant speed c_s:

$$t' = \frac{t - vx/c_s^2}{\sqrt{1 - v^2/c_s^2}}, \quad x' = \frac{x - vt}{\sqrt{1 - v^2/c_s^2}}, \quad x'_J = x_J.$$

If the speed v is such that $c_s/c < v/c_s < 1$, then the hypersurface of constant t' is *inside* the light cone (see Fig. 3.4) and it is obvious that one cannot formulate the Cauchy problem for the electromagnetic field on the hypersurface $t' = const$. Instead, the Cauchy problem for the electromagnetic field can be well posed on the hypersurface $\tilde{t} = const$ defined by the "correct" Lorentz transformation, with the invariant speed c:

$$\tilde{t} = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \quad \tilde{x} = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad \tilde{x}_J = x_J,$$

(see Fig. 3.4). This consideration is fully equivalent to those one above with the only replacement $c_s \leftrightarrow c$.

Thus we have shown that no physical paradoxes arise in the case when we have superluminal propagation of small perturbations on the background.

3.7 Chronology protection

It was claimed in [28] that the theories with superluminal propagation are plagued by closed causal curves (CCC). We will argue here that the superluminal propagation cannot be the sole reason for the appearance of CCC and moreover this problem can be avoided in this case in the same way as in General Relativity.

It is well know that General Relativity admits the spacetimes with the closed causal curves without involving any superluminal fields into consideration. Among examples of such spacetimes are: Gödel's cosmological model [58], Stockum's rotating dust cylinder [219, 220], wormholes [66], Gott's solution for two infinitely long strings [59] and others [60, 61, 62, 63, 64, 65, 67, 68]. A prominent time-machine model was suggested recently by Ori [60]. In this model, made solely of vacuum and dust, the spacetime evolves from a regular normal asymptotically flat state without CCCs and only later on develops CCCs without violating the weak, dominant and strong energy conditions. Thus, we see that initially "good" spacetime might in principle evolve to a state where the chronology is violated and the General Relativity does not by itself explains these strange phenomena. Therefore one needs to invoke some additional principle(s) to avoid the pathological situations with CCCs. With this purpose Hawking suggested the *Chronology Protection Conjecture*, which states that the laws of physics must prohibit the appearance of the closed timelike curves [54]. In [54] it was argued that in the situation when the timelike curve is ready to close, the vacuum polarization effects become very large and the backreaction of quantum fields prevents the appearance of closed timelike curves.

Similarly to General Relativity, one might assume that in the case of superluminal propagation the chronology protection conjecture is valid as well. For example, the chronology protection was already invoked to exclude the causality violation in the case of two pairs of Casimir plates [53], in which photons propagate faster than light due to the Scharnhorst effect [45, 46, 47, 48].

Once we employ the chronology protection principle, no constructions admitting CCCs, similar to those presented in [28], may become possible.

In fact, the first example in [28] with two finite fast moving bubbles made of superluminal scalar field (see Fig. 2 in Ref. [28]) is quite similar to the "time machine"involving two pair of Casimir plates. In the latter case the chronology protection excludes the existence of CCCs. Here the situation is a little bit more involved. In the example with the bubbles the background is not a free solution of the equation of motion (2.7). Indeed as it was pointed out in [28] the fast moving bubbles have to be separated in the direction orthogonal to the direction of motion. On the other hand they have to be connected by light. However, if this were a free solution, then the bubbles would expand with the speed of light and collide at the same moment of time, or even before the closed causal curve would be formed. Thus an external source $J(x)$ of the scalar field is required in order to produce this acausal background. However, without clear idea about the origin of this source and possible backreaction effects the physical interpretation of this "time machine"is obscure. It is well known that admitting all possible sources of gravitational field one can obtain almost any possible even acausal solutions in general relativity. Finally, generalizing the Hawking conjecture to the case of scalar field one can argue that the backreaction of quantum fluctuations of perturbations π around ϕ become large before CCC is formed thus destroying the classical solution imposed by the external source $J(x)$ and preventing the formation of CCC.

The other example considered in [28] involves non-linear electrodynamics. The electromagnetic field is created by charge currents, serving as a source. Thus, unlike the previous example, one can control the strength of the field, simply changing the configuration of charges. The electromagnetic part of the Lagrangian is the "wrong"-signed Euler-Heisenberg Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \alpha\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + ..., \tag{3.51}$$

with a small positive α. For such a system the propagation of light in a non-trivial background is superluminal. As a consequence, a cylindrical capacitor with the current-carrying solenoid leads to the appearance of the CCCs, provided the electrical and magnetic fields inside the capacitor are large enough (see Fig. 3 in [28]). In this case one may invoke a simplified version of the chronology protection conjecture. In fact, let us begin with some "good" initial conditions in the capacitor, namely, with electric and magnetic fields being not too large, so that no CCCs exist. Then we increase the current in the solenoid and the voltage between the plates of a capacitor in order to increase the strength of the fields. When the causal curves become almost closed, the expectation value of the energy-momentum tensor for the "quasi-photons" on this classical electromagnetic background becomes very large due to the quantum vacuum polarization effects. In the limit when the causal cone becomes horizontal, the energy density of the field in the capacitor tends to infinity and the capacitor will be broken before the CCCs will be formed.

Thus we conclude that concerning the causal paradoxes the situation in the theories with superluminal propagation on the non-trivial backgrounds is not much worse than in General Relativity. In fact, in this respect the similarity between these two theories goes even much deeper than it looks at the first glance. For example, let us imagine a time machine which is constructed with the help of superluminal propagation in non-trivial background produced by the external source $J(x)$, e.g., similar to those described in [28]. Then we can identify the *effective metric* $G^{\mu\nu}$ for this system with the *gravitational metric* $g^{\mu\nu}$ of some spacetime produced by an energy-momentum tensor $T^{(J)}_{\mu\nu}(x)$. Put differently, once having the effective metric, we can find spacetime where the gravitational metric is $G^{\mu\nu}$. In this spacetime the time machine exists as well. Remarkably, now the gravitation (or light) signals are used to make CCCs. The spacetime with the metric $G^{\mu\nu}$ is the solution of Einstein equations with the energy-momentum tensor $T^{(J)}_{\mu\nu}$ calculated substituting the metric in the Einstein equations. After that one could try to find such theories and such fields configurations on which their resulting energy momentum tensor is equal to $T^{(J)}_{\mu\nu}(x)$ consistently with equations of motion. One can, in principle, argue, that in the case when the CCCs exist the energy-momentum tensor mights have some undesired properties, for example, it would violate the Week Energy Condition (WEC). However, in several known examples with CCCs the WEC is satisfied, see, e.g. Refs. [58, 59, 219, 220]. Moreover, the system found by A. Ori [60] possesses CCCs and satisfies the week, dominant and strong energy conditions. Thus the violation of the energy conditions is not an inherent property of the spacetimes with CCCs. Therefore the question, whether the spacetime constructed by the procedure described above requires "bad" energy-momentum tensor or not, must be studied separately in each particular case.

Moreover the correspondence $G^{\mu\nu} \leftrightarrow g^{\mu\nu}$, $J \leftrightarrow T^{(J)}_{\mu\nu}$ can also be used to learn more about *Chronology Protection Conjecture* and time-machines in General Relativity with the help of more simple theory. It is well-known that Analogue Gravity [221, 222, 20, 223] gives more simple and intuitively clear way to investigate the properties of Hawking radiation, the effects of Lorentz symmetry breaking, transplanckian problem *etc.,* by using the small perturbations in the fluids instead of direct implication of General Relativity. In a similar way, *analogue time-machine* or analogue *Chronology Protection Conjecture* may provide one with a tool to check *Chronology Projection Conjecture* and the possibility of construction of time machines in General Relativity.

3.8 Is the gravitational metric universal?

It was argued in [24] that the same causal limits apply to all fields independent of the matter present, thus endowing the gravitational metric with the universal role. However, in the theories under consideration the causal limit is governed not by the gravitational metric $g_{\mu\nu}$, but by the effective acoustic metric $G^{-1}_{\mu\nu}$ and hence the gravity loses its universal role in this sense. Nevertheless, here we argue, that even in the case of the spontaneously broken Lorentz invariance with a superluminal propagation the gravitational metric $g_{\mu\nu}$ still keeps its universal role in the following sense. First in accordance with the discussion in 3.5 we remind that the Cauchy hypersurface for the field ϕ should anyway be a spacelike one in the gravitational metric. Thus in order to produce a background which breaks the Lorentz invariance one has to respect the usual causality governed by the gravitational metric $g_{\mu\nu}$. Moreover, if a clump of the scalar field is created in a finite region surrounded by a trivial background, then the boundaries of the clump will generically propagate with the speed of light.

Indeed, let us consider a finite lump of non-trivial field configuration with smooth boundaries (see Fig. 3.5) and assume that the initial data $\left(\phi(\mathbf{x}), \dot{\phi}(\mathbf{x})\right)$ are specified in some finite spatial region R. These initial data are smooth everywhere (see Fig. 3.5) and satisfy the conditions (3.31) and (3.29), in particular the first derivatives of the field ϕ are continuous everywhere including the boundaries of the clump. If the system described by action (2.1) has at least one trivial solution $\phi = \phi_{\text{triv}} = const$ with non-pathological acoustic geometry, then, as it follows from (2.7) and (3.7), the Lagrangian $\mathcal{L}(\phi, X)$ is at least twice differentiable at $(\phi, X) = (\phi_{\text{triv}}, 0)$ and moreover $\mathcal{L}_{,X}(\phi_{\text{triv}}, 0) \neq 0$. Thus for the theories of this type we have

$$\mathcal{L}(\phi, X) \simeq V(\phi) + K_1(\phi)X + K_2(\phi)X^2 + ... \tag{3.52}$$

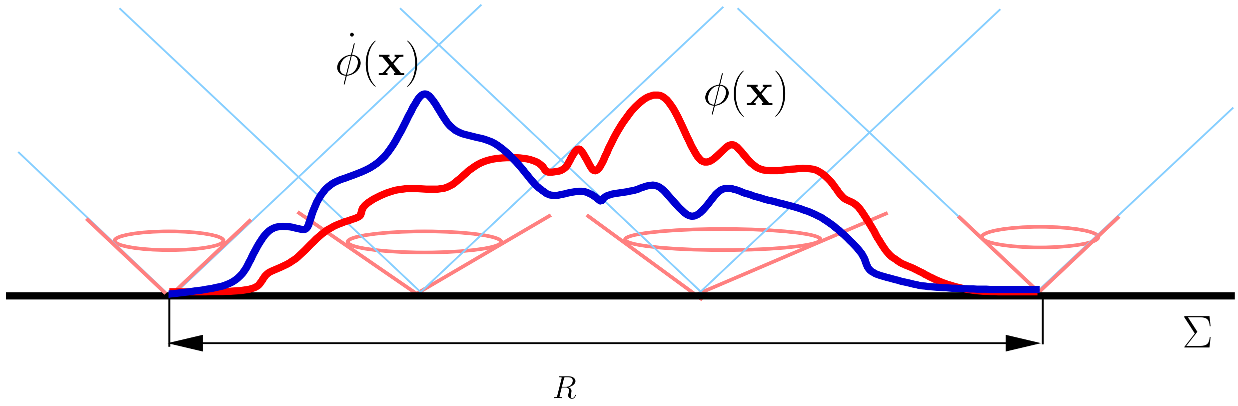

Figure 3.5: The figure shows that the gravitational metric $g_{\mu\nu}$ keeps its universal meaning even if the small perturbations on the non-trivial backgrounds propagate superluminally. If in the initial moment of time the non-trivial configuration of the field ϕ is localized in the finite region R on a spacelike in $g_{\mu\nu}$ hypersurface Σ, and beyond this region the field ϕ is in its vacuum state $\phi = const$, then the front of the solution always propagates with the speed of light. The blue lines correspond to the light rays. The pink cones represent the influence cones for k-*essence*. On the boundary of R the influence cones are equal to light cones.

in the vicinity of the trivial solution ϕ_{triv} [4]. And as expected we conclude that the speed of sound for the small perturbations is equal to the speed of light in the vicinity of ϕ_{triv} because any trivial solution and in particular a possible vacuum solution $\phi = 0$ does not violate the Lorentz invariance. Moreover, close to the boundaries of the clamp the initial data $\left(\phi(\mathbf{x}), \dot{\phi}(\mathbf{x})\right)$ can be considered as small perturbation around the trivial background and therefore the front of the clump propagates exactly with the speed of light in the vacuum. Thus, without preexisting nontrivial configuration of the scalar field the maximum speed of propagation never exceeds the speed of light and the causality is entirely determined by the usual gravitational metric only.

If we abandon the condition of the regularity of the emergent geometry $G^{-1}_{\mu\nu}$, but still require that the Lagrangian is analytic function of X in the neighborhood of $X = 0$, then the speed of propagation in vacuum is always *smaller* than the speed of light. Indeed in this case the speed of sound c_s is:

$$c_s^2 = \frac{1}{(1 + 2\,(n-1))} < 1,$$

where n is the power of the first non-zero kinetic term in (3.52).

To demonstrate explicitly the points stated above we will find now exact solitonic solutions in the purely kinetic k-*essence* theories with Lagrangian $\mathcal{L}\,(X)$ and verify that

[4] In particular, it was required in [74, 224, 225] that for models allowing topological k-*defects*, the asymptotic behavior near the trivial vacuum $X = 0$ (at the spatial infinity) is of the form (3.52)

these solitons propagate in the Minkowski spacetime with the speed of light. Assuming that the scalar field depends only on $\theta \equiv x + vt$ and substituting $\phi = \varphi(\theta)$ in equation (2.7) we find that this equation reduces to

$$\mathcal{L}_{,X}\varphi_{,\theta\theta}\left(v^2-1\right)+\mathcal{L}_{,XX}\varphi_{,\theta\theta}\varphi_{,\theta}^2\left(v^2-1\right)^2=0, \tag{3.53}$$

This equation is trivially satisfied for $v = \pm 1$, that is, there exist solitary waves $\varphi(x \pm t)$ propagating with the speed of light. They are solutions corresponding to rather special initial conditions $\phi_0(x) = \varphi(x)$ and $\dot{\phi}_0(x) = \pm\varphi(x)$. Note that the general solutions are not a superposition of these solitonic solutions because the equation of motion is nonlinear. Assuming that $v \neq \pm 1$ we find that (3.53) is satisfied by either nonlocalized solution $\phi = x \pm vt + const$, or it reduces to:

$$\mathcal{L}_{,X}+\mathcal{L}_{,XX}\varphi_{,\theta}^2\left(v^2-1\right)=\mathcal{L}_{,X}+2X\mathcal{L}_{,XX}=0, \tag{3.54}$$

This algebraic equation is trivially satisfied for all X if $\mathcal{L} = f(\phi)\sqrt{X} - V(\phi)$. This is the case when the perturbations propagate with the infinite speed on any background [226, 227]. For more general Lagrangians $\mathcal{L}(X)$ equation (3.54) can be solved algebraically to obtain a particular $X_0 = \frac{1}{2}\varphi_{,\theta}^2(v^2-1) = const$. The only solutions of this last equation are either $\varphi(x \pm t)$ or trivial solutions $\phi_{\text{triv}} = const$. For the Born-Infeld Lagrangian [1] the exact solutions of this type were found in [217, 218] (see also [228]). For more complicated Lagrangian, for example, of the form $\mathcal{L} = \mathcal{K}(X) + V(\phi)$ there exist solitonic solutions with $v < 1$ [74, 224, 225].

Thus, we have shown that under reasonable restrictions on the theory the field configurations localized in trivial vacuum never propagate faster than light. Therefore the causal limit for these localized configurations is always governed by the usual gravitational metric.

3.9 Discussion

In this Chapter we have considered the k-*essence*-like scalar fields with the Lorentz invariant action (2.1) and have studied the issues of causality and Cauchy problem for such theories. These questions are non-trivial because small perturbations π on backgrounds ϕ_0 can propagate faster-than-light. The perturbations "feel" the effective metric, $G^{\mu\nu}$ given by (3.7), which is different from the gravitational metric $g^{\mu\nu}$, if the Lagrangian $\mathcal{L}$ is a non-linear function of X and the background is nontrivial $\partial_\mu\phi_0 \neq 0$. We have derived the action for the perturbations on an arbitrary background and have shown that these perturbations "feel" the emergent geometry $G_{\mu\nu}^{-1}$. The influence cone determined by $G_{\mu\nu}^{-1}$ is larger than those one determined by metric $g_{\mu\nu}$ provided $\mathcal{L}_{,XX}/\mathcal{L}_{,X} < 0$ [26, 175, 80]. Thus perturbations can propagate with the speed exceeding the speed of light. In this case the background serves as a new *aether* and preselects the preferred reference frame. This is why the causal paradoxes arising in the presence of *tachyons* [5] (superluminal particles in the Minkowski vacuum) do not appear here. In particular, we have shown that in

[5]Do not confuse them with field theoretical tachyons with $m^2 < 0$.

physically interesting situations, namely, cosmological solutions and for the case of a black hole surrounded by an accreting fluid, the closed timelike curves are absent and hence we cannot send the signal to our own past using the superluminal signals build out of the "superluminal" scalar field perturbations. Thus, the k-*essence* models, which generically possess the superluminal propagation, *do not lead to the causal paradoxes*, contrary to the claim in [23, 22].

We have shown how to pose correctly the Cauchy problem for the k-*essence* fields with superluminal propagation, which sometimes might seem problematic [28]. The correct initial Cauchy hypersurface Σ must simultaneously be spacelike with respect to both gravitational metric $g_{\mu\nu}$ and the effective metric $G^{-1}_{\mu\nu}$. Because the effective metric $G^{-1}_{\mu\nu}$ itself depends on the values of the field ϕ and its first derivatives, the initial value problem must be set up in a self-consistent manner: in addition to the usually assumed hyperbolicity condition (2.9), one must require that the field ϕ and its derivative on Σ must satisfy the inequality (3.30). In particular, in the case of spacecraft which has very large velocity with respect to the homogeneous background of the k-*essence*, the latter conditions are violated on the hypersurface of constant astronaut proper time. Therefore no physical devices are able to produce an arbitrary configuration of perturbations on this hyperspace.

It was found in [28] that in the theories under consideration one can have the backgrounds possessing the closed causal curves (CCCs). However, as we have argued above, this is not directly related to the superluminal propagation. In fact, the situation here is very similar to the situation in General Relativity, where one can also have the manifolds with the closed causal curves although the speed of propagation is always limited by the speed of light. In this respect the situation in the theories with the superluminal propagation is not worse than in General Relativity. To avoid causal paradoxes in General Relativity, Hawking suggested the *Chronology Protection Conjecture*, which states that the quantum effects and, in particular, vacuum polarization effects can prevent the formation of the closed timelike curves [54]. Similarly to Hawking one may argue that in the case of the superluminal propagation the *Chronology Protection Conjecture* can be valid as well. In fact, this conjecture was already invoked to exclude the causality violation in the case of two pairs of Casimir plates [53]. Once we employ the *Chronology Protection Conjecture*, no constructions admitting CCCs, similar to those presented in [28], are possible.

Sometimes the "superluminal" theories are criticized in the literature on the basis of general, or better to say, aesthetic grounds. For example, Ref. [24] claims: "The spacetime metric is preferred in terms of clock measurements and free fall (geodesic) motion (including light rays), thus underlying General Relativity's central theme of gravity being encoded in spacetime curvature." Although this argument is not more than the matter of taste, we would rather prefer to have General Relativity as a theory which keeps its (restricted) universal meaning even in the presence of superluminal propagation. We have argued that under physically reasonable assumptions and without a preexisting nontrivial background the causality is governed by metric $g_{\mu\nu}$. Indeed, if initially the field ϕ is localized within some finite region of space surrounded by vacuum, then the border of this region propagates with the speed of light and it is impossible to send signals faster than light.

Chapter 4

Dynamical violation of the Null Energy Condition

In this Chapter we review the results from the work [105] and show that in the framework of the generic k-*essence* in the Friedmann universe the dynamical violation of Null Energy Condition (NEC) is physically implausible. For the recent discussion see Note that in this chapter we use units where $M_{_{Pl}} = (8\pi G_{_N})^{-1/2} = 1$.

4.1 Possible Mechanisms of the transition

As a warm up, it is worth to note that if we could consider our scalar dark energy as an "isentropic" fluid and express the pressure p only in terms of ε, then the the question "whether it is possible to evolve through the point ε_c, where $w(\varepsilon_c) = -1$" would be meaningless. Indeed, in that case we could rewrite the continuity equation (2.14) only in terms of ε

$$\dot{\varepsilon} = -\sqrt{3\varepsilon}\,(\varepsilon + p(\varepsilon))\,, \tag{4.1}$$

so that the system of Einstein equations (2.12), (2.13) could be reduced to the only one Eq. (4.1) and the values of energy density ε_c would be the fixed points of this equation. If, for example, $p = p(X)$ depends only on X, then $\varepsilon = \varepsilon(X)$. Let us further assume that there are some values X_c, where $w(X_c) = -1$. If Eq. (2.5) can be solved with respect to X in the neighborhoods of these points X_c then the transition is impossible, because in this case one can find $X(\varepsilon)$ and therefore $p(\varepsilon) = p(X(\varepsilon))$. From the theorem about the inverse function Eq. (2.5) is solvable with respect to X if $\varepsilon_{,X}(X_c) = (2Xp_{,XX}(X_c) + p_{,X}(X_c)) \neq 0$. So if $X_c = 0$ and $p_{,X}(X_c) \neq 0$ or $p_{,X}(X_c) = 0$ and $X_c p_{,XX}(X_c) \neq 0$, then X_c are fixed points and the transition through $w = -1$ is forbidden. In fact, as it was shown in Refs. [229, 17], there always exists the solution $X(t) \equiv X_c$ and what is more it is an attractor. Thus the transition is generally forbidden for the systems described by the purely kinetic Lagrangians $p(X)$. Nevertheless this easy example hints at conditions, under that the transition could be possible. In the general case, when $p = p(\phi, X)$, the pressure cannot be expressed only in terms of ε, since ϕ and X are independent.

There are two possibilities to evolve from the usual case of dark energy with $w \geq -1$ to a phantom one for such $w < -1$ (or vise versa), namely a continuous transition, in which the dark energy evolves through the point $w = -1$ and a discontinuous transition, occurring through the point where $\varepsilon = 0$. Assuming that the dark energy is the dominating source of gravitation, we will restrict ourself only to a consideration of the continuous transitions. The equation of state parameter can be rewritten in the following form, more convenient for our purposes:

$$w = -1 + \frac{2X}{\varepsilon} p_{,X} = -1 + \frac{\dot{\phi}}{\varepsilon} p_{,\dot{\phi}}. \tag{4.2}$$

If we remember that X is nonnegative and $\varepsilon > 0$, we find that $w < -1$ corresponds to $p_{,X} < 0$ or $\dot{\phi} p_{,\dot{\phi}} < 0$ and $w > -1$ to $p_{,X} > 0$ or $\dot{\phi} p_{,\dot{\phi}} > 0$, while at the points where either $X = 0$ or $p_{,X} = 0$ the equation of state parameter w takes the value -1. Suppose there exists a solution $\phi(t)$ and the moment of time t_c such that $w(t_c) = w(\phi_c, X_c) = -1$, where we denote $(\phi_c, \dot{\phi}_c) \equiv (\phi(t_c), \dot{\phi}(t_c))$ and $2X_c = \dot{\phi}_c^2$. As we have already seen the values of field ϕ_c and field's time derivative $\dot{\phi}_c$ build two disjunct sets:

A) the $\phi-$axis of the phase plot $(\phi, \dot{\phi})$ so $\dot{\phi}_c \equiv 0$.

B) the points $\Psi \equiv (\phi_c, \dot{\phi}_c)$ where $p_{,X}(\phi_c, \dot{\phi}_c) = 0$ and $\dot{\phi}_c \neq 0$. In this case the points Ψ generally build curves $\gamma(\lambda)$ (may be intersecting) and isolated points in the phase space $(\phi, \dot{\phi})$.

In the both cases $p_{,X}$ should change the sign during the dynamical evolution from the states $(\phi, \dot{\phi})$ where $w \geq -1$ to the sates with $w < -1$ (or vise versa).

As we have already learned from the example of a purely kinetic Lagrangian $p(X)$ and as we will see later, the following two subclasses of the candidates to the transition points Ψ are significant: $\varepsilon_{,X}(\Psi) \neq 0$ (or equivalently $p_{,XX}(\Psi) \neq 0$) and $\varepsilon_{,X}(\Psi) = 0$.

4.1.1 Transition at points $X_c = 0$

Here we will analyze, whether the transition is possible in the case A) $\dot{\phi}_c = 0$. To begin with, let us consider the enough differentiable function $p(\phi, X)$. We see that $\varepsilon_{,X}(t_c) \equiv p_{,X}(t_c)$ and the first derivative of equation of state parameter looks

$$\dot{w}(t_c) = \frac{2\dot{X}}{\varepsilon} p_{,X} = \frac{2\ddot{\phi} p_{,X}}{\varepsilon} \dot{\phi} = 0. \tag{4.3}$$

Then we have the relations $\ddot{X}(t_c) = \ddot{\phi}^2(t_c)$ and

$$\frac{dp_{,X}}{dt} = \dot{\phi} \left(p_{,\phi X} + \ddot{\phi} p_{,XX} \right). \tag{4.4}$$

Hence we infer that $dp_{,X}/dt(t_c) = 0$. Due to these relations the second derivative of the equation of state parameter reads

$$\ddot{w}(t_c) = \frac{2\ddot{X}}{\varepsilon} p_{,X} = \frac{2\ddot{\phi}^2}{\varepsilon} p_{,X}. \tag{4.5}$$

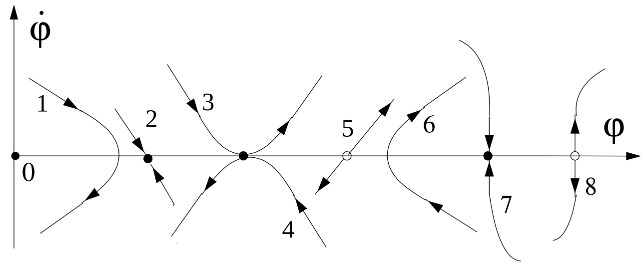

Figure 4.1: Possible phase curves in the neighborhood of the $\phi-$axis. Only on the curves 1 and 6 the system pass the $\phi-$axis. The curves 2,3,4,7 have an attractor as a shared point with the $\phi-$axis, whereas the curves 5,8 have an repulsor. These attractors and repulsors can be a fixed point solutions or singularities as well.

Using the equation of motion (2.16) we find

$$\ddot{\phi}(t_c)\varepsilon_{,X}(t_c) = -\varepsilon_{,\phi}(t_c). \tag{4.6}$$

If $\varepsilon_{,\phi}(t_c) = 0$ and $\varepsilon_{,X}(t_c) \neq 0$ then from the relation written above $\ddot{\phi}(t_c) = 0$. Therefore the considered solution $\phi(t)$ on which at time t_c: $w(\phi(t_c)\phi(t_c)) = -1$, is a fixed point solution $\phi(t) \equiv \phi_c \equiv const$ and therefore the transition is impossible. Taking into account that $\varepsilon > 0$, we see that this fixed point is obviously the de Sitter solution. Provided $\varepsilon_{,\phi}(t_c) \neq 0$ and $p_{,X}(t_c) \neq 0$ then as it follows from Eq. (4.5),(4.6) the equation of state parameter $w(t)$ has either minimum or maximum at the point t_c. Thus the transition is impossible in this case.

If, in addition, not only $X_c = 0$ but also $\varepsilon_{,X}(t_c) \equiv p_{,X}(t_c) = 0$ then $\varepsilon_{,\phi}(t_c) = 0$ and therefore, as it follows from the formula (2.5), $p_{,\phi}(t_c) = 0$. Moreover the equation of motion (2.16) is not solved with respect to the highest derivatives (namely with respect to $\ddot{\phi}$) and therefore do not need to have an unique solution. It happens because the point $(\phi_c, 0)$ on the phase plot $(\phi, \dot{\phi})$ does not directly define the $\ddot{\phi}$ via the equation of motion (. It is clear that in this case, the point-like (on the phase plot) solution $\phi(t) \equiv \phi_c \equiv const$ is the solution, but not necessary the unique one.

Below we will give the more general consideration of the phase curves geometry in the neighborhood of the $\phi-$axis. The phase flows are directed from right to left for the lower part of the phase plot $\dot{\phi} < 0$ and from left to right for the upper part $\dot{\phi} > 0$, see Fig (4.1). Therefore the system can pass the $\phi-$axis, only if the point of intersection is a turning point (curves 1 and 6 on Fig (4.1)). Otherwise the crossing is a fixed point (or a singularity). If there is a smooth phase curve on which the system pass the $\phi-$axis, then in a sufficient small neighborhood of the turning point $\phi \sim \dot{\phi}^{2n} \sim X^n$ where $n \geq 1$. But in this case the function $Xp_{,X}(\phi, X)$ depends only on X and therefore should preserve the sign. Hence as it follows from the formula (4.2) the system can not change the sign of $(w+1)$ passing the $\phi-$axis.

If a smooth phase curve do not cross but touches the $\phi-$axis at a point ϕ_c, (see Fig. (4.1), trajectories 3 and 4) then the following asymptotic holds: $\dot{\phi} \sim (\phi - \phi_c)^{2n}$, where $n \geq 1$.

Let us find the time that the system would need to reach the tangent point $(\phi_c, 0)$ in this case. We have

$$t \equiv \int_{\phi_{in}}^{\phi_c} \frac{d\phi}{\dot{\phi}(\phi)} \sim \int_{\phi_{in}}^{\phi_c} \frac{d\phi}{(\phi-\phi_c)^{2n}}, \tag{4.7}$$

where ϕ_{in} is a starting point on the phase curve. The last integral is obviously divergent. Therefore the system can not reach the tangent point in finite time and the transition is impossible.

Note that the analysis made above would also work for the such functions $p(\phi, X)$, for that not the first $X-$derivative $p_{,X}$ itself, but only the energy density ε (and therefore $Xp_{,X}(\phi, X)$) is well defined at $X = 0$.

Finally we come to conclusion: in the framework under consideration it is impossible to build a model with the desirable transition through the points $X_c = 0$.

4.1.2 Transition at points Ψ_c: $p_{,X}(\Psi_c) = 0$, $\varepsilon_{,X}(\Psi_c) \neq 0$, $X_c \neq 0$

From the theorem about the implicit function it follows that in the neighborhood of a point Ψ_c, at that the condition $\varepsilon_{,X}(\Psi_c) \neq 0$ (or equivalently the condition $p_{,XX}(\Psi_c) \neq 0$) holds one can find the function $\dot{\phi}_c(\phi_c)$: $p_{,X}(\phi_c, \dot{\phi}_c(\phi_c)) = 0$. One would anticipate that on the phase curves intersecting the curve $\dot{\phi}_c(\phi_c)$ the state of the dark energy changes to the phantom one (or vise versa). Let us express $p_{,X}$ from Eq. (4.2) and substitute it into the formula (2.24) for the sound speed of perturbations:

$$c_s^2 = \frac{(w+1)\varepsilon}{2X\varepsilon_{,X}}. \tag{4.8}$$

For the stability with respect to the general metric and matter perturbations the condition $c_s^2 \geq 0$ is necessary (see [69]). Indeed the increment of instability is inversely proportional to the wavelength of the perturbations, and hence the background models for that $c_s^2 < 0$ are violently unstable and do not have any physical significance. Note that the stability might be preserved if some higher derivative operators in the action become important, see e.g. [17] . Due to the continuity of $\varepsilon_{,X}$ there exists a neighborhood of the point Ψ_c where $\varepsilon_{,X} \neq 0$. Therefore, from the above expression for the sound speed (4.8) follows that if $(w + 1)$ change the sign then c_s^2 should change the sign as well. But if this is the case then the trajectories, realizing the transition violate the stability condition $c_s^2 \geq 0$ and therefore the model of the transition is not realistic.

4.1.3 Transition at points Ψ_c: $p_{,X}(\Psi_c) = 0$, $\varepsilon_{,X}(\Psi_c) = 0$, $X_c \neq 0$

In this case, as we have already mentioned, the equation of motion (2.16) is not solved with respect to the highest derivatives. Therefore the equation of motion not necessarily has a solution $\phi(t)$ such that $(\phi(t_c), \dot{\phi}(t_c)) = \Psi_c$ at some moment of time t_c. Or more harmless variants: there exists more than one solution passing through the exceptional point Ψ_c or there exists the desirable solution $\phi(t)$ but it does not possess the second derivative with respect to time at the point Ψ_c. Below we will analyze the behavior of the phase curves in the neighborhood of the points Ψ_c.

The the equation of motion (2.16) can be rewritten as a system of two differential equations of the first order:

$$\begin{aligned} \frac{d\dot{\phi}}{dt} &= -\dot{\phi}\frac{p_{,X}}{\varepsilon_{,X}}\sqrt{3\varepsilon} - \frac{\varepsilon_{,\phi}}{\varepsilon_{,X}} \\ \frac{d\phi}{dt} &= \dot{\phi} \end{aligned} \tag{4.9}$$

While the phase curves of this dynamical system are given by the following differential equation

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\dot{\phi}p_{,X}\sqrt{3\varepsilon} + \varepsilon_{,\phi}}{\dot{\phi}\varepsilon_{,X}}. \tag{4.10}$$

This equation for the phase curves follows from the system (4.9) and therefore all phase curves corresponding to the integral curves of system (4.9) are integral curves of the differential equation (4.10). But the reverse statement is false, so not each integral curve of Eq. (4.10) is necessarily a phase curve of the equation of motion (or of the system (4.9)). In the neighborhoods of the points where $\varepsilon_{,X} \neq 0$ the system (4.9) is equivalent to the $\tau-$system:

$$\begin{aligned} \frac{d\dot{\phi}}{d\tau} &= -\dot{\phi}p_{,X}\sqrt{3\varepsilon} - \varepsilon_{,\phi}, \\ \frac{d\phi}{d\tau} &= \dot{\phi}\varepsilon_{,X}, \end{aligned} \tag{4.11}$$

where we have introduced a new auxiliary time variable τ defined by

$$dt \equiv \varepsilon_{,X} d\tau. \tag{4.12}$$

The auxiliary time variable τ change the direction if $\varepsilon_{,X}$ change the sign. Note that the system (4.11) always possesses the same phase curves as the equation of motion (4.9).

In the case under consideration we have $\dot{\phi}_c \neq 0$ therefore from the formula

$$\frac{d\dot{\phi}}{d\phi} = \frac{\ddot{\phi}}{\dot{\phi}} \tag{4.13}$$

follows that $d\dot{\phi}/d\phi(t_c)$ should be finite, if $\ddot{\phi}(t_c)$ is finite. We have already assumed that

$$\varepsilon_{,X}(\Psi_c) = 0, \qquad p_{,X}(\Psi_c) = 0.$$

As one can see from the equation determining the phase curves (4.10), to obtain a finite $d\dot{\phi}/d\phi(t_c)$ it is necessary that at least $\varepsilon_{,\phi}(\Psi_c) = 0$. In the case if $\varepsilon_{,\phi}(\Psi_c) \neq 0$ the solution $\phi(t)$ do not posses the second t-derivative at the point t_c. Usually this can usually be seen as unphysical situation. But nevertheless for the general Lagrangian under consideration this does not necessarily lead to the unphysical incontinuity in the observed quantities ε, p, H and ϕ, $\dot{\phi}$. So for example if we impose that there exists $\lim_{t\to t_c}(\varepsilon_{,X}\ddot{\phi}-\varepsilon_{,\phi})=0$ then ε still to be differentiable. Probably one may face problems with the stability of such a solutions but let us nevertheless investigate the behavior of the phase curves in the case if $\varepsilon_{,\phi}(\Psi_c)\neq 0$. From Eq. (4.10) we obtain $d\phi/d\dot{\phi}=0$ at Ψ_c. Further we can parameterize the phase curve as $\phi=\phi(\dot{\phi})$ and bring the equation for phase curves (4.10) to the form

$$\frac{d\phi}{d\dot{\phi}}=\varepsilon_{,X}(\phi,\dot{\phi})F(\phi,\dot{\phi}), \tag{4.14}$$

where we denote:

$$F(\phi,\dot{\phi})\equiv-\frac{\dot{\phi}}{\dot{\phi}p_{,X}\sqrt{3\varepsilon}+\varepsilon_{,\phi}}. \tag{4.15}$$

If as we have assumed $\varepsilon_{,\phi}(\Psi_c)\neq 0$ and $\varepsilon(\Psi_c)\neq 0$ then $F(\phi,\dot{\phi})$ is differentiable in the neighborhood of the point Ψ_c and $F(\Psi_c)=-\dot{\phi}/\varepsilon_{,\phi}$. For the second $\dot{\phi}-$derivative at the point Ψ_c one obtains:

$$\frac{d^2\phi}{d^2\dot{\phi}}=-\frac{\dot{\phi}^2}{\varepsilon_{,\phi}}\varepsilon_{,XX}. \tag{4.16}$$

That is why the point Ψ_c is a minimum or a maximum for the function $\phi(\dot{\phi})$. In this case, as one can see from Fig. (4.2),(4.3), Ψ_c is a such exceptional point on the phase plot, where the continuous solution $\phi(t)$ can not exist. But if $\varepsilon_{,XX}(\Psi_c)=0$ then one can find the third $\dot{\phi}-$derivative of $\phi(\dot{\phi})$ at the point Ψ_c:

$$\frac{d^3\phi}{d^3\dot{\phi}}=-\frac{\dot{\phi}^3}{\varepsilon_{,\phi}}\varepsilon_{,XXX}. \tag{4.17}$$

In this case there exists a continuous solution $\phi(t)$ such that $(\phi(t_c),\dot{\phi}(t_c))=\Psi_c$ at some moment of time t_c and the only bad thing happening in this point is that $\ddot{\phi}(t_c)$ does not exists. Let us now investigate what happens with the equation of state at this point of time. Differentiating the both hand sides of the definition of $w=p/\varepsilon$ yields:

$$\frac{dw}{dt}(t_c)=\frac{\dot{\phi}}{\varepsilon}(p_{,\phi}-c_s^2\varepsilon_{,\phi}), \tag{4.18}$$

where we have used the equation of motion (2.16) at the point t_c and the definition (2.24) of c_s^2. Thus in general if $p_{,\phi}(t_c)\neq 0$ the transition could occur. Applying the L'Hospital rule for the $c_s^2(t_c)=\lim_{t\to t_c}p_{,X}/\varepsilon_{,X}$ we find that $c_s^2(t_c)=0$. Moreover using the L'Hospital rule for the derivative of c_s^2 at the point Ψ_c one can find that $dc_s^2/d\dot{\phi}\sim p_{,\phi}/\varepsilon_{,\phi}$. Thus the transition changes the sign of the sound speed and therefore, if the stability criteria are applicable to this case, then the transition must be unstable.

The necessary condition for the existence of $\ddot{\phi}$ during the transition is

$$\varepsilon_{,\phi}(\Psi_c)=2X_cp_{,X\phi}(\Psi_c)-p_{,\phi}(\Psi_c)=0. \tag{4.19}$$

This condition drastically reduces the set of the points Ψ_c, where the transition is possible. Namely they are the critical points of the function $\varepsilon(\phi, X)$ and on the other hand are the fixed points of the $\tau-$system (4.11). This fixed points are additional to the fixed points of the system (4.9) defined by $\dot{\phi} = 0$ and $\varepsilon_{,\phi}(\phi, 0) = 0$. From now on we will consider only those phase plot points Ψ_c^+ where the condition (4.19) holds. From the relation (4.19) it follows that if $p_{,\phi}(\phi_c, X_c) = 0$ then $p_{,X\phi}(\phi_c, X_c) = 0$. Because otherwise $X_c = 0$ and as we have already seen the transition can not happen via the points $X_c = 0$. Moreover, if $p_{,\phi}(\Psi_c^+) = 0$, then the points Ψ_c^+ are common critical points of the pressure $p(\phi, \dot{\phi})$, energy density $\varepsilon(\phi, \dot{\phi})$ and $p_{,X}(\phi, \dot{\phi})$. From the condition (4.19) follows that the points Ψ_c^+ are singular points equation (4.10) determining the phase curve of the equation of motion. In such points there can be more than one phase curve passing through this point. Moreover, the set of solutions $\phi(t)$, which pass through Ψ_c^+ with different $\ddot{\phi}$, could have a non-zero measure. For example, if Ψ_c^+ were a nodal point (see Fig. 4.4), there would be a continuous amount of the solutions passing through this point and therefore there were continuous amount of solutions on that the transition could occur. Let us investigate the type of the singular points Ψ_c^+. This will tell us about the amount of the solutions on which the transition is possible and their stability. For this analysis one can use the usual technique described for example in [230] and consider the integral curves of the equation (4.10). But here we will do this more intuitively, namely using the auxiliary $\tau-$system (4.11). The point is that for this system the singular points Ψ_c^+ are usual fixed points. As we have already mentioned the both systems have the same phase curves an therefore the analysis to perform is also applicable to the phase curves of the system (4.9). The only thing we should not forget, is the difference in the directions of the phase flows of this systems. If $\varepsilon(\Psi_c^+) \neq 0$ then one can linearize the right hand side of the $\tau-$system in the neighborhood of a point Ψ_c^+: $(\phi_c^+ + \delta\phi, \dot{\phi}_c^+ + \delta\dot{\phi})$. The linearized $\tau-$system (4.11) looks

$$\frac{d}{d\tau}\mathbf{V} = \mathbf{A}\mathbf{V}. \tag{4.20}$$

Here we denote

$$\mathbf{V} = \begin{pmatrix} \delta\phi \\ \delta\dot{\phi} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a & b \\ c & g \end{pmatrix}. \tag{4.21}$$

The elements of the matrix $\mathbf{A}$ are given by

$$\begin{aligned} a &= \varepsilon_{,X\phi}\dot{\phi}, \\ b &= 2X\varepsilon_{,XX}, \\ c &= -\left(3H\dot{\phi}p_{,X\phi} + \varepsilon_{,\phi\phi}\right), \\ g &= -\varepsilon_{,X\phi}\dot{\phi}, \end{aligned} \tag{4.22}$$

where all quantity should be calculated at Ψ_c^+. Here we have used the Friedmann equation (2.13). If Ψ_c^+ is an isolated fixed point of the $\tau-$system (4.11) (or equivalently the singular

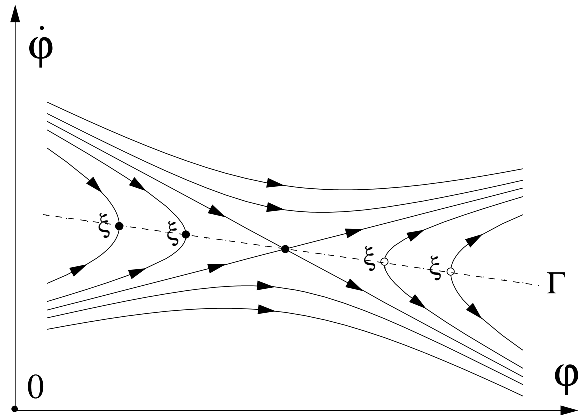

Figure 4.2: Phase curves in the neighborhood of the singular point Ψ_c^+ are plotted for the case of the real λ. At the points ξ the solutions $\phi(t)$ do not exist. These points together with Ψ_c^+ build the curve Γ on which $\varepsilon_{,X}(\Gamma) = 0$.

point of system (4.9)) then the following condition holds

$$\det \mathbf{A} = ag - bc \neq 0. \tag{4.23}$$

The type of the fixed point depends on the eigenvalues λ of the matrix $\mathbf{A}$ (for details see for example [231]). In the case under consideration $a = -g$ and therefore we have

$$\lambda^2 = bc + a^2 = -\det \mathbf{A}. \tag{4.24}$$

If $bc + a^2 > 0$ then eigenvalues λ are real and of the opposite signs. In accordance with the classification of the singular points, Ψ_c^+ is a saddle point (see Fig. (4.2)). Therefore the transition is absolutely unstable, there are only two solutions $\phi(t)$ on which the transition is allowed to occur.

But if $bc + a^2 < 0$ then λ are pure imaginary. Then the situation is a little bit more complicated: in accordance with [230] this fixed point of the nonlinear system can be either a focus or a center. But in this cases, as one can see from Fig. (4.3) there no solutions $\phi(t)$ passing through the point Ψ_c^+. Therefore, from now on we will consider only the first case - real λ of the opposite signs.

It is convenient to rewrite the expression for λ into a simpler form. Differentiating the continuity equation (2.14) yields

$$\ddot{\varepsilon}_c = -3H_c\dot{p}_c. \tag{4.25}$$

Here the index c denote that all quantities should be taken at t_c or equivalently at Ψ_c^+. Differentiating the pressure p as a composite function we have $\dot{p} = p_{,\phi}\dot{\phi} + p_{,X}\dot{X}$ and therefore assuming that $\ddot{\phi}_c$ is finite we obtain $\dot{p}_c = p^c_{,\phi}\dot{\phi}_c$. Thus the formula for $\ddot{\varepsilon}_c$ looks:

$$\ddot{\varepsilon}_c = -3H_c p^c_{,\phi}\dot{\phi}_c. \tag{4.26}$$

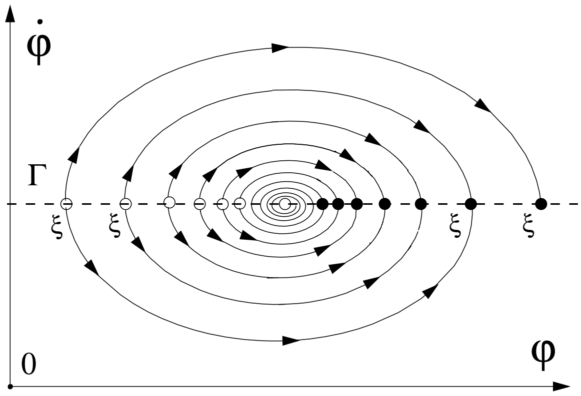

Figure 4.3: Phase curves in the neighborhood of the singular point Ψ_c^+ are plotted for the case of the pure imaginary λ. Her we assume that the singular point is a focus.

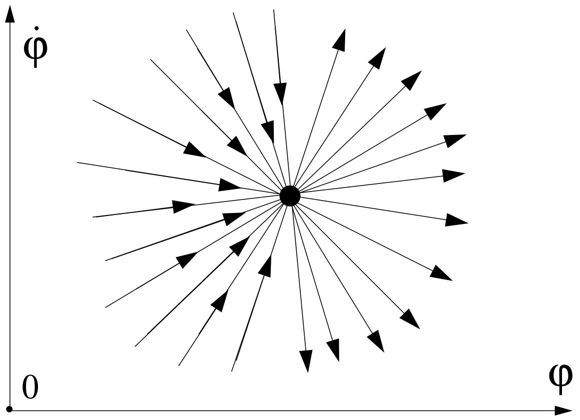

Figure 4.4: If **A** had the eigenvalues $\lambda_1 = \lambda_2$ then the singular point Ψ_c^+ would be a nodal point and there would be a continuous set of trajectories passing through it. To illustrate this we plot here the phase curves in the particular case of an degenerate nodal point. The form of the equation of motion excludes such type of the singular points and therefore prevents the possibility of such transitions.

Using the condition (4.19) and the last equation, we bring the element c to the following form:

$$c = -\varepsilon^c_{,\phi\phi} + \frac{\ddot{\varepsilon}_c}{2X_c}. \tag{4.27}$$

This relation allows one to rewrite the formula (4.24) as follows:

$$\lambda^2 = 2X_c\left(\left(\varepsilon^c_{,X\phi}\right)^2 - \varepsilon^c_{,\phi\phi}\varepsilon^c_{,XX}\right) + \ddot{\varepsilon}_c\varepsilon^c_{,XX}. \tag{4.28}$$

Here it is interesting to note that the expression $\varepsilon^c_{,\phi\phi}\varepsilon^c_{,XX} - \left(\varepsilon^c_{,X\phi}\right)^2$ from the previous formula is the determinant of the quadratic form arising in the Taylor set of ε in the neighborhood of the critical point Ψ^+_c. If this determinant is positive than the function $\varepsilon(\phi, X)$ has either a minimum or a maximum at the point Ψ^+_c. Otherwise there are either one ore two intersecting at Ψ^+_c curves on which the energy density ε takes the same value $\varepsilon(\Psi^+_c)$ in the neighborhood of Ψ^+_c (see [232]) . On the other hand, differentiating ε as a composite function we find

$$\ddot{\varepsilon}_c = 2X_c\left(\varepsilon^c_{,\phi\phi} + 2\varepsilon^c_{,X\phi}\ddot{\phi}_c + \varepsilon^c_{,XX}\ddot{\phi}^2_c\right). \tag{4.29}$$

Substituting this relation into the previous formula (4.28) for λ yields

$$\lambda^2 = 2X_c\left(\varepsilon^c_{,XX}\ddot{\phi}_c + \varepsilon^c_{,X\phi}\right)^2. \tag{4.30}$$

This formula provides the relation between $\ddot{\phi}_c$ at the moment of transition and λ. Note that λ depends on $\ddot{\phi}_c$ only in the case when $\varepsilon_{,XX}(t_c) \equiv 2X_c p_{,XXX}(t_c) \neq 0$. Moreover, comparing the formulas (4.26) and (4.29) for $\ddot{\varepsilon}_c$ one can obtain the equation on $\ddot{\phi}_c$:

$$\varepsilon^c_{,XX}\dot{\phi}_c\ddot{\phi}^2_c + 2\varepsilon^c_{,X\phi}\dot{\phi}_c\ddot{\phi}_c + \dot{\phi}_c\varepsilon^c_{,\phi\phi} + 3H_c p^c_{,\phi} = 0. \tag{4.31}$$

This equation is solvable in real numbers if the discriminant is positive. But as one can prove the discriminant is exactly the $4\lambda^2$ and therefore positive if we consider the saddle point. The same can be seen from the relation (4.30) as well.

Let us denote the positive and negative eigenvalues and the corresponding eigenvectors of $\mathbf{A}$ as λ_+, $\lambda_- = -\lambda_+$ and $\mathbf{a}_+$, $\mathbf{a}_-$ respectively.

If $b \neq 0$ (or equivalently $\varepsilon^c_{,XX} \neq 0$) then the eigenvectors can be chosen as $\mathbf{a}_+ = (1, (\lambda_+ - a)/b)$ and $\mathbf{a}_- = (1, -(\lambda_+ + a)/b)$. Therefore the separatrices build the saddle are

$$\delta\dot{\phi}_+ = \frac{\lambda_+ - a}{b}\delta\phi \quad \text{and} \quad \delta\dot{\phi}_- = -\frac{\lambda_+ + a}{b}\delta\phi.$$

The general solution for the phase curves in the neighborhood of Ψ^+_c looks

$$\left(\delta\dot{\phi} - \frac{\lambda_+ - a}{b}\delta\phi\right)\left(\delta\dot{\phi} + \frac{\lambda_+ + a}{b}\delta\phi\right) = const. \tag{4.32}$$

If $b = 0$ and additionally $a > 0$ then we have $\lambda_+ = a = \varepsilon_{,X\phi}\dot{\phi}$ and one can choose the eigenvectors as $\mathbf{a}_+ = (1, c/2a)$ and $\mathbf{a}_- = (0, 1)$. While for the negative a one can obtain the eigenvectors and eigenvalues by changing $\lambda_+ \leftrightarrow \lambda_-$ and $\mathbf{a}_+ \leftrightarrow \mathbf{a}_-$. The separatrices read:

$$\delta\phi = 0 \quad \text{and} \quad \delta\dot{\phi} = \frac{c}{2a}\delta\phi.$$

Thus, similarly to the previous case, the phase curves are given by

$$\delta\phi\left(\delta\dot{\phi} - \frac{c}{2a}\delta\phi\right) = const. \tag{4.33}$$

As we have already mentioned from the formula $d\dot{\phi}/d\phi = \ddot{\phi}/\dot{\phi}$ follows that at the points where the phase curves are parallel to the $\dot{\phi}-$ axis and where $\dot{\phi} \neq 0$ the second $t-$derivative does not exists. But if we look at the equations (4.32), (4.33) providing the phase curves in the neighborhood of Ψ_c^+ then we find that in the first case (when $\varepsilon^c_{,XX} \neq 0$) the phase curves lying on the right and left hand sides of the both separatrices should have a point ξ where they are parallel to the $\dot{\phi}-$ axis (see Fig. (4.2)). Therefore each of these phase curves consists of two solutions of the equation of motion (2.5) and the exceptional point ξ where the solution $\phi(t)$ does not exists. The same statement holds in the case of the pure imaginary λ (see Fig. (4.3)). This behavior is not forbidden because, as one can easily prove, the exceptional points ξ lie exactly on the curve Γ on which $\varepsilon_{,X} = 0$ and the equation of motion is not solved with respect to the highest derivatives. Note that we have already assumed $\lambda \neq 0$ and from this condition it follows that $\varepsilon^c_{,XX}$ and $\varepsilon^c_{,X\phi}$ can not vanish simultaneously. Therefore in the neighborhood of the point Ψ_c^+ there exists an implicit function $\dot{\phi}(\phi)$ (or $\phi(\dot{\phi})$) and its plot gives the above mentioned curve Γ on which $\varepsilon_{,X}(\Gamma) = 0$. But in the case if $\varepsilon^c_{,XX} = 0$ the separatrix $\delta\phi = 0$ locally coincide with Γ and therefore, except the point Ψ_c^+, this integral curve of the equation (4.10) does not correspond to any solution $\phi(t)$ of the equation of motion. Nevertheless, in virtue of the existence theorem, all phase curves obtained in the neighborhood of the separatrix $\delta\phi = 0$, correspond to the solutions of the equation of motion. Moreover if $\varepsilon^c_{,XX} = 0$ and $p^c_{,X\phi} \neq 0$ then the curves on which $p_{,X} = 0$ and $\varepsilon_{,X} = 0$ locally coincide with each other and with the curve $\delta\phi = 0$. The only phase curve intersecting the curve $\delta\phi = 0$ is the second separatrix $\delta\dot{\phi} = c\delta\phi/2a$. Thus the only one solution $\phi(t)$ on which the transition happens in the neighborhood of Ψ_c^+, corresponds to the separatrix $\delta\dot{\phi} = c\delta\phi/2a$. It can be also seen from the equation (4.31) which has the only one root $\ddot{\phi}_c$ in this case. In the section (4.2) we will illustrate this with a numerical example (see Fig. 4.5).

It is worthwhile to discuss cases fall out from the consideration made above. We have assumed that $\lambda \neq 0$ and therefore Ψ_c^+ is an isolated singular point of the equation (4.10). The most natural possibilities to drop out this condition are $\varepsilon^c_{,X\phi} = 0$ and either $\varepsilon^c_{,XX} = 0$ or $\dot{\phi}_c 3H_c p^c_{,X\phi} + \varepsilon^c_{,\phi\phi} = 0$. In the first case Ψ_c^+ is a critical point not only of ε but $\varepsilon_{,X}$ as well. And this can be obtained either for a very special kind of function p namely such that $p_{,X} = 0$, $p_{,XX} = 0$, $p_{,XXX} = 0$ and $(p - 4X^2 p_{,XX})_{,\phi} = 0$ at Ψ_c^+ or imposing the condition that the point Ψ_c^+ is a critical point not only of p but also of $p_{,X}$ and $p_{,XX}$: $p_{,X}=0$, $p_{,\phi}=0$, $p_{,XX}=0$, $p_{,X\phi}=0$, $p_{,XX\phi}=0$ and finally $p_{,XXX}=0$ at Ψ_c^+. In the second case Ψ_c^+ is a common critical point for ε and $\varepsilon_{,\phi}$. In terms of p this condition looks as follows: $p_{,X}=0$, $p_{,\phi}=0$, $p_{,X\phi}=0$, $p_{,XX}=0$, $p_{,\phi\phi}=0$, $p_{,XX\phi}=0$ and finally $p_{,X\phi\phi}=0$ at Ψ_c^+ . Thus the point Ψ_c^+ is a common critical point of p, $p_{,X}$ and $p_{,X\phi}$. Of course the analysis performed above does not work in the case if the function $p(\phi, X)$ is not differentiable enough.

Let us sum up the results obtained in this section. In the general case of linearizable functions $\varepsilon_{,X}$, $\varepsilon_{,\phi}$ and $p_{,X}$ the considered transitions are either occur through the points $\Psi_c^+=(\phi,\dot{\phi})$, where $p_{,X}=0$, $\varepsilon_{,X}=0$, $\varepsilon_{,\phi}=0$ and

$$\dot{\phi}\left[\left(\varepsilon^2_{,X\phi}-\varepsilon_{,\phi\phi}\varepsilon_{,XX}\right)\dot{\phi}-3H\varepsilon_{,XX}p_{,\phi}\right]>0$$

or, if ϕ dominates in the Friedmann universe, lead to an unacceptable instability with respect to the cosmological perturbations of the background. The points Ψ_c^+ are critical points of the energy density and are the singular points of the equation of motion of the field ϕ as well. These singular points are saddle points and the transition realized by the repulsive separatrix solutions, which build the saddle. Therefore the measure of these solutions is zero in the set of trajectories and the dynamical transitions from the states where $w>-1$ to $w<-1$ or vise versa are physically implausible.

4.2 Lagrangians linear in X

The simplest class of models, for that one could anticipate the existence of dynamical transitions, is the dark energy described by Lagrangians $p(\phi, X)$ linear in X:

$$p(\phi, X)=KX-\frac{1}{2}V=\frac{1}{2}\left(K(\phi)\dot{\phi}^2-V(\phi)\right). \tag{4.34}$$

For this models we always have $c_s^2=1$ and therefore, as it follows from our analysis, the transitions could occur only through the points where $\varepsilon_{,X}=0$. The energy density associated to this Lagrangian is

$$\varepsilon(\phi,\dot{\phi})=\frac{1}{2}\left(K(\phi)\dot{\phi}^2+V(\phi)\right). \tag{4.35}$$

If one takes $K(\phi)\equiv 1$ then the Lagrangian (4.34) is the usual Lagrangian density for scalar field with a self-interaction and if we take $K(\phi)\equiv -1$ then we obtain the so-called *Phantom field* from [19]. The case $K>0$ corresponds to $w\geq -1$, whereas $K<0$ corresponds to $w\leq -1$. The equation of motion (2.16) for the linear in X Lagrangians looks

$$\ddot{\phi}K+\dot{\phi}K\sqrt{3\varepsilon}+\varepsilon_{,\phi}=0. \tag{4.36}$$

While the equation determining the phase curves (4.10) reads in this particular case

$$\frac{d\dot{\phi}}{d\phi}+\sqrt{3\varepsilon}+\frac{1}{\dot{\phi}K}\varepsilon_{,\phi}=0. \tag{4.37}$$

If $K(\phi)$ is a sign-preserving function one can redefine field ϕ: $\sqrt{|K(\phi)|}d\phi = d\varphi$ (see also Ref. [144]). The equation of motion for the new field φ can be obtained from Eq. (4.36), through the formal substitutions $\phi \to \varphi$, $V(\phi) \to \tilde{V}(\varphi) \equiv V(\phi(\varphi))$ and $K(\varphi) \to \pm 1$, where the upper sign corresponds to a positive $K(\phi)$ and the lower one to a negative $K(\phi)$. After these substitutions the equation of motion (4.36) looks more conventionally:

$$\ddot{\varphi} + \dot{\varphi}\sqrt{\frac{3}{2}\left(\pm\dot{\varphi}^2 + \tilde{V}(\varphi)\right)} \pm \frac{1}{2}\left(\frac{\partial \tilde{V}(\varphi)}{\partial \varphi}\right) = 0. \tag{4.38}$$

moreover this equations is easier to dial with, because one can visualize the dynamic determined by it, as 1D classical mechanics of a point particle in a potential $\pm\tilde{V}(\varphi)/2$ with a little bit unusual friction force. If we were able to solve the equation of motion (4.38) for all possible $\tilde{V}(\varphi)$ and initial data, we could solve the problem of cosmological evolution for all linear on X Lagrangians with sign-preserving $K(\phi)$.

If the function $K(\phi)$ is not sign-preserving then at first sight it seems that the dark energy, described by such a Lagrangian, can realize the desirable transition. The function $K(\phi)$ can change the sign in the to ways: in the continuous one, then the function $K(\phi)$ takes the value zero for some values of field ϕ or in a discontinuous pole-like way.

4.2.1 Linearizable $K(\phi)$

Without loss of generality one can assume that $K(0) \equiv K_c = 0$, $K(\phi) < 0$ for the negative values of ϕ and $K(\phi) > 0$ for $\phi > 0$. The line $\phi = 0$ on the phase plot $(\phi, \dot{\phi})$ we will call "critical" line for the given class of Lagrangians. The phantom states $(\phi, \dot{\phi})$ of the scalar field lie on the left hand side, while the usual states with $w \geq -1$ are on the right hand side of the "critical" line. If there exists a solution $\phi(t)$ whose phase curve passes through the "critical" line, then the dark energy can change the sign of $(w+1)$ during the cosmological evolution. From now on we will investigate the behavior of the phase curves of the system in the neighborhood of the "critical" line.

First of all it is worth to consider the functions $K(\phi)$ such that $K'_c > 0$ (here we have denoted $K_{,\phi}(0) \equiv K'_c$), because in this case we can directly apply the outcome of our previous analysis made in the subsection 4.1.1. The condition (4.19) looks for the linear on X Lagrangians as follows:

$$\dot{\phi}_c^2 K'_c + V'_c = 0. \tag{4.39}$$

As we have already assumed $K'_c > 0$, therefore, if also $V'_c > 0$, then, as it follows from the condition (4.39), there are no twice differentiable solutions $\phi(t)$ whose phase curves would intersect or touch the "critical" line. Further (see the formula (4.47) and below) we will show, that for the linear on X Lagrangians, the condition (4.19) (or in our case condition (4.39) is necessary not only for the existence of the second $t-$derivative $\ddot{\phi}$ at the point of intersection with the "critical" line but for the existence of a solution $\phi(t)$ at this point as well. Thus we come to conclusion that, if $V'_c > 0$, then two regions $\phi < 0$, and $\phi > 0$ on the phase plot are not connected by any phase curves and accordingly the dark energy does not change the sign of $(w+1)$ during the cosmological evolution.

In the case when $V_c' < 0$ we can solve Eq. (4.39) with respect to $\dot{\phi}_c$:

$$\dot{\phi}_c = u_\pm \equiv \pm\sqrt{-\frac{V_c'}{K_c'}}. \tag{4.40}$$

The phase curves, lying in the neighborhoods of the singular points $(0, u_\pm)$, are to obtain from the relation (4.33), which gives:

$$\phi\left(\dot{\phi} - u_\pm - \frac{A_\pm}{2}\phi\right) = const, \tag{4.41}$$

where

$$A_\pm = -3H_c + \frac{V_c' K_c'' - V_c'' K_c'}{2u_\pm\left(K_c'\right)^2}. \tag{4.42}$$

For each singular point $(0, u_\pm)$, there is a corresponding solution $\phi_\pm(t)$ whose phase curve is the separatrix

$$\dot{\phi}_\pm = u_\pm + \frac{A_\pm}{2}\phi, \tag{4.43}$$

which intersects the "critical" line. These phase curves correspond to the $const = 0$ in the right hand side of Eq. (4.41). Another curve, which corresponds to $const = 0$ is $\phi = 0$. But as we have already mentioned in the end of the previous subsection this curve does not correspond to any solutions $\phi(t)$ of the equation of motion (4.36).

Considering the phase flow in the neighborhoods of $(0, u_\pm)$ (see Fig. (4.5)), we infer that the separatrices $\dot{\phi}_\pm$ are repulsors immediately before they intersect the "critical" line and attractors after the crossing. Hence the measure of the initial conditions $(\phi, \dot{\phi})$ leading to the transition to phantom field (or vice versa), is zero. In this sense the dark energy cannot change the sign of $K(\phi)$ (or equivalently the sign of $(w+1)$) during the cosmological evolution.

The typical behavior of the phase curves in the neighborhood of the singular points $(0,\, u_\pm)$, for the models under consideration ($K_c' > 0$, $V_c' < 0$), is showed in Fig. (4.5). Here, as an example, we have plotted the phase curves obtained numerically for a toy model with the Lagrangian density $p = \frac{1}{2}\phi\dot{\phi}^2 - \frac{1}{2}\left((\phi-1)^2 + \frac{1}{3}\right)$. For this model we have $u_\pm = \pm 1$, $A_+ = -\frac{3}{2}$, $A_- = -\frac{1}{2}$.

Let us now consider such potentials $V(\phi)$ that $V_c' = 0$. If $K(\phi)$ is a differentiable function, then, in the case under consideration, the equation of motion (4.36) obviously has the fixed-point solution $\phi(t) \equiv 0$ but this solution is not necessary the unique one. When $V_c' = 0$ and as we have already assumed $K_c' > 0$, then, as it follows from the condition (4.19), the only value $\dot{\phi}$, where a phase curve could have coinciding points with the critical line, is $\dot{\phi} = 0$. But from the analysis made in the subsection (4.1.1), we have already learned that the transition is impossible in this case. Nevertheless it is worth to show explicitly, how the phase curves looks at this case. Taking into consideration only

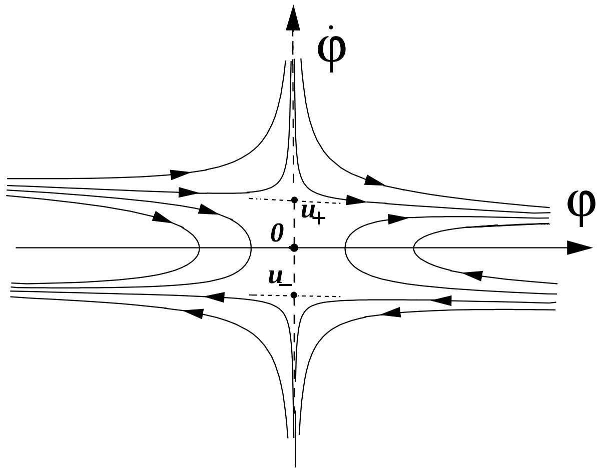

Figure 4.5: The typical behavior of the phase curves in the neighborhood of the "critical" line where $K(\phi) = 0$ (here $\dot{\phi}-$axis), is plotted for the case when $K_c' > 0$, $V_c' < 0$. Horizontal dashed lines are the analytically obtained separatrices $\dot{\phi}_{\pm}$, $(0, u_{\pm})$ are the points of transition.

the leading order in the numerator and denominator of the equation (4.37) and assuming that $V_c'' \neq 0$ we obtain:

$$\frac{d\dot{\phi}}{d\phi} \simeq -\frac{V_c''}{2\dot{\phi}K_c'}. \tag{4.44}$$

The solution of this equation, going through the point $(0,0)$ on the phase plot, is

$$\phi_s = -\dot{\phi}^2\left(\frac{K_c'}{V_c''}\right). \tag{4.45}$$

In Fig. (4.6) we have plotted the phase curves obtained numerically for the toy model with the Lagrangian density $p = \frac{1}{2}\phi\dot{\phi}^2 - \frac{1}{2}\left(\phi^2 + 2\right)$. As one can see from Fig. (4.6), the parabola-like phase curve ϕ_s, given by the formula (4.45), is the separatrix going through the fixed-point solution $\phi(t) \equiv 0$. Moreover this figure confirms that there are no phase curves intersecting the "critical" line.

4.2.2 General differentiable $K(\phi)$

The models we discussed above, belong to the more general class of models for that the function $K(\phi)$ has zero of an odd order $2n+1$ (where $n \geq 0$) at $\phi = 0$. For the linear on X Lagrangians we have $\varepsilon_{,XX} \equiv 0$, therefore, if $n > 0$ then $K_c' = 0$ and $\varepsilon^c_{,\phi X} = 0$. That is why the general analysis made in the subsection 4.1.3 does not work for this case. Therefore it is interesting to investigate on this simple example whether the desired transition could be

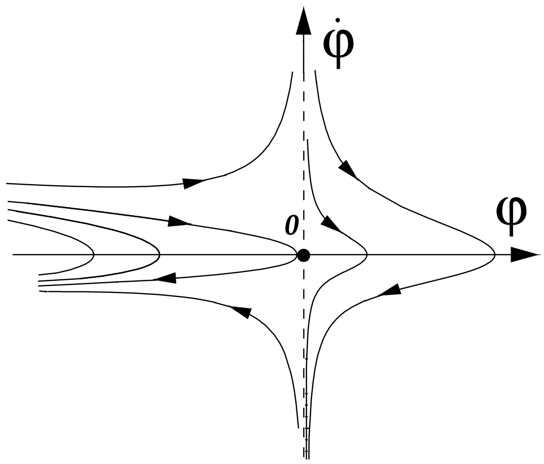

Figure 4.6: The typical behavior of the phase curves in the neighborhood of the "critical" line where $K(\phi) = 0$ (here $\dot{\phi}-$axis), is plotted for the case when $K_c' > 0$, $V_c' = 0$, $V_c'' > 0$.

possible for models not covered by our former analysis. If $K(\phi)$ is an enough differentiable function then for $|\phi| \ll 1$ we have

$$K(\phi) \simeq \frac{K_c^{(2n+1)}}{(2n+1)!}\phi^{2n+1}, \tag{4.46}$$

where $K_c^{(2n+1)}$ is the $(2n+1)$-th $\phi-$derivative of K at $\phi = 0$. If there is a phase curve, crossing the critical line at a finite nonvanishing $\dot{\phi}_c$ then integrating the both hand sides of the equation of motion (4.36) we obtain:

$$\dot{\phi}_c - \dot{\phi}_{in} = -3\int_{\phi_{in}}^{0} H(\phi)d\phi - \int_{\phi_{in}}^{0} \frac{\varepsilon_{,\phi}}{\dot{\phi}K}d\phi. \tag{4.47}$$

Here $(\phi_{in}, \dot{\phi}_{in})$ is a point on the phase curve in the neighborhood of critical line. The first integral on the right hand side of the equation (4.47) is always finite, whereas, as it follows from the singular behavior of $1/K$ given by relation (4.46), the second integral is definitely divergent, if $\varepsilon^c_{,\phi} \neq 0$. This divergence contradicts to our initial assumption: $\dot{\phi}_c$ - finite. Therefore we again obtain the condition (4.19), which restricts the possible intersection points on the "critical" line in the sense that in the other points, where the condition does not hold, not only the second derivative $\ddot{\phi}$ does not exists, but there are no solutions $\phi(t)$ at all. Moreover, it is clear, that the condition (4.39) is not enough for the existence of the solutions intersecting the "critical" line. Thus, if the order of $V'(\phi)$ exceeds the order of $K'(\phi)$ for $|\phi| \ll 1$, then one can neglect $V'(\phi)$ and the integral (4.47) has the logarithmic divergence (note, that we consider only the points $\dot{\phi}_c \neq 0$ because otherwise the transitions does not occur). When the order of $V'(\phi)$ is lower than $K'(\phi)$ (and therefore lower than the order of $K(\phi)$), we can neglect $\dot{\phi}^2 K'(\phi)$ and the integral (4.47) has a power-low divergence.

Finally if the functions $K'(\phi)$ and $V'(\phi)$ have the same order on ϕ for $|\phi| \ll 1$ and are of the the opposite signs in the sufficient small neighborhood of $\phi = 0$, then one can find the appropriate finite value $\dot\phi_c^2 \neq 0$ for which the divergence on the right hand side of the equation (4.47) is canceled. One would expect that at this points the phase curves intersect the "critical" line and the dark energy change the sign of $(w+1)$. Below we give the direct calculation of these $\dot\phi_c$ and the phase curves in the neighborhood of them. Suppose that the order of the functions $(V(\phi) - V_c)$ and $K(\phi)$ is $(2n+1)$ and there exist their derivatives of the order $(2n+2)$. Then for the $\phi-$derivative of the energy density we have in the neighborhood of the supposed intersection point $(0, \dot\phi_c)$:

$$\begin{aligned}\varepsilon_{,\phi} \simeq \frac{1}{2}\frac{\phi^{2n}}{(2n)!}\Big(\left(\dot\phi_c^2 K_c^{(2n+1)} + V_c^{(2n+1)}\right) + 2\dot\phi_c K_c^{(2n+1)}\delta\dot\phi \\ + \frac{\phi}{(2n+1)}\left(\dot\phi_c^2 K_c^{(2n+2)} + V_c^{(2n+2)}\right)\Big),\end{aligned} \tag{4.48}$$

whereas the denominator $\dot\phi K(\phi)$ of the second integral on the right hand side of Eq. (4.47) has the order $(2n+1)$ on ϕ. The only possibility to get rid of the divergence in the integral under consideration, is to assume that the first term in the brackets in the asymptotic (4.48) for $\varepsilon_{,\phi}$ is zero. Therefore the possible crossing points are given by

$$\dot\phi_c = u_\pm = \pm\sqrt{-\frac{V_c^{(2n+1)}}{K_c^{(2n+1)}}}. \tag{4.49}$$

Taking into account only the leading order on ϕ and $\delta\dot\phi$ in the denominator and the numerator of Eq. (4.37) we obtain differential equation for the phase curves in the neighborhood of the intersection points $(0, u_\pm)$

$$\frac{d\delta\dot\phi}{d\phi} = A_\pm - (2n+1)\frac{\delta\dot\phi}{\phi}, \tag{4.50}$$

where

$$A_\pm = -3H_c + \frac{K_c^{(2n+2)}V_c^{(2n+1)} - K_c^{(2n+1)}V_c^{(2n+2)}}{2\left(K_c^{(2n+1)}\right)^2 u_\pm}. \tag{4.51}$$

The solutions of this equation are given by the formula:

$$\left(\delta\dot\phi - \frac{A_\pm}{2n+2}\phi\right)\phi^{2n+1} = const, \tag{4.52}$$

that is the generalization of the formula (4.41). Similarly to the case $n = 0$ $(K_c' > 0)$ the solutions, on which the transition occur, have the measure zero in the phase curves set. Therefore we infer that the dynamical transition from the phantom states with $w \leq -1$ to the usual with $w \geq -1$ (or vice versa) is impossible.

Now we would like to mention the models, for that $V'(\phi)$ is one order higher on ϕ than $K'(\phi)$ for small ϕ. From the asymptotic expression for $\varepsilon_{,\phi}$ (4.48) and the relation, giving the possible values of $\dot{\phi}_c$ (4.47), we see that the only point on the critical line, which can be reached in a finite time is $\dot{\phi}_c = 0$. Therefore, as we have seen in the subsection 4.1.1, the transition is impossible. And the phase curve going trough the fixed-point solution $\phi(t) \equiv 0$ is a parabola given by the generalization of Eq. (4.45) :

$$\phi \simeq -\dot{\phi}^2 \left[\frac{K_c^{(2n+1)}}{V_c^{(2n+2)}} \right] . \tag{4.53}$$

If $V'(\phi)$ is more than one order higher on ϕ than $K'(\phi)$ than as we have already mentioned $\dot{\phi}_c = 0$ and the transition is impossible as well.

4.2.3 Pole-like $K(\phi)$

In this subsection we briefly consider the case when the function $K(\phi)$ has a pole of an odd order, so $K \sim \phi^{-2n-1}$, where $n > 0$, for $|\phi| \ll 1$. This kind of functions $K(\phi)$ are often discussed in the literature in connection with the k-*essence* models (see [13, 14, 15]). Let us keep the same notation as in the subsection 4.2.1. The potential $V(\phi)$ can not have a pole at the point $\phi = 0$, because, if it were the case, either the energy density ε or the pressure p would be infinite on the "critical" line. In order to obtain finite values of the energy density ε and pressure p it is necessary to assume that the system intersects the "critical" line at $\dot{\phi} = 0$. But, as we have already seen in the subsection 4.1.1, the dark energy cannot change the the sign of $(w + 1)$ at the points $\dot{\phi} = 0$.

Thus we have shown that in the particular case of the theories described by the linear in X Lagrangians $p(\phi, X) = K(\phi)X - V(\phi)$, which are differentiable in the neighborhood of $\Psi_c^+(K(\phi)$ and $V(\phi)$ differentiable but not necessary linearizable) the results, obtained for linearizable functions $\varepsilon_{,X}$, $\varepsilon_{,\phi}$ and $p_{,X}$, hold as well. This gives rise to hope that the same statement is true for the general non-linear in X Lagrangians as well. Especially we have proved that, if the construction of the Lagrangian allows the transition, then the transitions always realize on a couple of the phase curves. One phase curve corresponds to the transition from $w \geq -1$ to $w < -1$ while another one realizes the inverse transition. This couple of phase curves obviously has the measure zero in the set of trajectories of the system. Therefore we infer that the considered transition is physically implausible in this case.

4.3 k-*essence* in $k \neq 0$ universe and in the presence of other forms of matter

In the previous sections we have seen that the desirable transition from $w \geq -1$ to $w < -1$ is either impossible or dynamically unstable in the case when the scalar dark energy is a dominating source of gravity in the flat Friedmann universe. Let us now investigate whether this statement is true in the presence of other forms of matter and in the cases when the Friedmann universe has open and closed topology.

Following Ref. [69, 82] the effective sound speed c_s is given by the same Eq. (2.24) for the flat, open and closed universes. Therefore, if the dark energy is the dominating source of gravitation (in particular this means that the energy density of the dark energy $\varepsilon \neq 0$), than the analysis made in the subsection 4.1.2 is applicable to open and closed universes as well as to the flat universe.

If the dark energy under consideration interacts with ordinary forms of matter only through indirect gravitational-strength couplings, then the equation of motion (2.16) looks:

$$\ddot{\phi}\varepsilon_{,X} + 3\dot{\phi}Hp_{,X} + \varepsilon_{,\phi} = 0, \tag{4.54}$$

where merely the Hubble parameter depends on the spatial curvature and other forms of matter. This dependence is given by the Friedmann equation:

$$H^2 + \frac{k}{a^2} = \frac{1}{3}\left(\varepsilon + \sum \varepsilon_i\right) \tag{4.55}$$

where $\sum \varepsilon_i$ is the total energy density of all other forms of matter. It is obvious that the points on plot $(\phi, \dot{\phi})$ considered in the most of this paper, do not define the whole dynamics of the system anymore and therefore do not define the states of the whole system. But the analysis made in the subsections 4.1.1,4.1.3 and section 4.2 leans only on the behavior of the scalar filed ϕ and its first $t-$derivative $\dot{\phi}$ in the neighborhoods of their selected values namely such where some of the conditions $p_{,X} = 0$, $\dot{\phi} = 0$ $\varepsilon_{,X} = 0$ etc hold. For these conditions the contributions into the equation of motion (4.54), coming from the other forms of matter and spatial curvature would be of a higher order and therefore are not important for the local behavior of ϕ and $\dot{\phi}$ and the problem as a whole. In fact the value of the Hubble parameter did not change the qualitative futures of the phase curves considered in the subsections 4.1.1,4.1.3 and section 4.2. To illustrate this statement we plot the trajectories of the system $p = \frac{1}{2}\phi\dot{\phi}^2 - \frac{1}{2}\left((\phi-1)^2 + \frac{1}{3}\right)$(it is the same system that we considered in previous subsection) in presence of the dust matter for various values of the energy densities (see Fig. (4.7)). The only thing that is important is that $H \neq 0$. The universe should not change the expansion to the collapse and the plot of the scale factor $a(t)$ should not have a cusp directly at the time of the transition. Thus we infer that the most of our analysis is applicable to a more general physical situation of a Friedmann universe filled with various kinds of matter, which interact with the dark energy only through indirect gravitational-strength couplings. Moreover, if the interaction between the dark energy field ϕ and other fields does not include coupling to the derivatives $\nabla_\mu\phi$ then the obtained result holds as well.

4.4 Discussion

In this paper we have found that the transitions from $w \geq -1$ to $w < -1$ (or vice versa) of the dark energy described by a general scalar field Lagrangian $p(\phi, \nabla_\mu\phi)$ are either unstable with respect to the cosmological perturbations or realized on the trajectories of the measure zero. If the dark energy dominates in the universe, this result is still robust in the presence of other energy components interacting with the dark energy through non-kinetic couplings. In particular we have shown that under this assumption about interaction, the dark energy described by Lagrangians linear in $(\nabla_\mu\phi)^2$ Lagrangian cannot yield such transitions even if it is a sub-dominant source of gravitation.

Let us now discuss the consequences of these results. If further observations confirm the evolution of the dark energy dominating in the universe, from $w \geq -1$ in the close past to $w < -1$ to date, then it is impossible to explain this phenomenon by the classical dynamics given by an effective scalar field Lagrangian $p(\phi, \nabla_\mu\phi)$. In fact the models which allow such transitions have been already proposed (see e.g. [124, 233, 156] and other models from the Ref. [113]) but they incorporate more complicated physics then the classical dynamics of a one scalar field.

If observations reveal that $w < -1$ now and if we disregard the possibility of the transitions, then the energy density of the dark energy should grow rapidly during the expansion of the universe and therefore the coincidence problem becomes even more difficult. Thus from this point of view the transitions considered in this paper would be rather desirable for the history of the universe. But, as we have shown, to explain the transition under the minimal assumptions of the non-kinetic interaction of dark energy and other matter one should suppose that the dark energy was sub-dominating and described by a non-linear in $(\nabla_\mu\phi)^2$ Lagrangian. Thus some non-linear (or probably quantum) physics must be invoked to explain the value $w < -1$ in models with one scalar field.

The second application of our analysis is the problem of the cosmological singularity. To obtain a bounce instead of collapse the scalar field ϕ must change its equation of state to the phantom one before the bounce and should dominate in the universe at the moment of transition. Otherwise, if the scalar field was sub-dominant then it is still sub-dominant after the transition as well, because its energy density decreases during the collapse, while the other non-phantom forms of matter increase the energy density. The disappearing energy density of ϕ does not affect the gravitational dynamics and therefore does not lead the bounce. On the other hand, as we have already proved, a dominant scalar field ϕ described by the action without kinetic couplings and higher derivatives, cannot smoothly evolve to the phantom with $w < -1$. Therefore we infer that a smooth bounce of the non-closed Friedmann universe cannot be realized in this framework.

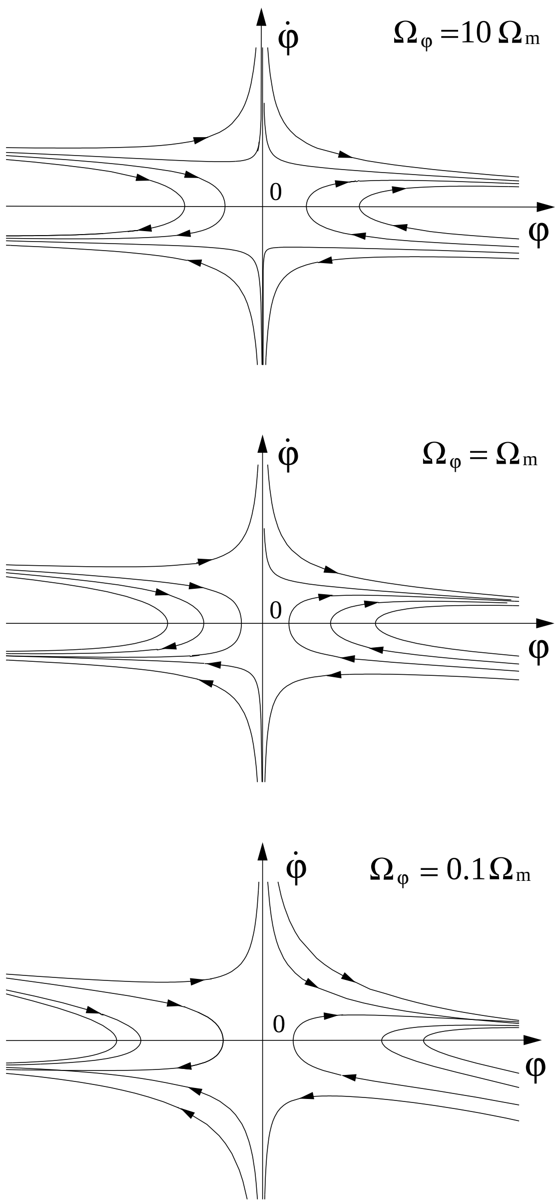

Figure 4.7: Numerically obtained trajectories of the dark energy described by a Lagrangian linear in X are plotted for the cases $\Omega_\phi = 10\Omega_m$, $\Omega_\phi = \Omega_m$ and $\Omega_\phi = 0.1\Omega_m$.

Chapter 5

Enhancing the tensor-to-scalar ratio in simple inflation

5.1 Generalized slow-roll inflation

It follows from (2.5) that inflation can be realized if the condition $Xp_{,X} \ll p$ is satisfied for a sufficiently long time interval. This can be done in two ways. Considering the canonical scalar field with $p = X - V(\phi)$, one can take a flat potential $V(\phi)$ so that $X \ll V$ for more than 75 e-folds. This is the standard slow-roll inflation [89] and in this case $c_s = 1$. The other possibility is the k-inflation [12], where p is a weakly dependent function of X, so that $p_{,X}$ is small. Here inflation is entirely based on the kinetic term and it can take place even if the field is running very fast (X is large); typically $c_s^2 \ll 1$ for k-*inflation*.

In this paper we consider a slow-roll inflation with a flat potential but in theories with a nontrivial kinetic term. To best of our knowledge this possibility has been ignored in the literature until now. However such models are very interesting because, as we will see, they allow us to have $c_S^2 > 1$ during inflation and thus enhance the tensor-to-scalar ratio for the perturbations produced.

For simplicity let us consider theories with Lagrangian

$$p = K(X) - V(\phi). \tag{5.1}$$

In this case

$$\varepsilon = 2XK_{,X} - K + V, \tag{5.2}$$

and the equation for scalar field 2.16 becomes

$$\ddot{\phi} + 3c_s^2 H\dot{\phi} + \frac{V_{,\phi}}{\varepsilon_{,X}} = 0. \tag{5.3}$$

It is clear that if the slow-roll conditions

$$XK_{,X} \ll V, \; K \ll V, \; \left|\ddot{\phi}\right| \ll \frac{V_{,\phi}}{\varepsilon_{,X}}$$

are satisfied for at least 75 e-folds then we have a successful slow-roll inflation due to the potential V. In contrast to ordinary slow-roll inflation one can arrange here practically any speed of sound c_s^2 by taking an appropriate kinetic term. For example, for

$$K\left(X\right)=\alpha X^{\beta},$$

we obtain from (3.1)

$$c_s^2=1/\left(2\beta-1\right).$$

and if $\beta\rightarrow 1/2$, then $c_s^2\rightarrow\infty$. This limiting case correspond to a non-dynamical theory [226, 227] which has been dubbed *Cuscuton.* Therefore by considering nontrivial kinetic terms $K\left(X\right)$ one can have, in principle, an arbitrarily large c_s, which thus becomes an additional free parameter of the theory. The crucial point is that the amplitude of the final scalar perturbations (during the postinflationary, radiation-dominated epoch) depends on c_s (see [82] Pages. 345 and 351):

$$\delta_{\Phi}^2\simeq\frac{64}{81}\left(\frac{\varepsilon}{c_s\left(1+p/\varepsilon\right)}\right)_{c_Sk\simeq Ha},\tag{5.4}$$

while the ratio of tensor to scalar amplitudes on supercurvature scales is given by

$$\frac{\delta_h^2}{\delta_{\Phi}^2}\simeq 27\left(c_s\left(1+\frac{p}{\varepsilon}\right)\right)_{k\simeq Ha}.\tag{5.5}$$

Here it is worthwhile reminding that all physical quantities on the right hand side of Eqs. (5.4) and (5.5) have to be calculated during inflation at the moment when perturbations with wave number k cross corresponding Horizon: $c_sk\simeq Ha$ for (5.4) and $k\simeq Ha$ for (5.5) respectively. The amplitude of the scalar perturbations δ_Φ is a free parameter of the theory which is taken to fit the observations. Therefore, in models where $c_s^2>1$, the energy scale of inflation must be higher than in ordinary slow-roll inflation with canonical kinetic term. Moreover, it follows from (5.5) that the tensor-to-scalar ratio can be arbitrarily enhanced in such models.

5.2 Simple model

As a concrete example from the class of theories with enhanced tensor contribution let us consider a simple model with the DBI-like Lagrangian

$$p(\phi,X)=\alpha\left[\sqrt{1+\frac{2X}{\alpha}}-1\right]-\frac{1}{2}m^2\phi^2,\tag{5.6}$$

where constant α is a free parameter. For $2X\ll\alpha$ one recovers the Lagrangian for the usual free scalar field. The function p is a monotonically growing concave function of X,

$$p_{,X}=\left(1+\frac{2X}{\alpha}\right)^{-1/2}>0,\qquad p_{,XX}=-\frac{1}{\alpha}\left(1+\frac{2X}{\alpha}\right)^{-3/2}<0,$$

and the corresponding energy density,

$$\varepsilon = \alpha \left[1 - \left(1 + \frac{2X}{\alpha} \right)^{-1/2} \right] + \frac{1}{2} m^2 \phi^2, \tag{5.7}$$

is always positive. The effective speed of sound,

$$c_s^2 = \frac{p_{,X}}{\varepsilon_{,X}} = 1 + \frac{2X}{\alpha}, \tag{5.8}$$

is larger than the speed of light, approaching it as $X \to 0$.

In the slow-roll regime and for p given in (5.6), equations (2.13), (2.16) simplify to

$$H \simeq \sqrt{\frac{4\pi}{3}} m\phi, \qquad 3p_{,X} H\dot{\phi} + m^2 \phi \simeq 0. \tag{5.9}$$

Taking into account that $\dot{\phi} = -\sqrt{2X}$, we infer that during inflation $\dot{\phi}$ is constant and

$$\frac{2X}{\alpha} = \left(\frac{12\pi\alpha}{m^2} - 1 \right)^{-1}. \tag{5.10}$$

The speed of sound

$$c_s \simeq c_\star = \left(1 - \frac{m^2}{12\pi\alpha} \right)^{-1/2} \tag{5.11}$$

can be arbitrarily large during inflation if we take $12\pi\alpha \to m^2$. Note, however, that $12\pi\alpha > m^2$ is necessary for the existence of the slow-roll solution. Hereafter we will use $c_\star$ as a parameter instead of α. One can easily find that during the slow-roll regime

$$\dot{\phi} \simeq -\frac{mc_\star}{\sqrt{12\pi}}, \tag{5.12}$$

and the pressure and energy density are given by

$$p \simeq m^2 \left(\frac{1}{12\pi} \frac{c_\star^2}{1 + c_\star} - \frac{\phi^2}{2} \right), \quad \varepsilon \simeq m^2 \left(\frac{1}{12\pi} \frac{c_\star}{1 + c_\star} + \frac{\phi^2}{2} \right), \tag{5.13}$$

respectively. The inflation is over when

$$\frac{\varepsilon + p}{\varepsilon} \simeq \frac{c_\star}{6\pi\phi^2} \tag{5.14}$$

becomes of order unity, that is, at $\phi \sim \sqrt{c_\star / 6\pi}$. After that the field ϕ begins to oscillate and decays. Here we restrict ourselves to illustration of the above analysis by a phase diagram Fig. 5.1 obtained numerically.

To determine $a(\phi)$ we use (5.12) to rewrite the first equation in (5.9) as

$$-\frac{mc_\star}{\sqrt{12\pi}} \frac{d \ln a}{d\phi} \simeq \sqrt{\frac{4\pi}{3}} m\phi, \tag{5.15}$$

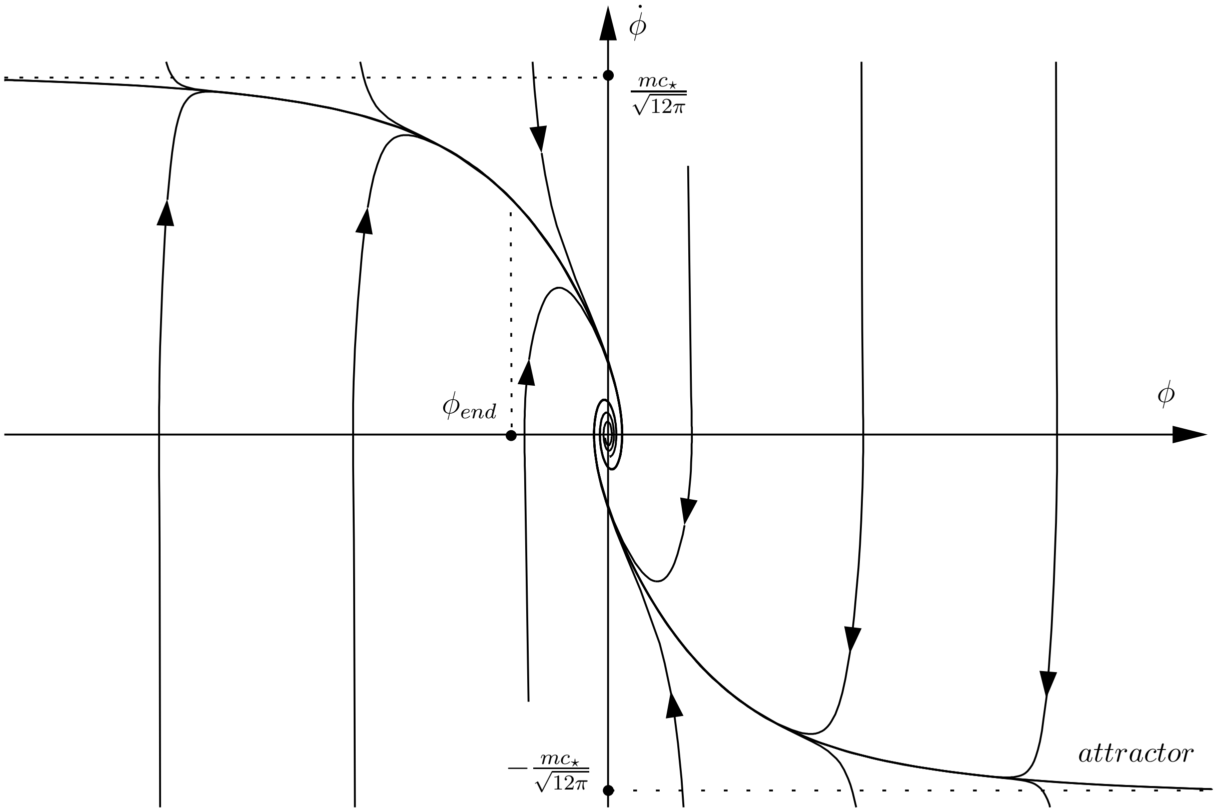

Figure 5.1: Numerically obtained phase portrait for the system with the parameters $m = 1.5 \cdot 10^{-7}$, $c_\star = 3.67$. The field value $\phi_{end} = 0.32$ corresponds to the end of acceleration when $w = -1/3$.

and obtain

$$a(\phi) \simeq a_f \exp\left(\frac{2\pi}{c_\star}\left(\phi_f^2 - \phi^2\right)\right), \tag{5.16}$$

where a_f and $\phi_f \sim \sqrt{c_\star/6\pi}$ are the values of the scale factor and the scalar field at the end of inflation. Given a number of e-folds before the end of inflation N, we find that at this time

$$\frac{2\pi\phi^2}{c_\star} \simeq N, \tag{5.17}$$

and, hence,

$$\frac{\varepsilon + p}{\varepsilon} \simeq \frac{1}{3N}$$

does not depend on $c_\star$. Thus, for a given scale, which crosses the Hubble scale N e-folds before the end of inflation, the tensor-to-scalar ratio is

$$\frac{\delta_h^2}{\delta_\Phi^2} \simeq 27c_\star\left(1 + \frac{p}{\varepsilon}\right) \simeq \frac{9c_\star}{N}. \tag{5.18}$$

It is clear that by choosing α close to the critical value $m^2/12\pi$ we can have a very large $c_\star$ and consequently enhance this ratio almost arbitrarily.

5.3 Discussion

We have shown above that in theories where the Lagrangian is a nontrivial, nonlinear function of the kinetic term, the scale of inflation can be pushed to a very high energies without coming into conflict with observations. As a result, the amount of produced gravitational waves can be much larger than is usually expected. If such a situation were realized in nature then the prospects for the future detection of the B-mode of CMBR polarization are greatly improved. Of course, the theories where this happens are somewhat "fine-tuned". Namely, the corresponding Lagrangian $p(X, \phi)$ must generically satisfy the condition, $p_{,X} + 2Xp_{,XX} \ll p_{,X}$, during inflation. For the particular model (5.6) this means that $12\pi\alpha - m^2 \ll 12\pi\alpha$. This fine-tuning of the parameters of the theory should not be confused, however, with the fine-tuning of the initial conditions. In fact, the true theory of nature is unique and, for example, Lagrangian (5.6), where $12\pi\alpha$ is not very different from m^2 looks even more attractive because it has fewer free parameters. Therefore, future observations of the CMBR fluctuations are extremely important since they will restrict the number of possible candidates for the inflaton.

Chapter 6

Looking beyond the Horizon

In this Chapter we follow our papers [72, 75, 73] (and on the work in preparation) and show how it is possible to obtain information from the spacetime region beyond the Schwarzschild horizon.

6.1 Model

Let us consider the simplest k-*essence* of DBI type (3.15) with the Lagrangian density

$$\mathcal{L}(X) = \alpha \left[\sqrt{1 + \frac{2X}{\alpha}} - 1 \right], \tag{6.1}$$

where α is a free parameters of the theory. Throughout this Chapter we use the natural units in which $G_{_N} = \hbar = c = 1$. The kinetic part of the action is the same as in [70] and for small derivatives, that is, in the limit $2X \ll \alpha$, it describes the usual massless free scalar field. As it follows from the discussion in the Section 3.3 for this theory we have $\sigma = +1$ in the generalized DBI Lagrangian (3.15) therefore the model for sure satisfies the Null Energy Condition. Therefore the *black hole area theorem* from the Ref. [21] holds and the area of the Black Hole horizon never decreases. Here we allow for both signs of the constant α, thus we consider both: the case of the *superluminal* propagation $\alpha > 0$ and standard DBI with $\alpha < 0$ and the sound speed c_i less than the speed of light.

In the case of field-independent k-*essence* the equation of motion for the scalar field (2.7) is

$$G^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0. \tag{6.2}$$

It is convenient to write energy density (3.17) and the pressure in the following form

$$\varepsilon = \alpha(1 - c_s^{-1}), \quad p = \alpha(c_s - 1). \tag{6.3}$$

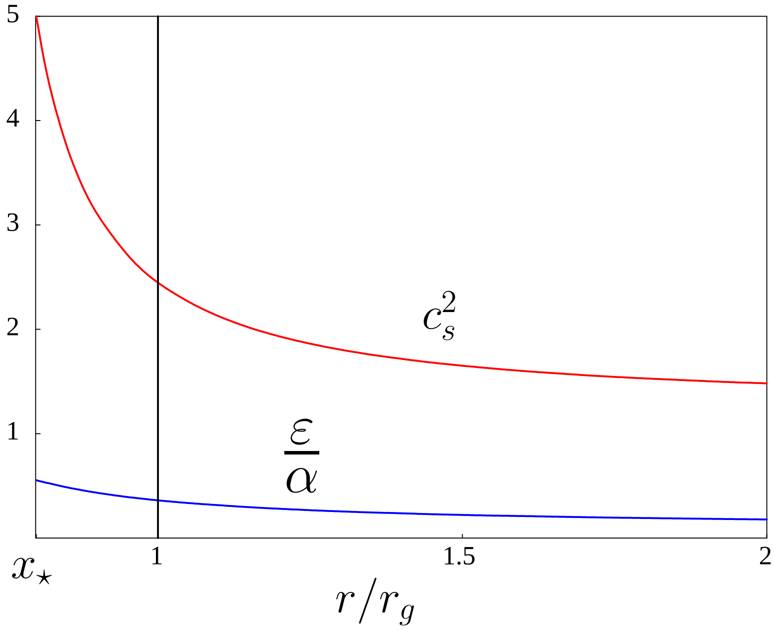

Figure 6.1: For the background solution in the case $c_i^2 = 5/4$ the squared sound speed (red) and the normalized energy density, ε/α, (blue) are shown as functions of radial coordinate $x = r/r_g$. The sound horizon $r_\star = \frac{4}{5} r_g$ is located inside the Schwarzschild horizon r_g.

6.2 Background solution

First we will find a stationary spherically symmetric background solution for the scalar field falling onto a Schwarzschild black hole. Keeping in mind the $\alpha > 0$ case where the superluminal propagation is possible it is convenient to consider the problem in the coordinates which are regular on r_g- the Schwarzschild horizon (gravitational radius). To describe the black hole we use the in-going Eddington-Finkelstein coordinates, in which the metric takes the form:

$$\mathrm{d}s^2 = f(r)\mathrm{d}V^2 - 2\mathrm{d}V\mathrm{d}r - r^2\mathrm{d}\Omega^2, \tag{6.4}$$

where

$$\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2, \tag{6.5}$$

is the line element on unit sphere and we have denoted $f(r) \equiv 1 - r_g/r$. The coordinate V is related to the Schwarzschild coordinates t and r as: $V \equiv t + r + r_g \ln|r/r_g - 1|$. For our further discussion it is convenient to introduce the rescaled radius coordinate $x \equiv r/r_g$. Let us assume that the in-falling field has a negligible influence on the metric, that is, we consider an accretion of the test fluid in the given gravitational field. The requirement of stationarity implies the following ansatz for the solution:

$$\phi(V, x) = \dot{\phi}_i \left(V + r_g \int F(x) dx \right), \tag{6.6}$$

where

$$\dot{\phi}_i = \sqrt{\alpha\left(c_i^2 - 1\right)}, \tag{6.7}$$

is the field derivative with respect to the Schwarzschild time t at spatial infinity and c_i is the speed of sound at infinity. Note that in this case the product $\alpha\left(c_i^2 - 1\right)$ is always positive. The overall factor in (6.6) is chosen to recover the cosmological solution at infinity: $\phi(V, x) \to t\sqrt{\alpha\left(c_i^2 - 1\right)}$ and r_g in front of the integral is for the further convenience. The function $F(x)$ must be determined by solving equations of motion (2.7) for appropriate boundary conditions. Substituting (6.6) into (6.2) and integrating over r we obtain the following equation for the function $F(x)$:

$$\frac{(fF+1)x^2}{\sqrt{1-\left(fF^2+2F\right)\left(c_i^2-1\right)}} = \frac{B}{c_i^4}, \tag{6.8}$$

where B is the constant of integration. The solution of (6.8), which is nonsingular at the black hole horizon, is given by:

$$F(x) = \frac{1}{f}\left(B\sqrt{\frac{c_i^2+f-1}{fx^4c_i^8+B^2\left(c_i^2-1\right)}} - 1\right). \tag{6.9}$$

The speed of sound speed can then be found using (6.9), (6.6) and (3.1):

$$c_s^2(x) = \frac{x^3c_i^8\left(xc_i^2-1\right)}{(x-1)\,x^3c_i^8+B^2\left(c_i^2-1\right)}. \tag{6.10}$$

Note that the speed of sound becomes infinite at some $x \equiv x_{sing}$ and this singularity is physical if the real regular solution (6.9) exists for all $x > x_{sing}$.

A constant of integration B, entering (6.9) and (6.10), determines the energy flux falling onto the black hole. To fix it we have to find the solution which is non-singular on the *sound horizon* and outside it. Below we consider the propagation of perturbations and find how the position of the sound horizon depend on B. Then, given c_i, and comparing the positions of the singularities and the sound horizon we determine the unique value for B.

6.3 Small perturbations

Let us now consider the small perturbations around background (6.6), (6.9). The characteristics (propagation vectors N^μ) for equation (6.2) are null vectors in the *emergent* geometry (see Appendix B formula (B.9))

$$G^{-1}_{\mu\nu}N^\mu N^\nu = 0, \tag{6.11}$$

where $G^{-1}_{\mu\nu}$ is given by Eqs. (3.7) and (3.19) calculated for the background solution Eqs. (6.6) and (6.9). For the *emergent* line element (3.8) $\mathrm{d}S^2 = G^{-1}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ we have

$$c_s^2\mathrm{d}S^2 = \left(c_i^2 - \frac{r_g}{r}\right)\mathrm{d}V^2 - 2\mathrm{d}V\mathrm{d}r\left(1-\left(c_i^2-1\right)F\right) + \left(c_i^2-1\right)F^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2 \tag{6.12}$$

From this line element one can guess that the there is *sonic horizon* at $r_\star = r_g/c_i^2$. In fact the constant B and consequently $F(r)$ and the background solution $\phi(V,r)$ can be determined by the Cosmic Censorship requirement applied to the *emergent* spacetime (6.12). Namely for physically meaningful solutions all possible singularities of the metric (6.12) should be inside the sonic horizon $r_\star$.

The vector N^μ describes the propagation of the wave front. After lengthy, but straightforward calculations, we obtain from (6.11) and (A.5) the following differential equation for the characteristics $N_\pm(x) \equiv dV/dx$:

$$N_\pm = \frac{1}{f} + \frac{1}{\xi_\pm}, \tag{6.13}$$

where

$$\xi_\pm = \pm f\sqrt{c_i^2 - \frac{1}{x}}\,\frac{\sqrt{B^2(c_i^2-1) + c_i^8 x^4 f}}{c_i^4 x^2 f \mp B(c_i^2-1)}. \tag{6.14}$$

It is worth mentioning that the equation $\xi_\pm = dx/dt$ determines the propagation of wave front in the Schwarzschild coordinates x and t.

Equation (6.13) does not specify the *direction* of the propagation completely. In addition to the value of dV/dx one has to choose a cone of *future* and a cone of *past* for every event. However, the position of the past and the future light cones helps us to fix the past and the future influence cones for the k-*essence* perturbations, or in other words, for the "sound". Using characteristics (6.13) we then select uniquely the sonic influence cones as follows: i) the past and the future sonic cones should not have overlapping regions; ii) the future sonic cone contains the future light cone, while the past sonic cone contains the past light cone. This last property can be justified because it holds at the spatial infinity and the sonic characteristics (6.13) nowhere coincide with the radial light geodesics (otherwise for the sonic signal $\mathrm{d}s^2$ would vanish somewhere and this is obviously not true). As a result we conclude: a signal propagating along N_+ points in the positive $V-$direction, while a signal corresponding to N_- points in the negative $V-$direction (see Fig. 6.2).

Having calculated the propagation vectors we can find the *sonic horizon.* The *sonic horizon* is defined as a surface, where the length of the spatial velocity vector is equal to the speed of sound. Outside this surface the signals can reach the spatial infinity, while sound cannot escape from inside because it is trapped by the supersonic motion of a fluid (in the same way as light is trapped inside the event horizon by the gravitational field). The acoustic signal directed *out* of the black hole corresponds to N_+ and therefore the *sound horizon* is located at $x \equiv x_\star$ where $N_+ \equiv (dV/dx)_+$ becomes infinite (see Fig. 6.2).

Now we can fix a constant of integration B, entering (6.6), (6.9). We simply demand that in the physically occurring situation there exists no singularity on the *sound horizon* and outside of it. This procedure is similar to that one arising in the problem of perfect fluid accretion where the physical solution does not diverge at the event horizon (see, e.g. [234]). Thus, fixing B reduces to the analysis of the mutual location of x_{sing} and $x_\star$. After some calculations we find the following:

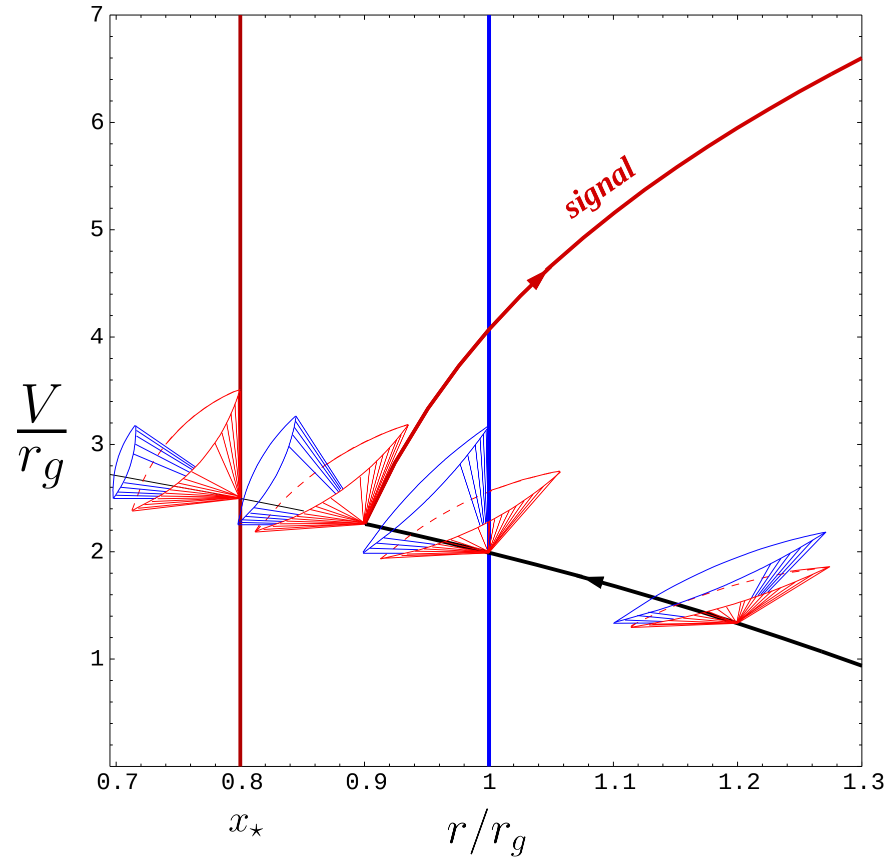

Figure 6.2: In the Eddington-Finkelstein coordinates the emission of a sound signal from the falling spacecraft is shown. The blue cones correspond to the future light cones and the red cones are the future sonic cones (6.13). The black curve represents the world line obtained numerically for the spacecraft which moves together with a falling background field. Being between the the Schwarzschild ($r = r_g$) and sound ($r = r_\star$) horizons the spacecraft emits an acoustic signal (shown by red) which reaches the distant observer in finite time. The trajectory of the signal is obtained by the numerical integration of Eq. (6.13).

- For $B \neq 1$ either the physical singularity coincides with the sound horizon or the speed of sound becomes imaginary (this means absolute instability) within some region outside the singular surface, for $x > x_{sing}$. In both cases the solution is nonphysical.

- For $B = 1$ and $c_i^2 > 4/3$ the speed of sound becomes imaginary before reaching of sound horizon or singularity. This solution is also nonphysical.

- For $B = 1$ and $c_i^2 < 4/3$ the sound horizon is located at $x_\star = 1/c_i^2$ and the singularity is hidden inside the sound horizon, $x_{sing} < x_\star$. This is the only physically relevant solution we are searching for.

Thus, we have to set $B = 1$ in (6.6), (6.9) and this ends the constructing of the background. It is worthwhile mentioning that using the new Schwarzschild-like coordinates (τ, r)

$$\mathrm{d}V = \mathrm{d}\tau - \left(G^{-1}_{rV}/G^{-1}_{VV}\right)\mathrm{d}r, \tag{6.15}$$

ne can write the emergent line element $\mathrm{d}S^2$ as

$$c_s^2 \mathrm{d}S^2 = H(r) c_s^2 \mathrm{d}\tau^2 - H^{-1}(r)\mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2, \tag{6.16}$$

where we have introduced

$$H(r) = \left(1 - \frac{r_g}{r}\right)\left(1 - \left(\frac{r_\star}{r}\right)^2\right) / \left(1 - c_s^2(r)\left(\frac{r_\star}{r}\right)^4\right), \tag{6.17}$$

and the sound speed is given by (6.10) with $B = 1$. Before we turn to the discussion of the signals propagation in the *emergent* geometry 6.16 we will briefly analyze the validity of the stationarity approximation when the backreaction can be neglected. Having fixed B the rate of the accretion can be easily evaluated as (see e.g. [235, 236]):

$$\dot{M} = 4\pi M^2 \alpha \left(c_i^2 - 1\right)/c_i^4, \tag{6.18}$$

here M is the mass of the Schwarzschild Black Hole. It is clear that for any fixed value of c_i we can choose a small enough α, so that the energy flux onto black hole is negligible. The propagation of perturbations (6.13) on the background (6.6) does not depend on α, but only on c_i. Therefore, we can always take sufficiently small α in (6.1) to ensure that during the gedanken experiment with sending signals from the interior of a black hole the background solution remains nearly unchanged.

After we have found the physically relevant background solution we will discuss whether the acoustic signals can really escape from the interior of the black hole. This becomes possible because in the case of $\alpha > 0$ the sound horizon ($x_\star = 1/c_i^2$) is located inside the Schwarzschild radius. If $\alpha < 0$ then $c_i < 1$ and obviously $x_\star > 1$ and the sonic horizon is outside the Schwarzschild sphere. As long as the signals are emitted at large enough x, namely, at $x > x_\star$, they reach the spatial infinity propagating along N_+. For example, at the Schwarzschild horizon r_g we have:

$$N_{\pm H} = \frac{1}{2}\frac{\left(c_i^4 \pm 1\right)^2}{c_i^2 - 1}. \tag{6.19}$$

The propagation vector N_{+H} is positive and so signals could freely penetrate the Schwarzschild horizon and move *outside* the black hole. The Fig. 6.2 shows how the acoustic signals go out from the interior of a black hole.

Let us calculate the redshift of the emitted signal. Suppose that a spacecraft moves together with the falling background field (such that in the spacecraft's system of coordinates $\partial_i\phi_0 = 0$) and sends the acoustic signals with the frequency ω_{em}. After simple calculations one can obtain that an observer at rest at the spatial infinity will detect these signals at the frequency ω_{inf}:

$$\frac{\omega_{inf}}{\omega_{em}} = \left(1 - \left(\frac{r_\star}{r}\right)^2\right)\sqrt{\frac{1 - r_g/r}{1 - c_s^2(r)\left(r_\star/r\right)^4}}. \tag{6.20}$$

This expression corrects our result from [73]. Note that the ratio is finite for any $r > r_\star$ and it vanishes for $r = r_\star$. In particular at the moment as the spacecraft passes through the Schwarzschild horizon r_g we have

$$\frac{\omega_{em}}{\omega_{inf}} = \frac{c_i^4\sqrt{1 + c_i^2 + c_i^4 + c_i^6}}{c_i^4 - 1}. \tag{6.21}$$

It is important to note that as it can be easily checked for the background $\phi\left(V, r\right)$ the derivatives are always timelike $X > 0$ outside the sonic horizon $r_\star$. Therefore as we have already mentioned in the Section 3.4 the emergent spacetime is stably causal, and consequently there are no causal paradoxes. Also we would like to stress that the solution obtained above is valid for both the positive and negative α parameter of DBI-like k-*essence.*

6.4 Discussion

The main result of this Chapter can be summarized as follows: if there exist a specific DBI-like k-*essence* , then during accretion of these field onto black hole one can send information from the interior of the Black Hole. We would like to stress that this result has a classical origin and no quantum phenomena are involved. The discussed effect changes the universal meaning of the Schwarzschild horizon as an event horizon and may have important consequences for the thermodynamics of black holes.

We consider the present work as simply an illustration of a concept. The particular theory examined does not have any justification from the point of view of particle physics.

However, for a wide class of nonlinear theories the situation can be similar and therefore it is quite possible that the information can really be send from inside the Black Hole.

Also we would like to point out that in our model the cosmic censorship hypothesis is holds because the singularity is hidden by the sound horizon. The Null Energy Condition is not violated as well. Hence the Schwarzschild horizon never decreases.

The recent paper [237] (see also later work [238]) deals with thermodynamics of black holes in the presence of *superluminal* propagation. The authors came to conclusion that the Second Low of the thermodynamics can be easily violated in this case. However, the model analyzed in this paper is completely different from ours, namely, the authors of [237] have considered two kinetically coupled fields, one of which is the *ghost condensate* [17]. We are sure that in our case the violation of the Second Low does not occur because the system is not closed and there is always external flux of energy entering the system from the spatial infinity. In addition it should be mentioned that the similar possibility of sending signals from the inside of a black hole opens in bigravity theories [188].

Appendix A

Effective Hydrodynamics

It is well-known that for timelike $\nabla_\nu \phi$ ($X > 0$ in our signature) one can employ the hydrodynamic approach to describe the system with the action (2.1). To do this one introduces an effective "four-velocity" as follows:

$$u_\mu \equiv \sigma \frac{\nabla_\mu \phi}{\sqrt{2X}}, \tag{A.1}$$

where $\sigma = \mathrm{sign}\,(\partial_0 \phi)$. Using (A.1) the energy momentum tensor (2.3) tensor can be rewritten in the perfect fluid form:

$$T_{\mu\nu} = (\varepsilon + p)\, u_\mu u_\nu - p g_{\mu\nu},$$

where the pressure coincides with the Lagrangian density, $p = \mathcal{L}(X, \phi)$, and the energy density is

$$\varepsilon\,(X, \phi) = 2X p_{,X} - p. \tag{A.2}$$

It should be stressed that the energy density ε and pressure p introduced in this way are scalars and correspond to T_0^0 and $-\frac{1}{3}T_i^i$ only in the *rest frame* where $u_i = 0$ and the stress is isotropy. For various cosmological applications it is convenient to introduce the equation of state parameter w

$$w \equiv \frac{p}{\varepsilon}. \tag{A.3}$$

Note that w defined in this way characterizes intrinsic properties of k-*essence* in the coordinate independent way. Following this hydrodynamical analogy further one can guess that the sound speed (3.1) can be expressed as

$$c_s^2 = \frac{p_{,X}}{\varepsilon_{,X}} = \left(\frac{\partial p}{\partial \varepsilon}\right)_\phi. \tag{A.4}$$

For the detailed derivation of this result see [69, 82]. Note that the general k-*essence* field theory with the Lagrangian $\mathcal{L}\,(X, \phi)$, which explicitly depends on ϕ, is not equivalent to the isentropic hydrodynamics, because ϕ and X are independent and therefore the pressure cannot be expressed though ε only. However, the hydrodynamical language provides a rather useful and physically intuitive description of the system. In particular the equation of motion for the homogeneous field configurations $\phi(t)$ is equivalent to the continuity equation (2.14). Note that the general equation of motion (2.7) cannot be obtained in the hydrodynamic setup for the ϕ-dependent k-*essence* Lagrangians $\mathcal{L}(X, \phi)$.

In what follows we restrict ourselves to the class of Lagrangians which do not depend of ϕ explicitly, $p = p(X)$ and in addition we require that $X > 0$. This class of models is precisely equivalent to perfect fluid models with zero vorticity and with the pressure being a function of the energy density only, $p = p(\varepsilon)$. Then the expressions (3.1) or (A.4) coincide with the usual definition of the sound speed for the perfect fluid: $c_s^2 = \partial p/\partial\varepsilon$. Apart from the energy density ε and pressure p one can also formally introduce the "concentration of particles":

$$n \equiv \exp\left(\int \frac{d\varepsilon}{\varepsilon + p(\varepsilon)}\right) = \sqrt{X} p_{,X}.$$

and the enthalpy

$$h \equiv \frac{\varepsilon + p}{n} = 2\sqrt{X}.$$

In particular the equation of motion (2.7) takes the form of the particle number conservation law: $\nabla_\mu (n u^\mu) = 0$.

Using these definitions we can rewrite the induced metric metric $G^{\mu\nu}$ and its inverse in terms of hydrodynamic quantities only:

$$G^{\mu\nu} = \frac{hc_s}{2n}\left[g^{\mu\nu} - \left(1 - c_s^{-2}\right) u^\mu u^\nu\right], \tag{A.5}$$

$$G^{-1}_{\mu\nu} = \frac{2n}{hc_s}\left[g_{\mu\nu} - \left(1 - c_s^2\right) u_\mu u_\nu\right]. \tag{A.6}$$

To our best knowledge these metrics (A.5) along with an action for the velocity potentials were introduced for the first time in [81], where the accretion of the perfect fluid onto black hole was studied. As it follows from the derivation in C, the metric (C.17) and the action (C.16) derived in our paper are applicable in the more general case of arbitrary nonlinear scalar field theories $\mathcal{L}(X, \phi)$ and for all possible (not only timelike $X_0 > 0$) backgrounds produced by any external sources.

Appendix B

Characteristics and superluminal propagation

Let us consider scalar field ϕ interacting with external source $J\left(x\right)$. The equation of motion for the scalar field is

$$\tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi+\varepsilon_{,\phi}=J \tag{B.1}$$

where metric $\tilde{G}^{\mu\nu}$ is given by (2.8) and for brevity we use the "hydrodynamic" notation $\varepsilon\left(X,\phi\right)=2X\mathcal{L}_{,X}-\mathcal{L}$. Suppose ϕ_0 is the background solution of (B.1) in the presence of source $J_0\left(x\right)$ and gravitational metric $g_{\mu\nu}\left(x\right)$. Let us consider a slightly perturbed solution $\phi=\phi_0+\pi$ of (B.1) with the source $J=J_0+\delta J$ and the original unperturbed metric $g_{\mu\nu}\left(x\right)$. The equation of motion for π is then

$$\tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\pi+\varepsilon_{,\phi\phi}\pi+\varepsilon_{,\phi X}\delta X+\delta\tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi_0=\delta J, \tag{B.2}$$

where

$$\delta X=\nabla_{\nu}\phi_0\nabla^{\nu}\pi\qquad\text{and}\qquad\delta\tilde{G}^{\mu\nu}=\frac{\partial\tilde{G}^{\mu\nu}}{\partial\phi}\pi+\frac{\partial\tilde{G}^{\mu\nu}}{\partial\nabla_{\alpha}\phi}\nabla_{\alpha}\pi. \tag{B.3}$$

This equation can be written as

$$\tilde{G}^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\pi+V^{\mu}\nabla_{\mu}\pi+\tilde{M}^{2}\pi=\delta J, \tag{B.4}$$

where

$$V^{\mu}\left(x\right)\equiv\frac{\partial\tilde{G}^{\alpha\beta}}{\partial\nabla_{\mu}\phi}\nabla_{\alpha}\nabla_{\beta}\phi_0+\varepsilon_{,\phi X}\nabla^{\mu}\phi_0, \tag{B.5}$$

and

$$\tilde{M}^{2}\left(x\right)\equiv\frac{\partial\tilde{G}^{\alpha\beta}}{\partial\phi}\nabla_{\alpha}\nabla_{\beta}\phi_0+\varepsilon_{,\phi\phi}. \tag{B.6}$$

Considering the eikonal (or short wavelength) approximation [239] we have

$$\pi\left(x\right)=A\left(x\right)\exp i\omega S\left(x\right), \tag{B.7}$$

where ω is a large dimensionless parameter and the amplitude $A(x)$ is a slowly varying function. In the limit $\omega \to \infty$ the terms containing no second derivatives, $V^\mu(x)\nabla_\mu \pi$ and $\tilde{M}^2(x)\pi$, become unimportant and (B.4) becomes

$$\tilde{G}^{\mu\nu}\partial_\mu S \partial_\nu S = 0. \tag{B.8}$$

The equation of motion in the eikonal approximation (B.8) is conformally invariant. The surfaces of constant eikonal S (constant phase) correspond to the wave front (characteristic surface) in spacetime. Thus the 1-form $\mathrm{d}S = \partial_\mu S \mathrm{d}x^\mu$ is orthogonal to the characteristic surface. The influence cone at point P is formed by the propagation vectors N^μ tangential to the characteristic surface $N^\mu \partial_\mu S = 0$ and positive projection on the time direction. Using (B.8) one can chose $N^\mu = \tilde{G}^{\mu\nu}\partial_\nu S$ and verify that this vectors are tangential to the characteristic surface. The metric $\tilde{G}^{\mu\nu}$ has an inverse $\tilde{G}^{-1}_{\mu\nu}$ due to the requirement of hyperbolicity (Lorentzian signature of $\tilde{G}^{\mu\nu}$). Therefore $\partial_\nu S = \tilde{G}^{-1}_{\mu\nu} N^\mu$ and we obtain the equation for the influence cone in the form

$$\tilde{G}^{-1}_{\mu\nu} N^\mu N^\nu = 0.$$

Thus the metric $\tilde{G}^{-1}_{\mu\nu}$ governs the division of acoustic spacetime into past, future and inaccessible “spacelike” regions (or in other words this metric yields the notion of causality). It is well known that this division is invariant under conformal transformations. From action (3.5) for perturbations π, which we derive in Appendix C, it follows that in four dimensions it is natural to consider a conformally transformed metric $G^{-1}_{\mu\nu} = \left(\mathcal{L}^2_{,X}/c_s\right)\tilde{G}^{-1}_{\mu\nu}$. Using this metric from (3.7) one obtains

$$G^{-1}_{\mu\nu} N^\mu N^\nu = \frac{\mathcal{L}_{,X}}{c_s} g_{\mu\nu} N^\mu N^\nu - c_s \mathcal{L}_{,XX}\left(\nabla_\mu \phi N^\mu\right)^2 = 0. \tag{B.9}$$

Therefore

$$g_{\mu\nu} N^\mu N^\nu = c_s^2 \left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\left(\nabla_\mu \phi N^\mu\right)^2,$$

and if $\mathcal{L}_{,XX}/\mathcal{L}_{,X}$ is negative, then $g_{\mu\nu}N^\mu N^\nu < 0$, that is, N^μ is spacelike and the cone of influence on this background is larger than the light cone: the wave front (or signal) velocity is larger then the speed of light. Note that this is a coordinate independent statement.

Appendix C

Action for perturbations in test field approximation

Here we derive of action (3.5) for π in the spacetime of arbitrary dimension $N > 2$. First of all we would like to investigate whether there exists a metric $G^{\mu\nu}$ for which the equation of motion for perturbations π takes a canonical (Klein-Gordon) form

$$G^{\mu\nu} D_\mu D_\nu \pi + M_{\text{eff}}^2 \pi = \delta I, \tag{C.1}$$

where D_μ is a covariant derivative with associated with the new metric $G^{\mu\nu}$: $D_\mu G^{\alpha\beta} = 0$. Note that the equations of motion (B.2) and (C.1) should have the same influence cone structure. Thus the metrics $G^{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ must be related by conformal transformation and if it is really possible to rewrite (B.2) in canonical form, then there must exist $\Omega\left(\phi_0, X_0\right)$, such that

$$G^{\mu\nu} = \Omega \tilde{G}^{\mu\nu}. \tag{C.2}$$

Therefore our first task is to find $\Omega\left(\phi_0, X_0\right)$. Note that this method makes sense for the dimensions $D > 2$ only. That happens because in $D = 2$ all metrics are conformally equivalent to $\eta_{\mu\nu}$ and the wave equation is conformally invariant, see e.g. Ref. [167], P. 447. Let us define the following covariant derivative

$$D_\mu A_\nu = \nabla_\mu A_\nu - L^\lambda_{\mu\nu} A_\lambda \tag{C.3}$$

which is compatible with the new metric whereas $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda$ denotes the standard covariant derivative associated with the gravitational metric: $\nabla_\mu g^{\alpha\beta} = 0$, as usual. Note, that the tensor $L^\lambda_{\mu\nu}$ introduced in (C.3) is the difference of the Christoffel symbols corresponding to the effective and gravitational metrics. Comparing (B.2) and (C.1) we infer that

$$\Omega \tilde{G}^{\mu\nu} D_\mu D_\nu \pi + M_{\text{eff}}^2 \pi = \Omega \tilde{G}^{\mu\nu} \nabla_\mu \nabla_\nu \pi - \Omega \tilde{G}^{\mu\nu} L^\lambda_{\mu\nu} \nabla_\lambda \pi + M_{\text{eff}}^2 \pi$$

must be equal (up to a multiplication by a scalar function Ω) to the l.h.s of (B.4). These can be true only if the following condition holds

$$\tilde{G}^{\mu\nu} L^\lambda_{\mu\nu} = -V^\lambda, \tag{C.4}$$

where V^λ is defined in (B.5). When this condition is satisfied we can always make the redefinition

$$M_{\text{eff}}^2 = \Omega \tilde{M}^2 \quad \text{and} \quad \delta I = \Omega \delta J,$$

where $\tilde{M}^2$ is defined in (B.6). The connection $L^\lambda_{\mu\nu}$ depends on the unknown function Ω (and its derivatives) which has to be obtained form (C.4). To solve (C.4) it is convenient to multiply its both sides by Ω. Then using (B.5) and (C.2) this condition takes the form:

$$G^{\mu\nu} L^\lambda_{\mu\nu} = -\Omega \left(\frac{\partial \tilde{G}^{\alpha\beta}}{\partial \nabla_\lambda \phi} \nabla_\alpha \nabla_\beta \phi_0 + \varepsilon_{,\phi X} \nabla^\lambda \phi_0 \right). \tag{C.5}$$

Let us now solve (C.5) with respect to Ω. In complete analogy with the formula (86,6) from Ref. [239] we have

$$G^{\mu\nu} L^\lambda_{\mu\nu} = -\frac{1}{\sqrt{-G}} \nabla_\alpha \left(\sqrt{-G} G^{\alpha\lambda} \right), \tag{C.6}$$

where $\sqrt{-G} = \sqrt{-\text{det} G^{-1}_{\mu\nu}} = \Omega^{-D/2} \sqrt{-\text{det} \tilde{G}^{-1}_{\alpha\beta}}$, and D is the number of dimensions of the spacetime. Using the formula (B14) from Ref. [16] one obtains

$$\text{det} \tilde{G}^{\alpha\beta} = (\mathcal{L}_{,X})^D c_s^{-2} \text{det}(g^{\mu\nu}), \quad \text{and} \quad \text{det} \tilde{G}^{-1}_{\alpha\beta} = (\mathcal{L}_{,X})^{-D} c_s^2 \text{det}(g_{\mu\nu}). \tag{C.7}$$

Finally we arrive to the relation,

$$\sqrt{-G} = c_s \sqrt{-g} \left(\Omega \mathcal{L}_{,X} \right)^{-D/2}. \tag{C.8}$$

It is convenient to introduce the auxiliary function

$$F = c_s \left(\Omega \mathcal{L}_{,X} \right)^{-D/2} \Omega. \tag{C.9}$$

and then using (C.6), we can rewrite equation (C.5) as:

$$\nabla_\alpha \left(F \tilde{G}^{\alpha\lambda} \right) = F \left(\frac{\partial \tilde{G}^{\alpha\beta}}{\partial \nabla_\lambda \phi} \nabla_\alpha \nabla_\beta \phi_0 + \varepsilon_{,\phi X} \nabla^\lambda \phi_0 \right). \tag{C.10}$$

Differentiating the metric $\tilde{G}^{\alpha\lambda}$ from the l.h.s. of the last equation in accordance with the chain rule we find:

$$\tilde{G}^{\alpha\lambda} \nabla_\alpha F = F \left(\left(\frac{\partial \tilde{G}^{\alpha\beta}}{\partial \nabla_\lambda \phi} - \frac{\partial \tilde{G}^{\alpha\lambda}}{\partial \nabla_\beta \phi} \right) \nabla_\alpha \nabla_\beta \phi_0 - \left(\frac{\partial \tilde{G}^{\alpha\lambda}}{\partial \phi} - \varepsilon_{,\phi X} g^{\lambda\alpha} \right) \nabla_\alpha \phi_0 \right). \tag{C.11}$$

Further we obtain

$$\frac{\partial \tilde{G}^{\alpha\lambda}}{\partial \phi} \nabla_\alpha \phi_0 = \left(\mathcal{L}_{,X\phi} + 2X \mathcal{L}_{,XX\phi} \right) \nabla^\lambda \phi_0 = \varepsilon_{,\phi X} \nabla^\lambda \phi_0. \tag{C.12}$$

For the first term in the brackets in (C.11) we have:

$$\frac{\partial \tilde{G}^{\alpha\beta}}{\partial \nabla_\lambda \phi} = \mathcal{L}_{,XX}\left(g^{\alpha\beta}\nabla^\lambda\phi_0 + g^{\lambda\alpha}\nabla^\beta\phi_0 + g^{\lambda\beta}\nabla^\alpha\phi_0\right) + \mathcal{L}_{,XXX}\nabla^\alpha\phi_0\nabla^\beta\phi_0\nabla^\lambda\phi_0, \tag{C.13}$$

and therefore

$$\frac{\partial \tilde{G}^{\alpha\beta}}{\partial \nabla_\lambda \phi} - \frac{\partial \tilde{G}^{\alpha\lambda}}{\partial \nabla_\beta \phi} = 0. \tag{C.14}$$

Thus the r.h.s. of (C.11) identically vanishes. Note that there exists the inverse matrix $\tilde{G}^{-1}_{\alpha\lambda}$ to $\tilde{G}^{\alpha\lambda}$. Therefore from (C.11) we conclude that $\nabla_\alpha F = 0$ or $F = const$ on all backgrounds and for all theories. Considering the linear case, $\mathcal{L}(\phi, X) = X - V(\phi)$, we infer that $F = c_s\left(\Omega\mathcal{L}_{,X}\right)^{-D/2}\Omega = 1$ or

$$\Omega = \left(c_s\mathcal{L}_{,X}^{-D/2}\right)^{1/(D/2-1)}. \tag{C.15}$$

Having calculated Ω we can formulate the main result of this Appendix as follows: the action from which one can obtain the equation of motion in the canonical Klein-Gordon form (C.1) is

$$S_\pi = \frac{1}{2}\int \mathrm{d}^D x\sqrt{-G}\,\left[G^{\mu\nu}\partial_\mu\pi\partial_\nu\pi - M_{\mathrm{eff}}^2\pi^2 + 2\pi\delta I\right], \tag{C.16}$$

where the emergent metric $G^{\mu\nu}$ is the conformally transformed eikonal metric $\tilde{G}^{\mu\nu}$, defined in (2.8),

$$G^{\mu\nu} \equiv \left(c_s\mathcal{L}_{,X}^{-D/2}\right)^{1/(D/2-1)}\tilde{G}^{\mu\nu} = \left(\frac{c_s}{\mathcal{L}_{,X}}\right)^{1/(D/2-1)}\left[g^{\mu\nu} + \left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\nabla^\mu\phi\nabla^\nu\phi\right]. \tag{C.17}$$

The inverse metric $G^{-1}_{\mu\nu}$ can be easily calculated using the ansatz $G^{-1}_{\mu\nu} = \alpha g_{\mu\nu} + \beta\nabla_\mu\phi_0\nabla_\nu\phi_0$ and is given by the formula

$$G^{-1}_{\mu\nu} = \left(\frac{c_s}{\mathcal{L}_{,X}}\right)^{-1/(D/2-1)}\left[g^{\mu\nu} - c_s^2\left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)\nabla^\mu\phi_0\nabla^\nu\phi_0\right]. \tag{C.18}$$

Finally the effective mass is

$$M_{\mathrm{eff}}^2 = \left(c_s\mathcal{L}_{,X}^{-N/2}\right)^{1/(D/2-1)}\left[2X\mathcal{L}_{,X\phi\phi} - \mathcal{L}_{,\phi\phi} + \frac{\partial\tilde{G}^{\mu\nu}}{\partial\phi}\nabla_\mu\nabla_\nu\phi_0\right], \tag{C.19}$$

and the effective source for perturbations is given by

$$\delta I = \left(c_s\mathcal{L}_{,X}^{-D/2}\right)^{1/(D/2-1)}\delta J. \tag{C.20}$$

For the reference we also list the formula

$$\sqrt{-G} = \sqrt{-g}\left(\frac{\mathcal{L}_{,X}^D}{c_s^2}\right)^{1/(D-2)}. \tag{C.21}$$

Appendix D

Action for Cosmological Perturbations

Here we compare the action (3.5) with the action for scalar cosmological perturbations from Refs. [69, 82]. In particular we show that cosmological perturbations propagate in the metric (3.7) but have an effective mass different from (3.4). Finally we derive the generally covariant action for the scalar cosmological perturbations.

To begin with let us consider the action (3.5) for a perturbations $\pi(\eta, \mathbf{x})$ around a homogeneous background $\phi(\eta)$ in the spatially flat Friedmann universe

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2(\eta)\left(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right) = a^2(\eta)\,\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{D.1}$$

where η is the conformal time $\eta = \int dt/a(t)$ and $\eta_{\mu\nu}$ is the standard Minkowski metric. Using Eq. (3.1) and Eq. (D.1) one can calculate the effective line element (3.8):

$$\begin{aligned}\mathrm{d}S^2 = G^{-1}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \frac{\mathcal{L}_{,X}}{c_s}\left[\mathrm{d}s^2 - a^2 c_s^2\left(\frac{\mathcal{L}_{,XX}}{\mathcal{L}_{,X}}\right)2X\mathrm{d}\eta^2\right] \\ &= \frac{\mathcal{L}_{,X}}{c_s}a^2\left(c_s^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right) \equiv c_s A^2\left(c_s^2\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2\right).\end{aligned} \tag{D.2}$$

where we have introduced the convenient variable

$$A \equiv \sqrt{\varepsilon_{,X}}a. \tag{D.3}$$

Note that for the models respecting the NEC ($\mathcal{L}_{,X} \geq 0$) the hyperbolicity condition (2.9) requires $\varepsilon_{,X} > 0$ and therefore A is always well defined. The factor $\sqrt{-G}$ can be then calculated either from the last expression above (D.2) or from the general expression (C.21):

$$\sqrt{-G} = \frac{\mathcal{L}^2_{,X}}{c_s}a^4 = c_s^3 A^4. \tag{D.4}$$

Using the formulas (3.2) and (3.1) we calculate the kinetic term

$$G^{\mu\nu}\partial_\mu\pi\partial_\nu\pi = \left(c_s a^2\mathcal{L}_{,X}\right)^{-1}\left((\pi')^2 - c_s^2(\vec{\nabla}\pi)^2\right). \tag{D.5}$$

Thus in the case when the perturbations π do not influence the metric $g_{\mu\nu}$ the action (3.5) takes the form

$$S_\pi = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\eta\,\left[a^2\varepsilon_{,X}\left((\pi')^2 - c_s^2(\vec{\nabla}\pi)^2\right) - M_{\text{eff}}^2\frac{\mathcal{L}_{,X}^2}{c_s}a^4\pi^2\right], \tag{D.6}$$

here we have used the definitions of the sound speed (3.1) and energy density (2.5). It is convenient to introduce the canonical normalization for the perturbations. This is achieved by the following field redefinition:

$$\nu = \sqrt{\varepsilon_{,X}}a\pi = \pi A. \tag{D.7}$$

Finally integrating by parts and dropping the total derivative terms we obtain the following "canonical" action

$$S_\pi = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\eta\,\left[(\nu')^2 - c_s^2(\vec{\nabla}\nu)^2 - m_{\text{eff}}^2\nu^2\right], \tag{D.8}$$

where the new effective mass m_{eff} is given by the following expression

$$m_{\text{eff}}^2 = M_{\text{eff}}^2\frac{\sqrt{-G}}{A^2} - \frac{A''}{A} = \frac{a^2}{\varepsilon_{,X}}\left[\varepsilon_{,\phi\phi} + \frac{\partial\tilde{G}^{\mu\nu}}{\partial\phi}\nabla_\mu\nabla_\nu\phi_0\right] - \frac{\left(\sqrt{\varepsilon_{,X}}a\right)''}{\sqrt{\varepsilon_{,X}}a}. \tag{D.9}$$

or in other terms

$$m_{\text{eff}}^2 = \frac{1}{\varepsilon_{,X}}\left[\varepsilon_{,X\phi}\phi'' + \mathcal{H}\phi'\left(3p_{,X\phi} - \varepsilon_{,X\phi}\right) + \varepsilon_{,\phi\phi}a^2\right] - \frac{\left(\sqrt{\varepsilon_{,X}}a\right)''}{\sqrt{\varepsilon_{,X}}a}. \tag{D.10}$$

Now let us consider the case of cosmological perturbations in the case where the field ϕ is responsible for the dynamics of the Friedmann universe. Following Refs. [69, 82] one introduces a canonical variable v

$$v \equiv \sqrt{\varepsilon_{,X}}a\left(\delta\phi + \frac{\phi'}{\mathcal{H}}\Psi\right) = A\left(\delta\phi + \frac{\phi'}{\mathcal{H}}\Psi\right), \tag{D.11}$$

and a convenient auxiliary variable z

$$z \equiv \frac{\phi'}{\mathcal{H}}\sqrt{\varepsilon_{,X}}a = \frac{\phi'}{\mathcal{H}}A, \tag{D.12}$$

where $\delta\phi$ is the gauge invariant perturbation of the scalar field, $\mathcal{H} \equiv a'/a$ and $\Psi = \Phi$ is the gauge invariant Newtonian potential. Using this notation the action for scalar cosmological perturbations takes the form:

$$S_{\text{cosm}} = \frac{1}{2}\int \mathrm{d}^3x\mathrm{d}\eta\,\left[(v')^2 - c_s^2(\vec{\nabla}v)^2 - m_{\text{cosm}}^2v^2\right] \tag{D.13}$$

where

$$m_{\text{cosm}}^2 \equiv -\frac{z''}{z}. \tag{D.14}$$

It is easily to check that for all cases besides canonical field without potential $\mathcal{L}(\phi, X) \equiv X$

$$m^2_{\mathrm{cosm}} \neq m^2_{\mathrm{eff}}. \tag{D.15}$$

However, comparing the action (D.8) with (D.13) one arrives to conclusion that the cosmological perturbations propagate in the same metric (3.2), (3.7). Further one can introduce the notation $\overline{\delta\phi}$ for the sometimes so-called "scalar perturbations on the spatially flat slicing"

$$\overline{\delta\phi} \equiv \delta\phi + \frac{\phi'}{\mathcal{H}}\Psi. \tag{D.16}$$

For this scalar field the action for cosmological perturbations (D.13) takes the form

$$S_{\mathrm{cosm}} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-G}\,\left[G^{\mu\nu}\partial_\mu\overline{\delta\phi}\partial_\nu\overline{\delta\phi} - M^2_{\mathrm{cosm}}\overline{\delta\phi}^2\right], \tag{D.17}$$

thus the cosmological perturbations $\overline{\delta\phi}$ live in the emergent acoustic spacetime with the metric (3.2), (3.7). Similarly as we have calculated in (D.9) we have

$$M^2_{\mathrm{cosm}}\frac{\sqrt{-G}}{A^2} - \frac{A''}{A} = M^2_{\mathrm{cosm}}a^2\mathcal{L}_{,X}c_s - \frac{\left(\sqrt{\varepsilon_{,X}}a\right)''}{\sqrt{\varepsilon_{,X}}a} = -\frac{z''}{z} \tag{D.18}$$

after some algebra the last expression reduces to

$$\chi'' + 2\left(\frac{A'}{A}\right)\chi' + A^2\left(M^2_{\mathrm{cosm}}c_s^3\right)\chi = 0 \tag{D.19}$$

where we have introduced a new auxiliary field

$$\chi(\eta) \equiv \frac{\phi'}{\mathcal{H}} = \left(\frac{3}{8\pi G_N}\right)^{1/2}\sqrt{\frac{2X}{\varepsilon}}. \tag{D.20}$$

The equation (D.19) is in turn the Klein-Gordon equation

$$\left(\Box_{\overline{g}} + \left(M^2_{\mathrm{cosm}}c_s^3\right)\right)\chi = 0 \tag{D.21}$$

for the field χ in the metric $\overline{g}_{\mu\nu} \equiv A^2\eta_{\mu\nu} = \varepsilon_{,X}g_{\mu\nu}$ conformally related to the gravitational metric $g_{\mu\nu}$. Thus we have

$$M^2_{\mathrm{cosm}} = -c_s^{-3}\chi^{-1}\Box_{\overline{g}}\,\chi. \tag{D.22}$$

One can rewrite this formula in terms of the gravitational metric $g_{\mu\nu}$. Using the rules of the conformal transformations we have

$$\Box_{\overline{g}}\chi = \frac{1}{\sqrt{-\overline{g}}}\nabla_\mu\left(\sqrt{-\overline{g}}\overline{g}^{\mu\nu}\nabla_\nu\chi\right) = \frac{1}{\varepsilon^2_{,X}}\frac{1}{\sqrt{-g}}\nabla_\mu\left(\varepsilon_{,X}\sqrt{-g}g^{\mu\nu}\nabla_\nu\chi\right) = \tag{D.23}$$

$$= -\nabla^\mu\chi\nabla_\mu\varepsilon^{-1}_{,X} + \varepsilon^{-1}_{,X}\Box_g\chi \tag{D.24}$$

Thus the effective mass for cosmological perturbations $\overline{\delta\phi}$ is

$$M_{\mathrm{cosm}}^{2}=-c_{s}^{-3}\varepsilon_{,X}^{-1}\left(\sqrt{\frac{\varepsilon}{X}}\Box_{g}\sqrt{\frac{X}{\varepsilon}}+\nabla_{\mu}\ln\left(\varepsilon_{,X}\right)\nabla^{\mu}\ln\sqrt{\frac{X}{\varepsilon}}\right). \tag{D.25}$$

Note that in the case of canonical kinetic terms $\mathcal{L}(\phi, X) = X - V(\phi)$ the last expression for M_{cosm}^{2} the simplifies to

$$M_{\mathrm{cosm,canonical}}^{2}=-\left(w+1\right)^{-1/2}\Box_{g}\left(w+1\right)^{1/2}. \tag{D.26}$$

where $w = p/\varepsilon$ is the equation of state parameter. In particular for the universe filled with the massless canonical scalar field $M_{\mathrm{cosm,}} = 0$.

Appendix E

Green functions for a moving spacecraft

Here we calculate the retarded Green's function for a moving spacecraft in the case of three spatial dimensions. First we calculate the retarded Green's function in the preferred (rest) frame and then we perform the Lorentz boost (with the invariant speed c) for the solution. We compare the result with one obtained by the direct calculation of Green's function for the Eq. (3.35). We will need the following formulas from Ref. [240] p.750:

$$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\sin(cx) = \frac{\cos\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}}, \quad \text{for} \quad 0<b<c \tag{E.1}$$

$$= 0, \quad \text{for} \quad 0<c<b \tag{E.2}$$

$$\int_a^\infty J_0\left(b\sqrt{x^2-a^2}\right)\cos(cx) = -\frac{\sin\left(a\sqrt{c^2-b^2}\right)}{\sqrt{c^2-b^2}}, \quad \text{for} \quad 0<b<c \tag{E.3}$$

$$= \frac{\exp\left(-a\sqrt{b^2-c^2}\right)}{\sqrt{b^2-c^2}}, \quad \text{for} \quad 0<c<b \tag{E.4}$$

$$\int_0^a J_0\left(b\sqrt{a^2-x^2}\right)\cos(cx) = \frac{\sin\left(a\sqrt{c^2+b^2}\right)}{\sqrt{c^2+b^2}}, \quad \text{for} \quad 0<b \tag{E.5}$$

In the preferred frame the Green function is (see e.g. [164])

$$G_R^{\mathrm{rf}}(t,x) = \frac{\theta(t)}{2c_s\pi}\delta\left(c_s^2t^2-|x|^2\right). \tag{E.6}$$

Performing the Lorentz transformation $x=\gamma(x'+vt')$, $t=\gamma(t'+vx')$, where $\gamma = (1-v^2)^{-1/2}$ we find the Green function in the moving frame:

$$G_R^{\mathrm{rf}}(t',x') = \frac{\theta(t'+vx')}{2c_s\pi}\delta\left[\gamma^2\left(c_s^2(t'+vx')^2-(x'+vt')^2\right)-y^2-z^2\right]. \tag{E.7}$$

We need to calculate the Fourier transform to the function (E.7). It is convenient to shift x' as follows:

$$x' = \overline{x} - vt'\left(\frac{1-c_s^2}{1-c_s^2v^2}\right). \tag{E.8}$$

Then the argument of the delta-function in (E.7) can be rewritten as

$$\gamma^2\left(c_s^2\left(t'+vx'\right)^2-\left(x'+vt'\right)^2\right)-y^2-z^2=\alpha c_s^2 t'^2-\alpha^{-1}\overline{x}^2-y^2-z^2,$$

where

$$\alpha=\frac{1-v^2}{1-c_s^2 v^2}. \tag{E.9}$$

Now we are ready to proceed with the Fourier transform of (E.7):

$$G_R^{\mathrm{rf}}(t',k')=\frac{\mathrm{e}^{i\varphi}}{2c_s\pi}\int_{-\infty}^{\infty}d\overline{x}dydz\,\theta\left(t'+vx'\right)\,\delta\left(\alpha c_s^2 t'^2-\alpha^{-1}\overline{x}^2-y^2-z^2\right)\mathrm{e}^{ik_{x'}\overline{x}+ik_y y+ik_z z} \tag{E.10}$$

where we introduced the notation:

$$\varphi=-k_{x'}vt'\left(\frac{1-c_s^2}{1-c_s^2v^2}\right). \tag{E.11}$$

Step-function in the integral implies that the integration over $\overline{x}$ is made from x_* to $+\infty$:

$$G_R^{\mathrm{rf}}(t',k')=\frac{\mathrm{e}^{i\varphi}}{2c_s\pi}\int_{x_*}^{\infty}d\overline{x}\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\,\delta\left(\alpha c_s^2 t'^2-\alpha^{-1}\overline{x}^2-y^2-z^2\right)\mathrm{e}^{ik_{x'}\overline{x}+ik_y y+ik_z z}, \tag{E.12}$$

$$x_*=vt'\left(\frac{1-c_s^2}{1-c_s^2v^2}\right)-\frac{t'}{v}=-\frac{t'}{v}\left(\frac{1-v^2}{1-c_s^2v^2}\right)=-\frac{\alpha}{v}t'. \tag{E.13}$$

Introducing $r\equiv\sqrt{y^2+z^2}$, ϕ as the angle between the vectors $\{k_y,k_z\}$ and $\{y,z\}$ and $k_\perp\equiv\sqrt{k_y^2+k_z^2}$ we obtain:

$$G_R^{\mathrm{rf}}(t',k')=\frac{\mathrm{e}^{i\varphi}}{2c_s\pi}\int_{x_*}^{\infty}d\overline{x}\int_0^{\infty}drr\int_0^{2\pi}d\phi\delta\left(\alpha c_s^2 t'^2-\alpha^{-1}\overline{x}^2-r^2\right)\mathrm{e}^{ik_{x'}\overline{x}+ik_\perp r\cos\phi}. \tag{E.14}$$

Integrating over r first gives:

$$G_R^{\mathrm{rf}}(t',k')=\frac{\mathrm{e}^{i\varphi}}{4c_s\pi}\int_{x_*}^{+\infty}d\overline{x}\int_0^{2\pi}d\phi\exp\left(ik_{x'}\overline{x}+ik_\perp\sqrt{\alpha c_s^2t'^2-\alpha^{-1}\overline{x}^2}\cos\phi\right) \tag{E.15}$$

for

$$\alpha c_s^2 t'^2-\alpha^{-1}\overline{x}^2>0, \tag{E.16}$$

otherwise it is zero. Integrating (E.15) over ϕ we find:

$$G_R^{\mathrm{rf}}(t',k')=\frac{\mathrm{e}^{i\varphi}}{2c_s}\int_{x_*}^{+\infty}d\overline{x}J_0\left(k_\perp\sqrt{\alpha c_s^2t'^2-\alpha^{-1}\overline{x}^2}\right)\exp\left(ik_{x'}\overline{x}\right), \tag{E.17}$$

where $J_0(x)$ is the Bessel function of the zeroth order. Now we need to integrate the expression (E.17) taking into account the condition (E.16).We consider two cases separately: the case of slow spacecraft, $v^2c_s^2 < 1$ ($\alpha > 0$), and the case of rapid spacecraft, $v^2c_s^2 > 1$ ($\alpha < 0$).

For the slow spacecraft we easily obtain from (E.17) and (E.16):

$$\begin{aligned} G_R^{\mathrm{rf}}(t',k') &= \frac{\mathrm{e}^{i\varphi}}{2c_s}\theta(t')\int_{-\alpha c_s t'}^{\alpha c_s t'} d\overline{x}\, J_0\left(k_\perp\sqrt{\alpha c_s^2 t'^2 - \alpha^{-1}\overline{x}^2}\right) e^{ik_{x'}\overline{x}} \\ &= \frac{\mathrm{e}^{i\varphi}}{c_s}\theta(t')\int_{0}^{\alpha c_s t'} d\overline{x}\, J_0\left(\frac{k_\perp}{\sqrt{\alpha}}\sqrt{\alpha^2 c_s^2 t'^2 - \overline{x}^2}\right)\cos\left(k_{x'}\overline{x}\right). \end{aligned}$$

Using (E.5) we then find the Green's function for slow moving spacecraft:

$$\begin{aligned} G_R^{\mathrm{rf}}(t',k') &= -\theta\left(t'\right)\frac{i\mathrm{e}^{i\varphi}}{2c_s\sqrt{k_{x'}^2 + k_\perp^2/\alpha}}\left(\mathrm{e}^{i\alpha c_s t'\sqrt{k_{x'}^2+k_\perp^2/\alpha}} - \mathrm{e}^{-i\alpha c_s t'\sqrt{k_{x'}^2+k_\perp^2/\alpha}}\right) \\ &= \theta\left(t'\right)\frac{1}{2ic_s}\left(k_{x'}^2 + k_\perp^2\frac{1-c_s^2v^2}{1-v^2}\right)^{-1/2}\left(\mathrm{e}^{i\omega_+ t'} - \mathrm{e}^{i\omega_- t'}\right). \end{aligned} \tag{E.18}$$

In the case of rapid spacecraft, $v^2c_s^2 > 1$ ($\alpha < 0$), one can verify that $\alpha^2 c_s^2 t'^2 > x_*^2$ for any t'. Thus (E.17) along with (E.16) can be rewritten as:

$$G_R^{\mathrm{rf}}(t',k') = \frac{\mathrm{e}^{i\varphi}}{2c_s}\int_{|\alpha c_s t'|}^{+\infty} d\overline{x}\, J_0\left(\frac{k_\perp}{\sqrt{|\alpha|}}\sqrt{\alpha^2 c_s^2 t'^2 - \overline{x}^2}\right)\left(\cos\left(k_{x'}\overline{x}\right) + i\sin\left(k_{x'}\overline{x}\right)\right). \tag{E.19}$$

Using (E.2) and (E.4) for $k_\perp^2 > |\alpha|\, k_{x'}^2$ and (E.1) and (E.3) for $k_\perp^2 < |\alpha|\, k_{x'}^2$ we obtain in both cases:

$$G_R^{\mathrm{rf}}(t',k') = \frac{\mathrm{e}^{i\varphi}}{2c_s}\frac{\exp\left(-\left|\alpha c_s t'\right|\sqrt{k_\perp^2\left|\alpha\right|^{-1} - k_{x'}^2}\right)}{\sqrt{k_\perp^2\left|\alpha\right|^{-1} - k_{x'}^2}}. \tag{E.20}$$

The last expression can be written as

$$G_R^{\mathrm{rf}}(t',k') = -\frac{1}{2ic_s}\left(k_{x'}^2 + k_\perp^2\frac{1-c_s^2v^2}{1-v^2}\right)^{-1/2}\left(\theta\left(t'\right)\mathrm{e}^{i\omega_+ t'} + \theta\left(-t'\right)\mathrm{e}^{i\omega_- t'}\right). \tag{E.21}$$

Thus the modes propagating in with

$$k_\perp^2 > k_{x'}^2\left|\alpha\right| = k_{x'}^2\left(\frac{1-v^2}{c_s^2v^2-1}\right) \tag{E.22}$$

are exponentially suppressed. The singular directions $k_\perp^2 = k_{x'}^2\left|\alpha\right|$ are unphysical because they have measure zero in the integral. This directions correspond to the sufficient but integrable singularities in the Green function.

If the Green's function is calculated directly from the Eq. (3.35) by means of standard approach then one can find, that the solution is:

$$G_R^{\mathrm{sc}}(t', k') \;=\; \theta\left(t'\right) \frac{1}{2ic_s}\left(k_{x'}^2 + k_\perp^2 \frac{1-c_s^2 v^2}{1-v^2}\right)^{-1/2}\left(\mathrm{e}^{i\omega_+ t'} - \mathrm{e}^{i\omega_- t'}\right), \tag{E.23}$$

which coincides with the Green's function (E.18) we calculated by applying the Lorentz transformation to the rest Green's function in the case of slow motion. Note, however, that the results differs for the case of fast moving spacecraft - compare (E.23) and (E.21). The function $G_R^{\mathrm{sc}}(t', k')$ from (E.23) contains exponentially growing modes for sufficiently high $k_\perp$, while correct way of calculation gave us a sensible result (E.21) - it contains only exponentially suppressed modes. This makes sense because the late time solution approaches the free wave which do not contain these high $k_\perp$.

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