# Scalar fields and higher-derivative gravity in brane worlds

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# Abstract

We consider the brane world picture in the context of higher-derivative theories of gravity and tackle the problematic issues fine-tuning and brane-embedding. First, we give an overview of extra-dimensional physics, from the Kaluza-Klein picture up to modern brane worlds with large extra dimensions. We describe the different models and their physical impact on future experiments.

We work within the framework of Randall-Sundrum models in which the brane is a gravitating object, which warps the background metric. We add scalar fields to the original model and find new and self-consistent solutions for quadratic potentials of the fields. This gives us the tools to investigate higher-derivative gravity theories in brane world models. Specifically, we take gravitational Lagrangians that depend on an arbitrary function of the Ricci scalar only, so-called $f(R)$-gravity. We make use of the conformal equivalence between $f(R)$-gravity and Einstein-Hilbert gravity with an auxiliary scalar field. We find that the solutions in the higher-derivative gravity framework behave very differently from the original Randall-Sundrum model: the metric functions do not have the typical kink across the brane. Furthermore, we present solutions that do not rely on a cosmological constant in the bulk and so avoid the fine-tuning problem.

We address the issue of brane-embedding, which is important in perturbative analyses. We consider the embedding of codimension one hypersurfaces in general and derive a new equation of motion with which the choice for the embedding has to comply. In particular, this allows for a consistent consideration of brane world perturbations in the case of higher-derivative gravity. We use the newly found background solutions for quadratic potentials and find that gravity is still effectively localized on the brane, i.e that the Newtonian limit holds.

# Contents

# Chapter 1

# Introduction and overview

In 1998 the physics of extra dimensions took an unexpected turn. First, Arkani-Hamed, Dvali and Dimopoulos published an article [1] on the physics of extra dimensions at the millimeter scale, in which they presented a new solution to the hierarchy problem in particle physics. Their model has us living on a domain wall, which henceforth we will call a brane. All known matter fields are confined to the brane, at least up to energy scales that are accessible in today's experiments, and only gravitons can leave the brane.

In this scenario, they did not take into account the gravitational backreaction of a physical brane-like object on spacetime. This problem was dealt with soon after by Randall and Sundrum in two articles [2] and [3]. The first was another attempt to tackle the hierarchy problem but now in self-consistent spacetime, which led to non-factorizable metric solutions. The latter article was more radical in that respect: instead of assuming a compact extra dimension that is bounded by two branes, they removed the second brane altogether. Thereby, they found a new scenario with an infinite extra dimension that still had the right gravitational properties at low energies on the brane. That is, a brane observer will see the Newtonian gravitational potential at distances large compared to the characteristic curvature scale of the full spacetime.

This was a radical change to the paradigm of extra dimensions. Until the work by Arkani-Hamed, Dvali and Dimopoulos, we had to use compactification at scales of about the Planck scale, which also was the standard way to deal with extra dimensions in string theory. The first two scenarios [1] and [2] within the new paradigm also offered a solution to the longstanding hierarchy problem, which allowed for a fundamental gravity scale in full spacetime of the order of the electroweak energy scale. This caused considerable upheaval in the physics community because for the first time string theoretic effects, which seemed unaccessible in direct observations, seemed to be reachable in future high energy collider experiments.

Now that the initial hype is over the picture looks somewhat different. Further table-top experiments have been done, in which no deviations from four-dimensional Newtonian gravity down to a scale of a hundredth of a millimeter have been found so far [5]. This makes a complete solution to the hierarchy problem only possible in models with at least three extra dimensions, in models with only one or two extra dimensions, the fundamental mass scale would be much higher. Therefore we cannot expect to see stringy effects in models with one or two extra dimensions in the next generation of collider experiments. On the other hand most models with extra dimensions are highly effective in shielding a brane observer from the higher-dimensional behavior of gravity. This makes it relatively easy to create models that do not contradict observation without unphysical or artificial assumptions. Moreover, fundamental theories with extra dimensions are still very popular and models with large extra dimensions are a reasonable and fascinating extension of the mechanisms that can hide the effects of higher-dimensional spacetime in current observations.

In this thesis, we tackle some of the many remaining problems of the brane world picture. One problem with higher-dimensional theories is caused by the lack of solutions for physically motivated choices of bulk scalar field potentials due to the complexity of the equations. In this thesis we will derive a new class of analytic solutions for scalar fields with quadratic potentials. We will mostly consider five-dimensional brane worlds, which are often treated as toy models of some underlying fundamental higher-dimensional theory. In this case, we have to regard the brane world scenarios as effective theories only and we should expect modifications to Einstein-Hilbert gravity. Therefore, we will consider higher-derivative gravity and we will have a closer look at the perturbations and the Newtonian limit on the brane. We will also examine the more conceptual problem of brane embedding, that has been causing considerable trouble in the perturbative investigation of brane worlds.

The outline of the thesis is as follows. In the second chapter, we look into the history and the basic set-up of extra-dimensional pictures, starting with Kaluza-Klein theories from more than eighty years ago that demanded small extra dimensions, up to modern brane world models with large extra dimensions. In chapter three, we incorporate scalar fields in our brane world models. We explain a convenient solution-generating technique that stems from supergravity and present new analytic solutions for quadratic bulk potentials. In chapter four, we consider cosmological issues in brane worlds, because certain cosmological features of brane worlds, particularly the reproduction of the standard Hubble expansion at low energies, contributed substantially to the popularity of the new scenarios.

Then we switch to models with modified gravity in chapter five. We consider gravity actions whose Lagrangians are arbitrary functions of the Ricci scalar, so-called higher-derivative gravity. In this case, we can capitalize on the equivalence of $f(R)$-gravity and usual Einstein gravity with a scalar field. We find as a general feature of higher-derivative-gravity solutions that the metric derivatives are smooth across the brane. This can be understood in terms of the structure of the equations of motion in higher-derivative theories. Furthermore, we find self-consistent models with gravitating branes that allow for a vanishing cosmological constant in the bulk.

In chapter six, we tackle the problem of consistent brane-embedding in the bulk. Firstly, we present the brane-bending mechanism found by Garriga and Tanaka in [4]. We derive an independent equation that describes possible embeddings of branes and, through that, offers a consistency check for the calculation of brane world scenarios.

We use the new solutions for bulk fields with quadratic potentials and apply the methods presented in the chapter about higher-derivative gravity to derive the Newtonian limit on the brane in chapter seven. For that, we also ensure that the embedding of the brane in spacetime with higher-derivative gravity is done consistently. Finally, we draw our conclusions and give an outlook on subsequent work in the last chapter.

# Chapter 2

# Extra dimensions

Extra dimensions in physics is not a new idea. Over eighty years ago, Kaluza and Klein independently began research into models with extra dimensions with the aim of unifying electromagnetism and gravity. Although their theory could not be extended to include later developments in particle physics, the basic idea had several revivals, for example in string theory, where the assumption of compactified dimensions is crucial to obtain consistent and viable models.

From 1998 on, there has been renewed interest in the physics of extra dimension due to ideas that allow for large or even infinite extra dimensions. In these models, only gravity can access the extra dimensions, nevertheless, we still obtain four-dimensional Newtonian gravity on an embedded hypersurface. All other fields are assumed to be constrained to the lower-dimensional hypersurface that is embedded in full spacetime in these models, the so-called brane.

There exist a number of different brane world models. In what follows, we will explain the set-ups and basic concepts of these models, and their advantages and shortcomings as descriptions of our universe.

## 2.1 Kaluza-Klein theory

As early as 1919, Kaluza brought the idea of unifying gravity and electromagnetism to Einstein's attention, but published it only in 1921 [6]. His aim was to unify the Einstein-Hilbert action (we neglect all matter terms for the moment)

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R, \tag{2.1}$$

where $g \equiv |\det(g_{\mu\nu})|$, and the action for the Maxwell theory of electromagnetism in four dimensions

$$S_{em} = -\frac{1}{4} \int d^4x \sqrt{|g|} F_{\mu\nu} F^{\mu\nu}. \tag{2.2}$$

Here, $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ stands for the electromagnetic field tensor. Instead of simply adding up the gravitational and the electromagnetic action, Kaluza considered a pure gravity theory in five-dimensional spacetime, i.e. with one extra spatial dimension

$$S = \int d^4x dy \sqrt{|\bar{g}|} \bar{R}, \tag{2.3}$$

where the standard definition for the Ricci scalar $\bar{R}$ applies. The indices of $\bar{g}_{AB}$ now run over $0, 1, 2, 3, 5$. Here and in the following we will use capital Roman letters as indices to describe full spacetime, while small Greek indices will run from $0, 1, 2, 3$ only, i.e. they are used to describe the four dimensional world that we seem to live in.

At this point, Klein's idea [7] becomes important. Kaluza only saw this unification procedure as a purely mathematical model. Therefore, he simply demanded that the metric components be independent of the $y$-coordinate. Klein, at that moment unaware of Kaluza's ideas, independently developed a similar theory. Instead of taking $y$-independent metric components, he compactified the extra dimension on a circle by identifying the points $y = 0$ and $y = 2\pi l$. Then, it is reasonable to write the metric as

$$g_{AB}(x^\mu, y) = \sum_{n=0}^{\infty} g_{AB}^{(n)}(x^\mu) e^{iny/l} + \text{c.c.}, \tag{2.4}$$

where we dropped the bars for convenience. Due to compactification $y$ runs from zero to $2\pi l$ only. Klein assumed a compactification scale $l$ of the order of the Planck length, i.e. $10^{-33}cm$, the fundamental constant of length derived from Newton's and Planck's constant and the speed of light. Klein considered this scale as the only natural length scale and therefore as the only reasonable cut-off parameter.

From a four-dimensional point of view, this splitting yields a massless mode plus an infinite tower of so-called Kaluza-Klein-modes, the masses of which are proportional to $n/l$ for the $n$th mode. This can be seen by considering the $n$th mode of a five-dimensional massless free scalar field $\chi$ with four-momentum $p_\mu$ on a cylinder with three infinite spatial dimensions

$$\chi_{p_\mu,n} \sim e^{ip_\mu x^\mu} e^{iny/l}, \tag{2.5}$$

where $n$ is an integer. Since this scalar field must obey the five-dimensional Klein-Gordon-equation for a massless particle, we easily read off the mass of the $n$th mode from

$$p_\mu p^\mu - \frac{n^2}{l^2} = 0 \tag{2.6}$$

Because of the extreme smallness of the Planck length, the massive modes are not excited at low temperatures and, therefore, we can neglect them. Then, it is possible to write down the remaining part of the metric in the convenient form

$$g_{AB} = \left( \begin{array}{c|c} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ \hline e^{2\beta\phi} A_\mu & e^{2\beta\phi} \end{array} \right). \tag{2.7}$$

After substituting for the metric (2.7) in the higher-dimensional action (2.3), demanding $2\alpha \equiv 1/\sqrt{3} \equiv \beta$ and performing the integration over the compact extra dimension, we obtain

$$S = l \int d^4x \sqrt{|g|} \left( R - \frac{1}{4} e^{-\sqrt{3}\phi} F^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right). \tag{2.8}$$

Obviously, this unified model contains Einstein and Maxwell theory, but it manifestly includes a scalar field, which is coupled to the electromagnetic field tensor. This extra scalar field is now known as the dilaton. Although we are now used to or even want to have fundamental scalar fields like the Higgs field, it caused considerable worries for Kaluza and Klein.

The procedure of reducing higher-dimensional theories to four dimensional ones can be generalized to models with more than one extra dimension. Moreover, the manifold over which we compactify does not necessarily have to be a circle. Indeed, we can compactify over an arbitrary manifold and the resulting effective theory will depend on the manifold. In string theory, the idea of Kaluza-Klein compactification has become popular again because bosonic string theory is only consistent in 26 dimensions. Also fermionic string theory and so-called M-theory that live in ten and eleven dimensions respectively, rely on such a mechanism.

For each of the extra dimensions, we typically demand that it is compactified over an extremely small manifold. From high energy collision experiments, we can deduce an upper bound for the size $l$ of the extra dimensions

$$l < \frac{1}{TeV} \approx 10^{-18} cm. \tag{2.9}$$

In this case the omission of the massive modes is a reasonable assumption and renders an effective low-energy theory that agrees with the standard four-dimensional picture. The smallness of the extra dimensions has been the standard lore for almost eighty years now. However, recently, an interesting ansatz has been found that allows for large extra dimensions in a way which scarcely affects the four-dimensional world we observe. This is the so-called brane world picture.

## 2.2 ADD brane worlds

The fundamental idea behind all different brane world models is that the Standard Model fields are confined to a lower-dimensional hypersurface, which is embedded in full spacetime, and only gravity can go off the brane. Therefore, one would naively expect that gravity shows higher-dimensional behavior, that is, for example, a Newtonian potential proportional to $1/r^{n-3}$. We will see in the following paragraphs that this is not the case for distances $r$ larger than the compactification scale.

Although the idea that lower-dimensional hypersurfaces could constitute the visible part of our world had shown up before (see [8] and particularly [9]), it only became popular in 1998. In that year, Arkani-Hamed, Dimopoulos and Dvali [1] (see also [10]) proposed a new scenario now known as the

ADD brane world model. Their main objective was to obtain a solution for the longstanding hierarchy problem without the help of supersymmetry or technicolor. The hierarchy problem comes from the observation that the electroweak mass scale is $M_{ew} \sim 10^3\,GeV$, while the Planck mass, which describes the energy scale at which gravity becomes comparable in strength, is about $M_{pl} \sim 10^{18}\,GeV \gg M_{ew}$.

In the ADD scenario, we assume that full spacetime has $n$ dimensions with $d$ compact extra dimensions, $n = 3 + 1 + d$. Standard model matter is confined to a brane and, therefore, does not have additional Kaluza-Klein-states. However, gravity lives in full spacetime and we have to consider the whole tower of KK modes. Because gravity is only measured down to length scales of about a tenth to a hundredth of a millimeter, the size of the compact dimensions can also be large compared to the allowed size in Kaluza-Klein theory.

In higher-dimensional theories, the four-dimensional Planck scale is not a fundamental parameter and, therefore, opens up opportunities for new solutions to the hierarchy problem. Take the action for $n$-dimensional gravity,

$$S = -\frac{1}{2\kappa_n^2} \int d^n x \sqrt{|g^{(n)}|} R^{(n)}, \tag{2.10}$$

where

$$\kappa_n^2 \equiv \frac{1}{M_n^{n-2}} \tag{2.11}$$

and $M_n$ denotes the fundamental mass that describes gravity in full spacetime. Let us now consider the long-range gravitational interaction, which will be mediated by the zero-mode, i.e. the massless mode, of the Kaluza-Klein tower of gravitons. Therefore, we can assume that the metric depends on the brane-coordinates only and, in this case, we can integrate out the extra dimensions

$$S_{eff} = -\frac{V_d}{2\kappa_n^2} \int d^4 x \sqrt{|g^{(4)}|} R^{(4)}, \tag{2.12}$$

where $V_d$ is the volume of the extra dimensions and $V_d \sim l^d$, if all extra dimensions are of the same size $l$. It follows that the four-dimensional Planck mass $M_{pl}$ depends not only on $M_n$, but also on the size of the compactified space

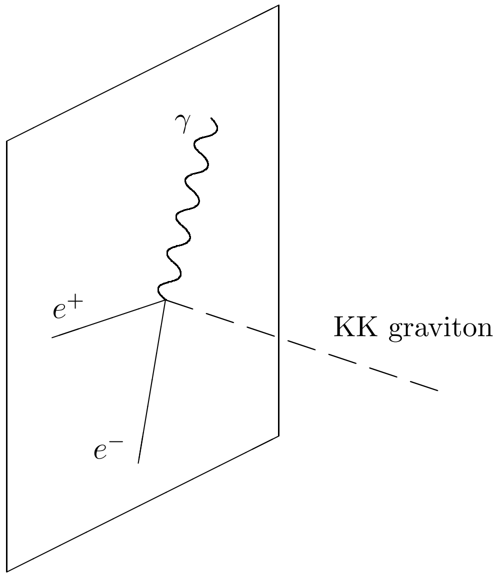


Figure 2.1: *emission of massive KK gravitons into the bulk in a typical collision experiment*

$$M_{pl} \sim M_n (M_n l)^{\frac{d}{2}}. \tag{2.13}$$

In order to solve the hierarchy problem, we demand that the fundamental mass $M_n$ is of the order of the electroweak mass $M_{ew} \sim 1$ TeV. Substituting in (2.13) gives

$$l \sim M_n^{-1-\frac{d}{2}} M_{pl}^{\frac{d}{2}} \sim 10^{\frac{32}{d}-17}\mathrm{cm}, \tag{2.14}$$

and we can read off that we need at least three additional compact dimensions to stay within the bounds from short-distance gravity measurements. These experiments are designed to detect deviations from the $1/r$ potential. In the ADD model any deviations at that distance are interpreted as arising from the excitations of massive KK modes. For $r \ll l$, this leads to a higher-dimensional Newtonian potential $V(r) \sim 1/r^{1+d}$, but so far no deviation from the four-dimensional Newtonian potential has been found.

Apart from the afore-mentioned change in the gravitational potential, there will also be new processes in particle physics which are not present in Standard Model physics. The most important one is the emission of gravitons into the bulk (see figure 2.1). In principle, these effects can add new constraints to the ADD model [11]. Emission of bulk gravitons could be detected in collision experiments, where massive KK graviton modes are excited and subsequently emitted into the bulk. Although the coupling of the massive gravitons to matter will be as weak as the coupling of the massless ones, the emission into the bulk will become important at energy scales comparable to the fundamental gravity scale $M_n$. This is due to the large number of massive graviton states that arise at high energies.

In an annihilation process of electrons and positrons, the cross section for the emission of a massive graviton is proportional to $\alpha/M_{pl}^2$ and the total cross section reads

$$\sigma \simeq \frac{\alpha}{M_{pl}^2} N(E), \tag{2.15}$$

where $N(E)$ is the number of KK graviton states with masses smaller than the center of mass energy $E$. Due to the quantization of the KK masses, we obtain $N(E) \sim (El)^d$. Therefore, the total cross section can be expressed as

$$\sigma \sim \frac{\alpha}{E^2} \left( \frac{E}{M_n} \right)^{d+2} . \tag{2.16}$$

If the fundamental gravity scale is low, we also expect a multitude of new particles to show up in this energy range. Some of these might even couple strongly to usual matter, but, so far, we have no observational hints of non-standard physics in collider experiments.

The new scenario has more serious implications in astrophysics and, particularly, in cosmology. High temperatures in the early universe allow for production of massive graviton states, which can change the thermal evolution of the universe fundamentally. For small temperatures far below the lowest massive state only the massless graviton will carry energy into the bulk. Because of the weak coupling this does not pose a problem. For temperatures much larger than $1/l$, we can estimate the rate of graviton production with masses smaller than or about the temperature $T$ (see [11])

$$\frac{dn}{dt} \sim \frac{T^6}{M_{pl}} (Tl)^d \sim T^4 \left( \frac{T}{M_n} \right)^{d+2} , \tag{2.17}$$

where we have used (2.13). The couplings of the massive modes are also suppressed by $1/M_{pl}$, but due to the high multiplicity of states this will give a significant contribution at high temperatures. When we assume standard expansion of the visible universe, i.e. the Hubble parameter

$$H \simeq \frac{T^2}{M_{pl}}, \tag{2.18}$$

then we obtain for the total density of KK modes created within a Hubble time $1/H$,

$$n(T) \sim T^2 M_{pl} \left(\frac{T}{M_n}\right)^{d+2}. \tag{2.19}$$

The typical mass of a graviton produced at temperature $T_0$ is about $T_0$. Therefore, we can assume for lower temperatures $T$ the non-relativistic number density proportional to $T^3$. Observationally, we can derive a stringent bound from the nucleosynthesis epoch at temperatures of about 1 MeV. The cumulative mass density of all excited massive graviton states

$$\rho\left(T_{nucleo}\right) \sim \left(\frac{T_{nucleo}}{T_0}\right)^3 T_0 n\left(T_0\right) \sim T_{nucleo}^3 M_{pl} \left(\frac{T_0}{M_n}\right)^{d+2} \tag{2.20}$$

has to be much lower than the energy density of the massless gravitons, which is proportional to $T_{nulceo}^4$. For two extra dimensions and a fundamental gravity scale $M_n \sim$TeV, we would obtain a maximum temperature of $T_0 \sim 10$ MeV. Even for six extra dimensions, which could be important in string theory, we find a maximum temperature of about 1 GeV. Although such low energy bounds for the early universe do not contradict observation directly, it is certainly difficult to construct theories of inflation at such low scales.

The initial popularity of the ADD model was also related to the fact that in the case of two extra dimensions and a fundamental gravity scale at the order of the electroweak scale there could be a wealth of new effects at energy scales reachable in the next generation of collider experiments. The same reasoning held for precision table-top gravity experiments. However, the cosmological bounds were a serious dent in this initial enthusiasm. These issues were tackled in subsequent scenarios, which we present in the next section.

## 2.3 Randall-Sundrum brane worlds

In the ADD set-up, the vacuum energy density of the brane is not taken into account, therefore, the extra dimensions are assumed to be Euclidean. By including the brane energy density, that is, by considering the effect of a gravitating brane, we find an interesting new kind of geometry. In 1998, Randall and Sundrum published two now famous papers, in which they proposed two different scenarios with extra dimensions. The first one [2] again describes a model with a compact extra dimension; the second set-up [3] contains only one brane and allows for an infinitely large extra dimension without destroying the effective localization of gravity on the brane.

New in both models was the inclusion of the so-called tension of the brane, describing the energy density per volume in three-dimensional space. In their papers Randall and Sundrum focused on models with only one extra dimension, but the basic concept can be generalized to an arbitrary higher-dimensional spacetime. The action $S$ for a single brane with tension $\lambda$ in higher-dimensional spacetime, where we additionally assume that the brane is infinitely thin, is

$$\begin{aligned} S &= S_{bulk} + S_{brane} \\ &= -\frac{1}{2\kappa_n^2}\int d^n x\sqrt{|g|}\,(R-2\Lambda) + \int d^{n-1}\sigma\sqrt{|\gamma|}\left(\lambda + \frac{1}{\kappa_n^2}[K]_\pm\right). \end{aligned} \tag{2.21}$$

where $g$ denotes the determinant of the full spacetime metric $g_{AB}$ and $R$ the five-dimensional Ricci scalar. The notation $[A]_\pm$ indicates that we evaluate the quantity $A$ on both sides of the brane and take the difference: $[A]_\pm \equiv A_+ - A_-$. We have included a cosmological constant term $\Lambda$ in the bulk, which will turn out indispensable in order to obtain viable physical solutions. By $\sigma_\mu$ we denote the coordinates on the embedded hypersurface, our brane, and $\gamma$ denotes the determinant of the induced metric $\gamma_{\mu\nu}$. The induced metric is defined by

$$\gamma_{\mu\nu} = g_{AB}\frac{\partial X^A}{\partial\sigma^\mu}\frac{\partial X^B}{\partial\sigma^\nu}, \tag{2.22}$$

where $X^A(\sigma^\gamma)$ describes the embedding of the brane in full spacetime.

A domain wall in spacetime leads to additional boundary conditions. Lancszos derived them as early as 1922 [12], but they became appreciated only after Israel's paper in 1966 [13] and they are now commonly known as the Israel junction conditions. In the brane action $S_{brane}$, we have an additional term known as the Gibbons-Hawking-term,

$$S_{GH} = \frac{1}{\kappa_n^2} \int d^{n-1}\sigma \sqrt{|\gamma|}[K]_{\pm}. \tag{2.23}$$

This term is needed to consistently derive the full set of equations in space-times with boundaries (see [14]). Here, $K$ is the trace of the extrinsic curvature $K_{AB}$ calculated from the unit normal $n_A$ to the brane, $n^A n_A = -1$. We can define the projected metric $q_{AB} \equiv g_{AB} + n_A n_B$, from which we obtain the extrinsic curvature

$$K_{AB} \equiv q^C_{\ A} \nabla_C n_B \quad \text{and} \quad K = q^{AB} K_{AB}. \tag{2.24}$$

Variation of the action with respect to the full metric renders the usual Einstein equations in the bulk,

$$R_{AB} - \frac{1}{2} g_{AB} R - \Lambda g_{AB} = 0. \tag{2.25}$$

These are supplemented by the Israel junction conditions, found by varying with respect to the induced metric on the hypersurface,

$$[K_{\mu\nu}]_{\pm} = -\kappa_n^2 \left( \tau_{\mu\nu} - \frac{1}{n-2} \gamma_{\mu\nu} \tau \right). \tag{2.26}$$

For the derivation of the junction conditions we have assumed a more general set-up, in which we included an arbitrary matter Lagrangian $\mathcal{L}_b$ on the brane. Then, the brane Lagrangian $\mathcal{L}_b$ gives rise to a stress-energy tensor on the embedded hypersurface

$$\tau_{\mu\nu} = \frac{2}{\sqrt{|\gamma|}} \frac{\delta\left(\sqrt{|\gamma|}\mathcal{L}_b\right)}{\delta\gamma^{\mu\nu}}. \tag{2.27}$$

Note that it is important for the derivation of the stress-energy tensor to vary with respect to the contravariant metric. After clarifying the basics of both Randall-Sundrum models, we will explain the two set-ups in detail in the following sections.

### 2.3.1 RS I scenario

In their first paper on extra dimensions [2], Randall and Sundrum described a scenario with one compact extra dimension, that is a five-dimensional spacetime with two $(3+1)$-branes as boundaries. As in the ADD model, their aim was a solution of the hierarchy problem; in the ADD scenario this was achieved with a flat bulk by decreasing the volume of the extra dimensions $V_d$ sufficiently. This brings down the fundamental gravity scale, but defines a new scale $1/V_d^{1/d}$ and, by that, we have a new hierarchy between the scale $1/V_d^{1/d}$ and the electroweak scale. As we will see, the non-trivial bulk in the RS I scenario allows for a solution to the hierarchy problem that does not imply a new energy scale.

The model has two $(3+1)$-branes in five-dimensional spacetime with tensions $\lambda_1$ and $\lambda_2$ respectively. Since the extra dimension is compact, we have to specify conditions at the boundaries of our spacetime. At first we assume $\mathbb{Z}_2$-symmetry, that is we identify points $(x^\mu, y)$ with $(x^\mu, -y)$, and take $y$ to be periodic with period $2l$. We locate the two branes at the orbifold fixed points $y_1 = 0$ and $y_2 = l$. Then it is enough to consider only the spacetime between $y_1$ and $y_2$ and thus the action reads

$$\begin{aligned} S &= -\frac{1}{2\kappa_5^2}\int d^5x\sqrt{|g|}\left(R + 2\kappa_5^2\Lambda\right) \\ &\quad + \sum_{i=1,2}\int_{y_i} d^4\sigma\sqrt{|\gamma|}\left(\lambda_i + \frac{1}{\kappa_5^2}\left[K|_{y_i}\right]_\pm\right). \end{aligned} \tag{2.28}$$

Although the introduction of branes breaks translational invariance in the direction of the extra dimensions, we want Lorentz symmetry preserved on the brane. Therefore, we choose the following ansatz for the metric

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2. \tag{2.29}$$

Substituting in the Einstein equations (2.26), we are free to choose the metric function on one of the branes, to be specific, a(0)=1, and obtain the solution

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2. \tag{2.30}$$

The effective cosmological constant $\Lambda_4$ on the brane derives from a combination of the bulk cosmological constant and the respective tension of the brane

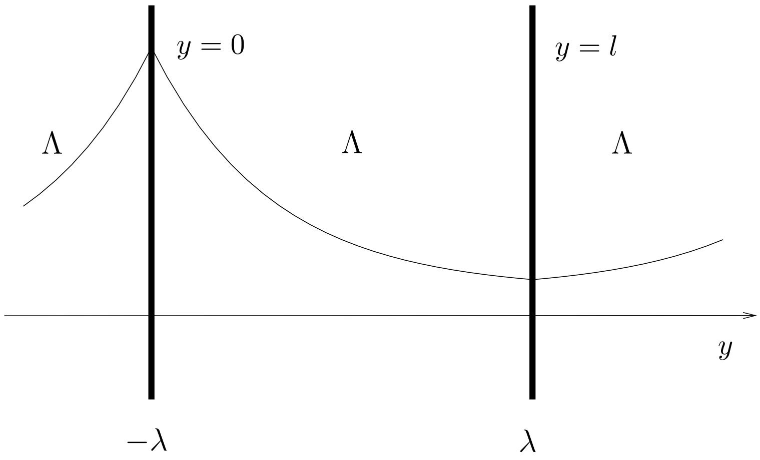


Figure 2.2: *warped geometry in RS I model*

$$\Lambda_4 = \frac{1}{2}\kappa_5^2 \left( \Lambda + \frac{1}{6}\kappa_5^2 \lambda^2 \right) \tag{2.31}$$

This result is established with the help of the Gauß-Codazzi-equations and the Israel junction conditions. We will show this in section 2.5. In order to obtain a Minkowski brane, we have to demand $\Lambda_4 = 0$ and, thereby, $k = \sqrt{-\Lambda/6} > 0$.

The main difference between the ADD and the RS models consists in the non-factorizable bulk geometry of the latter. The scale factor $a(y)$ is called the warp factor and decays exponentially away from the brane and is, by construction, $\mathbb{Z}_2$ symmetric. There is a Minkowskian solution on both branes only if the tension of the second brane is the exact negative of the tension on the visible brane. For physical solutions, the bulk cosmological constant $\Lambda$ has to be negative and hence the bulk has to be anti de-Sitter. This seems somewhat unphysical, because AdS space does not have an initial Cauchy hypersurface. Nevertheless, in some supersymmetric theories anti de-Sitter space can be a ground state of the system. We will now show that it is exactly the resulting exponential warp factor that leads to a different solution to the hierarchy problem.

We suppose the size of the extra dimension to be small, but still larger than $1/M_{ew}$. Therefore, we cannot access the extra dimension by gravity experiments. Then we can make use of the effective field theory description in four dimensions. We consider massless fluctuations of the Minkowskian background

$$dx^2 = e^{-2k|y|} \left(\eta_{\mu\nu} + h_{\mu\nu}(x^\alpha)\right) dx^\mu dx^\nu. \tag{2.32}$$

We set $h_{5A} = 0$ and impose the transverse and traceless conditions

$$\partial^\mu h_{\mu\nu} = 0 \quad \text{and} \quad h^\mu_{\ \mu} = 0. \tag{2.33}$$

Now we substitute the metric in the original action (2.28) to obtain the effective action. Since we are interested in the relation between $M_5$ and $M_{pl}$, we use $\kappa_5^2 = 1/M_5^3$ and focus on the gravitational term only:

$$S_{eff} = -M_5^3 \int_{-l}^{l} dy\, e^{-2k|y|} \int d^4x \sqrt{|g|} R^{(4)}(g_{\mu\nu}) + \dots\ , \tag{2.34}$$

where $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$ and $R^{(4)}$ is the four-dimensional Ricci scalar. We will show later in connection with the embedding of the brane, that at low energies the correct Newtonian limit is recovered (see also [4]). Performing the $y$-integration we find that the effective theory in four dimensions has a Planck mass that depends only weakly on the inter-brane distance $l$ for large enough $k$:

$$M_{pl}^2 = \frac{M_5^3}{k} \left(1 - e^{-2kl}\right). \tag{2.35}$$

For a more general treatment of scales and hierarchies in warped spacetimes see also [15].

Let us now investigate the physically observed masses of matter fields. For that, we assume a Higgs field with a fundamental mass $m$ on the hidden brane at $y = 0$. The metric on the hidden brane is $g_{\mu\nu}^{hidden} = g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, whereas the metric on the visible brane, which is located at $y = l$ reads $g_{\mu\nu}^{visible} = e^{-kl} g_{\mu\nu}$. Then the action for the Higgs field on the visible brane is given by

$$S = \int_{y=l} d^4x \sqrt{|g|} e^{-4kl} \left[ e^{2kl} g_{\mu\nu} \nabla_\mu H^\dagger \nabla_\nu H - \alpha \left(|H|^2 - m^2\right)^2 \right] \tag{2.36}$$

with $\alpha$ an arbitrary coupling constant and where $\nabla^\mu$ denotes the covariant derivative with respect to $g_{\mu\nu}$. An observer on the visible brane naturally chooses $a(l)$ to be unity for proper measurements, i.e. the Higgs field gets redefined, so that it absorbs the warp factor, and the effective action reads

$$S = \int_{y=l} d^4x \sqrt{|g|} \left[ g_{\mu\nu} \nabla_\mu \tilde{H}^\dagger \nabla_\nu \tilde{H} - \alpha \left( |\tilde{H}|^2 - e^{-2kl} m^2 \right)^2 \right], \qquad (2.37)$$

where $\tilde{H} \equiv e^{-kl} H$. Therefore, an observer on the visible brane measures the physical mass of the Higgs field $m_H = e^{-kl} m$. We can apply this result to all mass parameters except for the Planck mass.

This gives us the solution to the hierarchy problem. Assume there is no hierarchy between the unsuppressed Higgs mass $m$ and the fundamental gravity scale $M_5$ and both are at scales of $10^{19} GeV$. To find physical masses $m_H \sim 10^3 GeV$ and $M_{pl} \sim 10^{19} GeV$ we need $kl \sim 50$. Obviously, there is no new scale in this model and, therefore, no new hierarchy between $1/l$ and the AdS curvature scale $k$.

Of course, we can do the procedure in reverse and assume a low fundamental mass close to the electroweak scale on the negative tension brane. Changing the coordinates $x^\alpha \to e^{kl} x^\alpha$, leaves the Higgs mass at the positive tension brane almost unaffected, while the Planck mass blows up to the experimental value. This is also the right way to tackle the problem, because a negative tension on the visible brane would render Newtonian gravity with the wrong sign.

There is another fundamental difference between the ADD- and the RS scenarios. In the first case, where spacetime has a flat and factorizable structure, the KK gravitons will have relatively small masses. Therefore, they can be produced already at low energies, but they couple only weakly, i.e. like four-dimensional gravitons. Their effect will show up at collider experiments only due to the multiplicity of KK states. In the latter case, with a low fundamental gravity scale, the KK modes will only show up at energies close to the fundamental mass $M_5$, but they will couple strongly to standard model matter on the brane. The effects that trouble the ADD scenario, particularly in cosmological issues, are absent due to warped geometry. Furthermore, the production of strongly coupled massive gravitons opens up the possibility of observing quantum gravitational or stringy effects in future collider experiments directly.

So far we have not addressed the issue of stability of two brane scenarios. In principle, this can be achieved with a massive bulk scalar field, the so-called radion, whose potential fixes the inter-brane distance. We will come back to this in detail in chapter 3 about bulk scalar fields.

### 2.3.2 RS II scenario

Although the hierarchy problem is one of the most serious conceptual problems in theoretical physics, the other aspect of the RS scenario, namely large extra dimensions that are not directly observable, is tantalizing in itself. Therefore, Randall and Sundrum went one step further in their subsequent paper [3]. As we can see from equation (2.35), it is safe for the Planck mass to send off the second brane to infinity. This corresponds to removing the second boundary in spacetime. The fact that the Planck mass remains finite is a good hint that gravity might remain effectively four-dimensional on the brane. Nevertheless, we will have to show explicitly that the gravitational zero mode is somehow localized and mediates seemingly four-dimensional gravitation on the brane.

We already know the solution for the metric in the two brane context and the metric remains valid in the limit of infinite brane distance, as long as we stick to the fine-tuning between brane tension and bulk cosmological constant. The fine-tuning is one of the disadvantages of the RS-scenarios, but we will show later, that we can avoid it altogether by modifying the gravitational theory. The perturbed metric is

$$ds^2 = \left(e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(x^\alpha, y)\right) dx^\mu dx^\nu - dy^2, \tag{2.38}$$

where, for convenience, we have redefined $h_{\mu\nu}$ to include the warp factor $a(y)$. Again, we choose the so-called Randall-Sundrum gauge with

$$h_{5A} = 0 \quad \text{and} \quad h_{\mu\ ,\nu}^{\ \nu} = 0 = h^\mu_{\ \mu}. \tag{2.39}$$

We want to observe the gravitational potential between two static test particles on the brane. The equation of motion for the metric perturbations in the bulk then reads

$$\left(e^{2k|z|}\Box + \partial_z^2 - 4k^2\right) h_{\mu\nu} = 0, \tag{2.40}$$

where $\Box$ stands for the four-dimensional flat d'Alembert operator. As boundary conditions we have to use the Israel junction conditions (2.26). After combining the two equations, we perform a Kaluza-Klein reduction $h(x, y) = \sum \psi_m(y) e^{ipx}$, where $p$ and $x$ stand for the four-vectors that denote the directions parallel to the brane and $p^2 = m^2$ corresponds to the four-dimensional

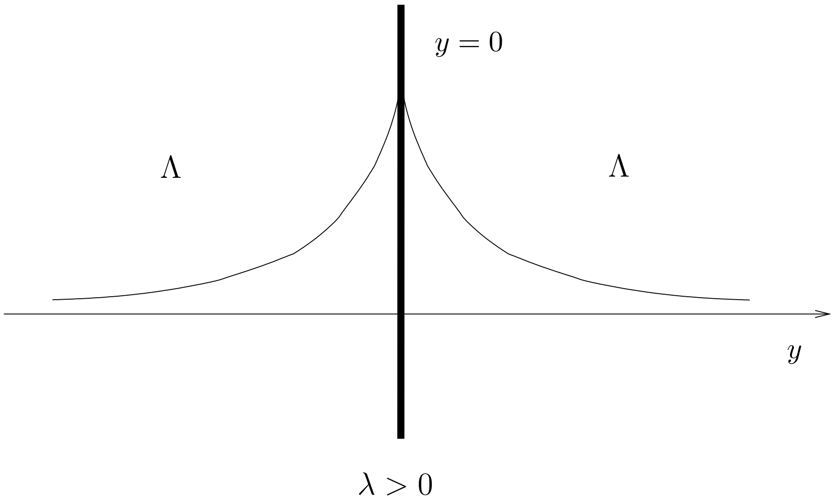


Figure 2.3: *warped geometry in spacetimes with a single brane*

mass of the modes. We omit all tensor indices because the equations agree for all components of the metric perturbations:

$$\left( \frac{-m^2}{2} e^{2k|y|} - \frac{1}{2}\partial_y^2 - 2k\delta(y) + 2k^2 \right) \psi_m(y) = 0. \tag{2.41}$$

Because of $\mathbb{Z}_2$-symmetry all odd functions drop out. To solve the equation, we transform to a new variable $z \equiv sgn(y)\left(e^{k|y|} - 1\right)/k$ and a new field $\phi_m(z) \equiv \psi_m(y)e^{k|y|}/2$ and end up with

$$\left( -\frac{1}{2}\partial_z^2 + \underbrace{\frac{15k^2}{8(k|z|+1)^2} - \frac{3}{2k}\delta(z)}_{\equiv V(z)} \right) \phi_m = m^2 \phi_m. \tag{2.42}$$

The $\delta$-function in the potential $V(z)$ allows for a normalizable bound state. This will be the massless graviton, which is effectively localized on the brane. Furthermore, we can see that the potential decays to zero for $|z| \to \infty$. On the other hand, this means that there is no gap in the continuum modes, i.e. they will start with $m = 0$.

The continuum modes as solutions to equation (2.42) are given in terms of Bessel functions. In order to satisfy the boundary conditions at the brane we have to choose the following combination for small masses, that is for modes that carry the long range interactions.

$$\phi_m \sim N_m \left(|z| + 1/k\right)^{\frac{1}{2}} \left( Y_2 \left(m(|z| + 1/k)\right) + \frac{4k^2}{\pi m^2} J_2 \left(m|z| + 1/k)\right) \right). \quad (2.43)$$

The normalization constant $N_m$, which can be approximately given in terms of the asymptotic expansions of the Bessel functions in the limit of large $m|z|$: $N_m \sim \pi m^{5/4}/(4k^2\sqrt{l})$. We can reintroduce the second brane, often referred to as regulator brane, without further problems. This will be necessary in most theories with bulk scalar fields, because there will be singularities in the bulk, from which we have to screen the observable universe to ensure predictability. This will only have an impact on the spectrum of massive modes, which will become quantized, because we also have to fulfill the boundary conditions on the second brane.

Having found the complete KK spectrum, we can calculate the static potential on the brane between two point masses $m_1$ and $m_2$. We present further details of the derivation in chapter 6. The zero mode gives us the expected Newtonian potential, while the continuum modes constitute the corrections to the Newtonian potential:

$$V(r) \sim G_{(4)} m_1 m_2 \left( \frac{1}{r} + \int_0^\infty dm \frac{m}{k^2} \frac{e^{-mr}}{r} \right). \quad (2.44)$$

The formula clearly shows that the massive modes are exponentially suppressed for distances larger than $1/m$, that is the corrections will only show up at small distances. Another interesting feature of the potential is the coupling of the second term which is proportional to the fundamental gravity scale $G_{(4)}/k \sim M_5^3$ and that means strongly coupled compared to the zero mode. We can integrate out the second term

$$V(r) \sim G_{(4)} \frac{m_1 m_2}{r} \left( 1 + \frac{1}{r^2 k^2} \right) \quad (2.45)$$

and obtain a potential that uncovers the corrections only at distances of about the inverse fundamental gravity scale. At this point, we have to mention that the answer has to be slightly modified. This is due to brane bending, an effect that shows up when we put matter on the brane. We will explain this effect in detail in the chapter on brane-embedding. Another way of understanding the corrections to this result is to consider the structure of the graviton propagator. In five dimensions, there is an extra degree of freedom included. In four dimensions, only the massive states will have the same number of degrees of freedom as a five-dimensional massless graviton, but at low energies they can be neglected. Then the propagator structure for the massless ground state differs from the truely four-dimensional propagator. This would change the bending of light significantly. To be more specific, it would imply a decrease in the bending of light down to a factor of only 3/4 of the usual four-dimensional value, which would have been detected more than eighty years ago. Nevertheless, the bending of the brane cancels exactly the effects of the extra degree of freedom, such that we obtain a ground state that looks like a four-dimensional massless graviton.

In principle, we could extend RS brane worlds such that bulk and brane have arbitrary vacuum expectation values. This simply renders our four-dimensional world de-Sitter or anti de-Sitter depending on the resulting $\Lambda_4$. In this context, it was also tried to tackle the cosmological constant problem as an effect of extra dimensions, but so far no convincing solutions have been found.

## 2.4 DGP model

In the previous sections, we described models whose extra dimensions have a finite volume. This is either realized by a finite size $l$ or in the RS II model by a warped geometry. The finite size of the volume allows for a massless bound state on the brane, which mediates the effective four-dimensional behavior of gravitation. If we introduce truly infinite extra dimensions, for instance an infinite Euclidean dimension, then the gravitational interaction on the brane has to be mediated by the continuum modes. Dvali, Gabadadze and Porrati considered such a model in [16] and found four-dimensional behavior of gravity only on intermediate scales, while on very short and on very long scales gravity becomes five-dimensional again. The five-dimensional behavior at long scales, although problematic, can in principle be hidden, because we can push up the second cross-over scale to the current day Hubble-size of the universe $1/H_0 \sim 10^{28} cm$. The four-dimensional graviton is no longer an eigenstate of the linearized theory, but we can imagine models where meta-stable graviton continuum modes exist. Newton's law will then be valid up to scales that correspond to the lifetime of the meta-stable states. Again, problems arise due to the different tensor structure of massive compared to massless gravitons, no matter how small the masses are. In the Green's functions for the two cases we can see the so-called Dam-Veltman-Zakharov discontinuity that mathematically describes the difference in the

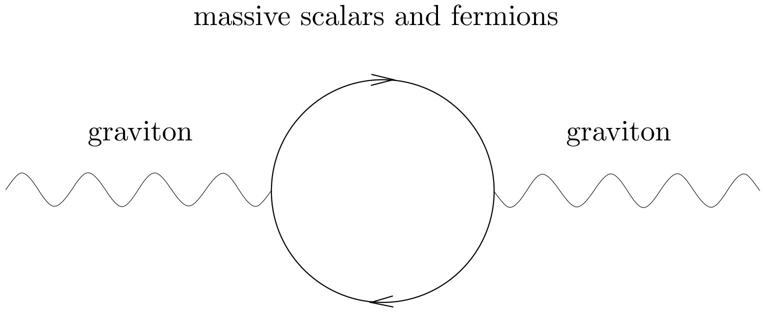


Figure 2.4: *one-loop diagram with massive scalars and fermions in the loop which generates the brane Ricci scalar*

tensor structures:

$$G_m^{\mu\nu\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{\frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\mu\alpha}) - \frac{1}{3}g^{\mu\nu}g^{\alpha\beta} + O(p)}{p^2 - m^2 - i\epsilon} e^{-ip(x-x')},$$

$$G_0^{\mu\nu\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{\frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\mu\alpha}) - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta} + O(p)}{p^2 - i\epsilon} e^{-ip(x-x')}. \quad (2.46)$$

Other models with truly infinity dimensions (see for example [17]) also experience problems with ghost fields. Therefore, Dvali, Gabadadze and Porrati introduced an extra term in the action [18], in which they described a model with a tensionless brane, so that they could obtain infinite bulk size. The action reads

$$S = -\frac{1}{\kappa_5^2}\int_{bulk}\sqrt{|g|}R - \frac{1}{\kappa_4^2}\int_{brane}\sqrt{|\gamma|}R^{(4)}, \quad (2.47)$$

omitting additional matter terms on the brane. The term that induces four-dimensional gravity on the brane need not necessarily be there at the classical level, but could be generated by one-loop quantum interactions (see figure 2.4).

The additional boundary term gives the massless graviton in the brane action the right structure. Nevertheless, there is still the five-dimensional graviton that brings in too many degrees of freedom. We can rewrite this in terms of scalar-tensor gravity theory and calculate the gravitational potential on the brane. This shows the right behavior at small distances. At larger distances a repulsive logarithmic term contributes and ultimately the potential shows the five-dimensional $1/r^2$ behavior. This model suffers from problems with the additional scalar mode. Although there is a four-dimensional Ricci scalar on the brane, an extra degree of freedom will arise due to the five-dimensional gravitational term in the bulk. Despite some successes, because the model is very complicated it is often problematic to solve the Einstein equations in higher-dimensional spacetime the viability of this model is in question.

## 2.5 Einstein equations on the 3-brane

Henceforth, we will work in the framework of Randall-Sundrum models exclusively. Under the notion Randall-Sundrum brane world we understand a wide range of models that take into account the gravitational effects of the brane. We will consider non-trivial background metrics in the bulk due to matter fields on the brane and in the bulk. We focus mostly on codimension one branes, although the scenarios can be generalized to objects with higher codimension (see for example [19], [20] and [21]).

It is very illustrative to look at the effective Einstein equations on the brane. These follow from the full set of equations by means of the Gauß-Codazzi relations. We will see, although an effectively four-dimensional description is not suitable in general, the effective Einstein equations on the brane usefully stage the effects of higher-dimensional spacetime from the perspective of a brane observer. Shiromizu, Maeda and Sasaki give a very good and detailed derivation of the equations in [22]. Let us start from the full Einstein equations in five dimensions,

$$ {}^{(5)}R_{AB} - \frac{1}{2} g_{AB} {}^{(5)}R = \kappa_5^2 T_{AB} \tag{2.48} $$

and apply the Gauß equation, which links the five-dimensional Riemann tensor with the four-dimensional one on the brane

$$ {}^{(4)}R^{A}{}_{BCD} = {}^{(5)}R^{K}{}_{LMN} q^{A}{}_{K} q^{L}{}_{B} q^{M}{}_{C} q^{N}{}_{D} + K^{A}{}_{C} K_{BD} - K^{A}{}_{D} K_{BC}. \tag{2.49} $$

The four-dimensional Riemann tensor finally reads as the remaining components on the brane ${}^{(4)}R^{\alpha}{}_{\beta\gamma\delta} \equiv \left({}^{(4)}R^{A}{}_{BCD}\right)^{\alpha}{}_{\beta\gamma\delta}$. In this derivation we understand all four-dimensional quantities that derive from the full spacetime tensors analogously. Again, $K_{AB}$ is the extrinsic curvature on the brane as given in equation (2.24). and $q_{AB}$ denotes the projected metric tensor. Additionally, we have the Codazzi equation

$$D_N K^N_M - D_M K = {}^{(5)}R_{RS} n^S q^R_M \tag{2.50}$$

with $n_A$ the unit normal to the brane and $D_A$ the covariant derivative with respect to the projected metric $q_{AB} \equiv g_{AB} + n_A n_B$. After taking the trace over the first and third index in the Gauß equation, we can write down the Einstein tensor on the brane

$$\begin{aligned} {}^{(4)}G_{MN} &= \left({}^{(5)}R_{RS} - \frac{1}{2} g_{RS} {}^{(5)}R\right) q^R_M q^S_N \\ &\quad + {}^{(5)}R_{RS} n^R n^S q_{MN} + K K_{MN} - K^R_M K_{NS} \\ &\quad - \frac{1}{2} q_{MN} \left(K^2 - K^{AB} K_{AB}\right) - D_{MN}, \end{aligned} \tag{2.51}$$

where $D_{MN} \equiv {}^{(5)}R^A_{BRS} n_A n^R q^B_M q^S_N$. Furthermore, we can always decompose the Riemann tensor in $n$ dimensions into the Weyl tensor, Ricci tensor and Ricci scalar: ${}^{(n)}R_{MANB} = {}^{(n)}C_{ABMN} + \frac{2}{n-2}\left({}^{(n)}R_{A[M} g_{N]B} - {}^{(n)}R_{B[M} g_{N]A}\right) - \frac{2}{(n-1)(n-2)} {}^{(n)}R g_{A[M} g_{N]B}$, where the square brackets stand for symmetrization. We end up with the effective Einstein equation

$$\begin{aligned} {}^{(4)}G_{MN} &= \frac{2\kappa_5^2}{3}\left(T_{RS} q^R_M q^S_N + \left(T_{RS} n^R n^S - \frac{1}{4} T\right) q_{MN}\right) \\ &\quad + K K_{MN} - K^S_M K_{NS} - \frac{1}{2} q_{MN} \left(K^2 - K^{AB} K_{AB}\right) \\ &\quad - \underbrace{{}^{(4)}C^A_{BRS} n_A n^R q^B_M q^S_N}_{\equiv E_{MN}}, \end{aligned} \tag{2.52}$$

where $E_{MN}$ is the electrical part of the Weyl tensor projected onto the brane. Substituting the full Einstein equations (2.48) in the Codazzi equation (2.50), we get

$$D_N K^N_M - D_M K = \kappa_5^2 T_{RS} n^S q^R_M. \tag{2.53}$$

In the following we will make further assumptions which will allow us to interpret equation (2.52) on the brane. As in the original RS scenario, we choose $\mathbb{Z}_2$ symmetry. Furthermore, we adopt Gaussian normal coordinates, in which the brane is located at a fixed point $y = 0$, and where the extra dimension is in $y$-direction: $n_A dx^A = dy$ and by this $ds^2 = q_{AB} dx^A dx^B - dy^2$. Going back to a scenario with matter only on the brane and only a cosmological constant in the bulk, we can write the stress-energy tensor

$$T_{AB} = \Lambda g_{AB} + \delta(y)(\underbrace{\lambda q_{AB} - \tau_{AB}}_{\equiv S_{AB}}), \tag{2.54}$$

where the latter part $S_{AB}$ is located on the brane and is perpendicular to the unit normal $S_{AB} n^A = 0$. Then we obtain the effective Einstein equation on the brane supplemented by the Israel junction conditions, in which the extrinsic curvature is determined by brane matter:

$$\begin{aligned} {}^{(4)}G_{\mu\nu} &= -\Lambda_4 q_{\mu\nu} + 8\pi G_{(4)} \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}, \\ [K_{\mu\nu}]_\pm &= -\kappa_5^2 \left( S_{\mu\nu} - \frac{1}{3} S q_{\mu\nu} \right) \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} \Lambda_4 &= \frac{1}{2}\kappa_5^2 \left( \Lambda + \frac{1}{6}\kappa_5^2 \lambda^2 \right) \\ G_{(4)} &= \frac{\kappa_5^4 \lambda}{48\pi} \\ \pi_{\mu\nu} &= -\frac{1}{4}\tau_{\mu\alpha}\tau_\nu{}^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu}\tau^2. \end{aligned} \tag{2.56}$$

At first, we see that the effective cosmological constant $\Lambda_4$ on the brane is given by a combination of the brane tension and the bulk cosmological constant. In order to obtain viable solutions, we need fine-tuning, by which brane worlds open up opportunities for new approaches to the cosmological constant problem. Furthermore, we see that Newton's constant $G_{(4)}$ on the brane depends linearly on the tension $\lambda$. Therefore, it is important that we live on the positive tension brane, otherwise we would observe anti-gravity, which would have been the case in the original RS I model [2]. There are two unusual terms in the equation. The term $\pi_{\mu\nu}$, which is quadratic in the stress-energy-tensor $\tau_{\mu\nu}$ on the brane can change the cosmological evolution significantly, as we will see in detail in the chapter on brane cosmology.

The first three terms on the right hand side of equation (2.56) are local and there is no problem with the four dimensional point of view. But $E_{\mu\nu}$ which deduces from the Weyl tensor in the bulk carries additional degrees of freedom, that do not show up in the usual four-dimensional Einstein equations. There is no equation to specify the additional degrees of freedom and, because of that, the system of equations (2.55) is not closed. Therefore, the effective equations are only good for a pure anti de-Sitter bulk, where the Weyl tensor vanishes completely. In all other cases, we will have to consider the full spacetime Einstein equations with boundary conditions on the branes. As a non-local term that carries bulk information only, we have to evaluate $E_{\mu\nu}$ close to, but not on, the brane. Although we cannot specify the projected part of the Weyl tensor in the $4d$ framework completely, there is still a constraint due to the Bianchi identity on the brane, which - as an algebraic relation - still holds. Taking the equation

$$D_\nu K^\nu_{\ \mu} - D_\mu K \propto \ D_\nu \tau^\nu_{\ \mu} = 0 \tag{2.57}$$

and applying the operator $D^\mu$ to the effective Einstein equations, we obtain

$$D^\mu E_{\mu\nu} = \kappa_5^4 D^\mu \pi_{\mu\nu}. \tag{2.58}$$

Astonishingly, the projected part of the Weyl tensor $E_{\mu\nu}$ is constrained by brane matter. We can further split $E_{\mu\nu}$ into a transverse-traceless and a longitudinal part. The longitudinal part is completely specified by the matter on the brane. The first part however, which corresponds to perturbations due to gravitational waves in the bulk, we can only specify in the full spacetime set of equations. Since we will consider non anti de-Sitter bulk background geometries as well as perturbations, we will henceforth work with the complete set of equations in the bulk.

## 2.6 Brane worlds and string theory

Initially, there was much talk going on about string inspired brane worlds. The brane worlds we focus on in this thesis are mostly not related to string or M-theory. Nevertheless, the idea that branes, particularly d-branes, show up in string theory has helped a lot in the acceptance of models with large extra dimensions, for which brane-like objects are crucial. We can construct string theoretical models, where d-branes act as the endpoints of open strings. This can be interpreted such that the endpoints of open strings, which are confined to branes, represent the Standard Model fields. On the other hand, closed strings that have the properties of the usual spin-2 graviton can go off the brane and propagate in the bulk. Often, stability of d-branes is an issue in string theory, therefore people mostly focus on BPS branes (see for instance [23]).

Of course, the five-dimensional view that most authors take can only represent an effective theory again. The missing dimensions for ten-dimensional string theory or respectively eleven-dimensional M-theory must be compactified on small scales according to the Kaluza-Klein procedure. In general, this will render so-called moduli fields in the effective theory which have to be treated as bulk scalar fields in brane worlds. We will consider such scenarios in chapter 3. Furthermore, the resulting brane worlds will, as effective models, in general include modifications of the gravitational theory, for example higher-derivative terms.

Therefore, we will treat modified gravity theories, whose Lagrangians include terms of type $f(R)$, in the course of this report. In spite of all these connections, the brane worlds we talk about are ad-hoc assumptions and we cannot derive them directly and consistently from fundamental theories. Also it is not clear so far, if there exist infinitely extended brane-like objects in string theory. Furthermore, d-branes have tension in string theory, but the quantitative strength of the tension is not specified either. Therefore, we will treat the brane world pictures in the following chapters as phenomenological models solely.

# Chapter 3

# Bulk scalar fields

So far we have only analyzed brane scenarios with an empty bulk. Compactifications of spacetimes with more than five dimensions to five-dimensional models will give rise to bulk scalar fields in general. In string theory these are known as dilaton fields or as moduli fields in the case of compactification on Calabi-Yau-spaces. There is also, a priori, no reason to exclude bulk scalar fields in general, although we have to take into account their effect on energy-momentum conservation on the brane. Moreover, we have to tackle the issue of stability of brane world scenarios with two branes. We will see that this can be done with bulk scalar fields.

The original Randall-Sundrum (RS) model consists of a Minkowski brane in a pure Anti-de-Sitter (AdS) bulk. This scenario is inappropriate for a cosmological description of our non-static, expanding universe and therefore has to be extended. In this chapter, we will mostly present static solutions and we address the topic of brane world cosmology in the following chapter. A further motivation for the study of scalar fields in brane world models is to evade the fine-tuning problem, which arises in both original RS models. Although there is no general solution, we will see examples that allow for satisfactory solutions without a bulk cosmological constant. In this chapter, we will focus on background solutions, perturbations will be included in later chapters.

## 3.1 Set-up and equations of motion

Here, we derive the equations of motion for an $n$-dimensional spacetime $\mathcal{M}_n$ with an $(n-1)$-dimensional embedded hypersurface $\mathcal{M}_{n-1}$. In the end, we will apply the results to the usual five-dimensional brane world set-up.

Consider a scalar field $\phi$ that lives in full spacetime $\mathcal{M}_n$ and an embedded brane $\mathcal{M}_{n-1}$ with matter confined to it. The action will take the form

$$S = \int d^n x \sqrt{|g|} \left( -\frac{R}{\kappa_n^2} + \mathcal{L}_B \right) + \int d^{n-1}\sigma \sqrt{|\gamma|} \left( \mathcal{L}_b + \frac{[K]_\pm}{\kappa_n^2} \right). \tag{3.1}$$

The Lagrangian density $\mathcal{L}_b$ on the brane is a function of the induced metric $\gamma_{\mu\nu}$ and possibly other matter fields $\psi_i$ that are confined to the brane. Furthermore, it includes the coupling of the bulk scalar $\phi$ to brane matter, $\mathcal{L}_b = \mathcal{L}_b(\phi, \gamma_{\mu\nu}, \psi_i)$. $\mathcal{L}_B$, on the other hand, denotes the Lagrangian density of the bulk scalar field and reads

$$\mathcal{L}_B = \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - V(\phi), \tag{3.2}$$

where $V(\phi)$ is a yet unspecified scalar field potential. Again, we have included the Gibbons-Hawking term in the action for consistency. The action $S$ has to be varied with respect to the metric as well as with respect to the scalar field $\phi$ separately in order to obtain the complete set of equations in the bulk. Variation with respect to the metric gives the Einstein equation in full spacetime and the usual Israel junction conditions

$$\begin{aligned} R_{AB} - \frac{1}{2} g_{AB} R &= \kappa_n^2 T_{AB}, \\ [K_{AB} - q_{AB} K]_\pm &= -\kappa_n^2 \tau_{AB}, \end{aligned} \tag{3.3}$$

where the energy-momentum tensor $T_{AB}$ is defined in terms of the bulk Lagrangian $\mathcal{L}_B$:

$$T_{AB} = \frac{2}{\sqrt{|g|}} \frac{\delta \left( \sqrt{|g|} \mathcal{L}_B \right)}{\delta g^{AB}}. \tag{3.4}$$

The energy-momentum tensor $\tau_{AB}$ is defined in the same way from the brane Lagrangian density $\mathcal{L}_b$ by variation with respect to the projected metric on the brane. Variation with respect to the bulk scalar field $\phi$ yields the Klein-Gordon equation in the bulk and a new boundary condition for the scalar field. The equations read

$$\begin{aligned} \nabla^2 \phi &= -\frac{dV}{d\phi}, \\ n^A \left[\partial_A \phi\right]_{\pm} &= \frac{\delta \mathcal{L}_b}{\delta \phi}. \end{aligned} \tag{3.5}$$

The boundary condition for the scalar field can be calculated explicitly only if the coupling of the bulk scalar field to matter fields that are confined on the brane is known.

Of course, one can think of a wide range of different coupling schemes between bulk and brane matter, since the nature of the bulk matter is completely unobserved. Nevertheless, we will focus on two different examples in the following. First, we consider a scalar field that is minimally coupled to the metric. We assume that the metric $\tilde{g}_{AB}$ is conformally related to the original metric by

$$\tilde{g}_{AB} = e^{2k(\phi)} g_{AB} \tag{3.6}$$

(see for example [24]). Then, variation of the brane Lagrangian with respect to the bulk field $\phi$ yields

$$\frac{\delta\left(|\gamma| \mathcal{L}_b\right)}{\delta \phi} = \frac{\partial \tilde{\gamma}^{\mu\nu}}{\partial \phi} \frac{\delta\left(|\gamma| \mathcal{L}_b\right)}{\delta \tilde{\gamma}^{\mu\nu}} = -\sqrt{|\gamma|} k'(\phi) \tau, \tag{3.7}$$

where $\tau$ denotes the trace of the brane stress-energy tensor. Then we have in the case of conformal coupling the following boundary condition

$$n^A \left[\partial_A \phi\right]_{\pm} = -k'(\phi) \tau. \tag{3.8}$$

Another possible coupling for the scalar field to matter is the so-called volume-element coupling. In this case, the dependence of the Lagrangian density $\mathcal{L}_b$ on $\phi$ can be completely split off as an overall function:

$$\mathcal{L}_b = e^{4\kappa_n \beta \phi} \tilde{\mathcal{L}}_b(\gamma_{\mu\nu}, \psi_i). \tag{3.9}$$

We have to bear in mind that volume-element coupling can lead to ambiguities, for instance, when matter on the brane is described as a perfect fluid. Since a perfect fluid is only an effective description of the matter fields on the brane, we can write down different Lagrangian densities that all lead to the same physical energy-momentum tensor. Therefore, (3.9) can effectively result in a number of different couplings between matter and the bulk field. Here, we will not discuss this in more detail, instead we will see in later chapters that we do not need volume-element coupling for our purposes.

## 3.2 Stabilization of brane worlds

We know that the original Randall-Sundrum scenario [2] is fine-tuned and does not include a mechanism to ensure the stability of the system. We also learned that the masses of elementary particles, which we observe on the brane, depend on the size $l$ of the extra dimensions. Now, consider a massless scalar perturbation in the bulk which is inherent in these models and is sometimes also referred to as a radion. The original Randall-Sundrum metric

$$ds^2 = e^{-2kl|y|}g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu - l^2 dy^2 \tag{3.10}$$

changes to

$$ds^2 = e^{-2kT(x^\alpha)|y|}g_{\mu\nu}(x^\alpha)dx^\mu dx^\nu - T^2(x^\alpha)dy^2. \tag{3.11}$$

Since the four-dimensional Planck mass $M_{pl}$ depends on the size of the extra dimension, which then can be interpreted as the vacuum expectation value of the radion field $< T(x^\alpha) >$, it becomes clear that a constant brane distance is crucial for all two-brane scenarios. Performing a Kaluza-Klein reduction of the full spacetime action for this metric yields the following action

$$S = 2M_5^3 \int d^4x dy \sqrt{|g|} e^{-2kT|y|} \left[6k|y|(\partial T)^2 - 6k^2|y|^2 T(\partial T)^2 + TR\right], \tag{3.12}$$

where $g$ denotes the determinant of the four-dimensional metric $g_{\mu\nu}$ and $R$ is the corresponding four-dimensional Ricci scalar. We can perform the $y$-integration explicitly and this leads to a cancellation of the first two terms in the action:

$$S = \frac{2M_5^3}{k} \int d^4x \sqrt{|g|} \left(1 - e^{-2k\pi T}\right) R + \frac{12M_5^3}{k} \int d^4x \sqrt{|g|} \left(\partial e^{-k\pi T}\right)^2 . \quad (3.13)$$

Defining $\phi \equiv \sqrt{24M_5^3/k}e^{-k\pi T}$, we end up with the action for a massless scalar field in four dimensions that is non-minimally coupled to the Ricci scalar,

$$S = \frac{M_5^3}{k} \int d^4x \sqrt{|g|} \left(1 - \frac{\phi^2}{24M^3/k}\right) R + \frac{1}{2} \int d^4x \sqrt{|g|} (\partial\phi)^2. \quad (3.14)$$

This action does not contain any mechanism for the massless, free scalar field to stabilize the size of the extra dimension. Furthermore, as long as we try to solve the hierarchy problem simultaneously the coupling is not of order the Planck scale, as one would think naively, but of order the weak scale. This behavior is at odds with observation.

The solution to this problem is the introduction of a massive scalar field, which lives in the bulk, the so-called modulus. Goldberger and Wise were the first to propose a stabilized model in [25] and [26], based on [27], in which bulk scalar fields were introduced (see also [28] for stabilization of inflating brane worlds). Their bulk scalar field has a small mass, therefore the resulting potential of the modulus field is relatively flat near its minimum for vacuum expectation values that solve the problems of the model. This could also lead to a modulus field that is lighter than the first excited massive Kaluza-Klein states and it could show up as the first observational signature of a higher-dimensional spacetime in collider experiments.

To show the stabilization mechanism in more detail, we take a scalar field $\psi$ with the following action

$$S = \frac{1}{2} \int d^4x \int_{-\pi}^{\pi} dy \sqrt{|g_{AB}|} \left(g^{AB} \partial_A \psi \partial_B \psi - m^2 \psi^2\right) \quad (3.15)$$

in addition to the gravitational action in the bulk. The metric $g_{AB}$ is given in (3.10). Furthermore, we need couplings of the scalar field $\psi$ to the boundary terms on the hidden and visible branes, located at $y = 0$ and $y = \pi$ respectively

$$S_{v/h} = -\int d^4x \int \sqrt{|g_{v/h}|} \lambda_{v/h} \left(\psi^2 - v_{v/h}^2\right)^2 , \quad (3.16)$$

where $g_{v/h}$ denotes the determinants of the metric on the visible and hidden brane respectively. For simplicity, the backreaction of the scalar field $\psi$ on the metric is neglected. We will see self-consistent solutions for brane worlds with bulk scalar fields in the next section. The terms on the branes make the vacuum expectation value of $\psi$ $y$-dependent, which we find by solving the classical equations of motion. We use the boundary conditions on both branes to specify the solution completely.

In order to obtain a more demonstrative result, we write down the solutions only in the limit of large brane tensions $\lambda_{v/h}$, although, in principle, the equations can be solved in full generality. We also take the limit for large $kl$, which we want of the order $\mathcal{O}(10)$ anyway. Finally, we obtain the effective potential that depends on the inter-brane distance $l$:

$$\begin{aligned} V_\psi(l) &= k\epsilon v_h^2 + 4ke^{-4kl\pi}\left(v_v - v_h e^{-\epsilon kl\pi}\right)^2 \\ &\quad - k\epsilon v_h e^{-(4+\epsilon)kl\pi}\left(2v_v - v_h e^{-\epsilon kl\pi}\right), \end{aligned} \tag{3.17}$$

where we have defined $\epsilon \equiv \frac{m^2}{4k^2}$. Neglecting terms proportional to $\epsilon$, because we want $m/k \ll 1$, but keeping terms proportional to $\epsilon kl$, we see immediately that the potential has a minimum at

$$kl = \left(\frac{4}{\pi}\right)\frac{k^2}{m^2}\ln\left(\frac{v_h}{v_v}\right). \tag{3.18}$$

Assuming the ratio $\frac{v_v}{v_h}$ to be of order unity, we only need $\frac{m^2}{k^2}$ of order $\frac{1}{10}$ to get $kl \sim 10$, which is the right order of magnitude for the extra dimension. Although the complete set-up seems somewhat artificial, we managed to stabilize the size of the extra dimension with the help of an additional scalar field. We will present more realistic solutions that also include the backreaction of the scalar field on the metric in the next section.

One can also check that we do not need unnaturally high values for the brane tensions $\lambda_{v/h}$ by computing the $\frac{1}{\lambda}$ corrections. The leading corrections to the potential have the same shape as the leading order of the potential itself and therefore does not change the location of the minimum significantly. Thus, even lower values of the tension do not pose a problem. There is no extreme fine-tuning involved, because the ratios for the tensions as well as for $m/k$ are of the order of one to ten.

## 3.3 Bulk scalar fields with superpotentials

So far we have not considered self-consistent solutions in which we incorporate the backreaction effects of the bulk scalar fields on the spacetime geometry. In most cases it is extremely difficult to find solutions for bulk scalar fields because the equations do not decouple as is often the case in homogeneous four-dimensional models. In the following, we will present a very useful solution-generating technique, based on ideas in supersymmetry.

This technique was first applied to the brane world picture by DeWolfe Freedman, Gubser and Karch in [29], where they considered scalar field potentials $V$ that derive from superpotentials. This allows for an easy splitting of the second order equations of motion into a set of only partially coupled first order equations. Let us start with the following set-up

$$\begin{aligned} S &= \int_{\mathcal{M}} d^n x \sqrt{|g|} \left( -\frac{1}{\kappa_n^2} R + \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - V(\phi) \right) \\ &\quad + \sum_k \int_{\partial \mathcal{M}_k} d^{n-1} \sqrt{|\gamma_k|} \left( \mathcal{L}_b^{(k)}(\phi, \gamma_{\mu\nu}, \psi_i) + \frac{1}{\kappa_n^2} [K]_{\pm} \right), \end{aligned} \tag{3.19}$$

where $\mathcal{M}$ denotes full spacetime and $\partial \mathcal{M}_k$ the respective boundaries with codimension one. For illustrative purposes we will henceforth neglect additional matter fields $\psi_i$ on the brane so that we have a Lagrangian that contains a tension coupled to the scalar field $\phi$ only: $\mathcal{L}_b^{(k)} = \lambda_k(\phi)$. Moreover, we will focus on a scenario that has $\mathbb{Z}_2$-symmetric behavior across the lower dimensional boundaries. To present a simpler model, we will work in a five-dimensional frame and also with a bulk scalar field that is only $y$-dependent. Using the following ansatz for the metric

$$ds^2 = e^{2X(y)} \eta_{\mu\nu} - dy^2, \tag{3.20}$$

it is straightforward to deduce the equations of motion for the metric functions as well as for the scalar field:

$$
\begin{aligned}
\phi'' + 4X'\phi' &= \frac{dV(\phi)}{d\phi} + \sum_k \frac{d\lambda_k(\phi)}{d\phi}\delta(y-y_k), \\
X'' &= -\frac{2}{3}\phi'^2 - \frac{2}{3}\sum_k \lambda_k(\phi)\delta(y-y_k), \\
X'^2 &= -\frac{1}{3}V(\phi) + \frac{1}{6}\phi'^2 \,. \qquad (3.21)
\end{aligned}
$$

Again, the prime denotes derivation with respect to the $y$-coordinate. The last equation is a constraint which follows identically, if the first two equations are satisfied. It can be regarded as the energy-conservation equation. Integrating the first two equations gives junction conditions for the metric and the scalar field derivatives on the boundaries:

$$
[X']_{\pm}\Big|_{y_k} = -\frac{2}{3}\lambda_k(\phi(y_k)) \quad \text{and} \quad [\phi']_{\pm}\Big|_{y_k} = 2\frac{d\lambda_k}{d\phi}(\phi(y_k)) \qquad (3.22)
$$

### 3.3.1 Solutions for superpotentials

The idea is to find a potential that allows for the separation of the set of equations (3.21) into a set of three first order equations. Suppose that we have a potential $V(\phi)$ that is derived from a superpotential $W(\phi)$

$$
V(\phi) = \frac{1}{8}\left(\frac{dW(\phi)}{d\phi}\right)^2 - \frac{1}{6}W(\phi)^2, \qquad (3.23)
$$

where we assumed five-dimensional spacetime $\mathcal{M}$. In this case, it is easy to check that

$$
\begin{aligned}
\phi' &= \frac{1}{2}\frac{dW(\phi)}{d\phi} \\
X' &= -\frac{1}{6}W(\phi) \qquad (3.24)
\end{aligned}
$$

is a solution to the system of equations (3.21). The junction conditions then read

$$
\begin{aligned}
\frac{1}{2}\left[W(\phi)\right]_{\pm}\Big|_{y_k} &= \lambda_k\left(\phi(y_k)\right), \\
\frac{1}{2}\left[\frac{dW(\phi)}{d\phi}\right]_{\pm}\Big|_{y_k} &= \frac{d\lambda_k}{d\phi}\left(\phi(y_k)\right).
\end{aligned}
\tag{3.25}
$$

We have to mention that this method works only for a single scalar field, otherwise we would end up with a set of complicated partial differential equations. We should also point out that this method does not allow for the solution of general quadratic potentials. Although we could choose a superpotential $W(\phi) = \alpha\phi + \beta$, with $\alpha$ and $\beta$ arbitrary real constants, we still have to fulfill the junction conditions and these constrain the two parameters $\alpha$ and $\beta$.

The fine-tuning problem, which arises as soon as we want a flat brane as in the Randall-Sundrum models, is not evaded in this scenario. The junction conditions fix the constants of the bulk solution, but the system of equations can be solved for arbitrary values of the brane tension.

With the help of this method some cosmological solutions have also been found. We will come back to it in the following chapter about brane world cosmology, because this method also generates solutions for non-critical branes, that is branes in de-Sitter or anti de-Sitter stage. Moreover, we will resort to this type of potentials in our work on higher-derivative gravity. There, we will find a solution that allows for an empty bulk with a gravitating brane in it.

Most solutions with scalar fields have singularities in the bulk and the superpotential solutions are no exception in this respect. Usually, one circumvents this problem by placing a second brane in the bulk that shields the singularity from the part of the universe in which we live. The tension of the so-called regulator brane is chosen such that the spacetime in between the two branes remains exactly the same as without the second brane.

## 3.4 Bulk fields with quadratic potentials

To solve for quadratic potentials we rewrite the equations of motion by substituting for the warp factor $a(y) \equiv e^{X(y)}$. Then the equations of motion (3.21) are subject to the following boundary conditions

$$\partial_y X\Big|_{y=0} = -\frac{1}{6}\,e^{-2\frac{\phi(0)}{\sqrt{3}}}, \qquad \partial_y \phi\Big|_{y=0} = -\frac{1}{\sqrt{3}}\,e^{-2\frac{\phi(0)}{\sqrt{3}}}. \tag{3.26}$$

We have made all quantities dimensionless by rescaling in terms of a characteristic length $\lambda^{-1}(\phi = 0)M_5^3$ and a characteristic scalar field value $M_5^{3/2}$. It follows from the equations of motion that $V(\phi(0)) = 0$. Now, we consider quadratic potentials of the simple form

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{3.27}$$

where $m^2$ can also be negative. Then $\phi(0) = 0$ and the boundary conditions become even simpler. We consider these type of potentials, because there is an equivalence between theories with scalar fields and higher-derivative gravity theories, which we will explain in chapter 5. Indeed, as long as the values of the scalar field are small enough, the quadratic potential model is related to the following type of higher-derivative gravity

$$f(R) = R + \alpha R^2 + \dots . \tag{3.28}$$

The complete equivalence, that is an $f(R)$ theory without terms of order $\mathcal{O}(R^3)$ and higher would demand an exponential scalar field potential, which is problematic as we will also see in chapter 5. Then, the significance of the mass parameter $m^2$ of our quadratic potential is that it is related to the coefficient $\alpha$ of the $R^2$-term in higher-derivative theory of gravity by

$$m^2 = -\frac{3}{16\alpha}. \tag{3.29}$$

### 3.4.1 Solutions for small mass parameters

Let us now investigate the case of small masses $|m^2|$ first. In this limit we take $\epsilon = m^2$ as our small parameter and consider solutions to first order in $\epsilon$ (see [30]). We use the so-called Lindstedt-Poincaré method with the strained variable $x = (1 + \omega\epsilon)^{-1}y$ and obtain the following equations of motion

$$
\begin{aligned}
-3X_{,xx} - 6X_{,x}^2 &= \frac{1}{2}\phi_{,x}^2 - \frac{3}{32}(1+\omega\epsilon)^2\epsilon\phi^2, \\
6X_{,x}^2 &= \frac{1}{2}\phi_{,x}^2 + \frac{3}{32}(1+\omega\epsilon)^2\epsilon\phi^2, \\
\phi_{,xx} + 4X_{,x}\phi_{,x} &= -\frac{3}{16}(1+\omega\epsilon)^2\epsilon\phi. \qquad (3.30)
\end{aligned}
$$

The boundary conditions imply $X_{,x}(0) = \frac{1}{6}(1+\omega\epsilon)$ and $\phi_{,x}(0) = -\frac{1}{\sqrt{3}}(1+\omega\epsilon)$, moreover, we are free to choose $X(0) = 0$. Then we take the ansatz $X = X_0(x)+\epsilon X_1(x)$ and $\phi = \phi_0(x)+\epsilon\phi_1(x)$, the boundary conditions are such that the values of the perturbative functions vanish at the brane. The derivatives of the functions at the brane position are $X_0'(0) = -\frac{1}{6} = \frac{X_1'(0)}{\omega}$ and similarly $\phi_0'(0) = -\frac{1}{\sqrt{3}} = \frac{\phi_1'(0)}{\omega}$. For the equations in zeroth order we immediately obtain the solution

$$X_0 = \frac{1}{4}\ln\left(1 - \frac{2}{3}x\right) \quad \text{and} \quad \phi_0 = 2\sqrt{3}X_0. \qquad (3.31)$$

Substituting the results in the first order equation, we end up with the following relation

$$\phi_1 = 2\sqrt{3}X_1 + \frac{1}{2}\int_0^x dx' \frac{\phi_0^2}{\phi_0'}, \qquad (3.32)$$

which can be integrated explicitly. Then, we eliminate $\phi_1$ from the equations of motion to obtain

$$X_1'' + 8X_0'X_1' = -\frac{1}{3}\phi_0^2. \qquad (3.33)$$

This has the particular solution $X_1 = A + \frac{B}{1-2x/3}$ and the boundary conditions render restrictions for the coefficients $A = \frac{9}{64}+\frac{1}{4}\omega$ and $B = -\frac{1}{24}-\frac{1}{4}\omega$. We now argue that $B = 0$, because we expect $e^X$ to be dominated by the zero order term $e^{X_0} = \left(1 - \frac{2}{3}x\right)^{1/4}$. However, if $B$ was not zero, the term proportional to $\left(1 - \frac{2}{3}x\right)^{-1}$ would become dominant close to $x = \frac{3}{2}$. Therefore, we choose

$$\omega \equiv -\frac{1}{6}. \qquad (3.34)$$

Because $\omega$ is nonzero it causes a shift of the position of the singularity and the solution reads

$$X_0 = \frac{1}{4}\ln\left(1 - \frac{\frac{2}{3}y}{1-\frac{1}{6}\epsilon}\right). \tag{3.35}$$

These are the analytic solutions for quadratic potentials which we will use in chapter 7 as background for gravitational perturbations in order to derive the Newtonian limit on the brane for brane worlds with higher-derivative gravity.

This was checked against numerical results, which are in conformance with the analytical solution derived in this section. For large $m^2$ we have seen that there is no shift of the singularity, which is also in very good agreement with numerical calculations.

### 3.4.2 Large negative and positive masses

We perform a perturbative analysis for the case of large negative $m^2$ by introducing the small parameter $\epsilon = |m|^{-1}$. Because the $\epsilon \to 0$ limit is problematic, it is not clear a priori, what perturbative expansion will work. Let us take the equations of motion with the boundary conditions $X(0) = 0$, $X_{,x}(0) = \frac{1}{6}$ and $\phi_{,x}(0) = -\frac{1}{\sqrt{3}}$. From numerical computations it is apparent that $X \sim \frac{1}{2}\ln(\left(1-\frac{1}{3}y\right)$, but that the derivatives of $X$ contain highly oscillatory terms. Therefore, we take the ansatz to quadratic order in $\epsilon$

$$\begin{aligned} X &= \frac{1}{2}\ln\left(1-\frac{x}{3}\right) + \epsilon^2\left(1-\frac{x}{3}\right)^{-2} G\left(\frac{x}{\epsilon}\right), \\ \phi &= \left(1-\frac{x}{3}\right)^{-1}\left[\epsilon F\left(\frac{x}{\epsilon}\right) + \epsilon^2 H\left(\frac{x}{\epsilon}\right)\right] \end{aligned} \tag{3.36}$$

and the boundary conditions become $F(0) = G(0) = H(0) = 0$ and the derivatives at the brane read $F'(0) = -\frac{1}{\sqrt{3}}$ and $G'(0) = -\frac{1}{6}\omega = \frac{1}{2\sqrt{3}}H'(0)$, where prime stands for the derivative with respect to the argument $\frac{x}{\epsilon}$.

Now we expand the equations of motion in powers of $\epsilon$ and obtain at order $\epsilon^{-1}$

$$F'' = -\frac{3}{16}F \quad \Rightarrow \quad F = -\frac{4}{3}\sin\left(\frac{\sqrt{3}}{4}\frac{x}{\epsilon}\right). \tag{3.37}$$

Then, we can solve for the equations at order $\epsilon^0$. These relate the functions $G$ and $H$ to the already known function $F$, from which it is easy to deduce the solutions

$$\begin{aligned} G &= -\frac{1}{6}\omega\frac{x}{\epsilon} - \frac{4}{27}\sin^2\left(\frac{\sqrt{3}}{4}\frac{x}{\epsilon}\right), \\ H &= \frac{1}{\sqrt{3}}\omega\frac{x}{\epsilon} - \frac{8}{3}\omega\,\sin\left(\frac{\sqrt{3}}{4}\frac{x}{\epsilon}\right). \end{aligned} \tag{3.38}$$

The terms linear in $x$ are what is normally called a secular term in oscillatory solutions. Usually, we seek to eliminate such terms by choosing $\omega = 0$. There is another reason to expel these terms, namely that the secular terms are of the wrong order: they are meant to be $\epsilon^2$ terms. This simplifies our solutions, because $x = y$ and $H = 0$. Our final solutions for the scalar field and the warp factor are

$$\begin{aligned} X &= \frac{1}{2}\ln\left(1-\frac{y}{3}\right) - \frac{1}{36}\epsilon^2\left(1-\frac{y}{3}\right)^{-2}\sin^2\left(\frac{y}{\epsilon}\right), \\ \phi &= -\frac{1}{\sqrt{3}}\epsilon\left(1-\frac{y}{3}\right)^{-1}\sin\left(\frac{y}{\epsilon}\right). \end{aligned} \tag{3.39}$$

For large positive $m^2$ the solution goes along the same lines and we obtain exactly the same solutions with all sine functions replaced by hyperbolic sines. Nevertheless, this is problematic, because the hyperbolic geometric functions are not bounded, therefore we cannot trust perturbation theory. Comparison with numerical results also reveals the breakdown of the perturbative scheme in this case and the sigularity moves to $y = 0$. Therefore, we cannot use this solution for large positive $m^2$.

We have derived new solutions for quadratic scalar field potentials. In chapter five, we will show the connection between bulk scalar fields and models with higher-derivative gravity. We will make use of these results in chapter 7, where we perform a perturbative expansion to derive the low-energy limit and, by that, the corrections to the Newtonian potential in the case of higher-derivative gravity.

# Chapter 4

# Brane world cosmology

In the beginning, much of the popularity of brane worlds and particularly the Randall-Sundrum brane worlds stemmed from the fact that they reproduce conventional Friedman-Robertson-Walker cosmology at low energies without unphysical assumptions. Having established that gravity could be localized on the brane, it was realized that cosmological models would permit additional tests for the existance of large extra dimensions. Even if the fundamental scale of gravity is not low enough to be seen in future collider experiments, we might still get a grip on fundamental physics with the help of cosmological observations.

There were some early efforts to investigate special topics of cosmology, particularly inflation on the brane (confer for example [31] and [32]), but a relatively complete treatment of five-dimensional brane cosmology was worked out in 1999 by Binetruy, Deffayet and Langlois in [33] and [34] as well as by Ida in [35]. At about the same time Kaloper provided a thorough description of brane inflation [36], which spurred even more interest in the new physical models, because more realistic scenarios seemed to be at hand. The formalism was further elaborated in subsequent papers (see e.g. [37] and [38]). In the following sections we will develop the new cosmological picture for brane worlds.

In the introductory chapter on extra dimensions we derived the effective four-dimensional equations and mentioned the problematic projection of the Weyl tensor that is not specified completely in four dimensions (see page 32ff). Although this framework would be sufficient for the empty bulk case, we will immediately start with the full spacetime framework, since it will be necessary for a closed description in the case of bulk scalar fields as well as for perturbations.

## 4.1 Modified Friedmann equations

Let us set up a scenario with a single brane with tension $\lambda$ in a bulk with a cosmological constant $\Lambda$ and no additional fields. Since we want a cosmological scenario, that is our brane should be Friedman-Robertson-Walker and not Minkowski, we need a time-dependent metric. We are looking for homogeneous and isotropic solutions on the brane with an open, flat or closed universe respectively corresponding to $k = -1, 0, +1$. Therefore, we take the ansatz

$$ds^2 = n^2(t,y)dt^2 - a^2(t,y)d\Omega_k^2 - b^2(t,y)dy^2, \tag{4.1}$$

in which $d\Omega_k$ is the volume element for the 3-space. Furthermore, we have a stress-energy tensor $T^L_M$, which we split into two parts for the bulk $T^L_M|_{Bulk}$ and for the brane $T^L_M|_{brane}$. For notational simplicity, we include the tension and the cosmological constant in the respective stress-energy tensors. Without loss of generality, we place the brane at $y = 0$. Then the brane stress-energy-tensor reads

$$T^L_M\Big|_{brane} = \frac{\delta(y)}{b} diag(\rho_b, p_b, p_b, p_b, 0), \tag{4.2}$$

where $\rho_b$ is the sum of the brane tension $\lambda$ and the matter density $\rho$ on the brane. Of course, $\rho_b$, $\rho$ and $p_b$ are functions of time.

Substituting the metric ansatz into the Israel junction conditions yields - in the case of $\mathbb{Z}_2$-symmetry that we assume here - the values for the derivatives of the metric functions at the brane position

$$\frac{[n']_\pm}{n_0 b_0} = \frac{\kappa_5^2}{3}\left(3p_b + 2\rho_b\right) \quad \text{and} \quad \frac{[a']_\pm}{a_0 b_0} = -\frac{\kappa_5^2}{3}\rho_b. \tag{4.3}$$

The subscript zero denotes the value of the functions on the brane, e.g. $n_0(t) \equiv n(y = 0, t)$. We still have the freedom to choose $b_0(t) = const.$.

Integrating the equations of motion and applying the junction conditions to get the quantities on the brane gives the first integral [33, 34]

$$\frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa_5^4}{36}\rho_b^2 - \frac{k}{a_0^2} + \frac{\kappa_5^2\Lambda}{6} + \frac{\mathcal{C}}{a_0^4}, \tag{4.4}$$

Here and in the following, we use the notation $\dot{f} \equiv \frac{\partial f}{\partial t}$. The constant of integration $\mathcal{C}$ results from the projected Weyl tensor. It can be interpreted in terms of the mass of a black hole in the bulk. The time variable has been chosen so that $n_0(t) = 1$. Then $H_0 \equiv \frac{\dot{a}_0}{a_0}$ is the Hubble parameter on the brane, and we can directly compare equation (4.4) to the standard Friedman equation in four-dimensional cosmology:

$$H^2 = \frac{8\pi G_4}{3}\rho - \frac{k}{a^2}. \tag{4.5}$$

We immediately see the difference in the dependence on the matter density, besides there is an additional term in brane cosmology that acts like radiation. This term is often referred to as dark radiation and vanishes identically for AdS space in the bulk. Hence, in general, we have an expansion behavior of the universe that differs from standard cosmology.

On the other hand, the energy conservation still holds on the brane, because no matter can go off the brane, nor is there a varying coupling to bulk quantities. From the contracted Bianchi identities $\nabla^A G_{AB} = 0$ we obtain

$$\dot{\rho}_b + 3H_0(\rho_b + p_b) = 0, \tag{4.6}$$

which is the same as in standard cosmology, since the tension remains constant and $\rho_b + p_b = \rho + p$.

## 4.2 Cosmological RS brane world solutions

### 4.2.1 Bulk solution

Solutions can be found with relative ease for Gaussian normal coordinates, since $b(t, y) = 1$. In that case we can integrate the $[0, 5]$-component of the Einstein equations, where we find

$$\frac{\dot{a}}{n} = \gamma(t), \tag{4.7}$$

with $\gamma(t)$ being a function of time only. The first integral of the $[0, 0]$-component implies the following first order equation

$$(aa')^2 - \gamma^2 a^2 - ka^2 + \frac{\kappa_5^2 \Lambda}{6} a^4 + \mathcal{C} = 0, \tag{4.8}$$

where for the right sign of gravity the bulk cosmological constant has to be negative. Then we can integrate this equation and obtain in the case of a vanishing constant $\mathcal{C}$

$$a(t, y) = a_0(t)[\cosh(\mu y) - \eta \sinh(\mu y)] \tag{4.9}$$

with $\mu = -\frac{\kappa_5^2 \Lambda}{6}$ and $\eta = \frac{\kappa_5 \rho_b}{\sqrt{-6\Lambda}}$ related to the brane and bulk vacuum energy densities and, as before, we chose $n_0(t) = 1$. Now we can specify $n(t, y)$ with the help of equation (4.7) and obtain

$$n(t, y) = \cosh(\mu y) - \left(\eta + \frac{\dot{\eta}}{H_0}\right) \sinh(\mu |y|). \tag{4.10}$$

If we put the bulk cosmological constant to zero, we end up with functions $a$ and $n$ that are only linear in $y$ and have a kink at the brane position. So far, the result is not very instructive for the situation on the brane.

### 4.2.2 Evolution on the brane

For a vanishing bulk cosmological constant $\Lambda$ and a flat 3-space on the brane the equations can be solved directly. To compare the behavior of fields in an expanding universe we assume an equation of state

$$p_b = w \rho_b \tag{4.11}$$

for constant $w$. Then the scale factor on the brane is

$$a_0(t) \propto t^{\frac{1}{3(1+w)}}. \tag{4.12}$$

This is different from the solution to the usual four-dimensional Friedmann equations, where $a \propto t^{\frac{2}{3(1+w)}}$. In the case of ultra-relativistic fields, $w = \frac{1}{3}$, this would cause the scale factor $a \sim t^{1/4}$ instead of $t^{1/2}$. This, of course, is not acceptable, since it would destroy the standard picture of nucleosynthesis, which crucially depends on the balance between reaction rates and the expansion rate of the universe.

Let us now consider the more general behavior with a cosmological constant $\Lambda \neq 0$. For this purpose, we split up the energy density on the brane $\rho_b$ in the matter density $\rho$ and the tension $\lambda$. Substituting this into the Friedmann equation on the brane (4.4) yields

$$H_0^2 = \frac{\kappa_5^4}{36}\lambda^2 + \frac{\kappa_5^2}{6}\Lambda + \frac{\kappa_5^4}{36}\rho^2 + \frac{\kappa_r^4}{18}\lambda\rho - \frac{k}{a_0} + \frac{\mathcal{C}}{a_0^4}. \tag{4.13}$$

To obtain a flat brane without effective cosmological constant we have to use the same fine-tuning that we needed in the effective four-dimensional description, $\lambda^2 = -\frac{6}{\kappa_5^2}\Lambda$. Following the Randall-Sundrum approach by identifying the constants in front of the term that is linear in the energy density with Newton's constant $8\pi G_4$, we can write the generalized Friedmann equation as

$$H_0^2 = \frac{8\pi G_4}{3}\rho\left(1 + \frac{\rho}{\lambda}\right) - \frac{k}{a_0^2} + \frac{\mathcal{C}}{a_0^4}. \tag{4.14}$$

In the low energy limit, that is $\rho \ll \lambda$, we obtain the standard expansion behavior in the universe. However, in the high energy regime, the term proportional to the square of the energy density becomes dominant. In this case, we can also neglect the bulk cosmological constant $\Lambda \approx 0$ and the scale factor shows the unconventional behavior which we derived in the beginning of this section. This means that the neglect of the bulk cosmological constant, i.e. the assumption of a flat metric in the bulk, cannot account for a viable low energy limit. The term proportional to the integration constant $\mathcal{C}$ is negligible at late times, because it drops off like radiation and, therefore, the non-relativistic matter density dominates. It is not problematic to choose the brane tension high enough to avoid possible bounds that arise from nucleosynthesis with an unconventional expansion of the universe. On the other hand, this behavior is potentially dangerous for high energy inflation, which takes place at early times in standard cosmology. Nevertheless, it is at least in principle no problem to put inflation at somewhat lower energy scales.

## 4.3 Brane inflation

The brane world picture does not offer sustantially different solutions to the basic problems of cosmology, namely the homogeneity and isotropy problem as well as the problem of causality. These have been overcome by inflation in the standard model. Therefore, we want to incorporate inflation in brane scenarios. A relatively complete treatment of this issue was presented by Kaloper in [36]. The basic idea is that instead of considering a time-dependent bulk solution which allows for cosmological evolution on the brane we can take a static bulk in which the brane is moving, i.e. the brane has position $y = y_b(t)$ (see e.g. [39] and [40]). Although we cannot apply this method in general, for most physical cases this equivalence avails. The technique was used by Chamblin and Reall [39] to generate solutions, in which they also include bulk matter.

### 4.3.1 Cosmological constant on the brane

Imagine a static bulk with Friedmann-Robertson-Walker metric on an open, flat or closed brane. Here we follow the outline of [35] and take the bulk point of view. This means that the coordinate system is no longer adapted to the brane and, therefore, is not a Gaussian coordinate system. The solutions for the metric can then be written as

$$ds^2 = F_k(y)dt^2 - y^2 d\Omega_k - F_k(y)^{-1}dy^2, \tag{4.15}$$

where

$$F_k(y) \equiv k + \frac{y^2}{l^2} - \frac{\mu}{y^2} \tag{4.16}$$

with $k = -1, 0, +1$ for an open, flat or closed universe, $d\Omega_k$ denotes the corresponding metric of a hyperboloid, a plane or a sphere. Metric (4.15) is often referred to as Schwarzschild-AdS metric. This is because the parameter $\mu$ defines the mass of a black hole in the bulk with a gravitational horizon of $r_h^2 = \frac{l^2}{2}(-k + \sqrt{k^2 + 4\mu/l^2})$. For $\mu = 0$ we have mere AdS-space in the bulk with $\Lambda \equiv -\frac{6}{l^2}$, that is $l$ gives the bulk curvature scale. For consistency, $k = 0, 1$ requires $\mu \leq 0$ and for $k = -1$ we have to demand $\mu \geq -\frac{l^2}{4}$.

Therefore, the brane has to live at a time-dependent position $y = y(\tau)$ with proper time $\tau$ so that the scale factor $a(\tau)$ on the brane will be determined by the junction conditions. Here, it is easier to rewrite the equations in a coordinate independent form. Let us define the velocity vector $u^A$ tangential to the domain wall with $u^A u_A = -1$. Then, the unit normal is orthogonal to the velocity vector: $n^A u_A = 0$. We have

$$u^t = -\frac{\sqrt{F_k + \dot{y}^2(\tau)}}{F_k}, \quad u^y = -\dot{y}(\tau)$$
$$n^t = \frac{\dot{y}(\tau)}{F_k}, \quad n^y = \sqrt{F_k + \dot{y}^2(\tau)}, \tag{4.17}$$

where a dot denotes the derivative with respect to proper time $\tau$. From this we get the junction condition for the spatial brane components, where we have assumed a brane with tension $\lambda$ but no additional matter terms

$$\sqrt{F_k + \dot{y}^2(\tau)} = -\frac{8\pi G_5}{6}\lambda y(\tau). \tag{4.18}$$

Henceforth, we will also assume $\mathbb{Z}_2$-symmetry across the brane. The equation for the component $K_{tt}$ simply leads to the time derivative of the above equation, therefore, we need not consider it any further. With the help of the junction equation, we can specify the metric on the brane

$$ds^2 = d\tau^2 - y^2(\tau)d\Omega_k \tag{4.19}$$

completely. Obviously, the equation for the scale factor $y(\tau)$ is quite different from the standard one in homogeneous and isotropic cosmology, where

$$\dot{y}^2(\tau) - \frac{8\pi G_4}{3}y^2(\tau) = -k\,. \tag{4.20}$$

Let us rewrite the junction equation (4.20) analogously to a particle in a one-dimensional potential

$$\frac{1}{2}\dot{y}^2 + V(y) = -\frac{k}{2} \tag{4.21}$$

with the potential

$$V(y) = \frac{1}{2}\left(1 - \left(\frac{\lambda}{\lambda_c}\right)^2\right)\frac{y^2}{l^2} - \frac{1}{2}\frac{\mu}{y^2} - \frac{1}{8}\left(\frac{\lambda}{\lambda_c}\right)^{-2}\frac{l^2\mu^2}{y^6}, \tag{4.22}$$

where $\lambda_c$ denotes the critical tension for which the brane is flat, $\lambda_c \equiv \frac{3}{4\pi G_5 l}$. The original Randall-Sundrum scenario arises for the fine-tuned tension $\lambda = \lambda_c$ and the parameters $\mu = 0 = k$. If we assume that the tension $\lambda$ is larger than the critical tension $\lambda_c$, we obtain de Sitter space on the brane

$$y(\tau) = y_0 e^{\pm \frac{1}{l}\sqrt{\left|(\lambda/\lambda_c)^2 - 1\right|}\tau}. \tag{4.23}$$

For a realistic scenario of inflation in the early universe we need dynamical matter, of course, otherwise we cannot exit the inflationary stage. The inclusion of brane matter does not pose a problem in this set-up, as we will see in the next section.

### 4.3.2 Matter on the brane

Neglecting dark energy components that drive the current day acceleration of the universe, we can describe our universe as a brane world with critical tension $\lambda_c$ such that no effective cosmological constant $\Lambda_4$ remains on the brane. We include matter which we assume to be a perfect fluid,

$$\tau_{\mu\nu} = -\lambda_c \gamma_{\mu\nu} + (\rho + p) u_\mu u_\nu + p\gamma_{\mu\nu} \tag{4.24}$$

with matter density $\rho$ and pressure $p$, whose equation of state reads $p = w\rho$ with constant $w$. In this case, energy conservation requires

$$\frac{d}{d\tau}\left(\rho y^3(t)\right) = -p\frac{d}{d\tau} y^3(t). \tag{4.25}$$

Then, we have the following junction conditions in which the term $\lambda$ in equation (4.18) is substituted by the sum of the brane tension $\lambda_c$ and the matter density $\rho$,

$$\left[\sqrt{F_k + \dot{y}^2(\tau)}\right]_\pm = -\frac{8\pi G}{3}(\lambda_c + \rho) y(\tau). \tag{4.26}$$

Let us consider the limit of matter density $\rho$ small compared to the brane tension $\lambda_c$ and for which we have seen that the Friedmann-equation is effectively the same as in standard cosmology. Rewriting the equations of motion in terms of a particle in a potential, as we did in the last section, yields the potential

$$V(y) = -\frac{\rho y^2}{l^2 \lambda_c} - \frac{1}{2}\frac{\mu}{y^2} - \frac{1}{8}\left(1 - \frac{2\rho}{\lambda_c}\right)\frac{l^2 \mu^2}{y^6}. \tag{4.27}$$

Assuming an empty bulk with vanishing black hole mass parameter $\mu$, we end up at exactly the same potential as in standard Friedmann-Robertson-Walker cosmology.

Again, we obtain the standard behavior of the expansion of the universe at energies scales below the brane tension. A flat, i.e. $k = 0$, radiation dominated universe with $w = \frac{1}{3}$ gives $y(\tau) = (4\rho_0/(l^2\lambda_c))^{1/4}\tau^{1/2}$. For matter domination, when the pressure $p$ is negligible, we obtain the typical $y(\tau) = (9\rho_0/(2l^2\lambda_c))^{1/3}\,\tau^{2/3}$ expansion. Interestingly the behavior for a matter dominated, flat universe does not change, when we assume $\mu \neq 0$. On the other hand we still have the $y(\tau) \propto \tau^{1/2}$ late time behavior for $\mu \neq 0$ in the radiation dominated case, but the energy density is shifted, $\rho_0 \to \rho_0 + l^2\lambda_c\mu/2$. For a fairly general solution to brane world cosmological problems in empty bulk set-ups see [45], which uses the bulk based approach.

## 4.4 Cosmology with bulk scalar fields

Brane worlds with scalar fields are, in principle, easy to understand, but it is very difficult to find solutions to the equations of motion. A general feature of such set-ups are bulk singularities that are not shielded by horizons. Since we do not want naked singularities in the physical picture, we will in general cut off the offending piece of the bulk by introducing a regulator brane, that does not change the background solution of the scale factor in between the branes. This is in accordance with the cosmic censorship conjecture in general relativity, because naked singularities make physics unpredictable. We could further assume that the two branes sit at orbifold fixed points, so that we end up with a compactified extra dimension. Most of the known solutions are generated by the method of superpotentials [41, 42, 43], that we presented in the previous chapter.

In the case of bulk scalar fields, we have to take care of the energy conservation on the brane. Since the bulk fields will in general couple to the metric and the matter on the brane, we will not have $\nabla^\mu \tau_{\mu\nu} = 0$ on the brane anymore, where $\tau_{\mu\nu}$ denotes the brane stress-energy tensor. Instead, we obtain

$$\dot{\rho} + 3\frac{\dot{a}_0}{a_0}(\rho + p) = 2T_{05}\Big|_{brane}, \tag{4.28}$$

where $\rho$ and $p$ denote the matter density and the pressure on the brane respectively and $T_{05}$ is the corresponding component of the bulk energy-momentum tensor. For the usual energy-momentum tensor for a scalar field, $T_{AB} = \partial_A\phi\partial_B\phi - \frac{1}{2}g_{AB}\left(\partial^C\phi\partial_C\phi - V(\phi)\right)$, the term on the right hand side is not problematic in the case of static bulk solutions that are only $y$-dependent. Till now, only solutions that seem extremely artificial have been found. This means the potentials are very complex and there is no reason to assume that such potentials are present in nature.

Langlois and Rodriguez-Martinez found in [24] some of the few known non-static solution in which the scalar field is proportional to the scale factor. Otherwise, there are mostly static bulk solutions in the literature. We will not describe these solutions in more detail, instead we refer the readers interested in cosmological solutions with bulk scalar fields to the papers mentioned in this section and the reviews [46],[47] and [48].

# Chapter 5

# Higher-derivative gravity

In the previous chapters, we have pointed out that brane world scenarios should be considered as effective models that arise from a more fundamental, but yet unknown, theory. Therefore, we should expect corrections, however small, to Einstein-Hilbert gravity. In string theory, the low-energy corrections to gravity lead to Gauß-Bonnet terms. Such brane worlds were considered in many papers, confer for example [50], [51], [52], [53] and [54] and references therein. Here, we will refer to the type of gravity theories in which the Lagrangian is an arbitrary scalar function of the Ricci scalar only as higher-derivative gravity:

$$S = -\frac{1}{\kappa_n^2}\int d^n x\sqrt{|g|}f(R). \tag{5.1}$$

The dimension of $f$ must be the same as the dimension of $R$. Then, variation with respect to the metric leads to the equations of motion in empty spacetime,

$$R_{AB} - \frac{1}{2}g_{AB}\frac{f}{f'} + g_{AB}\frac{\nabla^C\nabla_C f'}{f'} - \frac{\nabla_A\nabla_B f'}{f'} = 0, \tag{5.2}$$

where a prime denotes the derivative with respect to the Ricci scalar. Work on higher-derivative gravity in brane worlds was also done in conjunction with Dorothea Deeg (confer [55]). In most cases, it is impossible to solve those coupled, non-linear equations of motion, however, theories of type (5.1) are equivalent to usual Einstein gravity plus an effective matter field. This was first pointed out by Whitt [56] for the case of $R^2$-gravity and it was proven for general scalar functions $f(R)$ by Kofman and Mukhanov [57]. The heart of this theory is a conformal transformation. For recent work on conformal transformation, see [58, 59]. In what follows, we will show the procedure to obtain the effective scalar field theory from $f(R)$-gravity and we will apply it to our brane world scenario. It will turn out that even a small contribution from higher-derivative terms can change the scenario fundamentally and in some cases even allows for a conceptually simpler set-up.

## 5.1 Conformal equivalence

### 5.1.1 Conformal transformations

The basic idea is that we can rewrite an arbitrary higher-derivative gravity with Lagrangian $f(R)$ in terms of Einstein gravity plus matter fields with the help of conformal transformations. Conformal transformations change the metric by a conformal factor only

$$g_{AB} \to \bar{g}_{AB} = e^{2\omega} g_{AB}. \tag{5.3}$$

This leads to a transformation of the Ricci tensor and Ricci scalar in an $n$-dimensional spacetime with mainly negative metric signature,

$$\begin{aligned} R_{AB} \quad \to \quad & \bar{R}_{AB} = R_{AB} - g_{AB}\nabla^C\nabla_C\omega - (n-2)\Big[\nabla_A\omega_{,B} - \\ & \qquad\qquad -\omega_{,A}\omega_{,B} + g_{AB}\omega^{,C}\omega_{,C}\Big], \\ R \quad \to \quad & \bar{R} = e^{-2\omega}\left[R - 2(n-1)\nabla^C\nabla_C\omega - (n-1)(n-2)\omega^C\omega_C\right]. \end{aligned} \tag{5.4}$$

Note that the sign difference between these expressions and those of Birrell and Davies [60] brings us in line with the sign conventions of Mukhanov et al. [61]. In the following, a redefinition of the conformal factor will be useful,

$$e^{2\omega} \equiv \psi^a \quad \leftrightarrow \quad \omega = \frac{1}{2}a \ln \psi, \tag{5.5}$$

where $\psi$ is a new scalar field and $a$ a constant to be determined. We will choose the exponential factor $a$ such that it nullifies certain terms in the transformed frame. The redefinition of the conformal factor in terms of $\psi$ leads to the transformed quantities

$$\begin{aligned}\bar{R}_{AB} &= R_{AB} - \frac{1}{2} a g_{AB} \frac{\nabla^2 \psi}{\psi} - \frac{1}{2} a(n-2) \frac{\nabla_A \psi_{,B}}{\psi} \\ &+ \frac{1}{2} a \left( \frac{1}{2} a + 1 \right) (n-2) \frac{\psi_{,A} \psi_{,B}}{\psi^2} - \frac{1}{2} a \left[ \frac{1}{2} a(n-2) - 1 \right] g_{AB} \frac{\psi^{,C} \psi_{,C}}{\psi^2}, \\ \bar{R} &= \psi^{-a} \left[ R - a(n-1) \frac{\nabla^2 \psi}{\psi} + \left( \frac{1}{4} a(n-2) - 1 \right) \frac{\psi^{,C} \psi_{,C}}{\psi^2} \right]. \end{aligned} \tag{5.6}$$

Since we want to consider higher-dimensional spacetimes with boundaries, we have to specify the behavior of boundary quantities under conformal transformation as well. The projected metric $q_{AB} = g_{AB} + n_A n_B$, where $n_A$ denotes the unit normal to the boundary with $n_A n^A = -1$, transforms as

$$q_{AB} \rightarrow \bar{q}_{AB} = e^{2\omega} q_{AB} \quad \text{and} \quad n_A \rightarrow \bar{n}_A = e^{\omega} n_A. \tag{5.7}$$

This allows us to determine how the extrinsic curvature $K_{AB}$ given in equation (2.24) transforms

$$\begin{aligned} K_{AB} \rightarrow \bar{K}_{AB} &= e^{\omega} \left[ K_{AB} + q_{AB} n^C \omega_{,C} \right], \\ K \rightarrow \bar{K} &= e^{-\omega} \left[ K + (n-1) n^C \omega_{,C} \right]. \end{aligned} \tag{5.8}$$

Now, we seek the transformation to an appropriate frame in which the equations of motion (5.2) that come from the action (5.1) become the usual Einstein equation plus a scalar field matter component,

$$\bar{R}_{AB} - \frac{1}{2} \bar{g}_{AB} \bar{R} - \kappa_n^2 \bar{T}_{AB} = 0 \tag{5.9}$$

with

$$\begin{aligned} \bar{T}_{AB} &= \phi_{,A} \phi_{,B} - \bar{g}_{AB} \left( \frac{1}{2} \bar{g}^{CD} \phi_{,C} \phi_{,D} - V(\phi) \right) \\ &= \phi_{,A} \phi_{,B} - g_{AB} \left( \frac{1}{2} g^{CD} \phi_{,C} \phi_{,D} - \psi^a V(\phi) \right). \end{aligned} \tag{5.10}$$

Substituting for the transformed Ricci tensor and Ricci scalar in the Einstein equation, we obtain

$$\begin{aligned} 0 &= \frac{1}{2}g_{AB}\Bigg[\left(a(n-2)\frac{\nabla^2\psi}{\psi} - 2\frac{\nabla^2 f'}{f'}\right) + \left(\frac{f}{f'} - R - 2\kappa_n^2\psi^a V(\phi)\right) \\ &\quad - \left(-\frac{1}{4}a(n-2)[a(n-3)-4]\frac{\psi^{,C}\psi_{,C}}{\psi^2} - \kappa_n^2\phi^{,C}\phi_{,C}\right)\Bigg] \\ &\quad -\frac{1}{2}\left(a(n-2)\frac{\nabla_A\psi_{,B}}{\psi} - 2\frac{\nabla_A f'_{,B}}{f'}\right) \\ &\quad + \left(\frac{1}{2}(\frac{1}{2}a+1)(n-2)\frac{\psi_{,A}\psi_{,B}}{\psi^2} - \kappa_n^2\phi_{,A}\phi_{,B}\right). \end{aligned} \tag{5.11}$$

By defining a new scalar field $\phi \propto \ln\psi$, we can eliminate the third and the fifth groupings if we have $-\frac{1}{4}(a(n-2)[a(n-2)-4] = \frac{1}{2}a(\frac{1}{2}a+1)(n-2)$. Thereby, we are led to choose

$$a = \frac{2}{n-2}, \qquad \text{so that} \qquad \phi = \frac{1}{\kappa_n^2}\sqrt{\frac{n-1}{n-2}}\ln\psi. \tag{5.12}$$

Without loss of generality, we choose the positive root only. Taking $\psi = f'$, we eliminate the first and the fourth groupings separately. Finally, requiring the second grouping to vanish also, dictates the potential to take the form

$$V(\phi) = \frac{1}{2\kappa_n^2}\frac{f - Rf'}{(f')^{\frac{n}{n-2}}} \tag{5.13}$$

and specifies the effective theory with a scalar field completely.

We can think of this procedure alternatively as a reduction of higher-derivative gravity to a theory linear in the Ricci scalar $R$ via a Legendre transformation. It is possible to simplify the resulting Lagrangian to Einstein-Hilbert, non-minimally coupled or Brans-Dicke by an appropriate conformal transformation. For this purpose, we rewrite the Lagrangian for a general metric theory, which depends on only one metric quantity, the Ricci tensor,

$$S = \int d^n x\sqrt{|g|}\mathcal{L}_0(g_{AB}, \psi|R). \tag{5.14}$$

The set of additional matter fields and their derivatives are denoted by $\psi$, and the vertical dash divides the true dynamic degrees of freedom from quantities whose presence we only want to make explicit. That is, the derivatives of these quantities do not show up in the Lagrangian.

Now, we determine a Lagrangian $\mathcal{L}$ equivalent to $\mathcal{L}_0$ which is linear in $R$. This approach, due to Helmholtz, introduces a field $\chi$ canonically conjugate to $R$,

$$\chi = \frac{\partial \mathcal{L}_0}{\partial R}. \tag{5.15}$$

Then, we define the equivalent Lagrangian as

$$\mathcal{L}(g_{AB}, \psi | \chi) = \chi(R - r) + \mathcal{L}_0(g_{AB}, \psi | r), \tag{5.16}$$

where $r = r(g_{AB}, \psi|\chi)$ is the solution to the equation $\frac{\partial \mathcal{L}_0}{\partial R}(g_{AB}, \psi|r) = \chi$. In equation (5.16), we have essentially performed a Legendre transformation. The equivalence of the Lagrangians $\mathcal{L}$ and $\mathcal{L}_0$ is ensured because the equation of motion for $\chi$ yields $r = R$. It is easy to check that no derivatives of the field $\chi$ show up in $\mathcal{L}$, therefore, $\chi$ is not a truly dynamic field.

To simultaneously bring the Lagrangian $\mathcal{L}$ in standard form and to make $\chi$ dynamic, we perform a conformal transformation of the metric. We have already seen the transformation behavior of the Ricci scalar in equation (5.6). Let us additionally redefine the field $\chi \to -\sigma(\chi)/2\kappa_n^2$. Substituting in equation (5.6), we find

$$\begin{aligned} -\sigma R &= -\sigma e^{2\omega}\left[\bar{R} + 2(n-1)\bar{\nabla}^2\omega - (n-1)(n-2)(\bar{\nabla}\omega)^2\right] \\ &\to \sigma e^{2\omega}\left[-\bar{R} + (n-1)\bar{\nabla}\omega\left(2\frac{\bar{\nabla}\sigma}{\sigma} - (n-2)\bar{\nabla}\omega\right)\right] \end{aligned} \tag{5.17}$$

after a final integration by parts. This needs to be compared to a generalized scalar-tensor theory. The relevant portion of the Lagrangian is

$$-\frac{1}{2\kappa_n^2} f(\chi)\bar{R} + \frac{1}{2}k(\chi)\left(\bar{\nabla}\chi\right)^2. \tag{5.18}$$

From this we can read off that

$$\begin{aligned}\sigma(\chi) &= \exp\left(\pm\int d\chi\sqrt{\left(\frac{f'}{f}\right)^2+\frac{n-2}{n-1}\kappa_n^2\frac{k}{f}}\right),\\ \omega(\chi) &= \frac{1}{n-2}\ln\frac{\sigma(\chi)}{f(\chi)}. \end{aligned} \tag{5.19}$$

As an example of the procedure, we go back to $f(R)$-gravity and consider the action in the absence of matter terms. We have

$$\mathcal{L} = -\frac{1}{2\kappa_n^2}\Big[\sigma(\chi)(R-r)+f(r)\Big], \tag{5.20}$$

where $f'(r) = \sigma(\chi)$ defines $r = r(\chi)$. After the conformal transformation, we obtain

$$\sqrt{|g|}\mathcal{L} = \sqrt{|\bar{g}|}\left[-\frac{1}{2\kappa_n^2}f(\chi)\bar{R}+\frac{1}{2}k(\chi)(\bar{\nabla}\chi)^2-V(\chi)\right] \tag{5.21}$$

with

$$V(\chi) = -\frac{1}{2\kappa_n^2}e^{-n\omega(\chi)}\left[r(\chi)\tilde{f}'(r(\chi))-\tilde{f}(r(\chi))\right], \tag{5.22}$$

where $\tilde{f}(r(\chi)) \equiv f(\chi)$. Having specified the effective theory in the bulk, we will focus on the boundary terms in the following section.

### 5.1.2 Boundary terms

We have already seen that the Gibbons-Hawking boundary term is essential for the derivation of the correct junction conditions. First of all let us consider how to describe boundaries in general. The simplest and, in our case, sufficient way is to apply a scalar constraint amongst the $n$ coordinates $x^C$, namely

$$F\left(x^C\right) = 0. \tag{5.23}$$

The scalar function $F(x^C)$ also implies the definition of the unit vector normal to the brane

$$n_A = \left. \frac{F_{,A}}{\sqrt{-g^{CD}F_{,C}F_{,D}}} \right|_{F=0}. \tag{5.24}$$

A different way to specify the boundary is to consider it as an embedding. In particular we can suppose there are $n$ scalar functions $X^A$ depending on $n-1$ coordinates $\sigma^\mu$ such that

$$x^A = X^A(\sigma^\mu) \tag{5.25}$$

describes a time-like boundary. The induced metric is

$$\gamma_{\mu\nu} = g_{AB}\frac{\partial X^A}{\partial \sigma^\mu}\frac{\partial X^B}{\partial \sigma^\nu}, \tag{5.26}$$

where, of course, $g_{AB}$ has to be evaluated on the boundary (in most cases this is obvious and we will not mention it). Then $n_A$ is the properly normalized solution to the system of equations $n_A X^A_{,\mu} = 0$. Under a conformal transformation, the induced metric behaves just like the full one

$$\gamma_{\mu\nu} \longrightarrow \bar{\gamma}_{\mu\nu} = e^{2\omega}\gamma_{\mu\nu}. \tag{5.27}$$

This gives us an appropriate description of the boundary hypersurface. Let us now have a closer look at what happens when we go from the action in the scalar field frame, denoted by barred quantities, to the action in the unbarred, physical frame. We take

$$S = \int d^n x \sqrt{|\bar{g}|}\left[-\frac{\bar{R}}{2\kappa_n^2} + \bar{\mathcal{L}}\right], \tag{5.28}$$

where $\bar{\mathcal{L}}$ stands for the matter Lagrangian in the effective theory,

$$\bar{\mathcal{L}} = \frac{1}{2}\bar{g}^{AB}\phi_{,A}\phi_{,B} - V(\phi) = \frac{1}{2}\psi^{-a}\phi_C\phi^C - V(\phi). \tag{5.29}$$

Substituting for the barred quantities and using the results (5.12) and (5.13), we find

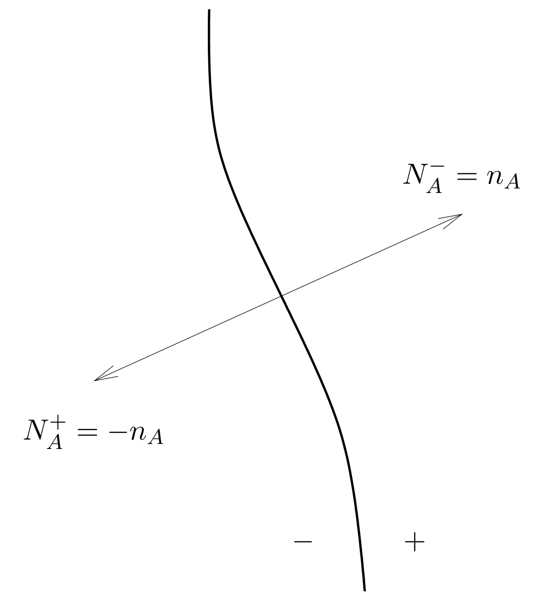


Figure 5.1: *$n_A$ is defined to point into the + region, thus the outward-pointing normal for the + region is $N_A^+ = -n_A$*

$$\begin{aligned} S &= -\frac{1}{2\kappa_n^2}\int d^n x\sqrt{|g|}\left(-f(R) + 2\frac{n-1}{n-2}\nabla^2\psi\right) \\ &= -\frac{1}{2\kappa_n^2}\int d^n x\sqrt{|g|} f(R) - \frac{1}{\kappa_n^2}\frac{n-1}{n-2}\int d^{n-1}\sigma\sqrt{|\gamma|}n^A\left[\psi_{,A}\right]_{\pm}. \end{aligned} \tag{5.30}$$

The surface term arises from the application of Gauß' theorem, which reads

$$\int \nabla^2\psi\sqrt{|g|}d^n x = \int \psi_{,A} N^A d^{n-1}\sigma, \tag{5.31}$$

where $N^A$ is the outward pointing normal. In our case, we get two terms denoted by ($\pm$) because we are integrating over volumes on either side of the boundary. The derivatives of the scalar field $\psi$ do not cancel if $\psi_{,A}$ is discontinuous across the boundary. As before, our notation is $[Z]_\pm \equiv Z_+ - Z_-$. Furthermore, we have to bear in mind to take the respective outward pointing normal, which is different on both sides of the brane. Our set-up is further illustrated in figure 5.1.

However, as we have pointed out in the introductory chapters, the presence of a boundary must be reflected in the appearance of the Gibbons-Hawking term in the action. That is, we have to add

$$S_{GH} = \frac{1}{\kappa_n^2} \int d^{n-1}\sigma \sqrt{|\bar{\gamma}|}\, [\bar{K}]_{\pm} \tag{5.32}$$

to the action (5.30). Transforming back to the unbarred frame, we find that the boundary term in (5.31) is canceled by the transformation behavior of the extrinsic curvature scalar $K$. The boundary term that remains we think of as the Gibbons-Hawking term appropriate for higher-derivative gravity. The full action is

$$S = -\frac{1}{2\kappa_n^2} \int d^n x \sqrt{|g|} f(R) + \frac{1}{\kappa_n} \int d^{n-1}\sigma \sqrt{|\gamma|} f'(R) [K]_{\pm} \tag{5.33}$$

in the physical frame. The boundary term for higher-derivative gravity corresponds to the one derived in [62]. We claim this is the right starting point for higher-derivative gravity. However, we do not compute the equations of motion from this action, because it would be unnecessarily tedious. Instead, we always go to the barred metric directly.

### 5.1.3 Equations of motion

Let us get back to physics and start over again in the higher derivative frame, which we distinguish with unbarred quantities. The boundary is a brane with matter terms confined to it, which we will specify later. We want an empty bulk, therefore, the complete action including the proper Gibbons-Hawking surface term, which we derived in the last section, reads

$$S = -\frac{1}{\kappa_n^2} \int d^n x \sqrt{|g|} f(R) + \frac{1}{\kappa_n^2} \int d^{n-1}\sigma \sqrt{|\gamma|} f'(R) [K]_{\pm} + S_b. \tag{5.34}$$

The matter term $S_b$ includes the tension of the brane $\lambda$ and all other fields that are confined to the boundary,

$$S_b = \int d^{n-1}\sigma \sqrt{|\gamma|} \mathcal{L}_b(\gamma_{\mu\nu}, \psi_i, \lambda). \tag{5.35}$$

In principle, we can derive the equations of motion directly by varying the action with respect to the metric, but this would lead to equations of higher than second order. Here, we exploit the conformal equivalence to derive solutions to higher-derivative gravity. For that, we also need to transform the brane Lagrangian into the barred scalar field frame, which will introduce coupling of brane matter to the effective bulk scalar field $\phi$ defined as in equation (5.12), $\mathcal{L}_b(\gamma_{\mu\nu}, \psi_i, \lambda) \to \bar{\mathcal{L}}_b(\bar{\gamma}_{\mu\nu}, \psi_i, \lambda, \phi)$. The barred brane Lagrangian is related to the original one by

$$\bar{\mathcal{L}}_b(\bar{\gamma}_{\mu\nu}, \psi_i, \lambda, \phi) = \mathcal{L}_b\left(\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} e^{-\frac{2\kappa_n\phi}{\sqrt{(n-1)(n-2)}}}, \psi_i, \lambda\right) e^{-\kappa_n\phi\sqrt{\frac{n-1}{n-2}}}. \quad (5.36)$$

This yields the action in the scalar field frame

$$\begin{aligned} S &= \int d^n x \sqrt{|\bar{g}|}\left[-\frac{\bar{R}}{2\kappa_n^2} + \frac{1}{2}\bar{g}^{AB}\phi_{,A}\phi_{,B} - V(\phi)\right] \\ &\quad + \int d^{n-1}\sigma\sqrt{|\bar{\gamma}|}\left[\frac{[\bar{K}]_\pm}{\kappa_n^2} + \mathcal{L}_b(\gamma_{\mu\nu}, \psi_i, \lambda)U(\phi)\right] \end{aligned} \quad (5.37)$$

with $U(\phi) = e^{-\kappa_n\phi\sqrt{\frac{n-1}{n-2}}}$. Variation with respect to the barred metric gives

$$0 \stackrel{!}{=} \delta_{\bar{g}} S_b = \frac{1}{2}\int d^{n-1}\sigma\sqrt{|\bar{\gamma}|}\bar{\tau}_{AB}\delta\bar{g}^{AB} \quad (5.38)$$

with an energy-momentum tensor $\bar{\tau}_{AB}$. Since $\bar{\tau}_{AB}$ is tangential to the boundary, we have $\bar{\tau}_{\mu\nu} = \bar{\tau}_{AB}X^A_{,\mu}X^B_{,\nu}$. Because the energy-momentum tensor in the transformed frame is only an effective one and because we know the behavior of matter only in the physical frame, we rewrite it in terms of the physical energy-momentum tensor in the higher derivative frame

$$\bar{\tau}_{AB} = e^{-\omega(n-3)}\tau_{AB}, \quad (5.39)$$

where $\omega$ comes from the conformal transformation $g_{AB} \to e^{2\omega}g_{AB}$. Hence, we have the standard Einstein equations with a bulk matter part in the barred frame and junction conditions of the special form

$$\left[\bar{K}^A_{\ B} - \delta^A_B\bar{K}\right]_\pm = -\kappa_n^2 e^{-\kappa_n\phi\sqrt{\frac{n-1}{n-2}}}\tau^A_{\ B}. \quad (5.40)$$

Variation with respect to the effective scalar field, which we treat as independent, gives the Klein-Gordon equation for scalar fields in curved spacetime (equation (3.5)), while the variation of the boundary matter term gives

$$\begin{aligned}\frac{\delta}{\delta\phi}\bar{\mathcal{L}}_b(\gamma_{\mu\nu}(\phi)) &= -\kappa_n\sqrt{\frac{n-1}{n-2}}\bar{\mathcal{L}}_b + e^{-\kappa_n\phi\sqrt{\frac{n-1}{n-2}}}\frac{\delta\gamma_{\mu\nu}}{\delta\phi}\frac{\delta\mathcal{L}_b}{\delta\gamma_{\mu\nu}} \\ &= \frac{\kappa_n}{\sqrt{(n-1)(n-2)}}\bar{\tau}, \end{aligned} \tag{5.41}$$

where $\bar{\tau} \equiv \bar{\gamma}^{\mu\nu}\bar{\tau}_{\mu\nu} = \bar{q}^{AB}\bar{\tau}_{AB}$. Thus, the solution to the Klein-Gordon equation for the scalar field has to satisfy the boundary condition

$$\bar{n}^A\left[\phi_{,A}\right]_{\pm} = \frac{\kappa_n}{\sqrt{(n-1)(n-2)}}\, e^{-\kappa_n\sqrt{\frac{n-1}{n-2}}\phi}\tau, \tag{5.42}$$

where $\tau \equiv \gamma^{\mu\nu}\tau_{\mu\nu}$ and $\bar{n}^A$ is the unit normal to the surface in the scalar field frame. Note that the boundary conditions here are different than the ones that we got for conformally coupled scalar fields in section 3.1.

### 5.1.4 $f(R)$ from the scalar field potential

After solving for the scalar field and the metric components, we have to calculate $f(R)$ from the scalar field potential. Here we take the second interpretation of the scalar field as a Legendre transform of the Ricci scalar $R$. Starting from the higher-derivative frame, we obtain with the help of the Legendre transformation

$$f_L(\psi) = \psi R - f(R), \tag{5.43}$$

where $\psi \equiv f'$ is the variable conjugate to $R$. The field $\psi$ is related to the effective scalar field $\phi$, which enters the Lagrangian, by

$$\psi = e^{\kappa_n\sqrt{\frac{n-2}{n-1}}\phi}. \tag{5.44}$$

Furthermore, the scalar field potential can be expressed in terms of $f_L$

$$V(\psi) = -\frac{1}{2\kappa_n^2}\psi^{\frac{n}{n-2}} f_L(\psi). \tag{5.45}$$

If the function $f(R)$ is a strictly increasing and concave function, i.e. $f' > 0$ and $f'' \neq 0$, then we automatically obtain a concave Legendre transform

$f_L(\psi)$. In this case, it is no problem to define the inverse of the Legendre transformation,

$$f(R) = R\psi - f_L(\psi), \tag{5.46}$$

where the Ricci scalar $R \equiv f_L'$. Otherwise, we can perform the transformation only in the region for which $f(R)$ fulfills the conditions of strict monotony and concavity.

In the following section, we will use the conformal equivalence to find solutions for higher-derivative theories by starting from the scalar field frame and transforming the solutions back to the physical frame. It will turn out that we have to cut off the transformation at certain values of $\psi$, because the functions $f$ will not be globally concave and monotonic.

## 5.2 Solutions for higher-derivative gravity

Now, we have all means at hand to obtain solutions for brane world models with modified gravitational theories. For that, we will take the set-up of the Randall-Sundrum II model that has one brane with tension $\lambda$. We neglect other matter fields on the brane and assume the bulk to be empty, too. Furthermore, we modify the Lagrangian so that we have higher-derivative gravity. The specific function $f(R)$ will be calculated at the end; we want to take advantage of the conformal equivalence and go straight to the scalar field frame. We will do all the calculations in the scalar field frame, therefore we drop the bars on all quantities without the risk of ambiguity. The junction conditions in the scalar field frame read

$$\begin{aligned} [K_{AB}]_\pm &= \frac{1}{3}\kappa_5^2\lambda U(\phi_O)q_{AB}, \\ n^A\left[\partial_A\phi\right]_\pm &= -\lambda \left.\frac{dU}{d\phi}\right|_{\phi_0}, \end{aligned} \tag{5.47}$$

where we now let $n = 5$ so that $U(\phi) = e^{-\frac{2}{\sqrt{3}}\kappa_5\phi}$ and $\phi_0$ denotes the value of the scalar field at the brane position.

In the scalar field frame, we take the following ansatz for the metric, where we restrict our model to flat space on the 3-brane for convenience,

$$ds^2 = e^{2X(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2. \tag{5.48}$$

Without loss of generality, we put the brane at $y = 0$. Then, the equations of motion are

$$\begin{aligned} -3(2X'^2 + X'') &= \kappa_5^2 \left( \frac{1}{2}\phi'^2 + V(\phi) \right), \\ 6X'^2 &= \kappa_5^2 \left( \frac{1}{2}\phi'^2 - V(\phi) \right), \\ \phi'' + 4X'\phi' &= \frac{dV}{d\phi}. \end{aligned} \tag{5.49}$$

Again, we assume $\mathbb{Z}_2$-symmetry across the brane, therefore, the junction conditions tell us not only the jump, but also the value of the $y$-derivatives of the metric functions and of the scalar field at the brane

$$\begin{aligned} X'\Big|_{\pm} &= \mp\frac{\kappa_5^2}{2}\lambda U(\phi_0) \\ \phi'\Big|_{\pm} &= \pm\frac{\lambda}{2}\frac{dU}{d\phi}\Big|_{\phi_0}, \end{aligned} \tag{5.50}$$

where '$\pm$' indicates evaluation on the respective side of the brane.

We will see that these models exhibit an interesting feature: due to the given potential $U(\phi)$ the brane always resides at points, where the scalar field potential $V(\phi)$ vanishes. Unfortunately, this excludes Liouville potentials, i.e. exponential potentials, for which the equations can often be solved with relative ease. For a general scalar field potential, equations (5.49) are difficult to solve. Therefore, we resort to the superpotential approach, which we described in chapter 3 (see page 41ff). Substitution of the potential $V(\phi)$ by the superpotential $W(\phi)$ in the junction conditions yields

$$\begin{aligned} W\Big|_{\pm} &= \pm\lambda U(\phi_0), \\ \frac{dW}{d\phi}\Big|_{\pm} &= \pm\lambda\frac{dU}{d\phi}\Big|_{\phi_0}, \end{aligned} \tag{5.51}$$

### 5.2.1 Linear superpotential

First, we consider the simplest potential that derives from a linear superpotential,

$$W(\phi) = 2(\alpha\phi + \beta), \tag{5.52}$$

$\alpha$, $\beta$ constant. This leads to the scalar field potential

$$V(\phi) = \frac{\alpha^2}{2} - \frac{2\kappa_5^2}{3}(\alpha\phi + \beta)^2. \tag{5.53}$$

The set of linear equations (3.24), which we have to solve, can be integrated easily:

$$\begin{aligned} \phi(y) &= \alpha|y| + \phi_0 \\ X(y) &= -\frac{\kappa_5^2\alpha^2}{6}y^2 + \frac{\kappa_5\alpha}{2\sqrt{3}}|y| + X_0. \end{aligned} \tag{5.54}$$

With the help of the junction conditions, we fix the value of the scalar field on the brane,

$$\phi_0 = -\frac{\sqrt{3}}{2\kappa_5}\ln\left(-\frac{\sqrt{3}\alpha}{\kappa_5\lambda}\right). \tag{5.55}$$

For a well defined solution for a brane with positive tension $\lambda$, we need a negative value of $\alpha$. We can rewrite the potential in terms of $\phi_0$

$$V(\phi) = -\frac{2}{3}\kappa_5^2\alpha(\phi - \phi_0)\left(\phi - \phi_0 - \sqrt{3}\kappa_5^{-1}\right). \tag{5.56}$$

Here we see the afore mentioned effect that the brane lives at a position for which the scalar field potential vanishes and that is not the minimum of the potential. From solution (5.54), we can calculate the corresponding higher-derivative gravity theory.

We take the metric ansatz in the physical frame, i.e. the frame of higher-derivative gravity, to be

$$ds^2 = e^{-2a(Y)}\eta_{\mu\nu}dx^\mu dx^\nu - dY^2 \tag{5.57}$$

with

$$a(Y) = \omega(Y) - X(Y) \qquad \text{and} \qquad Y = \int_0^y e^{-\omega(y')dy'}, \tag{5.58}$$

where the function $\omega(y)$ specifies the conformal transformation from the physical frame to our effective scalar field frame. In five dimensions, we have $\omega = \frac{\kappa_5}{2\sqrt{3}}\phi$. The choice of the integration constant for the definition of $Y$ is such that the brane remains at $Y = 0$. Furthermore, we choose the metric on the brane $a(0) = 0$ for convenience, which determines the remaining constant of integration completely,

$$X_0 = \frac{\kappa_5\phi_0}{2\sqrt{3}} = -\frac{1}{8}\ln\left(-\frac{\sqrt{3}\alpha}{\kappa_5\lambda}\right). \tag{5.59}$$

Then the solution for the scale factor in the physical frame reads

$$a(Y) = 2\left[\ln\left(1 + \sqrt{\frac{\kappa_5}{36\lambda}}\left(-\sqrt{3}\kappa_r\lambda\alpha\right)^{3/4}|Y|\right)\right]^2. \tag{5.60}$$

This solution has a very interesting feature, its $Y$-derivative is continuous across the brane. This stands in strong contrast to Einstein-Hilbert gravity, where a gravitating singular object like a brane causes a discontinuity of the $Y$-derivative of the metric functions. Indeed, it is easy to see from the junction conditions that this is a generic feature of higher-derivative gravity: the conformal transformation fixes the coupling of the brane tension to the scalar field as given in the junction conditions (5.47). Therefore, combination of the two junction conditions yields

$$X'\Big|_{y=0} = \frac{1}{2\sqrt{3}}\phi'\Big|_{y=0} = \omega'\Big|_{y=0}. \tag{5.61}$$

This forces the metric functions to be smooth across the brane in higher-derivative theories. However, this does not mean that the existence of the brane is not reflected in the geometric quantities. We have to bear in mind that $f(R)$-gravity results in equations of motion of higher than second order. For example, the Ricci scalar for the metric (5.57) is given by

$$R = 20a'^2 - 8a'' \tag{5.62}$$

where the prime now denotes derivative with respect to the $Y$-coordinate. Substituting the solution (5.60), we obtain the jump of the derivative of the Ricci scalar

$$[R']_{\pm} = \frac{8}{9}\left(\frac{\kappa_5}{\lambda}\right)^{\frac{3}{2}}\left(-\sqrt{3}\kappa_5\lambda\alpha\right)^{\frac{9}{4}}. \tag{5.63}$$

Thus, in our example, the discontinuity does not arise until the third derivatives of the metric. Due to the logarithm in the solution of the metric function $a(Y)$, the warp factor $e^{-2a(Y)}$ decays much slower than the warp factor of the original RS II model.

Now, we have to transform back to obtain the function $f(R)$ which we do with the help of the Legendre transformation. The particular form of $f_L(\psi)$ reads

$$\begin{aligned} f_L(\psi) &= \frac{\kappa_5^2\alpha^2}{9}\psi^{\frac{5}{3}}\left[4\,\ln\,\psi + 3\,\ln\left(-\frac{\sqrt{3}\alpha}{\lambda\kappa_5}\right)\right]\times \\ &\qquad \times\left[4\,\ln\,\psi + 3\,\ln\left(-\frac{\sqrt{3}\alpha}{\lambda\kappa_5}\right) - 6\right] \end{aligned} \tag{5.64}$$

with $\psi = e^{\frac{\sqrt{3}}{2}\kappa_5\phi}$. The derivative of the Legendre transform gives us the scalar curvature $R = f_L'(\psi)$. We have to bear in mind that the transformation is only feasible as long as $f_L'' \neq 0$. That restricts the validity of the function $f(R)$ to a certain region $(R_1, R_2)$, which corresponds to a range $(\psi_1, \psi_2)$.

We sketch the function $f(R)$ in figure 5.2, where we have chosen units so that $\lambda = 1 = \kappa_5$. Note that $f(0)$ does not vanish, because the brane lives at zeros of the potential, not at zeros of the field $\phi$. We determine the effective cosmological constant by Taylor expanding $f(R)$ around $R = 0$ and using the relations $f'(0) = \psi(R = 0)$ and $f'' = (f_L'')^{-1}(\psi(R = 0))$. We find

$$\Lambda_{eff} = -\frac{1}{2\kappa_5^2}f(0) = -0.217\left(\frac{-\alpha}{\kappa_5^2\lambda}\right)^{\frac{3}{4}}\kappa_5^4\lambda^2, \tag{5.65}$$

which is negative as in the original Randall-Sundrum model.

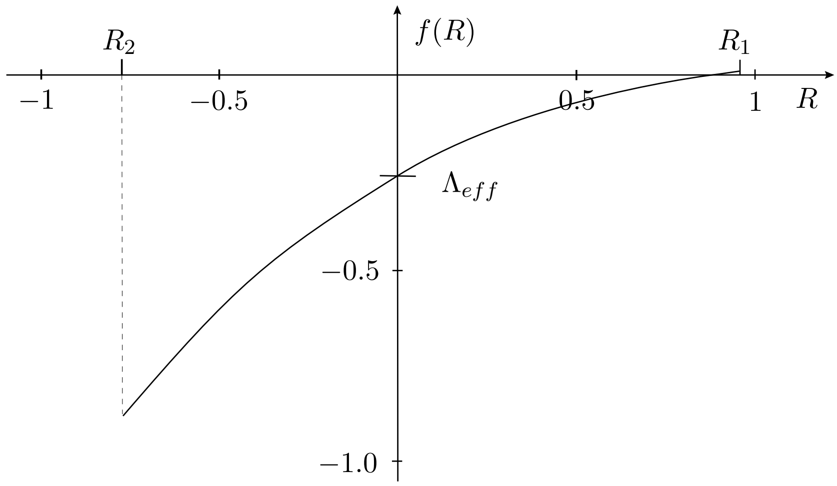


Figure 5.2: $f(R)$ *for a linear superpotential with* $\alpha = -\frac{1}{2}$

### 5.2.2 Quadratic superpotential

In the previous section with a linear superpotential, we were not able to evade an effective cosmological constant in the physical frame. Now, we show that for superpotentials with polynomials of higher than linear order in $\phi$ we can set up models without a cosmological constant. For that, we consider a quadratic superpotential of the most general form,

$$W(\phi) = 2\left(\alpha\phi^2 + \beta\phi + \gamma\right). \tag{5.66}$$

In this case, we have the freedom to choose one of the parameters which allows us to cancel the cosmological constant $\Lambda$ in the bulk entirely. A possible choice is

$$\gamma = \frac{3\alpha}{4\kappa_5^2}\left(1 + \left[\ln\left(\frac{3\alpha}{\lambda\kappa_5^2}\right) - 1\right]^2\right). \tag{5.67}$$

We read off that $\alpha$ has to be positive for a positive tension brane. Now, we perform the calculations along the lines of the previous section and obtain straightforwardly the following solutions to the coupled first order equations (3.24):

$$\begin{aligned}\phi(y) &= -\frac{\beta}{2\alpha} + \left(\phi_0 + \frac{\beta}{2\alpha}\right) e^{2\alpha|y|}, \\ X(y) &= -\frac{\kappa_5^2}{3}\left(\gamma - \frac{\beta^2}{4\alpha}\right)|y| - \frac{\kappa_5^2}{12}\left(\phi_0 + \frac{\beta}{2\alpha}\right)^2 e^{4\alpha|y|} + X_0 \,. \end{aligned} \tag{5.68}$$

The junction conditions (5.51) further restrict the choice of our potential and we obtain the value of the scalar field at the brane

$$\phi_0 = -\frac{\sqrt{3}}{2\kappa_5} \ln\left(\frac{3\alpha}{\lambda\kappa_5^2}\right). \tag{5.69}$$

and

$$\beta = \frac{-\sqrt{3}\alpha}{\kappa_5}\left[\ln\left(\frac{3\alpha}{\kappa_5^2\lambda}\right) - 1\right]. \tag{5.70}$$

This allows us to write the scalar field potential $V(\phi)$ that comes from $W(\phi)$ as in equation (3.23):

$$V(\phi) = -\frac{2\kappa_5^2\alpha^2}{3}(\phi - \phi_0)^2\left(\phi - \phi_0 - \sqrt{3}\kappa_5^{-1}\right)^2. \tag{5.71}$$

Moreover, we have a Minkowskian brane and, therefore, we are free to choose $a(0) = 0$ again. This determines the value of $X$ on the brane completely,

$$X_0 = \frac{\kappa_5\phi_0}{2\sqrt{3}} = -\frac{1}{4}\ln\left(\frac{3\alpha}{\lambda\kappa_5^2}\right) \tag{5.72}$$

and we can write down the solution for the scale factor $a(y) = \omega(y) - X(y)$ explicitly,

$$a(y) = \frac{1}{16}\left(3 + 4\alpha|y| - 4e^{2\alpha|y|} + e^{4\alpha|y|}\right). \tag{5.73}$$

Having solved for the full system, we transform back to the physical frame again. We have a metric analogous to the linear potential case, see

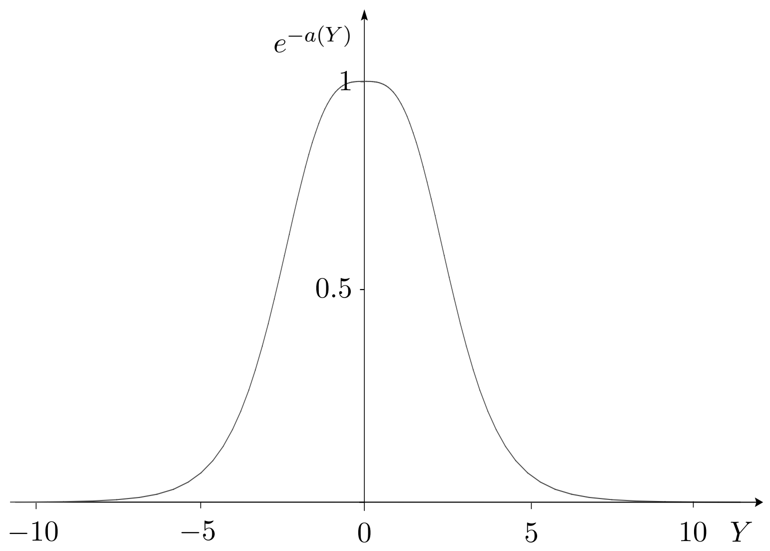

Figure 5.3: *scale factor* $e^{-a(Y)}$ *for a quadratic superpotential*

(5.57). Here, the rescaled coordinate of the extra dimension in the physical frame is given by

$$Y = \int_0^y e^{-\omega(y')dy'} = \left(\frac{3\alpha}{e\lambda\kappa_5^2}\right)^{\frac{1}{4}} \int_0^y e^{\frac{1}{4}e^2\alpha|y'|}dy' \,. \tag{5.74}$$

The solution, although well defined, cannot be written in terms of elementary functions, therefore, we specify $a(Y)$ numerically. In figure 5.3, we plot the scale factor $e^{-a(Y)}$, where we can see features similar to those for linear superpotentials, namely the smoothness across the brane as a general feature of higher-derivative gravity and a faster decay for large values of $Y$ than in the original Randall-Sundrum model.

We also determine the function $f(R)$, which characterizes our higher-derivative theory from the chosen scalar field potential, which we deduced from the quadratic superpotential. Again, the Legendre transformation back to the Ricci scalar is unique only in a certain range $R \in (R_1, R_2)$, for which we plot $f(R)$ in figure 5.4. As already stated in the beginning of the section, no effective cosmological constant term remains in the physical frame for an appropriate choice of parameters.

Figures 5.4 and 5.5 show that - different than for the solution for a linear superpotential - $R(Y)$ takes values outside the defined range $(R_1, R_2)$, in which we can compute $f(R)$ uniquely. Therefore, we have to introduce a

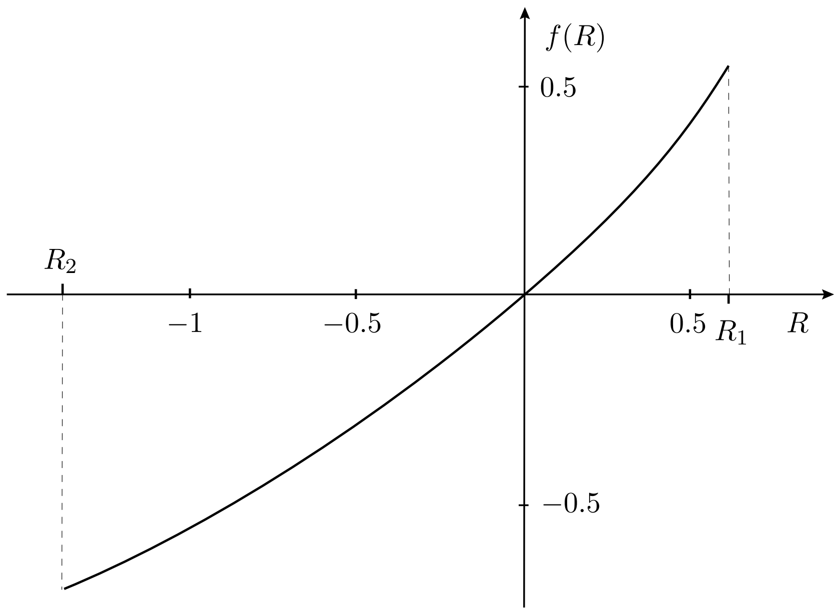


Figure 5.4: *$f(R)$ for a quadratic superpotential with $\alpha = \frac{1}{2}$*

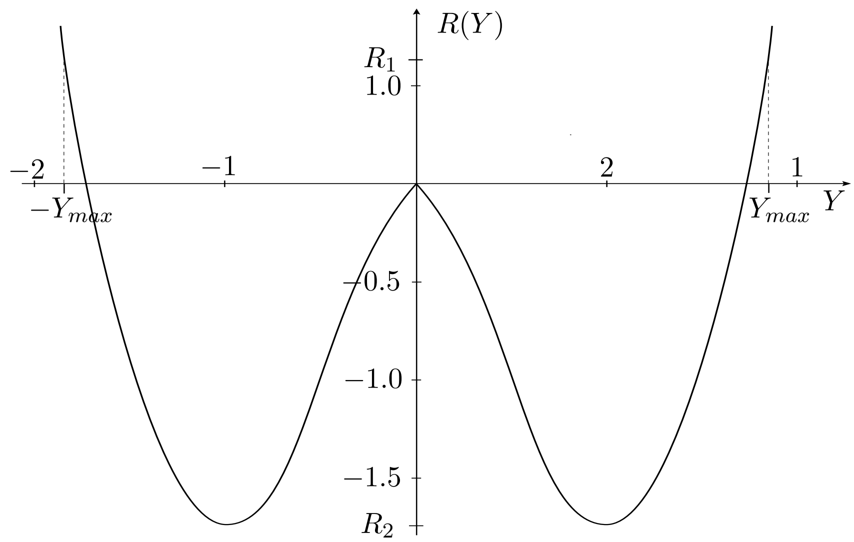


Figure 5.5: *scalar curvature $R$ for a quadratic superpotential*

regulator brane at position $Y_r < Y_{max}$ such that the Ricci scalar in between the branes remains within the allowed range. Although this procedure may seem ad hoc, it is analogous to the standard procedure to shield unwanted features, for example bulk singularities in scalar field models. In principle we can achieve stabilization of two-brane models at arbitrary distances. To conclude, we have introduced an unspecified parameter into the model again for which we have no equation. However, the model has the advantage that we need not fine-tune the bulk cosmoogical constant and the brane tension. We will see in the perturbative treatment that the brane distance has an effect on the spectrum of metric perturbations and, therefore, has to be chosen with care.

# Chapter 6

# A note on brane embedding

In the preceding chapters, we put the brane at an arbitrary, but fixed, position $y_b$ in full spacetime. This is always allowed for homogeneous and isotropic metrics like our background solutions. A relocation of a single brane in an infinite bulk does not change the physical situation, of course, but for a two brane scenario some variables may change, e.g. mass parameters (see chapter 2). Therefore, a stabilization of the inter-brane distance is indispensable, but does not exclude fluctuations of the brane position in spacetimes with arbitrarily perturbed metric quantities.

In the following, we will show why we have to take into account a perturbation of the brane position in order to obtain a consistent perturbative system. Solving for and reintroducing the perturbation of the brane position, henceforth also referred to as brane-bending, into the equations of motion, we will find the correct prefactor in the deviations from the Newtonian potential at small distances. The corrections turn out to be weaker than what was originally computed by Randall and Sundrum. After that, we will consider the embedding of hypersurfaces in higher-dimensional spacetime in a more general way. There, we will find an equation of motion, which describes the possible choices for the embedding of the brane. Finally, this will allow us to do the complete and self-consistent calculation of perturbed brane worlds.

## 6.1 Brane-bending

Again, we begin with a Randall-Sundrum II set-up with a single, positive tension brane in an infinitely extended five-dimensional bulk with negative cosmological constant and a fine-tuned relation such that the brane is Minkowskian. This model has the right sign of gravity on the brane, and the full spacetime metric reads

$$ds^2 = g^0_{AB} dx^A dx^B = a^2(y)\eta_{\mu\nu} dx^\mu dx^\nu - dy^2, \tag{6.1}$$

with $a(y) = e^{-k|y|}$ and $k$ being the inverse of the curvature radius of the bulk AdS spacetime, i.e. $\Lambda = -6k^2$. We consider the same type of transverse-traceless gravitational perturbations $h_{AB}$ as in chapter 2, again in the Randall-Sundrum gauge with $h_{5A} = 0$ and $h^\mu{}_{\nu,\mu} = 0 = h^\mu{}_\mu$. The full metric is given by $g_{AB} = g^0_{AB} + h_{AB}$. This brings us directly to the linearized equations of motion for the gravitational perturbations in the bulk,

$$\left(\frac{\Box}{a^2} + \partial_y^2 + 4k^2\right) h_{\mu\nu} = 0 \tag{6.2}$$

with the flat four-dimensional d'Alembertian $\Box \equiv \partial^\mu \partial_\mu$. This gauge choice is very advantageous in the bulk, because the equations for the components decouple and agree for all components of the perturbation. On the other hand, the brane will no longer be located at position $y_b = 0$. In general, the brane position will depend on the coordinates $x^\mu$ tangential to the brane such that

$$y_b = \zeta(x^\mu)\,. \tag{6.3}$$

Now we show that we cannot set $y_b$ to zero when we have matter on the brane. For the calculation, we change to Gaussian normal coordinates so that the brane in the new coordinate system, in which we denote all quantities with a bar, will be located at $\bar{y}_b = 0$. The components $\bar{h}_{5A}$ vanish also in the new coordinate system, but the transverse and traceless conditions are lost. Also for the perturbations we impose $\mathbb{Z}_2$-symmetry. The junction conditions in the barred system with matter on the brane, which is described by $\tau_{\mu\nu}$, read

$$\partial_y\left(\gamma_{\mu\nu} + \bar{h}_{\mu\nu}\right) = -\frac{\kappa_5^2}{3}\left(\left(\gamma_{\mu\nu} + \bar{h}_{\mu\nu}\right)\lambda + 3\tau_{\mu\nu} - \tau\gamma_{\mu\nu}\right), \tag{6.4}$$

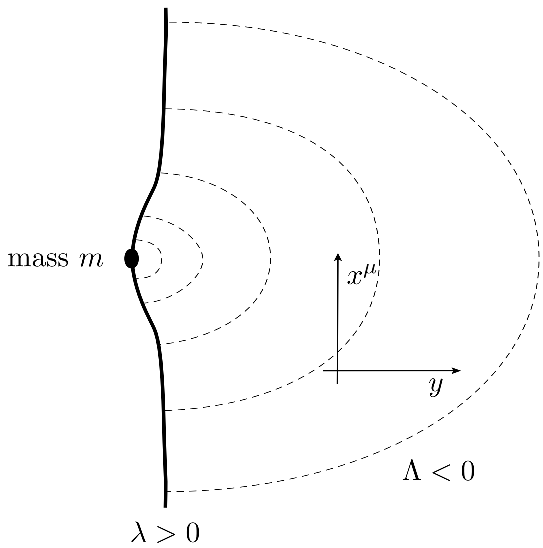


Figure 6.1: *brane-bending due to matter on the brane*

where $\gamma_{\mu\nu} = e^{-2k|y|}\eta_{\mu\nu}$ denotes the metric that is induced by the background metric $g^0_{AB}$. We assume $\tau_{\mu\nu}$ to be of first order in perturbation theory and because we only consider linearized theory, the linearized trace of the brane stress-energy tensor is $\tau \equiv \gamma^{\mu\nu}\tau_{\mu\nu}$. To proceed, we need the explicit form of the coordinate transformation between the unbarred and the Gaussian normal coordinate system. Since the $[5, A]$-components vanish in both systems, the most general coordinate transformation vector $(\xi^\mu, \xi^5)$ must be of the form

$$\begin{aligned} \xi^\mu &= -\frac{1}{2k}\gamma^{\mu\nu}\tilde{\xi}^5(x^\alpha)_{,\nu} + \hat{\xi}^\mu(x^\alpha) \\ \xi^5 &= \tilde{\xi}^5(x^\mu)\,. \end{aligned} \tag{6.5}$$

Using the gauge transformation, we can write down the junction conditions in terms of the unbarred quantities,

$$(\partial_y + 2k)\, h_{\mu\nu} = -\kappa_5^2 \left[ \left( \tau_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}\tau \right) + \frac{2}{\kappa_5^2}\hat{\xi}^5_{,\mu\nu} \right] \equiv -\kappa_5^2 \Sigma_{\mu\nu}. \tag{6.6}$$

This equation defines the location of the brane in the bulk. In the source term $\Sigma_{\mu\nu}$ on the right hand side not only the matter on the brane acts as source, but also the position of the brane. Taking the trace of (6.6) and identifying the brane location $\zeta(x^\sigma) \equiv -\hat{\xi}^5$, which effectively redefines the $y$-coordinate, we obtain

$$\Box\zeta = -\frac{\kappa_5^2}{6}\tau\,. \tag{6.7}$$

This result was found by Garriga and Tanaka in [4]. As one would naively expect, the bending leads to a change in the gravitational potential that we will derive in the next section.

## 6.2 Gravitational potential on the brane

To obtain the gravitational potential, we have to compute the Green's function for the perturbed equation of motion on the brane. Instead of writing the equations of motion (6.2) and the junction conditions (6.6) separately, we can merge them and enforce the discontinuity with a $\delta$-distribution. Then, the equation for the full spacetime retarded Green's function is given by

$$\left(\frac{\Box}{a^2} + \partial_y^2 - 4k^2 + 4k\delta(y)\right) G(x^A, x'^A) = \delta^{(5)}(x^A - x'^A)\,. \tag{6.8}$$

After performing a Fourier transformation in the tangential coordinates so that the d'Alembertian operator $\Box$ is substituted by $-m^2$, we can write down the solution to equation (6.8) in terms of the complete set of eigenstates. It reads

$$\begin{aligned} G(x^A, x'^A) &= -\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \left[\frac{ka(y)^2 a(y')^2}{\mathbf{k}^2 - (\omega - i\epsilon)^2}\right. \\ &\qquad \left. + \int_0^\infty dm \frac{u_m(y) u_m(y')}{m^2 + \mathbf{k}^2 - (\omega + i\epsilon)^2}\right], \end{aligned} \tag{6.9}$$

where the first term represents the only remaining discrete mode which is massless and the latter part the continuum states of the KK spectrum. The functions $u_m$ are a combination of Bessel functions $J_a$ and $Y_a$ of the order $a = 2$. Since the Newtonian potential describes a static source, we have to integrate over time and evaluate the Green's function for both points on the brane, that is at $y = 0 = y'$:

$$G(\mathbf{x}, 0, \mathbf{x}', 0) \approx -\frac{k}{4\pi}\frac{1}{r}\left(1 + \frac{1}{2k^2r^2}\right). \tag{6.10}$$

Here, bold letters stand for the spatial part of four-vectors and $r \equiv |\mathbf{x} - \mathbf{x}'|$ is the spatial distance on the brane at which we evaluate the Newtonian potential. The formal solution to equation (6.8) is

$$h_{\mu\nu}(x) = -2\kappa_5^2 \int d^4x' G(x, x')\Sigma_{\mu\nu}(x'). \tag{6.11}$$

After transforming back to the barred metric, we have $\bar{h}_{\mu\nu} = h_{\mu\nu}^{matter} - 2k\gamma_{\mu\nu}\zeta$, where the first term includes all matter terms $\tau_{\mu\nu}$ and the latter one the part that is caused by the bending of the brane.

Now, we can derive the weak field limit on the brane. For that we put spherically symmetric matter on the brane $\tau_{\mu\nu} \equiv \rho(r)u_\mu u_\nu$. With the help of equation (6.7), which describes the dislocation of the brane, and the solution for the Green's function, we calculate the Newtonian potential. Equation (6.11) gives that $h_{00} = -\frac{8}{3}V(r)$, where $V = \frac{\kappa_5^2}{2}\int d^3x' G(x, x')\rho(x')$. Since the appropriate point of view on the brane is that of the Gaussian normal coordinate frame, we obtain the following potential outside the mass distribution $\rho(r)$,

$$V(r) \approx -\frac{\kappa_5^2 k}{8\pi}\frac{M}{r}\left(1 + \frac{1}{2k^2r^2}\right), \tag{6.12}$$

where $M = \int d^3x\rho$. The brane displacement is $\zeta \approx -\kappa_5^2 M/(24\pi r)$. We want to stress that $V(r)$ is not the Newtonian potential, which, in contrast, is derived from

$$V_{Newton} = \frac{\bar{h}_{00}}{2} = \frac{G_4 M}{r}\left(1 + \frac{2}{3k^2r^2}\right), \tag{6.13}$$

where $G_4$ is the four-dimensional Newtonian constant as defined in section 2.5. Comparing this result with the one obtained in section 2.3, we find the correction terms to the Newtonian potential to be smaller by one third, but their structure remains the same. Thus, we have seen that we cannot neglect the brane displacement. Because of this, we will look into the issue of brane embedding from a more fundamental point of view in the following section.

## 6.3 Embedding of hypersurfaces

In the previous section, we saw that the embedding is of importance to perturbations in brane world models. The question, however, in which cases the displacement is a physical mode and when we can gauge it away, has not been clarified yet. In the absence of a common contradictory results in perturbative calculations can be found in the literature (see for example [63, 64, 65, 66, 67, 68]). A very different way to tackle the problem was taken by Mukohyama in [69, 70, 71], but his results have not dispelled the ambiguities completely either.

Since the brane is a physical object, we should extremize the action with respect to the variations of the embedding as we do for extended objects like cosmic strings. We will find an additional equation of motion for the brane, which sheds new light on the embedding procedure, and, by that, on perturbation theory in brane world models. The approach taken here goes back to Battye and Carter [72] and was used for the treatment of relativistic membranes and other extended objects in subsequent papers [73, 74, 75, 76, 77]. We adopt the formulation of fluctuations in terms of Lagrangian perturbation in which the displacement is already included. This allows for an easier variation of the action in the presence of brane-bending. Nevertheless, most of the calculations are rather long and tedious, but they lead to an interesting new result.

### 6.3.1 Notations and set-up

In what follows, we consider a codimension one brane in $n$ spacetime dimensions, that covers an $(n-1)$-dimensional, time-like sub-manifold. The embedding of the brane is given by $n$ functions $X^A(\sigma^\mu)$, where $\sigma^\mu$ denotes the coordinates on the sub-manifold. In the concurrent treatment of brane and bulk quantities, we have to deal with a mixing of full-spacetime indices and indices on the lower-dimensional hypersurface at the same time. In general, this leads to further complications and makes it harder to interpret the results. Therefore, we define full-spacetime tensors from the lower-dimensional ones by using the same procedure that relates the induced and the projected metric, $q_{AB} = \gamma_{\mu\nu} X^\mu{}_A X^\nu{}_B$, where $X^\rho{}_K$ is the inverse of $X^k_\rho \equiv \frac{\partial X^K}{\partial \sigma^\rho}$. For example, we define the brane energy-momentum tensor in full spacetime by

$$\tau_{AB} = \tau_{\mu\nu} X^\mu{}_A X^\nu{}_B. \tag{6.14}$$

This procedure amounts to a "deprojection" and we have to bear in mind that the tensors resulting from such an operation do not have full rank. We define a totally projected covariant derivative $D_A$ for which all indices are projected, i.e. $D_A F_B = q^C_{\ A} q^D_{\ B} \nabla_C F_D$. Then, the extrinsic curvature reads

$$K_{AB} = D_A n_B \tag{6.15}$$

with $n_C$ the unit normal to the hypersurface. Now, we go back to the framework of Einstein-Hilbert gravity with the most general set-up, in which the bulk and brane contain arbitrary matter fields. Then, the bulk and the brane action read

$$\begin{aligned} S_B &= -\frac{1}{2\kappa_n^2}\int_{bulk} d^n x\sqrt{|g|}\,(R+2\Lambda) + \int_{bulk} d^n x\sqrt{|g|}\mathcal{L}_B \\ S_b &= \frac{1}{\kappa_n^2}\int_{brane} d^{n-1}\sigma\sqrt{|\gamma|}[K]_\pm + \int_{brane} d^{n-1}\sigma\sqrt{|\gamma|}\mathcal{L}_b, \end{aligned} \tag{6.16}$$

where $[K]_\pm$ denotes the jump across the brane as in previous chapters and, by definition, the unit normal $n^A$ points in the '+'-region as sketched in figure 5.1. This gives an additional minus sign for the outward pointing normal of the '+'-region. From the action, we obtain the Einstein equations and junction conditions as given in equation (3.3) by variation with respect to the full metric and the induced metric. The energy-momentum tensors in the bulk and on the brane are defined in the usual way as in equation (3.4). Of course, the contracted Bianchi identities lead to energy-momentum conservation in the bulk, but we cannot expect $\nabla^\mu \tau_{\mu\nu} = 0$ to hold if there is some kind of interaction between bulk and brane matter.

### 6.3.2 Lagrangian perturbations

Normally, we suppose that the equations of motion for gravity and matter fields completely specify the evolution of the system. In our case, the embedding of the brane is an additional degree of freedom, therefore, it is our contention that we have to derive another equation by extremizing the action with respect to variations of the brane position. We argue that this is so, because the usual derivation of the equations of motion from the Nambu-Goto action for extended objects and also point particles goes along completely analogous lines.

But the straightforward computation of the variation $\delta S$ of the action for the variation of the embedding $X^A \to X^A + \delta X^A$ is a diabolical task, because

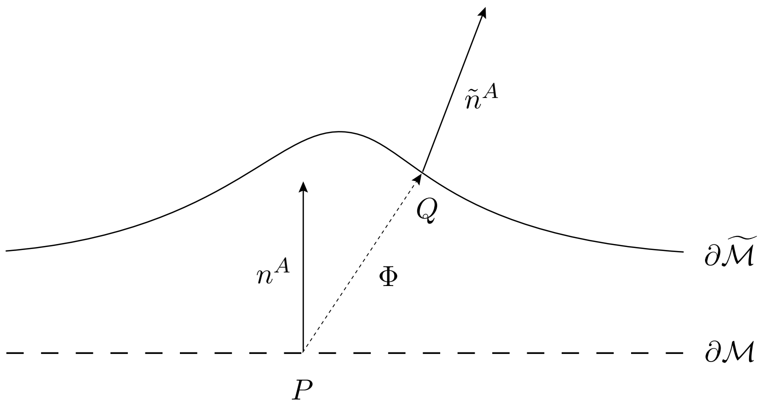


Figure 6.2: $\Phi$ *maps the unperturbed brane* $\partial\mathcal{M}$ *onto the perturbed brane* $\partial\widetilde{\mathcal{M}}$

we have to deal with the mixed tensor $X^A_\mu$ and its derivatives. Battye and Carter [72] described a different way of tackling the problem in terms of a bulk vector field. They switch to the so-called Lagrangian description of fluctuations, where one follows the displacement automatically.

To explain the idea of Lagrangian perturbations further, assume $\delta$ to be a perturbation of the metric, which we will henceforth refer to as Eulerian perturbation. As we have shown in the previous section, this Eulerian perturbation $\delta$ will in general bend the brane. This defines a mapping $\Phi$ from the unperturbed brane $\partial\mathcal{M}$ to the perturbed one $\partial\widetilde{\mathcal{M}}$ at which we have to evaluate the Eulerian fluctuation $\delta$ now (see figure 6.2). Because this would be extremely tedious, it is favorable to define a pull-back $\Phi^{-1}$ that brings the brane back to the unperturbed position and to include the additional correction caused by the bending of the brane in the perturbation itself. This enhanced perturbation is what we refer to as Lagrangian perturbation $\delta_L$. In mathematical terms, the difference between the Eulerian and the Lagrangian perturbation is given by the Lie derivative $\pounds_Y$ with respect to a differentiable vector field $Y^A$ that satisfies $Y^A(X^C) = \delta X^A$ at the position of the unperturbed brane. Hence the Lagrangian perturbation reads

$$\delta_L \equiv \delta + \pounds_Y, \tag{6.17}$$

Alternatively, we could say that in the Eulerian scheme the perturbations are defined with respect to a fixed coordinate system. Contrary to that, the Lagrangian scheme defines perturbations with respect to a coordinate system which is comoving with the perturbed position of the brane. By that, the coordinates of the brane remain unchanged and we can evaluate at the coordinates of the unperturbed brane.

Now we apply the Lie derivative to an arbitrary boundary and bulk action term

$$\begin{aligned} \pounds_Y \left[ \int d^{n-1}\sigma \sqrt{|\gamma|} \mathcal{L}_\pm \right] &= - \int d^{n-1}\sigma \sqrt{|\gamma|} \left[ \left( \pounds_n + K_\pm \right) \mathcal{L}_\pm \right] n_A \delta X^A \\ \pounds_Y \left[ \int d^n x \sqrt{|g|} \mathcal{L} \right] &= - \int d^{n-1}\sigma \sqrt{|\gamma|} \left[ \mathcal{L} \right]_\pm n_A \delta X^A, \end{aligned} \tag{6.18}$$

where $\pm$ stands for the evaluation on the respective side of the brane. Note that, although we apply the Lie derivative with respect to an arbitrary vector field $Y^A$, only the Lie derivative with respect to the normal vector $n^A$ survives.

### 6.3.3 A new equation of motion

Let us now apply the result of the previous section to the general action given in (6.16). The application of the Lie derivative to the trace of the extrinsic curvature gives

$$\pounds_n K = n^C \nabla_C K = \left( D^A + \alpha^A \right) \alpha_A - R_{AB} n^A n^B - K_{AB} K^{AB}, \tag{6.19}$$

where $\alpha_A \equiv n^C \nabla_C n_A$. Since we want to minimize the action under variations of the brane position, we demand that the Lie derivative of the action be zero and obtain

$$\begin{aligned} 0 &= \frac{1}{2\kappa_n^2} \left[ 2 \left( D^A + \alpha^A \right) \alpha_A - K_{AB} K^{AB} + K^2 \right]_\pm \\ &\quad + \left[ \mathcal{L}_B \right]_\pm + \left\{ \left( \pounds_n + K \right) \mathcal{L}_b \right\}_\pm . \end{aligned} \tag{6.20}$$

We have solved the ambiguity that is inherent in $\mathcal{L}_b$ by putting $\mathcal{L}_b \equiv \{\mathcal{L}_b\}_\pm$, where the curly brackets denote symmetrization, $\{u\}_\pm \equiv \frac{1}{2}(u_+ + u_-)$. Here it becomes obvious that an arbitrary displacement $\delta X^A$ that has five degrees of freedom in five-dimensional spacetime leads to only one equation. This is the manifestation of the fact that a brane displacement in tangential direction corresponds to a reparametrization of coordinates on the brane and not to a physical displacement. In the case of higher codimension objects, we would get the respective number of equations then.

Let us reformulate the terms in equation (6.20) in terms of quantities whose interpretation we know already. First, we tackle the projected derivative on $\alpha_A$. From the definition of the Christoffel symbols in general, as well as from the definition of the induced metric, we straightforwardly deduce the induced Christoffel symbols on the surface,

$$\Gamma^{\lambda}_{\ \mu\nu} = X_C^{\lambda}\left(\partial_\mu X^C_{\ \nu} + \Gamma^C_{\ AB} X^A_{\ \mu} X^B_{\ \nu}\right). \tag{6.21}$$

Applying the simple definition of the covariant derivative on the surface $\nabla_\mu X^C_{\ \nu} = \partial_\mu X^C_{\ \nu} - \Gamma^{\lambda}_{\ \mu\nu} X^C_{\ \lambda}$, we get

$$\begin{aligned}\nabla_\mu X^C_{\ \nu}\Gamma^C_{\ AB} X^A_{\ \mu} X^B_{\ \nu} &= (\delta^C_{\ D} - X^C_{\ \lambda} X^{\lambda}_D)(\partial_\mu X^D_{\ \nu} + \gamma^D_{\ AB} X^A_{\ \mu} X^B_{\ \nu}) \\ &= -n^C n_D(\partial_\mu X^D_{\ \nu} + \Gamma^D_{\ AB} X^A_{\ \mu} X^B_{\ \nu}.\end{aligned} \tag{6.22}$$

To proceed further, we contract the left and right hand sides of the equation with the unit normal $n_C$ and find for the extrinsic curvature

$$-K_{\mu\nu} = n_C(\nabla_\mu X^C_{\ \nu} + \Gamma^C_{\ AB} X^A_{\ \mu} X^B_{\ \nu}) = n_C(\partial_\mu X^C_{\ \nu} + \Gamma^C_{\ AB} X^A_{\ \mu} X^B_{\ \nu}), \tag{6.23}$$

where the first equality draws on the fact that $\nabla_\mu(n_C X^C_{\ \nu}) = 0$. Comparison with equation (6.21) and raising of the $\nu$-index as well as lowering of the $C$-index with the appropriate metrics gives

$$\nabla_\mu X^{\nu}_{\ C} = n_C K^{\nu}_{\ \mu} - \Gamma^B_{\ AC} X^A_{\ \mu} X^{\nu}_{\ B} \tag{6.24}$$

From this, we obtain for an arbitrary vector field $\nabla_\mu\left(A_C X^C_{\ \nu}\right) = X^B_{\ \mu} X^C_{\ \nu} \nabla_B A_C + n^C A_C K_{\mu\nu}$. Contraction with the induced metric then yields

$$\left[D^A \alpha_A\right]_\pm = -\left[n^A \alpha_A K\right]_\pm. \tag{6.25}$$

Next, we show the term $\alpha^A\alpha_A$ contains no derivatives in the normal direction. For an embedding that we describe by a scalar equation $F(X^A) = 0$, only terms proportional to $q_A^{\ B}\partial_B g_{CD}$ arise and the jump of this term vanishes. Now, take a brane Lagrangian that depends on the induced metric as well as on bulk matter fields and their derivatives, which we denote collectively by $M_i$ and which couple to brane quantities, i.e. $\mathcal{L}_b \equiv \mathcal{L}_b(\gamma_{\mu\nu}, M_i)$. Then, we obtain

$$(\pounds_n + K)\,\mathcal{L}_b = -\tau_{AB}K^{AB} + \frac{\delta\mathcal{L}_b}{\delta M_i}\pounds_n M_i. \tag{6.26}$$

Substituting the intermediate results into equation (6.20) and applying the standard Israel junction conditions, we obtain a strikingly simple new equation

$$[\mathcal{L}_B]_\pm + \left\{\frac{\delta\mathcal{L}_b}{\delta M_i}\pounds_n M_i\right\}_\pm = 0. \tag{6.27}$$

A magic cancellation of the term proportional to the energy-momentum tensor $\tau_{AB}$ on the brane occurs and $\kappa_n^2$ does not appear at all and no geometric tensors arise. Amazingly, this cancellation only appears in the case of gravity. The equation becomes more complicated, when we turn off the gravitational interaction: for $\kappa_n^2 \to 0$ we have

$$\left\{\tau_{AB}K^{AB}\right\}_\pm = [\mathcal{L}_B]_\pm + \left\{\frac{\delta\mathcal{L}_b}{\delta M_i}\pounds_n M_i\right\}_\pm. \tag{6.28}$$

Although this equation looks rather unfamiliar it results in the well known condition $K = 0$ in the case of an empty background with fixed geometry.

The new equation (6.27) is a constraint on the embedding of the brane. Therefore, we have to take it into account when we intend to transform to a gauge in which the brane is fixed. As long as the equation is fulfilled in a gauge that is convenient for the calculation, we are fine. There is an ambiguity in the case of a perfect fluid: there is no unique Lagrangian. Therefore, the constraint is problematic for an effective theory of matter. Only a description in terms of fundamental fields will yield unique results.

Let us check the constraint for configurations that we have considered. In the Randall-Sundrum case we have no additional bulk fields in the Lagrangian and the brane only carries a tension term, the derivative of which vanishes. There are no further matter terms that could contribute in equation (6.27) and it is satisfied trivially. Therefore, the procedure that Garriga and Tanaka applied in their derivation of the correct Newtonian limit, which we presented in the first part of this chapter, is consistent with equation (6.27).

Let us now consider the case of a bulk scalar field with Lagrangian $\mathcal{L}_B = \frac{1}{2}\phi_{,A}\phi_{,B} - V(\phi)$ and some form of coupling between the brane Lagrangian and the bulk scalar field, $\mathcal{L}_b \equiv \mathcal{L}_b(\gamma_{\mu\nu}, \phi)$. Varying the action with respect to $\phi$ gives the standard equations for a scalar field with the appropriate junction conditions on the brane as given in (3.5). Because the set of fields and their derivatives consists of $\phi$ only, $M_i = \{\phi\}$, it is easy to verify that the new equation (6.27) reduces to

$$\left[q^{AB}\phi_{,A}\phi_{,B}\right]_{\pm} = 0 \tag{6.29}$$

with the help of the junction condition for the bulk scalar field. Again, it is immediately obvious that the term does not contain normal derivatives and, therefore, the equation is trivially satisfied. This means that, in the scalar field case, the new equation is equivalent to the junction conditions of the bulk scalar field. We can understand this because a scalar field acts in a similar way to a matter source. But we have already shown that matter causes bending of the brane. This results in the equivalence of the new equation of motion and the junction conditions of the scalar field.

Although the new equation does not further restrict the solutions we have a look at, but the derivation of equation (6.27) includes new information because it makes clear that the consideration of the brane displacement is indispensable in any perturbative case. This cannot be seen from the junction conditions directly.

### 6.3.4 Perturbative equations of motion

Let us now treat perturbations in a general set-up in the description with Lagrangian perturbations. The perturbations in the bulk will take their usual form because the displacement of the brane location is irrelevant to them. However, the junction conditions will change in this set-up. In general, they can be written in the following form:

$$[H_{AB}]_{\pm} - h_{AB} = 0, \tag{6.30}$$

where $H_{AB}$ stands for the perturbed part of the respective quantity in the junction condition and $h_{AB}$ denotes the perturbative component of the metric and field functions. Again, we will replace $h_{AB}$ by the symmetrized quantity $\{h_{AB}\}_{\pm}$ to evade possible ambiguities. In linear order, we have for a smooth vector field $Y^A$, which is also assumed to be a perturbative quantity,

$$\pounds_Y \left([H_{AB}]_\pm - h_{AB}\right)\Big|_{y_0} = \left([\partial_C H_{AB}]_\pm - \{\partial_C h_{AB}\}_\pm\right)\delta X^C, \tag{6.31}$$

where we have to evaluate the left hand side at the position of the unperturbed brane. Note that we have made use of the background equations to simplify the expression. This is exactly, what we would naively expect for the junction conditions when we evaluate for the displaced brane. This result has recently been emphasized by Malik *et al.* in [68]. The advantage of the Lagrangian description is that the contribution is seen to be manifestly tensorial.

# Chapter 7

# Gravitational perturbations

Gravitational perturbations can and will be essential for the detection of extra dimensions. As already stated, it seems increasingly unlikely that the next generation of collider experiments will show higher-dimensional effects such as the Kaluza-Klein tower of massive states of particles. Therefore, much effort has been going into the investigation of gravitational fluctuations in general, see for example [79, 65, 80, 82, 83, 84] and references therein, and, particularly, in the cosmological context [85, 86, 87, 64, 88, 89, 90, 91]. It should be pointed out that the earlier articles do not take brane displacement into account and the results are contradictory. Maartens claims that there is a significant suppression of gravity waves during inflation [37], which could be detected in future CMB experiments.

In this chapter, we derive the gauge-invariant perturbations of brane worlds in the context of higher-derivative theory. We show the validity of Newtonian gravity on the brane. In the course of the derivation, we again exploit the conformal equivalence between higher-derivative gravity and usual Einsteinian gravity with an additional auxiliary scalar field. Our starting point will be the scalar field frame in which we solve the equations of motion. Then, we will transform the solutions back to the physical higher-derivative frame.

We want to remind the reader that the metric background functions in higher-derivative gravity do not have the typical kink at the brane in their first derivatives. In this case, kinks only arise in higher-derivatives of the metric functions and fields as pointed out in section 5.2. We saw that the absence of the jump in the first derivatives does not pose a problem per se, because the equations of motion for $f(R)$-gravity contain higher derivative of the metric functions than for Einstein-Hilbert gravity. In the course of this chapter, it will become clear that we can still have the standard Newtonian limit on the brane as long as it holds in the scalar field frame; the conformal transformations with only $y$-dependence cannot destroy it.

## 7.1 Set-up in scalar field frame

We follow the course of the calculations for higher-derivative gravity that were provided in chapter 5 and choose our set-up in the scalar field frame. This means we have a single (3+1)-brane embedded in a higher-dimensional bulk with usual Einstein gravity plus an auxiliary scalar field $\varphi$ as source,

$$\begin{aligned}\mathcal{L} \;=\; & \int d^5x\sqrt{|g|}\left(-\frac{1}{2\kappa_5^2}R+\frac{1}{2}g^{AB}\varphi_{,A}\varphi_{,B}-V(\varphi)\right) \\ & +\int d^4\sigma\sqrt{|\gamma|}\left((\lambda+\mathcal{L}_b)\,U(\varphi)+\frac{1}{\kappa_5^2}[K]_\pm\right). \qquad (7.1)\end{aligned}$$

To facilitate inspection, we treat the tension $\lambda$ separately from the otherwise arbitrary brane matter with Lagrangian $\mathcal{L}_b$. In $n$ dimensions, $U(\varphi) = e^{-\kappa_n\varphi\sqrt{\frac{n-1}{n-2}}}$ describes the coupling of brane matter to the bulk scalar field and is chosen such that we get a brane with tension only in the physical higher-derivative frame for vanishing $\mathcal{L}_b$. The same happens for additional matter on the brane that does not couple to $\varphi$. As usual, our set-up contains only one extra dimension. We split the metric into background and perturbative part

$$g_{AB} = {}^{(0)}g_{AB} + \delta g_{AB}. \qquad (7.2)$$

For the background we choose the conformal metric

$$ds^2 = a^2(y)(\eta_{\mu\nu}dx^\mu dx^\nu - dy^2), \qquad (7.3)$$

and for the perturbations we take the ansatz

$$\delta g_{AB} = a^2(y) \left( \begin{array}{c|c} 2\psi\eta_{\mu\nu} + E_{,\mu\nu} + h_{\mu\nu} & B_{,\mu} \\ \hline B_{,\mu} & 2\phi \end{array} \right) \tag{7.4}$$

where $h_{\mu\nu}$ denotes the transverse and traceless part of the metric perturbations, which, by definition, fulfills

$$h^{\mu}{}_{\nu,\mu} = 0, \qquad h^{\mu}{}_{\mu} = 0. \tag{7.5}$$

In the perturbations, we have neglected the vector parts, because to linear order there is no support for them in the bulk or on the brane, as long as we take the brane matter to be of scalar nature.

In order to avoid gauge artifacts and to consider only physical modes, we rewrite the equations of motion in a gauge invariant way. Considering the gauge transformations $x^A \to x^A + v^A$, where

$$v_A = a^2(y) \left( \xi\left(x^C\right)_{,\mu}, \xi_5\left(x^C\right) \right), \tag{7.6}$$

we obtain the transformation behavior for the perturbative quantities

$$\delta g_{AB} \to \delta g_{AB} + (a(y)^2 \xi_A)_{|B} + (a(y)^2 \xi_B)_{|A}, \tag{7.7}$$

where $_{|C}$ stands for covariant differentiation with respect to the background metric. From these transformations, we can easily find the two scalar gauge invariant variables

$$\begin{aligned} \Phi &\equiv \phi - \left(B - \frac{1}{2}E'\right)' - \frac{a'}{a}\left(B - \frac{1}{2}E'\right) \\ \Psi &\equiv \psi + \frac{a'}{a}\left(B - \frac{1}{2}E'\right). \end{aligned} \tag{7.8}$$

The tensor modes $h_{\mu\nu}$ are already gauge invariant. As usual, the partial differentiation with respect to the $y$-coordinate is denoted by a prime. By changing to the conformal gauge, we easily obtain the scalar part of the equations of motion for the gauge invariant quantities in the bulk:

$$
\begin{aligned}
5-5 \qquad & -6\frac{a'^2}{a^2} - 12\frac{a'^2}{a^2}\Phi + 3\Box\Psi - 12\frac{a'}{a}\Psi' = \\
& = \kappa_5^2\left(-\frac{1}{2}{}^0\varphi'^2 + a^2V({}^0\varphi) - {}^0\varphi'\delta\varphi' + a^2V'({}^0\varphi)\delta\varphi - {}^0\varphi'^2\Phi\right) \\
5-\mu \qquad & 3\frac{a'}{a}\Phi_{,\mu} + 3\Psi'_{,\mu} = -\kappa_5^2\,{}^0\varphi'\delta\varphi_{,\mu} \\
\mu-\nu \qquad & -\eta_{\mu\nu}\Big(3\frac{a''}{a} + \Box\Phi + 3\frac{a'}{a}\Phi' + 6\frac{a''}{a}\Phi - 2\Box\Psi + 9\frac{a'}{a}\Psi' + 3\Psi''\Big) + \\
& -\Phi_{,\mu\nu} + 2\Psi_{,\mu\nu} \quad = \quad \kappa_5^2\eta_{\mu\nu}\Big(\frac{1}{2}{}^0\varphi'^2 + a^2V({}^0\varphi) + {}^0\varphi'\delta\varphi + \\
& +a^2V'({}^0\varphi)\delta\varphi + {}^0\varphi'^2\Phi\Big), \qquad (7.9)
\end{aligned}
$$

where $\varphi$ denotes the scalar field in the bulk with an arbitrary potential $V(\varphi)$. The perturbation of the scalar field $\delta\varphi$ stands for the gauge invariant quantity. Although these equations seem rather complicated, we will show that the set of equations can be decoupled and consequently solved. In the $\mu-\nu$ component, we have an additional equation for the tensor part $h_{\mu\nu}$ of the metric, which in the linearized equations is totally decoupled from the scalar variables. Further simplification of the equations for the tensorial perturbations occurs because of the background equation and we obtain

$$\left(\partial_y^2 + 3\frac{a'}{a}\partial_y - \Box\right)h_{\mu\nu} = 0. \qquad (7.10)$$

We obtain two constraints from our set of equations: first, we read off from the off-diagonal $\mu-\nu$ components that $\Phi = 2\Psi$. Second, after substituting this equality into the $5-\mu$ equation, we find

$$3\Psi' + 6\frac{a'}{a}\Psi = -\kappa_5^2\,{}^0\varphi'\delta\varphi \qquad (7.11)$$

Plugging the first constraint into the sum of the $5-5$ and $0-0$ components of the Einstein equations (7.9), we end up with

$$3\Psi'' - 3\Box\Psi + 27\frac{a'}{a}\Psi' + \left(12\frac{a''}{a} + 24a'^2a^2\right)\Psi = -\kappa_5^2 2a^2V'({}^0\varphi)\delta\varphi. \quad (7.12)$$

This equation can be further simplified by substituting the right hand side by means of the second constraint (7.11) and the background equation of motion for the scalar field (see also reference [83]). Finally, we obtain a wave equation for the only remaining independent metric perturbation $\Psi$:

$$\Psi'' - \Box\Psi + \left(3H - 2\frac{\varphi''}{\varphi'}\right)\Psi' + 4\left(H' - H\frac{\varphi''}{\varphi'}\right)\Psi = 0, \tag{7.13}$$

where $H \equiv \frac{a'}{a}$.

We also have to take the junction conditions into account which give additional restrictions for the scalar field and the metric functions at the brane location. For the background they read

$$\begin{aligned} [a']_\pm &= -\frac{1}{3}\kappa_5^2\lambda U({}^0\varphi)a^2, \\ \left[{}^0\varphi'\right]_\pm &= a\lambda U'({}^0\varphi) \end{aligned} \tag{7.14}$$

and for the perturbed quantities we have

$$\begin{aligned} \left[\zeta_{,\mu\nu} + \tfrac{1}{2}h'_{\mu\nu}\right]_\pm &= \kappa_5^2 a\left(\tau_{\mu\nu} - \tfrac{1}{3}\,{}^0q_{\mu\nu}\tau\right), \\ [\delta\varphi']_\pm &= -a\lambda\left(2U'({}^0\varphi)\Psi - U''({}^0\varphi)\delta\varphi\right) + a\tfrac{1}{2\sqrt{3}}U({}^0\varphi)\tau, \\ [\Psi']_\pm &= \tfrac{1}{3}\kappa_5^2 a\lambda\left(2U({}^0\varphi)\Psi - U'({}^0\varphi)\delta\varphi\right). \end{aligned} \tag{7.15}$$

which we derived by means of the Lagrangian approach shown in the preceding chapter. In the first perturbed junction condition, we have included the perturbation of the brane position $y_b \to y_b + \zeta$, where $\zeta$ is defined as in section 6.1. We also substituted for $\delta\varphi$ using the second constraint from the Einstein equations. To derive the Newtonian potential for weak sources, we assume the matter term on the brane $\tau_{\mu\nu} = \rho(r)u_\mu u_\nu$ to be spherically symmetric, for example, a point-like mass. This means that $\rho(r)$ has to be a first order quantity and acts like a test particle. The term $\tau$ denotes the linearized trace of the brane stress-energy-tensor $\tau \equiv \gamma^{\mu\nu}\tau_{\mu\nu}$. Because the evaluated Lagrangian of a pressure-less perfect fluid vanishes, there is no additional term from the matter on the brane that enters the right hand side of the equation for $\delta\varphi'$.

## 7.2 Solutions for tensor perturbations

First, we focus on the tensor perturbations of the metric as was done by Randall and Sundrum and by Garriga and Tanaka, who included the effects from brane-bending in the RS I scenario. The junction conditions for the tensor perturbations $h_{\mu\nu}$ have no coupling to the scalar perturbations in linear order, of course. The stress-energy-tensor on the brane couples to the tensor mode of the metric as well as the perturbation of the scalar field. As in the preceding chapter we can combine equation (7.10) with the perturbed junction conditions in one equation such that the $y$-integral across the brane yields the junction conditions again. For a $Z_2$-symmetric set-up, we get

$$\left(-\frac{1}{2}\partial_y^2 - \frac{3}{2}\frac{a'}{a}\partial_y + \frac{1}{2}\Box\right) h_{\mu\nu} = -\delta(y)\kappa_5^2\Sigma_{\mu\nu}. \tag{7.16}$$

The source term is given by $\Sigma_{\mu\nu} \equiv \left(\tau_{\mu\nu} - \frac{1}{3}q_{\mu\nu}\tau\right) + \frac{1}{\kappa_5^2}\frac{1}{a}\zeta_{,\mu\nu}$. We again solve this equation by means of the Green's function. Additionally, we perform a Fourier transformation with respect to the coordinates $x^\mu$ tangential to the brane in order to obtain a one-dimensional problem:

$$\hat{D}_k G_k(y,y') \equiv \left(\partial_y^2 + 3\frac{a'}{a}\partial_y + k^2\right) G_k(y,y') = \delta(y-y') \tag{7.17}$$

This can be treated as an eigenvalue problem $\hat{D}_k\psi_m = (k^2 - m^2)\psi_m$. To get rid of the first derivative in $\hat{D}_k$, we redefine $\psi_m \equiv a^{3/2}\phi_m$ and obtain the eigenvalue equation

$$\left(\partial_y^2 - \frac{\left(a^{3/2}\right)''}{a^{3/2}} + m^2\right)\phi_m = 0. \tag{7.18}$$

We have to bear in mind that the second $y$-derivative of $a$ may have a discontinuous part across the brane, i.e. $a'' = a''_{cont.} + [a']_\pm\delta(y)$. The Green's function will then be given by

$$G_k(y,y') = \sum_{bound} \frac{a^{-3}\phi_m(y)\phi_m(y')}{k^2 - m^2} + \int_{continuum} dm \frac{a^{-3}\phi_m(y)\phi_m(y')}{k^2 - m^2}. \tag{7.19}$$

The normalization of $\phi_m$ is chosen such that

$$\int_{-\infty}^{\infty} a\phi_m(y)\phi_n(y) = \begin{cases} \delta_{mn} & \text{for bound states} \\ \delta(m-n) & \text{for continuum states.} \end{cases} \tag{7.20}$$

Because we are mostly interested in the Newtonian limit on the brane, we calculate the stationary limit of the Greens function at the brane position:

$$\begin{aligned} G^{(3)}(\mathbf{x}, \mathbf{x}') &= \int_{-\infty}^{\infty} dt' \int \frac{d^4k}{(2\pi)^4} G_k(0,0) e^{ik_\mu(x^\mu - x'^\mu)} \\ &= -\int_0^{\infty} \frac{dk}{2\pi^2} k^2 G_k^{(3)} \frac{\sin(kR)}{kR}, \end{aligned} \tag{7.21}$$

where $k$ now denotes the length of the spatial part of $k_\mu$, $R \equiv |\mathbf{x} - \mathbf{x}'|$ is the separation distance on the brane and

$$G^{(3)} = \sum_{bound} \frac{\psi_m^2(0)}{k^2 + m^2} + \int_{cont.} dm \frac{\psi_m^2(0)}{k^2 + m^2}. \tag{7.22}$$

We can perform the k-integration explicitly. This gives

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi R} \left[ \sum_{bound} e^{-mR} \psi_m^2(0) + \int_{cont.} dm\, e^{-mR} \psi_m^2(0) \right]. \tag{7.23}$$

A similar computation was performed in [92].

## 7.3 Newtonian limit

Inspection of equation (7.23) shows that we will have the right $1/R$ behavior as long as a zero mode exists. The contributions from the other modes can only spoil this limit if there are modes with small masses $mR << 1$ or tachyonic modes. The lowest massive eigenstate is model dependent, but we can show [90] that there is always a zero mode and no tachyonic mode, that is $m^2 \geq 0$.

To prove this contention, we rewrite equation (7.18) in terms of the operators $\hat{D}_\pm \equiv \partial_y \pm \frac{3}{2} X_y$. We find

$$\left(\hat{D}_+\hat{D}_- + m^2\right)\phi_m = 0. \tag{7.24}$$

Multiplying this equation with $\phi_m$ and integrating over $y$ gives

$$\begin{aligned} m^2 &= -2\int_0^{y_*} dy\,\phi_m\hat{D}_+\hat{D}_-\phi_m \\ &= -2\phi_m\hat{D}_-\phi_m\Big|_0^{y_*} + 2\int_0^{y_*} dy\,\left(\hat{D}_-\phi_m\right)^2. \end{aligned} \tag{7.25}$$

Note that, since we generally have a singularity in the bulk, we have assumed the existence of a regulator brane at $y = y_*$, which shields any singularity. The first term is killed by the boundary conditions, and the second one is non-negative and therefore we do not have tachyonic modes. The zero mode can be calculated from $\hat{D}_-\phi_0 = 0$ and reads

$$\phi_0 \propto a^{\frac{3}{2}}. \tag{7.26}$$

### 7.3.1 Quadratic potentials with small masses

In what follows, we want to focus on solutions for a scalar field with a quadratic potential, $V(\phi) = \frac{1}{2}M^2\phi^2$, which we derived in section 3.4 (confer also [30]). The reason for that is that square potentials are - for small values of the scalar field - approximately equivalent to gravity theories with a corrective term proportional to the square of the Ricci scalar (for more details on the approximate equivalence see chapter 5). For the quadratic potential, we derived the background solution for the warp factor in chapter 3 by means of perturbation theory. In the small $m^2$ limit, we cannot completely ignore the first order term in our solution of the warp factor $X$ because it is necessary for obtaining the right boundary conditions at $y = 0$. However, to a good approximation, we may account for the first order solution $X_1$ by simply rescaling $X_0$ to

$$X \simeq \frac{1}{4}A\,\ln\left(1 - \frac{2}{3}\frac{y}{A}\right), \tag{7.27}$$

where $A \equiv 1 - \frac{1}{6}\epsilon$. Then it follows that

$$a(y) = \left(1 - \frac{y}{y_s}\right)^{y_s/6} \equiv u^{y_s/6}, \tag{7.28}$$

where $y_s = 2 - \frac{9}{4}\epsilon$ to first order in $\epsilon$. Note that the background metric was deduced in a different frame with coordinate $\tilde{y}$, and, therefore, our coordinate is related to the old one by $y = \int_0^{\tilde{y}} e^{-X(y')}dy'$. As one can see in (7.28), the coordinate transformation does not change the structure of the solution $a(y)$. For the given $a(y)$, we substitute the ansatz $\phi_m = u^p\mathcal{C}(u^s)$ into equation (7.18) and obtain a linear combination of Bessel functions of the first and second kind for the function $\mathcal{C}(u^s)$. The order of the Bessel functions is given by $q \simeq \frac{1}{4}(y_s - 2) = -\frac{1}{9}\epsilon$. Again we have to deal with an obvious singularity in the bulk, which we shield with a regulator brane. The tension and position of the second brane are chosen such that the background remains unchanged in between the branes and such that the maximum value of the scalar field is low enough to guarantee good equivalence with $(R + \alpha R^2)$-gravity theories. Finally, we have the solution for the $m$th mode of the scalar field

$$\phi_m = (y_s - y)^{\frac{1}{2}}\mathcal{C}_q(m(y_s - y)), \tag{7.29}$$

where $\mathcal{C}_q \equiv AJ_q + BY_q$ is a linear combination of Bessel functions.

The introduction of a regulator brane gives rise to a second set of junction conditions. Therefore, the modified set-up will yield a discrete mass spectrum. From the junction condition at both branes we get $\mathcal{C}_{q+1}(6my_s) = 0$ at the visible brane ($y = 0$) and $\mathcal{C}_{q+1}(6m(y_s - y_*) = 0$ at the position $y = y*$ of the regulator brane. The finite distance between the branes allows for an orthogonality relation of the Bessel functions:

$$\begin{aligned}\int_{y_*}^{0} dy\, u\mathcal{C}_q(m(y_s - y))\mathcal{C}_q(m'(y_s - y)) = \\ = \delta_{mm'}\left[\frac{1}{2}\mathcal{C}_q^2(my_s) - \frac{1}{2}u_*^2\mathcal{C}_q^2(m(y_s - y_*))\right],\end{aligned} \tag{7.30}$$

where we also made use of the junction conditions. The coefficients in the linear combination of Bessel functions $\mathcal{C}_q$ are then

$$\begin{aligned} A &= N_{q+1}Y_{q-1}(m(y_s - y_*)) = -N_{q-1}J_{q+1}(my_s), \\ B &= -N_{q+1}J_{q-1}(m(y_s - y_*)) = N_{q-1}Y_{q+1}(my_s). \end{aligned} \tag{7.31}$$

The overall normalization coefficients $N_q$ are determined by normalizing expression (7.30) to one.

To proceed further, we make use of the large argument expansion for Bessel functions in terms of sine and cosine functions. This is justified because the order of the Bessel functions is low, i.e. $\mathcal{O}(1)$. For this case, the Bessel functions approach their large argument expansions very quickly and we only have to make sure that the minimal mass $m_0$ is not too small. To ensure this, we have to take a closer look at the mass gaps determined by the junction conditions and the minimal mass of the massive modes on the other hand. Combining the junction conditions with the large argument expansion, which we substitute for the Bessel functions $J_q$ and $Y_q$ as well as for the coefficients $A$ and $B$, we find the mass gaps of the discrete spectrum

$$\Delta m_n = \frac{n\pi}{y_*} \quad \text{with} \quad n \in \mathbb{N}. \tag{7.32}$$

But this equation does not fix the minimal mass $m_0$. Naively, we would expect that we can calculate it from the boundary conditions in the large argument expansion of the Bessel function as well. However, there is a problem with this consideration: although the large argument expansion is a good approximation of the Bessel functions in our case, it does not reproduce the zeros of the Bessel functions exactly, and a slightly different position of the zero mode will alter the minimal mass significantly.

Nevertheless, we can exclude modes with small masses in the limit of large $my_s$: from the junction conditions we find

$$\Delta m = \pi y_*^{-1} > \pi y_s^{-1} \gg r^{-1}, \tag{7.33}$$

where $r$ is the scale down to which we have measured gravity so far. Therefore, the infinite tower of states will not seriously alter the Newtonian limit significantly. The Newtonian limit will also not be affected by light modes, because there is no massive bound state in the limit of large $my_s$. We conclude that the lightest mode always satisfies

$$m_0 \geq \frac{1}{y_s}. \tag{7.34}$$

Numerical computations of the lowest mass show that it increases with decreasing distance of the branes.

Now, we can calculate the corrections to the Newtonian potential on the brane for a given background $a(y) = u^{y_s/6}$. For the allowed masses we substitute

$$m_n = m_0 + \frac{n\pi}{y_*}, \quad n = 0, 1, \dots \tag{7.35}$$

with a numerically computed $m_0$. In the large argument expansion we find at the visible brane

$$\phi_n\Big|_{y=0} = 2\frac{N_{q-1}}{m y_s} \cos\left(\pi k + \frac{2n\pi}{\frac{y_*}{y_s}}\right). \tag{7.36}$$

After substitution for the normalization constants, we find the amazingly simple result

$$\phi_m\Big|_{y=0} = 2. \tag{7.37}$$

With the help of equation (7.23) and bearing in mind that $\phi(u = 1) = \psi(y = 0)$ as long as we choose the warp factor $a(y)$ to be zero on the brane, furthermore, bearing in mind that we have discrete modes only, we can write down the Green's function explicitly:

$$G(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \sum_{discrete} 2\, e^{-m|\mathbf{x}-\mathbf{x}'|}, \tag{7.38}$$

where $m$ denotes the allowed masses including the zero mode. To obtain the Newtonian potential we assume a point-like mass $\mathcal{M}$ at rest at postition $\mathbf{x}' = 0$ on the brane. Then the Newtonian potential reads in analogy with equation (6.11)

$$V_{Newton} = \frac{h_{00}}{2} = -\frac{\kappa_5^2}{8\pi}\frac{\mathcal{M}}{|\mathbf{x}|}\left(1 + \underbrace{e^{-m_0|\mathbf{x}|}\frac{1}{1 - e^{\frac{\pi|\mathbf{x}|}{y_*}}}}_{\equiv \Delta}\right). \tag{7.39}$$

Restoring the units in the Newtonian potential results in an effective four-dimensional Newton's constant $G_4 \equiv \frac{\kappa_5^4\lambda}{8\pi}$. In figure 7.1 we show the structure of the corrective term $\Delta$ to the Newtonian potential which diverges for

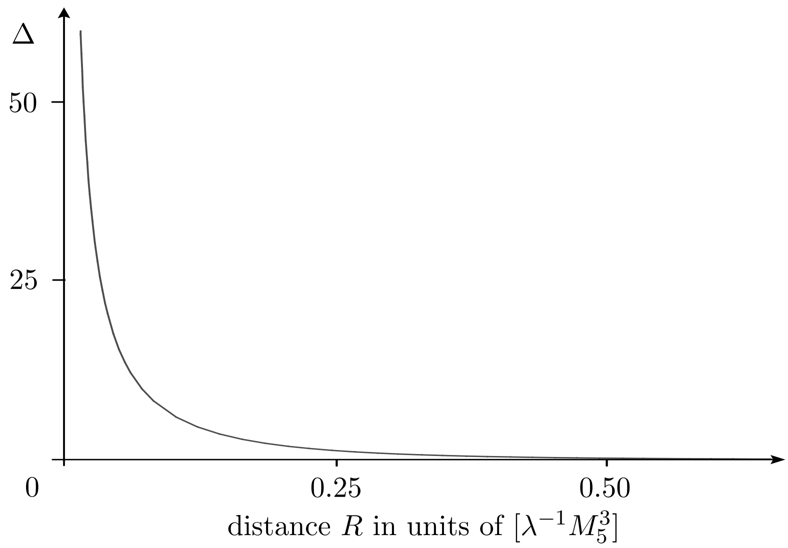


Figure 7.1: *correction term* $\Delta$ *in the Newtonian potential for* $m_0 = 5$

$|\mathbf{x}| \to 0$. From formula (7.39) it is obvious that we can always have corrections that drop off fast enough not to change the Newtonian potential in the experimentally observed range of gravity. Indeed, the corrections at large distances decay faster than in the original RS model (confer equation (2.45)). Today, the experimentally observed range of gravity goes down to scales of a hundredth of a millimeter. The fast decay of the corrections can always be achieved by choosing the minimal mass $m_0$ large enough, which, as we explained before, corresponds to placing the second brane close enough to the visible brane. Expanding the fraction in equation (7.39) for small values of $|x|$ yields $y_*/\pi|x|$. This shows that a position of the second brane close to the visible one does not counteract the exponential term $e^{-m_0 R}$, but, on the contrary, forces the corrections to drop off even faster with increasing distance. This corresponds to the result that the minimal mass rises with decreasing distance.

### 7.3.2 Large negative masses in the potential

Again, it would be no problem to compute our new $y$ variable by means of the zeroth order solution of the warp factor $X$, but we have to be careful with the derivatives of $X$. The second order term in equation (3.36), which is proportional to $G(x/\epsilon)$, is important for the computation of the effective potential in (7.18), as its second derivative is a zeroth order quantity. Therefore, it is more convenient in this limit to go back to the old coordinate system used in chapter 3, for which $ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$. Redefinition of the function $\chi_m \equiv a^{1/2}\phi_m$ gives

$$\left(\partial_y^2 + \frac{m^2}{1-\frac{y}{3}} + \frac{1}{12}\frac{\cos\frac{2y}{\epsilon}}{\left(1-\frac{y}{3}\right)^2}\right)\chi_m = 0. \tag{7.40}$$

This equation can be solved in the limit that the regulator brane is close enough to the visible brane. To be specific, we assume $y_* \sim \epsilon$, where $\epsilon$ is the small parameter proportional to $|M^2|^{-1}$. Defining $z \equiv y/\epsilon$, we immediately obtain a Mathieu equation

$$\left(\partial_z^2 + A - 2q\,\cos\,2z\right)\chi_m = 0, \tag{7.41}$$

with $A = \epsilon^2 m^2$ and $q = -\frac{1}{24}\epsilon^2$, for which the boundary conditions read $\partial_z\chi_m(0) = -\frac{1}{3}\epsilon = \partial_z\chi_m(z_*)$. In principle, we could have bound states with $my_* \ll mr \ll 1$, that is masses $m \sim \mathcal{O}(\epsilon^n)$ with $n < -1$, which could destroy the Newtonian limit. Let us show that this is not the case: the solutions to the Mathieu equation are linear combinations of periodic functions $F(\nu, \pm z)$, which for small $q$ can be expanded [93] to

$$F(\nu, z) \simeq e^{i\nu z}\left[1 - \frac{1}{4}q\left(\frac{e^{2iz}}{\nu+1} - \frac{e^{-2iz}}{\nu-1}\right)\right], \tag{7.42}$$

where $A \simeq \nu^2 + \frac{1}{2}q^2/(\nu^2-1)$ for $\nu$ real and non integer. Now we see that it is not possible to satisfy the boundary conditions for small, real $\nu$. This, on the other hand, implies that $\nu$ is imaginary or that $\nu \sim 1$, i.e. in an instability band of the Mathieu equation. For both cases we have that $A \sim 1$ and therefore no small mass states arise that would be dangerous for the Newtonian limit.

## 7.4 Scalar perturbations

In the case without scalar fields in the bulk, it is enough to consider only tensor perturbations. Obviously, $\Psi \equiv 0$ is a solution to equation (7.12), but it cannot fulfill the junction conditions (7.15) because the brane stress-energy-tensor enters the right hand side of the junction condition for the scalar field. This means a scalar mode might arise, which would change the Newtonian potential on the brane. To solve for $\Psi$ we define the new variable $u \equiv (a^{3/2}\varphi')\Psi$ and find

$$u'' + \Box u - \frac{\theta''}{\theta}u = 0 \qquad \text{with} \qquad \theta \equiv \frac{H}{a^{3/2}\varphi'}. \tag{7.43}$$

Again, we perform a Fourier transformation with respect to the tangential coordinates. It is enough to consider the massless mode only [94], because the massive modes cannot be excited in the low-energy regime, in which we are deriving the Newtonian limit. For the zero mode, it is possible to solve explicitly and we obtain

$$u_0(y) = c_1\theta + c_2\theta(y)\int_{y_r}^{y}\theta(\tilde{y})d\tilde{y} \tag{7.44}$$

with constants $c_1$ and $c_2$ and the position $y_r$ of the regulator brane. This translates into

$$\Psi_0(y) = \left[c_1 - 3\frac{a(y_r)^3}{H(y_r)}c_2\right]\frac{H(y)}{a(y)^3} + 3c_2\left[1 - 2\frac{H(y)}{a(y)^3}\int_{y_r}^{y}a(\bar{y})^3 d\bar{y}\right], \tag{7.45}$$

subject to the junction conditions. These can be rewritten [94] in the form

$$\left[g(\Psi' + 2H\Psi) + \Box\Psi\right]\Big|_{\text{brane}} = 0 \tag{7.46}$$

with

$$g \equiv \left[H - \frac{\varphi''}{\varphi} \pm \frac{1}{2}a\frac{d^2U(\varphi)}{d\varphi^2}\right]\Bigg|_{\text{brane}}. \tag{7.47}$$

The minus sign has to be taken on the visible brane. We can only comply with these junction conditions in some special cases, namely, either $g$ or $\varphi'$ has to be zero on one of the branes. This would only be the case if matter on the brane did not couple to the bulk scalar field $\varphi$ or has very special coupling like in Horava-Witten theory [94]. Here, the coupling $U(\phi)$ makes it impossible to fulfill the junction conditions, therefore, no zero mode of the scalar perturbations arises and the massive modes do not contribute at low energies. Therefore, we do not have an extra scalar contribution in the Newtonian limit in the scalar field frame.

## 7.5 Corrections in the physical frame

Now we have to check the weak field limit in the higher-derivative frame, in which we will experimentally observe gravity in the end. Unlike chapter 5, we have to transform back with the perturbed conformal factor $e^{\omega+\delta\omega}$. The quantity $\delta\omega$ will be a function of the perturbed Ricci scalar $\delta R$ only. In the scalar field frame $\delta R$ is expressed in terms of the perturbations of the scalar field $\delta\phi$. To specify this in more detail, we take a general $f(R)$-gravity theory and perturb the Ricci scalar. Then, we can rewrite the perturbation of the auxiliary scalar field to first order as

$$\delta\phi = \sqrt{\frac{3}{2}}\frac{\partial \ln F}{\partial R}\bigg|_{R_0}\delta R \tag{7.48}$$

with $F \equiv f'$ and $R_0$ the unperturbed Ricci scalar. On the other hand we also have to consider the transformation behavior of scalar perturbations, which could give us an extra contribution in the physical frame. To first order, the transformation reads

$$\bar{\Psi} = \Psi + \frac{1}{2}\frac{\partial \ln F}{\partial R}\bigg|_{R_0}\delta R. \tag{7.49}$$

The perturbation $\Phi$ transforms equivalently but with a minus sign. However, we have seen that no scalar perturbations arise at low energies and only the tensorial perturbations remain in the scalar field frame. It is a known result that tensor perturbations, which are transverse and traceless, are gauge invariant and do not cause perturbation of the Ricci scalar to linear order. Therefore, no additional scalar modes will appear after we have transformed back to the physical frame, i.e. $\bar{\Psi} = \bar{\Phi} = 0$. On the other hand, the tensor perturbations get rescaled by the conformal factor $e^{\omega}$, but their shape remains unchanged by the transformation. This means that the Newtonian limit holds for higher-derivative gravity with corrections to Einstein-Hilbert gravity, which are proportional to the square of the Ricci scalar, $f(R) = R+\alpha R^2$.

This is an intriguing result, because we have made no assumption for $\alpha$, which describes the strength of the corrective terms. Therefore, we can also choose an arbitrarily small corrective term, i.e. we can let $\alpha$ go to zero. We will still obtain a Minkowskian brane, on which gravity has the usual Newtonian limit, but we will not need an additional cosmological constant in the bulk, which is indispensable in the original Randall-Sundrum scenario.

That means, that higher-derivative gravity behaves fundamentally different than Einstein-Hilbert gravity and tends to a conceptually simpler model than the original Randall-Sundrum model.

# Chapter 8

# Conclusions

Let us now summarize and discuss the major results. We pointed out successes and problems of the still relatively new brane world scenario, and we tackled and provided solutions to some of the remaining problems. The treatment took place within the framework of Randall-Sundrum models in which the brane is a gravitating object which leads to a non-factorizable metric in the bulk and, thereby, allows for localization of gravity on the brane. We considered one-brane scenarios for simplicity and only introduced so-called regulator branes to shield unwanted features of the bulk.

Due to the complexity of the equations of motion, most of the known solutions for bulk scalar fields - and there are only a few - involve complicated and unnatural potentials. We should bear in mind that the early inflationary scenarios had to rely on quite unusual potentials, too. As we know now, there is a very simple and equally successfull scenario, chaotic inflation, which draws on simple quadratic potentials. Therefore, we sought for new solutions in the brane world picture with such simple potentials as the quadratic one, $V(\phi) = \frac{1}{2}m^2\phi^2$, in chapter 3. We made use of the method of strained parameters and obtained solutions for small mass parameters $m^2$ as well as for large negative $m^2$. The solutions expose an amazingly simple structure.

Since we often think of brane worlds as effective models of some fundamental theory, we expect modifications to Einstein-Hilbert gravity. In chapter 5, we considered higher-derivative gravity with Lagrangians that depend on the Ricci scalar only. We made use of the conformal equivalence between

$f(R)$-gravity and Einstein-Hilbert gravity with an effective scalar field to solve such systems. Then, we showed that the introduction of branes into the bulk does not cause a jump in the first derivatives of the metric functions and of the scalar field. Instead, jumps only arise in the third derivatives of the metric, i.e., for example, in the first derivative of the Ricci scalar. Moreover, we found that in models whose effective scalar field potentials have enough parameters, we can choose the parameters such that the effective cosmological constant in the physical higher-derivative frame vanishes. This is also possible for extremely flat potentials. Therefore, even a minimal deviation from Einstein-Hilbert gravity allows for a solution without a cosmological constant in the bulk. In this way, we were able to avoid the fine-tuning problem of the Randall-Sundrum models.

Because the kinks of the metric functions at the brane are missing in higher-derivative gravity theories, it is important to check that gravity is still effectively localized on the brane and that the Newtonian potential holds. We used the new background solutions for quadratic potentials to solve for the metric and scalar field perturbations. As usually, the background metric contains a singularity, which we have to screen with the help of a regulator brane. We found a discrete mass spectrum of the metric perturbations, which can be written as a linear combination of Bessel functions of the first and second kind. Then, we deduced the corrections to the four-dimensional Newtonian limit on the brane where the regulator brane is not too far away from the visible brane. Finally, we found that the corrections to Newton's potential decay exponentially and that the decay rate depends crucially on the mass of the first massive mode of the discrete spectrum. In most models, the first massive mode is heavy enough so that the standard Newtonian potential remains valid at distances larger than the typical curvature scale of the extra dimension.

In the treatment of perturbations, we argue that the embedding is not fixed a priori and, therefore, should be treated as an independent parameter of the theory. We derived the effect of brane-bending due to mass on the brane and showed that this changes the results of perturbations theory and, consequently, has to be included in perturbative considerations. Using Lagrangian perturbations which follow the brane displacement automatically, we derived a new equation of motion by varying the action with respect to the embedding. We showed that in the treatment of perturbations in higher-derivative gravity with brane-bending the additional equation of motion is satisfied.

With analytic solutions to quadratic scalar field potentials, and with results from models with higher-derivative gravity that do not rely on a cosmological constant in the bulk and hence can circumvent the problem of fine-tuning between brane tension and bulk cosmological constant, we have come a major step forward to finding simpler and more realistic brane world models. Nevertheless, much remains to be done and so far we have no observational signature of deviations from the standard four-dimensional picture. Aside from experiments that measure gravity at small scales and collider experiments, there is a chance to detect deviations in the cosmological background radiation.

Therefore, future work on higher-derivative gravity brane worlds will have to tackle cosmological issues as well. Nevertheless, the complexity of the equations does not often allow for an analytic solution of the problem and numerical computations will be indispensable. There are a number of numerical projects under way and some are already finished. The upcoming high-energy and, particularly, CMB experiments will further restrict possible scenarios and will help to spur the development of viable and realistic brane world models.

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