Background geometries in string and M-theory

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Munich 2005

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Dissertation
an der Fakultät für Physik
Ludwig-Maximilians-Universität München

vorgelegt von
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aus Marktoberdorf

München, den 14. Oktober 2005

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Datum der mündlichen Prüfung: 13.12.2005

Zusammenfassung

In dieser Arbeit betrachte ich Geometrien von Hintergründen, welche aus Kompaktifizierungen der Stringtheorie abgeleitet werden können. Im besonderen untersuche ich mittels klassischen und verallgemeinerten G-Strukturen supersymmetrische Vakuum-Räume von Supergravitationstheorien und topologisch getwisteten Sigma-Modellen.

Im ersten Teil kompaktifiziere ich aus phenomenologischen Gründen die 11d Supergravitation auf sieben-dimensionalen Mannigfaltigkeiten. Eine bestimmte Anzahl von Supersymmetrien zwingen den internen Hintergrund eine klassische $SU(3)$- or G_2-Struktur zu tragen. Im besonderen Fall eines vier-dimensionalen maximal symmetrischen Raumes und eines vier-Form Flusses berechne ich die Beziehung zur intrinsischen Torsion.

Der Hauptteil gliedert sich in zwei Abschnitte: Als erstes stelle ich fest, daß die verallgemeinerten Geometrien auf sechs-dimensionalen Mannigfaltigkeiten eine natürliche Umgebung bieten, um T-Dualität und Mirror-Symmetrie zu studieren. Dies gilt vor allem, wenn auch das B-Feld involviert ist. Ich gebe eine explizite Mirror-Abbildung an und wende diese Idee auf das topologisch getwistete Sigma-Modell an, welches verallgemeinert formuliert wird. Verschiedene Studien zur Mirror-Symmetrie hinsichtlich der Observablen und der topologischen A- und B-Branen werden gemacht.

Als zweites zeige ich, daß sieben-dimensionale NS-NS Hintergründe der Type II Supergravitationstheorien mit verallgemeinerten G_2-Strukturen beschrieben werden können. Eine Kompaktifizierung auf sechs-dimensionale Mannigfaltigkeiten führt zu einer neuen Struktur. Ich nenne diese Struktur eine *verallgemeinerte $SU(3)$-Struktur*. Ich untersuche die Beziehung zwischen verallgemeinerten $SU(3)$- und G_2-Strukturen auf sechs- und sieben-dimensionalen Räumen und generalisiere die Hitchin-Fluss-Gleichungen. Zum Schluss zeige ich, wie man das bekannte Variationsprinzip für verallgemeinerte $SU(3)$- und G_2-Strukturen mittels Zwangsbedingungen weiterentwickeln kann und dadurch die verbleibenden physikalischen R-R Felder beschreibt.

Abstract

In this thesis we consider background geometries resulting from string theory compactifications. In particular, we investigate supersymmetric vacuum spaces of supergravity theories and topological twisted sigma models by means of classical and generalised G-structures.

In the first part we compactify 11d supergravity on seven-dimensional manifolds due to phenomenological reasons. A certain amount of supersymmetry forces the internal background to admit a classical $SU(3)$- or G_2-structure. Especially, in the case that the four-dimensional space is maximally symmetric and four form fluxes are present we calculate the relation to the intrinsic torsion.

The second and main part is two-fold. Firstly, we realise that generalised geometries on six-dimensional manifolds are a natural framework to study T-duality and mirror symmetry, in particular if the B-field is non-vanishing. An explicit mirror map is given and we apply this idea to the generalised formulation of a topological twisted sigma model. Implications of mirror symmetry are studied, e.g. observables and topological A- and B-branes.

Secondly, we show that seven-dimensional NS-NS backgrounds in type II supergravity theories can be described by generalised G_2-geometries. A compactification on six manifolds leads to a new structure. We call this geometry a *generalised $SU(3)$-structure*. We study the relation between generalised $SU(3)$- and G_2-structures on six- and seven-manifolds and generalise the Hitchin-flow equations. Finally, we further develop the generalised $SU(3)$- and G_2-structures via a constrained variational principle to incorporate also the remaining physical R-R fields.

Acknowledgements

I want to acknowledge the Graduiertenkolleg "Strukturuntersuchungen, Präzessionstests und Erweiterungen des Standardmodells der Elementarteilchenphysik" at Humboldt Universität zu Berlin and Max Planck Institut für Physik in Munich for financial support.

I want to thank Helga Baum (Humboldt Universität zu Berlin) who introduced me to spin geometry and further thanks go to Thomas Friedrich (Humboldt Universität zu Berlin) who taught me a lot about G-structures.

I am thankful to my PhD fellows with which I had enlightening discussions over various topics of string theory and life. Thanks go to Stefan Stieberger, Ralph Blumenhagen and Dimitrios Tsimpis for sharing with me their huge experience in string theory. Moreover, I am indebted to my collaborators Klaus Behrndt, Stefano Chiantese, Florian Gmeiner and especially to Frederik Witt who explained me the idea of generalised geometries. Our frequent Cafe visits are fond memory.

Special thanks go to Klaus Behrndt, Florian Gmeiner, Timo Weigand and Frederik Witt for proofreading the thesis.

I am very grateful to Melanie Zauner who made my PhD period richer.

Finally, I want to express my deep gratitude to my supervisor Dieter Lüst who never failed to support me and for his continuous interest in the progress of my work.

Contents

Chapter 1

Introduction

1.1 Preliminaries

The present Standard Model of particle physics is based on fundamental particles that come in two classes - quarks and leptons. Those can be gathered into three families and the interactions can be described by gauge bosons that are associated to the Standard Model group $SU(3)\times SU(2)\times U(1)$. This model agrees very well with a huge amount of experiments. However, there are seminal remaining questions. Why do there exist precisely three families? Can we explain the various parameters of the Standard Model and why do the parameters have just these measured values? Another fundamental question concerns the gravititional interaction that is not captured by the Standard Model. Gravity is described by general relativity, which is a classical theory in the sense that it has no adequate quantum mechanical extension. Furthermore, general relativity faces the problem of singularities. And since the theory is based on the geometrical idea of manifolds it cannot deal with them. Thus, discussing particle scenarios nearby a black hole, where the space-time is highly curved, it becomes obvious that the Standard Model and general relativity should be treated within one consistent framework.

The most promising theory up to know that incorporates all four fundamental forces is called *string theory* [70, 88, 42, 43, 75, 76]. This setup is based on the assumption that the fundamental objects are no longer point particles as in the Standard Model but one-dimensional extended objects, the so-called *strings*. This generalisation, however, leads to a model that deals from the beginning with a lot of additional properties. One new feature is that strings can be open and closed, i.e. the ends for a closed string are glued together, and reflects that strings can sweep out topologically non-trivial two-dimensional surfaces in space-time. At low energies we find the graviton of spin two that realises the gravitational interaction. By taking the possible field theory limit, the length of the strings can be neglected and physics can again be described by a usual quantum field theory.

After quantising the string it turns out that the model has to be formulated in ten space-time dimensions. Moreover, the procedure of building an adequate supersymmetric string theory is not unique and therefore five superstring theories can be consistently formulated, namely type IIA, type IIB, type I and two heterotic theories. Those can be non-trivially related to each other by duality maps [78].

Everyday life experience tells us that our real world should be modeled by four large dimensions only. On the other side string theory predicts space-time to be ten-dimensional. However, this is not in contradiction since we did not specify the space-time to have ten large dimensions. We resolve this problem by allowing for a large four-dimensional space and assume the remaining six-dimensional space to be invisibly tiny. This assumption is called *compactification* and, unfortunately, we have to do it by hand, i.e. it is not proven if string theory does prefer specific spaces. We are faced even with a further problem, because the amount of supersymmetry in four dimensions cannot be derived from string theory and has to be assumed in addition. With respect to the present status of measurements we do not have supersymmetry in our real world and can argue that it is in principle realised at higher energies and broken at enegies relevant for our life. We therefore mostly assume the presence of minimal supersymmetry in four dimensions. Furthermore, we first investigate the so-called *vacuum* or *background*, which is a supersymmetric manifold that allows only bosons. So, we are interested in phenomenological motivated backgrounds, i.e. we compactify the ten-dimensional space-time and its bosons to obtain a four-dimensional space that admits a minimal amount of supersymmetry.

The first approach to tackle this problem was done in 1981 by Witten [85]. He realised that the amount of supersymmetry in the external four-dimensional space can be captured by investigating the supersymmetry variations of supergravity - being the low energy effective description of string theory. He splitted the ten-dimensional variations into an external and an internal part and found out that the external supersymmetry is governed by the number of internal covariantly constant spinors. In 1985 Candelas, Horowitz, Strominger and Witten [18] compactified the supergravity descriptions of the heterotic theories on six-dimensional spaces. Assuming minimal supersymmetry in the external space and setting all internal physical fields to zero lead to internal spaces with $SU(3)$ holonomy - Calabi-Yau manifolds.

1.2 Classical geometries

Strominger [80] developed the idea further (see also [59]) by focusing on the space-time supersymmetry variations of the appropriate action which are formulated via spinors. Supersymmetry can be achieved if the variations vanish. Additionally, he took all NS-NS bosons into account, i.e. the internal metric g, the dilaton function ϕ and the 3-form flux H (neglecting the gaugino condensate). These objects appear as extra terms in the two supersymmetry variations which are relevant for the vacuum. The first variation is a Killing spinor equation where the metric corresponds to the Levi Civita connection whereas H can be understood as a modification. The second variation is a differential condition for the dilaton ϕ. Using spin geometry Strominger found that the additional H term can be captured by the torsion tensor. With the idea of torsion and the supersymmetry variations one can search for background solutions that allow for a specific amount of supersymmtry in the external space.

It is this claim for supersymmetry that constraints, in general, the internal physical fields. Gauntlett et al. [39, 38] and Cardoso et al. [20] realised for heterotic and type II theories that the participating internal fields can be characterised by means of G-structures (see also [34]). This mathematical tool allows to adjust the number of parallel spinors with respect to a special group G and considers the physical objects as representations of this group. If one assumes $G = SU(3)$ and that all physical fields but the metric vanish the structure becomes torsion-free and brings us back to the abovementioned Calabi-Yau spaces. In this thesis we refer to G-structures as classical structures. I.e. the parallel transport of all objects on the manifold is governed by a single connection that is associated to the structure group G.

In 1978 Cremmer, Julia and Scherk [25] discoverd a further supergravity theory that is called *11d supergravity* and is formulated on an 11-dimensional space-time manifold. Since the graviton, which realises the gravitational interaction, has spin two and the fact that we do not allow for higher spin fields, the dimension of the supergravity theory can at most be eleven. This theory, as it is believed today, appears as the low energy limit of the mostly unknown fundamental M-theory. Since the five known string theories are only defined in the perturbative regime they should also appear as different limits of the underlying M-theory.

In chapter 2 of this thesis we apply the above ideas to 11d supergravity and study compactifications on seven-dimensional spaces, such that the corresponding external space is four-dimensional. The field content of the considered theory is given by a metric, a gravitino (spin 3/2) and a 3-form potential with associated 4-form field strength. The supersymmetry variations for all fields are given in [25]. Since we are interested in the vacuum only the bosons are present and thus only the gravitino variation remains. According to Witten's idea [85] the amount of supersymmetry in four dimensions can be adjusted by the number of internal spinors. We adopt the idea of G-structures and study supersymmetric compactifications on seven dimensions, or in other words, phenomenologically interesting background manifolds.

The compactification program for of 11d supergravity was first done by Candelas and Raine [19]. They chose the eleven-dimensional space-time to be a direct product of Minkowski space an internal space and allowed for internal 4-from fluxes. This assumptions turned out to be very restrictive and force the internal four form flux to vanish. Thus, the Levi Civita connection in the internal supersymmetry variation is not modified and should parallel transport precisely one spinor in order to preserve minimal supersymmetry in the external space. These seven-dimensional spaces are the spaces analogous to Calabi-Yau manifolds in dimension six and are called *G_2-manifolds*. The structure is torsion-free and $G = G_2$.

After the idea of G-structures was introduced in string theory [39, 38, 20, 34] it became clear that one can even handle, in principal, internal 4-form fluxes in 11d supergravity. An obvious next step to obtain these fluxes is that some of the assumptions made bei Candelas and Raine can be relaxed. Work in this direction was done by e.g. [68, 14, 5, 1, 26, 27, 2, 7, 71], see also [6, 29, 30, 28]. In chapter 2 we follow [8, 9, 10, 11] and compactify in the first part the gravitino variations where we introduce additional objects: a warp function, a one-parameter external 4-form flux and allow the external space to be maximally symmetric, e.g. Minkowski, dS or AdS. Also direct and non-direct spinor decompositions are taken into account.

Considerations of this setup lead to algebraic constraints and relations between the involved fields in the external part of the gravitino variation. The internal part, however, does not result in the obstruction that all participating fields, as for instance the 4-form flux, have to vanish. We implement the external algebraic constraints into the internal Killing spinor equation and provide solutions by using G-structures.

Before we discuss explicit solutions we give an introduction to G-structures. This theory deals with two pictures. The *spinor picture* that also appears in the supersymmetry variations and the equivalent *form picture*. In principle, the group G is the structure group and measures roughly how "flat" the manifold is, i.e. how many spinors are parallel. Once the group is fixed and only the metric as a physical field is present the object that is responsible for the parallel transportation is the Levi Civita connection. In the case where the torsion tensor is non-vanishing the Levi Civita connection gets modified and the additional torsion must be decomposed into G-representations in order to provide that the spinors are still parallel. The G-structure allows to measure torsion, e.g. fluxes, by means of torsion classes that are G-representations in the form picture. It is important to note that this does not mean that we only have to decompose fluxes into G-representations within the spinor picture and compare if the same representations appear as a torsion class in the form picture. This is in general not true and therefore we have to translate the spinor picture into the form picture via fierzing to make a serious comparison. We review useful results of G_2- and $SU(3)$-structures.

We first discuss $G = G_2$ and get again hard obstructions since only the external 4-form flux parameter combined with a non-flat four-dimensional space can appear in the torsion class. In mathematical terms, the background admits a weak G_2-structure (see also [14]). Using the compactification method we usually decompose the supersymmetry parameter, that is a space-time spinor, into a direct product. An expansion of the internal part into invariant G_2-structure forms is done but it is proven that the space of solutions cannot be extended in this way.

Furthermore, we give some unpublished results, where we only consider the metric in the internal Killing spinor equation, i.e. we start with a G_2-manifold. The authors in [69] showed that the Levi-Civita connection of this manifold can get modified by α'-corrections since the curvature gets modified by these corrections. We prove that the modifications of the new connection can be fully classified via G_2 torsion classes. The result is that α'-corrections will never allow for weak G_2-manifolds but all remaining three classes pick up contributions and, moreover, the resulting G_2-structure becomes non-integrable in general.

For the case of a $G = SU(3)$-structure, which we consider as two perpendicular G_2-structures on the internal manifold, supersymmetry allows for a richer physical background. In this framework, besides a non-vanishing warp-factor and general external spaces, also the internal fluxes are present. The participating fields are classified within G_2-structures but it turns out that it is more convenient to characterise them directly by $SU(3)$-torsion classes, e.g. [26].

1.3 Generalised geometries and mirror symmetry

We now introduce a new mathematical structure and show its physical relevance by discussing T-duality, mirror symmetry, twisted topological models and type II supergravity theories.

One of the key statements we learned from G-structures is that we can understand flux modifications of the Levi Civita connection in terms of torsion classes. This picture is obviously useful but a couple of questions still remain, which we address in the following. In general, torsion is a $(1, 2)$-tensor, i.e. an object in the space $\Lambda^1 \otimes \Lambda^2$. The H-flux, however, is a 3-form and we manage the correspondence to the torsion tensor by choosing the latter totally skew symmetric. Taking as a next step also the remaining R-R bosons into account [13], which are given by forms of all degrees, we also have to measure them with the torsion tensor. This can be done in principle, since the G-structure measures only G-representations of the R-R fields that fit into torsion classes. But this would immediately imply from a G-structure point of view that the torsion classes are blind with respect to their origin, i.e. they treat NS-NS fluxes and R-R fluxes on equal footing. This point of view is too rough from a physical perspective since physics does distinguish the NS-NS and R-R sectors. We come back and resolve this puzzle within a new framework later on.

Furthermore, we know that different string- and supergravity theories are connected to each other by duality maps. By means of G-structures we can characterise the background manifolds of e.g. type IIA and type IIB and, moreover, these backgrounds should correspond to each other via *T-duality* or *mirror symmetry* [58]. This duality is geometrical. We note that the T-duality rules (or Buscher rules) are known, see e.g. [64], but not geometrically well understood. From Strominger, Yau and Zaslow [81, 58] we know that *mirror symmetry is T-dualtiy* and that the two algebraic structures, the complex- and the symplectic structure, on the two involved spaces get exchanged. Note that mirror symmetry works for T^3-fibred manifolds where T-duality can act on. The complex structure is characterised via the holomorphic $(3,0)$-form. In [32] it was shown how to mirror dualise the NS-NS backgrounds of type IIA to type IIB in a two step process: firstly, characterise the background by a G-structure and, secondly, apply the T-duality rules. In other words, the G-structure is not powerful enough to provide maps that transform backgrounds to each other and keep the property of the G-structure. In view of mirror symmetry, the demanded map must exchange complex- and symplectic structures.

We will briefly sketch why G-structures will not motivate dualities. In principle, the real part of a holomorphic $(3,0)$-form reduces the structure group to $SL(3,\mathbb{C})$, whereas the symplectic structure reduces the structure group to $Sp(6,\mathbb{R})$. Relating the two structures algebraically reduces the structure group to the common group $SU(3)$. Let us use this understanding to focus once more on mirror symmetry. The mirror map transforms the $SU(3)$-structure, viewed as the intersection of two structures, to its mirror $SU(3)$-structure, where the involved holomorphic $(3,0)$-form and the symplectic form get exchanged. This implies that the duality map should act on the complex- and symplectic structure in a setup where they are considered on equal footing. However, the G-structures cannot achieve this, because they treat them as different. This demanded "generalisation" is provided by the so-called *generalised structures* which goes back to the seminal work of Hitchin [55] and will us occupy us next.

The basic idea in the author's articles [60, 23] is to work within the generalised structures to find a geometrical description of mirror symmetry and verify it by applications. In mathematics G-structures were introduced in the problem of parallel transporting a tangent vector, i.e. an element in T. The group $G \subset SO(n)$ measures the difference of the initial and final vector when parallel transported along a closed curve on an oriented n-dimensional manifold. We now want to parallel transport elements in the space $T \oplus T^*$. In an informal way we can think about this as a "doubling" of the tangent space. And since this space has a natural metric of signature (n,n) the structure group will be $SO(n,n)$. This group has dimension $2n$ and its Lie algebra decomposes into three different pieces, where one part is given by a 2-form. This intrinsic 2-form, as one can show, behaves like the B-field given in string theory. We therefore call the mathematical 2-form the *B-field b*. Moreover, the B-field has an even more special property. In the generalised structure the Lie-bracket, being diffeomorphism invariant, gets substituted by the Courant bracket [55, 57], which is even invariant under B-field transformations. This implies that, once we found an integrable structure, a further action of the B-field will not violate integrability.

We introduce some more refinements. It is also possible to endow the generalised manifold with a generalised metric G on $T \oplus T^*$ [45], which is completely characterised by the Riemannian metric and the B-field. This generalised metric decomposes $SO(n, n)$ to $SO(n) \times SO(n)$ and, moreover, the space $T \oplus T^*$ decomposes in a direct product of two n-dimensional spaces, a space- and light-like . An even dimensional generalised manifold also allows for a *generalised complex structure* (GCS) $\mathcal{J}$, which is a map from $T \oplus T^*$ to $T \oplus T^*$ and has the properties of being complex *and* symplectic. It reduces the structure to $U(\frac{n}{2}) \times U(\frac{n}{2})$. A *generalised Kähler structure* (GKS) admits two GCSs that are compatible with the generalised metric. Naturally, a usual Kähler structure (g, J, ω) can be discussed in the generalised picture and is a special case of a GKS, where g is the metric, J is the complex structure and ω is the symplectic form. We embed these by using matrices that act on $T \oplus T^*$,

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^T \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}. \tag{1.1}$$

A generalised Calabi-Yau metric admits an $SU(\frac{n}{2}) \times SU(\frac{n}{2})$-structure.

We investigate mirror symmetry by embedding a usual Calabi-Yau structure into the GKS. Since mirror symmetry uses T^3-fibred spaces we discuss in section 3.2.1 the example of a six-torus. The key idea is to split the indices of the participating structures into base and fibre type, i.e. the $(2, 2)$-matrices in (1.1) become $(4, 4)$-matrices. We define a mirror map $\mathcal{M}$ such that it acts only in the fibre and interchanges the T-fibre part with the T^*-part of the structures. This might seem simple, but it is precisely this action which we prove to exchange the complex with the symplectic structure. By also involving the B-field, which can be done by matrix multiplication, the GCSs become more complicated but the mirror map $\mathcal{M}$ acts in the same way and the results are the following: On the mirror side the B-field vanishes and it is completely absorbed in the Riemannian mirror metric. We get as a result that the mirror complex- and symplectic structures are precisely determined by the initial structures that resolve a puzzle in the literature [46]. The mirror metric is exactly the same as the T-duality rules [64] demand. We verify this by using the more involved pure spinor line picture. The idea of the mirror map $\mathcal{M}$ suggests that this procedure is true for more complicated manifolds and has been proven by the mathematician Ben-Bassat [12]. The basic result is that generalised structures provide us with a framework where we can have G-structure and *also* duality properties.

We use the generalised structures and show that they are a natural framework for twisted topological models. The six-dimensional target space geometry for an $\mathcal{N} = (2, 2)$ non-linear sigma model admits a bi-hermitian structure, i.e. it allows for two independent almost complex structures, as was proven by Gates, Hull and Rocek [37] in 1984. This model investigates all possible embeddings of the two-dimensional string into the target space. The fields which appear in the action are the bosonic embedding coordinate X and two other objects $\psi_\pm$, which are spinors on the worldsheet, and, additionally, carry a target space index. The two complex structures are associated to ψ_+ and ψ_-. Due to the lack of understanding of the general bi-hermitian structure, the two complex structures were identified and the resulting Kähler manifolds were discussed over a long time in the literature. Based on that, Witten [86] twisted the world-sheet fermions with the axial/vector R-currents and observed that the "fermions" become world-sheet scalars and 1-forms. He furthermore considered only the scalars and realised that the two twists reduce the model to two topological subsectors called *A- and B-model*. Kapustin discussed in [65] T-duality properties of the two topological models and found out that it is necessary to allow for two independent structures. He treats them in the sense of GCSs in a generalised description. It was proven by Gualtieri [45] that bi-hermitian structures are equivalent to GKSs.

Motivated by the seminal work of Kapustin [65](see also [66]) we adopt the idea and discuss the twisted A- and B-model [86] on general target spaces, where we use GKSs. Kapustin discussed in [65] the B-model and we extend this to the A-model and call them *generalised topological A- and B-model* [23]. Guided by the mirror symmetry results for GKS [60, 12] we discuss the mirror map $\mathcal{M}$ on T^6 for GCSs, which is characterised by two independent complex structures [45]. Gualtieri proved that the $T \oplus T^*$ bundle of a GKS decomposes under the two GCSs into four subbundles. Since the mirror map acts on these subbundles it is shown how sections of these bundles transform, which gives a deeper understanding of the properties of mirror symmetry [23]. The two GCS of the GKS gets interchanged by $\mathcal{M}$ [12]. Having this at hand we mirror transform the observables, the instantons and the generalised first Chern class and make contact with the literature. Note, it is argued that the observables are elements in a BRST-complex. This complex is isomorphic to a generalised complex which is governed by a Lie-algebroid structure (see also [65]). Zabzine [87] formulated the topological A- and B-branes by means of GCSs and we show that the mirror map $\mathcal{M}$ transforms them into each other [23]. Furhermore, we allow for a 2-form F on the brane. In case the brane has at least a two-dimensional fibre part, we realise that the fibre part of F gets mapped via $\mathcal{M}$ to a new object - a bi-vector - that could suggest a non-commutative structure. It is also valuable to mention that there exists, apart from the know A-branes, also co-isotropic branes [67] and isotropic branes [22].

It was suggested [31] that the target space of Witten's B-model can equivalently be captured by the classical Hitchin functional [53], that defines at its critical point a holomorphic $(3,0)$-form sitting inside a certain cohomology class. The authors [74] proved for the B-model, that the genus one free energy *does not* coincide with the one-loop free energy of the quantised classical Hitchin functional. They checked this by evaluating the Ray-Singer torsion for the quantised classical Hitchin functional and compared it to the known result from the B-model. Furthermore, they substituted the classical Hitchin functional by the generalised Hitchin functional [55], quantised it and realised that the one-loop free energy matches precisely the known B-model results. This result not only suggests that generalised geometries are preferable and convenient, it proves that these geometries are even necessary at 1-loop level.

1.4 Generalised geometries and type II theories

We argue that generalised structures are the proper setup to characterise supersymmetric type II supergravity backgrounds. The low energy limit of type II string theories is given by type IIA and type IIB supergravity. A supersymmetric vacuum background for type II theories consists of bosons that come in two flavours; the NS-NS fields and the R-R fields. To the former class belong the metric g, the B-field b and the scalar field ϕ - the dilaton. The latter class is given by differential forms of degree p, the R-R fields F^p, which satisfy the duality relation $F^p = \star\widehat{F^{10-p}}$, where $\wedge$ is a sign changing operator depending on p. The space-time supersymmetry is realised by two supersymmetry parameters $\varepsilon_{1,2}$, which are independent spinors and non-physical.

The theory is said to be of type IIA or IIB, if $\varepsilon_{1,2}$ are of opposite or equal chirality. The background is supersymmetric if these spinors satisfy additional conditions, namely the vanishing of the supersymmetry variations. In the so-called *democratic formulation* of Bergshoeff et al. [13], the vanishing of these is equivalent to

$$\begin{aligned} 0 &= \big(\nabla_X + \frac{1}{4}X \lrcorner H \cdot \mathcal{P}\big)(\varepsilon_1, \varepsilon_2) + \frac{1}{16}e^{\phi}F^{ev,od} \cdot X \cdot \mathcal{P}^{ev,od}(\varepsilon_1, \varepsilon_2) \\ 0 &= (d\phi \cdot + \frac{1}{2}H \cdot \mathcal{P})(\varepsilon_1, \varepsilon_2) \mp \frac{1}{8}e^{\phi}\Big(5F^{ev,od} - \sum_{p=ev,od} pF^p\Big) \cdot \mathcal{P}^{ev,od}(\varepsilon_1, \varepsilon_2), \end{aligned} \tag{1.2}$$

where H is the field strength of the B–field and $\mathcal{P}$ and $\mathcal{P}^{ev,od}$ are certain projection operators. Note that in type IIA, only R-R–fields of even degree are present, while in type IIB they are odd.

We compactify type II theories on seven-dimensional spaces to find backgrounds that describe solutions of wrapped NS5-branes as was shown by Gauntlett et al. [38]. The explicit compactification is done in absence of R-R fields, so only the NS-NS fields occur. We assume the following: the ten-dimensional space-time is a direct product of a three-dimensional Minkowski space and a seven-dimensional internal space, the H-field and the dilaton live only on the internal space, the internal space allows for two independent spinors only. Requiring that the vanishing of the supersymmetry variations (1.2) (see also [50, 51, 52]) vanish leads only to internal equations, and we realise in [61], that they are equivalent to a generalised G_2-structure with structure group $G_2 \times G_2$. Witt introduced these structures in [84, 83], showed that all NS-NS fields appear and proved that they completely characterise the so-called *topological data*. The generalised G_2-structures arise as critical points of a purly topological variational principle in dimension seven. The resulting critical points realise the integrability conditions in the *form picture*. The structure forms are of mixed degree which is in contrast to structure forms for G-structures. Moreover, Witt proved a one-to-one correspondence of the integrability conditions to a *spinor picture*. The special feature is that the two spinors are independent and can even coincide over a subset of the manifold which is not allowed in terms of G-structures. Moreover, the property of independence is essential and thus each spinor reduces the structure group independently which, roughly, results in the structure group $G_2 \times G_2$. It is precisely this spinor picture that is identical with the internal supersymmetry conditions. So, with the physical assumptions given above, we completely characterise the topological data of the internal background. Alternatively we can say that the realisation of supersymmetry is achieved if the form picture is integrable. Note that the internal supersymmetry variations already appeared in Gauntlett et al. [38] but there the solution was given within classical G-structures. We show that these solutions appear as a special case within the generalised G_2-structures, namely, by forcing the two spinors to be globally orthogonal the structure becomes effectively a classical $SU(3)$-structure.

Next we review and explain important facts about generalised G_2-structures [84, 83]. Since we learned that a compactification on seven manifolds leads to a generalised structure we do afterwards an analogous compactification on six-dimensional manifolds. The resulting internal supersymmetry variations look similar to the seven-dimensional case but there was no comparable generalised structure for this six-dimensional case in the mathematical literature. We argued that the proofs of Witt [84, 83] can be adjusted to the case of a $SU(3) \times SU(3)$-structure group and define what we call a *generalised* $SU(3)$*-structure* [61]. Having the generalised $SU(3)$- and G_2-structures in dimension six and seven at hand we study their relation. With the inclusion $SU(3) \subset G_2$ it is possible to study classical $SU(3)$- and G_2-structures on six- and seven-dimensional manifolds [24]. Motivated by this we realise the inclusion $SU(3) \times SU(3) \subset G_2 \times G_2$ and investigate the relation between generalised $SU(3)$- and G_2-structures in dimension six and seven. After we discuss the general embedding we specialise to the following example. A six-dimensional manifold is endowed with a generalised $SU(3)$-structure and we let it flow over a line to obtain on the seven-dimensional total space a generalised G_2-structure. This generalises the so-called *Hitchin flow equations* [53].

It is also interesting to consider D-branes and we give here unpublished results. Within a pure geometrical description these D-branes can be viewed as submanifolds and in case the space admits a certain G-structure, the consistent submanifolds are not arbitrary. By assuming to have supersymmetric branes it is well-known that they have to wrap calibrated cycles. In case of a classical G_2-structure the supersymmetric three-dimensional submanifolds can be calibrated by the G_2 three-from φ [49, 63]. We will sketch how D-branes can be calibrated in the presence of a generalised G_2-structure. The method we develop is perfectly general and does not rely on the specific dimension. The idea is that a generalised G_2-structure admits two G_2 three forms $\varphi_\pm$ which can, naively, calibrate the same three-dimensional submanifold differently. We prove that this is not the case and show explicitly that the two calibrations, when restricted to the submanifold, can be related to each other via the physical gluing matrix R. This gluing matrix can even be understood as an object that defines the Dbrane. Strictly speaking, our result gives a conditon on R with respect to $\varphi_\pm$ and thus singles out calibrated D-branes. If a Dbrane is calibrated with respect to, say φ_+, a calibration with the pullback $R^*\varphi_-$ *must* be identical.

We showed that the internal space and their NS-NS fields of type II backgrounds can be completely characterised by generalised $SU(3)$- and G_2-structures. But what about the remaining R-R fields that additionally appear in the supersymmetry variations given in (1.2)? A few unproven statements are already given in [41]. Recall that we first did a compactification and figured out the internal spinor equations. We compared these with the mathematical literature. In the same line of arguing we can also incorporate the R-R fields but here a problem arises. A comparison with the mathematical literature cannot be done since the generalised $SU(3)$- and G_2-structures only capture the NS-NS fields. Even more is true. Witt showed that the topological data is already exhausted by the NS-NS fields and so these generalised structures do not provide further degrees of freedom to allow for R-R fields as well. In other words, to make a serious comparison of the internal supersymmetry equations we have to work on the mathematical side first. This means that we have to extend the generalised structures from a non-topological point of view. Remember, the generalised structures appear as a critical point of a variational principle. We start here and modify this principle to a constrained variational principle. This constraint is purely *geometrical* and can be achieved by an even or odd form $F^{ev/od}$. It modifies the former integrability condition in a way that $F^{ev/od}$ appears as an inhomogeneous part. One might guess now that $F^{ev/od}$ of the constraint are the R-R fields? The answer is "yes", but this has to be proven in a rigorous manner [62]. In this mathematical article we prove that the modified integrability condition translates one-to-one to a modified spinor picture. Having now this mathematical spinor picture at hand we can do again a serious comparison with the result we derived from the compactification. The basic result is that the spinor pictures are *precisely* the same, i.e. we have now a full picture of the physical NS-NS and R-R fields in terms of *extended* generalised $SU(3)$- and G_2-structures (we omit in the following the word extended). So, the NS-NS fields characterise the topological data and the R-R fields are responsible for the geometrical data. Note that a classical torsion-free G_2-manifold is governed by the differential conditions $d\varphi = 0$ and $d \star \varphi = 0$. In case the manifold picks up torsion the equations get modified to $d\varphi = T_1 + T_7 + T_{27}$ and $d \star \varphi = T_7 + T_{14}$, i.e. the specific torsion classes T_i appear as inhomogeneous terms. The same situation appears in the just discussed case of generalised geometries and we therefore call the R-R fields the *generalised torsion*. By using $G \times G$-representations it must be possible to establish *generalised torsion classes* but this is for future work. As an application of this we study T-duality for generalised structures with an S^1-fibration.

Chapter 2

Classical geometries and supergravity in 11d

It is believed that the five existing 10d superstring theories are all ruled by a 11d theory, called M-theory. This theory is in principle not known but the low energy formulation goes under the name of 11d supergravity and was found by Cremmer, Julia and Scherk [25] in 1978.

We assume that 11d supergravity is formulated on a spinnable manifold $(M^{1,10}, g^{(11)})$ of Minkowski signature. The theory involves the 3/2-spin gravitino $\Psi_X \in \Delta \otimes TM$, where Δ denotes the spin bundle, the metric $g^{(11)}$ and a 3-form potential $C \in \Lambda^3 M$. The field strenght of C is the 4-form $\hat{F} \in \Lambda^4 M$. For the bosonic action we write

$$S \sim \int \sqrt{g}R - \frac{1}{2}\hat{F} \wedge \star\hat{F} - \frac{1}{6}C \wedge \hat{F} \wedge \hat{F}.$$

In the following we are only interested in the background manifold - the vacuum - and so allow only the metric $g^{(11)}$ and the 3-form potential C to be present.

From a phenomenological perspective we want to compactify the theory on a seven-dimensional internal Riemannian manifold (M^7, h), i.e. $M^{1,10} = M^{1,3} \times M^7$. This was first discussed by Candelas and Raine [19] in the presence of 4-form fluxes and by assuming that the external space is not warped. A further assumption arise by claiming to preserve a minimal amount of supersymmetry in the external 4-dimensional space $(M^{1,3}, g)$. This constraint can be achieved by bringing the supersymmetry variations of the fields into play. Since we are interested in the vacuum manifold we only have to focus on the supersymmetry variation of the gravitino [25],

$$0 = \delta_\eta \Psi_X = \nabla_X^{g^{(11)}} \eta + \frac{1}{144}\Big(X \wedge \hat{F} \cdot - 8\, X \lrcorner \hat{F} \cdot \Big)\eta\,, \qquad X \in TM\,, \tag{2.1}$$

where $\cdot$ and $\lrcorner$ means Clifford multiplication and contraction. The supersymmetry parameter $\eta \in \Delta$ is a Majorana spinor, globally defined and not a physical field. The first solutions for $\mathcal{N} = 1$ supersymmetry appeared in [6, 29].

To become more explicit with the compactification we introduce a non-trivial warping and the 4-dimensional external space $(M^{1,3}, g)$ admits a Minkowski signature. Since our 4-form flux should be consistent with the maximally symmetric external space the most general $\hat{F}$ is given by the well known Freund-Rubin Ansatz [33],

$$\begin{aligned} ds^2 &= e^{2A}(\, g^{(4)} \,+\, h^{(7)}) \;, \\ \hat{F} &= dC = m \,\mathrm{vol}_4 \,+\, F \;, \end{aligned} \tag{2.2}$$

where $A \in C^\infty(M^7)$ is a warp function on the 7-manifold (M^7, h), m is the Freund-Rubin parameter and $F \in \Lambda M^7$.

We specified the dimension of the 7-manifold by hand with the experience that we live in 4-dimensions. The internal 7-dimensinal space is argued to be (mostly) compact and invisibly tiny. Furthermore, physics lack up to now of a satisfactory mechanism to adjust the right amount of supersymmetry - so we have to specify it by hand. Since we do not experience supersymmetry in everyday life but strongly believe in its existence we assume here (at least) minimal supersymmetry.

Before we start with our main discussion in the next section we want to give the reader a first flavour of the problem to find physically relevant minimal supersymmetric vacua. The external space is choosen to be 4d-Minkowski $\mathbb{R}^{1,3}$. Let us set for convenience the warp factor A and the 4-form field strength $\hat{F}$ to zero. We attack the given compactification by decomposing the 11d spinor η such that $\eta = \epsilon \otimes \theta$, where $\epsilon \in \Delta_{\mathbb{R}^{1,3}}$ and $\theta \in \Delta_{M^7}$. We note that Δ_{M^7} is eight dimensional and real. By taking all the given data into account we treat (2.1) by seperating it with respect to the external and internal part. The external part for $X \in T\mathbb{R}^{1,3}$ is trivially satisfied since for $\mathbb{R}^{1,3}$ we have $\nabla^{g^{(4)}}_X \epsilon = 0$, i.e.

$$\delta_\eta \Psi_X = \nabla^{g^{(4)}}_X \eta = \nabla^{g^{(4)}}_X (\epsilon \otimes \theta) = (\nabla^{g^{(4)}}_X \epsilon) \otimes \theta = 0 \,.$$

The internal part for $X \in TM^7$ yields

$$\delta_\eta \Psi_X = \nabla^h_X \eta = \nabla^h_X (\epsilon \otimes \theta) = \epsilon \otimes (\nabla^h_X \theta) = 0$$

and implies that we can solve the present problem by only focusing on the internal space. Since we want to have minimal supersymmetry, i.e. four unbroken supercharges in 4d, there must be precisely one internal covariantly constant spinor. Or in other words, if we parallel transport eight independent spinors with respect to ∇^h over all possible loops only one spinor should remain the same. The discrepancy of the initial and final spinors, after the parallel transport around the closed loop, can be measured by group theory and goes under the name of holonomy theory or the notion of G-structures. The G refers the the structure group. In our simple example the holonomy group is G_2.

This chapter is mainly based on the author's articles [8, 9, 10, 11]. We first seperate the 11d supersymmetry variations (2.1) into an external and an internal part. The external part leads only to algebraic relations on the involved physical fields, e.g. the 4-form $\hat{F}$. However, the internal equation is a Killing spinor equation that mainly characterises the internal background manifold. This characterisation is governed by assuming supersymmetry. The amount of supersymmetry we claim is equivalent to the question of the number of parallel spinors on the 7-fold with respect to the connection. Note, this is not the Levi Civita connection since we assume non-vanishing fluxes.

We tackle this problem by first reviewing some fundamental facts about G-structures that provide us to find solutions to our physical problem. We explicitly discuss G_2- and $SU(3)$-structures. This brings us in the position to classify the internal manifolds and thus we give solutions to the initial problem (2.1). The basic result is that the internal 4-form flux F, the Freund-Rubin parameter m and the differential of the warp-factor dA can be captured by various torsion classes of the underlying G-structure. We also ask the question if different 11d spinor decompositions can drop some constraints on the fluxes but the answer here is negative.

In addition, we provide some unpublished results. Let us focus only on the internal Killing spinor equation where we set all physical fields but the metric to zero. So, we deal with the Levi Civita connection like the above given example and the 7-space is a torsion-free G_2-manifold. The authors in [69] showed that the connection of a torsion-free G_2-structure can pick up $\alpha^{\backprime}$-corrections, since the curvature gets modified by these corrections. We prove that the modifications of the initial connection can be completely captured by G_2 torsion classes. As a result we obtain that $\alpha^{\backprime}$-corrections will never lead to a weak G_2-manifold but all remaining three classes get contributions. It is even possible that the resulting G_2-structure becomes non-integrable.

2.1 Compactification of 11d supergravity

In the following we suppress the explicit notation of eleven-, seven- and four-dimensional objects which should be clear from the context. We also prefer the index free notation of Clifford multiplication, i.e.

$$\hat{F}\cdot \equiv \hat{F}_{MNPQ}\Gamma^{MNPQ}\,, \qquad (X \lrcorner \hat{F})\cdot \equiv X^M\, \hat{F}_{MNPQ}\Gamma^{NPQ}\,.$$

Choosing the diagonal metric η and using the convention $\{X, Y\} = 2\eta(X, Y)$ with $\eta = \mathrm{diag}(-, +, + \ldots +)$, we decompose the 11d Γ-matrices as usual

$$\Gamma^{\mu} = \hat{\gamma}^{\mu} \otimes \mathbb{I}\,, \qquad \Gamma^{a+3} = \hat{\gamma}^5 \otimes \gamma^a\,,$$

where $\mu = \{0, 1, 2, 3\}$ and $m = \{1, 2, \ldots 7\}$. The 4-d $\hat{\gamma}^{\mu}$-matrices are real and $\hat{\gamma}^5$ as well as the 7-d γ^a-matrices are purely imaginary. We also denote

$$\hat{\gamma}^5 = i\, vol_4\cdot\,, \quad vol_7\cdot = -i\,\mathbb{I} \tag{2.3}$$

which implies the following identities in components [5]

$$i\hat{\gamma}^5\hat{\gamma}^{\mu} = \frac{1}{3!}\epsilon^{\mu\nu\rho\lambda}\hat{\gamma}_{\nu\rho\lambda} \quad , \qquad \frac{i}{3!}\epsilon^{abcdmnp}\gamma_{mnp} = \gamma^{abcd}\,.$$

Having this in hand we write the 4-form field $\hat{F}$ as

$$\hat{F}\cdot = -i\, m\,\hat{\gamma}^5 \otimes \mathbb{I} \,+\, \mathbb{I} \otimes F\cdot$$

and we also denote

$$\begin{aligned} (X \lrcorner \hat{F})\cdot \;&=\; \tfrac{1}{4}\, i\, m\hat{\gamma}^5 X\cdot \otimes \mathbb{I}, \quad X \in TM^{1,3}, \\ (X \lrcorner \hat{F})\cdot \;&=\; \hat{\gamma}^5 \otimes (X \lrcorner F)\cdot, \quad X \in TM^7. \end{aligned} \tag{2.4}$$

Since our metric is warped, we use

$$ds^2 = e^{2A}\widetilde{ds}^2 \quad \rightarrow \quad \nabla_X = \tilde{\nabla}_X + \frac{1}{2}(X \wedge dA)\cdot\,, \quad X \in TM^{11}.$$

According to our initial example we consider the gravitino variation (2.1) and split it in an external part, $X \in TM^{1,3}$,

$$0 \;=\; \Big[\nabla_X \otimes \mathbb{I} + X \cdot \hat{\gamma}^5 \otimes \Big(\tfrac{1}{2}\, dA \cdot + \tfrac{im}{36}\Big) + \tfrac{1}{144} e^{-3A}\, X \otimes F\Big]\eta\,, \tag{2.5}$$

and an internal part, $X \in TM^7$,

$$0 \;=\; \Big[\mathbb{I} \otimes \Big(\nabla_X + \tfrac{1}{2} X \wedge dA \cdot + \tfrac{im}{144}\, X\Big) + \tfrac{1}{144} e^{-3A}\, \hat{\gamma}^5 \otimes \Big(X \cdot F \cdot -12\,(X \lrcorner F)\cdot\Big)\Big]\eta\,, \tag{2.6}$$

where we used: $\Gamma_M \Gamma^{N_1 \cdots N_n} = \Gamma_M{}^{N_1 \cdots N_n} + n\,\delta_M^{[N_1}\Gamma^{N_2 \cdots N_n]}$.

By calculating the external Dirac operator $D = \hat{\gamma}^\mu \nabla_\mu$ from (2.5) and multiplying this equation by $\frac{1}{4}\hat{\gamma}^5 \otimes X\cdot$, $X \in TM^7$, we can eliminate the term $\sim X \cdot F\eta$ from the internal gravitino variation and obtain,

$$0 = \Big[\mathbb{I} \otimes \Big(\nabla_X - \frac{1}{2} dA \cdot + \frac{im}{48} X\cdot\Big) - \frac{1}{4}\hat{\gamma}^5 D \otimes X \cdot - \frac{1}{12}\, e^{-3A}\, \hat{\gamma}^5 \otimes X \lrcorner F \cdot\Big)\Big]\eta\;. \tag{2.7}$$

The next step in solving the external and internal equation is that we decompose the 11d Majorana spinor η and discuss possible geometrical Ansätze for the external equation. This will occupy us in the next section.

2.2 Decomposition of the 11d spinor

We decompose from a phenomenological point M^{11} into $M^{1,3} \times M^7$ and also split the 11d Γ-matrices. It is obvious that we also should decompose the 11d Majorana spinor. Since the Spin(7) module on a 7-manifold has dimension eight, $\dim(\Delta) = 8$, we can choose a basis θ_i in Δ_{M^7} and expand the 11d Majorana spinor by

$$\eta = \sum_{i=1}^{\dim(\Delta)} \left(\epsilon^i \otimes \theta^i + cc\right)\,, \tag{2.8}$$

where $\epsilon^i \in \Delta_{M^{1,3}}$ and θ^i (eventually complexified) denote the 4- and 7-d spinors and are globally defined. Note, the spin module of $M^{1,3}$ does not have dimension eight. For example, in case of $\hat{F} = 0$, $A = 0$ and by further assuming the 11-manifold to be flat, all θ^i are covariantly constant with respect to the Levi-Civita connection. Therefore, $\dim(\Delta) = 8$ gives the resulting extended supersymmetries in four dimensions. In general, if we allow more structure not all θ^i are any longer parallel with respect to the Levi-Civita connection. But it can be possible that we can find a different connection by that we are able to parallel transport at least a fraction of eight. And this number with respect to a general connection gives a classification of the internal space and counts the number of unbroken supersymmetries in four dimensions. On the other side, to preserve at least a minimal amount of supersymmetry it is necessary to find a connection which parallel transport at least one spinor. We address this classifying problem in the next sections and solve it with the mathematical theory of G-structures, i.e. we classify the problem from a group theoretical point of view. We can also ask about parallel spinors in 4d. Here we will make a geometical Ansatz by means of the Killing spinor equation in 4d to discuss also non-Minkowski spaces. If the 4-d spinors are covariantly constant, the resulting vacuum will be a 4-d flat Minkowski space, but for an anti deSitter vacuum the spinors satisfy

$$\nabla_X \epsilon^i \ \sim \ X \cdot (W_1 + i\hat{\gamma}^5 \, W_2)\epsilon^i \,, \quad X \in TM^{1,3} \,,$$

where $W_{1/2} \in \mathrm{End}(\Delta)$. Note, in case that $W_{1/2}$ are just functions the resulting 4-d cosmological constant will be $-|W|^2$. In [9, 10, 11] the object W is considered as the superpotential. In what follows we only refer to W in a geometrical sense. If there is only a single spinor that is Majorana on the internal 7-manifold , i.e. i only has value equal to one, this equation simplifies to

$$\nabla_X \epsilon \ \sim \ X \cdot (W_1 + i \, \hat{\gamma}^5 \, W_2) \, \epsilon \,, \quad X \in TM^{1,3} \,,$$

where $W_{1/2}$ are now functions and ϵ is a Majorana spinor. In case of two internal spinors or equivalently of having one complex (and conjugate) spinor the external spinor ϵ is a complex Weyl spinor. We obtain

$$\nabla_X \epsilon = X \cdot \bar{W} \epsilon^\star$$

where $W = W_1 + i \, W_2$. Plugging these 4d Ansätze in the 11d framework we have

$$\Big[\nabla_X \otimes \mathbb{I}\Big]\eta = (X \cdot \otimes \mathbb{I})\widetilde{\eta} \qquad \text{where :} \qquad \widetilde{\eta} = \begin{cases} [(W_1 + i \, \hat{\gamma}^5 W_2) \otimes \mathbb{I}] \, \eta & \mathrm{M} \\ W\epsilon \otimes \theta^\star + cc & \mathrm{W} \end{cases} \tag{2.9}$$

where M/W refers to a 4-d Majorana or Weyl spinor ϵ. In what follows we only discuss the two cases where the internal 7-spinor is either real or complex which corresponds to the external Ansätze M(ajorana) or W(eyl). Until now we do not even know if such cases are mathematical well defined, e.g. if we investigate the most trivial case by having a 4d Minkowski space, $W_1 = W_2 = 0$, vanishing flux $\hat{F} = 0$, and warp factor, $A = 0$. The external equation is satisfied trivially. However, the internal equation reduces effectively to a equation on the 7-manifold but stays non-trivial. This is because supersymmetry tells us that we want to find a solution for exactly one real or complex spinor. The question thus reduces to the problem: Do there exist 7-manifolds that admit precisely one real or complex parallel spinor (to a certain connection)? How is such a space characterised. We want to address these questions in the following section.

2.3 G-structures

We investigate the M/W case in the following section and show that the external equation (2.5) leads only to an algebraic constraint for the flux, the differential of the warp factor and W_1, W_2. But from the internal part (2.7) we get a complicated differential equation for the involved spinors where the number of parallel spinors predicts the number of supersymmetry. Since the number of supersymmetries cannot be derived from the physical theory we will fix it by hand. We will introduce the mathematical theory of G-structures to solve the Killing spinor equation for a fixed number of parallel internal spinors.

Let us consider the phenomenological Ansatz $M^{1,3} \times M^7$ that is accompanied by the splitting of the 11d spinor (2.8). Moreover, the internal space is supposed to be very tiny (roughly at Planck scale) and compact.

We want to collect some known facts about those internal spaces from the mathematical literature e.g. [34, 36, 17, 77]. Since we assume to have a compact, spinnable manifold M^7 it admits a Spin(7)-structure. This spin bundle Δ has real dimension equal eight, $\dim(\Delta) = 8$. We therefore often use the notation $\Delta = \mathbf{8}$. The biggest compact subgroup of SO(7) is G_2 which appears in representations e.g. $\mathbf{1}$, $\mathbf{7}$, $\mathbf{14}$ and $\mathbf{27}$. Since G_2 sits also in Spin(7) a reduction of the structure group to G_2 is equivalent to the decompositon $\mathbf{8} \to \mathbf{1} \oplus \mathbf{7}$. Let us call the singlet spinor θ. Furthermore, on an orientable compact 7-manifold there exist a nowhere vanishing vector field X from a topologically point of view [82]. And since the map $X \cdot \theta \in \Delta$ is an isomorphism the spinors θ, $X \cdot \theta$ are linearly independent in any point on M^7 and define a topological SU(3)-structure. Note that there exists also a second linearly independent vector field that even defines a topological reduction to a SU(2)-structure.

This only means that having a compact, spinnable 7-manifold we have a topological reduction even to a SU(2)-structure but this does not imply that we always have a geometrical reduction to the structure group SU(2). The geometrical reduction can only be achieved by satisfying certain differential conditions for the participating spinors. For instance, we have a topological reduction to SU(2) but by imposing only differential conditions for the G_2-spinor θ we can have at most a geometrical G_2-structure. If θ also satisfies the differential constraint then we have precisely a G_2-reduction. The geometrical reduction is achieved if the internal Killing spinor equation preserves the number of singlet spinors. In other words the singlet spinors must exist globally on M^7 to reduce the structure group geometrically. Before we analyse the complicated Killing spinor equations given by 11d supergravity we collect facts from the mathematical theory of G-structures that are useful for our following investigations.

2.3.1 The mathematical idea of G-structures

We introduce the mathematical concept of holonomy or G-structures and find it useful to motivate and discuss it in a naive and informal way. The precise definitions and proofs can be found e.g. in [34, 36, 17, 77, 63, 24].

Let M be a generic, oriented n-dimensional manifold and let $\vec{v}$ be a specific vector at $x \in M$. We parallel transport $\vec{v}$ (with respect to a certain connection ∇) around a closed curve γ. In general, this transport does not bring it back to itself (see figure 2.1). A deeper investigation of the initial vector $\vec{v}_i$ and the final vector $\vec{v}_f$ yields the basic result that the final vector $\vec{v}_f$ is the image of the initial vector $\vec{v}_i$ under certain transformations that depend on the the curve γ and even on $x \in M$. Since we only deal here with connected manifolds we can drop the dependence of $x \in M$. It is a basic result that these transformations satisfy the axioms of a Lie group. We will call this Lie group the holonomy group $\mathrm{Hol}(\nabla)$. For a general oriented n-manifold we have $\mathrm{Hol}(\nabla) = SO(n)$. For instance, the holonomy group of $\mathbb{R}^n$ is simply the identity. Furthermore, $SO(n)$ preserves the length of the vector $\vec{v}$ that is parallel transported and so does the holonomy group. In case of having a Riemannian manifold (M, g), i.e. M also admits a metric g, there is a natural parallel transport that is given by the Levi-Civita connection ∇^{LC}. This connection is special since it has the property $\nabla^{LC} g = 0$.

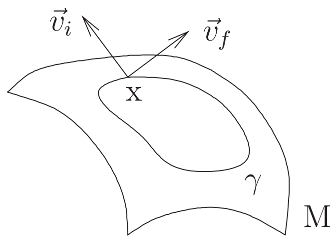


Figure 2.1: Parallel transport

In 1955 Berger proved that only certain Lie groups can appear in special dimensions as holonomy groups. The possible holonomy groups are shown in the following list [63], e.g.

$$\begin{array}{ll} SO(n) & \text{in general}\,, \\ U(m) \subset SO(n) & n = 2m > 2\,, \\ SU(m) \subset SO(n) & n = 2m > 2\,, \\ Sp(k) \subset SO(n) & n = 4k > 4\,, \\ G_2 \subset SO(7) & n = 7\,, \\ Spin(7) \subset SO(8) & n = 8\,, \end{array}$$

This list only proves that these groups could appear as holonomy groups in principle but Berger did not prove if they really occur. For instance, Bryant [16] discoverd no more than 1987 the existence of metrics with holonomy of G_2 and $Spin(7)$.

We mentioned that the existing holonomy groups are Lie groups and are contained in $SO(n)$. This implies that the concept of holonomy can neatly be formulated in the notian of principle fibre bundles. In general, the base space is the manifold itself and the fibre over $x \in M$ is isomorphic to the Lie group G that denotes the structure group. Roughly speaking, the structure group is the group that characterises how to glue the patches of the manifold together. So we can say that the holonomy group is the "minimal" structure group. For instance, let us take the manifold $\mathbb{R}^n$. The principle fibre bundle has structure group e.g. $G = SO(n)$ but the "necessary" structure group, the holonomy group, is the identity. In the following we always identify G with $\mathrm{Hol}(\nabla)$ and talk about manifolds having a G-structure.

Let us go back to the idea of parallel transporting a vector over a loop by using now the principle fibre bundle picture. In figure 2.2 we parallel transport an object over the closed path γ. This path can be lifted via a connection to define a horizontal path γ^* in the principle fibre bundle. Let us parametrise the curves γ and γ^* by $t = [0, 1]$, where $\gamma(0)$ and $\gamma(1)$ are identified in case of a closed path γ. Remember, γ is a path on M whereas we can denote γ^* (in a local trivialisation) by $(\gamma_t, g(\gamma_t))$, $g \in SO(n)$. We assign to the point $\gamma(t) \in M$ precisely one group element of $g \in SO(n)$. This picture illustrates that the lifted path γ^* does not close in general and the discrepancy is also captured by a group element of G that maps $\gamma^*(1)$ to $\gamma^*(0)$. In case we only need a subgroup $g \in G \subset SO(n)$ to achieve $g \cdot \gamma^*(1) = \gamma^*(1)$ for all γ the structure group is reduced to G. The manifold admits a G-structure.

Until now we gave a very abstract and formal picture of G-structures in terms of a principle fibre bundle. The question that arise is: How can we geometrically characterise the cases

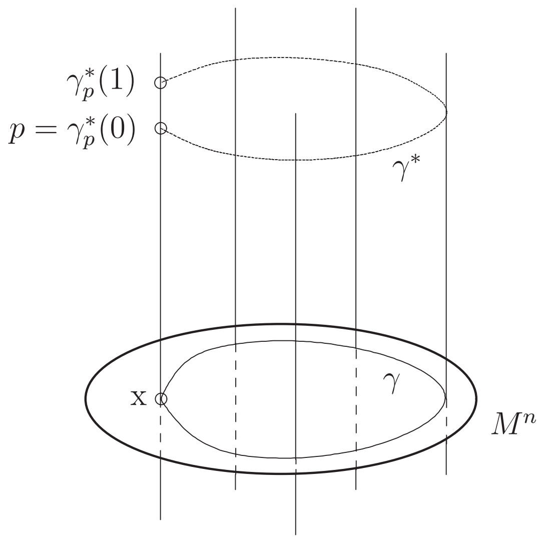


Figure 2.2: Lift of a curve γ from the base space

$G \subset SO(n)$ - situations where the structure group is a proper subgroup of $SO(n)$, e.g. $G = G_2$? The basic idea is the following. We can associate to a principle fibre bundle several bundles, e.g. the tangent bundle, the bundle of 3-forms or even the bundle of all exterior-forms $\Lambda^\bullet M^n$. If we assume that the manifold is spinnable we can lift the general structure group $SO(n)$ to $Spin(n)$. Therefore we can also associate the spinor bundle Δ to the principal fibre bundle. This implies that we can parallel transport by a given connection not only a vector but also e.g. a spinor or a tensor. The fundamental observation is that certain tensors or spinors are invariant under a proper subgroup $G \subset SO(n)$. This means that $G \subset SO(n)$ stabelises specific objects. Or the other way round, if we know that a manifold has parallel sections of a certain associated bundle then the structure group is probably reduced. E.g. $G = G_2$ stabelises a certain 3-form φ or equivalently exactly one spinor θ. In other words, by claiming $\nabla\varphi = 0$ we reduce the structure group to G_2.

We are now in a position to look back to our initial problem (2.1) and the subsequent discussed example ($M^{1,10} = \mathbb{R}^{1,3} \times M^7, \hat{F} = 0, dA = 0$). Here it becomes immediately clear that we want to parallel transport spinors and the number of independent spinors that solve the equation predicts the number of supersymmetry. Regarding our example, if we are able to classify all possible parallel spinors solutions we thus have all supersymmetry results. This question reduces at first to the decomposition of the spinor bundle Δ under possible subgroups of $SO(7)$ or rather $Spin(7)$. Let us denote the common result for M^7 that is useful for us later,

	Δ
G_2	$\Delta^{\mathbb{R}} = \mathbf{1} \oplus \mathbf{7}$
$SU(3)$	$\Delta^{\mathbb{C}} = \mathbf{1} \oplus \overline{\mathbf{1}} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$

where we note that $\Delta^{\mathbb{R}} = \Delta \otimes \mathbb{R} = \mathbf{8}$ and $\Delta^{\mathbb{C}} = \Delta \otimes \mathbb{C} = \mathbf{4} \oplus \overline{\mathbf{4}}$.

The question that will occupy us next is: What will happen to the holonomy group if we add terms to the Levi-Civita connection, e.g. flux terms in (2.1)?

Let us now motivate how to bring the flux $\hat{F}$ in (2.1) into the game of G-structures. By assuming $M^{1,10} = M^{1,3} \times M^7$ we splitted (2.1) in an external (2.5) and an internal part (2.7). Later on we solve the external equation by pushing it to an algebraic constraint for the flux. Taking this into account the interesting part is given by the internal equation. There is not only the Levi-Civita connection involved but also additional flux terms appear. If we assume to have an internal spinor $\theta \in \Delta_{M^7}$ we schematically will obtain for a pure internal equation,

$$0 = \nabla_X \theta + (\text{flux-terms})(X) \cdot \theta \,. \tag{2.10}$$

Strominger [80] observed for a similar situation ($G = SU(3)$, heterotic theory) that the shape of the flux term can be captured by the torsion term. Later it was realised that the mathematical framework of G-structures deals with the similar equation,

$$0 = \nabla_X \theta + (X \lrcorner T) \cdot \theta \,,$$

where, in general, the torsion tensor T is an object in the space $\Lambda^1 \otimes \Lambda^2$. This is the crucial idea that we will pick up in the following, i.e. we want to achieve a equivalence between the physical flux terms and the mathematical torsion tensor. This correspondence can be seen by solving a 2-fold problem that we address now.

Firstly, one part of the flux terms is given by the 4-form F. Let us suppress for the moment the terms that are given e.g. by dA. We can make contact with the flux F to the torsion by first assuming T to be totally skew symmetric, i.e. $T \in \Lambda^3 \subset \Lambda^1 \otimes \Lambda^2$. Secondly, since we are on M^7 we could treat instead of $F \in \Lambda^4$ the hodge dual $\star F$ that is a 3-form like T.

This results on M^7 (having the above assumptions) in the following possible interpretation

$$\text{flux} \leftrightarrow \text{torsion} \leftrightarrow \text{geometry} \,.$$

Let us focus once more on general mathematical aspects of G-structures and consider the reduction of the structure group from $SO(n)$ to G, where we denote the Lie algebras by $\mathfrak{so}(n)$ and $\mathfrak{g}$. Since $\mathfrak{so}(n) \cong \Lambda^2$, a structure group reduction to G is given by

$$\mathfrak{so}(n) \;=\; \mathfrak{g} \,\oplus\, \mathfrak{g}^{\perp} \,.$$

Note, that $\mathfrak{g}^{\perp}$ is in general not G-invariant and characterises the torsion by $T \in \Lambda \otimes \mathfrak{g}^{\perp}$. But since we want to have parallel spinors also T must be G-invariant. The idea of torsion-full G-structures is to decompose T into G-modules, i.e. irreducible G-representations, that are called torsion classes T_i,

$$T \in \Lambda \otimes \mathfrak{g}^{\perp} = \bigoplus_i T_i \,. \tag{2.11}$$

Remember, we are nevertheless intersted in having a skew-symmetric torsion tensor.

The parallel spinors we are now looking for are no longer parallel with respect to the Levi-Civita connection but instead must be parallel by the equation (2.10). We want to regard the additional flux (or the torsion) as a deformation of the Levi-Civita connection and introduce a new (torsion-full) connection $\widetilde{\nabla}$ by

$$0 = \widetilde{\nabla}_X = \nabla_X \theta + (\text{flux-terms})(X) \cdot \theta \,.$$

So, we are interested in parallel spinors with respect to the connection $\widetilde{\nabla}$. This means that the basic question about parallel spinors remain and only the connection changed. E.g. space time supersymmetry is only sensible to parallel spinors and not in a certain connection. In the next sections we discuss more explicit the groups G_2 and $SU(3)$ and give useful algebraic relations.

2.3.2 G_2-structure

A topological reduction to a G_2-structure allows for exactly one singlet in the spinor bundle $\mathbf{8} \to \mathbf{1} \oplus \mathbf{7}$. Let us define this G_2-spinor by

$$\theta = e^Z \, \theta_0 \in \mathbf{8}\,, \quad Z \in C^\infty,$$

where θ_0 has norm one with respect to the metric q on the spin bundle $\mathbf{8}$, i.e. $q(\theta_0, \theta_0) = 1$. The corresponding 11d spinor η is thus given by $\eta = e^Z \, \epsilon \otimes \theta_0$, where ϵ is a Majorana spinor in 4d. Since η is globally defined so is ϵ and θ_0. By the well known procedure of fierzing we can produce differential forms $\Omega \in \Lambda^\bullet$ of certain degrees. The coefficients of the forms $\Omega \in \Lambda^\bullet$ can be computed by

$$g(\Omega, e_I) = q(e_I \cdot \theta_0, \theta_0),$$

where I denotes a multi-index, q is the metric in the spin bundle and g here is the extended metric in the bundle of exterior forms. The only G_2-invariant forms we can produce are $1 \in \Lambda^0$, $\varphi \in \Lambda^3$, $\psi \in \Lambda^4$ and $\mathrm{vol}_g \in \Lambda^7$. In components we have (the gamma matrices on the 7-manifold are chosen imaginary)

$$\begin{aligned} 1 &= q(\theta_0, \theta_0)\ , \\ i\,\varphi_{abc} &= q(\theta_0, \gamma_{abc}\theta_0)\ , \\ -\psi_{abcd} &= q(\theta_0, \gamma_{abcd}\theta_0)\ , \\ i\,\epsilon_{abcdmnp} &= q(\theta_0, \gamma_{abcdmnp}\theta_0)\ . \end{aligned} \tag{2.12}$$

It is also possible to construct out of the G_2 3-form φ the metric on the 7-manifold. By using the metric g we can calculate the hodge dual of φ and find $\star\varphi = \psi \in \Lambda^4$. Similarly, $\star 1 = \mathrm{vol}_g \Lambda^7$. Since we learned that the Lie algebra $\mathfrak{so}(7)$ is isomorphic to Λ^2 and a reduction of the structure group on a general M^7 from SO(7) to the subgroup G_2 implies the following splitting:

$$\mathfrak{so}(7) \ = \ \mathfrak{g}_2 \ \oplus \ \mathfrak{g}_2^\perp \ .$$

This induces a decomposition of the space of 2-forms Λ^2 in the following irreducible G_2-modules, $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$, where the subscripts denote G_2 representations, e.g. $\mathbf{1}$, $\mathbf{7}$, $\mathbf{14}$ and $\mathbf{27}$. Some usefull identities in components are given by

$$\begin{aligned} \gamma_{ab}\theta_0 &= i\varphi_{abc}\gamma^c\theta_0\ , \\ \gamma_{abc}\theta_0 &= \big(i\varphi_{abc} + \psi_{abcd}\gamma^d\big)\,\theta_0\ , \\ \gamma_{abcd}\theta_0 &= \big(-\psi_{abcd} - 4i\varphi_{[abc}\gamma_{d]}\big)\theta_0\,. \end{aligned} \tag{2.13}$$

By using the spin picture and the identities (2.13) we can calculate the decomposition of the exterior forms and get

- $\Lambda^0 = C^\infty(M^7)$,
- $\Lambda^1 = \Lambda^1_7 = TM^7$,
- $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

$$\begin{aligned} \Lambda^2_7 &= \{w \lrcorner \varphi | w \in \Gamma(TM^7)\} = \{\alpha \in \Lambda^2 \,|\, \star(\varphi \wedge \alpha) - 2\alpha = 0\}\,, & (2.14)\\ \Lambda^2_{14} &= \{\alpha \in \Lambda^2 \,|\, \star(\varphi \wedge \alpha) + \alpha = 0\} \cong \mathfrak{g}_2\,, & (2.15)\end{aligned}$$

- $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where

$$\begin{aligned} \Lambda^3_1 &= \{f \cdot \varphi | t \in C^\infty(M^7)\}\,, & (2.16)\\ \Lambda^3_7 &= \{w \lrcorner \psi | w \in \Gamma(TM^7)\}\,, & (2.17)\\ \Lambda^3_{27} &= \{\alpha \in \Lambda^3 | \alpha \wedge \varphi = \alpha \wedge \psi = 0\}\,, & (2.18)\end{aligned}$$

where by hodge duality e.g. $\star\Lambda^3_k = \Lambda^4_k$.

We already mentioned that a geometrical reduction to a G_2-structure comes along by a differential equation on the spinor $\theta_0 \in \Delta$ or equivalently for the 3-form φ. Let us now recapitulate the ideas given in [34]. We are interested in the connection

$$\widetilde{\nabla} \equiv \nabla^{LC} + \frac{1}{4}(X \lrcorner T)\cdot\,,$$

where T is the torsion that measures the failure of the connection to be Levi-Civita.

In [34] the authors showed that in case of a totally skew symmetric torsion the connection is unique and can be characterised from a group theoretical point of view by $T \in \Lambda^1 \otimes \mathfrak{g}_2^\perp$. These components decompose under G_2, by using $\Lambda^1 \cong \mathfrak{g}_2^\perp \cong \mathbf{7}$, as

$$\Lambda^1 \otimes \mathfrak{g}_2^\perp = \mathbf{7} \otimes \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27} = T_1 + T_7 + T_{14} + T_{27} \qquad (2.19)$$

where the subscript in T_i denotes the dimension of the G_2-modules.

The parallel transport via $\widetilde{\nabla}$ does not violate the spinor decomposition $\mathbf{8} = \mathbf{1} \oplus \mathbf{7}$ globally iff the G_2-structure is integrable. The G_2-structure is integrable iff T_{14} vanishes [35].

Since the spinor $\theta \in \Delta$ defines φ and ψ, these torsion classes T_i can be measured by $d\varphi$ and $d\psi$ (see e.g. [34]) as follows

$$\begin{aligned} d\varphi &\in \Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}\,,\\ d\psi &\in \Lambda^5 = \Lambda^5_7 \oplus \Lambda^5_{14}\,, \end{aligned} \qquad (2.20)$$

where the $\mathbf{7}$ in Λ^4_7 is the same as in Λ^5_7 up to a multiple.

The different G_2-invariant torsion modules can be explicitly extracted from the forms $d\varphi$ and $d\psi$ by,

$$\begin{aligned} T_{(1)} &\longleftrightarrow \psi \lrcorner d\varphi\,, & T_{(7)} &\longleftrightarrow \varphi \lrcorner d\varphi\,,\\ T_{(14)} &\longleftrightarrow *d\psi - \tfrac{1}{4}(*d\psi) \lrcorner \psi\,, & T_{(27)} &\longleftrightarrow (d\varphi_{cde\{a}\psi_{b\}}{}^{cde})_0\,, \end{aligned} \qquad (2.21)$$

where in $(\cdot)_0$ the trace is removed. T_{14} and T_{27} have to satisfy: $\varphi_3 \wedge \Lambda^3_{27} = 0$.

2.3.3 SU(3)-structure

The existence of two globally defined real spinors which are perpendicular reduce topologically the structure group to SU(3). By the topological fact of having a nowhere vanishing vector field $v \in TM^7$ on a spinnable compact 7-manifold we can also start from a G_2-structure by using the singlet spinor θ_0. We construct a second perpendicular spinor by the isomorphism $v \cdot \theta_0$. Since we have chosen the gamma matrices to be purely imaginary the spinor $v \cdot \theta_0$ is imaginary and no longer an object in the real spin module Δ. We define the complex (and the complex conjugate) spinor θ to be an object in $\Delta \otimes \mathbb{C}$ by

$$\theta = \frac{1}{\sqrt{2}} e^Z (\mathbb{I} + v\cdot)\theta_0 \quad , \quad Z \in C^\infty(M^7), \tag{2.22}$$

where the vector field v has norm one, $\|v\| = 1$. Thanks to that property and since the vector field v is globally well-defined we get a foliation of M^7 by a 6-manifold M^6. We can also think about the vector field v to serve as a complex structure of the complexified spin module $\Delta \otimes \mathbb{C}$[1]. In general the complexified spin bundle can be decomposed as $\Delta \otimes \mathbb{C} = \mathbf{4} \oplus \bar{\mathbf{4}}$ that is the spin representation of Spin(6) = SU(4). The spinor θ and its complex conjugate $\theta^\star$ are therefore chiral spinors on M^6, induce e.g. the decomposition $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$ and reduce the structure group to SU(3).

We write the 11d Majorana spinor in case of an internal SU(3)-structure by the folloiwng decomposition

$$\eta = \epsilon \otimes \theta + \epsilon^\star \otimes \theta^\star \ , \tag{2.23}$$

where ϵ and $\epsilon^\star$ are two 4-d Weyl spinors of opposite chirality.

Due to the choice of a globally non-vanishing vector field v we induced a splitting in the 7d tangent space such that $TM^7 = TM^6 \oplus \mathbb{R} \cdot v$. In other words we obtain a SU(3)-structure on TM^6 that we lift via v to a G_2-structure on TM^7.

Before we talk about the embedding of the structures we first clarify the properties and the notation of SU(3)-structures on 6-manifolds [53]. It is usefull to understand the appearance of the SU(3)-structure group from the following arguments. Let M^6 be a 6-manifold. On one hand the existence of a complex structure J reduces the structure group from SO(6) to SL(3, $\mathbb{C}$) and on the other hand the choice of a symplectic form ω forces the structure group to reduce to Sp(6, $\mathbb{R}$). The intersection of the two groups is U(3) = SL(3, $\mathbb{C}$) $\cap$ SP(6, $\mathbb{R}$) if we can construct a positive definite metric g by $g = \omega\, J$ and ω lives in $\Lambda^{1,1}$ only. Using the induced metric g we can measure the lenght of forms. A holomorphic (3, 0)-form $\Omega^{(3,0)}$ has real dimension two and can be written by the real 3-from ψ_+ such that $\Omega^{(3,0)} = \psi_+ + iJ\psi_+$. By forcing the real form ψ_+ to have unit length the structure reduce from U(3) to SU(3).

Let us introduce two standart notations [77, 24] in the mathematical literature that deals with complex representations. Let TM^6 be the real tangent space of a 6-manifold. By the choice of a complex structure we can introduce complex coordinates z_1, z_2, z_3 and complexified forms $\oplus_{k=0}^3 \Lambda^k \otimes \mathbb{C}$. The k-forms decompose such that $\oplus_{k=p+q} \Lambda^{p,q}$ and have the property $\overline{\Lambda^{p,q}} = \Lambda^{q,p}$. Let V be a complex representation space and $\overline{V}$ its complex conjugate. In case V is given by the complexification of a real vector space then we denote it by $[V]$, i.e.

$$[V] \otimes \mathbb{C} = V.$$

[1] This can be easily seen if one chooses the 7d-gamma matrices real.

As complex representations we have the property $V \cong \overline{V}$ and $\Lambda^{p,p}$ is an example. An instance where $V \ncong \overline{V}$ is given by $TM^6 \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}$. Here we obtain a real vector space $[\![V]\!]$ if we forget the complex structure. We get the property $[\![V]\!] = [\![\overline{V}]\!]$ and the complexification is

$$[\![V]\!] \otimes \mathbb{C} = V \oplus \overline{V}.$$

For the above example we thus get $TM^6 = [\![\Lambda^{1,0}]\!]$.

The exterior forms decompose at each point under SU(3) and can be denoted in real representations, e.g.

$$\begin{array}{rcl} \Lambda^1 & = & [\![\Lambda^{1,0}]\!]\,, \\ \Lambda^2 & = & [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}\omega\,, \\ \Lambda^3 & = & \mathbb{R}\psi_+ \oplus \mathbb{R}\psi_- \oplus [\![S^{2,0}]\!] \oplus [\![\Lambda^{1,0}]\!]\,, \end{array} \tag{2.24}$$

where $[\Lambda_0^{1,1}] \cong \mathfrak{su}(3)$ and $[\![S^{2,0}]\!] \cong [\![\Lambda_0^{2,1}]\!]$.

The forms ω, ψ_+ and ψ_- satisfy the relations

$$\begin{array}{rcl} \omega \wedge \psi_\pm & = & 0\,, \\ \psi_+ \wedge \psi_- & = & \frac{2}{3}\omega^3\,. \end{array} \tag{2.25}$$

One can now follow the same arguments as for the G_2-structure to analyse the differantial conditions for the structure forms. Since we do not need the explicit form of the five $SU(3)$ torsion classes we suppress here the detailed formulas and go over to discuss the embedding of the SU(3)-structure into a G_2-structure. We use later on the G_2 torsion classes to characterise M^7.

Using the internal spinors we can fierz certain differential forms $\Omega, \widetilde{\Omega} \in \Lambda^\bullet$ invariant under the structure group SU(3). We obtain the coefficients of the forms $\Omega, \widetilde{\Omega} \in \Lambda^\bullet$ by

$$\begin{array}{rcl} g(\Omega, e_I) & = & q(e_I \cdot \theta, \theta^\star)\,, \\ g(\widetilde{\Omega}, e_I) & = & q(e_I \cdot \theta, \theta)\,, \end{array}$$

where I denotes a multi-index and g denotes the extended metric in $\Lambda^\bullet$. The forms are given by

$$\begin{array}{rcl} \Omega^0 & = & e^{2\,\mathrm{Re}(Z)}\,, \\ \Omega^1 & = & e^{2\,\mathrm{Re}(Z)}\, v\,, \\ \Omega^2 & = & i\, e^{2\,\mathrm{Re}(Z)}\, v \lrcorner \varphi = i\, e^{2\,\mathrm{Re}(Z)}\, \omega\,, \\ \Omega^3 & = & i\, e^{2\,\mathrm{Re}(Z)} \big[v \wedge (v \lrcorner \varphi)\big] = i\, e^{2\,\mathrm{Re}(Z)}\, v \wedge \omega\,, \\ \Omega^4 & = & -e^{2\mathrm{Re}(Z)} \big[\psi - v \wedge (v \lrcorner \psi)\big] = -\frac{1}{2} e^{2\,\mathrm{Re}(Z)}\, \omega \wedge \omega\,, \\ \tilde{\Omega}^3 & = & i\, e^{2\,\mathrm{Re}(Z)} \big[e^{2i\,\mathrm{Im}(Z)} \big(\varphi - v \wedge \omega - i\, v \lrcorner \psi\big)\big] = i\, e^{2\,\mathrm{Re}(Z)}\, \Omega^{(3,0)} \\ \tilde{\Omega}^4 & = & e^{2\mathrm{Re}(Z)} \big[v \wedge (v \lrcorner \psi) - i\, v \wedge \varphi\big] = -i\, e^{2\,\mathrm{Re}(Z)} v \wedge \bar{\Omega}^{(3,0)}\,. \end{array} \tag{2.26}$$

where we used for Ω^4 the identity,

$$-\psi + v \wedge (v \lrcorner \psi) = -\frac{1}{2}\,(v \lrcorner \varphi) \wedge (v \lrcorner \varphi)\,.$$

We used the SU(3)-structure forms, i.e. the symplectic two-form ω and the holomorphic $(3,0)$-form $\Omega^{(3,0)}$. We constructed these by using the vector field v and the underlying G_2-structure forms φ and $\star\varphi$. This means on the other side that we can thus embed the SU(3)-structure (not uniquely) in the G_2-structure. Setting $Z=0$ we write

$$\begin{aligned} \varphi &= \psi_+ + v\wedge\omega\ , \\ \star\varphi &= \psi_-\wedge v + \tfrac{1}{2}\omega^2\ , \end{aligned} \tag{2.27}$$

where the identities (2.25) are satisfied,

$$\begin{aligned} \Omega^{(3,0)}\wedge\omega &= (\psi_+ + i\,\psi_-)\wedge\omega = 0\,, \\ \psi_+\wedge\psi_- &= \tfrac{2}{3}\,\omega^3\ , \end{aligned}$$

and we defined $\Omega^{(3,0)}=\psi_+ + i\,\psi_-$.

Finally, we give the following identities by which we can simplify later calculations,

$$\begin{aligned} \gamma_a\theta &= \tfrac{e^Z}{\sqrt{2}}(\gamma_a + v_a + i\varphi_{abc}v^b\gamma^c)\theta_0\ , \\ \gamma_{ab}\theta &= \tfrac{e^Z}{\sqrt{2}}(i\varphi_{abc}\gamma^c + i\varphi_{abc}v^c + \psi_{abcd}v^c\gamma^d - 2v_{[a}\gamma_{b]})\theta_0\ , \\ \gamma_{abc}\theta &= \tfrac{e^Z}{\sqrt{2}}(i\varphi_{abc} + \psi_{abcd}\gamma^d + 3iv_{[a}\varphi_{bc]d}\gamma^d - \psi_{abcd}v^d - 4i\varphi_{[abc}\gamma_{d]}v^d)\theta_0\,, \\ \gamma_{abcd}\theta &= \tfrac{e^Z}{\sqrt{2}}(-\psi_{abcd} - 4i\varphi_{[abc}\gamma_{d]} - 5\psi_{[abcd}\gamma_{e]}v^e \\ &\qquad -4iv_{[a}\varphi_{bcd]} - 4v_{[a}\psi_{bcd]e}\gamma^e)\theta_0\ , \\ \gamma_{abcde}\theta &= \tfrac{e^Z}{\sqrt{2}}(-5\psi_{[abcd}\gamma_{e]} - i\varepsilon_{abcdefg}\gamma^g v^f - 5v_{[a}\psi_{bcde]} - 20iv_{[a}\varphi_{bcd}\gamma_{e]})\theta_0\,, \\ \gamma_{abcdef}\theta &= \tfrac{e^Z}{\sqrt{2}}(-i\varepsilon_{abcdefg}\gamma^g + \varepsilon_{abcdefg}v_h\gamma_j\varphi^{ghj} - i\varepsilon_{abcdefg}v^g)\,\theta_0\ . \end{aligned} \tag{2.28}$$

2.4 The Killing spinor equation

Let us come back to our initial problem in solving the external and internal Killing spinor equations (2.5),(2.7). We also take into account the geometrical Ansätze (2.9) for the two cases of having one real or one complex spinor on the 7-manifold. But these two cases which we have seen in the last chapter correspond either to a G_2- or a SU(3)-structure on M^7. Therefore, we want to distinguish these two cases in the sequel by using the language of structures.

Before we discuss the two structures independently we can find for the external equation

$$0 = \widetilde{\eta} + \big[\hat{\gamma}^5\otimes\big(\frac{1}{2}dA + \frac{im}{36}\big) + \frac{1}{144}e^{-3A}\,(\mathbb{I}\otimes F)\big]\eta \tag{2.29}$$

and defining $\widehat{\eta}=e^{-\frac{A}{2}}\eta$, the internal equation (2.7) yields

$$0 = \mathbb{I}\otimes\big(\nabla_X + \frac{i\,m}{48}\,\gamma_a\big)\widehat{\eta} - e^{-\frac{A}{2}}\,\hat{\gamma}^5\,X\cdot\widetilde{\eta} - \frac{1}{12}\,e^{-3A}\,\hat{\gamma}^5\otimes(X\lrcorner F)\widehat{\eta}\ . \tag{2.30}$$

where we distinguish the spinors $\eta,\widetilde{\eta}$ and $\widehat{\eta}$.

Since $\widetilde{\eta}$ includes only W_1, W_2 and by considering dA as a vector field we only get algebraic constraints from the external equation (2.29) which we can use to solve the internal differential equation.

Since we analyse the two structures in the following two sections separately we should also mention that by reducing the structure group to G also the exterior forms decompose into G-modules. E.g. by having a G_2-structure the 4-forms decompose into

$$\Lambda^4 = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \tag{2.31}$$

where the subscript indicates the G_2 representations. This also implies that the 4-form flux on the 7-manifold M^7 splits into three G_2-invariant parts,

$$F \in \Lambda^4 \longrightarrow F^{(1)} \oplus F^{(7)} \oplus F^{(27)} \,, \tag{2.32}$$

where obviously e.g. $F^{(7)} \in \Lambda^4_7$.

Let us spend some words about the three different representations which we can consider from two points of view. Firstly, we can project out of the general 4-form the specific modules or secondly we can ask about embeddings of the modules into the 4-form.

One can extract the components $\mathcal{F}^{(1)}, \mathcal{F}^{(7)}, \mathcal{F}^{(27)}$ of the modules from the 4-form by using the structure forms φ, ψ,

$$\begin{array}{ccc} \mathcal{F}^{(1)} & \longleftrightarrow & \psi \lrcorner F \in \Lambda^4_1 \,, \\ \mathcal{F}^{(7)} & \longleftrightarrow & \varphi \lrcorner F \in \Lambda^4_7 \,, \\ \mathcal{F}^{(27)} & \longleftrightarrow & \psi \lrcorner F - \mathrm{tr} \in S^2_0 \end{array} \tag{2.33}$$

where S^2 denotes the symmtric 2-tensors and $(\cdot)_0$ indicates that the trace, tr, is removed. It should be clear from the context how many legs were contracted, e.g. $\psi \lrcorner F \in S^2$ means in components $F_{cde\{a}\psi_{b\}}{}^{cde}$ which is an symmetric 2-index object. This becomes plausible since the symmetric 2-tensors decompose under G_2 by $S^2 \to S^2_0 \oplus \mathrm{tr}$. Note, by the projection onto the 2-forms we obtain

$$\psi \lrcorner F \in \Lambda^2 \longleftrightarrow \mathcal{F}^{(7)} \lrcorner \varphi \in \Lambda^2 \tag{2.34}$$

We can also embed the modules of interest into the 4-form by

$$F = \mathcal{F}^{(1)}\, \psi + \mathcal{F}^{(7)} \wedge \varphi + \mathcal{F}^{(27)} \,. \tag{2.35}$$

Here the symbol $\mathcal{F}^{(27)}$ indicates already the embedding into the 4-froms, $\mathcal{F}^{(27)} \hookrightarrow \Lambda^4_{27}$, which can be achieved by (in components) $\mathcal{F}^{(27)}_{k[a}\, \psi^k{}_{bcd]}$.

2.4.1 G_2-manifolds

Direct product spinor

The solution we try to figure out is based on the fact that the internal manifold M^7 admits exactly one spinor $\theta \in \Delta$ and we characterise the result by a geometrical G_2-structure. The 11d spinor we use is a direct product spinor, $\eta = \epsilon \otimes \theta$.

Since we introduced in the last section the decomposition of the internal flux into G_2-modules we now can use this splitting to derive the algebraic constraints from the external equation (2.29). Making extensively use of the decomposition we also use the identities (2.13) to obtain

$$\begin{aligned} F\cdot\theta_0 &= \Big(\mathcal{F}^{(1)}+i\mathcal{F}^{(7)}\cdot\Big)\,\theta_0\ , \\ (X\lrcorner F)\cdot\theta_0 &= \Big(-\mathcal{F}^{(1)}\,X\cdot+i\,X\lrcorner\mathcal{F}^{(7)}+\,X\lrcorner(\mathcal{F}^{(7)}\lrcorner\varphi)\cdot+X\lrcorner\mathcal{F}^{(27)}\cdot\Big)\,\theta_0\ , \end{aligned} \tag{2.36}$$

where we suppressed the numerical pre-factors.

We attack the external equation (2.29), involve the first identity in (2.36) and recapitulate that the 4d spinor ϵ is of Majorana type. This implies that the terms $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\hat{\gamma}^5\epsilon)$ have to vanish separately,

$$\begin{aligned} \mathcal{O}(\hat{\gamma}^5\epsilon):&\qquad 0=\Big(i\,W_2+\tfrac{1}{2}dA\cdot+\tfrac{im}{36}\Big)\,\theta_0\ , \\ \mathcal{O}(\epsilon):&\qquad 0=W_1\theta_0-\tfrac{1}{6}e^{-3A}\Big(\mathcal{F}^{(1)}+i\mathcal{F}^{(7)}\cdot\Big)\,\theta_0\ . \end{aligned}$$

Due to the independence of the terms $\mathcal{O}(\theta_0)$ and $\mathcal{O}(X\cdot\theta_0)$, the final solution for the external equation is

$$A=const.\ ,\qquad m=-36\,W_2\qquad \mathcal{F}^{(7)}=0\ ,\qquad \frac{e^{-3A}}{6}\mathcal{F}^{(1)}=W_1\ . \tag{2.37}$$

For the internal equation (2.30) we can procede in a similar way and include also the just derived algebraic constraints (2.37),

$$\begin{aligned} \mathcal{O}(\epsilon):&\qquad 0=\Big(\nabla_X+i\,\tfrac{7m}{144}\,X\cdot\Big)\,e^Z\,\theta_0\ ,\quad X\in TM^7\ , \\ \mathcal{O}(\hat{\gamma}^5\epsilon):&\qquad 0=\Big(W_1-\tfrac{2}{7}\,e^{-3A}\mathcal{F}^{(1)}\Big)\,X\cdot+\tfrac{1}{2}\,e^{-3A}\,X\lrcorner\mathcal{F}^{(27)}\cdot\theta_0\ ,\quad X\in TM^7\ . \end{aligned} \tag{2.38}$$

The second equation yields an algebraic constraint which must be satisfied for all $X\in TM^7$. Taking (2.37) into account we obtain $0=W_1=\mathcal{F}^{(1)}=\mathcal{F}^{(27)}$ and thus the internal 4-form flux has to vanish identically,

$$0=F\in\Lambda^4\ . \tag{2.39}$$

The equation which is left comes from the $\mathcal{O}(\epsilon)$ constraint and it is not difficult to see that also $dZ=0$, we furthermore choose $Z=1$. This remaining differential equation of the spinor $\theta_0\in\Delta$ characterises a 7-manifold admitting a nearly parallel (or weak) G_2-structure (see also [14]). Such a manifold has torsion only in the singlet which in our case is given by the parameter of the external 4-form flux m or equivalently by W_2. It is thus possible to solve the 11d Killing spinor equation by having a non-trivial Freund-Rubin parameter but, additionally, this forces the 4d external space to be non-Minkowskian. Note, the non-trivial parameter W_2 measures the non-flatness of the curvature in 4d.

The differential condition of the spinor can also be re-formulated in terms of differential forms where the non-trivial torsion classes can be figured out easily,

$$d\varphi=-\frac{7\,m}{18}\,\psi\quad ,\qquad d\psi=0 \tag{2.40}$$

i.e. using (2.21) only $T^{(1)}$ is non-zero. A further differentiation of the first equation in (2.40) yields $0=dd\varphi\sim dm\wedge\psi$ by which we conclude that the Freund-Rubin paramter m has to be constant over the 7-manifold.

Non-direct product spinor

In the last section we used a specific Ansatz for the 11d spinor to solve the Killing spinor equations. Let us go back again before we used this Ansatz and consider again the external and internal equations (2.29),(2.30). Both equations can be seperated in terms including the operator $\hat{\gamma}^5$ or not. This formal seperation becomes manifest for the external and internal equations by using the direct product Ansatz as done in the last section. This is due to the fact that the 4d spinor is Majorana and thus the terms $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\hat{\gamma}^5\epsilon)$ become independent. In other words the direct spinor Ansatz does not mix terms, including $\hat{\gamma}^5$ or not, within the Killing spinor equations. The just mentioned mixing can only be achieved by using a spinor Ansatz for η that contains at least one $\hat{\gamma}^5$-term. For the remainder of this section we want to discuss a non-direct product Ansatz which give rise to a mixing.

Let $\Omega \in \Lambda^\bullet$ be a formal sum of exterior forms $\Omega^{(n)} \in \Lambda^n$ on M^7 accompanied by constants c_n, where n labels the degree of the forms, $n \in \{0, \dots, 7\}$. By Clifford multiplication we obtain the non-direct product spinor Ansatz,

$$\widehat{\eta} = \Omega \cdot \epsilon \otimes \theta \tag{2.41}$$

Since we want to classify the 11d Killing spinor equations with respect to a G_2 structure the internal spinor and also the forms $\Omega \in \Lambda^\bullet$ must be G_2-invariant. This restricts the formal sum to be

$$\Omega = c_0 \oplus c_3\varphi \oplus c_4\psi \oplus c_7 \,\mathrm{vol}_7 \tag{2.42}$$

where φ, ψ are the G_2 structure forms.

This yields the 11d spinor $\widehat{\eta}$,

$$\begin{aligned} \widehat{\eta} &= (c_0 + c_3\varphi + c_4\psi + c_7 \,\mathrm{vol}_7) \cdot \epsilon \otimes \theta \,, \\ &= \Big(c_0(\mathbb{I}\otimes\mathbb{I}) + c_3(\hat{\gamma}^5 \otimes \varphi\cdot) + c_4(\mathbb{I}\otimes\psi\cdot) + c_7(\hat{\gamma}^5 \otimes -i\mathbb{I})\Big) \,, \epsilon \otimes \theta \end{aligned} \tag{2.43}$$

where we used (2.3), $vol_7\cdot = -i$, in the last term. By a further short computation using (2.13) we find

$$\widehat{\eta} = \Big[(c_0 - 7c_4)(\mathbb{I}\otimes\mathbb{I}) - i\,(c_7 - 7c_3)(\hat{\gamma}^5 \otimes \mathbb{I})\Big]\epsilon \otimes \theta \tag{2.44}$$

where we finally have the 11d non-direct product spinor Ansatz. The two terms which appear are responsible for the above mentioned non-trivial mixing of the terms.

We use this spinor in the following to investigate a modified 11d Killing spinor equation with respect to (2.29),(2.30). Firstly, we take the 4d space to be Minkowski which implies that $W_1 = W_2 = 0$, set the Freund-Rubin parameter m to zero and use a more complicated (compare with (2.45)) 11d metric by,

$$ds^2 = e^{2A}\, g^{(4)} \,+\, e^{-2B}\, h^{(7)} \,, \tag{2.45}$$

where $A, B \in C^\infty(M^7)$ are functions on the 7-manifold (M^7, h).

For convenience we denote the modified external equation,

$$0 \;=\; \partial_X\eta + \tfrac{1}{2}e^{A+B}\Big[X\cdot\hat{\gamma}^5 \otimes dA\cdot + \tfrac{1}{72}e^{3B}\, X \otimes F\Big]\eta\,, \quad X \in TM^{1,3}, \tag{2.46}$$

and the internal equation , where $X \in TM^7$,

$$0 \;=\; \Big[\mathbb{I}\otimes\Big(\nabla_X - \tfrac{1}{2}X\wedge dB\,\cdot\Big) + \tfrac{1}{144}e^{3B}\,\hat{\gamma}^5\otimes\Big(X\cdot F\cdot - 12\,(X\lrcorner F)\,\cdot\Big)\Big]\eta\,. \tag{2.47}$$

Since the external 4d space is Minkowski the integrability condition for the spinor η must vanish,

$$\Big[\nabla_X,\nabla_Y\Big] = \Big[\partial_X,\partial_Y\Big] = 0\,,\quad X\in TM^{1,3}, \tag{2.48}$$

which yields using (2.46) the condition,

$$0 = \Big[\|dA\|(\mathbb{I}\otimes\mathbb{I}) - \frac{1}{9}e^{3B}(\mathbb{I}\otimes F^2\cdot) + \frac{1}{9}e^{3B}(\hat{\gamma}^5\otimes\, dA\lrcorner F\cdot)\Big]\eta\,. \tag{2.49}$$

This constraint can be in components re-written in the following more convenient form

$$0 = \Big[(\mathbb{I}\otimes\gamma^{pq})\partial_q A - \frac{1}{12}e^{3B}\hat{\gamma}^5\otimes F^p\Big]\gamma_p\gamma_m\Big[(\mathbb{I}\otimes\gamma^{mn})\partial_n A + \frac{1}{12}e^{3B}\hat{\gamma}^5\otimes F^m\Big]\eta\,, \tag{2.50}$$

which can be solved by

$$0 = \Big[\mathbb{I}\otimes X\wedge dA\,\cdot \pm \frac{1}{12}e^{3B}\hat{\gamma}^5\otimes X\lrcorner F\,\cdot\Big]\eta\,,\quad X\in TM^7. \tag{2.51}$$

Attacking this condition by $X\in TM^7$ via Clifford multiplication and plugging it back into the external equation we obtain, $\partial_X\eta = 0$. This just means that the 11d spinor η has to be constant on $M^{1,3}$ and we can drop the term $\partial_X\eta$ from the external equation (2.46) which yields,

$$0 \;=\; \Big[\hat{\gamma}^5\otimes dA\,\cdot + \tfrac{1}{72}e^{3B}\,\mathbb{I}\otimes F\Big]\eta\,. \tag{2.52}$$

Next we have to solve the internal equation (2.47) where we follow [5]. Note, in this equation there appear the two flux terms containing F and $X\lrcorner F$ but with the two above equations (2.52) and (2.51) we can substitute these by using only the warp factor A.

The final internal equation becomes,

$$0 = \mathbb{I}\otimes\Big(\nabla_X - \frac{1}{2}X\wedge d(A+B)\,\cdot\Big)\widehat{\eta}\,,\quad X\in TM^7, \tag{2.53}$$

where we also rescaled the spinor by $\eta = e^{\frac{A}{2}}\widehat{\eta}$ and only took the $+$-equation of (2.51) into account.

So far, we derived the algebraic constraints from the external equation and differential condition from the internal equation. But before we want to apply the 11d non-direct spinor we want to see how the 11d direct spinor Ansatz behaves under the modified situation compared to the last section. Since the external spinor ϵ is Majorana the external constraint (2.51) yields,

$$\begin{aligned}\mathcal{O}(\epsilon):&\qquad 0 = X\wedge dA\cdot\theta\,,\quad X\in TM^7\,,\\ \mathcal{O}(\hat{\gamma}^5\epsilon):&\qquad 0 = X\lrcorner F\cdot\theta\,,\quad X\in TM^7\,,\end{aligned} \tag{2.54}$$

which must be satisfied for all $X\in TM^7$. We immediately get $dA = 0$ and thanks to (2.36) also $F = 0$. But we still have the function B which remains unconstraint. If we identify $B = -A$ we also have $dB = 0$ and get back to the result of the last section. But this is not necessary and thus the internal equation become $(\nabla_X - \frac{1}{2}X \wedge dB\cdot)\theta$, $X \in TM^7$. Such a internal 7-manifold admits a so-called conformal G_2-structure. This structure carries only the Lee-one-form torsion class, $T^{(7)}$. And in the above case the Lee-form is also exact, dB. In other words we can always start from the more general metric Ansatz from above (2.45) but this only results in a exact Lee-form which modifies the torsion class $T^{(7)}$.

We want to investigate the non-direct product spinor (2.44) in the following. Let us analyse first the algebraic constraint (2.51). We obtain,

$$\begin{aligned} \mathcal{O}(\epsilon): &\qquad (c_0 - 7c_4)(X \wedge dA)\cdot\theta &=& \pm i\tfrac{1}{12}(c_7 - 7c_3)e^{3B}(X \lrcorner F)\cdot\theta\,, \\ \mathcal{O}(\hat{\gamma}^5\epsilon): &\qquad i(c_7 - 7c_3)(X \wedge dA)\cdot\theta &=& \pm\tfrac{1}{12}(c_0 - 7c_4)e^{3B}(X \lrcorner F)\cdot\theta\,, \end{aligned} \tag{2.55}$$

for all $X \in TM^7$.

By means of (2.36) a short calculation shows that the flux $F \in \Lambda^4$ and the differential of the function $A \in C^\infty(M^7)$ have to vanish identically. We conclude that also for the non-direct spinor Ansatz it is only possible to have a non-trivial function B. This function is responsible that a torsion-free G_2-holonomy manifold picks up a exact Lee-one-form and jumps to the class of conformal G_2-structures having torsion in $T^{(7)}$ only.

2.4.2 A note on α'-corrections

The correction and the curvature

In this chapter we discussed so far compactifications of 11d supergravity on 7-dimensional spaces. These 7-manifolds have a certain G-structure to capture the right amount of supersymmetry. But we also know that in general supergravity theories are the low-energy limits of string and M-theory. Let us consider string theory from an effective action point of view. An expansion in powers of the inverse string tension α' then yields the supergravity as its leading term by having integrated out the massiv modes.

We are given for this section with a supergravity theory compactified on a 7-dimensional space M^7 having exactly one parallel spinor and no fluxes at all. Since we assume the external space to be Minkowski and set the warp factor to zero the internal space is torsion free and has G_2-holonomy. A further property of a precise G_2-manifold is the Ricci-flatness. The idea of the authors in [69] (see also [79]) is based on the fact that at tree level in string perturbation theory the Ricci tensor can pick up corrections. It is known that the first non-trivial contribution can be captured by calculating amplitudes with four external gravitinos that is of order α'^3. The authors [69] showed on one hand that the correction of the Ricci tensor of a G_2-manifold given at d=10 string tree-level is equal to a deformation of 11d M-theory. On the other hand the authors explicitly calculated the α'^3 correction and formulated it via the curvature tensor R as an additional term that has to be added to the internal Killing spinor equation,

$$D_i = \nabla_i^{LC} - \frac{3}{4}\,\alpha'^3\,(\nabla^j R_{ikm_1m_2})R_{jlm_3m_4}\,R^{kl}{}_{m_5m_6}\,\Gamma^{m_1m_2m_3m_4m_5m_6}\,. \tag{2.56}$$

We have to keep in mind that we started with a G_2-holonomy manifold that means that the curvature tensor R is not perfectly general. Let us shortly discuss the curvature properties of G_2-manifolds where we follow strictly [17].

The curvature tensor R in general is given by a 4-tensor where two pairs of components, the first and second and the third and forth, are antisymmetric. Due to the fact that these two pairs are symmetric the curvature tensor R is an element in $R \in S^2(\Lambda^2)$. Moreover, under the torsion free G_2-structure the tensors decompose into G_2 representations that means to the curvature tensor R,

$$R \in S^2(\Lambda^2) \cong \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{77} \cong \mathrm{Scal} \oplus \mathrm{Ric}_0 \oplus S$$

where we called Scal the scalar curvature and Ric_0 the trace free Ricci-curvature. It is also a common fact [17] that the curvature R can be written as

$$R_{ijkl} = S_{ijkl} + \varphi_{ij}{}^{p}\, T_{pkl}$$

where T originates from derivatives of the torsion only i.e. non vanishing torsion does never affect the S-term in R. The property of S given by $\varphi_m{}^{ij} S_{ijkl} = 0$ reflects the fact that the first pair of components are restricted from $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14} \to \Lambda^2_{14}$ i.e. $S \in \mathfrak{g}_2 \otimes \Lambda^2 \cap S^2(\Lambda^2)$.

By means of the Bianchi-identity we have to put 28 constraints on T_{pkl}. The Ricci-tensor $\mathrm{Ric} = \mathrm{Scal} \oplus \mathrm{Ric}_0 \in S^2(\Lambda^1)$ can be derived as usual from the curvature tensor R by the contraction $\mathrm{Ric}_{ij} = R^k{}_{ikj}$ and measures the symmetric part $\mathbf{1} \oplus \mathbf{27}$ of R. We write,

$$\mathrm{Ric}_{ij} = \varphi^{pq}{}_i T_{pqj}.$$

This equation proves what we already mentioned above. A G_2-holonomy manifold is torsion free has vanishing Ricci-tensor. This yields that the curvature tensor of a G_2-holonomy manifold is given by

$$R_{ijkl} = S_{ijkl}.$$

If we substitute this back into (2.56) we get as a first result the actual curvature contribution since we are interested in the correction of a G_2-holonomy manifold, i.e.

$$D_i = \nabla_i^{LC} - \frac{3}{4}\,\alpha'^3\,(\nabla^j S_{ikm_1m_2}) S_{jlm_3m_4}\, S^{kl}{}_{m_5m_6}\, \Gamma^{m_1m_2m_3m_4m_5m_6}\,.$$

The authors in [69] also showed that it is possible to express (2.56) by means of the G_2-invariant 3-form φ,

$$D_i = \nabla_i^{LC} - \frac{i}{2}\,\alpha'^3\,\varphi_{ikl} \nabla^k Z^{lj}\,\Gamma_j\,, \tag{2.57}$$

$Z \in S^2(\Lambda^1)$, where the Bianchi identity implies $\nabla_i Z^{ij} = 0$.

Expression (2.57) has the advantage that it does not involve complicated contractions as it is the case in (2.56). On one side it remains a highly non-trivial object since it includes derivatives of Z but on the other side a group theoretical analysis is now more convenient.

The basic idea is to carefully analyse the correction term given in (2.57) in terms of G_2-representations and achieve a relation to G_2-torsion. We show that (2.57) can be fully captured by distinct torsion classes. We started with a torsion free G_2-manifold and found that $T_{pkl} = 0$. But since we prove that the correction means torsion the object T_{pkl} gets modified by α'^3 and is no longer zero. This immediately yields to a contribution to the Ricci-tensor as well. In other words, the corrected 7-manifold is no longer Ricci-flat at tree-level that was also found by [69]. We are interested in the following only in the group theoretical analysis of (2.57) and relate it to G_2-structures.

The correction and torsion classes

Let us denote the correction term given in (2.57) by Q. Becoming more precise, we consider

$$D_X = \nabla^{LC}_X + Q(X)\cdot$$

where ∇^{LC} is the Levi-Civita connection in the spin-bundle. The object $Q(X)$ is a vector and can be considered as the map $Q : TM^7 \to TM^7$. Equivalently, Q is an object in $\Lambda^1 \otimes TM^7$ and is at most an element in the space

$$\Lambda^1 \otimes TM^7 \cong \mathbf{7} \otimes \mathbf{7} \cong \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27}$$

But on the other side we know from (2.19) that a manifold admitting a G_2-structure has torsion in the space

$$(\mathfrak{so}(7)/\mathfrak{g}_2) \otimes \mathbb{R}^7 \cong \mathfrak{g}_2^\perp \otimes \mathbf{7} \cong \mathbf{7} \otimes \mathbf{7} \cong \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27} \tag{2.58}$$

This means that we can interpret the $(1,1)$-tensor Q as an object describing torsion. Let us investigate $Q(X)\cdot$,

$$Q_i{}^j \Gamma_j = -\frac{i}{2}\,\alpha'^3\,\varphi_{ikl} \nabla^k Z^{lj}\,\Gamma_j$$

from a group theoretical perspective only, where $Z \in S^2(\Lambda^1)$ and φ ist the G_2-3-form.

By using G_2-modules we attack $\nabla^k Z^{lj}$ and compute

$$\begin{aligned} \Lambda^1 \otimes S^2(\Lambda^1) &\cong \mathbf{7} \otimes (\mathbf{1} \oplus \mathbf{27})\,, && (2.59)\\ &\cong \mathbf{77} \oplus \mathbf{27} \oplus 2\cdot\mathbf{7} \oplus \mathbf{64} \oplus \mathbf{14}\,, && (2.60)\end{aligned}$$

where we note that the multiplicity of $\mathbf{7}$ is 2 but one can easily show that one is coming from the derivative of the singlet while the other arise from the derivative of $\mathbf{27}$. We write,

$$\nabla^k Z^{lj} \in \mathbf{77} \oplus \mathbf{27} \oplus 2\cdot\mathbf{7} \oplus \mathbf{64} \oplus \mathbf{14}.$$

We can represent the two high dimensional G_2-modules on the 3-tensors where we use [17]. The representation of $\mathbf{64}$ on the 3-tensors is given by having the first two indices antisymmetric while the last two are symmetric. A totally symmetric G_2-invariant tensor represents the $\mathbf{77}$. The contraction by φ maps $\mathbf{64}$ and $\mathbf{77}$ to the space of 2-tensors. Since we do it with the G_2-equivariant map $\varphi \in \Lambda^3_1$ this is an isomorphism. But the 2-tensors are represented by $\mathbf{1} \oplus \mathbf{7} \oplus \mathbf{14} \oplus \mathbf{27}$ only i.e. the $\mathbf{64}$ and $\mathbf{77}$-module drops out that can also be seen by the Lemma of Schur.

The remaining G_2 modules are $\mathbf{7}$, $\mathbf{14}$ and $\mathbf{27}$ which can be represented on the 2-tensors. We collect them together in $F \in S^2_0(\Lambda^1) \oplus \Lambda^2$. Let us now embed $F = F^7 \oplus F^{14} \oplus F^{27}_0$ into the space $\Lambda^1 \otimes S^2(\Lambda^1)$ given by $\nabla^k Z^{lj}$, where the second and third index is symmetric. This is achieved by the formula $\varphi_{i\{j}{}^r F_{k\}r}$. After doing the contraction by φ we find for $Q = Q_7 \oplus Q_{14} \oplus Q_{27}$,

$$Q \cong 2\cdot\Lambda^2_7 \oplus \Lambda^2_{14} \oplus S^2_0(\Lambda^1).$$

We want to be more explicit and give a full description for $Q(X)\cdot$. Let us use (2.14) and (2.15) for a representation of the two-forms and write F^{27}_0 for the symmetric and traceless matrix. The result is given by,

$$\begin{aligned} Q(X)\cdot &\cong X \lrcorner\, (\,F^7 \lrcorner\, \varphi)\cdot + X \lrcorner\, F^{14}\cdot + X \lrcorner\, F^{27}\cdot\,, && (2.61)\\ &\cong X^a\,(\varphi_{abc}\,(F^7)^b\,\gamma^c + (F^{14})_{ab}\,\gamma^b + (F^{27}_0)_{ab}\,\gamma^b\,)\,. && (2.62)\end{aligned}$$

We give several comments,

- Q lacks of having a singlet that is due to that fact that the singlet torsion class vanishes i.e. the correction will never lead to a weak G_2-structure.
- We actually have two **7** representations. But since one is coming from the derivative of the singlet (2.59), say a function f, we write $F^7 = df + w$, where $w \in \Lambda^1$.
- The correction term Q can be interpreted as torsion but it is not possible to characterise it by a totally skew symmetric torsion 3-form [34, 35], because it is not possible to embed F^{14} into Λ^3. Furthermore the appearance of F^{14} prevents the G_2-structure being induced by α'^3-corrections to be integrable [35].

2.4.3 SU(3)-manifolds

In the last section the 11d Killing spinor equation was solved by using a G_2-structure. But for this case it was not possible to have a non-trivial $F \in \Lambda^7$. In this section we want to reduce the structure group further to SU(3) and try to charactrise solutions admitting a non vanishing 4-form flux $F \in \Lambda^4$. The explicit reduction to obtain the SU(3)-structure can be claimed by the assumption of having two globally defined spinors being perpendicular. In section 4.4 we gave a complex formulation and the corresponding explicit construction (2.22). Remember, we defined the 11d spinor Ansatz in (2.23) by

$$\eta = \epsilon \otimes \theta + \epsilon^\star \otimes \theta^\star \ , \tag{2.63}$$

where the 4d Weyl spinors $\epsilon, \epsilon^\star$ have opposite chirality. We use $(1 - \hat{\gamma}^5)\epsilon = 0$.

Let us first consider the external equation (2.29) by using the above spinor (2.63) and also the W-Ansatz from (2.9). The external equation can be decomposed into the terms $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^\star)$. Due to having the property $(\mathcal{O}(\epsilon))^\star = \mathcal{O}(\epsilon^\star)$ it is sufficinet to consider the algebraic constraints coming from

$$\mathcal{O}(\epsilon): \quad 0 = e^{K/2}\, W\theta^\star + \Big(\frac{im}{36} + \frac{1}{2}\partial A + \frac{1}{144}\, e^{-3A}\, F\Big)\theta \, , \tag{2.64}$$

where we also introduced a Kähler potential by the substitution $W \to e^{K/2}\, W$.

The following analysis is based on the fact that we consider the SU(3)-structure in such a way that it is build up by the G_2 spinor θ_0 and the vector field $v \in TM^7$ via the spinor (2.22). Thus, we can use the methods of a G_2-structure but additionally have to involve differential and algebraic constraints for the non-vanishing v making the reduction to SU(3) manifest.

Therefore, equation (2.64) yields eight algebraic conditions on the components of θ in the compexified spin module $\Delta \otimes \mathbb{C}$. It decomposes into the independent terms $\mathcal{O}(\theta_0), \mathcal{O}(X \cdot \theta_0)$, where $X \in TM^7$, and can furthermore seperated into real and imaginary parts,

$$\begin{array}{llcl}
\mathcal{O}(\theta_0): & 0 & = & e^{K/2}\, W_1 - \frac{1}{6}\, e^{-3A}\, \mathcal{F}^{(1)} + \frac{1}{2} dA(v) \ , \\
 & 0 & = & e^{K/2}\, W_2 + \frac{m}{36} - \frac{1}{6}\, e^{-3A}\, \mathcal{F}^{(7)}(v) \ , \\
\mathcal{O}(X \cdot \theta_0): & 0 & = & [-e^{K/2}\, W_1 + \frac{1}{42}\, e^{-3A}\, \mathcal{F}^{(1)}]\, v + \frac{1}{2} dA - \frac{1}{3}\, e^{-3A}\, \mathcal{F}^{(27)}(v) \ , \\
 & 0 & = & \big[- e^{K/2}\, W_2 + \frac{m}{36} \big]\, v + \frac{1}{6}\, e^{-3A}\, \mathcal{F}^{(7)} + \frac{1}{2}\varphi(dA, v) \ ,
\end{array} \tag{2.65}$$

where we abused notation in the last equation.

Contracting the last two equations by v and appropriately adding and substracting them from the first two equations we conclude,

$$\begin{aligned} W = W_1 + i\,W_2 &= \frac{1}{6}\Big(\frac{4}{7}\mathcal{F}^{(1)} - \mathcal{F}^{(27)}(v,v) + i\mathcal{F}^{(7)}(v)\Big) \\ m &= 0\,, \end{aligned} \tag{2.66}$$

where we identified $K = -6A$. The conditions on the flux become,

$$\mathcal{F}^{(7)} - \mathcal{F}^{(7)}(v)v = d(e^{3A}) \lrcorner \varphi(v)\,, \tag{2.67}$$
$$= 2\,\mathcal{F}^{(27)}(v) \lrcorner \varphi(v)\,, \tag{2.68}$$
$$2\,\mathcal{F}^{(27)}(v) = \Big(-\frac{3}{7}\mathcal{F}^{(1)} + \mathcal{F}^{(27)}(v,v)\Big)v + d(e^{3A})\,. \tag{2.69}$$

For the internal equation (2.30) we take the 11d spinor (2.63) into account and note that it is sufficient to consider,

$$\mathcal{O}(\epsilon): \qquad 0 = \nabla_X\hat{\theta} - e^{K/2}\,W\,X\cdot\hat{\theta}^\star - \tfrac{1}{12}\,e^{-3A}\,(X \lrcorner F)\cdot\hat{\theta}\,, \tag{2.70}$$

where $\hat{\theta} = \frac{1}{\sqrt{2}}e^{-\frac{A}{2}+Z}(\mathbb{I}+v)\theta_0$.

The last term including the internal flux $F \in \Lambda$ in the just obtained equation (2.70) can be decomposed under G_2 by using (2.31), $F = F^{(1)} + F^{(7)} + F^{(27)} \in \Lambda^4$,

$$\begin{aligned} X \lrcorner F^{(1)}\cdot\hat{\theta} &= \tfrac{24}{7}\mathcal{F}^{(1)}(X\lrcorner - X\cdot)\theta_0\;, \\ X \lrcorner F^{(7)}\cdot\hat{\theta} &= 3\Big[2i\mathcal{F}^{(7)}(X) - \varphi(X,v,\mathcal{F}^{(7)}) - \varphi(X,\mathcal{F}^{(7)})\cdot \\ &\qquad - i\,(X\lrcorner v)\mathcal{F}^{(7)}\cdot - i\,\mathcal{F}^{(7)}(v)X\cdot\Big]\theta_0\;, \\ X \lrcorner F^{(27)}\cdot\hat{\theta} &= 6\Big[-X\lrcorner\mathcal{F}^{(27)}(v)\cdot + X\lrcorner\mathcal{F}^{(27)}\cdot + i\varphi(X,v)\lrcorner\mathcal{F}^{(27)}\cdot \\ &\qquad - i\mathcal{F}^{(27)}(v)\lrcorner\varphi(X)\cdot\Big]\,\theta_0\;. \end{aligned}$$

where we suppressed the function $\frac{1}{\sqrt{2}}e^{-\frac{A}{2}+Z}$.

Due to the fact that we can embed the strict SU(3)-structure into a G_2-structure (2.27) we can calculate the torsion classes of this G_2-structure $T^{(i)}, i \in \{1,7,14,27\}$,

$$\begin{aligned} T^{(1)} &\longleftrightarrow W_2 \in \Lambda^0\;, \\ T^{(7)} &\longleftrightarrow 48\,W_1\,v - \tfrac{24}{7}\,\mathcal{F}^{(1)}\,v - \tfrac{3}{2}\,\varphi(v,\mathcal{F}^{(7)}) + 27\mathcal{F}^{(27)}(v) \in \Lambda^1\,, \\ T^{(27)} &\longleftrightarrow \mathcal{F}^{(7)}\odot v \,-\, 2\,\mathcal{F}^{(27)}\lrcorner\varphi(v) \in S_0^2\;, \\ T^{(14)} &\longleftrightarrow \psi(v,\mathcal{F}^{(7)}) \,+\, 4\,\mathcal{F}^{(7)}\wedge v \,-\, 2\,\mathcal{F}^{(27)}(v)\lrcorner\varphi \,-\, 12\,\mathcal{F}^{(27)}\lrcorner\varphi(v) \in \Lambda^2\;, \end{aligned}$$

where the contractions are done in the obvious way and $\odot$ denotes the symmetric product. We also used here the constraint

$$0 = d[-A + 2\mathrm{Re}(Z)]\,,$$

which can be derive by calculating the differential of the norm of $\hat{\theta} \in \Delta \otimes \mathbb{C}$,

$$d\|\hat{\theta}\| = e^{-A+2\mathrm{Re}(Z)}\, d[-A + 2\,\mathrm{Re}(Z)] \;=\; \Big[(\nabla_a \hat{\theta})^T \hat{\theta}^* + \hat{\theta}^T \nabla_a \hat{\theta}^*\Big] dx^a = 0$$

Due to the fact that the underlying structure is effectively characterised by SU(3) the just derived four torsion classes $T^{(i)}, i \in \{1, 7, 14, 27\}$ does not give the full characterisation of the 7-manifold. It is shown in [24] that the four G_2 torsion classes can be re-constructed by using the five SU(3) torsion classes. Thus, the above four G_2 invariant torsion classes can be decomposed under SU(3) or on the other side one can directly calculate the five SU(3) torsion classes to give the actual classification of the 7-manifold with respect to the structure group. Work in this direction was first done by [26] where the assumption of the external space being Minkowski was made, i.e. $W = 0$.

In the sequel of this section we only calculate the differential of $v \in TM^7$ and re-write the algebraic constraints from above by using SU(3)-structure invariants.

The differential condition for the normalised vector field $v \in TM^7$, necessary for reducing the structure group geometrically to SU(3) is given by,

$$d(e^{3A} v) = 0 \ .$$

This result can be obtained by re-writting the object dv using the spinor language, the corresponding covariant derivative given in (2.70) and certain algebraic constraints from above. Compare also [26, 68], but there the authors used a 11d metric Ansatz where the function A only appears in front of the external metric.

Let us now re-formulate the algebraic constraints of W, A and the 4-form flux $F \in \Lambda^4 M^7$ given in (2.66-2.69). This we will do by considering the 7-manifold as a foliation by the normalised vector field $v \in TM^7$ and the associated 6-manifold M^6. Therefore, we split the algebraic constraints in a part parallel to v and in a horizontal part restricted to M^6. Since we are not focusing in the following on $F \in \Lambda^4 M^7$ with respect to the G_2 decomposition we split the 4-form flux in horizontal parts,

$$F \in \Lambda^4 M^7 \longrightarrow v \wedge H + G \,, \tag{2.71}$$

where $v \lrcorner F = H \in \Lambda^3 M^6, F|_{M^6} = G \in \Lambda^4 M^6$, and decompose the 3-form H and the 4-form G under SU(3), we use (2.24)

$$\begin{aligned} H \in \Lambda^3 M^6 \;&=\; \mathbb{R}\psi_+ \oplus \mathbb{R}\psi_- \oplus [\![S^{2,0}]\!] \oplus [\![\Lambda^{1,0}]\!] \,, \\ G \in \Lambda^4 M^6 \;&=\; [\![\Lambda^{2,0}]\!] \oplus [\Lambda_0^{1,1}] \oplus \mathbb{R}\omega \,. \end{aligned} \tag{2.72}$$

We are now investigating how the algebraic constraints give conditions on the horizontal fluxes H, G. The contraint of W given in (2.66) can alternatively be calculated by attacking the external equation (2.64) with respect to $q(\theta, \cdot)$, where we also use $\tilde{\Omega}^4$ from (2.26) and (2.71),

$$W = \frac{i}{36}\, \bar{\Omega}^{(3,0)} \lrcorner H \,,$$

which identifies $W_{1,2}$ of W with the part of H given by $\mathbb{R}\psi_+ \oplus \mathbb{R}\psi_-$. We also make extensively use of (2.26) in the following.

Similarly, we can hit the external equation (2.64) by $q(\theta^\star, \cdot)$ to obtain the contribution of dA in the direction parallel to v,

$$v \lrcorner d(e^{3A}) = \frac{1}{144}\omega^2 \lrcorner G\,,$$

which re-phrases the constraint given in (2.69), $v \lrcorner d(e^{3A}) = \frac{3}{7}\mathcal{F}^{(1)} + \mathcal{F}^{(27)}(v,v)$. Moreover, this implies that the SU(3)-module $\mathbb{R}$ of G is given by $v \lrcorner d(e^{3A})$ since the 4-form ω^2 is a basis for this module.

Next we consider equation (2.67) where we use the definition of $\mathcal{F}^{(7)}$ given in (2.33) and (2.71),

$$\mathcal{F}^{(7)}|_{M^6} = \frac{1}{6}\Big(-(\chi_+) \lrcorner G + 3\omega \lrcorner H\Big) = d(e^{3A}) \lrcorner \omega \tag{2.73}$$

We use also the definition (2.33) of $\mathcal{F}^{(27)}$ when investigating (2.68),

$$\mathcal{F}^{(7)}|_{M^6} = 2\mathcal{F}^{(27)}(v) \lrcorner \omega = \frac{1}{6}\Big((\chi_+) \lrcorner G + 3\omega \lrcorner H\Big) \tag{2.74}$$

The last two equations imply immediately $(\chi_+) \lrcorner G$.

In the same vain, we also attack (2.69) and get

$$2v \lrcorner \mathcal{F}^{(27)}|_{M^6} = -\frac{1}{6}\Big((\chi_-) \lrcorner G + \frac{1}{2}H \lrcorner \omega^2\Big) = d(e^{3A})|_{M^6} \tag{2.75}$$

which finally results in the constraints

$$\Omega^{(3,0)} \lrcorner G = 0\,, \qquad \text{and} \qquad \frac{1}{2}\omega \lrcorner H = d(e^{3A}) \lrcorner \omega\,. \tag{2.76}$$

The first constraint set the $[\![\Lambda^{2,0}]\!]$ part of G to zero and the second condition identifies the non-primitive part $[\![\Lambda^{1,0}]\!]$ of H with the horizontal part of dA, $d(e^{3A})|_{M^6}$.

Since we already put all conditions on the fluxes into game the primitive part $[\![S^{2,0}]\!]$ of H and the $[\Lambda_0^{1,1}]$ part of G remains unconstrained. However, one can show that the $[\lambda_0^{1,1}]$-module appears in the symmetrised version of the covariant derivative of v. One can also prove that the 6-manifold M^6 is also not complex.

Let us schematically summarise the conditions on the fluxes H, G,

$$\begin{aligned} H \in \Lambda^3 M^6 &= (W_1, W_2) \oplus [\![S^{2,0}]\!] \oplus d(e^{3A})|_{X_6}\,, \\ G \in \Lambda^4 M^6 &= [\Lambda_0^{1,1}] \oplus v \lrcorner d(e^{3A})\,. \end{aligned} \tag{2.77}$$

Chapter 3

Generalised geometries, mirror symmetry and topological models

In the last chapter we used the technique of G-structures to obtain non-trivial solutions for the 11d supergravity Killing spinor equation. The 4-form flux was interpreted as torsion and with respect to the G-structure we decomposed F into G-modules. We characterised the solutions by asking about the number of space-time supersymmetry. This question, as we showed, can be rephrased: For a given connection, what is the holonomy group that leaves invariant exactly e.g. one spinor? In our previous discussion we asked for covariantly constant spinors, but originally, the concept of holonomy or G-structures deals with the parallel transport of vectors, i.e. elements in T. We attacked this by introducing the abstract theory of principal fibre bundles and considered several associated bundles, in particular, the tangent bundle where the vector fields live in. Let us consider a G-structure as the *classical structure* and let us introduce a new type of structure that goes under the name of a *generalised structure* and was introduced by Hitchin [55].

The principle idea is that we want to parallel transport not only elements in T, but elements in $T \oplus T^*$. The dimension of this space is $2d$ and, as we will see later on, this bundle admits a signature of type (d, d), where d is the dimension of the manifold [55]. By considering once more the principle fibre bundle picture the group of our present interest (for an oriented manifold) is $SO(d, d)$. For usual vector fields we use the Lie bracket to ask about integrability. In the $T \oplus T^*$ case the Lie bracket will be substituted by the Courant bracket [55], which is not only invariant under diffeomorphisms but also under a 2-form B, that appears naturally in the Lie algebra of $SO(d, d)$. We refer to this object as the B-field. Furthermore, it is possible to twist the Courant bracket by a closed 3-form H.

Remember, that the general structure group on 6-manifolds is $SO(6)$. Introducing an algebraic structure such as an almost complex structure $J \in \mathrm{End}(T)$ and/or a symlectic structure $\omega : T \to T^*$ the group $SO(6)$ will be reduced to either $U(3)$, $Sp(6)$ or even $SU(3)$.

For generalised geometries one can also introduce an algebraic structure, called a generalised complex structure (GCS) $\mathcal{J}$ [45], which is a map $\mathcal{J} : T \oplus T^* \to T \oplus T^*$. But in this case one crucial property occurs. The classical algebric structures J and ω appear as two special cases of GCSs. This is why the formalism is called *generalised*. By further introducing a generalised metric G on $T \oplus T^*$ it is possible to define a second GCS. This setup goes under the name of a *generalised Kähler structure* (GKS). It was proven by Gualtieri [45] that this structure is equivalent to a bi-hermitian geometry that consists of a hermitian metric g, a closed 3-form H and two almost complex structures J_+ and J_- that are both compatible with the metric.

The authors in [37] proved already in 1984 that the target space of a non-linear sigma model in two dimensions admits a bi-hermitian structure. The model describes all possible embeddings of a two dimensional string into a 6-manifold, requiring a certain amount of worldsheet supersymmetry. But since the bi-hermitian structure was only poorly understood physicists used Kähler manifolds to handle the target space. Based on this assumption Witten [86] introduced a neatly twist of the worldsheet fermions with the axial/vector R-currents. The resulting "fermions" are scalars and 1-forms and by further neglecting the 1-forms Witten defined two distinct topological field theories - the A- and B-model. One might guess that the twisted models are physically irrelevant and serve only as a toy model but there are observables, the Yukawa couplings, that coincide for the twisted (unphysical) and untwisted (physical) models. First Kapustin [65] realised from the physical perspective that the twisted non-linear sigma model can be formulated via generalised geometries. He even showed that T-duality arguments come along with a GKS structure. Moreover, the A-model and the B-model can be mapped onto each other via mirror symmetry.

What is mirror symmetry? Let the target 6-manifold be T^3-fibred. As was proven by Strominger, Yau and Zaslow (SYZ) [81], mirror symmetry is T-duality in case we apply T-duality only in the fibre. It is known that mirror symmetry exchanges all observables and ensures that the two involved target spaces, one for each topological model, are mirror pairs. Furthermore, mirror symmetry relates the involved complex and symplectic structures of the two mirror target spaces to each other. There was no simple and appropriate tool to describe this exchange since there was no mathematical theory that considers the two structures on equal footing. But generalised structures can achieve that [60, 23, 12, 32].

This chapter is based on the author's articles [60] and [23]. We introduce the idea of generalised geometries, characterise some basics and define the notion of a GKS. Subsequently, we use this type of manifolds and assume that the 6-manifolds are T^3-fibred. As a toy model we take a 6-torus. We define the mirror map $\mathcal{M}$, by using pure differential geometry only. Applying this map to a usual Kähler manifold, described in terms of a GKS, the underlying GCSs are interchanged. This happens even if the B-field enters the game and verifies the already known Buscher rules (see for instance [64]). In case of a non-trivial GKS, i.e. having independent I_+ and I_-, the GCSs get also mapped into each other. We also discuss this in terms of involutive subbundles inside $T \oplus T^*$ and thus understand in more detail the mirror map. Having this at hand we discuss the twisted topological models within the generalised setup and study various aspects of mirror symmetry, e.g. observables, the anomaly and also topological branes.

3.1 Mathematical preliminaries

The theory of generalised complex (and Calabi-Yau) structures was introduced by Hitchin [55]. Based on this seminal work, Gualtieri [45] introduced the notion of generalised Kähler structures (GKS) in his thesis where he discussed also integrability conditions and torsion. In what follows we will stick (almost) to the definitions given there. In the following we will remember the concepts we use later on.

3.1.1 Basic definitions

Let T be a 6-dimensional real vector space and T^* its dual. Since we are basically interested in manifolds we also use (in abuse of notation) the same symbols for the (co-)tangent bundle. By introducing local coordinates and using the canonical basis for T and T^* we have the natural pairings,

$$dx^\mu(\partial_\nu) = \delta^\mu{}_\nu\,, \qquad \partial_\mu(dx^\nu) = \delta_\mu{}^\nu\,. \tag{3.1}$$

Using this fact we define the non-degenerate, symmetric bilinear form of signature $(6,6)$ on the vector space $T \oplus T^*$ by

$$\langle X+\xi, Y+\eta\rangle = \frac{1}{2}(\xi(Y)+\eta(X))\,, \tag{3.2}$$

where $X, Y \in T$, $\xi, \eta \in T^*$. The group preserving this bilinear form and the orientation is the non-compact special orthogonal group $SO(6,6)$. The Lie algebra $\mathfrak{so}(6,6)$ can be decomposed into a direct sum of three terms, $\mathfrak{so}(6,6) = \mathrm{End}(T) \oplus \Lambda^2 T^* \oplus \Lambda^2 T$, and an element is given by

$$g = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}\,, \tag{3.3}$$

$A \in \mathrm{End}(T), B \in \Lambda^2 T^*, \beta \in \Lambda^2 T$. We observe that there exists an intrinsic two-form which we will further call the (physical) B-field. The B-transformation is given by

$$e^B = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix}\,, \tag{3.4}$$

which acts as: $\exp(B)(X+\xi) = X + \xi + X \lrcorner B$. Correspondingly, we obtain for the β-transformation

$$e^\beta = \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}\,, \tag{3.5}$$

where: $\exp(\beta)(X+\xi) = X + \xi + \xi \lrcorner \beta$.

3.1.2 Spinors and associated bilinear form

Let us act with $X + \xi \in T \oplus T^*$ on $\varphi \in \Lambda T^*$ by

$$(X+\xi)\cdot\varphi = X \lrcorner \varphi + \xi \wedge \varphi \tag{3.6}$$

and note that $(X+\xi)^2 \cdot \varphi = \langle X+\xi\rangle\varphi$, where we used (3.2). In other words, this action means Clifford multiplication for a $T \oplus T^*$ element and the corresponding spin representation is the exterior algebra ΛT^*. Shortly, forms are spinors in the $T \oplus T^*$ language. Taking the argument of dimension and signature into account the spin representation splits into two chiral irreducible parts $S = S^+ \oplus S^-$. We consider elements of $S^+(S^-)$ as even(odd) forms. Let us define an invariant bilinear form on the spinor bundles $S^\pm$

$$\langle\,\cdot\,,\,\cdot\,\rangle : \; S \otimes S \to \det T^* . \tag{3.7}$$

Because we are dealing with manifolds of real dimension $n = 6$ the bilinear form is skew-symmetric

$$\langle \varphi, \psi \rangle = (\widehat{\varphi} \wedge \psi)_{top} \,, \tag{3.8}$$

where we used the involution $\wedge$ by

$$\begin{array}{c|cccc} p \bmod 4 & 0 & 1 & 2 & 3 \\ \hline \wedge & + & - & - & + \end{array} .$$

3.1.3 Purity

In this subsection we introduce (less familiar for physicists) the powerfull tool of pure spinors. Let $L_\varphi \subset T \oplus T^*$ be defined by using the Clifford multiplication such that

$$L_\varphi = \{ X + \xi \in T \oplus T^* | (X + \xi) \cdot \varphi = 0 \} \,. \tag{3.9}$$

It can be easily checked that L_φ is isotropic. Spinors having an associated maximally isotropic annihilator L_φ (or null space) are called pure, i.e. φ is pure when $\dim(L_\varphi) = 6$. The power of pure spinors comes into play by using the fact that every maximal isotropic subspace of $T \oplus T^*$ is generated by a unique pure spinor line. Using the bilinear form on the spinor bundle we can distinguish two maximal isotropics L_φ, L_ψ:

$$L_\varphi \cap L_\psi = 0 \qquad \Leftrightarrow \qquad 0 \neq \langle \varphi, \psi \rangle = (\widehat{\varphi} \wedge \psi)_{top} \tag{3.10}$$

where φ, ψ are pure spinors. Note that every maximal isotropic subspace L of type k has the form

$$L(E, \varepsilon) = \{ X + \xi \in E \oplus T^* \; : \xi|_E = \varepsilon(X) \} \,, \tag{3.11}$$

where $E \subset T$, $\varepsilon \in \Lambda^2 T^*$ and k is the codimension of its projection onto T. In what follows we only consider the special case when $\varepsilon = B$ and only introduce the extremal isotropics of lowest and highest type. It is possible to extend our previous facts by complexification to $(T \oplus T^*) \otimes \mathbb{C}$. This provides us also with the action of complex conjugation. We are now prepared to define one of the important tools we use later on to construct generalised Calabi-Yau structures: The complex maximal isotropic subspace $L(E, B) \subset (T \oplus T^*) \otimes \mathbb{C}$ (E denotes its projection on $T \otimes \mathbb{C}$) is defined by the complex spinor line $U_L \subset \Lambda(T^* \otimes \mathbb{C})$ which is generated by

$$\varphi_L = c \cdot \exp(B + i\,\omega)\theta_1 \wedge \ldots \wedge \theta_k, \tag{3.12}$$

where $c \in \mathbb{C}$, $(B + i\,\omega) \in \Lambda^2(T^* \otimes \mathbb{C})$ and θ_i are linearly independent complex one-forms. Note: If $\theta_1 \wedge \ldots \wedge \theta_k$ is pure then one can easily check that Clifford multiplication by $\exp(B + i\,\omega)$ respects the property of purity. Obviously, we may single out different isotropics. The reader might wonder if the real form $\omega \in \Lambda^2 T^*$ can be the symplectic form. The answer in our case is 'yes'. Later on we define two spinors, one of type $k = 0$ (symplectic type) and one of type $k = 3$ (complex type). We do not need here non-extremal types.

3.1.4 Integrability

In the ordinary sense (at least in differential geometry) we call a structure integrable if smooth vector fields are closed under the Lie bracket. The structure of the Lie bracket, however, is invariant only under diffeomorphisms. The situation changes if we ask for integrability of smooth sections of $(T \oplus T^*) \otimes \mathbb{C}$. The answer to this question is the Courant bracket. Our main concern are smooth sections of maximal isotropic sub-bundles which are closed under this bracket. Hitchin showed [55] that the Courant bracket is additionally invariant under B-field transformations iff $dB = 0$. Clifford multiplication of $X + \xi \in T \oplus T^*$ on a spinor is a map taking $\Lambda^{ev/od} \to \Lambda^{od/ev}$ (see (3.6)), i.e. we map ev/od forms to od/ev forms. Also the exterior derivative is such a map. There is the following correspondence between isotropic L being involutive (closed under the Courant bracket) and smooth sections of the spin bundle:

$$L_\rho \text{ is involutive} \qquad \Leftrightarrow \qquad \exists (X + \xi) \in C^\infty(T \oplus T^*) \otimes \mathbb{C} : \, d\rho = (X + \xi) \cdot \rho \tag{3.13}$$

for any local trivialization ρ. We can also extend this definition by twisting the Courant bracket with a gerbe [57]. Integrability forces the substitution of the ordinary differential operator d by the twisted differential operator d_H:

$$d\cdot \to d_H \cdot = d\ \cdot + H \wedge \cdot \tag{3.14}$$

where $H \in \Lambda^3 T^*$ is real and closed.

3.1.5 Generalised Kähler structures

Remember, that a manifold often admits additional structures, e.g. a metric, an almost complex structure or a symplectic form. We consider these structures usually from the tangent bundle point of view. In case that the structures exist globally, the structure group reduces to $SO(6), U(3)$ or $Sp(6)$. In this section we define similar structures in the $T \oplus T^*$ bundle and show how the structure group $O(6, 6)$ can be reduced.

Consider the natural indefinite metric (3.2) on $T \oplus T^*$. In what follows we are interested not only in involutive but additionally in positive(negative) definite subbundles, called $C_\pm$. This forces the structure group to reduce globally to the maximal compact subgroup $O(6) \times O(6)$. Therefore we get the splitting $T \oplus T^* = C_+ \oplus C_-$. This serves to define the positive definite metric G on $T \oplus T^*$. The metric G has two properties, it is symmetric ($G^* = G$) and it squares to one ($G^2 = 1$). (Note that G is an automorphism).

Let us do this more explicit (in the complexified case) by using the projectors $P_\pm = \frac{1}{2}(1 \pm G)$, where the associated subbundles $C_\pm \otimes \mathbb{C}$,

$$(T \oplus T^*) \otimes \mathbb{C} = (C_+ \oplus C_-) \otimes \mathbb{C}\,, \tag{3.15}$$

have eigenvalues ± 1 and carry a positive/negative definite metric (see figure 3.1). We will give an explicit formulation in (3.23).

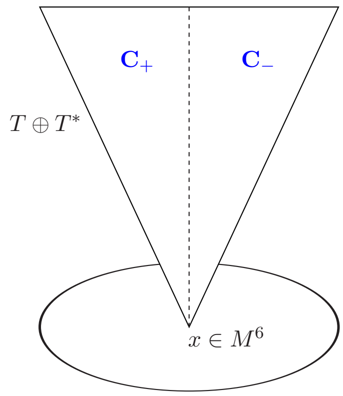


Figure 3.1: The positive and negative definite subbundles $C_\pm$

A generalised complex structure (GCS) is an endomorphism $\mathcal{J}$ on $T \oplus T^*$ which commutes (is compatible) with G. So $C_\pm$ is stable under the action of $\mathcal{J}$. It satisfies $\mathcal{J}^2 = -1$ and its dual $\mathcal{J}^*$ is symplectic ($\mathcal{J}^* = -\mathcal{J}$). Using the properties of G and $\mathcal{J}$ we can define another GCS. Additionally, requiring that the two commuting GCSs are integrable we have a generalised Kähler structure (GKS) and G is given by

$$G = -\mathcal{J}_1 \mathcal{J}_2. \tag{3.16}$$

This reduces the structure group to $U(3) \times U(3)$.

Let us now explain the relation between the structures $G, \mathcal{J}_1, \mathcal{J}_2$ of the generalised Kähler geometry with the pure spinor lines (defining the maximal isotropics). This we will explain for the case of having a usual Kähler manifold that is endowed by a metric g, a complex structure J and a symplectic form ω satisfying $\omega = gJ$. In this case, we have the additional identities,

$$J\omega^{-1} = g^{-1} \qquad \text{and} \qquad J^T \omega = g\,. \tag{3.17}$$

We embed the complex and symlectic structures in the GCSs by defining

$$\mathcal{J}_J = \begin{pmatrix} J & \\ & -J^T \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} & -\omega^{-1} \\ \omega & \end{pmatrix}. \tag{3.18}$$

Both algebraic structures, $\mathcal{J}_J, \mathcal{J}_\omega$, have the properties of a GCS. Using the above identities (3.17) we can calculate the metric G in the $T \oplus T^*$ bundle by,

$$G = -\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}. \tag{3.19}$$

The triple $(\mathcal{J}_J, \mathcal{J}_\omega, G)$ provides us with a simple example of a GKS.

We also want to include the B-field, which turns out to be simple, since we only have to multiply matrices,

$$(\mathcal{J}_J^B, \mathcal{J}_\omega^B, G^B) = (e^B\, \mathcal{J}_J\, e^{-B}, e^B\, \mathcal{J}_\omega\, e^{-B}, e^B\, G\, e^{-B}) \tag{3.20}$$

where e^B acts via (3.4). This provides us with the generalised complex structures [45]

$$\mathcal{J}_J^B = \begin{pmatrix} J & \\ BJ + J^T B & -J^T \end{pmatrix}, \qquad \mathcal{J}_\omega^B = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}, \tag{3.21}$$

and the generalised metric

$$G^B = \begin{pmatrix} -g^{-1}B & -g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix}. \tag{3.22}$$

The generalised metric G^B motivates the orthogonal spaces C_+ and C_-,

$$C_+ = \{X + (B+g)X | X \in T\}, \qquad C_- = \{X + (B-g)X | X \in T\}. \tag{3.23}$$

In other words, if we are given with a section X in T, we can construct the T^*-part by setting $(b \pm g)X \in T^*$ and get in this way the full element in $C_\pm$.

Example 3.1.1. *Let $\psi_+ \in T$ and $\psi_- \in T$ be two sections. We define the $\pm$ signs such that the two sections originate from the T-part of two sections in $C_\pm$. This means that we can seperate the two independent sections $\psi_\pm$ in T by lifting them into the two orthogonal bundles C_+ and C_-. See figure 3.2.*

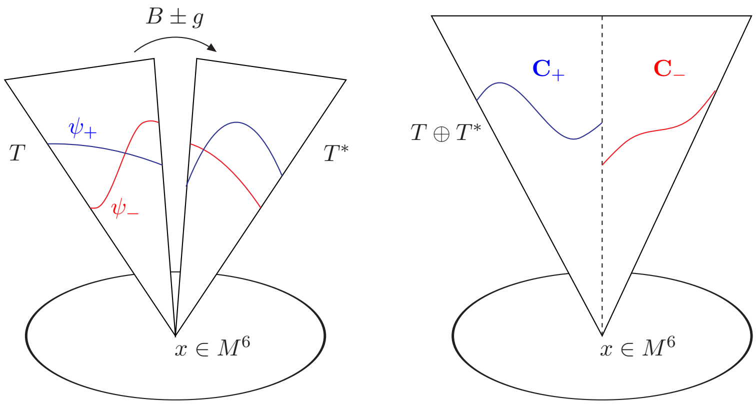


Figure 3.2: The positive and negative definite subbundles $C_\pm$

For the above GCSs, $(\mathcal{J}_J, \mathcal{J}_\omega)$, we have two corresponding spinor lines $(\varphi_J, \varphi_\omega)$. Thus, the maximal isotropics $(L_{\varphi_J}, L_{\varphi_\omega})$ are stable under $(\mathcal{J}_J, \mathcal{J}_\omega)$ and are generated by the pure spinor lines $(\varphi_J, \varphi_\omega)$,

$$L_{\varphi_J} \;:\; \text{stable under } \mathcal{J}_J \text{ and generated by } \varphi_J = \Omega^{(3,0)}, \tag{3.24}$$

$$L_{\varphi_\omega} \;:\; \text{stable under } \mathcal{J}_\omega \text{ and generated by } \varphi_\omega = e^{i\omega}, \tag{3.25}$$

where

$$L_{\varphi_J} = T^{0,1} \oplus T^{*1,0}, \qquad \text{and} \qquad L_{\varphi_\omega} = \{X - i\, X \lrcorner\, \omega : X \in T \otimes \mathbb{C}\}. \tag{3.26}$$

A generalised Calabi-Yau structure (GCYS) is a GKS with the following additional constraint for the generating spinor lines,

$$(\widehat{\varphi}_1 \wedge \bar{\varphi}_1)_{top} = c \cdot (\widehat{\varphi}_2 \wedge \bar{\varphi}_2)_{top} \qquad \text{for each point}, \tag{3.27}$$

where $c \in \mathbb{R}$. We can understand the constraint by first remembering that the property

$$(\widehat{\varphi}_{1/2} \wedge \bar{\varphi}_{1/2})_{top} \neq 0 \tag{3.28}$$

trivialises each determinant bundle of the involved pure spinors. And via the constraint (3.27) the two determinant bundles are equal up to a constant at any point. This also reduces the structure group to $SU(3) \times SU(3)$.

If we take the above generating spinor lines $\varphi_J, \varphi_\omega$ we obtain according to (3.27)

$$\omega^3 = \frac{i\,3!}{2^3}\,\Omega \wedge \bar{\Omega}\,, \tag{3.29}$$

where $c = 1$.

In this section we have shown how a usual Kähler structure can be naturally embedded into a GKS. But note that this is a very specific case since the framework of generalised Kähler geometry is much richer. We will see later that the most general framework is equivalent to a bi-hermitian geometry [45]. A bi-hermitian geometry appears in physics by treating a full non-linear sigma model [37] and is characterised by the fact that the target manifold allows for two independent complex structures.

3.2 Mirror symmetry

The puzzle of having several distinct, but physically relevant, superstring theories formulated in $\mathbb{R}^{1,9}$ was solved by duality maps. Let us consider for a moment type II superstring theories, i.e. IIA and IIB, each compactified on a Calabi-Yau 6-manifold. Using moduli space investigations of D-branes, it was conjectured by Strominger, Yau and Zaslow (SYZ) [81] that these theories are mirror symmetric to each other if the Calabi-Yau spaces are T^3 fibered. The crucial idea is that mirror symmetry is just a T-dualtiy transformation in the fibre. The basic message can be denoted by - *mirror symmetry is T-duality* (in the fibre T^3). This means that only a very restricted subspace of the huge moduli space of Calabi-Yau spaces is relevant (being compatible with the duality). It is known that mirror symmetry exchanges topological data, e.g. $h^{1,1}$ and $h^{1,2}$. And since also the complex and symplectic structures of the two involved Calabi-Yaus get mapped onto each other, one often uses this property to define mirror symmetry. In the following it might be comfortable to think about mirror symmetry in a purely geometrical picture that interchanges algebraic structures.

During the last years mirror symmetry was often studied by Calabi-Yaus. But this is a simplification of the actual full physically interesting problem. In principle one should also take e.g. the background fluxes into account, i.e. the B-field and the Ramond-Ramond fields. Note, already the Calabi-Yau case is difficult enough since the exchange of the participating algebraic structures can neither be motivated nor understood from a pure G-structure point of view. In other words, there did not exist a more abstract theory that, firstly, can handle complex and symplectic structures on equal footing and, secondly, allow them to transform into each other. Generalised structures will close that gap.

Let us pick up this last thought and consider the physical perspective. Physicists are able to partly characterise with the help of G-structures, even in case of background fluxes, the target space. We know that different string theories and their target spaces can be mapped to each other via duality transformations. Until know these dualtiy maps are not connected to G-structures and are therefore unrelated distinct tools. Furthermore, the duality level remained up to now only on a physical concept with no fundamental mathematical background. This is why it was difficult to find a simple and neatly mirror map during the last years that solves relevant problems. This immediately implies that studying generalised structures could be a first step towards a better understanding of T-duality (see also [65]). Generalised geometries have the power of the already known classical structures but moreover they are also able to capture two known geometries - the complex and the symplectic.

We argued that a GCS can model both, an almost complex structure and a symplectic structure. The main idea for a better understanding of mirror symmetry is to use the concept of a GKS on a 6-manifold that is T^3-fibered. The mirror map $\mathcal{M}$, which we conjecture and explicitly construct, is given by interchanging the T and T^* part of the fibre. Performing this map $\mathcal{M}$ the two GCSs, given by a GKS, get interchanged and the Buscher rules are reproduced. Note, the Buscher rules tell us how the NS-NS fields - the metric g, the B-field and dilaton ϕ - and the R-R fields that are forms get mapped under T-duality. Note, NS-NS and R-R denotes two different sectors in string theory, where NS stands for Neveau-Schwarz and R stands for Ramond. Furthermore, the Buscher rules only mix the NS-NS fields, but never the R-R degrees of freedom, i.e. the R-R fields get only mapped into each other. This means that T-duality conserves the degrees of freedom of the NS-NS and R-R sector separately. Since the metric, the B-field and the dilaton transform non-trivially, the Buscher rules look quite messy in the tangent space picture, compare for instance [64]. But in the $T \oplus T^*$ picture, i.e. within generalised geometries, we will see that these rules can be achieved and understood more systematically [60, 23, 12, 32]. Moreover, on the mirror side it was not quite clear if the complex structure is completely fixed by the data of the original one (see e.g. [46]). Our construction of the mirror map resolves this puzzle.

In this section we explicitly construct the mirror map $\mathcal{M}$ in the $T \oplus T^*$ setup. We apply it to the toy manifold of T^6 and argue to hold for more general T^3-fibred Calabi-Yau manifolds. By including also the B-field the Buscher rules can be found. We further introduce the mirror map for pure spinors and show how the Calabi-Yau relevant pure spinor lines $e^{i\omega}$ and $\Omega^{3,0}$ get mapped to each other. Afterwards we perform the mirror map to non-trivial GCSs, i.e. GCS that are not usual Kähler structures. We will use them to study mirror symmetry for twisted topological models in the next section. Finally, via the GCS one can decompose the bundle $T \oplus T^*$ into four subbundles and investigate the influence under the map $\mathcal{M}$.

3.2.1 The mirror map $\mathcal{M}$ and the T-duality rules

Let us define a six-torus T^6 endowed with the triplet (g, J, ω) and a vanishing B-field. We consider this manifold as a trivial fibration of $T^3 \hookrightarrow T^6$ over the base space $\mathcal{B} = T^3$. Later on we investigate manifolds having also non-trivial T^3 fibrations (and also a non-vanishing B-field). The basic idea is to embed the given structures into generalised ones and consider their behavior under a special map, the mirror symmetry map $\mathcal{M}$.

Because we are interested in dealing with the generalised tangent bundle $T \oplus T^*$, it was natural to denote the GCSs in (3.18) by (2×2) matrices. But since we furthermore split the tangent bundle (and similar the co-tangent bundle) into a base $\mathcal{B}$ and fibre $\mathcal{F}$ part we also denote the usual algebraic structures (g, J, ω) by (2×2)-matrices. Therefore, the GCSs have the shape of (4×4) matrices.

In local coordinates we have

$$g = \begin{pmatrix} \delta_{ij} & \\ & \delta_{\alpha\beta} \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} \delta^{ij} & \\ & \delta^{\alpha\beta} \end{pmatrix}, \tag{3.30}$$

where (y_i, x_α) $((i,\alpha) \in \{1,2,3\})$ denotes the coordinates on the base $\mathcal{B}$ and fibre $\mathcal{F}$, respectively.

The complex and symplectic structures we use are choosen to be trivial, which is already sufficient to see the properties we are looking for. Therefore we denote,

$$J = \begin{pmatrix} & -\delta_\alpha{}^i \\ \delta_i{}^\alpha & \end{pmatrix}, \qquad J^T = \begin{pmatrix} & \delta^\alpha{}_i \\ -\delta^i{}_\alpha & \end{pmatrix}, \tag{3.31}$$

and

$$\omega = \begin{pmatrix} & -\delta_{\alpha i} \\ \delta_{i\alpha} & \end{pmatrix}, \qquad \omega^{-1} = \begin{pmatrix} & \delta^{\alpha i} \\ -\delta^{i\alpha} & \end{pmatrix}. \tag{3.32}$$

By using (3.18) we embed the objects into generalised geometry and the just defined space admits a trivial GCYS.

This fixes our setup and we are now prepared to define a map $\mathcal{M}$ which assigns to a GKS another GKS. Since we specialise only on spaces which are T^3 fibrations over the base $\mathcal{B}$ and act only on the fibre, we will call this map $\mathcal{M}$ a *mirror symmetry map*. This map is an isomorphism of the bundle $T\oplus T^*$ and also transforms the triple $(\mathcal{J}_J, \mathcal{J}_\omega, G)$ in a well defined way. This means that on the mirror side this triple is completely fixed.

Let the mirror map $\mathcal{M} : T\oplus T^* \to T\oplus T^*$ be given by

$$\mathcal{M} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}, \tag{3.33}$$

where we distinguish the vielbeins (as above) in T and T^*, more precisely,

$$\mathcal{M} : T_\mathcal{B} \oplus T_\mathcal{F} \oplus T^*_\mathcal{B} \oplus T^*_\mathcal{F} \to T_\mathcal{B} \oplus T^*_\mathcal{F} \oplus T^*_\mathcal{B} \oplus T_\mathcal{F}\,. \tag{3.34}$$

This is simply a map that acts on the base as the identity and on the fibre as a “flip”. Note: The identity maps in $\mathcal{M}$ are tensors of adequate type to make $\mathcal{M}$ a well defined isomorphism. The property $\mathcal{M} = \mathcal{M}^{-1}$ makes the mirror map an involution, $\mathcal{M}^2 = 1$.

The action of the mirror map on a GKS $(\mathcal{J}_J, \mathcal{J}_\omega, G)$ is defined by,

$$(\hat{\mathcal{J}}_J, \hat{\mathcal{J}}_\omega, \hat{G}) = \mathcal{M}\,(\mathcal{J}_J, \mathcal{J}_\omega, G)\,\mathcal{M}^{-1}\,. \tag{3.35}$$

Let us act by $\mathcal{M}$ on the generalised structures defined on the T^6. Note also the property $\mathcal{M} = \mathcal{M}^{-1}$. We obtain the mirror metric $\hat{G}$ as,

$$\hat{G} = \mathcal{M}\, G\, \mathcal{M}^{-1} = \begin{pmatrix} & & \delta^{ij} & \\ & & & \delta_{\alpha\beta} \\ \delta_{ij} & & & \\ & \delta^{\alpha\beta} & & \end{pmatrix} . \tag{3.36}$$

The Riemannian mirror metric $\hat{g}$ on the tangent bundle is therefore given by

$$\hat{g} = \begin{pmatrix} \delta_{ij} & \\ & \delta^{\alpha\beta} \end{pmatrix} , \tag{3.37}$$

where we have the inverse metric in the trivial fibre, which is expected using the Buscher rules (see [64]). This result suggests strongly that we have found a way to reproduce the Buscher rules by multiplying matrices only. Or in other words, to do a mirror transformation we do not have to take the Buscher rules explicitly into account, they are included already in the matrix multiplication, i.e. using Buscher rules is nothing else but matrix multiplication. As the mirror transformation of $(\mathcal{J}_J, \mathcal{J}_\omega)$ we get

$$(\widehat{\mathcal{J}}_J, \widehat{\mathcal{J}}_\omega) = (\mathcal{M}\, \mathcal{J}_J\, \mathcal{M}^{-1}, \mathcal{M}\, \mathcal{J}_\omega\, \mathcal{M}^{-1}) \,, \tag{3.38}$$

where we obtain

$$\widehat{\mathcal{J}}_J = \begin{pmatrix} & & & -\delta^{\alpha i} \\ & & \delta^{i\alpha} & \\ & -\delta_{\alpha i} & & \\ \delta_{i\alpha} & & & \end{pmatrix} , \qquad \widehat{\mathcal{J}}_\omega = \begin{pmatrix} & -\delta_\alpha{}^i & & \\ \delta_i{}^\alpha & & & \\ & & & -\delta^\alpha{}_i \\ & & \delta^i{}_\alpha & \end{pmatrix} . \tag{3.39}$$

Considering these structures as an embedding of usual complex and symplectic structures (acting in the tangent bundle) we see immediately that $\widehat{\mathcal{J}}_J$ is of pure symplectic type ($k = 0$) while $\widehat{\mathcal{J}}_\omega$ is of pure complex type ($k = 3$). This verifies the well known statement that mirror symmetry exchanges the complex and symplectic structures:

$$\begin{aligned} \mathcal{J}_J \quad &\leftrightarrow \quad \widehat{\mathcal{J}}_J = \mathcal{J}_\omega \,, \\ \mathcal{J}_\omega \quad &\leftrightarrow \quad \widehat{\mathcal{J}}_\omega = \mathcal{J}_J \,, \\ g_{\mathcal{B}} + g_{\mathcal{F}} \quad &\leftrightarrow \quad g_{\mathcal{B}} + g_{\mathcal{F}}^{-1} \,, \\ \text{trivial GCYS} \quad &\leftrightarrow \quad \text{trivial GCYS} \,. \end{aligned} \tag{3.40}$$

We verify (for T^6) therefore the work of [12, 32]. In [32] the authors had to use the Buscher rules explicitly for there construction, in contrast to the mappings above, where these rules are intrinsically included.

As a next step let us introduce a globally defined and flat B-field. We are considering only a closed B-field and therefore we do not affect integrability. The presence of a B-field can be well understood by focusing on the B-transformed generalised metric G^B. It is not hard to see that all elements of $C_\pm$ are captured taking an arbitrary element $X \in T$, whereas the elements in T^* are already given by $(B \pm g)X$. Or in other words, $C_\pm$ is the graph of $B \pm g : T \to T^*$, i.e. including a B-field only "rotates" the positive/negative definite subspaces $C_\pm$ inside $T \oplus T^*$.

We do not consider general B-fields in the following. Let $B = B_{i\alpha} dy^i \wedge dx^\alpha$ be the B-transformation of our interest:

$$e^B = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & B_{\alpha i} & 1 & \\ B_{i\alpha} & & & 1 \end{pmatrix}, \qquad e^{-B} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -B_{\alpha i} & 1 & \\ -B_{i\alpha} & & & 1 \end{pmatrix}. \tag{3.41}$$

The B-transformed metric G of the T^6 can be computed via (3.20) to be

$$G^B = e^B \, G \, e^{-B}\,, \tag{3.42}$$

by a subsequent mirror transformation of this metric we obtain

$$\hat{G}^B = \mathcal{M} G^B \mathcal{M}^{-1}, \tag{3.43}$$

where we suppress the explicit form. We only note that it is completely off-diagonal. One can prove that it is again a generalised metric. More important, it is of pure Riemannian type, i.e. not B-transformed, and thus can be constructed by embedding of a Riemannian metric $\hat{g}$ only,

$$\hat{g} = \begin{pmatrix} g_{\mathcal{B}} - B g_{\mathcal{F}}^{-1} B & B g_{\mathcal{F}}^{-1} \\ -g_{\mathcal{F}}^{-1} B & g_{\mathcal{F}}^{-1} \end{pmatrix}, \tag{3.44}$$

where $g_{\mathcal{B}} = \delta_{ij}$ is the metric in the base $\mathcal{B}$ and $g_{\mathcal{F}} = \delta_{\alpha\beta}$ denotes the metric in the fibre $\mathcal{F}$. These transformation rules do verify the Buscher rules exactly and mean that on the mirror side there exists no $\hat{B}$-field and the old one is completely absorbed into the metric $\hat{G}^B$ (or $\hat{g}$).

Example 3.2.1. *Let $B = (B_{i\alpha} dy^i) \wedge dx^\alpha$ (in local coordinates) only depends on coordinates y on the base. The mirror metric $\hat{g}$ has a shape of the following form*

$$\hat{g} = (g_{\mathcal{B}})_{ij} dy^i dy^j + (g_{\mathcal{F}}^{-1})_{\alpha\beta} (dx^\alpha + A^\alpha)(dx^\beta + A^\beta)\,, \tag{3.45}$$

where A is a local connection-one-form. Thus, the B-field transforms into the metric and re-appears in a non-trivial fibration.

Moreover, because of $\mathcal{M}$ being an isomorphism and an involution, we can reverse this procedure. The initial data, a non-trivial fibration and vanishing B-field, produces a trivial fibred T^6 with non-trivial B-field as it's mirror. Naturally, the combination of both "effects" is possible and independent of each other.

The B-field transformation of the GCSs $(\mathcal{J}_J, \mathcal{J}_\omega)$ can be calculated by (3.20)

$$(\mathcal{J}_J^B, \mathcal{J}_\omega^B) = (e^B \, \mathcal{J}_J \, e^{-B}, e^B \, \mathcal{J}_\omega \, e^{-B})\,, \tag{3.46}$$

which are no longer diagonal/off-diagonal, respectively. An action of the mirror map $\mathcal{M}$ on these is given by

$$(\widehat{\mathcal{J}}_J, \widehat{\mathcal{J}}_\omega) = (\mathcal{M}\, \mathcal{J}_J^B\, \mathcal{M}^{-1}, \mathcal{M}\, \mathcal{J}_\omega^B\, \mathcal{M}^{-1}). \tag{3.47}$$

It can also be shown that the mirror structures are GCSs and become again completely off-diagonal/diagonal, or equivalently, structures of pure symplectic and complex type (see also [12]). The embedded structures are given in components by

$$\hat{J} = \begin{pmatrix} \delta B & -\delta \\ B\delta^{-1}B + \delta & -B\delta^{-1} \end{pmatrix}, \qquad \hat{\omega} = \begin{pmatrix} B\delta + \delta B & -\delta \\ \delta & 0 \end{pmatrix}. \tag{3.48}$$

We used a condensed notation where δ denotes the appropriate tensors (see (3.31),(3.32)) and we also suppressed the identity maps coming from $\mathcal{M}$. Note: The object $\hat{J}$ is embedded in $\widehat{\mathcal{J}}_\omega$ (which makes it a complex structure of type $k = 3$) and should therefore not be mixed up with the subscript ω which denotes the symplectic form on the original T^6.

The concept we explained above holds in more generality, and we applied it only to a simple example to clarify the mappings of the generalised structures (see also [12]). One can immediately use this framework for more complicated generalised structures, what we will do later on. We will apply $\mathcal{M}$ to GCSs that describe two usual but independent complex structures.

We also have to discuss integrability in more detail. Focusing on the cases considered above, these are obviously integrable and torsion-less and so is its mirror. The case of non-vanishing NS-NS fluxes was at first worked out in [32], where the authors also used the generalised concept but they used the Buscher rules explicitly. There it was shown how torsion-full 6-manifolds are interchanged, and the authors gave a precise description of each $SU(3)$ torsion component.

3.2.2 The mirror map $\mathcal{M}$ and pure spinors

As we learned in (3.9), for a pure spinor φ there exists an associated maximal isotropic L. We defined this by using Clifford multiplication. We use [44] (see also [45]) and denote the graded Clifford algebra by $CL(T \oplus T^*)$. The objects CL^k are generated by products of k elements of $(T \oplus T^*) \otimes \mathbb{C}$. Moreover, if we have a GCS it is known that the space $(T \oplus T^*) \otimes \mathbb{C}$ decomposes as,

$$(T \oplus T^*) \otimes \mathbb{C} = L \oplus L^*\,,$$

where L^* is the dual. Using this fact we can use now the Clifford algebra and the pure spinor φ to obtain an alternative grading for all differential forms $\Lambda^\bullet T^* \otimes \mathbb{C}$. Let us spend a few words about the 'alternative' in the last sentence.

Usually, differential forms are elements of a graded exterior algebra and we have the degree k of the form as an index of the graduation. Here we use a different concept, i.e. the index k do not denote the degree of the form. Let us build up the graduation by starting with the pure spinor line which we take to sit inside U_0 and to be of even or odd degree. U_0 denotes the canonical bundle. Since it is pure $L \subset (T \oplus T^*) \otimes \mathbb{C}$ annihilates U_0. So, the only non-vanishing objects we can generate are $L^* \cdot U_0 = U_1$. We can proceed in the obvious way, consequently $U_k = \Lambda^k L^* \cdot U_0$ for $k = 1, \ldots, 6$,

$$\Lambda^* T^* \otimes \mathbb{C} = U_0 \oplus U_1 \oplus \cdots \oplus U_6. \tag{3.49}$$

Note also that by Clifford multiplying $U_0 \subset \Lambda^{od/ev} \otimes \mathbb{C}$ by an element $CL^{od/ev}$ we get $CL^{od} \cdot U_0 \subset \Lambda^{ev/od} \otimes \mathbb{C}$, whereas $CL^{ev} \cdot U_0 \subset \Lambda^{od/ev} \otimes \mathbb{C}$. Furthermore, we also have the property $\overline{U_k} = U_{6-k}$ and using the generalised metric we can identify $\overline{L} = L^*$.

There is a correspondence between the just introduced index k for U_k and eigenvalues of the GCS. We can see this if we act on the GCS via the Lie algebra representation on differential forms, i.e. spinors. This action associates to each U_k the eigenvalue $i(3-k)$. Since $k = 0, \dots, 6$, this makes clear that we can also use U_{3-k} for the graduation which is more convenient in what follows, i.e. the notation of (3.49) becomes

$$\Lambda^* T^* \otimes \mathbb{C} = U_3 \oplus U_2 \oplus U_1 \oplus U_0 \oplus U_{-1} \oplus U_{-2} \oplus U_{-3}. \tag{3.50}$$

Let us discuss this in more detail for the case of having a usual complex structure J on the 6-manifold which we embed into a GCS using (3.18). The generating spinor line is just the holomorphic $(3,0)$-form $\Omega^{3,0}$, which we can denote in local coordinates choosing $z^i = dx^i + idy^i, i = 1, \dots 3$ by,

$$\Omega^{3,0} = (dx^1 + idy^1) \wedge (dx^2 + idy^2) \wedge (dx^3 + idy^3) = z^{123}\,,$$

where the maximal isotropic is $L = T^{0,1} \oplus T^{*1,0}$ ($\overline{L} = T^{1,0} \oplus T^{*0,1}$).

Using $J = i\partial_{z^i} \otimes dz^i - i\partial_{z^{\bar{i}}} \otimes dz^{\bar{i}}$, we act via the generalised structure $\mathcal{J}_J$ on $\Omega^{3,0}$ by

$$\mathcal{J}_J \cdot \Omega^{3,0} = -J^* \cdot \Omega^{3,0} = idz^i \wedge (\partial_{z^i} \lrcorner\, \Omega^{3,0}) - idz^{\bar{i}} \wedge (\partial_{z^{\bar{i}}} \lrcorner\, \Omega^{3,0}) = 3\,i\,\Omega^{3,0}\,,$$

which means that $\Omega^{3,0} \in U_3$. This shows immediately that the eigenvalue counts the number of holomorphic minus anti-holomorphic indices. E.g. all forms $\alpha \in \bigoplus_{p=0}^{3} \Lambda^{p,p}$ have eigenvalue equal to zero, i.e. $U_0 = \bigoplus_{p=0}^{3} \Lambda^{p,p}$, see figure 3.3. We also want to mention that starting with $\Omega^{3,0} \in U_3$ we can generate all elements in $U_k, k = -3, \dots, 3$ by $\Lambda^{3-k}\overline{L} \cdot \Omega^{3,0}$.

In case of a symplectic structure on a 6-manifold we can also embed the usual symplectic form ω according to (3.18) into a GCS $\mathcal{J}_\omega$. The corresponding pure spinor line is the even form $e^{i\omega}$. Also in this case the differential forms, i.e. the spinors, decompose under the spin action of

$$\mathcal{J}_\omega \cdot \varphi = (\,\omega \wedge - \,\omega^{-1} \lrcorner)\,\varphi\,, \tag{3.51}$$

where $\varphi \in \Lambda^{ev,od}$. This is a more complicated decomposition and according to [56] U_k is isomorphic to Λ^{3+k}. If we consider a GKS the two GCSs commute and we can first decompose the spinors with respect to $\mathcal{J}_J$ and further split U_k by $\mathcal{J}_\omega$ to get $U_{k,j}$, where $k, j \leq -3, \dots, 3$.

We are now prepared to develop the mirror map $\mathcal{M}$ for pure spinors. Let us kick of by choosing, for convenience only, the same setup as in section (3.2.1), i.e. a T^6 and a vanishing B-field. The local coordinates in the base $\mathcal{B}$ and the fibre $\mathcal{F}$ are given by y^i, x^α, where $i, \alpha = 1, \dots 3$.

We conjecture the mirror map $\mathcal{M} : \Lambda^{ev/od} \to \Lambda^{od/ev}$ for pure spinor lines φ by (given in real local coordinates),

$$\mathcal{M} \cdot \varphi = \prod_{\alpha=1}^{3} (\partial_{X_\alpha} + dx^\alpha) \cdot \varphi\,, \tag{3.52}$$

$$= (\partial_{X_3} + dx^3) \cdot (\partial_{X_2} + dx^2) \cdot (\partial_{X_1} + dx^1) \cdot \varphi\,, \tag{3.53}$$

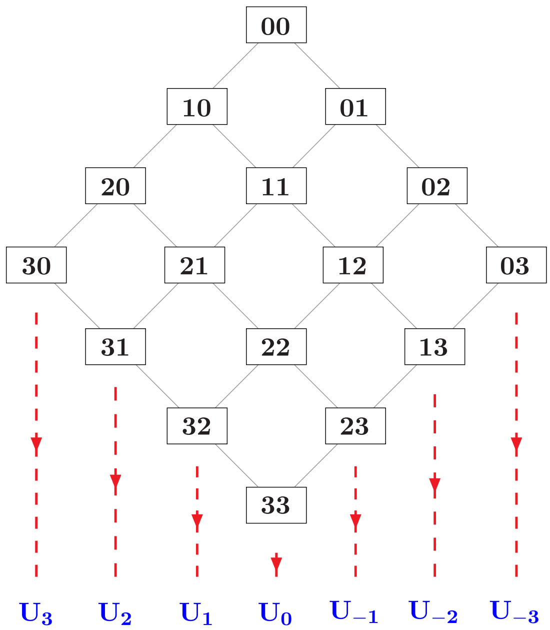


Figure 3.3: The space of differential forms on M^6 decompose under the almost complex structure J. We visualise this by the hodge diamond. Let J be embedded into a generalised complex structure $\mathcal{J}_J$. All differential forms on M^6 decompose under the Lie algebra action of $\mathcal{J}_J$ into U_k-spaces, where the eigenvalue k counts the number of holomorhphic minus anti-holomorphic indices.

where in complex coordinates this reads

$$\mathcal{M} \cdot \varphi = \prod_{i=1}^{3} (\partial_{z_i} + \partial_{z_{\bar{i}}} + \frac{1}{2} dz^i + \frac{1}{2} dz^{\bar{i}}) \cdot \varphi \,. \tag{3.54}$$

The map $\mathcal{M}$ acts only in the fibre and we will show in the following that it exchanges the two pure spinor lines responsible for a usual Kähler 6-manifold. We can check this by defining the pure spinors in real local coordinates by

$$\Omega^{(3,0)} = (dx^1 + i\, dy^1) \wedge (dx^2 + i\, dy^2) \wedge (dx^3 + i\, dy^3) \,, \tag{3.55}$$

$$e^{i\,\omega} = 1 \,+\, i\, dx^i dy^i \,+\, dx^{12} dy^{12} \,+\, dx^{23} dy^{23} \,+\, dx^{13} dy^{13} \,+ i\, dx^{123} dy^{123} \,, \tag{3.56}$$

and in local complex coordinates,

$$\Omega^{(3,0)} = dz^{123} \,, \tag{3.57}$$

$$e^{i\,\omega} = 1 + \frac{1}{2} dz^{\bar{i}i} + \frac{1}{4}(dz^{\bar{3}\bar{2}23} + dz^{\bar{3}\bar{1}13} + dz^{\bar{2}\bar{1}12}) + \frac{1}{8} dz^{\bar{3}\bar{2}\bar{1}123} \,, \tag{3.58}$$

where $\omega = -\frac{i}{2} dz^{\bar{i}i} = -\frac{i}{2}(dz^{\bar{1}1} + dz^{\bar{2}2} + dz^{\bar{3}3})$.

Let us verify this for the pure spinor $\Omega^{3,0} = dz^{123}$ first. Note, that the maximal isotropic for this spinor is given by $T^{0,1} \oplus T^{*1,0}$ and therefore the mirror map $\mathcal{M}$ condenses to $\mathcal{M} \cdot dz^{123} =$

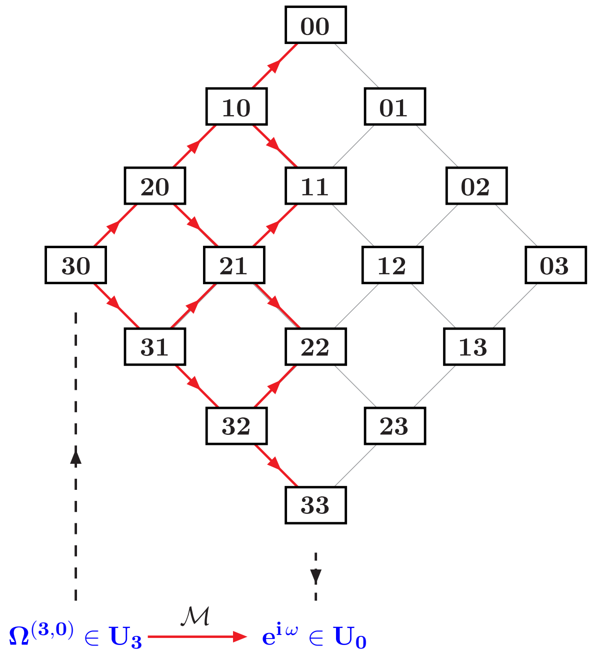


Figure 3.4: The form $\Omega^{(3,0)}$ sits in U_3 with respect to $\mathcal{J}_J$. The action of the mirror map $\mathcal{M}$ is a mechanism to transport e.g. $\Omega^{(3,0)}$ via three single Clifford multiplications to a form in $\oplus_p \Lambda^{p,p}$. This resulting form is precisely $e^{i\,\omega}$ and sits in U_0.

$\prod_{i=1}^{3} (\partial_{z_i} + \frac{1}{2} dz^{\bar{i}}) \cdot dz^{123}$ and so we obtain,

$$\mathcal{M} \cdot dz^{123} = \prod_{i=1}^{3} (\partial_{z_i} + \frac{1}{2} dz^{\bar{i}}) \cdot dz^{123}\,, \tag{3.59}$$

$$= (\partial_{z_3} + \frac{1}{2} dz^{\bar{3}}) \cdot (\partial_{z_2} + \frac{1}{2} dz^{\bar{2}}) \cdot (dz^{23} + \frac{1}{2} dz^{\bar{1}123})\,, \tag{3.60}$$

$$= (\partial_{z_3} + \frac{1}{2} dz^{\bar{3}}) \cdot (dz^{3} + \frac{1}{2} dz^{\bar{2}23} + \frac{1}{2} dz^{\bar{1}12} + \frac{1}{4} dz^{\bar{2}\bar{1}123})\,, \tag{3.61}$$

$$= e^{i\,\omega}\,. \tag{3.62}$$

We have found that $\Omega^{3,0}$ gets mapped to $e^{i\,\omega}$ under the mirror map $\mathcal{M}$, see figure 3.4. Furthermore, we see that $\mathcal{M}$ changes the eigenvalue by 3 with respect to $\mathcal{J}_J$, due to $\Omega^{3,0} \in U_3$ and $e^{i\,\omega} \in U_0$. This is obvious, since we constructed the mirror map such that $\mathcal{M} \in \Lambda^3 \overline{L}$. A further short calculation shows that $\mathcal{M} \cdot e^{i\,\omega} = -\Omega^{3,0}$. This completes the proof that the two spinor lines get exchanged (up to a sign).

3.2.3 The mirror map $\mathcal{M}$ and Generalised Complex Structures

In this section we further develop the investigations done in section 3.2.1. There we used the GKS framework for a usual Kähler structure on a 6-manifold. Now we want to start by using the full setup given by a GKS setup [45], while setting the B-field to zero. Note that the target space geometry of $(2, 2)$-supersymmetric non-linear sigma model is determined by a bi-hermitian geometry [37]. In [45] the author proved that the data of a GKS, $(G, \mathcal{J}_1, \mathcal{J}_2)$, is equivalent to a bi-hermitian structure, (g, J_+, J_-), i.e. we can use the generalised Kähler geometry to discuss properties of the target space of a $(2,2)$-non-linear sigma models (even in the topological twisted case).

Generically, the two GCS of a GKS are given in the $T \oplus T^*$ basis by [45]

$$\mathcal{J}_{1/2} = \frac{1}{2} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^T \pm J_-^T) \end{pmatrix}, \tag{3.63}$$

where the complex structures J_+ and J_- are independent sections ($\forall p \in M^6$) in the twistor space $\mathcal{Z}M^6$. Note that we assume in the following integrability for the two complex structures. We can also define a generalised metric by $G = -\mathcal{J}_1 \mathcal{J}_2$.

Since our interest is to show how certain properties of the two GCS behave under the mirror map, a specific bundle isomorphism, we do not need general 6-manifolds and choose once more (3.2.1) the manifold to be $M^6 = T^3 \oplus T^3$ (with fibre $\mathcal{F} = T^3$ over the base space $\mathcal{B} = T^3$).

Therefore we have the following splitting of the generalised tangent space:

$$(T \oplus T^*) \otimes \mathbb{C} = (T_{\mathcal{B}} \oplus T_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}}) \otimes \mathbb{C}. \tag{3.64}$$

This choice is for computational convenience, but one can consider a more general M^6 as a nontrivial T^3 torus fibration over a general three dimensional base space without changing the essence of our argument [81]. Furthermore, we want to consider only GCS which are adapted in the sense of [12], i.e.

$$\mathcal{J}_{1/2} : T_F \oplus T^*_F \to T_B \oplus T^*_B. \tag{3.65}$$

Respecting additionally the algebraic properties of GCS (3.1.5) we take

$$J_+ \pm J_- = \begin{pmatrix} & -(\tilde{J}_+ \pm \tilde{J}_-) \\ \tilde{J}_+ \pm \tilde{J}_- & \end{pmatrix}, \tag{3.66}$$

$$\omega_+ \mp \omega_- = \begin{pmatrix} & -(\tilde{\omega}_+ \mp \tilde{\omega}_-) \\ \tilde{\omega}_+ \mp \tilde{\omega}_- & \end{pmatrix}. \tag{3.67}$$

Note that $\tilde{J}_+, \tilde{J}_-$ and $\tilde{\omega}_+, \tilde{\omega}_-$ are not complex structures and Kähler forms and, moreover, to satisfy the properties $I_\pm^2 = -1$ and $\omega_\pm^T = -\omega_\pm$ one has to require $\tilde{I}_\pm^2 = 1$ and $\tilde{\omega}_\pm^T = \tilde{\omega}_\pm$.

We are now prepared to write the specific GCS by

$$\mathcal{J}_{1/2} = \frac{1}{2} \begin{pmatrix} & -(\tilde{J}_+ \pm \tilde{J}_-) & & -(\tilde{\omega}_+^{-1} \mp \tilde{\omega}_-^{-1}) \\ \tilde{J}_+ \pm \tilde{J}_- & & \tilde{\omega}_+^{-1} \mp \tilde{\omega}_-^{-1} & \\ & -(\tilde{\omega}_+ \mp \tilde{\omega}_-) & & -(\tilde{J}_+^T \pm \tilde{J}_-^T) \\ \tilde{\omega}_+ \mp \tilde{\omega}_- & & \tilde{J}_+^T \pm \tilde{J}_-^T & \end{pmatrix}, \tag{3.68}$$

where the transpose and inverse operation only indicate that the indices are up/down, appropriately.

Let us now apply the mirror map $\mathcal{M}$, given in (3.33), to get the conjugated GCS in the following way:

$$\hat{\mathcal{J}}_{1/2} = \mathcal{M} \circ \mathcal{J}_{1/2} \circ \mathcal{M}^{-1} : T_{\mathcal{B}} \oplus T^*_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T_{\mathcal{F}} \to T_{\mathcal{B}} \oplus T^*_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T_{\mathcal{F}}. \tag{3.69}$$

Applying this construction explicitly we get

$$\hat{\mathcal{J}}_{1/2} = \frac{1}{2} \begin{pmatrix} & -(\tilde{\omega}_+^{-1} \mp \tilde{\omega}_-^{-1}) & & -(\tilde{J}_+ \pm \tilde{J}_-) \\ \tilde{\omega}_+ \mp \tilde{\omega}_- & & \tilde{J}_+^T \pm \tilde{J}_-^T & \\ & -(\tilde{J}_+^T \pm \tilde{J}_-^T) & & -(\tilde{\omega}_+ \mp \tilde{\omega}_-) \\ \tilde{J}_+ \pm \tilde{J}_- & & \tilde{\omega}_+^{-1} \mp \tilde{\omega}_-^{-1} & \end{pmatrix} . \tag{3.70}$$

To compare $\hat{\mathcal{J}}_{1/2}$ with $\mathcal{J}_{1/2}$ we reinterpret $\hat{\mathcal{J}}_{1/2}$ as a map $T_{\mathcal{B}} \oplus T_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}} \to T_{\mathcal{B}} \oplus T_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}}$ instead of (3.69). We then use the fiber metric g_F and its inverse and we write them back into $\hat{\mathcal{J}}_{1/2}$. By using the identity $\omega = gI$, we get finally

$$\hat{\mathcal{J}}_{1/2} = \frac{1}{2} \begin{pmatrix} & -(\tilde{J}_+ \mp \tilde{J}_-) & & -(\tilde{\omega}_+^{-1} \pm \tilde{\omega}_-^{-1}) \\ \tilde{J}_+ \mp \tilde{J}_- & & \tilde{\omega}_+^{-1} \pm \tilde{\omega}_-^{-1} & \\ & -(\tilde{\omega}_+ \pm \tilde{\omega}_-) & & -(\tilde{J}_+^T \mp \tilde{J}_-^T) \\ \tilde{\omega}_+ \pm \tilde{\omega}_- & & \tilde{J}_+^T \mp \tilde{J}_-^T & \end{pmatrix} , \tag{3.71}$$

where now $\hat{\mathcal{J}}_{1/2}$ are again maps

$$\hat{\mathcal{J}}_{1/2} : T_{\mathcal{B}} \oplus T_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}} \to T_{\mathcal{B}} \oplus T_{\mathcal{F}} \oplus T^*_{\mathcal{B}} \oplus T^*_{\mathcal{F}}. \tag{3.72}$$

These are the mirror transformed GCSs. In the following we will denote by $\mathcal{M}$ the mirror map, which is the combined operation of $\mathcal{M}$ and the reinterpretation of maps. We see immediatley that mirror symmetry interchanges the two GCSs:

$$\begin{array}{ccc} \mathcal{J}_{1/2} & \longleftrightarrow & \hat{\mathcal{J}}_{1/2} = \mathcal{J}_{2/1} \\ (J_+, J_-) & \longleftrightarrow & (\hat{J}_+, \hat{J}_-) = (J_+, -J_-) \, . \end{array} \tag{3.73}$$

When M^6 is a nontrivial torus fibration, using the same remark above, also the mirror manifold $\hat{M}^6$ is a nontrivial torus fibration.

Let us assume for the moment that we have a generic GKS ($B = 0$) on a 6-manifold M^6 with two generic commuting (integrable) GCS (see 3.63), $\mathcal{J}_{1/2}$. Using the generalised metric G and the two GCSs we get a decomposition of $(T \oplus T^*) \otimes \mathbb{C}$ into a direct sum of four subbundles, as it is shown in [45]. Let us shortly review this decomposition to understand how these subbundles get mapped under the mirror map $\mathcal{M}$.

We already mentioned in section 3.1.5 that the generalised metric G gives a decomposition into two subbundles of dimension $3_{\mathbb{C}}$ each, i.e. by using the projectors $P_{\pm} = \frac{1}{2}(1 \pm G)$ we obtained the two subbundles (having eigenvalues ± 1) $C_{\pm} \otimes \mathbb{C}$,

$$(T \oplus T^*) \otimes \mathbb{C} = (C_+ \oplus C_-) \otimes \mathbb{C} \, . \tag{3.74}$$

It can be shown that elements of $C_\pm \otimes \mathbb{C}$ can be written as ($B = 0$)

$$C_+ \otimes \mathbb{C} = \{X + gX | X \in T \otimes \mathbb{C}\}\,, \qquad C_- \otimes \mathbb{C} = \{X - gX | X \in T \otimes \mathbb{C}\}\,, \tag{3.75}$$

where the generalised metric G is purely Riemannian,

$$G = \begin{pmatrix} & g^{-1} \\ g & \end{pmatrix}\,. \tag{3.76}$$

Moreover, since the GCS commute with G, we can also decompose the generalised tangent bundle with respect to the GCS $\mathcal{J}_{1/2}$. This we will do by the useful formulae,

$$\begin{aligned} \mathcal{J}_1 &= \pi|_{C_+}^{-1} J_+ \, \pi \, P_+ + \pi|_{C_-}^{-1} J_- \, \pi \, P_- \,, \\ \mathcal{J}_2 &= \pi|_{C_+}^{-1} J_+ \, \pi \, P_+ - \pi|_{C_-}^{-1} J_- \, \pi \, P_- \,, \end{aligned} \tag{3.77}$$

where $\pi : C_\pm \to T$ is a projection. Note, by explicitly using the projectors $P_\pm, \pi$ and the generalised metric G in (3.76), we can reconstruct, via (3.77), the generic GCS given in (3.63). Strictly speaking, (3.63) is equal to (3.77).

We will denote the i eigenbundle of $\mathcal{J}_{1/2}$, or equivalently the graphs of the maps $-i\mathcal{J}_{1/2}$, by $L_{1/2}$,

$$\begin{aligned} L_1 &= \{X + gX | X \in T_+^{1,0}\} \oplus \{X - gX | X \in T_-^{1,0}\}\,, \\ L_2 &= \{X + gX | X \in T_+^{1,0}\} \oplus \{X - gX | X \in T_-^{0,1}\}\,. \end{aligned} \tag{3.78}$$

The generalised tangent bundle decomposes therefore in

$$(T \oplus T^*) \otimes \mathbb{C} = L_1 \oplus \overline{L_1} = L_2 \oplus \overline{L_2}. \tag{3.79}$$

Since the two GCS commute we can decompose $L_{1/2}$ further by $\mathcal{J}_{2/1}$. We indicate with $\pm$ the eigenvalues $\pm i$ corresponding to the second splitting, e.g.

$$L_1 \oplus \overline{L_1} = L_1^+ \oplus L_1^- \oplus \overline{L_1^+} \oplus \overline{L_1^-}, \tag{3.80}$$

where

$$\begin{aligned} L_1^+ &= \{X + gX | X \in T_+^{1,0}\}\,, \qquad & L_1^- &= \{X - gX | X \in T_-^{1,0}\}\,, \\ L_2^+ &= \{X + gX | X \in T_+^{1,0}\}\,, \qquad & L_2^- &= \{X - gX | X \in T_-^{0,1}\}\,. \end{aligned} \tag{3.81}$$

We see that $L_2 = L_1^+ \oplus \overline{L_1^-}$ and

$$C_\pm \otimes \mathbb{C} = L_{1/2}^\pm \oplus \overline{L_{1/2}^\pm}\,. \tag{3.82}$$

In (3.73) we summarised how the GCSs of a GKS behave under mirror symmetry. Let us take this result and the above observations of the subbundles into account. Thus, by changing $J_- \to -J_-$ we do not affect the C_+-bundle and moreover only exchange in the C_--bundle holomorphic with anti-holomorphic objects with respect to J_-. Thus, mirror symmetry interchanges the subbundles $L_1^- \leftrightarrow \overline{L_1^-}$. See figure 3.5.

In the last section we already derived that maximal isotropics are associated to pure spinor lines. Using the mirror map $\mathcal{M}$ for spinors is therefore equivalent to mirror map the associated maximal isotropics. In the remainder of this section we shortly verify the just derived

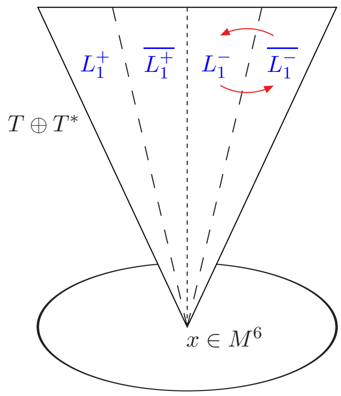


Figure 3.5: The mirror map $\mathcal{M}$ is effective only in C_-. It exchanges the bundles $L_1^- \leftrightarrow \overline{L_1^-}$ and means the interchange of holomorphic and anti-holomorphic partners.

results by the picture of pure spinor lines.

The maximal isotropics L_1, L_2 are basically given by the isotropics L_1^+, L_1^- and their complex conjugated partners. We will use in the following the complex conjugated objects $\overline{L_1^+}, \overline{L_1^-}$, which can be described by the following four pure spinor lines ϕ_i, $i \in \{1, \ldots, 4\}$, schematically,

$$\begin{aligned} 0 &= \overline{L_1^+} \cdot \phi_1 = \overline{L_1^+} \cdot \Omega_+^{(3,0)}, \qquad & 0 &= \overline{L_1^-} \cdot \phi_2 = \overline{L_1^-} \cdot \Omega_-^{(3,0)}, \\ 0 &= \overline{L_1^+} \cdot \phi_3 = \overline{L_1^+} \cdot e^{i\,\omega_+}, \qquad & 0 &= \overline{L_1^-} \cdot \phi_4 = \overline{L_1^-} \cdot e^{-i\,\omega_-}, \end{aligned} \tag{3.83}$$

where $\Omega_\pm^{(3,0)} \in \Lambda^{od}$ are holomorphic top degree forms with respect to I_+, I_- and $\omega_\pm \in \Lambda^{ev}$ are the Kähler forms.

Before we can apply the mirror map $\mathcal{M}$ to the spinors we first choose local coordinates to write down the spinor lines explicitly. Let us identify the spinors $\Omega_+^{(3,0)}$ and $e^{i\,\omega_+}$ with those given explicitly in (3.55). Let us construct $\Omega_-^{(3,0)}$ and $e^{i\,\omega_-}$ for the case $J_- = -J_+$ where we use (3.55) once more.

We apply the mirror map $\mathcal{M}$ (3.52) to the pure spinor lines ϕ_i to get

$$\begin{aligned} \hat{\phi}_1 &= \mathcal{M} \cdot \Omega_+^{(3,0)} = e^{i\,\omega_+}, \qquad & \hat{\phi}_2 &= \mathcal{M} \cdot \Omega_-^{(3,0)} = e^{i\,\omega_-}, \\ \hat{\phi}_3 &= \mathcal{M} \cdot e^{i\,\omega_+} = -\,\Omega_+^{(3,0)}, \qquad & \hat{\phi}_4 &= \mathcal{M} \cdot e^{-i\,\omega_-} = -\,\overline{\Omega_-^{(3,0)}}. \end{aligned} \tag{3.84}$$

Let us now focus on the isotropics which are associated to these mirror transformed pure spinor lines $\hat{\phi}_i$, $i \in \{1, \ldots, 4\}$. We see immediately that L_1^+ is untouched by the map $\mathcal{M}$ but in the C_--bundle it interchanged L_1^- with $\overline{L_1^-}$. Thus, we verified our previous result exactly.

3.3 Topological models and generalised Kähler geometry

In 1984 Gates, Hull and Rocek [37] investigated the enhancement of $(1,1)$ world sheet supersymmetry to $(2,2)$ in the non-linear sigma model for a 6-dimensional target space X. The sigma model describes maps $\phi : \Sigma \to X$ from the Riemann surface Σ to the target space X. They showed that the additional supercharges can be constructed using a complex structure on the target space X. Moreover, they recognised that it is possible to achieve the enhancement of supersymmetry by treating the left and right moving supercharges differently. Strictly speaking, they introduced two independent complex structures, one for the left moving and one for the right moving sector. Since both almost complex structures must be compatible with the metric g on the target space X, the geometry is called bi-hermitian and is denoted by the triple (g, J_+, J_-). Since the integrable model also allows for the skew-symmetric part of the metric, the B-field b (being a 2-form), and a additional 3-form field strenght H the data can be enhanced to be $(g, J_+, J_-, H + db)$.

Since bi-hermitian geometry was almost unknown at that time, even for mathematicians, people usually treated this model afterwards by identifying the two indepentent complex structures $J_+ = J_-$ and used the well known Kähler geometry. Even in this simplified version, the model stayed quite complicated until Witten introduced the topological twist [86] in 1988. He still kept the property of identifying the two complex structures, but the big step was that he twisted the worldsheet fermions by the axial/vector R-currents to end up with worldsheet scalars and 1-forms. By focusing only on the constructed scalars the model turned out to solely describe the topological sector of the field theories. Having those topological field theories at hand it was possible to explicitly construct physically relevant objects, e.g. observables, which proved to be useful even in the untwisted theory.

In the original work Witten twisted the $\mathcal{N} = (2,2)$ sigma model in two different ways. He introduced the topological A/B model by twisting with the vector/axial current. In case of Kähler geometry the A model localises on holomorphic maps and depends only on the Kähler moduli of the target space X. The B model localises on constant maps from Σ to a Calabi-Yau manifold X and depends only on the complex moduli of X. Note that one has to restrict in the B-model the target space geometry from Kähler to Calabi-Yau, which is necessary to cancel the axial anomaly.

Since mirror symmetry exchanges symplectic and complex structures of two different manifolds, as we already explained in former sections, it is obvious to ask, if the topological A and B model are mirror duals? We learned from [45] that bi-hermitian geometry is equivalent to generalised Kähler geometry. Thus, we can alternatively use the GKS by also relaxing the constraint $J_+ = J_-$ to $J_+ \neq J_-$ that was already done by Kapustin [65]. He introduced the generalised twisted B-model. The advantage of using the GKS picture and not the bi-hermitian geometry picture becomes evident since we know form the previous section how to explicitly do mirror symmetry for GCS (3.73).

We kick of this section to provide the reader with the basic definitions of a topological twisted $(2,2)$ non-linear sigma model. Since we use a GKS we formulate the generalised topological A and B model and define the corresponding BRST operators $Q_{A/B}$ and their variations in the $T \oplus T^*$ picture. These objects depend in a given model only on one of the two present GCSs. A mirror symmetry consideration shows that the two generalised topological models get interchanged. We verify that by focusing on the observables, the instantons, the anomalies and the branes. We make contact to Witten's A/B models by identifying the two independent complex structures J_+ and J_- appropriately.

Before we start let us spend a view words about the observables of the A and B model of Witten [86]. In the A and B model he constructed the observables out of the BRST invariant field configurations. He explicitly used the BRST invariant field configurations as a base and introduced the corresponding coordinate functions. Due to the nilpotence property of the BRST variations one establishes a BRST complex for the observables. The associated cohomology is called BRST cohomology. The basic idea is that Witten found a map from the BRST complex of the A or B model to more convenient complexes. He achieved this correspondence since he assumed that the coordinate functions of the observables are also coordinate functions for other objects. For the A model these objects are just the usual differential forms and for the B model these objects are given by elements of the space $\Lambda^q T^{1,0} \otimes \Lambda^{0,p} T^*$, i.e. $(0,p)$ forms with values in the qth power of the holomorphic tangent space. The BRST operator translates therefore for the A model to the usual differential operator d and for the B model to the Dolbeault operator $\bar{\partial}$. Witten showed that the observables are elements of the BRST complex, i.e. they are closed, iff they are also closed in the related complex. Thus, the A model BRST cohomology is isomorphic to the deRham cohomology and the B model BRST cohomology is isomorphic to the Hochschild cohomology $\oplus_{p,q} H^p(M^6, \Lambda^q T^{1,0})$ (roughly, a extended Dolbeault cohomology). Since we consider the observables in the generalised setup it is convenient to introduce the notion of a Lie algebroid. We will see that this is the proper tool to capture the complexes for both - the old A and B model.

3.3.1 Introduction of the generalised A and B model

We start with the nonlinear sigma model formulated in the $\mathcal{N} = (1,1)$ superfield formalism:

$$S = \frac{1}{2} \int \mathrm{d}^2\sigma \, \mathrm{d}^2\theta \, (g + B)(D_+\Phi, D_-\Phi), \quad \text{where} \tag{3.85}$$

$$D_\pm = \frac{\partial}{\partial\theta^\pm} + i\theta^\pm \partial_\pm \quad ; \quad \partial_\pm := \partial_0 \pm \partial_1 \, . \tag{3.86}$$

The $\mathcal{N} = (1,1)$ SUSY transformations are generated by $Q_\pm$, defined as

$$Q_\pm^{(1)} := \frac{\partial}{\partial\theta^\pm} - i\theta^\pm \partial_\pm \, , \tag{3.87}$$

and the chiral superfield can be expanded in components as

$$\Phi = \phi + \theta^+ \psi_+ + \theta^- \psi_- + \theta^- \theta^+ F. \tag{3.88}$$

An additional supersymmetry can be defined by [37]

$$Q_\pm^{(2)} := J_\pm D_\pm. \tag{3.89}$$

This is a well defined (1,1) supersymmetry, if $J_\pm$ is a pair of integrable almost complex structures on the tangent space T and the metric g is hermitian with respect to both J_+ and

J_-. Furthermore the almost complex structures have to be covariantly constant with respect to covariant derivatives with connection:

$$\Gamma^a_{\pm bc} := \Gamma^a_{bc} \pm g^{ad} H_{dbc}, \tag{3.90}$$

where Γ is the Levi-Civita connection and H is the 3-form field strenght associated to B. We get the following relation between the two connections

$$\Gamma^a_{+\,bc}\psi^b_+\psi^c_- = -\Gamma^a_{-\,bc}\psi^b_-\psi^c_+ \,. \tag{3.91}$$

The variations of the superfield (3.88) can be written in components as

$$\begin{array}{rclcrcl}
\delta^{(1)}_+\phi &=& \psi_+\,, & & \delta^{(1)}_-\phi &=& \psi_-\ , \\
\delta^{(1)}_+\psi_+ &=& -i\partial_+\phi\,, & & \delta^{(1)}_-\psi_+ &=& F\ , \\
\delta^{(1)}_+\psi_- &=& -F\,, & & \delta^{(1)}_-\psi_- &=& -i\partial_-\phi\ , \\
\\
\delta^{(2)}_+\phi &=& J_+\psi_+\,, & & \delta^{(2)}_-\phi &=& J_-\psi_-\ , \\
\delta^{(2)}_+\psi_+ &=& iJ_+\partial_+\phi\,, & & \delta^{(2)}_-\psi_+ &=& J_-F\ , \\
\delta^{(2)}_+\psi_- &=& -J_+F\,, & & \delta^{(2)}_-\psi_- &=& iJ_-\partial_-\phi\ .
\end{array} \tag{3.92}$$

We can integrate out the auxiliary field F using the equations of motion

$$F^a = \Gamma^a_{+bc}\psi^b_+\psi^c_-. \tag{3.93}$$

Furthermore, we define combinations of the supersymmetry generators

$$\begin{array}{rclcrcl}
Q_+ &=& \frac{1}{2}(Q^{(1)}_+ - iQ^{(2)}_+)\,, & & \overline{Q}_+ &=& \frac{1}{2}(Q^{(1)}_+ + iQ^{(2)}_+)\,, \\
Q_- &=& \frac{1}{2}(Q^{(1)}_- - iQ^{(2)}_-), & & \overline{Q}_- &=& \frac{1}{2}(Q^{(1)}_- + iQ^{(2)}_-)\,.
\end{array} \tag{3.94}$$

With these we make contact to the definitions of [58].

We are now prepared to define the generalised topological A/B model. We twist the spin of the fermionic fields by the vector/axial $U(1)$ current. The charges of the fields are given in the following table,

	q_V	q_A	J	J_A	J_B
$\mathcal{P}_+\psi_+$	-1	-1	$-\frac{1}{2}$	-1	-1
$\overline{\mathcal{P}}_+\psi_+$	$+1$	$+1$	$-\frac{1}{2}$	0	0
$\mathcal{P}_-\psi_-$	-1	$+1$	$+\frac{1}{2}$	0	$+1$
$\overline{\mathcal{P}}_-\psi_-$	$+1$	-1	$+\frac{1}{2}$	$+1$	0

(3.95)

where $q_{V/A}$ indicate the vector/axial charge. J and $J_{A/B} = J + q_{V/A}/2$ define the spins before and after the twist and we used projectors on the (anti-)holomorphic parts of the fields with respect to $J_\pm$,

$$\mathcal{P}_\pm = \frac{1}{2}(1 - iJ_\pm), \quad \overline{\mathcal{P}}_\pm = \frac{1}{2}(1 + iJ_\pm). \tag{3.96}$$

As BRST operators for the generalised A/B model we take[1]

$$Q_A := \overline{Q}_+ + Q_-, \quad Q_B := \overline{Q}_+ + \overline{Q}_-, \tag{3.97}$$

[1]Note that [86] uses a different definition for the world sheet fermions, which leads to a different BRST operator for the A model, $Q_A = Q_+ + \overline{Q}_-$.

which act on the scalar fields of the twisted models like

$$\begin{array}{rclrcl} \delta_A\phi &=& \overline{\mathcal{P}}_+\psi_+ + \mathcal{P}_-\psi_- \ , & \delta_B\phi &=& \overline{\mathcal{P}}_+\psi_+ + \overline{\mathcal{P}}_-\psi_- \ , \\ \delta_A\overline{\mathcal{P}}_+\psi_+ &=& \Gamma_+\overline{\mathcal{P}}_+\psi_+\mathcal{P}_-\psi_- \ , & \delta_B\overline{\mathcal{P}}_+\psi_+ &=& \Gamma_+\overline{\mathcal{P}}_+\psi_+\overline{\mathcal{P}}_-\psi_- \ , \\ \delta_A\mathcal{P}_-\psi_- &=& \Gamma_-\mathcal{P}_-\psi_-\overline{\mathcal{P}}_+\psi_+ \ , & \delta_B\overline{\mathcal{P}}_-\psi_- &=& \Gamma_-\overline{\mathcal{P}}_-\psi_-\overline{\mathcal{P}}_+\psi_+ \ . \end{array} \tag{3.98}$$

3.3.2 Generalised A and B models and mirror symmetry

In section (3.2.3) we investigated mirror symmetry with GCSs from a mathematical point of view. In this section we apply the results to generalised topological sigma models as defined in section (3.3.1).

We start by rewriting the BRST operators defined in (3.97) in the $T \oplus T^*$ bundle. Let us define the fermionic basis

$$\psi := (\psi_+ + \psi_-) \in T, \qquad \rho := g(\psi_+ - \psi_-) \in T^*, \qquad \Psi := \begin{pmatrix} \psi \\ \rho \end{pmatrix} , \tag{3.99}$$

where we stress that $\psi_\pm$ are the resulting fermionic scalars coming from the twist and are furthermore independent sections in T. This definition corresponds to the previous given example 3.1.1. Then the BRST operators of the generalised A and B model take the form [65]

$$\begin{aligned} Q_A &= \left\langle \begin{pmatrix} \partial_1\phi \\ g\partial_0\phi \end{pmatrix}, (1+i\mathcal{J}_2)\Psi \right\rangle , \\ Q_B &= \left\langle \begin{pmatrix} \partial_1\phi \\ g\partial_0\phi \end{pmatrix}, (1+i\mathcal{J}_1)\Psi \right\rangle , \end{aligned} \tag{3.100}$$

where $\langle\, , \rangle$ is the natural metric on $T \oplus T^*$ (3.2). It is necessary to note the following. The generalised B model yields the old B and A models under the identifications $J_+ = J_-$ and $J_+ = -J_-$. Under the same identifications the generalised A model yields the old A and B model.

We can reformulate the BRST variations (3.98) in the $T \oplus T^*$ basis. The BRST variations which vanish because of the property (3.91) are the following

$$\delta_A \frac{1}{2}(1+i\mathcal{J}_1)\Psi = 0, \quad \delta_B \frac{1}{2}(1+i\mathcal{J}_2)\Psi = 0\,. \tag{3.101}$$

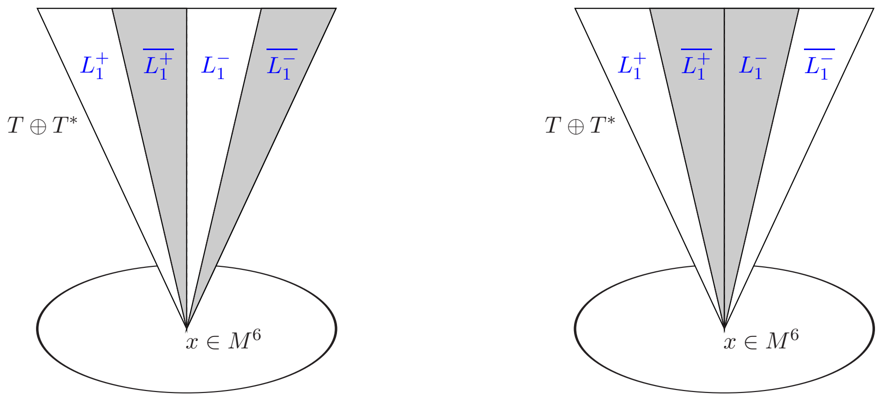
The classical $U(1)_{A/V}$ symmetry can be broken by quantum effects. This anomaly is given in terms of the first Chern class of the $L_{1/2}$ bundle for the B/A model [66]. The cancellation of this anomaly constraints the target space geometry via $c_1(L_{1/2}) = 0$. Remember, in the Witten model the first Chern class is calculated in the tangent bundle. But since we deal here with its extension to $T \oplus T^*$ the relevant first Chern class is in principle given by $c_1(T \oplus T^*, \mathcal{J}) = c_1(C_+) \oplus c_1(C_-)$ [45], i.e. there are two classes (compare also with (3.82)).

We are now in the position to show how the relevant quantities of the generalised B model with the target space M^6 (being actually T^6) are mapped to the ones of the generalised A model[2] with the mirror target space $\hat{M}^6$. From section 3.2.3 we know that $\mathcal{M} : \mathcal{J}_1 \to \mathcal{J}_2$ so that $\mathcal{M} : Q_B \to Q_A$. We also know that (J_+, J_-) is mapped to $(J_+, -J_-)$ under the mirror map and equation (3.81) tells us that $\mathcal{M} : L_1 \to L_2$. Therefore, $\mathcal{M} : c_1(L_1) \to c_1(L_2)$ and the anomaly cancellation condition of the generalised B model gets mapped to that of the generalised A model.

The next step is to show that the observables of the generalised B and A model are mirrors of each other. We show this for the local observables of the closed topological sector, but first let us remember how they were constructed in [65](see also [66]). Following [86], one has to construct scalar BRST invariant field configurations. Writing the BRST variations in the $T \oplus T^*$ bundle, we get[3]

$$\delta_{B/A}\Phi = \Psi_{1/2} := \frac{1}{2}(1 + i\mathcal{J}_{1/2})\Psi \in \overline{L_{1/2}}\,, \qquad\qquad \Phi := \begin{pmatrix} \phi \\ g\phi \end{pmatrix}\,. \tag{3.102}$$

The nilpotence properties $\delta^2_{B/A} = 0$ of the BRST variations then yield $\delta_{B/A}\Psi_{1/2} = 0$, which was also obtained in (3.101). Thus, $\Psi_{1/2}$ are the configurations we are looking for in the generalised B/A model. Explicitly, we have $\Psi_1 = \overline{L_1} = \overline{L_1^+} \oplus \overline{L_1^-}$ and $\Psi_2 = \overline{L_2} = \overline{L_1^+} \oplus L_1^-$ (see figure 3.6).



$a)$ Ψ_1 for the generalised B-model $\qquad$ $b)$ Ψ_2 for the generalised A-model

Figure 3.6: The field configurations for the generalised B/A model are given by $\Psi_{1/2} \in \overline{L_{1/2}}$. The relevant subbundles for $\Psi_{1/2}$ are gray colored. Mirror symmetry exchange $\Psi_1 \leftrightarrow \Psi_2$.

Before we proceed, we introduce the notion of a Lie algebroid [45]. Remember: Let T be the tangent bundle (being a vector bundle). By taking the dual T^* we can generate the associated exterior k-forms, being elements in $\Lambda^k T^*$, which we can differentiate by the differential

[2] This choice is of course arbitrary, one could as well start with the generalised A model on M^6 and map it to the B model on $\hat{M}^6$

[3] This Φ is an element of $T \oplus T^*$ and should not be confused with the chiral superfield defined in section 3.3.1.

operator d. We can measure the topological non-triviality of the manifold by using exterior forms and set up, due to $d^2 = 0$, the deRham complex.

We generalise this setup and define a Lie algebroid L to be a vector bundle on a manifold that comes together with a Lie bracket $[\,,\,]$ on L. The manifold also admits a smooth map $a : L \to T$, called the anchor, which is a homomorphism and satisfies for a function $f \in C^\infty(M)$ the Leibniz rule $[X, f\,Y] = f[X,Y] + (a(X)f)\,Y$, $\forall\, X,\, Y \in C^\infty(L)$.

The corresponding first order Lie algebroid derivative $d_L : C^\infty(\Lambda^k L^*) \to C^\infty(\Lambda^{k+1}L^*)$ is defined by

$$\begin{aligned}(d_L\alpha)(X_0,\ldots,X_k) = &\sum_i (-1)^i\, a(X_i)\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)\,,\end{aligned}$$

where $\alpha \in C^\infty(\Lambda^k L^*)$, $X_i \in C^\infty(L)$. Shortly, we differentiate the coordinate functions α (depending on the coordinates only) by $a(X) \in T$, since $X \in C^\infty(L)$. For instance, if $L = T$ and $a = id$ then d_L is the usual deRham differential operator and we have the deRham complex. When $L = T^{0,1}$ and a is the inclusion map then the corresponding forms are elements in $\Lambda^k T^{*0,1}$ and $d_L = \bar{\partial}$, i.e. we have the Dolbeault complex.

Our aim in the following is to achieve a correspondence between the BRST complex and the Lie algebroid complex. This will be useful since a Lie algebroid complex can be handled much easier. We do this explicitly by using the coordinate functions α for both complexes. Let us now apply the Lie algebroid picture to the BRST invariant configurations $\Psi_{1/2} \in \overline{L_{1/2}}$ (in the B/A model). We associate to them a Lie algebroid $\overline{L_{1/2}}$ where the anchor is the projection $\pi : \overline{L_{1/2}} \to T$. Let $\alpha^k \in C^\infty(\Lambda^k \overline{L^*_{1/2}})$ and let us use the derivative $d_{\overline{L}_{1/2}}$ to setup a $\overline{L_{1/2}}$-complex, since $d^2_{\overline{L}_{1/2}} = 0$. Note also that $\Lambda^k \overline{L^*_{1/2}} \simeq \Lambda^k L_{1/2}$.

We use the coordinate functions α to define the space of observables by

$$(\mathcal{O}_\alpha)_{B/A} = \alpha_{a_1\cdots a_n}(\phi)\Psi^{a_1}_{1/2}\cdots\Psi^{a_n}_{1/2}\,. \tag{3.103}$$

If we now perform the BRST variation of the observables, where we make use of the Lie algebroid derivative, we realise that

$$\delta_{B/A}(\mathcal{O}_\alpha)_{B/A} = (\mathcal{O}_{d_{\overline{L}_{1/2}}\alpha})_{B/A}. \tag{3.104}$$

This implies that the observables are closed under the BRST variation iff $\alpha^k \in C^\infty(\Lambda^k \overline{L^*_{1/2}})$ is closed under the Lie algebroid derivative $d_{\overline{L}_{1/2}}$, i.e. the BRST-complex is isomorphic to the $\overline{L_{1/2}}$-complex.

We can even be more precise, which we sketch roughly in the following. Actually, we do have two GCSs on the manifold. According to section 3.2.3, the vector bundle $\overline{L_{1/2}}$ and so the Lie algebroid decomposes further with respect to $\mathcal{J}_{2/1}$, $\overline{L_{1/2}} = \overline{L^+_{1/2}} \oplus \overline{L^-_{1/2}}$. Let us also define $\Lambda^{p,q}\overline{L^*_{1/2}} = \Lambda^p\overline{L^{+*}_{1/2}} \otimes \Lambda^q\overline{L^{-*}_{1/2}}$, where the Lie algebroid derivative can be denoted by [45]

$$d_{\overline{L}_{1/2}} = \partial^+_{\overline{L}_{1/2}} + \partial^-_{\overline{L}_{1/2}}\,, \tag{3.105}$$

due to the fact that $\overline{L^\pm_1}$ is closed under the Lie bracket. For instance, in the generalised B model $\overline{L_1}$ equals $\overline{L^+_1} \oplus \overline{L^-_1}$ and the differential operator becomes $\partial^+_{\overline{L}_1} + \partial^-_{\overline{L}_1}$. This means that we (schematically) differentiate the coordinate functions of α by, $\pi(\overline{L}^+_1 \oplus \overline{L}^-_1) = \partial^+_{\bar{z}} + \partial^-_{\bar{z}}$, i.e. both zs are 'bared'. Here we used the explicit notation given in (3.81). For the generalised A model we have $\pi(\overline{L^+_2} \oplus \overline{L^-_2}) = \partial^+_{\bar{z}} + \partial^-_{z}$, where in this case only one z is 'bared'. Note, that one has to distinguish the "bars" on top of z which correspond to the appropriate complex structures $J_\pm$. We can easily recover Witten's old B and A model results, where we take the generalised B model and do the identifications $J_+ = J_-$ and $J_+ = -J_-$. For instance, for the old A model ($J = J_+ = -J_-$) we obtain $\overline{L_1} = \overline{L^+_1} \oplus L^-_1$,

$$\overline{L^+_1} = \{X + gX | X \in T^{0,1}\}_{J_+ = J}\,, \qquad L^-_1 = \{X - gX | X \in T^{1,0}\}_{J_- = -J}\,. \tag{3.106}$$

The Lie algebroid derivative becomes $\pi(\overline{L^+_1} \oplus L^-_1) = \partial^+_{\bar{z}} + \partial^-_{z}$, where now the local coordinates z correspond to only one complex structure J. Witten denoted this Lie algebroid derivative by the usual deRham operator d and found the deRham complex for his A model. Note that we have nevertheless two independent sections $\psi_\pm \in T$.

Coming back to our initial interest in mirror symmetry we get as a final result for the observables: Since $\mathcal{M} : \overline{L_1} \to \overline{L_2}$, the cohomologies of the differential complexes for the generalised A and B models are mirror pairs.

We want to do the same for the generalised instantons [65]. The instantons are the fixed points of the BRST transformations. Performing the Wick rotation $\partial_0\phi \to i\partial_2\phi$ on the Riemann surface, one gets the instanton equations

$$\delta_{B/A}\Psi = (1 - i\mathcal{J}_{1/2}) \begin{pmatrix} i\partial_2\phi \\ g\partial_1\phi \end{pmatrix} = 0\,. \tag{3.107}$$

These equations tell us that the instantons of the generalised B model are mapped to those of the generalised A model under the mirror map.

3.3.3 Topological branes and their mirrors

In this section we want to investigate how topological branes behave under the mirror map $\mathcal{M}$. We will strongly follow the notation and conventions used in [87] and references therein.

Branes in the topological A/B model (A/B branes) can be defined by a gluing matrix $R : T \to T$, which encodes information about the mapping of left- and right-moving fermions at the boundary $\partial\Sigma$ [3, 4]. The gluing conditions read

$$\psi_- = R\psi_+\,. \tag{3.108}$$

In the generalised picture this translates to [87]

$$\mathcal{R} : T \oplus T^* \to T \oplus T^*, \qquad \mathcal{R}\Psi = \Psi\,, \tag{3.109}$$

where Ψ is defined in (3.99). Locally we can write $\mathcal{R}$ in adapted coordinates,

$$\mathcal{R} = \begin{pmatrix} r & \\ & -r^t \end{pmatrix}\,, \quad \text{where} \quad r = \begin{pmatrix} 1_N & \\ & -1_D \end{pmatrix}\,. \tag{3.110}$$

We denoted the identity matrices for the Neumann- and Dirichlet boundary conditions by 1_N and 1_D. One important property of the gluing operator $\mathcal{R}$ is that it singles out the generalised tangent bundle of the p-dimensional submanifold $\mathcal{D}$ by $\mathcal{R}\Psi = \Psi$, i.e. we have p Neumann directions that define the brane. We will later see which p-submanifolds can appear in the generalised topological A/B model.

Furthermore, $\mathcal{R}$ respects the natural metric $\langle\cdot,\cdot\rangle$ on $T \oplus T^*$, squares to one, i.e. $\mathcal{R}^2 = 1$, and anticommutes with G, i.e. $G\,\mathcal{R} + \mathcal{R}\,G = 0$.

In the (physical) gluing framework the operator $\mathcal{R}$ contains the information about Dirichlet and Neumann boundary conditions (bc). It defines a smooth distribution $\mathcal{D} \subset T$ which has rank equal to the dimension of the brane. In case of an integrable distribution we even have (Frobenius) a maximal integral submanifold $\mathcal{D}$.

From a different point of view, the above properties of $\mathcal{R}$ serve to consider the projection operator $\frac{1}{2}(1 + \mathcal{R})$ to define a special almost Dirac structure $\tau^0_{\mathcal{D}}$ (a real, maximal isotropic sub-bundle),

$$\tau^0_{\mathcal{D}} = T\mathcal{D} \oplus \mathrm{Ann}(T\mathcal{D}) \subset T \oplus T^*\,, \tag{3.111}$$

which is (Courant) integrable iff $\mathcal{D}$ is integrable.

The extension of $\mathcal{R}$ by a closed two-form $F \in \Omega^2(\mathcal{D})$, $dF = 0$, on the submanifold $\mathcal{D}$ corresponds to [87]

$$\tau^F_{\mathcal{D}} = \{\, \frac{1}{2}(1 + \mathcal{R})(X + \xi) = (X + \xi) \,:\, (X + \xi) \in T\mathcal{D} \oplus T^*M|_{\mathcal{D}}\,,\ \xi|_{\mathcal{D}} = X \lrcorner F \,\} \tag{3.112}$$

and is equivalent to the definition of a generalised tangent bundle given in [45]. This gluing matrix is given by

$$\mathcal{R} = \begin{pmatrix} 1 & \\ F & 1 \end{pmatrix} \begin{pmatrix} r & \\ & -r^t \end{pmatrix} \begin{pmatrix} 1 & \\ -F & 1 \end{pmatrix} = \begin{pmatrix} r & \\ F\,r + r^t\,F & -r^t \end{pmatrix}\,. \tag{3.113}$$

We use the above observations to define submanifolds for the generalised topological A and B model. Submanifolds for the A/B model are called A/B branes and are characterised by the the $U(1)$ currents $j_\pm = \omega_\pm(\psi_\pm,\, \psi_\pm), \psi_\pm \in T$, where $\omega_\pm$ are the symplectic forms. Here we used the common notation from the literature that is a little bit sloppy. The two sections we plugged in have actually to be different ones. In a precise way it is better to think about two sections $X_\pm, Y_\pm \in T$ and write $j_\pm = \omega_\pm(X_\pm,\, Y_\pm)$.

The $U(1)$ currents have to fulfill the matching conditions

$$0 = j_+ \,\pm\, j_- = \frac{1}{2}\left\langle \Psi\,, \mathcal{J}_{2/1}\Psi \right\rangle \tag{3.114}$$

for the A/B model.

Combining this with the gluing conditions for the fermions (3.109), we obtain a stability condition for $\mathcal{R}$, or equivalently, a stability condition for $\tau^F_{\mathcal{D}}$. Using also $\{G, \mathcal{R}\} = 0$, one gets:

$$\begin{array}{lrclcrcl} \text{A branes:} & \mathcal{R}\mathcal{J}_1 & = & -\mathcal{J}_1\mathcal{R} & \text{and} & \mathcal{R}\mathcal{J}_2 & = & \mathcal{J}_2\mathcal{R} \\ \text{B branes:} & \mathcal{R}\mathcal{J}_1 & = & \mathcal{J}_1\mathcal{R} & \text{and} & \mathcal{R}\mathcal{J}_2 & = & -\mathcal{J}_2\mathcal{R}\,. \end{array} \tag{3.115}$$

We will call the (anti)commuting constraints $\mp$-stability with respect to a certain GCS. Thus, the A/B model is $\mathcal{J}^-_{1/2}/\mathcal{J}^+_{2/1}$ stable. This reflects the fact that the generalised tangent bundle $\tau^F_{\mathcal{D}}$ in the A/B model splits into $\pm i$ eigenbundles of $\mathcal{J}_{2/1}$, or in other words, it becomes a stable subbundle of $L_{2/1} \oplus \overline{L_{2/1}}$, respectively:

$$\text{A/B model:} \quad \tau^F_{\mathcal{D}} = \tau^{F\,+}_{\mathcal{D}} \oplus \tau^{F\,-}_{\mathcal{D}}\,, \quad \text{with respect to} \quad L_{2/1} \oplus \overline{L_{2/1}}\,. \tag{3.116}$$

We are now prepared to apply the mirror map $\mathcal{M}$ to the branes we just introduced. The gluing operator $\mathcal{R}$ gets mapped to $\hat{\mathcal{R}} = \mathcal{M}\mathcal{R}\mathcal{M}^{-1}$ and one can show that the properties for $\hat{\mathcal{R}}$ are the same as they were for $\mathcal{R}$. As before, we take M^6 which has a T^3 fibration, then mirror symmetry interchanges Neumann bc with Dirichlet bc in the fibre[4]. Remember that under mirror symmetry the GCS (also the A/B model) gets interchanged. Therefore it is easy to see that the $U(1)$ current conditions get mapped to each other and A/B branes get naturally mapped to B/A branes. But note that on the mirror side the stability conditions are formulated with $\hat{\mathcal{R}}$.

Furthermore, in case of non-vanishing $F \in \Omega^2(\mathcal{D})$, let us focus on the part of (3.113), where only F appears. Then, applying $\mathcal{M}$, we obtain the following symbolical shape of $\hat{\mathcal{R}}$

$$\mathcal{R} = \begin{pmatrix} r & \\ \square & -r^t \end{pmatrix} \qquad \longrightarrow \qquad \hat{\mathcal{R}} = \begin{pmatrix} \hat{r} & \square \\ & -\hat{r}^t \end{pmatrix} \tag{3.117}$$

with a bi-vector $\beta = F^{-1}$ in the upper triangular part. Thinking of F in components this means that the indices get raised. We will denote this by

$$\tau^\beta_{\hat{\mathcal{D}}} = \{\, \frac{1}{2}(1 + \hat{\mathcal{R}})(X + \xi) = (X + \xi) \, : \, (X + \xi) \in T\hat{M}|_{\hat{\mathcal{D}}} \oplus N^*\hat{\mathcal{D}}\,,\, X|_{N\hat{\mathcal{D}}} = \beta(\xi)\,\}, \tag{3.118}$$

where $N^*\hat{\mathcal{D}}$ is the conormal bundle to $\hat{\mathcal{D}}$. For example, if we start with a brane which has only a worldvolume in the fibre directions and a non-vanishing two-form F, it will be mapped to a brane which corresponds to a "point". But on the mirror side F disappears and we find a bi-vector β in the fibre directions instead. This can be interpreted as a noncommutative deformation of $\hat{M}$, as has been argued in [65].

This brings us immediately to the proposal to investigate the case of having at the same time both independent structures, a two-form and a bi-vector. This would correspond to B and β transformations in the sense of [45] and therefore we get a natural extension of the generalised tangent bundle.

[4]The mirror map $\mathcal{M}$ is only a special case of the more general T-duality transformation and therefore this statement can be extended.

Chapter 4

Generalised geometries and supergravity in 10d

From a duality and a phenomenological point of view, the idea of compactifying supergravity theories is a rather appealing one. It also points to interesting geometrical issues as requiring a certain amount of supersymmetry to be preserved puts constraints on the internal background geometry and thus leads to special G-structures. As we already discussed in the first chapter, a compactification of 11d supergravity on 7-manifolds in conjunction with a single internal single parallel spinor leads to G_2-structure spaces.

Supergravity theories are physical concepts that describe the low energy effective action of the corresponding superstring theories. The five existing theories are the most promising canditates for a unified description of all physically fundamental forces. In the present chapter we investigate 10d type IIB supergravity vacua by compactifying on seven and six dimensional manifolds. We shall focus on the geometrical structure of the vacuum space admitting a certain amount of supersymmetry in the external space, which can be achieved by a spinorial formulation of the supersymmetry variations. An additional analysis of the equations of motion single out the physical vacua.

The situation where only the massless bosonic fields are present at low energies characterises what is called the vaccum. The fundamental bosonic objects that appear come in two flavours: the NS-NS fields and the R-R fields. The NS-NS fields consist of a metric g, a 2-form B-field b and the dilaton function ϕ. The odd/even forms $C^{od/ev}$ describe the R-R potentials in type IIA/B and the corresponding field strength tensors can be denoted by $F^{ev/od}$. In type II one further imposes that the 10d R-R fields have to be anti-self-dual with respect to $\Box_{M^{1,9}}$, i.e.

$$F^{ev/od} = -\Box_{M^{1,9}} F^{ev/od}. \tag{4.1}$$

The field content of the NS-R and R-NS fields is purely fermionic and is given by the gravitino Ψ_X (spin 3/2) and the dilatino λ (spin 1/2). Although these fermionic fields being absent in the vacuum the supersymmetry variations are non-vanishing. A supersymmetric vacuum can be achieved in case the fermionic supersymmetry variations of the gravitino Ψ_X and dilatino λ, that depend on the two bosonic sectors and two independent spinors $\varepsilon = (\varepsilon_1, \varepsilon_2)$, vanish,

$$\delta_\varepsilon \Psi_X = 0, \qquad \delta_\varepsilon \lambda = 0\,.$$

The spinors $\varepsilon_{1/2}$ are non-physical and represent the globally well defined parameters necessary to establish supersymmetry.

This chapter reviews the authors' results given in [61, 62]. We first discuss the case of vanishing R-R fields by doing an explicit supersymmetric compactification on 7-manifolds. In general, the two 10d supersymmetry parameters result in precisely two internal spinors on the 7-manifold and induce a topological generalised structure. This is due to the 10d supersymmetry parameters being independent and this property can be transported to the internal space. We furthermore discuss a solution where we have two real spinors on M^7. Since the two spinors are independent each spinor reduce the structure group to two distinct G_2s, say $G_{2\pm}$. The supersymmetry variations ask for integrability of this topological structure and thus achieve the geometrical reduction. This correspondence - the physical supersymmetry constraint versus the mathematical reduction of the structure group - can be seen on the spinorial level since the physical supersymmetry variations are formulated from this point of view. By relating the spin picture to the picture of mixed degree forms one even obtain the opportunity to characterise the supersymmetry conditions in the language of forms.

By calculating the supersymmetry constraints for the internal space we observe that the resulting two spinorial 7d equations, the gravitino and the dilatino equation, are precisely the equations that characterise what is called a generalised G_2-structure. This type of structures were first defined in mathematics by F. Witt [84, 83]. This means that the structure group reduces from $\mathbb{R}_0 \times SO(7,7)$ to $G_2 \times G_2$ and all internal defined NS-NS degrees of freedom can be modeled by this geometrical structure.

A short review about generalised G_2-structures is given. Here we also point on the fact that generalised structures can be understood as the holonomy theory for elements in $T \oplus T^\star$. Since this space naturally admits a signature of $(7,7)$ it becomes obvious that, in general, one has to deal with the structure group $SO(7,7)$. Remember, in the classical case one parallel transports elements in T and here the interesting holonomy groups are subgroups of $SO(7)$. The two internal spinorial equations were also derived by Gauntlett et al. [38] from the perspective of wrapped NS5-branes in IIB supergravity. There, the authors found a solution by assuming that the two real internal spinor are globally orthogonal and therefore this corresponds to a classical $SU(3)$-structure. We rederive this result as a special case within the generalised structures.

We further do the analogous compactification on a 6-manifold and assume to have two complex internal spinors. We introduced this new structure in the literature under the name of a generalised $SU(3)$-structure [61] and here the structure group is $SU(3) \times SU(3)$. The basic ideas of the generalised G_2-structures can also be applied here. The inclusion $SU(3) \subset G_2$ relates $SU(3)$- to G_2-structures on 6- and 7-manifolds. This motivates to consider $SU(3) \times SU(3) \subset G_2 \times G_2$ and we investigate the connection between generalised $SU(3)$- and G_2-structures in dimension six and seven. We discuss the general setup and explitly calculate the case where the generalised $SU(3)$-structure on the 6-manifold is fibred over a line to obtain on the 7-dimensional total space a generalised G_2-structure. Our result generalises the well known Hitchin flow equations.

It is also interesting to ask about submanifolds on generalised structure spaces. Since a generalised G_2-structure admits two independent spinors they also define two independent G_2 3-forms $\varphi_\pm$. It is known for classical G_2-structures that those are calibration forms. Strictly speaking, they measure e.g. the volume of a 3-submanifold and judge if it is supersymmetric. This property is essential if supersymmetric branes are considered from a geometrical point of view. We develop a dimension independent view on those submanifolds within the setup of generalised G_2-structures. It turns out that the two calibrations restricted to the submanifold can be related to each other via the physical gluing operator R and must coincide. This verifies the naive understanding that the submanifold can be calibrated by both forms $\varphi_\pm$ but they measure the same volume of this space. Or in other words, the data of the submanifolds are encoded in R and we only allow for those R, i.e. for those submanifolds, that ensure that the two calibrations are identical up to B- or F-field transformations.

At the end of this chapter we also take the R-R fields into account. But there is one crucial discrepancy. In the physical supersymmetry variations both, the NS-NS and R-R fields, appear while in mathematics only the the NS-NS fields can be described in terms of a generalised geometry. More concretely, in mathematics the NS-NS fields characterise the topological data of the generalised structure and their integrability conditions are governed by the variational principle that is purely topological. All degrees of freedom that the generalised structure provides are fully occupied by the NS-NS fields, so there are no degrees of freedom left to implement the R-R fields too. We therefore extend the generalised geometries in dimension six and seven by also including the R-R sector and give them a mathematical meaning in terms of an extendend generalised $SU(3)$- and G_2-structure. The result is 4-fold. Firstly, the basic idea is to introduce the R-R fields as a constrained variational principle that generalises the results of [53, 55, 84, 83]. Secondly, the critical points of the functional define the integrability condition of generalised structures including the R-R fields. Thirdly, these new integrability conditions were translated to the classical spinor picture and result in the gravitino and dilatino equation where now the R-R fields appear. Fourthly, we compactify type IIA/B supergravity given in the democratic formulation of Bergshoeff et al. [13] and prove the equivalence of the internal equations to those we found from pure mathematics.

4.1 Supersymmetry variations of type II theories

We want to use this section to set up the basic definitions of the 10d supersymmetry variations for type II theories. We follow the democratic idea given in [13] and therefore consider the R-R fields for the type IIA/B theory as even/odd, $F^{ev/od}$. For the vacuum background the two supersymmetry variations, one for the gravitino Ψ_X and one for the dilatino λ, are given by

$$\delta_\varepsilon \Psi_X = \Big(\nabla_X + \frac{1}{4} X \lrcorner H \cdot \mathcal{P}\Big)\varepsilon + \frac{1}{16}\, e^\phi F^{ev/od} \cdot X \cdot \mathcal{P}_{ev/od}\, \varepsilon\,, \tag{4.2}$$

$$\delta_\varepsilon \lambda = \Big(d\phi \cdot + \frac{1}{2} H \cdot \mathcal{P}\Big)\varepsilon \mp \frac{1}{8}\, e^\phi \Big(5\, F^{ev/od} - \sum_{ev/od} p\, F^p\Big) \cdot \mathcal{P}_{ev/od}\, \varepsilon\,, \tag{4.3}$$

where ϵ (also Ψ_X and λ) is a vector that includes the two supersymmetry parameters for the type II theories, and $X \in TM^{1,9}$. For instance, in type IIB spinorial parameters are non-chiral, $\Gamma^{11}\varepsilon = \varepsilon$. We define the operators $\mathcal{P}$ for the ev/od (or IIA/B) case as $\Gamma^{11}/{-\sigma_3}$

and for $\mathcal{P}_{ev/od} = \sum_{ev/od} \mathcal{P}^p$ we write

p mod 4	0	1	2	3
$\mathcal{P}^p$	σ_1	$i\,\sigma_2$	$\Gamma^{11}\sigma_1$	σ_1

where we used the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Having these variations in hand we can figure out the space-time supersymmetric vacuum background by claiming

$$0 = \delta_\varepsilon \Psi_X \,, \qquad 0 = \delta_\varepsilon \lambda \,. \tag{4.4}$$

We will develop a first solution to these equations by ignoring the R-R fields, i.e. we set all R-R fields to zero. Later on we also take these fields into account. Another step in finding a solution to the supersymmetry variations (4.4) is that we use the tool of compactification. This means, we assume that $M^{1,9}$ is given either by $\mathbb{R}^{1,3} \times M^6$ or by $\mathbb{R}^{1,2} \times M^7$ where the internal spaces M^6 and M^7 are very tiny (say of order Planck length) in case of being compact.

4.2 Compactification on M^7

In this section, we explicitly compactify type IIB on a 7-manifold M^7, that is we consider the direct product model $\mathbb{R}^{1,2} \times M^7$ where H and ϕ take non-trivial values only over M^7 and we set all R-R fields to zero, $F^{od} = 0$. We want to determine the constraints on the underlying geometry of the internal space M^7 imposed by the vanishing of the supersymmetry variations (4.4).

To that end, we are given the supersymmetry parameters $\varepsilon = (\varepsilon_+, \varepsilon_-)$, where $\varepsilon_\pm \in \Delta^+_{M^{1,9}}$ since $\Gamma^{11}\varepsilon = \varepsilon$. We decompose them accordingly, that is

$$\varepsilon_\pm = \sum_N \xi^N_\pm \otimes \eta^N_\pm \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $N \leq \dim \mathbf{8} = 8$ and ξ and η live in the irreducible spin representation $\Delta_{\mathbb{R}^{1,2}}$ and $\mathbf{8}$ of $Spin(1,2)$ and $Spin(7)$ respectively.

We fix the 10-dimensional space-time coordinates X^M (M=0,. . . ,9) and assume the background fields to be independent of X^μ ($\mu = 0, 1, 2$). Coordinates on the internal space will be labeled by X^a for $a = 3, \ldots, 9$. We use the convention

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\, \mathbb{I}_{32\times 32}$$

with signature $(-, +, \ldots, +)$. We choose the explicit gamma matrix representation

$$\Gamma^M = \begin{cases} \gamma_\mu \otimes \mathbb{I}_{8\times 8} \otimes \sigma_2 & : \quad \mu = 0, \ldots, 2 \\ \mathbb{I}_{2\times 2} \otimes \gamma_a \otimes \sigma_1 & : \quad a = 3, \ldots, 9 \end{cases}$$

where the (8×8)-matrices γ_a are imaginary. The $SO(1,2)$ gamma matrices γ_μ are given by

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, we note the relations

$$\prod_\mu \gamma_\mu = -\mathbb{I}_{2\times 2} \qquad \prod_a \gamma_a = -i\,\mathbb{I}_{8\times 8}.$$

The chirality operator Γ^{11} is therefore $\Gamma^{11} = \mathbb{I}_{2\times 2}\otimes \mathbb{I}_{8\times 8}\otimes \sigma_3$.

With these splittings at hand we want to carry out the supersymmetry variations (4.2) and (4.3). The external part of the dilatino variation vanishes trivially. For the internal part, we first note the useful identity

$$\Gamma^{M_1M_2M_3}H_{M_1M_2M_3} = (\mathbb{I}_{2\times 2}\otimes \gamma^{abc}\otimes \sigma_1)\,H_{abc}$$

by means of which we immediately obtain the dilatino variations

$$\delta_{\varepsilon\pm}\lambda = \mathbb{I}_{2\times 2}\xi^N_\pm \otimes (\gamma^a\partial_a\phi \mp \frac{1}{2}\gamma^{abc}\,H_{abc})\eta^N_\pm \otimes \sigma_1 \begin{pmatrix}1\\0\end{pmatrix}.$$

The condition $\delta_{\varepsilon_\pm}\lambda = 0$ is then equivalent to

$$\Big(\gamma^a\partial_a\phi \mp \frac{1}{2}\gamma^{abc}\,H_{abc}\Big)\eta^N_\pm = 0. \tag{4.5}$$

Next we focus on the variation of the gravitinos $\delta_{\varepsilon\pm}\Psi_M$. The flatness of $\mathbb{R}^{1,2}$ implies

$$\nabla_\mu \xi^N_\pm = 0.$$

This solves the external part, and consequently we are left with

$$\delta_{\varepsilon_\pm}\Psi_a = \mathbb{I}_{2\times 2}\,\xi^N_\pm \otimes \Big(\nabla_a \mp \frac{1}{4}H_{abc}\gamma^{bc}\Big)\eta^N_\pm \otimes \begin{pmatrix}1\\0\end{pmatrix}.$$

Imposing the condition $\delta_{\varepsilon\pm}\Psi_X = 0$ finally yields

$$\Big(\nabla_a \mp \frac{1}{4}H_{abc}\gamma^{bc}\Big)\eta^N_\pm = 0. \tag{4.6}$$

In this article we shall deal with the case $N = 1$, i.e. with exactly two internal spinors $\eta_\pm$. Hence a solution consists of the internal background data $(M^7, g, H, \eta_\pm, \phi)$ satisfying (4.5) and (4.6), where g is a metric, H is a closed 3-form, $\eta_\pm$ are two unit spinors in the associated irreducible spin representation **8** and ϕ is a scalar function.

Note that the considerations above can be easily modified to tackle the case of non-chiral type IIA theory which results in similar geometric conditions.

4.3 Generalised G_2-structures

In the previous section we derived from the physics point of view the data and constraints that fully characterise the geometry of the internal space M^7. We achieved this by using the language of spin geometry. The same characterisation of the above geometry appeared also in the mathematical literature, where Frederik Witt introduced the theory of generalised G_2 structures [83] (see also [84]). We merely outline here the definitions and results given there, being essential in the following. For further results, details and examples the reader should consult [83].

Let us start by considering the data of the previous section, that is a spinnable Riemannian 7-manifold (M^7, g) admitting two unit spinors η_+ and η_- in the spin bundle $\Delta = \mathbf{8}$, the irreducible $Spin(7)$-representation. It is well-known in the theory of G-structures, that each spinor induces a reduction of the structure group G to G_2. Since we have two independent spinors $\eta_\pm \in \mathbf{8}$, we denote the associated structure groups by $G_{2\pm}$. Note, by assuming the spinors to be globally orthogonal we effectively have a global $SU(3)$-structure. Strictly speaking, we can only treat the spinors $\eta_\pm$ as independent if we have the possibility of two G_2 copys, say $G_{2\pm}$. A basic instance of such an effective $SU(3)$-structure was considered in [38]. As usual in physics, the authors constructed via fierzing all possible forms of pure degree, i.e. $\eta_-\gamma_{ijk\ldots}\eta_+$. In mathematical language this means that one fierzes the tensor product of the spinors, $\eta_+ \otimes \eta_-$, and project onto $\Lambda^p T^{*7}$. The differential conditions on the spinors then translate into differential conditions on the forms and characterise the specific torsion classes of the underlying $SU(3)$-structure. We will generalise this procedure in the following and show that two new features arise. These ingredients are essential and bring us to the theory of generalised G_2-structures.

Firstly, let us consider the most general setup, where we have two independent and globally defined unit spinors and their associated structure groups $G_{2\pm}$. Altough the spinors induce in general an $SU(3)$-structure, since $SU(3) = G_{2+} \cap G_{2-} = G_2$, it can break down over some subset on the manifold, where the two spinors coincide, i.e. the $SU(3)$-fibre bundle becomes singular. In other words, over the subset we only have one G_2-structure, since here $G_2 = G_{2+} = G_{2-}$, see schematical figure 4.1. This feature does not appear in the case of having two orthogonal spinors, because they define a global $SU(3)$-structure. Note, the subset can be measured by the zero locus of a certain vector field.

Secondly, let us fierz the tensor product $\eta_+ \otimes \eta_-$ and collect all even(odd) forms together in one abstract form of mixed even(odd) degree. We always think about the tensor product $\eta_+ \otimes \eta_-$ as a collection of even or odd forms and denote it by $(\eta_+ \otimes \eta_-)^{ev,od}$. Moreover, since we identify the tensor product $\eta_+ \otimes \eta_-$ with an even or odd form $\rho_0^{ev,od}$, we can regard it as a spinor for $T^7 \oplus T^{7*}$. In the case of generalised 6-manifolds we also considered forms as spinors, see section 3.1.2, and therefore we take over some arguments for 7-manifolds.

Let us consider the $T \oplus T^*$-bundle on M^7, that admits a natural inner product of signature $(7,7)$ and a spinnable $SO(7,7)$-structure. An element $X \oplus \xi \in T \oplus T^*$ acts on a form $\tau \in \Lambda^*$

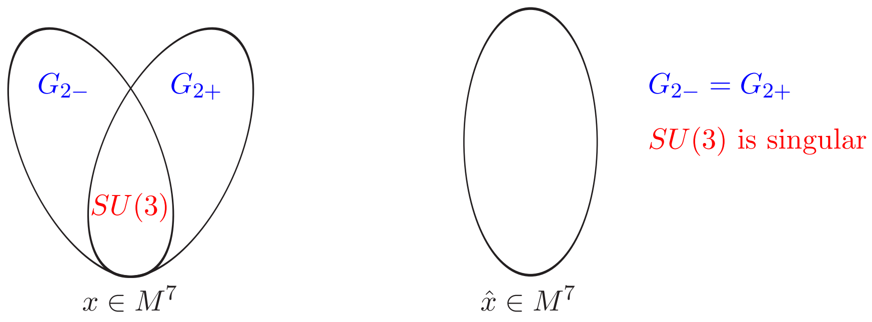


Figure 4.1: Let there be given two independent spinors $\eta_\pm$. Each spinor reduces the structure group to a certain G_2, say G_{2+} and G_{2-}. In general, as can be seen on the left hand side, the groups intersect in $SU(3)$ at $x \in M^7$. It can happen that the two spinors coincide, e.g. at $\hat{x} \in M^7$, as can be seen on the right hand side. This results in $G_{2+} = G_{2-}$ and the $SU(3)$-fibre becomes singular. Such a scenario is typical for generalised structures.

by

$$(X \oplus \xi) \bullet \tau = X \lrcorner \tau + \xi \wedge \tau. \tag{4.7}$$

As this squares to minus the identity,[1] we obtain an isomorphism between $Cliff(T \oplus T^*)$ and $\mathrm{End}(\Lambda^*)$. Since we also allow for a metric g, the structure group reduces to $SO(7) \times SO(7)$, and, moreover, the irreducible spin representations of $Spin(7) \times Spin(7)$ are given by

$$S^\pm = \Lambda^{ev,od} T^*.$$

The given metric g also induces a generalised metric G, compare with the GKS section 3.1.5 on a 6-manifold, and therefore the orthogonal splitting $T \oplus T^* = C_+ \oplus C_-$ is given by,

$$C_+ = \{X \oplus (b-g)X | X \in T\}, \qquad C_- = \{X \oplus (b+g)X | X \in T\}. \tag{4.8}$$

Note, we have different signs compared to section 3.1.5, since our natural metric is defined here by the mathematical convention. With the same spirit as for 6-manifolds, we consider the 2-form b, sitting inside the Lie algebra $\mathfrak{so}(7,7)$, as a B-field. It acts on $S^\pm$ by wedging with the exponential $e^b \bullet \tau = (1 + b + b^2/2 + \ldots) \wedge \tau$.

Having a metric g, we identify the tangent vector X with its dual and act on a form τ by $X \lrcorner \tau + X \wedge \tau$. We can also act with tangent vectors and their duals, via the usual Clifford multiplication, on the spinors $\eta_\pm \in \mathbf{8}$, and, moreover, on the tensor product $\eta_+ \otimes \eta_-$. The tensor product can be considered, via fierzing, as an even or odd form. As a next step, it is obvious that we compare the two actions on $\mathbf{8} \otimes \mathbf{8}$ and $S^\pm$. For the comparison we take $X, Y \in T$ and use the fact, that one can lift them isomorphically via (4.8) into the subbundles $C_\pm$. We write[2]

$$Cliff(T,g) \widehat{\otimes} Cliff(T,-g) = Cliff(T \oplus T^*) \tag{4.9}$$

[1] We follow the usual convention in mathematics where unit elements in the Clifford algebra square to -1, cf. the definition (3.2) for 6-manifolds.

[2] The symbol $\widehat{\otimes}$ means the tensor product of Z_2-graded Clifford algebras.

where the isomorphism is given by extension of the map

$$X\widehat{\otimes}Y \mapsto (X \oplus -X \lrcorner g) \bullet (Y \oplus Y \lrcorner g), \tag{4.10}$$

and $\bullet$ now also denotes multiplication in *Cliff*$(T \oplus T^*)$. Let $\cdot$ denote Clifford multiplication on $\mathbf{8}$. One can show that

$$\begin{array}{rcl} (X \cdot \eta_+ \otimes \eta_-)^{ev,od} & = & X\widehat{\otimes}1 \bullet (\eta_+ \otimes \eta_-)^{od,ev} \\ (\eta_+ \otimes Y \cdot \eta_-)^{ev,od} & = & \pm 1\widehat{\otimes}Y \bullet (\eta_+ \otimes \eta_-)^{od,ev} \end{array} \tag{4.11}$$

for any η_+, $\eta_- \in \mathbf{8}$. Hence the $G_{2+} \times G_{2-}$-invariant tensor product $\eta_+ \otimes \eta_-$ induces elements $\rho_0^{ev,od} \in S^{\pm}$ whose stabiliser inside $Spin(7,7)$ is conjugate to $G_2 \times G_2$.

Conversely, a $G_2 \times G_2$-invariant spinor ρ in S^+ (or S^-) can be uniquely written (up to a sign) as

$$\rho^{ev,od} = e^{-\phi} e^b \wedge (\eta_+ \otimes \eta_-)^{ev,od} \in S^{\pm}. \tag{4.12}$$

We call the pair (M^7, ρ) a generalised G_2-structure, i.e. the structure group of $T \oplus T^*$ inside $\mathbb{R}_0 \times Spin(7,7)$ is precisely $G_2 \times G_2$. This definition uses the form-picture and is equivalent to the data $(M^7, g, b, \eta_\pm, \phi)$. Since our background manifold of interest is characterised by the latter data, the vaccum space can be modeled by a generalised G_2-structure. We will postpone the discussion about the differential conditions to the end of this section. Usually, we assume the $G_2 \times G_2$-invariant spinor ρ to be even and write $\rho_b = e^b \wedge (\eta_+ \otimes \eta_-)^{ev}$ and ρ_0 if $b = 0$.

Let us make a comment that does not concern our investigations later on. Hitchin [53] showed that certain geometries appear as critical points of a variational principle. This principle only involves the topological data. Therefore, the variational principle for non-generalised structures gained some attraction with a view towards a topological M-theory [31, 73]. In [74] the authors showed, that it is even necessary to use the quantised generalised framework for 6-manifolds to capture the genus one free energy of the B-model. Let us focus on 7-manifolds. Note that the

$$g : 28, \quad b : 21, \quad \eta_+ : 7, \quad \eta_- : 7, \quad \phi : 1$$

degrees of freedom sum to $64 = \dim \Lambda^{ev,od}$, so that this data effectively parametrises the open orbit of a $G_2 \times G_2$-invariant form under the action of $\mathbb{R}_{>0} \times Spin(7,7)$. Following the language in [53] such a spinor is called stable. Stability is the key feature for the variation of the spinors and therefore allows us to consider a certain variational principle introduced by Hitchin [53] and formulated for generalised G_2-structures by Witt [83].

After these more abstract definitions let us focus next on the practical techniques that we extensively use afterwards. Firstly, let us derive an explicit description of $\rho^{ev,od}$ in terms of the underlying $G_{2+} \times G_{2-}$-invariants. The coefficients of the form $\eta_+ \otimes \eta_-$ can be computed by

$$g(\eta_+ \otimes \eta_-, e_I) = q(e_I \cdot \eta_+, \eta_-), \tag{4.13}$$

where q denotes a suitably scaled $Spin(7)$-invariant inner product on $\mathbf{8}$ and $e_I = e_{i_1 \dots i_p}$ is an orthonormal basis for $\Lambda^p T^*$. In physics language this means, using $e_I \cdot = e_{i_1 \dots i_p} \cdot = \gamma_{i_1 \dots i_p}$, that we define the components of the form $\eta_+ \otimes \eta_-$ by calculating bi-linears, i.e. $(\eta_+ \otimes \eta_-)_{i_1 \dots i_p} = \eta_-^T \gamma_{i_1 \dots i_p} \eta_+$. More explicitly, we can also decompose $\eta_- = \cos(a)\eta_+ + \sin(a)\eta_+^{\perp}$, where η_+

and $\eta_+^\perp$ are perpendicular to each other and $a = \sphericalangle(\eta_+, \eta_-)$ describes the angle between the spinors. Since $Spin(7)$ acts transitively on the set of pairs of orthonormal spinors, we may choose an orthonormal basis in $\mathbf{8}$ such that

$$\eta_+ = (1,0,0,0,0,0,0,0)^T\,, \quad \text{and} \quad \eta_- = (\cos(a),\sin(a),0,0,0,0,0,0)^T\,.$$

If the spinors η_+ and η_- are linearly independent, their isotropy groups G_{2+} and G_{2-} intersect in $SU(3)$, which act on T^7 by leaving invariant a 1-form α, a symplectic form ω and two 3-forms ψ_+ and ψ_- which are the real and the imaginary part of the $SU(3)$-invariant holomorphic $(3,0)$-form given in section 2.3.3 (see also [24]). We then find

$$\rho_0^{ev} \;=\; c + s\omega - c(\psi_- \wedge \alpha + \frac{\omega^2}{2}) + s\psi_+ \wedge \alpha - s\frac{\omega^3}{6}\,, \tag{4.14}$$

$$\rho_0^{od} \;=\; s\alpha - c(\psi_+ + \omega\wedge\alpha) - s\psi_- - s\frac{\omega^2}{2}\wedge\alpha + c\,vol_g\,, \tag{4.15}$$

where c and s are shorthand for $\cos(a)$ and $\sin(a)$. Let us write this in the shape

$$\rho_0^{ev} = f_0 + \alpha\wedge f_1\,, \qquad \rho_0^{od} = g_1 + \alpha\wedge g_0\,,$$

where we set

$$\begin{array}{rclcl} f_0 & = & c + s\omega - c\frac{\omega^2}{2} - s\frac{\omega^3}{6} & = & \mathrm{Re}(e^{-ia}e^{i\omega})\,, \\ f_1 & = & c\psi_- - s\psi_+ & = & \mathrm{Im}(e^{-ia}\Omega)\,, \end{array} \tag{4.16}$$

and

$$\begin{array}{rclcl} g_1 & = & -c\psi_+ - s\psi_- & = & -\mathrm{Re}(e^{-ia}\Omega)\,, \\ g_0 & = & s - c\omega - s\frac{\omega^2}{2} + c\frac{\omega^3}{6} & = & -\mathrm{Im}(e^{-ia}e^{i\omega})\,. \end{array} \tag{4.17}$$

by means of

$$\begin{array}{rcl} e^{-ia}\Omega & = & (c - i\,s)\cdot(\psi_+ + i\,\psi_-)\,, \\ & = & (c\psi_+ + s\psi_-) + i(c\psi_- - s\psi_+)\,, \\ e^{-ia}e^{i\omega} & = & (c - i\,s)\cdot(1 + i\omega - \frac{\omega^2}{2} - i\,\frac{\omega^3}{6})\,, \\ & = & (c + s\omega - c\frac{\omega^2}{2} - s\frac{\omega^3}{6}) + i(-s + c\omega + s\frac{\omega^2}{2} - c\frac{\omega^3}{6})\,. \end{array} \tag{4.18}$$

Accordingly, the normal forms (4.14) and (4.15) become

$$\rho_0^{ev} \;=\; \mathrm{Re}(e^{-ia}e^{i\omega}) \,+\, \alpha\wedge\mathrm{Im}(e^{-ia}\Omega)\,, \tag{4.19}$$

$$\rho_0^{od} \;=\; -\mathrm{Re}(e^{-ia}\Omega) \,-\, \alpha\wedge\mathrm{Im}(e^{-ia}e^{i\omega})\,, \tag{4.20}$$

and taking also the B-field b and the dilaton ϕ into account the most general $G_2\times G_2$-invariant spinors $\rho^{ev,od}$ are

$$\rho^{ev} \;=\; \mathrm{Re}(e^{-\phi-ia}e^{b+i\omega}) \,+\, \alpha\wedge\mathrm{Im}(e^{-\phi-ia}e^{b}\wedge\Omega)\,, \tag{4.21}$$

$$\rho^{od} \;=\; -\mathrm{Re}(e^{-\phi-ia}e^{b}\wedge\Omega) \,-\, \alpha\wedge\mathrm{Im}(e^{-\phi-ia}e^{b+i\omega})\,, \tag{4.22}$$

The underlying $SU(3)$-structure fluctuates with a and breaks down when $s = 0$, i.e. where the spinors are parallel. Since $s\alpha\eta_+ = s\eta_+^\perp$, this happens precisely over the zero locus of the vector field dual to $s\alpha$. Consequently, only the forms $s\alpha$, $s\omega$ etc. are globally defined over

M^7 and it follows that in general the $Spin(7)$-structure does not reduce to a global $SU(3)$-structure. Even if the $SU(3)$-structure is not singular it varies in general over M^7 with a parameter a and is therefore dynamical. This should be compared to the case of orthogonal spinors of unit length, where the $SU(3)$-structure is even static. Moreover, at a point where $a = 0$, i.e. $\eta = \eta_+ = \eta_-$, we have

$$\rho^{ev} = 1 - \star\varphi \tag{4.23}$$
$$\rho^{od} = -\varphi + vol\,, \tag{4.24}$$

with φ denoting the invariant 3-form of a classical G_2-structure defined by η. In other words, at certain points where the $SU(3)$-structure breaks down, we get a classical G_2-structure (M^7, φ). Going one step ahead, in case of having even $a = 0$ globally, we get a global classical G_2-structure and the formulae (4.23) and (4.24) tells us how to embed a classical G_2-structure into a generalised one .

We already used implicitly in (4.23) our knowledge about the Hodge-operator for classical G_2-structures. This explicit description also reveals how to relate the $G_2 \times G_2$-invariant forms ρ_0^{ev} and ρ_0^{od}. Let us introduce the canonical involution $\wedge$ of a Clifford algebra, given on elements of degree p by

$p \bmod 4$	0	1	2	3
$\wedge$	$+$	$-$	$-$	$+$

A general formula in 7d can be denoted by $\star(\tau^{ev,od})^\wedge = (\star\tau^{ev,od})^\wedge$. Using this we obtain

$$\star(\widehat{\eta_+ \otimes \eta_-})^{ev,od} = (\eta_+ \otimes \eta_-)^{od,ev}.$$

Next we want to explicitly verify this result by also incorporating the B-field b to consider a general $G_2 \times G_2$-form ρ (4.12) where we note the identity $(e^b \wedge \tau)^\wedge = e^{-b} \wedge \widehat{\tau}$. Let us introduce the generalised Hodge- or box operator $\Box_{g,b} : \Lambda^{ev,od} \to \Lambda^{od,ev}$ defined by

$$\Box_{g,b}\rho^{ev,od} = e^b \wedge \star(e^{-b} \wedge \rho^{ev,od})^\wedge\,, \tag{4.25}$$

such that

$$\Box_{\rho^{ev,od}}\rho^{ev,od} = \rho^{od,ev}\,. \tag{4.26}$$

Let us check this formula by calculating $\Box_{g,b}\rho^{ev/od}$,

$$\begin{aligned}
\Box_{g,b}\rho^{ev/od} &= \Box_{g,b}(e^{-\phi}\, e^b\, \wedge \rho_0^{ev/od})\,,\\
&= e^b \wedge \star(e^{-b} \wedge (e^{-\phi}\, e^b\, \rho_0^{ev/od}))^\wedge\,,\\
&= e^b \wedge \star(e^{-b} \wedge\, e^b\, e^{-\phi}\rho_0^{ev/od})^\wedge\,,\\
&= e^{-\phi}\, e^b \wedge \star(\rho_0^{ev/od})^\wedge\,,
\end{aligned} \tag{4.27}$$

where we used $e^{-b} \wedge e^b = 1$. Thus, the $\Box_{g,b}$-operator is only sensible to the normal form and it remains to calculate $\star(\rho_0^{ev/od})^\wedge$. By means of

$$\begin{array}{rclrclrclrcl}
\star 1 &=& vol_g, & \star\alpha &=& \frac{\omega^3}{6}, & \star\omega &=& \alpha \wedge \frac{\omega^2}{2}, & \star\frac{\omega^2}{2} &=& \alpha \wedge \omega,\\
\star(\alpha \wedge \psi_+) &=& \psi_-, & \star(\alpha \wedge \psi_-) &=& -\psi_+, & \star\psi_+ &=& -\alpha \wedge \psi_-, & \star\psi_- &=& \alpha \wedge \psi_+\,,
\end{array}$$

we obtain e.g. for $\star(\rho_0^{ev})^\wedge$,

$$
\begin{aligned}
\star(\rho_0^{ev})^\wedge &= \star\Big(c + s\omega - c(\psi_- \wedge \alpha + \tfrac{\omega^2}{2}) + s\psi_+ \wedge \alpha - s\tfrac{\omega^3}{6}\Big)^\wedge , \\
&= \star\Big(c - s\omega - c(\psi_- \wedge \alpha + \tfrac{\omega^2}{2}) + s\psi_+ \wedge \alpha + s\tfrac{\omega^3}{6}\Big) , \\
&= c\, vol_g - s\alpha \wedge \tfrac{\omega^2}{2} - c(\psi_+ + \alpha \wedge \omega) - s\psi_- + s\alpha , \\
&= \rho_0^{od} .
\end{aligned}
\tag{4.28}
$$

At the end of this section we come back to integrability of a generalised G_2-structure. Remember, we already made sure that the physical data $(M^7, g, b, \eta_\pm, \phi)$, steming from the Killing spinor equations which are equal to the supersymmetry variations, define a most generic generalised G_2-structure. We furher explained how this data can be rephrased in terms of forms $\rho^{ev,od}$, being spinors for the $T \oplus T^*$-bundle. Next we have to fully consider the supersymmetry variations, or put differently, we must analyse the differential conditions for the spinors $\eta_\pm$ and how these can be related to differential conditions for the spinors $\rho^{ev,od}$.

Let us mention before we proceed that we have to make sure what the B-field physically means. In the supersymmetry variations only the corresponding 3-form H appears and we can think about it in two different ways. Firstly, the case if H is globally exact, i.e. globally we have a 2 form b such that $H = db$. Secondly, H can only be defined locally by $H = db$, implying that b is rather a connection 2-form. Both situations can be captured (even at the same time) by the generalised framework. In case of having the B-field only defined locally this setup goes in the mathematical literature under the name of a gerbe [54, 57]. We prefer in the following this interpretation and it can be implemented for the generalised G_2 structures by taking the twisted differential operator $d_H = d + H$ (compare with (3.14)) instead of the usual operator d. In contrast to that, i.e. if we have from a physical point of view a globally defined B-field, we identify it with the 2-form b sitting in the Lie algebra as already mentioned.

The detailed precise proof of the correspondence is given in [83] and is roughly based on the fact that the twisted Dirac operator over $\mathbf{8} \otimes \mathbf{8}$ transforms into $d + d^\star$ under fierzing. A more general argument taking into account the possible action of the B-field b, where H is the closed NS-NS 3-form flux, can then be invoked to show that $(M^7, g, H, \eta_\pm, \phi)$ satisfies (4.5) and (4.6) if and only if the corresponding $G_2 \times G_2$-invariant spinors ρ and $\Box\rho$ satisfies

$$
\begin{aligned}
0 &= d_H e^{-\phi}\rho_0 &&= de^{-\phi}\rho_0 + H \wedge e^{-\phi}\rho_0 , \\
0 &= d_H \Box_{\rho_0} e^{-\phi}\rho_0 &&= d\Box_{\rho_0} e^{-\phi}\rho_0 + H \wedge \Box_{\rho_0} e^{-\phi}\rho_0 ,
\end{aligned}
\tag{4.29}
$$

If H is globally exact, i.e. $H = db$, (4.29) can be written in the more succinct form

$$
\begin{aligned}
0 &= de^{-\phi}(e^b \wedge \rho_0) , \\
0 &= de^{-\phi}\Box_{g,b}(e^b \wedge \rho_0) .
\end{aligned}
\tag{4.30}
$$

4.4 Recovering the classical $SU(3)$-case

Equations (4.5) and (4.6) were first derived by Gauntlett et al. [38] from a quite different point of view. Starting with IIB supergravity they studied wrapped NS5-branes over calibrated submanifolds inside an internal 7-manifold with an $SU(3)$-structure. As an illustration of the previous section, we reconsider their setup which turns out to be described by a "static" generalised G_2-structure with $a \equiv \pi/2$ (that is, the structure group reduces to a fixed $SU(3)$), together with the closed NS-NS 3-form flux H.

Under this assumption the form ρ defining the generalised G_2-structure becomes

$$\rho_0 = \omega + \psi_+ \wedge \alpha - \frac{\omega^3}{6}$$

with associated odd form

$$\Box_{\rho_0}\rho_0 = \alpha - \psi_- - \frac{\omega^2}{2} \wedge \alpha.$$

The supersymmetry equations are equivalent to

$$d_H e^{-\phi}\rho_0 = 0 \quad \text{and} \quad d_H e^{-\phi}\Box_{\rho_0}\rho_0 = 0$$

which written in homogeneous components can then be rephrased by

$$\begin{aligned} d\omega &= d\phi \wedge \omega\,, \\ \psi_+ \wedge d\alpha &= -d\phi \wedge \psi_+ \wedge \alpha + d\psi_+ \wedge \alpha - H \wedge \omega\,, \\ \frac{1}{2} d\omega \wedge \omega^2 &= d\phi \wedge \frac{\omega^3}{6} - H \wedge \psi_+ \wedge \alpha \end{aligned}$$

and

$$\begin{aligned} d\alpha &= d\phi \wedge \alpha\,, \\ d\psi_- &= d\phi \wedge \psi_- + H \wedge \alpha\,, \\ d\omega \wedge \omega \wedge \alpha &= d\phi \wedge \alpha \wedge \frac{\omega^2}{2} - H \wedge \psi_-\,. \end{aligned}$$

We finally conclude

$$\begin{array}{rclcrcl} d(e^{-\phi}\alpha) &=& 0\,, & & d(e^{-\phi}\omega) &=& 0\,, \\ \alpha \wedge d\psi_+ &=& H \wedge \omega\,, & & \frac{1}{3} d\omega \wedge \omega &=& \alpha \wedge H \wedge \psi_+\,, \\ d(e^{-\phi}\psi_-) &=& H \wedge \alpha\,, & & d\omega \wedge \omega \wedge \alpha &=& \psi_- \wedge H\,. \end{array} \tag{4.31}$$

The equations of motion are solved since H is closed, i.e. $dH = 0$, as proved in [38]. Therefore (4.31) characterises the physical vacua.

4.5 Compactification on M^6 and generalised $SU(3)$-structures

Let us use next IIB theory to make contact for phenomenogical reasons to our four dimensional world. Following the procedure of section 4.2 we achieve this if we compactify on a 6-dimensional manifold M^6. Also here we set the R-R fields to zero. Recall that we have $Spin(6) = SU(4)$ and that the irreducible spin representations of positive and negative chirality $\Delta_\pm$ are just the $SU(4)$-vector representation $\mathbf{4}$ and its conjugate $\overline{\mathbf{4}}$. The supersymmetry equations compactified on M^6 thus become

$$\begin{array}{rclcrcl} \nabla_X \Xi_+ + \frac{1}{4} X \lrcorner H \cdot \Xi_+ &=& 0, & & (d\phi + \frac{1}{2}H) \cdot \Xi_+ &=& 0, \\ \nabla_X \Theta_+ - \frac{1}{4} X \lrcorner H \cdot \Theta_+ &=& 0, & & (d\phi - \frac{1}{2}H) \cdot \Theta_+ &=& 0, \end{array} \tag{4.32}$$

for two complex spinors $\Theta_+, \Xi_+ \in \mathbf{4}$ that are related in type II theories because Θ_+ and Ξ_+ are real Majorana-Weyl spinors. Note, since we work in type IIB theory both complex spinors Θ_+ and Ξ_+ are of positive chirality. Similarly, we can consider type IIA theory by choosing the spinors to be non-chiral.

Recall that $SU(4)/SU(3) = S^7$, hence the choice of two unit spinors $\Theta, \Xi \in \mathbf{4}$ induces a reduction to two $SU(3)_\pm$-subbundles. The $SU(4)$-representations $\Delta_\pm$ decompose into $\mathbf{3}_\pm \oplus \mathbf{1}_\pm$ and $\overline{\mathbf{3}}_\pm \oplus \overline{\mathbf{1}}_\pm$. Consequently, we can also consider the corresponding $SU(3)_\pm$-invariant spinors, e.g. $\Theta_- = \overline{\Theta}_+ \in \overline{\mathbf{4}}$. This setup is similar to that given in section 4.3 which motivates

Definition 4.5.1. [61]
Let M^6 be a 6-dimensional, spinnable manifold. A (topological) generalised $SU(3)$-structure is defined by the 6-tuple $(M^6, g, b, \phi, \Xi_+, \Theta_+)$, where g is a Riemannian metric on M^6, b a 2-form, ϕ a smooth function and Ξ_+, $\Theta_+ \in \Delta_+$ are two (complex) unit spinors of positive chirality. We denote the stabiliser of $\Xi_\pm$ and $\Theta_\pm$ by $SU(3)_l$ and $SU(3)_r$ respectively.

The proofs are analogous to that given in [83] and carry over without difficulty. Again we content ourselves with a brief outline of the corresponding results.

Rather than working with the complex spinors we will consider the real $SU(4)$-module S obtained by forgetting the complex structure on $\mathbf{4}$ or $\overline{\mathbf{4}}$, that is the complexification of S is just $S \otimes \mathbb{C} = \mathbf{4} \oplus \overline{\mathbf{4}}$. We prefer here the real objects and develop the results with the view towards the next section. In a later section we also include the R-R fields and use complex spinors only.

Let us define

$$\begin{aligned} \Theta_+ &= \varphi_\Theta + i\widehat{\varphi}_\Theta \in \mathbf{4}, \qquad & \Theta_- &= \varphi_\Theta - i\widehat{\varphi}_\Theta \in \overline{\mathbf{4}}, \\ \Xi_+ &= \varphi_\Xi + i\widehat{\varphi}_\Xi \in \mathbf{4}, \qquad & \Xi_- &= \varphi_\Xi - i\widehat{\varphi}_\Xi \in \overline{\mathbf{4}}, \end{aligned}$$

As the Riemannian volume element vol_g induces a complex structure on S and acts on $\mathbf{4}$ and $\overline{\mathbf{4}}$ by multiplication with i and $-i$ respectively, we can specify $\widehat{\varphi}_\Theta = \mathrm{Im}(\Theta)$ and $\widehat{\varphi}_\Xi = \mathrm{Im}(\Xi)$ by, e.g.

$$vol_g \cdot \varphi_\Theta = vol_g \cdot \frac{1}{2}(\Theta_+ + \Theta_-) = \frac{1}{2}(i\,\Theta_+ - i\,\Theta_-) = -\widehat{\varphi}_\Theta.$$

Since S carries an $SU(4)$-invariant Riemannian inner product, we can identify $S \otimes S$ with $\Lambda^* T^{6*}$ through fierzing so that (4.11) holds. This yields two forms $\varphi_\Theta \otimes \varphi_\Xi$ and $\varphi_\Theta \otimes \widehat{\varphi}_\Xi$ which we can interpret as $SU(3) \times SU(3)$-invariant spinors and which we want to decompose into an even and an odd part. Note that under complexification of this isomorphism, the components $\mathbf{4} \otimes \mathbf{4}$ and $\overline{\mathbf{4}} \otimes \overline{\mathbf{4}}$ get mapped onto odd complex forms, while the off-diagonal components $\overline{\mathbf{4}} \otimes \mathbf{4}$ and $\mathbf{4} \otimes \overline{\mathbf{4}}$ become even since $\mathbf{4}$ and $\overline{\mathbf{4}}$ are dual to each other. Let us calculate the odd forms

$$\begin{aligned} \Xi_+ \otimes \Theta_+ &= (\varphi_\Xi \otimes \varphi_\Theta - \widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta) + i\,(\varphi_\Xi \otimes \widehat{\varphi}_\Theta + \widehat{\varphi}_\Xi \otimes \varphi_\Theta)\,, \\ \Xi_- \otimes \Theta_- &= (\varphi_\Xi \otimes \varphi_\Theta - \widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta) - i\,(\varphi_\Xi \otimes \widehat{\Theta}_\Xi + \widehat{\varphi}_\Xi \otimes \varphi_\Theta)\,, \end{aligned}$$

where we see the properties

$$\begin{aligned} \overline{\Xi_+ \otimes \Theta_+} &= \Xi_- \otimes \Theta_-\,, \\ \mathrm{Re}(\Xi_+ \otimes \Theta_+) &= \mathrm{Re}(\Xi_- \otimes \Theta_-)\,, \\ \mathrm{Im}(\Xi_+ \otimes \Theta_+) &= -\,\mathrm{Im}(\Xi_- \otimes \Theta_-)\,. \end{aligned}$$

Evaluating the even forms

$$\begin{aligned}\Xi_+ \otimes \Theta_- &= (\varphi_\Xi \otimes \varphi_\Theta + \widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta) + i\,(-\varphi_\Xi \otimes \widehat{\varphi}_\Theta + \widehat{\varphi}_\Xi \otimes \varphi_\Theta)\,,\\ \Xi_- \otimes \Theta_+ &= (\varphi_\Xi \otimes \varphi_\Theta + \widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta) - i\,(-\varphi_\Xi \otimes \widehat{\varphi}_\Theta + \widehat{\varphi}_\Xi \otimes \varphi_\Theta)\,,\end{aligned}$$

we obtain the properties

$$\begin{aligned}\overline{\Xi_+ \otimes \Theta_-} &= \Xi_- \otimes \Xi_+\,,\\ \mathrm{Re}(\Xi_+ \otimes \Theta_-) &= \mathrm{Re}(\Xi_- \otimes \Theta_+)\,,\\ \mathrm{Im}(\Xi_+ \otimes \Theta_-) &= -\,\mathrm{Im}(\Xi_- \otimes \Theta_+)\,.\end{aligned}$$

Furthermore, applying these properties to e.g. $\varphi_\Xi \otimes \varphi_\Theta = (\Xi_+ + \Xi_-) \otimes (\Theta_+ + \Theta_-)/4$ we get

$$\begin{aligned}(\varphi_\Xi \otimes \varphi_\Theta)^{ev} &= ((\Xi_+ + \Xi_-) \otimes (\Theta_+ + \Theta_-))^{ev}/4\,,\\ &= (\Xi_+ \otimes \Theta_- + \Xi_- \otimes \Theta_+)/4\,,\\ &= (\Xi_+ \otimes \Theta_- + \overline{\Xi_+ \otimes \Theta_-})/4\,,\\ &= \tfrac{1}{2}\,\mathrm{Re}(\Xi_+ \otimes \Theta_-)\,,\end{aligned}$$

and obtain moreover

$$\begin{aligned}\tau_0 = (\varphi_\Xi \otimes \varphi_\Theta)^{ev} &= (\widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta)^{ev} = \tfrac{1}{2}\,\mathrm{Re}(\Xi_+ \otimes \Theta_-)\,,\\ \widehat{\tau}_0 = (\widehat{\varphi}_\Xi \otimes \varphi_\Theta)^{ev} &= -(\varphi_\Xi \otimes \widehat{\varphi}_\Theta)^{ev} = \tfrac{1}{2}\,\mathrm{Im}(\Xi_+ \otimes \Theta_-)\,,\\ \upsilon_0 = (\widehat{\varphi}_\Xi \otimes \widehat{\varphi}_\Theta)^{od} &= -(\varphi_\Xi \otimes \varphi_\Theta)^{od} = -\tfrac{1}{2}\,\mathrm{Re}(\Xi_+ \otimes \Theta_+)\,,\\ \widehat{\upsilon}_0 = (\varphi_\Xi \otimes \widehat{\varphi}_\Theta)^{od} &= (\widehat{\varphi}_\Xi \otimes \varphi_\Theta)^{od} = \tfrac{1}{2}\,\mathrm{Im}(\Xi_+ \otimes \Theta_+)\,,\end{aligned}$$

where we defined the four real fundamental $SU(3) \times SU(3)$-forms τ_0, $\widehat{\tau}_0$, υ and $\widehat{\upsilon}$.

To see how these forms relate to each other, we note that in dimension 6 the $\Box_{g,b}$-operator respects the parity of the forms and satisfies $\Box_{g,b}^2 = -Id$, that is $\Box_{g,b}$ induces a complex structure on $\Lambda^* T^*$. We then have

$$\Box_{g,b}\tau_b = \widehat{\tau}_b, \quad \Box_{g,b}\upsilon_b = \widehat{\upsilon}_b\,,$$

where $\tau_b = e^b \wedge \tau_0$ etc.. We can write

$$\frac{1}{2}\Xi_+ \otimes \Theta_- = \tau_0 + i\,\Box_{g,0}\tau_0\,, \tag{4.33}$$

and

$$\frac{1}{2}\Xi_+ \otimes \Theta_+ = -\upsilon_0 + i\,\Box_{g,0}\upsilon_0\,. \tag{4.34}$$

As in section 4.3 we can compute a normal form description which we can express in terms of the underlying $SU(2) = SU(3)_+ \cap SU(3)_-$-invariants if the unit spinors Ξ_+ and Θ_+ are linearly independent. Using again the complexified isomorphism $S^{\mathbb{C}} \otimes S^{\mathbb{C}} \cong \Lambda^* T^{6*\mathbb{C}}$ and decomposing $\Theta_+ = c_1\Xi_+ + c_2\Xi_+^\perp$ with two complex scalars c_1, $c_2 \in \mathbb{C}$ we find

$$\Xi_+ \otimes \Theta_+ = i\bar{Z} \wedge (c_1\Omega + c_2 e^{i\omega_1}) \tag{4.35}$$

and

$$\Xi_+ \otimes \Theta_- = e^{i\alpha\wedge\beta} \wedge (\bar{c}_1 e^{i\omega_1} + \bar{c}_2\Omega) \tag{4.36}$$

where expressed in a suitable local orthonormal basis $e_1, \dots, e_6$ we have the two real 1-forms $\alpha = e_5$, $\beta = e_6$, the complex 1-form $Z = e_5 + ie_6$, the self-dual 2-forms $\omega_1 = e_{12} + e_{34}$, $\omega_2 = e_{13} - e_{24}$, $\omega_3 = e_{14} + e_{23}$ and the complex symplectic form $\Omega = \omega_2 - i\omega_3$. The normal forms of $\Xi_- \otimes \Theta_+$ and $\Xi_- \otimes \Theta_-$ are obtained by complex conjugation in $\Lambda^* T^{6*\mathbb{C}}$.

Let us relate the above normal forms to the special case of a classical $SU(3)$-structure on a 6-manifold which is equivalent to identify the spinors, $\eta = \Theta_+ = \Xi_+$, i.e. $c_1 = 1$ and $c_2 = 0$. The normal forms (4.35) and (4.36) condense to

$$\eta \otimes \eta = i\bar{Z} \wedge \Omega = \psi_- + i\,\psi_+ = i\overline{\Omega^{3,0}}\,, \tag{4.37}$$

and

$$\eta \otimes \overline{\eta} = e^{i(\alpha\wedge\beta+\omega_1)} = e^{i\omega} \tag{4.38}$$

in the language of [24] and accordingly

$$\begin{array}{rclcrcl} \upsilon_0 & = & -\frac{1}{2}\psi_-\,, & & \widehat{\upsilon}_0 & = & \frac{1}{2}\psi_+\,, \\ \tau_0 & = & \frac{1}{2}(1-\frac{\omega^2}{2})\,, & & \widehat{\tau}_0 & = & \frac{1}{2}(\omega - \frac{\omega^3}{6})\,. \end{array} \tag{4.39}$$

Finally we wish to state the supersymmetry equations (4.32) in terms of the $SU(3) \times SU(3)$-invariant forms τ_0, $\widehat{\tau}_0$, υ and $\widehat{\upsilon}$. The real version of (4.32) is given by

$$\nabla^{LC}_X \varphi_{\Xi/\Theta} \pm \frac{1}{4} X \lrcorner H \cdot \varphi_{\Xi/\Theta} = 0, \quad (d\phi \pm \frac{1}{2}H)\cdot \varphi_{\Xi/\Theta} = 0$$

and

$$\nabla^{LC}_X \widehat{\varphi}_{\Xi/\Theta} \pm \frac{1}{4} X \lrcorner H \cdot \widehat{\varphi}_{\Xi/\Theta} = 0, \quad (d\phi \pm \frac{1}{2}H)\cdot \widehat{\varphi}_{\Xi/\Theta} = 0.$$

The same computation as in the generalised G_2-case shows that this is equivalent to

$$\begin{array}{rclcl} d_H e^{-\phi}\tau_0 & = & d_H e^{-\phi}\widehat{\tau}_0 & = & 0\,, \\ d_H e^{-\phi}\upsilon_0 & = & d_H e^{-\phi}\widehat{\upsilon}_0 & = & 0, \end{array} \tag{4.40}$$

that is

$$d_H e^{-\phi}\Xi_+ \otimes \Theta_+ = 0, \quad d_H e^{-\phi}\Xi_- \otimes \Theta_+ = 0.$$

If H is globally exact, that is $H = db$, we can write these equations more succinctly as

$$\begin{array}{rclcl} de^{-\phi}\tau_b & = & de^{-\phi}\Box_{g,b}\tau_b & = & 0\,, \\ de^{-\phi}\upsilon_b & = & de^{-\phi}\Box_{g,b}\upsilon_b & = & 0. \end{array} \tag{4.41}$$

4.6 Dimension 6 vs. 7

The inclusion $SU(3) \subset G_2$ allows one to pass from an $SU(3)$-structure in dim $= 6$ to a G_2-structure in dim $= 7$. In the same vein, the inclusion $SU(3) \times SU(3) \subset G_2 \times G_2$ relates generalised $SU(3)$- to generalised G_2-structures. In this section we want to render this link explicit in both the spinorial and the form picture of a generalised structure. We first discuss the algebraic setup before we turn to integrability issues.

To start with, assume that we are given a generalised G_2-structure $(T, g, b, \eta_\pm, \phi)$ over the 7-dimensional vector space $T^7 = T$ together with a preferred unit vector α. We want to induce a generalised $SU(3)$-structure on $T^6 = \widetilde{T}$ defined by $T = \widetilde{T} \oplus \mathbb{R}\alpha$. Since $\alpha \cdot \alpha = -1$, the choice of such a vector induces a complex structure on the irreducible $Spin(7)$-module $\mathbf{8}$ which is compatible with the spin-invariant Riemannian inner product. Hence the complexification of $\mathbf{8}$ is

$$\mathbf{8} \otimes \mathbb{C} = \Delta^{1,0} \oplus \Delta^{0,1},$$

where

$$\Delta^{1,0/0,1} = \{\eta \mp i\alpha \cdot \eta \mid \eta \in \Delta\}.$$

The choice of α also induces a reduction from $SO(7)$ to $SO(6)$ which is covered by $Spin(6) = SU(4)$, and as an $SU(4)$-module we have $\Delta^{1,0} = \mathbf{4}$ and $\Delta^{0,1} = \overline{\mathbf{4}}$. We define

$$\psi_\pm = \eta_\pm - i\alpha \cdot \eta_\pm$$

and let $\widetilde{g} = g_{|\widetilde{T}}$ and $\widetilde{b} = \alpha \lrcorner (\alpha \wedge b)$. Then a generalised $SU(3)$-structure over $\widetilde{T}$ is given by $(\widetilde{T}, \widetilde{g}, \widetilde{b}, \psi_\pm, \phi)$. Moreover, we get a (possibly zero) 1-form $\beta = \alpha \lrcorner b \in \Lambda^1\widetilde{T}^*$. It is clear that we can reverse this construction by defining a metric $g = \widetilde{g} + \alpha \otimes \alpha$, $b = \widetilde{b} + \alpha \wedge \beta$ and two spinors $\eta_\pm \in \mathbf{8}$ through $\eta_\pm = \mathrm{Re}(\psi_\pm)$.

To see what happens in the form picture, we start with the special $G_2 \times G_2$-invariant form

$$\rho_0 = (\eta_+ \otimes \eta_-)^{ev} = f_0 + \alpha \wedge f_1,$$

where $f_0 \in \Lambda^{ev}\widetilde{T}^*$ and $f_1 \in \Lambda^{od}\widetilde{T}^*$. It follows from (4.11) that

$$\begin{aligned}
\alpha \wedge (\eta_+ \otimes \eta_-)^{ev,od} &= \frac{1}{2}(\alpha \cdot \eta_+ \otimes \eta_- \mp \eta_+ \otimes \alpha \cdot \eta_-)^{od,ev} \\
\alpha \lrcorner (\eta_+ \otimes \eta_-)^{ev,od} &= \frac{1}{2}(-\alpha \cdot \eta_+ \otimes \eta_- \mp \eta_+ \otimes \alpha \cdot \eta_-)^{od,ev}.
\end{aligned}$$

Therefore the forms f_0 and f_1 can be expressed by

$$f_0 = \alpha \lrcorner (\alpha \wedge \rho_0) = \frac{1}{2}(\eta_+ \otimes \eta_- + \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-)^{ev}$$

and

$$f_1 = \alpha \lrcorner \rho_0 = -\frac{1}{2}(\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-)^{od}.$$

Using the spinors $\psi_\pm$ as defined above we find

$$\begin{aligned}
\psi_+ \otimes \psi_- &= (\eta_+ \otimes \eta_- - \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-) - i(\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-) \\
\psi_+ \otimes \overline{\psi}_- &= (\eta_+ \otimes \eta_- + \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-) + i(-\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-).
\end{aligned}$$

Using the knowledge from the previous section, we have

$$f_0 = \frac{1}{2}\,\mathrm{Re}(\psi_+ \otimes \overline{\psi}_-) = \tau_0$$

and

$$f_1 = \frac{1}{2}\,\mathrm{Im}(\psi_+ \otimes \psi_-) = \widehat{\upsilon}_0.$$

In the same vein, decomposing $\Box_{g,0}\rho_0 = g_1 + \alpha \wedge g_0$ yields

$$g_0 = \frac{1}{2}\operatorname{Im}(\psi_+ \otimes \overline{\psi}_-) = \widehat{\tau}_0$$

and

$$g_1 = \frac{1}{2}\operatorname{Re}(\psi_+ \otimes \psi_-) = -\upsilon_0.$$

In presence of a non-trivial B-field $b \in \Lambda^2 T^*$ we write $b = \widetilde{b} + \alpha \wedge \beta$. Since $e^{\widetilde{b}+\alpha\wedge\beta} = e^{\widetilde{b}} \wedge (1 + \alpha \wedge \beta)$ we obtain for the general case the expressions

$$\rho_b = e^{-\phi}\tau_{\widetilde{b}} + \alpha \wedge (e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} + \beta \wedge e^{-\phi}\tau_{\widetilde{b}}) \tag{4.42}$$

and

$$\Box_{g,b}\rho = -e^{-\phi}\upsilon_{\widetilde{b}} + \alpha \wedge (e^{-\phi}\widehat{\tau}_{\widetilde{b}} - \beta \wedge e^{-\phi}\upsilon_{\widetilde{b}}). \tag{4.43}$$

Conversely, if (T, g, b, ρ_0, ϕ) defines a generalised G_2-structure and $\alpha \in T$ is a unit vector, then the forms $\widetilde{b} = \alpha \lrcorner b$, $\tau_0 = \alpha \lrcorner (\alpha \wedge \rho_0)$ and $\upsilon_0 = -\alpha \lrcorner \Box_{g,0}\rho_0$ define a generalised $SU(3)$-structure $(\widetilde{T}, \widetilde{g}, \widetilde{b}, \tau_0, \upsilon_0, \phi)$ with $\widetilde{g} = g_{|T}$.

To see how the integrability conditions relate to each other over the manifolds $M^7 = M$ and $M^6 = \widetilde{M}$, consider a smooth family $(\widetilde{g}(t), \widetilde{b}(t), \tau_0(t), \upsilon_0(t), \phi(t))$ of metrics $\widetilde{g}(t)$, of 2-forms $\widetilde{b}(t)$, of even and odd forms $\tau_0(t)$ and $\upsilon_0(t)$ and of scalar functions $\phi(t)$ which we assume to define a generalised $SU(3)$-structure for any t lying in some open interval I. Moreover, we consider a curve of 1-forms $\beta(t) \in \Omega^1(\widetilde{M})$. In order to obtain an integrable generalised G_2-structure over $\widetilde{M} \times I$ defined by $(\widetilde{M} \times I, g, b, \rho_0, \phi)$ where $g = \widetilde{g}_t \oplus dt \otimes dt$, $b = \widetilde{b}(t) + dt \wedge \beta(t)$, $\rho_0 = \tau_0(t) + dt \wedge \widehat{\upsilon}_0(t)$ and $\phi = \phi(t)$, we need to solve the equations

$$d\rho = 0, \quad d\Box_{g,b}\rho = 0.$$

We decompose the exterior differential d over $M = \widetilde{M} \times I$ into

$$d_{|\Omega^{ev,od}}\cdot \to d_{|\Omega^{ev,od}}\cdot = \widetilde{d}_{|\Omega^{ev,od}} \cdot \pm \partial_{t|\Omega^{ev,od}} \cdot \wedge dt,$$

where $\widetilde{d}$ is the exterior differential on $\widetilde{M}$. From (4.42) we conclude the first equation to be equivalent to

$$d\rho = \widetilde{d}e^{-\phi}\tau_{\widetilde{b}} + dt \wedge (\partial_t e^{-\phi}\tau_{\widetilde{b}} - \widetilde{d}e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} - \widetilde{d}(\beta \wedge e^{-\phi}\tau_{\widetilde{b}}))$$

so that

$$\widetilde{d}e^{-\phi}\tau_{\widetilde{b}} = 0, \qquad \partial_t e^{-\phi}\tau_{\widetilde{b}} = \widetilde{d}e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} + \widetilde{d}\beta \wedge e^{-\phi}\tau_{\widetilde{b}}. \tag{4.44}$$

By (4.43) the second equation reads

$$d\Box\rho = -\widetilde{d}e^{-\phi}\tau_{\widetilde{b}} + dt \wedge (-\partial_t e^{-\phi}\upsilon_{\widetilde{b}} - \widetilde{d}e^{-\phi}\widehat{\tau}_{\widetilde{b}} - \widetilde{d}(\beta \wedge e^{-\phi}\upsilon_{\widetilde{b}}))$$

and therefore yields

$$\widetilde{d}e^{-\phi}\upsilon_{\widetilde{b}} = 0, \qquad \partial_t e^{-\phi}\upsilon_{\widetilde{b}} = -\widetilde{d}e^{-\phi}\widehat{\tau}_{\widetilde{b}} + \widetilde{d}\beta \wedge e^{-\phi}\upsilon_{\widetilde{b}}. \tag{4.45}$$

If we let $\bar{\beta}(t) = \int_0^s \beta(s)ds$ we can bring (4.44) and (4.45) into Hamiltonian form, that is the generalised G_2-structure is integrable if and only if

$$\begin{array}{rclcrcl} \widetilde{d}(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \upsilon_{\widetilde{b}}) & = & 0\,, & \qquad & \widetilde{d}\widehat{\upsilon} & = & \partial_t(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \tau_{\widetilde{b}})\,, \\ \widetilde{d}(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \tau_{\widetilde{b}}) & = & 0\,, & & \widetilde{d}\widehat{\tau} & = & -\partial_t(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \upsilon_{\widetilde{b}})\,. \end{array} \tag{4.46}$$

We illustrate the previous discussion by considering a classical $SU(3)$-structure defined by a unit spinor η and taking the generalised $SU(3)$-structure given by (M^6, g, η) with trivial B-field and vanishing dilaton, i.e. $b = 0$ and $\phi = 0$. Using the invariant forms given in (4.39) then the equations (4.46) become the Hitchin flow equations

$$\begin{aligned} \widetilde{d}\psi_- &= 0\,, & \widetilde{d}\psi_+ &= -\partial_t\omega \wedge \omega\,, \\ \widetilde{d}\omega \wedge \omega &= 0\,, & \widetilde{d}\omega &= \partial_t\psi_-\,, \end{aligned}$$

which appeared in [24] and go back to [53]. Note that although equations (4.46) are, like the Hitchin flow equations, in Hamiltonian form we have not shown yet that if the data $(\widetilde{g}(t), \widetilde{b}(t), \tau_0(t), \upsilon_0(t), \phi(t))$ defines a generalised $SU(3)$-structure at $t = t_0$ and satisfies (4.46), then it automatically defines a generalised $SU(3)$-structure for $t > t_0$, as it is the case for classical $SU(3)$-structures evolving along the Hitchin flow.

4.7 D-branes and generalised G_2-structures

In case D-branes can be described by submanifolds it is well known that these objects must minimise the energy functional [48, 47] and have to be calibrated. Let us assume to have a generalised G_2-manifold and we ask within this section about calibrated p-dimensional submanifolds L that are compatible with the underlying $G_2 \times G_2$-structure. In the literature there is so far nothing known about this problem and we sketch here an idea of how calibrated submanifolds L can may be characterised. The idea we develop is perfectly general and can also be applied to e.g. 6-manifolds.

A concept that is applicable in various dimensions is based on the gluing operator $\mathcal{R}$ given in (3.109) and (3.113). Previously, we used it in dimension six to describe A- and B-branes. Let us now focus on $\mathcal{R}$ once more and start by setting $F = 0$. In adapted coordinates,

$$\mathcal{R} = \begin{pmatrix} r & \\ & -r^t \end{pmatrix}\,, \quad \text{where} \quad r = \begin{pmatrix} 1_N & \\ & -1_D \end{pmatrix}\,. \tag{4.47}$$

We denoted the identity matrices for the Neumann- and Dirichlet boundary conditions by 1_N and 1_D.

It is important to remember that the properties of the gluing operator $\mathcal{R}$ are 2-fold. First of all it singles out the generalised tangent bundle of the p-dimensional submanifold L by $\mathcal{R}\Psi = \Psi$, i.e. we have p Neumann directions that defines the brane. Secondly, the condition $\mathcal{R}\Psi = \Psi$ carries more information. We denoted $\mathcal{R} : T \oplus T^* \mapsto T \oplus T^*$ but actually there is also the isomorphism, see (3.108), that maps $C_\pm \to C_\mp$. Thus, if we respect the ordering, $\mathcal{R} : C_+ \oplus C_- \to C_- \oplus C_+$. This tells us in case of a vanishing field strength F that two elements $X_+, X_- \in T$ coming from $X_\pm \pm gX_\pm \in C_\pm$ are equal on the brane. If $F \neq 0$ we can put all the gluing information about F in an operator $R|_L$ living on the brane, $R|_L = (g+F)^{-1}(g-F)|_L$, and acting on T only, $R|_L X_+|_L \to X_-|L$. This is precisely the general object given in (3.108). Strictly speaking, if we have a non-vanishing F we only have to think in the T-picture about substituting in r the identity matrix 1_N by $R|_L$.

For a classical G_2-structure we know that the defining 3-form φ is a calibration i.e. the restriction of φ to the 3-dimensional submanifold is the volume form [49]. Similar arguments hold for $*\varphi$ and the corresponding 4-dimensional submanifolds. Equivalently, we can characterise p-dimensional submanifolds by using the spinor picture. Let ξ^p be a p-form on a p-dimensional submanifold and $\psi \in \mathbf{8}$ be a G_2-spinor. The p-submanifolds, compatible with the classical G_2-structure satisfy the algebraic spinor equation

$$\xi^p \cdot \psi = \psi\,. \tag{4.48}$$

Let us understand this equation from a G_2-representation point of view. The existence of the G_2-structure decomposes the spin bundle i.e. $\mathbf{8} \to \mathbf{1} \oplus \mathbf{7}$ and also the bundle of exterior forms decompose into G_2-invariant spaces, e.g. $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where the subscripts denote the dimensions of the irreducible G_2-representations. Let us analyse the constraint (4.48) in case of ξ^3 where we use the identity given in (2.13).

$$\begin{aligned} \psi &= \xi^3 \cdot \psi\,, \\ &= \xi^{abc}\gamma_{abc}\psi\,, \\ &= \xi^{abc}\big(i\varphi_{abc} + \psi_{abcd}\gamma^d\big)\,\psi\,. \end{aligned} \tag{4.49}$$

It is not hard to see that the constraint forces ξ^3 in $\Lambda^3_7 \oplus \Lambda^3_{27}$ to vanish. In other words (4.48) only accepts the singlet. Furthermore the length of the singlet, $\xi^3 \in \Lambda^3_1$, is forced to one which is the calibration condition. By the same arguments we can consider all p-forms ξ^p. And since we saw that the form must have a singlet part a G_2-manifold can only have submanifolds L of dimension 0,3,4 and 7.

We now want to consider submanifolds L inside 7-manifolds admitting a generalised G_2-structure. Also in this case we first focus on 3-dimensional submanifolds where in general p-dimensional manifolds can be analysed similarly. The generalised G_2-structure form is $\rho^{ev/od}$ and we choose two 3-forms $\xi^3_\pm = X_\pm \wedge Y_\pm \wedge Z_\pm$. One can think about e.g $X_\pm$ as a dualised vector that origins from $X_\pm \mp gX_\pm \in C_\pm$. The strategy we follow is to act in a convenient way via Clifford multiplication on $\rho^{ev/od}$ and rewrite this as an action on the two linearly independent spinors η_+ and η_-. We achieve this by using the isomorphism (4.11) and (4.10) and define the operator $\mathcal{G}$,

$$\mathcal{G}_\mp \equiv (X_+ \cdot Y_+ \cdot Z_+ \cdot \,\widehat{\otimes}\, 1) \mp (1 \,\widehat{\otimes}\, X_- \cdot Y_- \cdot Z_- \cdot)\,. \tag{4.50}$$

This operator realises that we act with ξ^3 via Clifford multiplication in C_+ and C_-. Furthermore, we apply this action to $\rho^{ev/od} = (\eta_+ \otimes \eta_-)^{ev/od}$,

$$\begin{aligned} \mathcal{G}_\mp \bullet \rho^{ev/od} &= ((X_+ \cdot Y_+ \cdot Z_+ \cdot \,\widehat{\otimes}\, 1) \mp (1 \,\widehat{\otimes}\, X_- \cdot Y_- \cdot Z_- \cdot)) \bullet (\eta_+ \otimes \eta_-)^{ev/od}\,, \\ &= (X_+ \cdot Y_+ \cdot Z_+ \cdot \eta_+ \otimes \eta_- \mp \eta_+ \otimes \pm(X_- \cdot Y_- \cdot Z_- \cdot \eta_-))^{od/ev}\,. \end{aligned} \tag{4.51}$$

We first note that this results in a parity switch of the forms and the participating 3-forms $\xi^3_\pm$ only act on the appropriate spinors $\eta_\pm$. This means that we seperated the action of the C_+ and C_- objects ξ^3_+ and ξ^3_-. But since we have a generalised G_2-manifold we do have two calibration forms $\varphi_\pm$ and the corresponding identities

$$\gamma_{abc}\eta_\pm = i\varphi_{+abc}\eta_\pm + \psi_{+abcd}\gamma^d\eta_\pm.$$

Also here we assume that the parts of $\xi^3_\pm$ in $\Lambda^3_{\pm 7} \oplus \Lambda^3_{\pm 27}$ vanishes that is similar to the classical G_2 case. Note, we now do not assume that the singlet of ξ has norm one and thus do not assume to have a calibrated 3-submanifold according to $\varphi_\pm$ analogous to the formula (4.49).

We apply the above identities by suppressing the imaginary unit i, take the just made assumptions into account and conclude

$$\begin{aligned}\mathcal{G}_{\mp} \bullet \rho^{ev/od} &= (\varphi_+(X_+,Y_+,Z_+)\cdot\eta_+\otimes\eta_- - \eta_+\otimes\varphi_+(X_-,Y_-,Z_-)\eta_-)^{ev/od}\,,\\ &= (\varphi_+(X_+,Y_+,Z_+) - \varphi_-(X_-,Y_-,Z_-))\cdot(\eta_+\otimes\eta_-)^{ev/od}\,,\end{aligned} \tag{4.52}$$

Since the vectors $X_\pm$,$Y_\pm$,$Z_\pm$ are pullbacks to L we want to use the gluing matrix $\mathcal{R}$ in the following. The matrix $\mathcal{R}$ does not only project onto a certain p-dimensional subbundle inside $T\oplus T^*$ it also includes a map $C_\pm \to C_\mp$ i.e. actually we have $\mathcal{R}: C_+\oplus C_- \to C_-\oplus C_+$. Let us apply $\mathcal{R}$ to the consideration above where we prefer the R operator in T-picture given in (3.108) instead of the $\mathcal{R}$ operator in the $T\oplus T^*$-picture. The vectors $X_\pm$,$Y_\pm$,$Z_\pm$ live on the p-submanifold L but on this submanifold we can map the vectors (and the 1-forms) from the C_+ bundle to the C_- bundle via the $\mathcal{R}$ map. I.e. in the tangent bundle we simply have $RX_+ = X_-|_L$. We conclude

$$\begin{aligned}\mathcal{G}_{\mp} \bullet \rho^{ev/od} &= (\varphi_+(X_+,Y_+,Z_+) - \varphi_-(RX_+,RY_+,RZ_+))\cdot(\eta_+\otimes\eta_-)^{ev/od}\,,\\ &= (\varphi_+(X_+,Y_+,Z_+) - (R^*\varphi_-)(X_+,Y_+,Z_+))\cdot(\eta_+\otimes\eta_-)^{ev/od}\,,\\ &= (\varphi_+ - R^*\varphi_-)(X_+,Y_+,Z_+)\cdot(\eta_+\otimes\eta_-)^{ev/od}\,,\end{aligned} \tag{4.53}$$

where R^* defines the pullback of the gluing operator R on the brane. Demanding

$$\mathcal{G}_{\mp} \bullet \rho^{ev/od} = 0 \tag{4.54}$$

yields the following condition:

$$0 = \varphi_+ - R^*\varphi_-\,, \quad \text{on } L\,. \tag{4.55}$$

We see that the two G_2-structure 3-forms $\varphi_\pm$ have to be equal via R^* on the 3-dimensional submanifold. This means that if $\varphi|_L = \mathrm{vol}$ then also $R^*\varphi_-|_L = \mathrm{vol}$. It becomes now clear that the operator $\mathcal{G}_\mp$ applied to the G_2-structure forms measures the difference between the volume that is measured in C_+ and the volume that is measured in C_-. The condition says that these two measured volumes must be identical on L.

In adapted coordinates and by having a vanishing F- and B-field we simply have $R = \mathbb{I}$ and we conclude $\varphi_+ = \varphi_-$ on the submanifold. By having a classical G_2-structure, i.e. $\varphi = \varphi_+ = \varphi_-$, the constraint is satisfied in general.

Note, if we start with $\mathcal{G}_\pm$ instead of $\mathcal{G}_\mp$ in (4.50) then we end up with $0 = \varphi_+ + R^*\varphi_-$, on the submanifold. But the opposite sign can be absorbed in the operator R which just means, having even $F = B = 0$, that we interchange the former Neumann conditions with Dirichlet conditions and thus do not characterise 3-submanifolds but 4-submanifolds.

4.8 Generalised geometries and R-R fields

In this section we consider the supersymmetry variations for type II theories as given in (4.2) and (4.3) and take also the R-R fields into account. Remember, in case of vanishing R-R fields we compactified on a 6- and 7-manifold and found that we could capture all NS-NS fields, i.e. the metric, the B-field and the dilaton, via generalised $SU(3)$- and G_2-structures. Even more is true: We found a one-to-one correspondence between the physical data and the generalised structures. This immediately implies one important fact. By additionally taking also the R-R fields into account we run into trouble. In the generalised picture there are no more mathematical degrees of freedom left. Or in other words, the one-to-one correspondence of generalised structures can only cover the NS-NS fields. This suggests that we have to work first on the mathematical side to extend the generalised geometries where we have to find an adequate mathematical describtion for the physical R-R fields. But before we do that let us go back to physics. First of all we translate the 10d supersymmetry condition for the dilatino (4.3) into a equivalent one that is called modified dilatino variation. As a next step we give the explicit compactification on a 6-manifold where we take the R-R fields and also the 10d Hodge duality into account. Afterwards we fix the mathematics and give the main results.

4.8.1 The modified dilatino equation

During the last sections we achieved supersymmetry by claiming

$$0 = \delta_\varepsilon \Psi_X \,, \qquad 0 = \delta_\varepsilon \lambda \,. \tag{4.56}$$

But since we take now the R-R fields into account both variations include R-R fields. The aim of this section is to reformulate the variations such that only one includes the R-R fields.

Instead of requiring the above supersymmetry conditions we will discuss in the following the equivalent couple of equations

$$0 = \delta_\varepsilon \Psi_X \,, \qquad 0 = \sum_i e_i \cdot \delta_\varepsilon \Psi_{e_i} - \delta_\varepsilon \lambda \,. \tag{4.57}$$

Since we only modified the dilatino variation of (4.3) we have to calculate the additional term $\sum_i e_i \cdot \delta_\varepsilon \Psi_{e_i}$.

We tackle this problem by first noting three identities that we use in the following intensively,

1. $X \cdot \alpha^k \cdot \psi = (X \wedge \alpha^k) \cdot \psi - (X \lrcorner \alpha^k) \cdot \psi$
2. $\alpha^k \cdot X \cdot \psi = (-1)^k (X \wedge \alpha^k + X \lrcorner \alpha^k) \cdot \psi$
3. $X \lrcorner (\alpha^k \wedge \beta) = (X \lrcorner \alpha^k) \wedge \beta + (-1)^k \alpha^k \wedge (X \lrcorner \beta)$

where $X \in T$, $\alpha^k \in \Lambda^k$ and $\psi \in \Delta$.

Let us kick off our calculation of $\sum_i e_i \cdot \delta_\varepsilon \Psi_{e_i}$ by evaluating the two NS-NS terms first,

$$\begin{aligned} \sum_i e_i \cdot \Big(\nabla_{e_i} + \frac{1}{4} e_i \lrcorner H \cdot \mathcal{P}\Big)\varepsilon &= \mathrm{D}\varepsilon + \frac{1}{4}\sum_i \Big(e_i \wedge (e_i \lrcorner H) - e_i \lrcorner (e_i \lrcorner H)\Big) \cdot \mathcal{P}\,\varepsilon \\ &= (\mathrm{D} + \frac{3}{4} H\,\mathcal{P}\,\cdot)\varepsilon \end{aligned} \tag{4.58}$$

where D denotes the Dirac operator. Next we focus on the R-R part that results in

$$\begin{aligned}\sum_i e_i \cdot F^{ev/od} \cdot e_i \cdot \mathcal{P}_{ev/od}\,\varepsilon &= \sum_i \pm e_i \cdot \Big[(e_i \wedge F^{ev/od}) \cdot \mathcal{P}_{ev/od}\,\varepsilon + (e_i \lrcorner F^{ev/od}) \cdot \mathcal{P}_{ev/od}\,\varepsilon\Big] \\ &= \Big[\sum_i \mp e_i \lrcorner (e_i \wedge F^{ev/od}) \pm \sum_i e_i \wedge (e_i \lrcorner F^{ev/od})\Big] \cdot \mathcal{P}_{ev/od}\,\varepsilon\,.\end{aligned}$$

Making use of the above third identity for the first summand

$$\sum_i \mp e_i \lrcorner (e_i \wedge F^{ev/od}) \cdot \mathcal{P}_{ev/od}\,\varepsilon = \sum_i \mp(e_i \lrcorner e_i) \wedge F^{ev/od} \cdot \mathcal{P}_{ev/od}\,\varepsilon \pm e_i \wedge (e_i \lrcorner F^{ev/od}) \cdot \mathcal{P}_{ev/od}\,\varepsilon$$

yields, by noting $\sum_i e_i \lrcorner e_i = \dim(M^{1,9}) = 10$, for the R-R term

$$\begin{aligned}\frac{1}{16}\,e^{\phi} \sum_i e_i \cdot F^{ev/od} \cdot e_i \cdot \mathcal{P}_{ev/od}\,\varepsilon &= \frac{1}{16}\,e^{\phi}\Big(\mp 10\,F^{ev/od} \pm 2 \sum_{ev/od} p\,F^p\Big) \cdot \mathcal{P}_{ev/od}\,\varepsilon \\ &= \mp\frac{1}{8}\,e^{\phi}\Big(5\,F^{ev/od} - \sum_{ev/od} p\,F^p\Big) \cdot \mathcal{P}_{ev/od}\,\varepsilon\,.\end{aligned}$$

But this object is precisely the object that also appears in the physical dilatino variation (4.3). This basically motivates to consider instead of $\delta\lambda$ the equation $\sum_i e_i \cdot \delta\Psi_{e_i} - \delta\lambda$ since it does not include R-R terms at all,

$$\sum_i e_i \cdot \delta\Psi_{e_i} - \delta\lambda = (\mathrm{D} - d\phi \cdot + \frac{1}{4} H\,\mathcal{P}\,\cdot)\varepsilon.$$

This equation goes in the literature under the name of a modified dilatino variation [40, 72, 41].

In summary, the supersymmetry variations of IIA/B supergravity being in our interest are given by (4.57) and can be written as

$$0 = \Big(\nabla_X + \frac{1}{4} X \lrcorner H \cdot \mathcal{P}\Big)\varepsilon + \frac{1}{16}\,e^{\phi} F^{ev/od} \cdot X \cdot \mathcal{P}_{ev/od}\,\varepsilon\,, \tag{4.59}$$

$$0 = (\mathrm{D} - d\phi \cdot + \frac{1}{4} H\,\mathcal{P}\,\cdot)\varepsilon\,, \tag{4.60}$$

where $X \in TM^{1,9}$. To find a solution to (4.59) and (4.60), we assume the space $M^{1,9}$ to be $\mathbb{R}^{1,3} \times M^6$ or $\mathbb{R}^{1,2} \times M^7$. This specific assumption is called compactification.

4.8.2 Compactification on M^6

As a next step we compactify type IIB on $M^{1,9} = \mathbb{R}^{1,3} \times M^6$ where we suppress the IIA case which is analogous. For a compactification on $M^{1,9} = \mathbb{R}^{1,2} \times M^7$ the tools can be found in [61] where we drop the explicit compactification but give a few comments later on.

We choose 10d local coordinates X^M $(M = 0, \ldots, 9)$ and label the external coordinates by X^μ $(\mu = 0, \ldots, 3)$ whereas the internal coordinates are denoted by X^a $(a = 1, \ldots, 6)$. Let the 10d Gamma matrix Γ^M be given by

$$\Gamma^M = \begin{cases} \gamma_\mu \otimes \mathbb{I} & : \quad \mu = 0, \ldots, 3 \\ \gamma^5 \otimes \gamma_a & : \quad a = 1, \ldots, 6 \end{cases}$$

where

$$\gamma^5 = i \ \mathrm{vol}_4 \cdot , \qquad \gamma^7 = -i \ \mathrm{vol}_6 \cdot , \tag{4.61}$$

and therefore $\Gamma^{11} = \mathrm{vol}_{10} \cdot = \gamma^5 \otimes \gamma^7$. Note also $(\gamma^5)^2 = \mathbb{I}$.

In type IIB the supersymmetry vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ is chiral. In other words, the two spinors $\varepsilon_{1/2}$ are defined in $\Delta_{+M^{1,9}}$, i.e. $\Gamma^{11}\varepsilon_{1/2} = \varepsilon_{1/2}$. Since we are interested in a minimal supersymmetric vacuum background we choose the splitting to be

$$\varepsilon_1 = \zeta_+ \otimes \Theta_+ + \zeta_- \otimes \Theta_- , \tag{4.62}$$

$$\varepsilon_2 = \zeta_+ \otimes \Xi_+ + \zeta_- \otimes \Xi_- , \tag{4.63}$$

where $\zeta_\pm \in \Delta_{\pm \mathbb{R}^{1,3}}$ and $\zeta_- = \overline{\zeta}_+$. For the internal spinors we have $\Theta_\pm, \Xi_\pm \in \Delta_{\pm M^6}$ and e.g. $\Theta_- = \overline{\Theta}_+$.

Let us attack the 10d R-R part by introducing two sets of internal R-R fields F_1 and F_2 which we combine to preserve 4d Poincar invariance,

$$F^{(p)} = \mathrm{vol}_4 \wedge F_1^{(p-4)} + F_2^{(p)}. \tag{4.64}$$

Since we additionally have to satisfy the 10d hodge duality constraint (4.1) one can rephrase it in 6d by imposing

$$\begin{array}{rclcrcl} F_1^{ev} & = & \Box F_2^{ev}, & & \widehat{F_1^{ev}} & = & -\Box \widehat{F_2^{ev}}, \\ F_1^{od} & = & -\Box F_2^{od}, & & \widehat{F_1^{od}} & = & -\Box \widehat{F_2^{od}}. \end{array} \tag{4.65}$$

We are now in a position where we do the explicit compactification on $M^{1,9} = \mathbb{R}^{1,3} \times M^6$. Let us start with the gravitino variation (4.59) and focus on the external part first. Since we define the H-field and the dilaton, only on the internal space the R-R part remains. E.g. we find for R-R 1-form term

$$F^1 \cdot \Gamma_\mu \cdot \mathcal{P}_1 \, \varepsilon = (\gamma^5 \otimes F_2^1)(\gamma_\mu \otimes \mathbb{I}) i\sigma_2 \varepsilon = (-\gamma_\mu \gamma^5 \otimes F_2^1) i\sigma_2 \varepsilon$$

and the other terms follow analogously. By applying ε_1 we obtain

$$\gamma_\mu \zeta_+ \otimes \Big(F_2^1 - F_2^3 + F_2^5 + i(F_1^1 - F_1^3 + F_1^5) \Big) \Theta_+ \tag{4.66}$$

$$\gamma_\mu \zeta_- \otimes \Big(-F_2^1 + F_2^3 - F_2^5 + i(F_1^1 - F_1^3 + F_1^5) \Big) \Theta_- \tag{4.67}$$

$$= \ -\gamma_\mu \zeta_+ \otimes \Big(\widehat{F_2^{od}} + i\widehat{F_1^{od}} \Big) \Theta_+ + \gamma_\mu \zeta_- \otimes \Big(\widehat{F_2^{od}} - i\widehat{F_1^{od}} \Big) \Theta_- . \tag{4.68}$$

We can implement the 6d hodge duality constraints (4.65) and get

$$-\gamma_\mu \zeta_+ \otimes \Big(\widehat{F_2^{od}} - i\Box\widehat{F_2^{od}} \Big) \Theta_+ + \gamma_\mu \zeta_- \otimes \Big(\widehat{F_2^{od}} + i\Box\widehat{F_2^{od}} \Big) \Theta_- .$$

By considering also ε_2 and claiming $\delta_\varepsilon \Psi_\mu = 0$ (4.57) we can summarise the external algebraic equations to be

$$\begin{array}{rclcrcl} 0 & = & (F_2^{od} - i\Box F_2^{od}) \cdot \Xi_+ , & & 0 & = & (F_2^{od} + i\Box F_2^{od}) \cdot \Xi_- , \\ 0 & = & (\widehat{F_2^{od}} - i\Box\widehat{F_2^{od}}) \cdot \Theta_+ , & & 0 & = & (\widehat{F_2^{od}} + i\Box\widehat{F_2^{od}})\Theta_- . \end{array} \tag{4.69}$$

These equations do not necessarily imply constraints onto the R-R fields. But here it is the case what we prove for, for instance, the first equation. We show this by using the identity

$$\star\alpha \cdot \psi_\pm = \widehat{\alpha} \cdot \mathrm{vol} \cdot \psi_\pm = \pm i\widehat{\alpha} \cdot \psi_\pm \tag{4.70}$$

where $\alpha \in \Lambda^\bullet$, $\psi_\pm \in \Delta_\pm$. We use this to obtain e.g.

$$(F_2^{od} - i\Box F_2^{od}) \cdot \Xi_+ = (F_2^{od} - i \star \widehat{F_2^{od}}) \cdot \Xi_+ = (F_2^{od} - iF_2^{od} \cdot \mathrm{vol} \cdot) \cdot \Xi_+ = 2F_2^{od} \cdot \Xi_+.$$

The external equations (4.69) including the $\wedge$-operator can be considered similarly but we note that this operator acts complex anti-linearly. Therefore, all above external equations (4.69) constrain the R-R fields. They can be summarised to

$$0 = F_2^{od} \cdot \Xi_\pm, \quad 0 = \widehat{F_2^{od}} \cdot \Theta_\pm.$$

Next we investigate the internal part of the gravitino variation. For the NS-NS part we find

$$\Big(\mathbb{I} \otimes \nabla_a + \frac{1}{4} H_{abc}\gamma^{bc}(-\sigma_3)\Big)\varepsilon\,,$$

where we conclude for ϵ_1

$$\zeta_+ \otimes \Big(\nabla_a - \frac{1}{4}H_{abc}\gamma^{bc}\Big)\Theta_+ + \zeta_- \otimes \Big(\nabla_a - \frac{1}{4}H_{abc}\gamma^{bc}\Big)\Theta_-\,. \tag{4.71}$$

and for ϵ_2

$$\zeta_+ \otimes \Big(\nabla_a + \frac{1}{4}H_{abc}\gamma^{bc}\Big)\Xi_+ + \zeta_- \otimes \Big(\nabla_a + \frac{1}{4}H_{abc}\gamma^{bc}\Big)\Xi_-\,. \tag{4.72}$$

The internal R-R part can be discussed similarly to the external part. We note that $\mathrm{vol}\cdot = -i\gamma^5$ and, for instance, the R-R 5-form term is given by

$$F^5\Gamma_a\,\mathcal{P}_5\,\varepsilon = (-i\gamma^5 \otimes F_1^1 + \mathbb{I} \otimes F_2^5)(\varepsilon_2, -\varepsilon_1)\,.$$

By gathering all R-R terms together we get

$$\begin{aligned} & \zeta_+ \otimes \Big(F_2^{od} - iF_1^{od}\Big)\gamma_a\Xi_+ + \zeta_- \otimes \Big(F_2^{od} + iF_1^{od}\Big)\gamma_a\Xi_- && (4.73)\\ = \;& \zeta_+ \otimes \Big(F_2^{od} + i\Box F_2^{od}\Big)\gamma_a\Xi_+ + \zeta_- \otimes \Big(F_2^{od} - i\Box F_2^{od}\Big)\gamma_a\Xi_- && (4.74)\end{aligned}$$

where we also denote the ε_1 part

$$\zeta_+ \otimes \Big(\widehat{F_2^{od}} + i\Box\widehat{F_2^{od}}\Big)\gamma_a\Theta_+ + \zeta_- \otimes \Big(\widehat{F_2^{od}} - i\Box\widehat{F_2^{od}}\Big)\gamma_a\Theta_-.$$

We are now in the position to put the the NS-NS and R-R terms of the internal gravitino variation together. Note, for example we have to combine the NS-NS term including ε_1 and the R-R terms including ε_2. This is because $\mathcal{P}_{ev/od}$ switches the spinors in ε. Moreover, we use the identity (4.70) and follow the same arguments as in the external case to finally write,

$$\begin{aligned} 0 &= \nabla_X\Theta_+ - \tfrac{1}{4}(X \lrcorner H) \cdot \Theta_+ + \tfrac{e^\phi}{8}F_2^{od} \cdot X \cdot \Xi_+,\\ 0 &= \nabla_X\Xi_+ + \tfrac{1}{4}(X \lrcorner H) \cdot \Xi_+ + \tfrac{e^\phi}{8}\widehat{F_2^{od}} \cdot X \cdot \Theta_+,\end{aligned} \tag{4.75}$$

where $X \in TM^6$.

Next we turn to the modified dilatino equation (4.60). Remember that our external space is $\mathbb{R}^{1,3}$, i.e. the external Dirac operator is given by $\sum_{i=0}^{3} e_i \cdot \nabla_{e_i} = 0$. Since also the dilaton and H is only defined on the internal space we only get an internal contribution. Thus, the modified dilatino variation can be written by

$$0 = (\mathrm{D} - d\phi - \tfrac{1}{4}H) \cdot \Theta_+, \quad 0 = (\mathrm{D} - d\phi + \tfrac{1}{4}H) \cdot \Xi_+. \tag{4.76}$$

Let us summarise the type IIB results. We note the external algebraic constraints

$$\text{(i)} \qquad 0 = F_2^{od} \cdot \Xi_\pm, \quad 0 = \widehat{F_2^{od}} \cdot \Theta_\pm,$$

the gravitino equations

$$\text{(ii)} \qquad \begin{array}{rcl} 0 &=& \nabla_X \Theta_+ - \frac{1}{4}(X \lrcorner H) \cdot \Theta_+ + \frac{e^\phi}{8} F_2^{od} \cdot X \cdot \Xi_+ \\ 0 &=& \nabla_X \Xi_+ + \frac{1}{4}(X \lrcorner H) \cdot \Xi_+ + \frac{e^\phi}{8} \widehat{F_2^{od}} \cdot X \cdot \Theta_+ \end{array} \tag{4.77}$$

and the modified dilatino equations

$$\text{(iii)} \qquad 0 = (\mathrm{D} - d\phi - \tfrac{1}{4}H) \cdot \Theta_+, \quad 0 = (\mathrm{D} - d\phi + \tfrac{1}{4}H) \cdot \Xi_+ \tag{4.78}$$

where $X \in TM^6$. Since we suppressed the explicit compactification for type IIA we nevertheless give the reader the results. We write the external equations by

$$\begin{array}{rclrcl} 0 &=& (F_2^{ev} + i\Box F_2^{ev}) \cdot \Theta_+\,, & 0 &=& (F_2^{ev} - i\Box F_2^{ev}) \cdot \Theta_-\,, \\ 0 &=& (\widehat{F_2^{ev}} - i\Box \widehat{F_2^{ev}}) \cdot \Xi_-, & 0 &=& (\widehat{F_2^{ev}} + i\Box \widehat{F_2^{ev}}) \Xi_+. \end{array} \tag{4.79}$$

For the gravitino variation we obtain

$$\begin{array}{rcl} 0 &=& \nabla_X \Theta_+ + \frac{1}{4}(X \lrcorner H) \cdot \Theta_+ + \frac{e^\phi}{16}(\widehat{F_2^{ev}} + i\Box \widehat{F_2^{ev}}) \cdot X \cdot \Xi_- \\ 0 &=& \nabla_X \Xi_+ - \frac{1}{4}(X \lrcorner H) \Xi_+ - \frac{e^\phi}{16}(F_2^{ev} + i\Box F_2^{ev}) \cdot X \cdot \Theta_- \end{array} \tag{4.80}$$

and for the modified dilatino variation we find

$$0 = (\mathrm{D} - d\phi + \tfrac{1}{4}H) \cdot \Theta_+, \quad 0 = (\mathrm{D} - d\phi - \tfrac{1}{4}H) \cdot \Xi_+, \tag{4.81}$$

where $X \in TM^6$. But here a problem arises since the first and second external equations vanish on its own by using (4.70)

$$(F_2^{ev} + i\Box F_2^{ev}) \cdot \Theta_+ = (F_2^{ev} + i \star \widehat{F_2^{ev}}) \cdot \Theta_+ = (F_2^{ev} + iF_2^{ev} \cdot \mathrm{vol}\,\cdot) \cdot \Theta_+ = 0.$$

This means that the algebraic equation is satisfied trivially and does not constrain the R-R field. Even worse, applying the same argument for the second gravitino equation the R-R terms even drop out completely unlike the IIB case. Since we assume that the supersymmetry variations by which we started with are basically true we strongly believe that there is only a sign mismatch for IIA that should be put in the right order.

Let us spend a view words about the compactification on $M^{1,9} = \mathbb{R}^{1,2} \times M^7$ that we already mentioned earlier. First of all the internal and also the external spinors are not chiral since we are in odd dimensions. Analogous to the 6d case we start with a similar definition as in (4.64). But here, for IIA and IIB, we have to introduce an even and an odd set of internal RR-fields $F_{1/2}$. By using (4.1) the two sets $F_{1/2}$ are identified as in the 6d case via the 7-dimensional $\Box$-operator.

4.8.3 The constrained variational principle and R-R fields

Let us go to the basics of generalised structures. In [53] Hitchin motivated that certain classical geometrical structures can be understood as a critical point of a variational principle. Let us roughly review how this variational principle works. Given an n-dimensional manifold and an exterior form ρ that is invariant under a certain holonomy group that fits to n, e.g. φ for G_2 and $n = 7$. Let us assume that ρ is closed and thus fixes a certain cohomology class $[\rho]$. He further showed that one can construct out of ρ the volume element $\rho \wedge \hat{\rho}$. The form $\hat{\rho}$ depends on ρ and is also invariant of the stabeliser of ρ. For example, $\hat{\varphi} = \star\varphi$, where the Hodge operator $\star_g$ depends on the metric. With this volume element Hitchin sets up a variational problem where he assumes that ρ is stable, i.e. it lies in an oper orbit so that it can be varied over this open orbit. Since we want to vary the functional we want to vary with respect to $[\rho]$, i.e. ρ within a fixed class. We remember that $[\rho]$ defines the affine space $\rho_{fix} + da$ where we denote the fixed element by ρ_{fix} and parametrise the affine space by the exact forms da. Hitchin showed that by varying the functional with respect to $[\rho]$ the relevant term in the volume element is da. Using Stokes theorem we get a critical point of the functional if also the form $\hat{\rho}$ is closed. Therefore, a certain geometric structure is given if it is a critical point, that is, the structure forms satisfy

$$d\rho = 0, \qquad d\hat{\rho} = 0.$$

For instance, for a classical G_2 holonomy manifold we have the two forms $\rho = \star\varphi$ and $\hat{\rho} = \varphi$ that satisfy $d \star \varphi = d\varphi = 0$. It is essential that the metric is not fixed beforehand. This implies that the variational principle is purely topological. Note that it is also possible to start with $[\hat{\rho}]$ and the critical point is then given by condition $d\rho = 0$. This property is useful with a view towards including the R-R fields later on.

This concept translates into the generalised picture as shown by Hitchin [55] and Witt [84, 83]. Furthermore the setup can be twisted by a closed 3-form H which results in the fact that we have to substitute the usual differential operator d by the twisted differential operator $d_H = d + H$ as already defined.

More subtle, here a couple of questions arises immediately: Are the critical points isolated? Can the set of critical points be described by a manifold? This results in the question: What is the moduli space of critical points? Hitchin realised that the critical points cannot be isolated. Let us assume to have a critical point. We can always act by the diffeomorphism group and B-field transformations to move to other critical points in the neighbourhood. Thus, he divided these two transformation out and found that transverse to this action the critical points are non-degenerate. He uses this to define local charts from the moduli space of critical points to an oben set in cohomology.

Let us mention that the even and odd structure forms given by $\Xi_+ \otimes \Theta_+$, $\Xi_+ \otimes \Theta_-$ and $(\eta_+ \otimes \eta_-)^{ev/od}$ for the already discussed generalised $SU(3)$- and G_2-structure can define critical points. It is important to note that $\hat{\rho}$ is given by $\Box\rho$. This gives us a full picture of the generalised structures,

- Let us have a topological generalised structure $(M, g, \rho, \Box\rho)$. The variational principle characterise the integrability condition of the underlying generalised structure by

$$d_H\rho = 0, \qquad d_H\Box\rho = 0,$$

i.e. ρ and $\Box\rho$ are d_H-closed.

- The structure forms ρ and $\Box\rho$ are spinors for the $T\oplus T^*$-bundle and represent the form picture. In this form picture the most general structure form ρ can be fully characterised by
$$\rho = e^{-\phi}\, e^b \wedge \rho_0,$$
where ρ_0 denotes the normal form, ϕ the dilaton and b the B-field.

- The form picture and the integrability condition can be translated one-to-one into a spinor picture. There we have two real or complex spinors and also the metric g, the dilaton ϕ and the B-field. The integrability condition is given by a Killing spinor equation and a dilaton equation.

We are now prepared to implement the R-R fields. As we already mentioned on the mathematical side there are no degrees of freedom left to bring the R-R fields into play. We therefore need a new mathematical object that is able to give the R-R fields a conceptual mathematical meaning. We only review the basic ideas and give the results. The precise mathematical proofs can be found in [62]. We bring the R-R fields into the game by formulating a constrained variational principle. Let us assume to start with the closed real structure form e.g. $[\rho^{ev/od}]$ as e.g. given for a generalised $SU(3)$-structure. The constrained is given by a real even form $F^{ev/od}$ and the critical points are in principle given by $d\Box\tau^{ev/od} = F^{od/ev}$. Observe that $\Box\rho^{ev/od}$ is even/odd. Note, that it is only important to have an even/odd form as the constraint and this implies that we can even substitute for $F^{ev/od}$ the form $-\Box F^{od/ev}$. We do this, as we will see later on, to make contact with the notation for the R-R fields in the democratic formulation [13]. This result motivates

Definition 4.8.1. *Let $d^{\Box} = \Box\, d\, \Box$.*
(i) *Let $(M^6, \rho^{ev/od})$ be a generalised $SU(3)$-structure. The form $\rho^{ev/od}$ is real, H a 3-form and $F \in \Omega^*(M^6)$. The structure is said to be* integrable with respect to H and F *if and only if*
$$d_H\rho^{ev/od} = 0, \quad d_H^{\Box_\rho}\rho^{ev/od} = F^{od/ev}. \tag{4.82}$$
Equivalently, we can write more succinctly by using a complex notation
$$d_H\tau = -i\Box_\tau\widetilde{F},$$
where $\tau = \rho^{ev} - i\Box_\rho\rho^{ev} \oplus \rho^{od} + i\Box_\rho\rho^{od}$.

(ii) *Let $(M^7, \rho^{ev,od})$ be a generalised G_2-structure, H a 3-form and $F^{ev,od} \in \Omega^{ev,od}(M)$. Then the structure is said to be* integrable *with respect to H and $F^{od,ev}$ if and only if*
$$d_H\rho = F^{od,ev},$$
where $\rho = \rho^{ev/od} \oplus \Box_{\rho^{ev/od}}\rho^{\bullet}$. An integrable structure is said to be of even *or* odd *type according to the parity of $F^{ev/od}$.*

(iii) *An integrable generalised G-structure is* parallel *if $F = 0$.*

We spend a view words about the notation for the 6d case. The object $d^{\square} = \square\, d\, \square$ is a kind of co-differential. Let us rewrite the second constraint of the generalised $SU(3)$-structure by

$$d_H^{\square}\rho^{ev/od} = \square\, d_H \square \rho^{ev/od} = F^{od/ev},$$

where we use in 6d the property $\square^2 = -1$ to obtain

$$d_H \square\, \rho^{ev/od} = -\square F^{od/ev}.$$

By using now the integrability conditions, i.e. $d_H \rho^{ev/od} = 0$ and $d_H \square\, \rho^{ev/od} = -\square F^{od/ev}$, we can easily verify the complex notation by using τ. We further use the $\sim$-operator to write $\tilde{F} = F^{ev} - F^{od}$.

Since we skip afterwards the proof of the main result of this section we nevertheless point to one ingredient that is important. We first go back to the first chapter where we compactified 11d supergravity on a 7-manifold. Remember, this theory comes with a 4-form flux F. Since we reduced the structure group to e.g. G_2 it was useful to decompose the internal $F \in \Lambda^4 M^7$ into G_2-irreducible parts $F \to F_1 + F_7 + F_{27}$ (2.32).

Let us adopt this idea to our forms $F^{ev/od}$ within the generalised geometries. In general we now have to decompose $F^{ev/od}$ into $G \times G$-irreducible parts, where here $G = SU(3), G_2$. In the following we are interested to translate the integrability condition into the classical spinor picture. This means that each of the two spinors (real or complex) reduce the structure group. We will call these associated groups by G_l and G_r where $G = SU(3), G_2$. It can be shown that this observation is useful and yields in the end

$$\begin{array}{lcll} F &=& F_1 + F_{3l} + F_{3r} + F_9, & \text{for } SU(3) \times SU(3), \\ F &=& F_1 + F_{7l} + F_{7r} + F_{49}, & \text{for } G_2 \times G_2. \end{array}$$

where $F \in \Delta_+ \otimes \Delta_-$ can be even or odd. The subscripts here denote the representations with respect to the group $G \times G$ and not only to representations of G_l or G_r.

Let us now come to the case of a generalised $SU(3)$-structure where we have the spinors Ξ_+, Θ_+ and their complex conjugates Ξ_-, Θ_-. By means of them we characterise

$$F = \lambda_F \Xi_+ \otimes \Theta_- + \alpha_F \cdot \Xi_- \otimes \Theta_- + \Xi_+ \otimes \overline{\beta}_F \cdot \Theta_+ + \Gamma_F(\Xi_- \otimes \Theta_+)$$

where $\lambda \in \mathbb{C}$ is the singlet $\mathbf{1}$. The vectors $\alpha_F \in \mathbb{C}^3$ and $\overline{\beta}_F \in \overline{\mathbb{C}^3}$ determine $\mathbf{3}l$ and $\mathbf{3}r$. The sesquilinear form $\Gamma_F \in \mathbb{C}^3 \otimes \overline{\mathbb{C}^3}$ describes the representation $\mathbf{9}$.

The similar arguments can be applied for F in case of a underlying generalised G_2-structure. We denote the two spinors by η_+ and η_- and use the objects $\lambda \in \mathbf{1}$, $\alpha_F \in \mathbf{7}l$, $\beta_F \in \mathbf{7}r$ and $\Gamma_F \in \mathbf{49}$ to write

$$F = \lambda_F \eta_+ \otimes \eta_- + \alpha_F \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \beta_F \cdot \eta_- + \Gamma_F(\eta_+ \otimes \eta_-).$$

We now come to the main results of this section

Theorem 4.8.1. *A generalised $SU(3)$–structure (M^6, τ) is integrable with respect to the forms H_0 and F, i.e.*

$$d_{H_0}\tau = -i\square_\tau \widetilde{F},$$

if and only if the following conditions hold, where $\tau = e^{-\phi}([\Xi_+ \otimes \Theta_-]_b \oplus [\Xi_+ \otimes \Theta_+]_b)$ *and* $H = db/2 + H_0$.

(i) *The* algebraic constraints*: Seen as an endomorphism* $\Delta \to \Delta$, $F_{|\Delta_a \otimes \Delta_b}$ *preserves the decomposition of* $\Delta_{a,b}$ *into irreducible* $SU(3)_{l,r}$*–modules, i.e. for all combinations* $a, b \in \{+, -\}$,

$$(F_{|\Delta_a \otimes \Delta_b})_{3r,l} = 0.$$

Moreover, the 1*– and* $\widehat{1}$*–components of* $F^{ev,od}$ *and* $\widehat{F^{ev,od}}$ *couple via*

$$F^{ev} \cdot \Theta_{\pm} = F^{od} \cdot \Theta_{\mp}, \quad \widehat{F^{ev}} \cdot \Xi_{\mp} = -\widehat{F^{od}} \cdot \Xi_{\mp}.$$

(ii) *The* generalised Killing equations

$$\begin{aligned} \nabla_X \Xi_+ + \tfrac{1}{4}(X \llcorner H) \cdot \Xi_+ - e^{\phi} F^{ev}_{-b} \cdot X \cdot \Theta_- + e^{\phi} F^{od}_{-b} \cdot X \cdot \Theta_+ &= 0 \\ \nabla_X \Theta_+ - \tfrac{1}{4}(X \llcorner H) \cdot \Theta_+ + e^{\phi} \widehat{F^{ev}_{-b}} \cdot X \cdot \Xi_- + e^{\phi} \widehat{F^{od}_{-b}} \cdot X \cdot \Xi_+ &= 0, \end{aligned}$$

(iii) *The* dilatino equations

$$\begin{aligned} D\Xi_+ - d\phi \cdot \Xi_+ + \tfrac{1}{4} H \cdot \Xi_+ + e^{\phi} F^{ev}_{-b} \cdot \Theta_- + e^{\phi} F^{od}_{-b} \cdot \Theta_+ &= 0 \\ D\Theta_+ - d\phi \cdot \Theta_+ - \tfrac{1}{4} H \cdot \Theta_+ - e^{\phi} \widehat{F^{ev}_{-b}} \cdot \Xi_- + e^{\phi} \widehat{F^{od}_{-b}} \cdot \Xi_+ &= 0. \end{aligned}$$

Theorem 4.8.2. *A generalised* G_2*-structure* $(M^7, \rho^{od,ev})$ *is integrable with respect to the forms* H_0 *and* $F^{od,ev}$*, i.e.*

$$d_{H_0} \rho^{od,ev} = F^{ev,od}$$

if and only if the following conditions hold, where $\rho^{ev,od} = e^{-\phi}[\eta_+ \otimes \eta_-]_b^{ev,od}$, $H = db/2 + H_0$ *and* $F = F^{ev,od} + \Box F^{ev,od}$.

(i) *The algebraic constraints: Seen as an endomorphism* $\Delta \to \Delta$, F *preserves the decomposition of* Δ *into irreducible* $G_{2l,r}$*–modules, i.e.* $F_{7l,r} = 0$.

(ii) *The generalised Killing equations*

$$\begin{aligned} \nabla_X \eta_+ + \tfrac{1}{4}(X \llcorner H) \cdot \eta_+ \pm e^{\phi} F_{-b} \cdot X \cdot \eta_- &= 0 \\ \nabla_X \eta_- - \tfrac{1}{4}(X \llcorner H) \cdot \eta_- - e^{\phi} \widehat{F}_{-b} \cdot X \cdot \eta_+ &= 0. \end{aligned}$$

(iii) *The dilatino equations*

$$\begin{aligned} D\eta_+ - d\phi \cdot \eta_+ + \tfrac{1}{4} H \cdot \eta_+ - e^{\phi} F \cdot \eta_- &= 0 \\ D\eta_- - d\phi \cdot \eta_+ - \tfrac{1}{4} H \cdot \eta_+ \pm e^{\phi} \widehat{F} \cdot \eta_+ &= 0. \end{aligned}$$

Let us compare the results we got on one hand from compactifying II theories on 6-manifolds and on the other hand from mathematics only. Here we assume that the IIA case can be rectified. We first observe that we have to interchange $\widehat{F}^{ev/od}$ and $F^{ev/od}$ that is just a conventional definition. The different factors in front of the R-R fields in the gravitino equations can be simply adjusted. One can absorb the factor $1/8$ appearing in the physical gravitino equations into the R-R fields. The comparison of the modified dilatino equations tell us that physics force the R-R terms in this equation to vanish. But this constraint is exactly the external constraint we got from physics. Thus, both constraints are equivalent and only one remains.

The external constraint implies that the singlets of the R-R fields have to vanish. In addition, we have the constraint $(F_{|\Delta_a \otimes \Delta_b})_{3r,l} = 0$ that leads to the fact that only the F_9 part of the R-R fields is present.

A compactification on 7-manifolds can be done analogously. Similar constraints can be found by following the same arguments as given in remark 1, i.e. the vacuum background of the internal 7-fold in presence of R-R fields can be characterised by a generalised G_2-structure.

If the generalised structure is parallel, then the Theorems 4.8.1 and 4.8.2 assert the spinors (Ξ_+, Θ_+) and (η_+, η_-) to be parallel with respect to the lift of Hitchin's connections in Theorem 2 of [57]. From this point of view, F is most naturally interpreted as the "torsion" of these connections. As in the classical case, we obtain obstructions to integrability in the form of algebraic constraints on the "torsion" components.

4.8.4 T-duality

In this section we discuss the device of T-duality and take the NS-NS as well as the R-R fields into account. We study this within the generalised structures where the result for the NS-NS fields was already given in [21, 84]. Remember, we already learned in the previous chapter: T-duality is mirror symmetry. We investigated this problem in using generalised structures and T-dualised in the T^3-fibre. Here we study T-duality in the S^1-fibre.

In general, the well known T-duality rules are denoted in the literature by component notation of the involved NS-NS and R-R fields (see e.g. [64]). Let us assume to have a manifold that is S^1-fibred and we thus decompose all components of the involved fields with respect to this fibration. In principle, the T-duality rules show how to map the fields to the T-dual side that is also S^1-fibred space. With this notation it is simple to see that the NS-NS fields transform only into each other and so behave the R-R fields. By investigating e.g. the NS-NS rules further the underlying concept of the transformation was not fully understood. The geometrical idea was first realised by [15] where the authors now focus on the integrability conditions of the involved objects. Let us roughly summarise the result. The authors in [15] showed that the assumption of the S^1-fibration of the manifold lead to two first Chern classes. The first is given by the S^1-fibration of the manifold itself, i.e. the curvature, and the other first Chern class is encoded in the NS-NS 3-form field strength H. Since both Chern classes are integral the classes only specify integer numbers that represent the different topologies. It turned out that T-duality interchange these two numbers.

Let us now discuss what T-duality means in a more abstract geometrical picture. This duality assumes that the manifold M^n of interest is special in the way that it can be described by a, in general, non-trivial S^1-fibration over an $(n-1)$-dimensional base manifold M^{n-1} that is compact. We further assume that there exists an S^1-invariant generalised G-structure (M^n, ρ). The non-triviality of the S^1-fibration can be captured by the connection form θ and we denote by X_θ the corresponding dual vertical vector field, i.e. $X_\theta \lrcorner \theta = 1$. The curvature of the S^1-bundle is given by ω. Moreover, assume to be given a closed, integral and S^1-invariant

3-form H such that the 2-form ω^t defined by $\omega^t = X \lrcorner H$ is also integral. Integrality of ω^t ensures the existence of another principal S^1-bundle, the *T-dual* of the former defined by the choice of a connection form θ^t with $d\theta^t = -\omega^t$. Writing $H = \theta \wedge \omega^t + \mathcal{H}$ where $\mathcal{H} \in \Omega^3(M)$, we define the T-dual of H by

$$H^t = -\theta^t \wedge \omega + \mathcal{H}.$$

Since the $G \times G$-invariant spinor ρ is also S^1-invariant, we can decompose ρ into forms ρ_0, ρ_1 living on the base manifold M^{n-1}, $\rho = \theta \wedge \rho_0 + \rho_1$. The T-dual of ρ is then defined to be

$$\rho^t = \theta^t \wedge \rho_1 + \rho_0,$$

so T-duality is enacted by multiplication with the element $X_\theta \oplus \theta$ on ρ followed by the substitution $\theta \to \theta^t$. Let us note that the object $X_\theta \oplus \theta$ is an element of $Pin(n,n)$, which covers $O(n,n)$ like $Spin(n,n)$ covers $SO(n,n)$.

The crucial feature of the just described operation is that it preserves the $Spin(n,n)$-orbit structure on $\Lambda^{ev,od}TM^n$ and in particular, ρ^t is also $G \times G$–invariant. Consequently, $\Box_{\rho^t}\rho^t$ and $(\Box_\rho \rho)^t$ are both $G \times G$-invariant spinors which are therefore equal up to a universal scalar which we henceforth ignore. In the same vein, we decompose an S^1-invariant form of mixed degree as we do in the following for the R-R fields F.

In summary, the generalised structure form ρ and the R-R-form F together with the T-dual partners can be denoted by

$$\rho = \theta \wedge \rho_0 + \rho_1\,, \qquad \rho^t = \theta^t \wedge \rho_1 + \rho_0\,,$$

and

$$F = \theta \wedge F_0 + F_1\,, \qquad F^t = \theta^t \wedge F_1 + F_0\,.$$

We now formulate with the definitions above that the integrability condition can be T-dualised

Proposition 4.8.3. *Let ρ and F be S^1-invariant. Then*

$$d_H\rho = F \iff d_{H^t}\rho^t = -F^t.$$

We find it useful to provide a proof for the reader.

Proof: We start by considering the left hand side first

$$\begin{aligned}
d_H\rho &= d_H(\theta \wedge \rho_0 + \rho_1) \\
&= d\theta \wedge \rho_0 - \theta \wedge d\rho_0 + d\rho_1 + \theta \wedge \omega^t \wedge \rho_1 + \mathcal{H} \wedge \theta \wedge \rho_0 + \mathcal{H} \wedge \rho_1 \\
&= \theta \wedge (-d\rho_0 + \omega^t \wedge \rho_1 - \mathcal{H} \wedge \rho_0) + \omega \wedge \rho_0 + d\rho_1 + \mathcal{H} \wedge \rho_1 \\
&= \theta \wedge F_0 + F_1
\end{aligned}$$

that yields

$$F_0 = -d\rho_0 + \omega^t \wedge \rho_1 - \mathcal{H} \wedge \rho_0 \text{ and } F_1 = \omega \wedge \rho_0 + d\rho_1 + \mathcal{H} \wedge \rho_1\,. \tag{4.83}$$

On the T-dual side we obtain

$$\begin{aligned}
d_{H^t}\rho^t &= d_{H^t}(\theta^t \wedge \rho_1 + \rho_0) \\
&= d\theta^t \wedge \rho_1 - \theta^t \wedge d\rho_1 + d\rho_0 - \theta^t \wedge \omega \wedge \rho_0 + \mathcal{H} \wedge \theta^t \wedge \rho_1 + \mathcal{H} \wedge \rho_0 \\
&= \theta^t \wedge (-d\rho_1 - \omega \wedge \rho_0 - \mathcal{H} \wedge \rho_1) - \omega^t \wedge \rho_1 + d\rho_0 + \mathcal{H} \wedge \rho_0 \\
&= -\theta^t \wedge F_1 - F_0
\end{aligned}$$

where we used (4.83). ■

Corollary 4.8.4. *If ρ defines an integrable $SU(3)$–structure with respect to H and F, then so does ρ^t with respect to H^t and $-F^t$. Similarly, if $\rho^{ev,od}$ defines an integrable G_2-structure of odd or even type with respect to H and $F^{od,ev}$, then $\rho^{od,ev\,t}$ defines an integrable structure of even or odd type with respect to H^t and $-F^{od,ev\,t}$.*

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