MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN

Investigations on the structural properties of Carlson's $<_1$-relation

Dissertation an der Fakultät für Mathematik, Informatik und Statistik der
Ludwig-Maximilians-Universität München

von Parménides García Cornejo

Doktorvater: Prof. Dr. Wilfried Buchholz

München, den 31. Oktober 2011

Erster Berichterstatter: Prof. Dr. W. Buchholz
Zweiter Berichterstatter: Prof. Dr. H. Schwichtenberg
Dritter Berichterstatter: Prof. Dr. W. Pohlers

Tag der Disputation: 2. Februar 2012.

I dedicate this, the best of what I have done, to the four pillars that have been supporting me since I can remember:

My aunt Lucía Cornejo Barrera
My uncle Dédalo García Arellano
My father Parménides García Arellano
My mother Julia Cornejo Barrera

Acknowledgments

There are a lot of people that have helped me, directly or indirectly, to carry out this dissertation. I want to thank all of them.

I thank my family, which has been the very foundation on which I have always sustained.

I thank all my friends; they are the benign environment in which this work progressed.

I acknowledge the mexican Consejo Nacional de Ciencia y Tecnologia CONACYT and the mexican society for their economical support.

There are some people that I have to mention specifically.

I want to thank Prof. Schwichtenberg, Prof. Buchholz and Prof. Sachs for everything what I have learned from them along these years I have been studying in Munich.

My most sincere thanks to Mrs. Schehrer and Mrs. Sousa von Blume for the immense help and kindness they have granted to me.

I want to thank my girlfriend Margarethe and her family. To her family, because of all the considerations they have had toward me and my family. To Margarethe, because she is the person that, after me, most has "felt in the flesh" the hard and good times of the development of this dissertation.

This document has been produced using GNU $\mathrm{T_EX_{MACS}}$ (see `http://www.texmacs.org`).

Abstract

This work studies Carlson's $<_1$-relation, where $\alpha <_1 \beta$ stands for $\alpha < \beta$ and that for any finite subset Z of β, there exists an $(<, +, <_1)$-embedding h: $Z \longrightarrow \alpha$ with $h|_{Z\cap\alpha} = \mathrm{Id}_{Z\cap\alpha}$.

The key ideas for the study of $<_1$ presented here are the introduction of the class $\mathrm{Class}(n)$, the intervals $[\alpha, \alpha(+^n))$, the space of functions $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$ and the relation $<^n$. The main results provide $(<, <_1, +)$-isomorphism-like properties of the space $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$, cofinality properties for $<^n$ and $<_1$, the fact that the class $\mathrm{Class}(n)$ is κ-club for any non-countable regular ordinal κ and the fact that certain subclasses of $\mathrm{Class}(n)$ (the "space of solutions of a condition $\langle \alpha, \eta(n, \alpha, t) + 1\rangle$") are κ-club for any non-countable regular ordinal κ bigger than α.

In the last chapter, after the results for $\mathrm{Class}(n)$, $\mathrm{Class}(\omega)$ is considered. This class is κ-club for any non-countable regular ordinal κ too and it is seen that its elements constitute the class of ordinals α such that $\forall \beta > \alpha. \alpha <_1 \beta$. From this fact and the work of Carlson follows that $O_\omega = \min\mathrm{Class}(\omega) = |\Pi^1_1\text{-}\mathrm{CA}_0|$.

The second part of the last chapter shows that, for Buchholz collapsing functions ψ_i, $\forall n \in \omega. \psi_n(\Omega_{n+2}) = \Omega_n(+^2)$; this means, particularly, that $|\mathrm{ID}_1| = \psi_0(\Omega_2) = \Omega_0(+^2) = \min\mathrm{Class}(2) = O_2$ as was already shown by Wilken.

The final conjecture is $\forall n \in \omega \forall m \in [1, \omega). \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$. In particular, this would mean $\forall m \in [1, \omega). |\mathrm{ID}_m| = \psi_0(\Omega_{m+1}) = \Omega_0(+^{m+1}) = \min\mathrm{Class}(m+1) = O_{m+1}$ and as an easy corollary of this fact it would follow another proof of the equality $O_\omega = |\Pi^1_1\text{-}\mathrm{CA}_0|$. The general statement $\forall n \in \omega \forall m \in [1, \omega). \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ remains, for $n \geqslant 3$, as a conjecture. The author of this thesis gives a sketch of a proof that in his opinion should be the essential argument for a proof of the conjecture in case one is able to provide certain version ψ'_i of the ψ_i functions satisfying some rather technical conditions.

Abstract

Diese Arbeit befasst sich mit Carlsons $<_1$-Relation, in der $\alpha <_1 \beta$ bedeutet, dass $\alpha < \beta$ und dass für jede endliche Teilmenge Z von β eine $(< , <_1 , +)$-Einbettung $h\colon Z \longrightarrow \alpha$ mit $h|_{Z \cap \alpha} = \mathrm{Id}_{Z \cap \alpha}$ existiert.

Die hier gegebenen Hauptideen für die Untersuchungen von $<_1$ stellen die Einführung der Klasse $\mathrm{Class}(n)$, die Intervale $[\alpha, \alpha(+^n))$, den Funktionsraum $\{x \longrightarrow x[g(n, \alpha, \gamma)] | \alpha, \gamma \in \mathrm{Class}(n)\}$ und die Relation $<^n$ dar. Die Hauptergebnisse zeigen dem Leser $(< , <_1 , +)$-isomorphisme Eigenschaften des Raumes $\{x \longrightarrow x[g(n, \alpha, \gamma)] | \alpha, \gamma \in \mathrm{Class}(n)\}$, Kofinalität Eigenschaften für $<^n$ und $<_1$, sowie die Tatsache, dass die Klasse $\mathrm{Class}(n)$ κ-club für jede nicht abzählbare reguläre Ordinalzahl κ ist und dass bestimmte Unterklassen von $\mathrm{Class}(n)$ κ-club für jede nicht abzählbare reguläre Ordinalzahl κ grösser als α sind.

Im letzten Kapitel, nach den Ergebnisen für $\mathrm{Class}(n)$, wird $\mathrm{Class}(\omega)$ untersucht. Diese Klasse ist auch κ-club für jede nicht abzählbare reguläre Ordinalzahl κ und es wird gezeigt, dass deren Elemente die Klasse der Ordinalzahlen α darstellt, die $\forall \beta > \alpha . \alpha <_1 \beta$ erfüllen. Aus dieser Ergebnis und der Arbeit von Carlson folgt, dass $O_\omega = \min \mathrm{Class}(\omega) = |\Pi^1_1 - \mathrm{CA}_0|$.

Der zweite Teil des letzten Kapitels zeigt, dass für Buchholz Kollabierungsfunktionen ψ_i, $\forall n \in \omega . \psi_n(\Omega_{n+2}) = \Omega_n(+^2)$ gilt. Insbesondere heisst das $|\mathrm{ID}_1| = \psi_0(\Omega_2) = \Omega_0(+^2) = \min \mathrm{Class}(2) = O_2$, wie es schon bei Wilken gezeigt wurde.

Die letzte Vermutung ist $\forall n \in \omega \forall m \in [1, \omega) . \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$. Das wurde insbesondere bedeuten, dass $\forall m \in [1, \omega) . |\mathrm{ID}_m| = \psi_0(\Omega_{m+1}) = \Omega_0(+^{m+1}) = \min \mathrm{Class}(m+1) = O_{m+1}$ und als ein einfaches Korollar aus diesem Ergebnis folgte die Gleichung $O_\omega = |\Pi^1_1 - \mathrm{CA}_0|$. Die allgemeine Aussage $\forall n \in \omega \forall m \in [1, \omega) . \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ bleibt, für $n \geqslant 3$, als eine Vermutung. Der Author dieser Dissertation skizziert in seiner Arbeit einen Beweis, der seiner Meinung nach das Hauptargument für den Beweis der oben genannten Vermutung wäre, sollte eine version ψ'_i der ψ_i Funktionen geben, die gewisse technische Voraussetzungen erfüllen.

Table of contents

Prologue

This work has as purpose the study of the $<_1$-relation. The main motivation to study $<_1$ are the works of T. Carlson and G. Wilken. The first version $\prec_1$ of $<_1$ was used by Carlson as a tool to show Reinhardt's conjecture: The Strong Mechanistic Thesis is consistent with Epistemic Arithmetic (see [8]); moreover, Carlson showed a characterization of ε_0 in terms of $\prec_1$ (see [9]) and indeed, set up a new approach to ordinal notation systems based on these ideas (see [10]). $<_1$ is a binary relation in the class of ordinals and in it's original form, $\alpha <_1 \beta$ asserts that the structure $(\alpha, <, +, <_1)$ is a Σ_1-substructure of $(\beta, <, +, <_1)$ (see Carlson [10] or Wilken [18]). In this work, for the study of $<_1$, I do not depart from it's original definition; instead, I use an equivalent notion that follows from standard theorems of model theory: $\alpha <_1 \beta$ means $\alpha < \beta$ and the following assertion: for any finite subset Z of β, there exists an $(<, +, <_1)$-embedding h: $Z \longrightarrow \alpha$ with $h|_{Z\cap\alpha} = \mathrm{Id}_{Z\cap\alpha}$ (see definition 1.1). Moreover, $\alpha \leqslant_1 \beta$ stands for $\alpha = \beta$ or $\alpha <_1 \beta$.

The study of $<_1$, as done here, is then a study of (a sort of) isomorphisms between the finite subsets of an arbitrary ordinal. Being more specific, there are several interrelated aspects that are considered along the whole work and whose understanding provide, in the end, a description of the behavior of $<_1$ in the whole class of ordinals: In a rather informal way, these are

0. The functional $m\colon \mathrm{OR} \longrightarrow \mathrm{OR} \cup \{\infty\}$,
   $m(\alpha) := \begin{cases} \max\{\xi \in \mathrm{OR} | \alpha \leqslant_1 \xi\} & \text{if there exists } \beta \in \mathrm{OR} \text{ such that } \alpha < \beta \text{ and } \alpha \not<_1 \beta \\ \infty & \text{otherwise, that is, } \forall \beta \in \mathrm{OR}. \alpha < \beta \Longrightarrow \alpha <_1 \beta \end{cases}.$
1. The classes $(\mathrm{Class}(n))_{n \in [1,\omega)}$, where
   $\mathrm{Class}(n) := \{\alpha \in \mathrm{OR} | \exists (\alpha_n, \ldots, \alpha_1) \in \mathrm{OR}^n. \alpha = \alpha_n <_1 \alpha_{n-1} <_1 \alpha_3 <_1 \ldots <_1 \alpha_1 <_1 \alpha_1 2\}$.
2. The study of $<_1$ in intervals of the form $[\alpha, \alpha(+^n))$, where $\alpha \in \mathrm{Class}(n)$ and $\alpha(+^n)$ is the successor of α in $\mathrm{Class}(n)$.
3. For $t \in [\alpha, \alpha(+^n))$, the space of isomorphisms h that are witnesses of $\alpha \leqslant_1 t$.
4. For $t \in [\alpha, \alpha(+^n))$, the space of ordinals γ that are solutions of the expression $\alpha \leqslant_1 t$, where $\alpha \leqslant_1 t$ is seen as a condition (**the pair $\langle \alpha, t\rangle$** is seen as a condition) that may be fulfilled by many ordinals.

The main results will show that the nature of $<_1$ is such that, for $n \in [1, \omega)$, we are able to describe a space of substitutions $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$ satisfying many $(<, <_1, +)$-isomorphism-like properties (we denote as **$\mathrm{Ep}(x)$** to the set of epsilon numbers appearing in the Cantor Normal Form of x and $x[g(n, \alpha, \gamma)]$ to ordinal obtained by the substitution of all the $e \in \mathrm{Ep}(x)$ by $g(n, \alpha, \gamma)(e)$ in the Cantor Normal Form of x); moreover, for $\alpha \in \mathrm{Class}(n)$ and $t \in [\alpha, \alpha(+^n))$, this isomorphisms-space allows us to consider $\{\gamma \in \mathrm{Class}(n) \mid \mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n, \alpha, \gamma) \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$, which is our formalization of the space of ordinals that are solutions of the condition $\langle \alpha, t\rangle$. It turns out that our isomorphisms-space will provide us with (canonical) witnesses for $\alpha \leqslant_1 t$ in important cases and that, for the condition $\langle \alpha, t\rangle$, it's space of solutions is κ-club for any non-countable regular ordinal κ bigger than α.

With respect to the work of Wilken [18], [19], [20] (our main bibliographic reference), this thesis develops in a quite different direction: In broad terms, Wilken defines certain (Skolem-hulling-based) notation systems such that from the form in which an ordinal α is denoted he can read off $m(\alpha)$, which Wilken denotes as $\mathrm{lh}(\alpha)$ and calls "the $\leqslant_1$-reach of α" (see definition 1.5). I do not embark upon the development of adequate and general enough notation systems having these characteristics (development that is quite intricate and full of complexities as shown in Wilken's work): The point of view taken here is that $<_1$ induces κ-club classes of ordinals (solutions of conditions $\langle \alpha, t\rangle$) and that given one of these classes and through a limit procedure, we can get the "higher classes" induced by $<_1$.

The phenomena that $<_1$ induces κ-club classes of ordinals as solutions of $\leqslant_1$-conditions occurs from the first attempts to understand how $<_1$ behaves. Chapter 1 is an introductory chapter showing this: After the basic definitions, conditions $\langle\alpha, t\rangle$ with $t\in[\alpha,\alpha\omega)$ are studied using the relation $<^0$ and it's cofinality properties. This, the simplest study of $<_1$ carried out with a first version of the general notions introduced in chapter 3, should give the reader a sense of how the whole work up to chapter 6 proceeds.

In chapter 2, as in [18], we study conditions $\langle\alpha,t\rangle$ with $\alpha\in\mathrm{Class}(1)$ and $t\in[\alpha,\alpha(+^1))$ (as a comment aside, $\mathrm{Class}(1)=\{\alpha\in\mathrm{OR}\mid\alpha<_1\alpha 2\}$ turns out to be the class of epsilon numbers). The techniques to solve the difficulties we encounter here resemble the ones of the first chapter:

- Introduction of the substitutions $\{x\longmapsto x[\alpha:=c]\mid\alpha,c\in\mathbb{E}\}=\{x\longmapsto x[g(1,\alpha,c)]\mid\alpha,c\in\mathbb{E}\}$ and it's $(<,<_1,+)$-isomorphism-like properties: corollary 2.17.

- The definition of the $<^1$-relation and it's relation with the space $\{x\longmapsto x[g(1,\alpha,c)]\mid\alpha,c\in\mathbb{E}\}$.

- Cofinality properties for $<^1$ (propositions 2.23 and 2.24): For $t\in[\alpha,\alpha(+^1))$, $\alpha<^1 t+1\Longleftrightarrow\alpha=\sup\{\gamma\in\alpha\cap\mathrm{Class}(1)\mid\mathrm{Ep}(t)\subset\mathrm{Dom}(g(1,\alpha,\gamma))\wedge\gamma\leqslant_1 t[g(1,\alpha,\gamma)]\}$. A version of this result appears already in the work of Wilken (see [18], lemmas 3.11 and 3.12).

- The solution of a main problem not occurring with the $<^0$-relation: Since for $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha(+^1))$ it is NOT always the case that $\alpha<_1 t+1\Longrightarrow\alpha<^1 t+1$, the ordinal $\eta t=\eta(1,\alpha,t)$ is introduced. Then, a major result showing crucial properties of ηt is shown: the covering theorem (theorem 2.33). The covering theorem has interesting consequences:

- Proof that the minimal $<_1$-witness of a cover $\Delta(\alpha,B)$ is a substitution whenever $\Delta(\alpha,B)\subset t<\alpha(+^1)$ and $\alpha<_1 t$ (corollary 2.34).
- Proof of $\alpha<_1\eta t+1\Longleftrightarrow\alpha<^1\eta t+1$ and the subsequent gain of cofinality properties for the $<_1$-relation in Class(1) (corollaries 2.35 and 2.40).

- Hierarchy theorem (theorem 2.45). Here the idea is that, for $\alpha\in\mathrm{Class}(1)$ and $t\in[\alpha,\alpha(+^1))$, the set $G(t):=\{\gamma\in\mathrm{Class}(1)\cap(\alpha+1)\mid\mathrm{Ep}(t)\subset\mathrm{Dom}\,g(n,\alpha,\gamma)\wedge\gamma\leqslant_1\eta t[g(n,\alpha,\gamma)]+1\}$ can be generated through a thinning procedure.

- As soon as we know that $G(t)$ is generated through a thinning procedure, it is shown that for $t\in[\kappa,\kappa(+^1))$, $G(t)$ is κ-club for each non-countable regular ordinal κ (proposition 2.49).

- Proof that Class(2) is κ-club for any non-countable regular ordinal κ (prop. 2.59).

Just as the study of conditions of the form $\langle\alpha,t\rangle$ with $t\in[\alpha,\alpha 2)$ leads naturally to the study of conditions $\langle\alpha,t\rangle$ with $\alpha\in\mathrm{Class}(1)$ and $t\in[\alpha,\alpha(+^1))$, we have that the studies carried out in chapter 2 lead to the study of conditions $\langle\alpha,t\rangle$ with $t\in[\alpha,\alpha(+^2))$ and $\alpha\in\mathrm{Class}(2)$. It is in this moment that we encounter a big problem that we didn't really care before: What do the elements of $[\alpha,\alpha(+^2))$ look like?. Previously, while studying, for example, conditions $\langle\alpha,t\rangle$ with $t\in[\alpha,\alpha 2)$, t could be simply written as $t=\alpha+\xi$ and $\alpha+\xi$ was a satisfactory way to "represent t in terms of α"; similarly, while studying conditions $\langle\alpha,t\rangle$ with $t\in[\alpha,\alpha(+^1))$, t could be "represented in terms of α" by it's Cantor Normal Form. Now, for $t\in[\alpha,\alpha(+^2))$, the Cantor Normal Form of t does not suffice anymore.

The way to describe an arbitrary $t\in[\alpha,\alpha(+^2))$ provided here is based on $m(t)$ and the cofinality properties developed while studying the intervals $[\delta,\delta(+^1))$ (where $\delta\in\mathrm{Class}(1)$). The basic idea is the following:
If $t=\alpha$, then "α is the description of t in terms of α". If $t\in(\alpha,\alpha(+^2))$, then $t\notin\mathrm{Class}(2)$ and we have two possibilities:

- $t\notin\mathrm{Class}(1)$. Then the cantor normal form of t and the description of the epsilon numbers appearing in such normal form provide the desired description of t.

- $t \in \mathrm{Class}(1)$. Considering $t^- := \sup(\{\alpha\} \cup \{e \in (\alpha, t) \cap \mathrm{Class}(1) \mid m(e)[e := t] \geqslant m(t)\})$ we have that, by the cofinality properties, $t^- < t$ and therefore, provided the description of t^-, we describe t as "t is the smallest epsilon number e in $(t^-, \alpha(+^2))$ such that $m(e)[e := t] \geqslant m(t)$".

The description of an ordinal $t \in [\alpha, \alpha(+^2))$ allow us to think of the pair $\langle \alpha, t\rangle$ as a condition that may be fulfilled by ordinals in Class(2); intuitively, given $\gamma \in \mathrm{Class}(2)$ and $s \in [\gamma, \gamma(+^2))$, γ is a solution of the condition $\langle \alpha, t\rangle$ if and only if:

- The "description of s in terms of γ" is the "same" as "the description of t in terms of α"
- $\gamma \leqslant_1 s$.

The reader may notice that the previous lines actually mean that we have some kind of isomorphism $H: [\alpha, \alpha(+^2)) \longrightarrow [\gamma, \gamma(+^2))$ and that, to tell that $\gamma \in \mathrm{Class}(2)$ is a solution of $\langle \alpha, t\rangle$ is just to tell that $\gamma = H(\alpha) \leqslant_1 H(t)$. This idea is important and through the careful development of it one gets that the collection of those H's constitutes $\{x \longmapsto x[g(2, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(2)\}$, which we may call "our class of isomorphisms for Class(2)".

Let us take a closer look at $\{x \longmapsto x[g(2, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(2)\}$. This is a class of substitutions build up from the $g(2, \alpha, \gamma)$ functions. The intuition of how $g(2, \alpha, \gamma)$ is defined was given above: For $\alpha, \gamma \in \mathrm{Class}(2)$ with $\alpha < \gamma$, $g(2, \alpha, \gamma): \alpha(+^2) \cap \mathbb{E} \longrightarrow \gamma(+^2) \cap \mathbb{E}$ is the function:

- $\forall e \in \mathbb{E} \cap \alpha . g(2, \alpha, \gamma)(e) := e$,
- $g(2, \alpha, \gamma)(\alpha) := \gamma$,
- For $t \in (\alpha, \alpha(+^2)) \cap \mathbb{E}$,
  $g(2, \alpha, \gamma)(t) := \min\{e \in (g(2, \alpha, \gamma)(t^-), \gamma(+^2)) \cap \mathrm{Class}(1) | m(e) \geqslant m(t)[g(2, \alpha, \gamma)|_t][t := e]\}$,
  where $t^- := \sup(\{\alpha\} \cup \{e \in (\alpha, t) \cap \mathrm{Class}(1) \mid m(e)[e := t] \geqslant m(t)\})$.

It takes quite a bit of work to show that the previous notions are well defined, but the idea is that based on them one can develop:

- Isomorphism-like properties of $\{x \longmapsto x[g(2, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(2)\}$.
- The relation $<^2$ based on the space $\{x \longmapsto x[g(2, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(2)\}$ and cofinality properties for $<^2$.
- Covering theorem for Class(2) and it's consequences.
- Hierarchy theorem for Class(2).
- Canonical sequence for an ordinal $\alpha(+^2) \in \mathrm{Class}(2)$.
- Non-countable regular ordinals and consequences for Class(2).
- $\mathrm{Class}(3) = \{\alpha \in \mathrm{Class}(2) \mid \alpha <_1 \alpha(+^2)\}$ is κ-club for any non-countable regular ordinal κ.

Upper classes induced by $<_1$

The reason to give the previous closer view to Class(2) is because the most general form of these ideas and it's formalization introduced in chapter 3 (the classes Class(n), the intervals $[\alpha, \alpha(+^n))$, the space of functions $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$, the ordinals $\eta(n, \alpha, t)$ and $l(n, \alpha, t)$ and the relation $<^n$) is quite elaborated and it is easier to explain them "with an example" (i.e., with Class(2)). Indeed, chapter 3 consists of such ample formalization and the statement of the generalization of the results obtained in chapter 2 we are striving for: Theorem 3.25 (or it's more technical version: Theorem 3.26).

The proof of theorem 3.26 is very very long and it is finished until chapter 6. The reason for such a big proof is that the propositions stated there are dependent to each other and therefore it is necessary to show them simultaneously.

Chapter 4 shows that Class(n) is κ-club for any non-countable regular ordinal κ. The proof that Class(n) is closed in κ follows without many complications from our induction hypothesis (in particular, from the cofinality properties for the relation $<^{n-1}$). To show that Class(n) is unbounded in κ is much more complicated: The idea is that, by induction hypothesis, the space $\{x \longmapsto x[g(n-1,\alpha,\gamma)] \mid \alpha,\gamma \in \mathrm{Class}(n-1)\}$ is already defined, and therefore, for $\alpha \in \mathrm{Class}(n-1)$ and $t \in [\alpha, \alpha(+^{n-1}))$, the set
$G^{n-1}(t) := \{\gamma \in \mathrm{Class}(n-1) \mid \mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n-1,\alpha,\gamma) \wedge \alpha \geqslant \gamma \leqslant_1 \eta(n-1,\alpha,t[g(n-1,\alpha,\gamma)]) + 1\}$ is well defined. Then one shows, through a generalized hierarchy theorem, that $G^{n-1}(t)$ can be generated through a thinning procedure; after that, picking $\alpha = \kappa$ as a non-countable regular ordinal, one gets that $G^{n-1}(t)$ is κ-club (proposition 2.49). Finally, one shows that for any $r \in \kappa \cap \mathrm{Class}(n-1)$ and for $M^{n-1}(r,\kappa) := g(n-1, r, \kappa)[[r, r(+^{n-1}))]$, the set $\bigcap_{s \in M^{n-1}(r,\kappa)} G^{n-1}(s)$ is κ-club and is contained in Class(n) (propositions 4.19 and 4.20). As a final commentary, for the proof of the contention $\bigcap_{s \in M^{n-1}(r,\kappa)} G^{n-1}(s) \subset \mathrm{Class}(n)$ it is used a fundamental sequence for the ordinal $r(+^{n-1}) \in \mathrm{Class}(n-1)$ (definition 4.16 and proposition 4.17).

Chapter 5 deals with the construction of the $\{x \longmapsto x[g(n,\alpha,\gamma)] \mid \alpha,\gamma \in \mathrm{Class}(n)\}$ space. The work carried out here is very heavy (if not, sometimes, overwhelming), dealing with a lot of technical problems that arise while trying to construct the functions $g(n,\alpha,\gamma)$. It also contains what, in the opinion of the author of this thesis, the cornerstone of these studies is: **The extension theorem** (theorem 5.10).

The problem starts as follows: For $\alpha, \gamma \in \mathrm{Class}(n)$, we would like to obtain $g(n,\alpha,\gamma)$ in an analogous way as the $g(2,\beta,\zeta)$ functions were sketched before and particularly, we want to be sure that the equality $m(\delta[g(n,\alpha,\gamma)]) = m(\delta)[g(n,\alpha,\gamma)]$ holds for any $\delta \in (\alpha, \alpha(+^n))$. However, for example for $n \geqslant 6$ and given $\alpha, \gamma \in \mathrm{Class}(n)$ with $\alpha < \gamma$, what should be the value of $g(n,\alpha,\gamma)$ in some $\delta \in (\alpha,\alpha(+^n)) \cap \mathrm{Class}(3)$?. When we were working in Class(2) we essentially had the problem to define the values of $g(2,\alpha,\gamma)$ in elements of Class(1) (that is, for $\alpha \in \mathrm{Class}(2)$, $(\alpha, \alpha(+^2)) \cap \bigcup_{i \in [1,\omega)} \mathrm{Class}(i) = (\alpha,\alpha(+^2)) \cap \mathrm{Class}(1)$), but in general, for $n \in [1,\omega)$, the function $g(n,\alpha,\gamma)$ has to take values in elements of Class(i), for $i \in [1,n)$. One way to tackle the problem of defining $g(n,\alpha,\gamma)$ in all these different kinds of ordinals is the following: Noticing that $[\alpha,\alpha(+^n))$ is a union of intervals of the form $[\beta, \beta(+^{n-1}))$, one could define $g(n,\alpha,\gamma)$ in "the first one of these intervals" $[\alpha,\alpha(+^{n-1}))$ as $g(n-1,\alpha,\gamma)$; then one could try to extend $g(n,\alpha,\gamma)|_{\alpha(+^{n-1})}$ adequately to the next interval $[\alpha(+^{n-1}), \alpha(+^{n-1})(+^{n-1}))$ and in general, continue this process until one has defined $g(n,\alpha,\gamma)$ in the whole of $[\alpha,\alpha(+^n))$. One should notice that, if this procedure is going to work, then one needs to guarantee that the extension of some already extended function behaves well; that is, one needs to work with a space of functions bigger than the space $\{g(i,\beta,\zeta) | i \in [1,n) \wedge \beta,\zeta \in \mathrm{Class}(i)\}$, since an extension of some $g(i,\beta,\zeta)$ does not necessarily belong to $\{g(i,\beta,\zeta) | i \in [1,n) \wedge \beta,\zeta \in \mathrm{Class}(i)\}$.

The previous paragraph explains intuitively why the extension theorem does not express directly anything about the $g(n,\alpha,\gamma)$ functions, but rather makes a more general claim: It essentially states that, for $\alpha,\gamma \in \mathrm{Class}(n)$ with $\alpha < \gamma$ and an arbitrary strictly increasing function $p\colon \alpha \cap \mathbb{E} \longrightarrow \gamma \cap \mathbb{E}$, we can always extend p to the interval $[\alpha, \alpha(+^n)) \cap \mathbb{E}$ such that the resulting extension $\Phi(n,\alpha,\gamma,p)$ induces the $(<, <_1, +)$-embedding $H\colon (\alpha,\alpha(+^n)) \longrightarrow (\gamma,\gamma(+^n))$, $H(x) := x[\Phi(n,\alpha,\gamma,p)]$. The proof of this fact is very long and with many technicalities, but in the end one gets the following recursive definition of the $\Phi(n,\alpha,\gamma,p)$ functions:

Base case
For arbitrary $\alpha,\gamma \in \mathrm{Class}(1)$ with $\alpha \leqslant \gamma$ and $p\colon \alpha \cap \mathbb{E} \longrightarrow \gamma \cap \mathbb{E}$ a strictly increasing function,
$\Phi(1,\alpha,\gamma,p) := \begin{cases} e \longmapsto p(e) & \text{if } e \in \alpha \cap \mathbb{E} \\ \alpha \longmapsto \gamma & \end{cases}$.
Moreover, $\Phi(1,\gamma,\alpha,p) := (\Phi(1,\alpha,\gamma,p))^{-1}$.

Inductive case.
Let $n \in [2, \omega)$. By induction hypothesis $\Phi(m, \alpha', \gamma', p')$ and $\Phi(m, \gamma', \alpha', p')$ are already defined for $m < n$ and for arbitrary, $\alpha', \gamma' \in \mathrm{Class}(m)$ and $p' : \alpha' \cap \mathbb{E} \longrightarrow \gamma' \cap \mathbb{E}$ a strictly increasing function.
Now, for any $\alpha, \gamma \in \mathrm{Class}(n)$ with $\alpha \leqslant \gamma$ and $p : \alpha \cap \mathbb{E} \longrightarrow \gamma \cap \mathbb{E}$ a strictly increasing function,
$\Phi(n, \alpha, \gamma, p) : \alpha(+^n) \cap \mathbb{E} \longrightarrow \gamma(+^n) \cap \mathbb{E}$ is given by a (side)-recursion on $(\alpha(+^n) \cap \mathbb{E}, <)$ as:
$\Phi(n, \alpha, \gamma, p)(e) := p(e)$ if $e \in \alpha \cap \mathbb{E}$;
$\Phi(n, \alpha, \gamma, p)(\alpha) := \gamma$;
$\Phi(n, \alpha, \gamma, p)(t) := \Phi(n-1, \xi, \Phi(n, \alpha, \gamma, p)(\xi), \Phi(n, \alpha, \gamma, p)|_{\xi})(t)$ if
$t \in (\xi, \xi(+^{n-1})) \cap \mathbb{E} \wedge \xi \in [\alpha, \alpha(+^n)) \cap \mathrm{Class}(n-1)$;
$\Phi(n, \alpha, \gamma, p)(\xi) := \min\{\delta \in (\gamma, \gamma(+^n)) \cap \mathrm{Class}(n-1) \mid \Phi(n, \alpha, \gamma, p)(\xi^-) < \delta \wedge$
$m(\delta)[g(n-1, \delta, \gamma(+^n))] \geqslant m(\xi)[\Phi(n-1, \xi, \gamma(+^n), \Phi(n, \alpha, \gamma, p)|_{\xi})]\}$, where
$\xi \in \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n))$ and
$\xi^- := \sup(\{\alpha\} \cup \{e \in \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)) \cap \xi \mid m(e)[g(n-1, e, \xi)] \geqslant m(\xi)\})$.
Moreover, $\Phi(n, \gamma, \alpha, p) := (\Phi(n, \alpha, \gamma, p))^{-1}$.

It is after the proof of the extension theorem that the function $g(n, \alpha, \gamma)$ is finally defined as the function $\Phi(n, \alpha, \gamma, \mathrm{Id}_\alpha)$. Of course, it still rests to show that the $g(n, \alpha, \gamma)$ functions have the properties we need. Some of these properties follow directly from the extension theorem, for instance:

g1. $g(n, \alpha, \gamma)$ is strictly increasing.

g2. $\forall e \in \alpha \cap \mathbb{E}. g(n, \alpha, \gamma)(e) = e$ and $g(n, \alpha, \gamma)(\alpha) = \gamma$.

g3. $\forall x, y \in (\alpha, \alpha(+^n)) \cap \mathbb{E}. x <_1 y \Longleftrightarrow x[g(n, \alpha, \gamma)] <_1 y[g(n, \alpha, \gamma)]$.

g4. $g(n, \alpha, \alpha) = \mathrm{Id}_{\alpha(+^n) \cap \mathbb{E}}$.

however, other require still much work, for example:

g5. For $\delta \in \mathrm{Class}(n)$ with $\alpha < \gamma < \delta$, $g(n, \alpha, \delta) \circ g(n, \alpha, \gamma) = g(n, \alpha, \delta)$.

g6. $\mathrm{Dom}\, g(n, \alpha, \gamma) = \{e \in \mathbb{E} \cap \alpha(+^n) | T(n, \alpha, e) \cap \alpha \subset \gamma\}$, where $T(n, \alpha, e)$ is certain finite set defined for $\alpha \in \mathrm{Class}(n)$ and $e \in \alpha(+^n)$.

Indeed, in the way to obtain these theorems one actually shows the more general results

- $g(n-1, b_1, b_2) \circ \Phi(n-1, a_1, b_1, \Phi(n, \alpha, \gamma, p)|_{a_1}) = \Phi(n-1, a_2, b_2, \Phi(n, \alpha, \gamma, p)|_{a_2}) \circ g(n-1, a_1, a_2)$ (see proposition 5.15).
- $\mathrm{Im}\, \Phi(n, \alpha, \gamma, p) = \{s \in \gamma(+^n) \cap \mathbb{E} | T(n, \gamma, s) \cap \gamma \subset \mathrm{Im}\, p\}$ (proposition 5.20).

Chapter 6 has as motivation to see how the cofinality properties look in $\mathrm{Class}(n)$. Specifically, based on $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$, the binary relation $<^n$ is defined as
$\alpha <^n t :\Longleftrightarrow$

1. $\alpha \in \mathrm{Class}(n)$, $t \in [\alpha, \alpha(+^n)]$

2. $\alpha < t$

3. $\forall B \subset_{\mathrm{fin}} t. \exists \delta \in \mathrm{Class}(n) \cap \alpha$ such that

i. $\forall x \in B. \mathrm{Ep}(x) \subset \mathrm{Dom}\, g(n, \alpha, \delta)$

ii. The function h: $B \longrightarrow h[B]$ defined as $h(x) := x[g(n, \alpha, \delta)]$ is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$.

Later, in propositions 6.5 and 6.6, it is seen that $<^n$ satisfies cofinality properties:

- For $t \in [\alpha, \alpha(+^n))$,
$\alpha <^n t + 1 \Longleftrightarrow \alpha = \sup\{\gamma \in \alpha \cap \mathrm{Class}(n) \mid \mathrm{Ep}(t) \subset \mathrm{Dom}(g(n, \alpha, \gamma)) \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$

Moreover, through the generalized covering theorem (theorem 6.17), one gets cofinality properties for $<_1$ in Class(n) through the equivalence $\alpha <_1 \eta(n, \alpha, t) + 1 \Longleftrightarrow \alpha <^n \eta(n, \alpha, t) + 1$ (which follows easily from corollary 6.20). Finally, corollaries 6.18 and 6.19 explain how the space $\{x \longmapsto x[g(n, \alpha, \gamma)] \mid \alpha, \gamma \in \mathrm{Class}(n)\}$ is also a class of canonical witnesses of $\alpha <_1 t$ for t closed under the covering construction.

After the conclusion, in chapter 6, of the proof of theorem 3.25 (actually, of it's more technical version, theorem 3.26), we have plenty of information about the $<_1$ relation. Now the idea is to use all that and link it with known proof theoretic concepts. Chapter 7 introduces $\mathrm{Clas}(\omega) := \bigcap_{i \in [1,\omega)} \mathrm{Class}(i)$ and the ordinals $O_i := \min \mathrm{Class}(i)$, for $i \in [1, \omega]$, and shows that Class(ω) (which is κ-club for any non-countable regular ordinal κ) consists of those ordinals α satisfying $\alpha <_1 \infty$. This and the work of Carlson (see [10]) mean $O_\omega = \mathrm{Core}(R_1) = |\Pi^1_1\text{-}\mathrm{CA}_0|$. It is also shown here that any non-countable cardinal belongs to Class(ω).

Knowing that $O_\omega = |\Pi^1_1\text{-}\mathrm{CA}_0|$ and the fact that the $(O_i)_{i \in [1,\omega)}$ are cofinal in O_ω (proposition 7.11) leads easily to the inquiries:

1. Trying to obtain a notation system for the segment $[0, O_\omega)$ based on the work done in the previous 6 chapters.
2. Trying to tell what the O_i ordinals are in terms of known proof theoretic functions.

Inquiry 1, obtaining a notation system for the segment $[0, O_\omega)$ based on the theorems obtained up to this point (particularly, based on the "description of t in terms of α" for an ordinal $t \in [\alpha, \alpha(+^n)) \cap \mathrm{Class}(n-1)$ and $\alpha \in \mathrm{Class}(n)$), was a task "done" by the author of this thesis that does not appear in the thesis. The reason is that the notation system obtained in such way was complicated and in the end, after a meeting with Prof. Buchholz, it was decided not to include that stuff in this work (in fact, since it was decided to stop working in that direction, the proofs that the obtained system of notations is indeed a notation system for the segment $[0, O_\omega)$ were never completed).

It is around inquiry 2 that the rest of chapter 7 develops. The second part of this chapter introduces Buchholz $(\psi_n)_{n \in \omega}$ functions (as given in [4]) and ultimately provides a (complete) proof of the statement $\forall n \in \omega. \psi_n(\Omega_{n+2}) = \Omega_n(+^2)$ (see corollary 7.44); in particular, this means $|\mathrm{ID}_1| = \psi_0(\Omega_2) = \Omega_0(+^2) = O_2$ as was already shown by Wilken in [18].

The final conjecture is $\forall n \in \omega \forall m \in [1, \omega). \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$. In particular, this would mean $\forall m \in [1, \omega). |\mathrm{ID}_m| = \psi_0(\Omega_{m+1}) = \Omega_0(+^{m+1}) = O_{m+1}$ and as an easy corollary of this fact, we would get another proof of $O_\omega = |\Pi^1_1\text{-}\mathrm{CA}_0|$. The general statement $\forall n \in \omega \forall m \in [1, \omega). \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ remains, however, as a conjecture[1]: The problem is, for $\alpha \in [\Omega_{n+m}, \Omega_{n+m}(+^1))$, to provide suitable lower and upper bounds for the ordinal $m(\psi_n(\alpha))$. The author of this thesis gives what he thinks is an ALMOST complete proof of such an upper bound of $m(\psi_n(\alpha))$ (see "lemma 7.46"). But it turns out that for such proof one requires that some rather technical conditions hold; in particular, one needs to know already a lower bound for $m(\psi_j(\beta))$, where $\beta \in [\Omega_{j+i}, \Omega_{j+i}(+^1)) \wedge j \in [1, \omega) \wedge i \in [1, m)$. This suggests that one needs to provide simultaneously the upper and lower bounds of $m(\psi_n(\alpha))$ and at the same time ensure that the other conditions are satisfied. The completion of such a proof will have to be, for reasons of time, a task for a future work.

1. The statement $\forall n \in \omega \forall m \in [1, 2]. \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ holds by the theorems proven in this work. So the actual remaining conjecture is to see that $\forall n \in \omega. \psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ holds for $m \geqslant 3$.

Basic conventions used throughout this work.

We use the standard logical symbols in it's standard way: $\wedge, \vee, \Longrightarrow, \Longleftrightarrow, \forall, \exists, \neg$, etc.
We use the standard set theoretical symbols in it's standard form: $\emptyset, \cup, \cap, \subset, =, \in$, etc.
By $B \subset_{\text{fin}} A$ we mean B is a finite subset of A.
$h: A \longrightarrow B$ denotes that h is a functional with domain A and codomain B.
For a functional $h: A \longrightarrow B$ and $C \subset A$, we define $h[C] := \{h(x) \mid x \in C\}$.
For a functional $h: A \longrightarrow B$, we denote $\operatorname{Dom} h := A$ and $\operatorname{Im} h := h[A]$.

By OR we denote the class of ordinals.
$0, 1, 2, ...$ denote, as usual, the finite ordinals.
ω denotes the first infinite ordinal.
Lim denotes the class of limit ordinals.
$\mathbb{P}$ denotes the class of additive principal ordinals.
$\mathbb{E}$ denotes the class of epsilon numbers.
$<$, $+$, $\lambda x.\omega^x$ denote the usual order, the usual addition and the usual ω-base-exponentiation in the ordinals, respectively.

For an ordinal $\alpha \in \mathrm{OR}$, ε_α denotes the α-th epsilon number.

Let $A \subset \mathrm{OR}$ be a class of ordinals such that $A \neq \emptyset$.
$\min A$ denotes the minimum element of A (with respect to the order $<$).
$\max A$ denotes the maximum element of A (with respect to $<$ and in case such maximum exists).
In case $\exists \alpha \in \mathrm{OR}. A \subset \alpha$, then $\sup A$ denotes the minimal upper bound of A with respect to $<$ (the supremum of A).
$\operatorname{Lim}(A) := \{\alpha \in \mathrm{OR} \mid \alpha = \sup(A \cap \alpha)\}$.
By $(\xi_i)_{i \in I} \subset A$ we mean $(\xi_i)_{i \in I}$ is a sequence of elements of A.

Given an ordinal $\alpha \in \mathrm{OR}$ and a sequence $(\xi_i)_{i \in I} \subset \mathrm{OR}$, we say that $(\xi_i)_{i \in I}$ is cofinal in α whenever $I \subset \mathrm{OR}$, $\forall i \in I \forall j \in I. i \leqslant j \Longrightarrow \xi_i \leqslant \xi_j$, $\forall i \in I \exists j \in I. i < j \wedge \xi_i < \xi_j$ and $\sup\{\xi_i \mid i \in I\} = \alpha$. By $\xi_i \underset{\text{cof}}{\hookrightarrow} \alpha$ we mean that the sequence $(\xi_i)_{i \in I}$ is cofinal in α.

Whenever we write $\alpha =_{\text{CNF}} \omega^{A_1} a_1 + ... + \omega^{A_n} a_n$, we mean that $\omega^{A_1} a_1 + ... + \omega^{A_n} a_n$ is the cantor normal form of α, that is: $\alpha = \omega^{A_1} a_1 + ... + \omega^{A_n} a_n$, $a_1, ..., a_n \in \omega \backslash \{0\}$, $A_1, ..., A_n \in \mathrm{OR}$ and $A_1 > ... > A_n$.

Given two ordinals $\alpha, \beta \in \mathrm{OR}$ with $\alpha \leqslant \beta$, we denote:
$[\alpha, \beta] := \{\sigma \in \mathrm{OR} \mid \alpha \leqslant \sigma \leqslant \beta\}$
$[\alpha, \beta) := \{\sigma \in \mathrm{OR} \mid \alpha \leqslant \sigma < \beta\}$
$(\alpha, \beta] := \{\sigma \in \mathrm{OR} \mid \alpha < \sigma \leqslant \beta\}$
$(\alpha, \beta) := \{\sigma \in \mathrm{OR} \mid \alpha < \sigma < \beta\}$

Given $\alpha \in \mathbb{E}$, we denote by α^+ or by $\alpha(+^1)$ to $\min\{e \in \mathbb{E} | \alpha < e\}$.

For a set A, $|A|$ denotes the cardinality of A; the only one exception to this convention is done in chapter 6, where we denote as $|\mathrm{ID}_n|$ and $|\Pi^1_1\text{-}\mathrm{CA}_0|$ to the proof theoretic ordinals of the theories ID_n and $\Pi^1_1\text{-}\mathrm{CA}_0$ respectively.

Part I

The lower classes

Chapter 1

Class(0)

1.1 The $<_1$-relation

Our purpose is to study the (binary) relation $<_1$ defined by recursion on the ordinals as follows

Definition 1.1. *Let $\beta \in \mathrm{OR}$ be arbitrary and suppose $\alpha' <_1 \beta'$ has already been defined for any $\beta' \in \beta \cap \mathrm{OR}$ and for any $\alpha' \in \mathrm{OR}$. Let $\alpha \in \mathrm{OR}$ be arbitrary. Then*
$\alpha <_1 \beta :\Longleftrightarrow \alpha < \beta$ *and* $\forall Z \subset_{\text{fin}} \beta \exists \tilde{Z} \subset_{\text{fin}} \alpha . \exists h$ *such that:*

(i) $h: (Z, +, <, <_1) \longrightarrow (\tilde{Z}, +, <, <_1)$ is an isomorphism, that is:
- *$h: Z \longrightarrow \tilde{Z}$ is a bijection.*
- *For any $a_1, a_2 \in Z$*
  - *$a_1 + a_2 \in Z \Longleftrightarrow h(a_1) + h(a_2) \in \tilde{Z}$*
  - *If $a_1 + a_2 \in Z$, then $h(a_1 + a_2) = h(a_1) + h(a_2)$.*
- *For any $a_1, a_2 \in Z$,*
  - *$a_1 < a_2 \Longleftrightarrow h(a_1) < h(a_n)$.*
  - *$a_1 <_1 a_2 \Longleftrightarrow h(a_1) <_1 h(a_n)$.*

(ii) $h|_{Z \cap \alpha} = \mathrm{Id}|_{Z \cap \alpha}$, where $\mathrm{Id}|_{Z \cap \alpha}: Z \cap \alpha \longrightarrow Z \cap \alpha$ is the identity function.

By $\alpha \leqslant_1 \beta$ we mean that $\alpha <_1 \beta$ or $\alpha = \beta$. Moreover, to make our notation simpler, we will write $h|_\alpha = \mathrm{Id}|_\alpha$ instead of $h|_{Z \cap \alpha} = \mathrm{Id}|_{Z \cap \alpha}$.

Remark 1.2. We will eventually use functions $f: Z \longrightarrow \tilde{Z}$ that are $\lambda x.\omega^x$-isomorphisms; of course, by this we mean the analogous situation as the one we had with $+$ above:
For any $a \in Z$,
- $\omega^a \in Z \Longleftrightarrow f(\omega^\alpha) \in \tilde{Z}$
- If $\omega^a \in Z$, then $f(\omega^a) = \omega^{f(a)}$.

Some of the most basic properties that $\leqslant_1$ satisfies are the following

Proposition 1.3. *Let $\alpha, \beta, \gamma \in \mathrm{OR}$.*

a) $\alpha \leqslant_1 \beta \Longrightarrow \{x \in \mathrm{OR} | \alpha \leqslant_1 x \leqslant \beta\} = [\alpha, \beta]$.

b) Let $(\xi_i)_{i \in I} \subset \mathrm{OR}$ be a sequence such that $\xi_i \underset{\text{cof}}{\longrightarrow} \beta$. Then $[\forall i \in I . \alpha \leqslant_1 \xi_i] \Longrightarrow \alpha \leqslant_1 \beta$.

c) $\alpha \leqslant_1 \beta \leqslant_1 \gamma \Longrightarrow \alpha \leqslant_1 \gamma$.

d) *Let* $(\xi_i)_{i\in I}\subset\mathrm{OR}$ *be a sequence such that* $\xi_i \underset{\text{cof}}{\longrightarrow} \beta$. *Then* $[\exists i_0\in I.\alpha\not\leqslant_1\xi_{i_0}\wedge\alpha<\xi_{i_0}]\Longrightarrow\alpha\not\leqslant_1\beta$.

Proof. The proofs of *a*), *b*) and *c*) follow direct from definition 1.1. Moreover, *d*) follows easily from *a*). □

We call $\leqslant_1$**-connectedness** (or just connectedness) to the property *a*) of previous proposition 1.3; moreover, we call $\leqslant_1$**-continuity** (or just continuity) and $\leqslant_1$**-transitivity** (or just transitivity) to the properties *b*) and *c*) (respectively) of the same proposition. We will make use of the three of them over and over along all our work.

Proposition 1.4. *Let* $\alpha,\beta\in\mathrm{OR}$ *with* $\alpha<\beta$ *and* $\alpha\not\leqslant_1\beta$. *Then there exists* $\gamma\in[\alpha,\beta)$ *such that*

a) $\{x\in\mathrm{OR}|\alpha\leqslant_1 x\}=[\alpha,\gamma]$.

b) $\{x\in\mathrm{OR}|\alpha<x,\alpha\not\leqslant_1 x\}=[\gamma+1,\infty)$.

c) *For any* $\sigma>\gamma$, $\gamma\not\leqslant_1\sigma$.

Proof. Let $k:=\min\{r\in\mathrm{OR}|r>\alpha\not\leqslant_1 r\}$. Then $k\leqslant\beta$. Moreover, since $\forall\sigma\in[\alpha,k).\alpha\leqslant_1\sigma$, then k must be a successor (otherwise, by $\leqslant_1$-continuity would follow $\alpha<_1 k$). So $k=\gamma+1\leqslant\beta$ for some $\gamma\in\mathrm{OR}$ and therefore $\{x\in\mathrm{OR}|\alpha\leqslant_1 x\}=[\alpha,\gamma]$. This shows *a*).

On the other hand, note that for any $\sigma\geqslant k$, it is not possible that $\alpha\leqslant_1\sigma$ (otherwise, by $\leqslant_1$-connectedness, one gets the contradiction $\alpha<_1 k$). This proves *b*).

Finally, observe it is not possible that for some $\sigma>\gamma$, $\gamma<_1\sigma$, otherwise, from $\alpha\leqslant_1\gamma\leqslant_1\sigma$ and $\leqslant_1$-transitivity follows $\alpha<_1\sigma$, which is contradictory with *b*) (because $\sigma\geqslant k=\gamma+1$). □

For an ordinal α, the ordinal γ referred in previous proposition 1.4 will be very important for the rest of our work. Because of that we make the following

Definition 1.5. *(The maximum* $\leqslant_1$*-reach of an ordinal). Let* $\alpha\in\mathrm{OR}$. *We define*

$$m(\alpha):=\begin{cases}\max\{\xi\in\mathrm{OR}|\alpha\leqslant_1\xi\} & \textit{iff there exists } \beta\in\mathrm{OR} \textit{ such that } \alpha<\beta \textit{ and } \alpha\not\leqslant_1\beta\\ \infty & \textit{otherwise, that is, } \forall\beta\in\mathrm{OR}.\alpha<\beta\Longrightarrow\alpha<_1\beta\end{cases}.$$

Note that when $m(\alpha)\in\mathrm{OR}$, *then it is the only one* $\gamma\in\mathrm{OR}$ *satisfying* $\alpha\leqslant_1\gamma$ *and* $\alpha\not\leqslant_1\gamma+1$. *Because of this* ***we call*** $\boldsymbol{m(\alpha)}$ ***the maximum*** $\leqslant_1$***-reach of*** $\boldsymbol{\alpha}$.

1.2 Characterization of the ordinals α such that $\alpha<_1\alpha+1$

Up to this moment we do not know whether there are ordinals α, β such that $\alpha<_1\beta$; however, in such a case, since $\alpha<\alpha+1\leqslant\beta$, then by $\leqslant_1$-connectedness we would conclude that the relation $\alpha<_1\alpha+1$ must hold. This shows that the simplest nontrivial case when we can expect that something of the form $\alpha<_1\beta$ holds is for $\beta=\alpha+1$. Then, for this simplest case, what should α satisfy?. The answer to this question is the purpose of this subsection.

Proposition 1.6. *Let* $\alpha,\beta\in\mathrm{OR}$, $\alpha=_{\mathrm{CNF}}\omega^{\alpha_1}a_1+\ldots+\omega^{\alpha_n}a_n$, *with* $n\geqslant 2$ *or* $a_1\geqslant 2$. *Moreover, suppose* $\alpha<\beta$. *Then* $\alpha\not\leqslant_1\beta$.

Proof. Case $n\geqslant 2$.
Since $\alpha<\beta$, then $\{\omega^{\alpha_1}a_1,\ldots,\omega^{\alpha_n}a_n\}\subset\alpha\cap\beta$, but $\beta\ni\omega^{\alpha_1}a_1+\ldots+\omega^{\alpha_n}a_n=\alpha\notin\alpha$, and so there is no $+$-isomorphism $h\colon Z\to\tilde{Z}$ from $Z:=\{\omega^{\alpha_1}a_1,\ldots,\omega^{\alpha_n}a_n,\alpha\}\subset_{\mathrm{fin}}\beta$ in some $\tilde{Z}\subset_{\mathrm{fin}}\alpha$ such that $h|_\alpha=\mathrm{Id}|_\alpha$, since any of such isomorphisms should accomplish
$h(\omega^{\alpha_1}a_1+\ldots+\omega^{\alpha_n}a_n)=h(\omega^{\alpha_1}a_1)+\ldots+h(\omega^{\alpha_n}a_n)=\alpha\notin\alpha$.

The same argument works for the case $n=1, a_1 \geqslant 2$. □

Corollary 1.7. *Let $\alpha, \beta \in \mathrm{OR}$. If $\alpha <_1 \beta$, then $\alpha =_{\mathrm{CNF}} \omega^\gamma \in \mathbb{P} \subset \mathrm{Lim}$, for some $\gamma \in \mathrm{OR}, \gamma > 0$.*

Proof. Direct from previous proposition 1.6. The only left cases are $\alpha = 0$ or $\alpha = 1$ but for those cases it is very easy to see that $\alpha \not<_1 \alpha + 1$, since $\alpha + 1$ has $\alpha + 1$ elements and α has only α elements, and so for those cases $\alpha \not<_1 \beta$ for any $\beta > \alpha$. □

Proposition 1.8. *If $\alpha = \omega^n$, $n \in \omega$, then $\alpha \not<_1 \alpha + 1$.*

Proof. Not hard. But we will give a more general proof of this fact in the next propositions. □

Corollary 1.9. *Let $\alpha, \beta \in \mathrm{OR}$. If $\alpha <_1 \beta$, then $\alpha =_{\mathrm{CNF}} \omega^\gamma$ for some $\gamma \in \mathrm{OR}, \gamma \geqslant \omega$.*

Proof. From previous proposition and previous corollary. (This will be proved in the next three propositions in a more general way). □

Proposition 1.10. *Let $\alpha \in \mathrm{OR}$, $1 < \alpha \in \mathrm{Lim}$ and let $\mathbb{P}$ be the class of additive principal ordinals. Suppose $\alpha \cap \mathbb{P}$ is not cofinal in α. Then $M := \max(\mathbb{P} \cap \alpha)$ exists.*

Proof. Since $\mathbb{P}$ is a closed class of ordinals, then $\sup(\mathbb{P} \cap \alpha) \in \mathbb{P} \cap \alpha$. So $M = \sup(\mathbb{P} \cap \alpha)$. □

Proposition 1.11. *Let α, $p \in \mathrm{OR}$, $1 < \alpha <_1 p + 1$, with $p \in \mathbb{P}$ an additive principal number. Then:*

(i) $\alpha \cap \mathbb{P}$ is confinal in α.

(ii) $\alpha \in \mathrm{Lim}\,\mathbb{P} \subset \mathbb{P}$, (or equivalently, (ii') $\alpha = \omega^\gamma$, for $\gamma \in \mathrm{Lim}$.)

Proof. (i). By corollary 1.7 we know $\alpha \in \mathrm{Lim}$. Now, suppose $\alpha \cap \mathbb{P}$ is not confinal in α. Then by previous proposition 1.10, let $M := \max \alpha \cap \mathbb{P} \in \alpha$.

Then $M + p = p$, but on the other hand, $\forall \gamma \in \alpha. M + \gamma > \gamma$. Therefore, for $Z := \{M, p\} \subset_{\mathrm{fin}} p + 1$ and for any $\tilde{Z} \subset \alpha$ there is no $+$ -isomorphism $h \colon Z \to \tilde{Z}$, such that $h|_\alpha = \mathrm{Id}|_\alpha$, since any such function would satisfy $h(p) = h(M + p) = h(M) + h(p) = M + h(p) > h(p)$ (Contradiction!).

Thus $\alpha \cap \mathbb{P}$ is confinal in α.

(ii). Clear from (i). □

Corollary 1.12. *Let $\alpha, \beta \in \mathrm{OR}$ such that $\alpha <_1 \beta$. Then $\alpha \in \mathrm{Lim}\,\mathbb{P}$.*

Proof. From corollary 1.7 we have that $\alpha <_1 \beta$ implies $\alpha \in \mathbb{P}$. Moreover, from $\alpha <_1 \beta$ we know $\alpha < \alpha + 1 \leqslant \beta$ and then $\alpha <_1 \alpha + 1$ by $<_1$-connectedness. Finally, from $\alpha <_1 \alpha + 1$, $\alpha \in \mathbb{P}$ and the previous proposition 1.11, $\alpha \in \mathrm{Lim}\,\mathbb{P}$. □

Proposition 1.13. *Let $\alpha \in \mathrm{OR}$. The following are equivalent:*

$a)$ $\alpha <_1 \alpha + 1$

$b)$ $\alpha \in \mathrm{Lim}\,\mathbb{P}$

c) $\alpha = \omega^{\gamma}$ *for some* $\gamma \in \mathrm{Lim}$.

d) $\alpha = \omega^{\gamma}$ *and* $\gamma =_{\mathrm{CNF}} \omega^{A_1} a_1 + \ldots + \omega^{A_n} a_n$ *with* $A_n \neq 0$.

Proof. The proof of $b) \Longleftrightarrow c) \Longleftrightarrow d)$ is a standard fact about ordinals.

$a) \Longrightarrow b)$ is previous corollary 1.12.

So let's prove $b) \Longrightarrow a)$.
Let $\alpha \in \mathrm{Lim}\mathbb{P}$. Take $B \subset_{\mathrm{fin}} \alpha + 1$. If $\alpha \notin B$, then $l: B \longrightarrow \alpha$, $l(x) := x$ is an $(< , <_1 , +)$-isomorphism such that $l|_{\alpha} = \mathrm{Id}_{\alpha}$. So suppose $B = \{a_0 < \ldots < a_n = \alpha\}$ for some natural number n. Let $A := \{m(a) | a \in (B \cap \alpha) \wedge m(a) < \alpha\}$. Since $\alpha \in \mathrm{Lim}\mathbb{P}$ and A is finite, then there exists $\rho \in (a_{n-1}, \alpha) \cap (\max A, \alpha) \cap \mathbb{P}$. Let $h: B \longrightarrow h[B] \subset \alpha$ be the function
$h(x) := \begin{cases} x & \text{iff } x < \alpha \\ \rho & \text{otherwise} \end{cases}$. It is clear that $h|_{\alpha} = \mathrm{Id}_{\alpha}$.

We assure that h is an $(< , <_1 , +)$-isomorphism.
The details are left to the reader. □

1.3 The ordinals α satisfying $\alpha <_1 t$, for some $t \in [\alpha, \alpha\omega)$.

We have seen previously that the "solutions of the $<_1$-inequality" $x <_1 x + 1$ are the elements of $\mathrm{Lim}\mathbb{P}$. It is natural then to ask himself about the solutions of $x <_1 x + 2$ or of $x <_1 x + \omega$. In general, this question can be informally stated as: What are the solutions of $x <_1 \beta$, where "we pick β as big as we can"?. The descriptions of such solutions in a certain way is a main purpose of this work: we will describe them as certain classes of ordinals obtained by certain thinning procedure. The rest of this chapter is devoted to our investigations concerning this question for $x \in \mathbb{P}$ and $\beta \in [x, x\omega]$. We will introduce various concepts that at the first sight may look somewhat artificial; however, these concepts and the way to use them is just "the most basic realization" of the general tools and methodology developed from chapter 3 to chapter 6 that will allow us to understand the $<_1$-relation in the whole class of ordinals.

1.3.1 Class(0)

Definition 1.14. *Let* $\mathrm{Class}(0) := \mathbb{P}$.

Definition 1.15. *For* $\alpha, \beta \in \mathrm{OR}$*, let*
$$-\alpha + \beta := \begin{cases} \text{the only one ordinal } \sigma \text{ such that } \alpha + \sigma = \beta & \text{iff } \alpha \leqslant \beta \\ \\ -1 & \text{otherwise} \end{cases}$$

Definition 1.16. *Let* $\alpha, c \in \mathrm{Class}(0)$ *with* $\alpha \leqslant c$.
We define $g(0, \alpha, c): \alpha\omega \longrightarrow c\omega$ *as:*
$g(0, \alpha, c)(x) := x$ *iff* $x < \alpha$.
$g(0, \alpha, c)(x) := cn + l$ *iff* $x \in [\alpha n, \alpha n + \alpha) \wedge x = \alpha + l$ *for some* $l \in \alpha$.
Moreover, we define $g(0, c, \alpha) := g(0, \alpha, c)^{-1}$.

Proposition 1.17. *Let* $\alpha, c \in \mathrm{Class}(0)$. *Then*

1. $\mathrm{Dom}\, g(0,\alpha,c) = (\alpha \cap c) \cup \bigcup_{n\in[1,\omega)} \{t \in [\alpha n, \alpha n + \alpha) \mid -\alpha n + t < c\}$.
2. $\mathrm{Im}\, g(0,\alpha,c) = (\alpha \cap c) \cap \bigcup_{n\in[1,\omega)} \{t \in [cn, cn + c) \mid -cn + t < \alpha\}$.
3. $g(0,\alpha,c)\colon \mathrm{Dom}\, g(0,\alpha,c) \longrightarrow \mathrm{Im}\, g(0,\alpha,c)$ *is an* $(<,+)$*-isomorphism and* $g(0,\alpha,c)|_\alpha = \mathrm{Id}_\alpha$.

Proof. Left to the reader. □

Proposition 1.18. *Let* $\alpha, c \in \mathrm{Class}(0)$ *and* $X := (\alpha \cap c) \cup \bigcup_{n\in[1,\omega)} \{t \in [\alpha n, \alpha n + \alpha) \mid -\alpha n + t < c\}$. *Then the function* $H\colon (\alpha, \alpha\omega) \cap X \longrightarrow H[(\alpha,\alpha\omega) \cap X] \subset (c, c\omega)$, $H(x) := g(0,\alpha,c)(x)$ *is an* $(<, <_1, +)$*-isomorphism.*

Proof. Let α, c, X and H be as stated. By previous proposition 1.17 follows easily that H is an $(<,+)$-isomorphism. Moreover, H is also an $<_1$-isomorphism because by proposition 1.13 and $<_1$-connectedness it follows that $\forall a, b \in (\alpha, \alpha\omega). a \not<_1 b$ and $\forall a, b \in (c, c\omega). a \not<_1 b$. □

Definition 1.19. *Consider* $\alpha \in \mathrm{Class}(0)$ *and* $t \in \alpha\omega$.

We define $T(0,\alpha,t) := \begin{cases} \{t\} & \textit{iff } t < \alpha \\ \{t, -\alpha n + t\} & \textit{iff } t \in [\alpha n, \alpha n + \alpha) \textit{ for some } n \in [1,\omega). \end{cases}$

Proposition 1.20. $\forall \alpha, c \in \mathrm{Class}(0). \forall t \in \alpha\omega. t \in \mathrm{Dom}(g(0,\alpha,c)) \Longleftrightarrow T(0,\alpha,t) \cap \alpha \subset c$

Proof. Direct from definition 1.19 and proposition 1.17. □

Definition 1.21. *Let* $\alpha \in \mathrm{Class}(0)$ *and* $t \in [\alpha, \alpha\omega]$. *By* $\alpha <^0 t$ *we mean*

1. $\alpha < t$
2. $\forall B \subset_{\mathrm{fin}} t. \exists \delta \in \mathrm{Class}(0) \cap \alpha$ *such that*
   i. $(\bigcup_{t\in B} T(0,\alpha,t) \cap \alpha) \subset \delta$;
   ii. *The function* $h\colon B \longrightarrow h[B]$ *defined as* $h(x) := g(0, \alpha, \delta)(x)$ *is an* $(<, <_1, +)$*-isomorphism with* $h|_\alpha = \mathrm{Id}_\alpha$.

As usual, $\alpha \leqslant^0$ *just means* $\alpha <^0 t$ *or* $\alpha = t$.

Proposition 1.22. *Let* $\alpha \in \mathrm{Class}(0)$, $(\xi_i)_{i\in I} \subset [\alpha, \alpha\omega] \ni \beta, \gamma$. *Then*

1. $\alpha \leqslant^0 \beta \Longrightarrow \alpha \leqslant_1 \beta$.
2. *If* $\alpha \leqslant \beta \leqslant \gamma \wedge \alpha \leqslant^0 \gamma$ *then* $\alpha \leqslant^0 \beta$. ($\leqslant^0$*-connectedness)*
3. *If* $\forall i \in I. \alpha \leqslant^0 \xi_i \wedge \xi_i \xrightarrow[\mathrm{cof}]{} \beta$ *then* $\alpha \leqslant^0 \beta$. ($\leqslant^0$*-continuity)*

Proof. Left to the reader. □

Proposition 1.23. *(First fundamental cofinality property of* $<^0$*).*
Let $\alpha \in \mathrm{Class}(0)$ *and* $t \in [\alpha, \alpha\omega)$.
Then $\alpha <^0 t + 1 \Longrightarrow \alpha \in \mathrm{Lim}\{\beta \in \mathrm{Class}(0) \mid T(0,\alpha,t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0,\alpha,\beta)(t)\}$.

Proof. Let α, t be as stated.

Suppose $\alpha <^0 t+1$. **(*1)**

Let $\gamma \in \alpha$ be arbitrary and consider $B_\gamma := \{\gamma, \alpha, t\} \subset_{\text{fin}} t+1$. By (*1) there exists $\delta_\gamma \in \alpha \cap \text{Class}(0)$ such that $(\bigcup_{q \in B} T(0, \alpha, q) \cap \alpha) \subset \delta_\gamma$ and the function $h\colon B \longrightarrow h[B] \subset \alpha$, $h(x) := g(0, \alpha, \delta_\gamma)(x)$ is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \text{Id}_\alpha$. In particular, note:

1. $\gamma < \delta_\gamma$ because $\gamma \in (\bigcup_{q \in B} T(0, \alpha, q) \cap \alpha) \subset \delta_\gamma$.
2. $\delta_\gamma = g(0, \alpha, \delta_\gamma)(\alpha) \leqslant_1 g(0, \alpha, \delta_\gamma)(t)$ because $T(0, \alpha, t) \cap \alpha \subset \delta_\gamma$ and $\alpha \leqslant_1 t \Longleftrightarrow h(\alpha) \leqslant_1 h(t)$.

Since the previous was done for arbitrary $\gamma < \alpha$, 1 and 2 show that
$\forall \gamma \in \alpha \exists \delta_\gamma \in \{\beta \in \text{Class}(0) \mid \gamma < \beta \wedge T(0, \alpha, t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t)\}$. Thus
$\alpha \in \text{Lim}\{\beta \in \text{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t)\}$. □

Proposition 1.24. *(Second fundamental cofinality property of $<^0$).*
Let $\alpha \in \text{Class}(0)$ and $t \in [\alpha, \alpha\omega)$.
Then $\alpha <^0 t+1 \Longleftarrow \alpha \in \text{Lim}\{\beta \in \text{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t)\}$.

Proof. Let α, t be as stated.

Suppose $\alpha \in \text{Lim}\{\beta \in \text{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t)\}$. **(*1)**

We prove by induction: $\forall s \in [\alpha, t+1].\alpha \leqslant^0 s$. **(*2)**

Let $s \in [\alpha, t+1]$ and suppose $\forall q \in s \cap [\alpha, t+1].\alpha \leqslant^0 q$. **(IH)**

Case $s = \alpha$.
Then clearly (*2) holds.

Case $s \in \text{Lim} \cap (\alpha, t+1]$.
Since by our (IH) $\forall q \in s \cap [\alpha, t+1].\alpha \leqslant^0 q$, then $\alpha \leqslant^0 s$ follows by $\leqslant^0$-continuity.

Suppose $s = l+1 \in (\alpha, t+1]$.
Let $B \subset_{\text{fin}} l+1$ be arbitrary. Consider $A := \{\alpha, l\} \cup \{m(a) \mid a \in B \cap \alpha \wedge m(a) < \alpha\}$. Then the set $\bigcup_{q \in B \cup A} T(0, \alpha, q) \cap \alpha$ is finite and then, by (*1), there is some $\delta \in \text{Class}(0) \cap \alpha$ such that $(\bigcup_{q \in B \cup A} T(0, \alpha, q) \cap \alpha) \subset \delta \wedge \delta \leqslant_1 g(0, \alpha, \delta)(t)$. **(*3)**

Consider the function $h\colon B \longrightarrow h[B] \subset \alpha$ defined as $h(x) := g(0, \alpha, \delta)(x)$. From (*3) and propositions 1.20 we know that h is well defined; moreover, from proposition 1.17 it follows that h is an $(<, +)$-isomorphism with $h|_\alpha = \text{Id}_\alpha$. **(*4)**

Before showing that h is an $<_1$-isomorphism, we do two observations:

Let $b \in B$ with $b \geqslant \alpha$. Then $\alpha \leqslant b \leqslant l$, which, together with $\alpha \underset{\text{by (IH)}}{\leqslant^0} l$, imply by $\leqslant^0$-connectedness that $\alpha \leqslant^0 b$; subsequently, $\alpha \leqslant_1 b$. This shows $\forall b \in B.\alpha \leqslant b \Longrightarrow \alpha \leqslant_1 b$ **(*5)**

Let $b \in B$ with $b \geqslant \alpha$. Then $\alpha \leqslant b \leqslant t$ implies
$\delta = g(0, \alpha, \delta)(\alpha) \underset{g(0,\alpha,\delta) \text{ strictly increasing}}{\leqslant} g(0, \alpha, \delta)(b) \underset{g(0,\alpha,\delta) \text{ strictly increasing}}{\leqslant} g(0, \alpha, \delta)(t)$; the latter together with $\delta \underset{\text{by (*3)}}{<_1} g(0, \alpha, \delta)(t)$ imply by $\leqslant_1$-connectedness that
$g(0, \alpha, \delta)(\alpha) = \delta \leqslant_1 g(0, \alpha, \delta)(b)$. All this shows $\forall b \in B.\alpha \leqslant b \Longrightarrow \delta \leqslant_1 g(0, \alpha, \delta)(b)$ **(*6)**.

Now we show that h is an $<_1$-isomorphism. **(*7)**

Let $a, b \in B$ with $a < b$.
Case $\boldsymbol{\alpha < a < b}$.
Then $a <_1 b \underset{\text{by proposition 1.18}}{\Longleftrightarrow} h(a) = g(0, \alpha, \delta)(a) <_1 g(0, \alpha, \delta)(b) = h(b)$.
Case $\boldsymbol{a = \alpha < b}$.
By (*5) and (*6) we have that $\alpha <_1 b$ and $h(\alpha) = g(0, \alpha, \delta)(\alpha) = \delta <_1 g(0, \alpha, \delta)(b) = h(b)$.
Case $\boldsymbol{a, b < \alpha}$.
Then $a <_1 b \underset{\text{by (*4)}}{\Longleftrightarrow} a = h(a) <_1 b = h(b)$.
Case $\boldsymbol{a < \alpha \leqslant b}$.

- $a <_1 b \underset{\text{by } \leqslant_1\text{-connectedness and (*5)}}{\Longrightarrow} a <_1 \alpha \leqslant_1 b \underset{\text{by proposition 1.17 and by (*6)}}{\Longrightarrow}$
  $a = g(0, \alpha, \delta)(a) < g(0, \alpha, \delta)(\alpha) = \delta < \alpha \wedge a <_1 \alpha \wedge \delta \leqslant_1 g(0, \alpha, \delta)(b) \underset{\text{by } \leqslant_1\text{-connectedness}}{\Longrightarrow}$
  $a = g(0, \alpha, \delta)(a) <_1 g(0, \alpha, \delta)(\alpha) = \delta \wedge \delta \leqslant_1 g(0, \alpha, \delta)(b) \underset{\text{by } \leqslant_1\text{-transitivity}}{\Longrightarrow}$
  $h(a) = g(0, \alpha, \delta)(a) <_1 g(0, \alpha, \delta)(b) = h(b)$.
- $a \not<_1 b \Longrightarrow a \not<_1 \alpha$ (because $a <_1 \alpha$ implies, using (*5), that $a <_1 b$), that is, $a \in B \cap \alpha$ with $m(a) < \alpha$. Then, $m(a) \underset{\text{by (*3)}}{<} \delta = g(0, \alpha, \delta)(\alpha) \underset{g(0,\alpha,\delta) \text{ is strictly increasing}}{\leqslant} g(0, \alpha, \delta)(b)$, that is, $h(\alpha) = a \not<_1 g(0, \alpha, \delta)(b) = h(b)$.

The previous shows that (*7) holds. In fact, (4*) and (7*) show that (2*) also holds for the case $s = l + 1 \subset (\alpha, t + 1]$ and with this we have concluded the proof of (*2). Hence, the proposition holds. ☐

The idea now is that $<_1$ and $<^0$ have something to do with each other. The relation between $<_1$ and $<^0$ is very direct (see next proposition 1.25); however, when we introduce Class(1) (or in general Class(n) for $n \in [1, n]$), the way to relate $<_1$ with a relation $<^1$ (or in general $<^n$ for $n \in [1, n]$) will be much harder and will be done through the covering theorems (theorem 2.33 for Class(1)). So, said in other words, the covering theorem for Class(0) is trivial and therefore we can prove the next proposition 1.25 without anymore preparations.

Proposition 1.25. *Let $\alpha \in \mathrm{Class}(0)$ and $t \in [\alpha, \alpha\omega)$. Then $\alpha <^0 t + 1 \Longleftrightarrow \alpha <_1 t + 1$*

Proof.
$\Longrightarrow$). Clear by the definition of $<^0$.
$\Longleftarrow$). Suppose $\alpha <_1 t + 1$. **(*1)**

Note (*1) and proposition 1.13 imply that $\alpha \in \mathrm{Lim}\mathbb{P}$ **(*2)**.

Case $t = \alpha$.
Let $B \subset_{\mathrm{fin}} t + 1 = \alpha + 1$ be arbitrary. Since $B \cap \alpha$ is finite and (2*) holds, then there exists $\delta \in \mathbb{P}$ such that $B \cap \alpha \subset \delta$. This way, note
$(\bigcup_{t \in B} T(0, \alpha, t) \cap \alpha) \subset B \cap \alpha \subset \beta$, and then, by proposition 1.20, the function $h: B \longrightarrow h[B] \subset \alpha$, $h(x) := g(0, \alpha, \delta)(x)$ is well defined. Finally, note that from propositions 1.17 and 1.18 it follows that the function h is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$.

Case $t > \alpha$.
Let $B \subset_{\mathrm{fin}} t + 1$ be arbitrary. Consider
$C := B \cup \{\alpha, 1, \alpha + 1\} \cup \{\alpha m, l, \alpha m + l \mid \alpha n + l \in B \wedge m \in [1, n] \wedge l \in [0, \alpha)\} \subset_{\mathrm{fin}} t + 1$. So, by (*1), there exists $k: C \longrightarrow k[C] \subset \alpha$ an $(<, <_1, +)$-isomorphism with $k|_\alpha = \mathrm{Id}_\alpha$. **(*3)** Then:

1. $\alpha <_1 \alpha + 1 \Longleftrightarrow k(\alpha) <_1 k(\alpha + 1) = k(\alpha) + k(1) = k(\alpha) + 1$, i.e., $k(\alpha) \underset{\text{proposition 1.13}}{\in} \mathrm{Lim}\mathbb{P}$.
2. $\forall s \in C \cap \alpha . s < \alpha \Longleftrightarrow s = k(s) < k(\alpha)$
3. $\forall n \in [1, \omega) \forall s \in C \cap [\alpha n, \alpha n + \alpha) . -\alpha n + s < \alpha \Longleftrightarrow -\alpha n + s = k(-\alpha n + s) < k(\alpha)\}$

From 1, 2 and 3 follows that $\delta := k(\alpha) \in \text{Class}(0) \cap \alpha$, $(\bigcup_{t\in C} T(0, \alpha, t) \cap \alpha) \underset{\text{propositions 1.17 and 1.20}}{\subset} \delta$ and that the function $H: C \longrightarrow H[C] \subset \alpha$, $H(x) := g(0, \alpha, \delta)(x)$ is well defined. Moreover, by propositions 1.17 it follows that H is an $(<, +)$-isomorphism with $H|_{\alpha} = \text{Id}_{\alpha}$. **(*4)**

Now we show that H is also an $<_1$-isomorphism. **(*5)**

Let $a, b \in C$ with $a < b$.

Case $a = \alpha \wedge b \in [\alpha n, \alpha n + \alpha)$ for some $n \in [1, \omega)$. Then $\alpha <_1 t+1$ and $\alpha < b < t+1$ imply by $\leqslant_1$-connectedness that $\alpha <_1 b$.

On the other hand, note $H(\alpha) = k(\alpha) \underset{\text{by (*3)}}{<_1} k(b) = k(\alpha n + (-\alpha n + b)) \underset{\text{by (*3)}}{=} k(\alpha n) + k(-\alpha n + b) = k(\alpha)n + (-\alpha n + b) = H(\alpha)n + H(-\alpha n + b) \underset{\text{by (*4)}}{=} H(\alpha n) + H(-\alpha n + b) \underset{\text{by (*4)}}{=} H(\alpha n + (-\alpha n + b)) = H(b)$. **(*6)**

Case $a, b < \alpha$. Then $a <_1 b \Longleftrightarrow a = H(a) <_1 b = H(b)$.

Case $a < \alpha \leqslant b$. Then $a <_1 b \underset{\leqslant_1\text{-connectedness and } \leqslant_1\text{-transitivity}}{\Longleftrightarrow} a <_1 \alpha \leqslant_1 b \underset{\text{by (*3) and (*6)}}{\Longleftrightarrow} a = H(a) = k(a) <_1 k(\alpha) = H(\alpha) \leqslant_1 k(b) = H(b)$.

Case $\alpha < a < b$. Then $a <_1 b \underset{\text{by proposition 1.18}}{\Longleftrightarrow} H(\alpha) <_1 H(b)$.

The previous shows that (*5) holds.

Finally, from (*4), (*5) and the fact that $B \subset C$ we conclude, by proposition A.1 in the appendices section, that the function $H|_B: B \longrightarrow H|_B[B] \subset \alpha$, $H|_B(x) = g(0, \alpha, \delta)(x)$ is an $(<, <_1, +)$-isomorphism with $H|_{\alpha} = \text{Id}_{\alpha}$.

All the previous shows that $\alpha <^0 t+1$. □

Corollary 1.26. *Let $\alpha \in \text{Class}(0)$ and $t \in [\alpha, \alpha\omega)$. The following are equivalent:*

1. $\alpha <^0 t+1$
2. $\alpha <_1 t+1$
3. $\alpha \in \text{Lim}\{\beta \in \text{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t)\}$

Proof. Direct from previous propositions 1.25, 1.23 and 1.24. □

1.3.2 A hierarchy induced by $<_1$ and the intervals $[\omega^{\gamma}, \omega^{\gamma+1})$.

In this subsection we show theorem 1.28 which is our way to link "solutions of the conditions $\alpha <_1 t+1$, with $\alpha \in \text{Class}(0)$ and $t \in [\alpha, \alpha\omega)$" (what below is defined as the $G^0(t)$ sets) with a thinning procedure (the sets $A^0(t)$, also defined below). After that, we will see that, for $\alpha = \kappa$ a regular non-countable ordinal, the set of "solutions of the condition $\kappa <_1 t+1$" is club in κ.

Definition 1.27. *By recursion on $([\omega, \infty), <)$, we define $A^0: [\omega, \infty) \longrightarrow \text{Subclasses(OR)}$ in the following way: Let $t \in [\omega, \infty)$ be arbitrary. Let $\alpha \in \text{Class}(0)$ be such that $t \in [\alpha, \alpha\omega)$. Then*

$$A^0(t) := \begin{cases} (\text{Lim}\text{Class}(0)) \cap (\alpha+1) & \textit{iff} \quad t = \alpha \\ \text{Lim}\, A^0(l+1) & \textit{iff} \quad t = l+1 \\ \text{Lim}\{r \in \text{Class}(0) \cap (\alpha+1) \mid T(0, \alpha, t) \cap \alpha \subset r \in \bigcap_{s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset r\}} A^0(s)\} & \textit{iff} \quad t \in [\alpha, \alpha\omega) \cap \text{Lim} \end{cases} =$$

$$\begin{cases} (\text{Lim}\text{Class}(0)) \cap (\alpha+1) & \textit{iff} \quad t = \alpha \\ \text{Lim}\, A^0(l+1) & \textit{iff} \quad t = l+1 \\ \text{Lim}\{r \in \text{Class}(0) \cap (\alpha+1) \mid -\alpha n + t < r \in \bigcap_{s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset r\}} A^0(s)\} & \textit{iff} \quad \begin{cases} t \in [\alpha n, \alpha n + \alpha) \cap \text{Lim} \\ \textit{for some } n \in [1, \omega) \end{cases} \end{cases}$$

On the other hand, we define G^0: $[\omega, \infty) \longrightarrow$ Subclasses(OR) *as follows: Let* $t \in [\omega, \infty)$ *be arbitrary. Let* $\alpha \in \mathrm{Class}(0)$ *and* $n \in [1, \omega)$ *be such that* $t \in [\alpha n, \alpha n + \alpha)$. *Then*
$G^0(t) := \{\beta \in \mathrm{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^0 g(0, \alpha, \beta)(t) + 1\}$
$= \{\beta \in \mathrm{Class}(0) \mid -\alpha n + t < \beta \leqslant \alpha \wedge \beta \leqslant^0 g(0, \alpha, \beta)(t) + 1\}$
$=$, *by proposition 1.25,*
$= \{\beta \in \mathrm{Class}(0) \mid -\alpha n + t < \beta \leqslant \alpha \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t) + 1\}$.
$= \{\beta \in \mathrm{Class}(0) \mid T(0, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t) + 1\}$

Theorem 1.28. $\forall t \in [\omega, \infty). G^0(t) = A^0(t)$.

Proof. We show $\forall t \in [\omega, \infty). G^0(t) = A^0(t)$ by induction on $([\omega, \infty), <)$.

Let $t \in [\omega, \infty)$ be arbitrary and consider $\alpha \in \mathrm{Class}(0)$ and $n \in [1, \omega)$ such that $t \in [\alpha n, \alpha n + \alpha)$.

Suppose $\forall s \in t \cap [\omega, \infty). G^0(s) = A^0(s)$. **(IH)**

Case $t = \alpha$.
Then $G^0(\alpha) = \{\beta \in \mathrm{Class}(0) \mid -\alpha + \alpha < \beta \leqslant \alpha \wedge \beta \leqslant_1 g(0, \alpha, \beta)(\alpha) + 1\} =$
$= \{\beta \in \mathrm{Class}(0) \mid \alpha \geqslant \beta \leqslant_1 \beta + 1\} \underset{\text{proposition 1.13}}{=} (\mathrm{Lim}\,\mathrm{Class}(0)) \cap (\alpha + 1) = A^0(\alpha)$.

Case $t = l + 1$ for some $l \in [\alpha n, \alpha n + \alpha)$.
Then $G^0(l+1) = \{\beta \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \beta \leqslant \alpha \wedge \beta \leqslant_1 g(0, \alpha, \beta)(l+1) + 1\} \underset{\text{corollary 1.26}}{=}$
$\{\beta \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \beta \leqslant \alpha \wedge$
$\beta \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\beta n + g(0, \alpha, \beta)(l+1) < \gamma \wedge \gamma \leqslant_1 g(0, \beta, \gamma)(g(0, \alpha, \beta)(l+1))\}\} =$
$\{\beta \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \beta \leqslant \alpha \wedge$
$\beta \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\beta n + (\beta n + (-\alpha n + l + 1)) < \gamma \wedge$
$\gamma \leqslant_1 \gamma n + (-\beta n + (\beta n + (-\alpha n + l + 1)))\}\} =$
$\{\beta \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \beta \leqslant \alpha \wedge$
$\beta \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \gamma \wedge \gamma \leqslant_1 \gamma n + (-\alpha n + l + 1)\}\} =$
$\mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \gamma \leqslant \alpha \wedge \gamma \leqslant_1 \gamma n + (-\alpha n + l + 1)\} =$
$\mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\alpha n + (l+1) < \gamma \leqslant \alpha \wedge \gamma \leqslant_1 g(0, \alpha, \gamma)(l+1)\} =$
$\mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\alpha n + l < \gamma \leqslant \alpha \wedge \gamma \leqslant_1 g(0, \alpha, \gamma)(l) + 1\} =$
$\mathrm{Lim} G^0(l) \underset{\text{by (IH)}}{=} \mathrm{Lim} A^0(l) = A^0(l+1)$.

Case $\alpha < t \in [\alpha n, \alpha n + \alpha) \cap \mathrm{Lim}$.
In order to show $G^0(t) = A^0(t)$, we make some preparations first. Note
$G^0(t) = \{\beta \in \mathrm{Class}(0) \mid -\alpha n + t < \beta \leqslant \alpha \wedge \beta \leqslant_1 g(0, \alpha, \beta)(t) + 1\} =$, as in the previous case,
$= \mathrm{Lim}\{\gamma \in \mathrm{Class}(0) \mid -\alpha n + l < \gamma \leqslant \alpha \wedge \gamma \leqslant_1 g(0, \alpha, \gamma)(t)\}$. **(*0)**

On the other hand, let's show
$\forall \xi \in \mathrm{Class}(0). -\alpha n + t < \xi \leqslant \alpha \wedge \xi \leqslant_1 g(0, \alpha, \gamma)(t)\} \Longrightarrow \xi \in \bigcap_{s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset \xi\}} A^0(s)$
(*1)

Let $\xi \in \mathrm{Class}(0)$ be such that $-\alpha n + t < \xi \leqslant \alpha \wedge \xi \leqslant_1 g(0, \alpha, \gamma)(t)\}$. **(*2)**
Let $s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset \xi\}$ be arbitrary and let $m \in [1, n]$ be such that $s \in [\alpha m, \alpha m + \alpha)$. Then clearly $-\alpha m + s < \xi \leqslant \alpha$ and $\xi \leqslant \xi m + (-\alpha m + s + 1) \leqslant \xi n + (-\alpha n + t) = g(0, \alpha, \gamma)(t)$; the latter implies, by (*2) and $\leqslant_1$-connectedness, $\xi \leqslant_1 \xi m + (-\alpha m + s + 1) = (\xi m + (-\alpha m + s)) + 1 = g(0, \alpha, \gamma)(s) + 1$. This shows $\xi \in \{\gamma \in \mathrm{Class}(0) \mid -\alpha m + s < \gamma \leqslant \alpha \wedge \gamma \leqslant_1 g(0, \alpha, \gamma)(s) + 1\} = G^0(s) \underset{\text{by our (IH)}}{=} A^0(s)$ and since this was done for arbitrary $s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset \xi\}$, it follows $\xi \in \bigcap_{s \in \{q \in (\alpha, t) \mid T(0, \alpha, q) \cap \alpha \subset \xi\}} A^0(s)$. Hence (*1) holds.

Now we show $\{\gamma\in\text{Class}(0)\mid -\alpha n+l<\gamma\leqslant\alpha\wedge\gamma\leqslant_1 g(0,\alpha,\gamma)(t)\}=$
$\{r\in\text{Class}(0)\cap(\alpha+1)\mid -\alpha n+t<r\in\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset r\}}A^0(s)\}$ **(*3)**

Note from (*1) follows immediately that the contention $''\subset''$ of (*3) holds. Let's see that the contention $''\supset''$ also holds:

Let $\beta\in\{r\in\text{Class}(0)\cap(\alpha+1)\mid -\alpha n+t<r\in\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset r\}}A^0(s)\}$ be arbitrary. Then $\beta\in\text{Class}(0)\wedge-\alpha n+l<\beta\leqslant\alpha$ **(*4)** and
$\beta\in\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset\beta\}}A^0(s)\underset{\text{by (IH)}}{=}\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset\beta\}}G^0(s)=$
$\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset\beta\}}\{\gamma\in\text{Class}(0)\mid T(0,\alpha,s)\cap\alpha\subset\gamma\leqslant\alpha\wedge\gamma\leqslant_1 g(0,\alpha,\gamma)(s)\}$ **(*5)**.
This way, for the sequences $(\delta_s)_{s\in I}$ and $(\xi_s)_{s\in I}$ defined as
$I:=\begin{cases}(0,-\alpha n+t) & \text{iff } t>\alpha n\\ (0,\beta) & \text{iff } t=\alpha n\end{cases}$,
$\delta_s:=\begin{cases}\alpha n+s & \text{iff } t>\alpha n\\ \alpha(n-1)+s & \text{iff } t=\alpha n\end{cases}$
and
$\xi_s:=\begin{cases}\beta n+s & \text{iff } t>\alpha n\\ \beta(n-1)+s & \text{iff } t=\alpha n\end{cases}$,
we have that, by (*4) and (*5),
$\forall s\in I.T(0,\alpha,\delta_s)\cap\alpha\subset\beta\leqslant_1 g(0,\alpha,\beta)(\delta_s)=\xi_s$ and
$\xi_s\underset{\text{cof}}{\hookrightarrow}\left\{\begin{matrix}\beta n+(-\alpha n+t) & \text{iff } t>\alpha n\\ \beta n & \text{iff } t=\alpha n\end{matrix}\right\}=g(0,\alpha,\beta)(t)$. From all this and using $\leqslant_1$ -continuity, we conclude $\alpha\geqslant\beta\in\text{Class}(0)\wedge-\alpha n+t<\beta\leqslant_1=g(0,\alpha,\beta)(t)$, that is,
$\beta\in\{\gamma\in\text{Class}(0)\mid -\alpha n+t<\gamma\leqslant\alpha\wedge\gamma\leqslant_1 g(0,\alpha,\gamma)(t)\}=G^0(t)$. Since this was done for arbitrary $\beta\in\{r\in\text{Class}(0)\cap(\alpha+1)\mid -\alpha n+t<r\in\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset r\}}A^0(s)\}$, then $''\supset''$ of (*3) also holds.

Finally, it is now very easy to see that $G^0(t)=A^0(t)$ holds:
$G^0(t)\underset{\text{by (*0)}}{=}\text{Lim}\{\gamma\in\text{Class}(0)\mid -\alpha n+t<\gamma\leqslant\alpha\wedge\gamma\leqslant_1 g(0,\alpha,\gamma)(t)\}\underset{\text{by (*3)}}{=}$
$=\text{Lim}\{r\in\text{Class}(0)\cap(\alpha+1)\mid -\alpha n+t<r\in\bigcap_{s\in\{q\in(\alpha,t)\mid T(0,\alpha,q)\cap\alpha\subset r\}}A^0(s)\}=A^0(t)$. □

Proposition 1.29. *Let κ be a regular non-countable ordinal. Then $\forall t\in[\kappa,\kappa\omega).A^0(t)$ is closed unbounded in κ.*

Proof. By induction on $([\kappa,\kappa\omega),<)$. One needs to work a little bit with the usual properties of closed unbounded classes. □

To finish this chapter, we show that there are ordinals $\alpha\in\text{Class}(0)$ such that $\alpha<_1\alpha\omega$.

Proposition 1.30. *Let κ be a regular non-countable ordinal and $\alpha:=\min\text{Class}(0)=\omega$. Then*

1. $\bigcap_{t\in[\kappa,\kappa\omega)\wedge T(0,\kappa,t)\cap\kappa\subset\alpha}A^0(t)=\{\gamma\in\text{Class}(0)\cap(\kappa+1)\mid\gamma<_1\gamma\omega\}$.
2. $\{\gamma\in\text{Class}(0)\mid\gamma<_1\gamma\omega\}$ *is closed unbounded in κ.*

Proof. Let κ and α be as stated
1.
To show $\bigcap_{t\in[\kappa,\kappa\omega)\wedge T(0,\kappa,t)\cap\kappa\subset\alpha}A^0(t)\subset\{\gamma\in\text{Class}(0)\cap(\kappa+1)\mid\gamma\leqslant_1\gamma\omega\}$. **(*0)**

Let $\beta \in \bigcap_{t \in [\kappa, \kappa\omega) \wedge T(0,\kappa,t) \cap \kappa \subset \alpha} A^0(t) = \bigcap_{t \in [\kappa, \kappa\omega) \wedge T(0,\kappa,t) \cap \kappa \subset \alpha} G^0(t) = \bigcap_{t \in [\kappa, \kappa\omega) \wedge T(0,\kappa,t) \cap \kappa \subset \alpha} \{\gamma \in \text{Class}(0) \mid T(0, \kappa, t) \cap \kappa \subset \gamma \leqslant \kappa \wedge \gamma \leqslant_1 g(0, \kappa, \gamma)(t) + 1\}$. Notice from this follows that $\forall n \in [1, \omega).T(0, \kappa, \kappa n) \cap \kappa \subset \alpha \leqslant \beta \leqslant \kappa \wedge \beta \leqslant_1 g(0, \kappa, \beta)(\kappa n) + 1 = \beta n + 1$; therefore, since the sequence $(\beta n + 1)_{n \in [1,\omega)}$ is cofinal in $\beta\omega$, we get, by $\leqslant_1$-continuity, $\kappa \geqslant \beta \leqslant_1 \beta\omega$. Since this was done for arbitrary $\beta \in \bigcap_{t \in [\kappa, \kappa\omega) \wedge T(0,\kappa,t) \cap \kappa \subset \alpha} A^0(t)$, then (*0) follows.

To show $\bigcap_{t \in [\kappa, \kappa\omega) \wedge T(0,\kappa,t) \cap \kappa \subset \alpha} A^0(t) \supset \{\gamma \in \text{Class}(0) \cap (\kappa+1) \mid \gamma \leqslant_1 \gamma\omega\}$. **(*1)**

Let $\beta \in \{\gamma \in \text{Class}(0) \cap (\kappa+1) \mid \gamma \leqslant_1 \gamma\omega\}$. **(*2)**

Let $t \in [\kappa, \kappa\omega) \wedge T(0, \kappa, t) \cap \kappa \subset \alpha$ be arbitrary and let $n \in [1, \omega)$ be such that $t \in [\kappa n, \kappa n + \kappa)$. Then $T(0, \kappa, t) \cap \kappa = \{-\kappa n + t\} \subset \alpha \leqslant \beta \leqslant \beta n + (-\kappa n + t) + 1 < \beta(n+1) < \beta\omega$ and then, by (*2) and $\leqslant_1$-connectedness, we get $T(0, \kappa, t) \cap \kappa \subset \beta \leqslant \kappa \wedge \beta \leqslant_1 \beta n + (-\kappa n + t) + 1 = g(0, \kappa, \beta)(t) + 1$, that is, $\beta \in G^0(t) = A^0(t)$. Since this was done for arbitrary $\beta \in \{\gamma \in \text{Class}(0) \cap (\kappa+1) \mid \gamma \leqslant_1 \gamma\omega\}$ and for arbitrary $t \in [\kappa, \kappa\omega) \wedge T(0, \kappa, t) \cap \kappa \subset \alpha$, then we have shown that (*1) holds.

Hence, by (*0) and (*1) the theorem holds.

2.

Left to the reader. See proposition 2.59 to get a hint for a proof of this fact. $\square$

Chapter 2

Class(1)

The previous chapter exemplifies the way in which we will be studying the $<_1$ relation up to chapter 6. However, our subsequent work will be based not on the class of additive principal numbers $\mathrm{Class}(0)=\mathbb{P}$ (as done previously), but on the class of epsilon numbers $\mathbf{Class(1)}:=\mathbb{E}$. The reason for this is merely circumstantial: the main ideas used in this work for the study of the $<_1$ relation were discovered by the author of this thesis considering $\mathbb{E}$ as our "base class" and it has been after the successful development of these ideas up to their most general form (chapters 3, 4, 5 and 6) that it was clear that one could make the whole treatment of the study of the $<_1$ relation based on $\mathbb{P}$.[2.1] This chapter contains, then, the original considerations that eventually lead to the point of view used from chapter 3 ahead.

2.1 The ordinals α satisfying $\alpha <_1 \alpha+\xi$, for some $\xi \in [1,\alpha]$.

We will first show a theorem and a corollary appearing in [18]. The proof of the theorem we present here is slightly different than the one given by Wilken. Primarily, let's state the next proposition.

Proposition 2.1. *(Cofinality properties for the easiest case).*
Let $\alpha\in\mathrm{OR}$ and $t\in[\alpha,\alpha 2)$. Let $l\in[0,\alpha)$ be such that $t=\alpha+l$. Then
$\alpha <_1 t+1=(\alpha+l)+1 \Longleftrightarrow$ *there exists a strictly increasing sequence* $(\xi_i)_{i\in I}\subset\alpha\cap\mathbb{P}$ *such that* $\xi_i \underset{\mathrm{cof}}{\longrightarrow} \alpha$ *and* $\forall i\in I. l<\xi_i\leqslant_1 \xi_i+l$.

Proof. Direct from corollary 1.26. □

Remark 2.2. Let $\alpha\in\mathbb{P}$, $t\in[\alpha,\alpha 2)$ and $l\in[0,\alpha)$ be such that $t=\alpha+l$. For an additive principal number $\beta\in\mathbb{P}$ with $\beta>l$, let's denote $t/\alpha:=\beta/$ to the ordinal $\beta+l$; that is, $t/\alpha:=\beta/$ is simply the replacement of α by β in $t=\alpha+l$. With this convention we can enunciate previous proposition 2.1 as:
$\alpha <_1 t+1 \Longleftrightarrow \alpha\in\mathrm{Lim}\{\beta\in\mathbb{P}\mid l<\beta\wedge\beta\leqslant_1 t/\alpha:=\beta/\}$.

2.1. My supervisor, Prof. W.Buchholz, noticed that the ideas used to study the upper classes (chapter 3 to chapter 6) could be already applied for Class(1); in fact, he provided me a draft where he presented all this in a very nice way and suggested me to make such changes. Ultimately, I decided to add what is now the first chapter of this thesis in order to give the reader a sense of how the most general theorems are done (as suggested by my supervisor) and leave the results about Class(1) as they were, since making changes in them imply the need to make plenty of changes in the subsequent chapters in order to get a consistent work.

Now we present the theorem of Wilken that we mentioned before.

Theorem 2.3. *(Wilken).*
$\forall\alpha\in\mathrm{OR}\forall\xi\in[1,\alpha).\alpha<_1\alpha+\xi\Longleftrightarrow\alpha=\omega^A$ *with* $A=\omega^{\xi}\cdot s$ *for some* $s\in\mathrm{OR}$, $s\neq 0$.
Remark: In the previous line, we are NOT saying that the Cantor Normal Form of A *is* $\omega^{\xi}\cdot s$.

Proof. We prove the theorem by induction on $(\mathrm{OR},<)$.

Let $\alpha\in\mathrm{OR}$ and suppose the claim of the theorem holds for any $\beta<\alpha$. **(IH)**

We continue the proof by a side induction on $[1,\alpha)$.

Let $\xi\in[1,\alpha)$ and assume the claim holds for any $z\in[1,\alpha)\cap\xi$. **(SIH)**

Case $\xi=1$. Then the claim holds by proposition 1.13.

Case $\xi\in(1,\alpha)\cap\mathrm{Lim}$.

$\Longrightarrow$) Assume $\alpha<_1\alpha+\xi$. Then $\alpha<_1\alpha+y$ for any $y\in[1,\xi)$ by $\leqslant_1$-connectedness. Then, by our side induction hypothesis, for any $y\in[1,\xi)$, $\alpha=\omega^A$, where $A=\omega^y\cdot s_y$ for some $s_y\neq 0$. From this follows that $A=\sup\{\omega^y\cdot s_y|y\in[1,\xi)\}\geqslant\omega^{\xi}$. Now, by Euclid's division algorithm for ordinals, there exist $s,\rho\in\mathrm{OR}$ with $\rho<\omega^{\xi}$ such that $A=\omega^{\xi}\cdot s+\rho$. But since $\rho<\omega^{\xi}$ and $\xi\in(1,\alpha)\cap\mathrm{Lim}$, then there exists $\delta\in[1,\xi)$ such that $\omega^{\delta}>\rho$. All this means that $A\underset{\text{by our SIH}}{=}\omega^{\delta}\cdot s_{\delta}=\omega^{\xi}\cdot s+\rho$, which implies that ω^{δ} divides $\rho<\omega^{\delta}$; therefore $\rho=0$ and $A=\omega^{\xi}\cdot s$.

$\Longleftarrow$). Assume $\alpha=\omega^A$ with $A=\omega^{\xi}\cdot s$ for some $s\in\mathrm{OR}$, $s\neq 0$. Then, for any $y\in[1,\xi)$, we can write $A=\omega^y\cdot s_y$ for some $s_y\neq 0$. Then by our SIH we get $\forall y\in[1,\xi).\alpha<_1\alpha+y$. But the sequence $(\alpha+y)_{y\in[1,\xi)}$ is cofinal in $\alpha+\xi$, so by $\leqslant_1$-continuity we conclude that $\alpha<_1\alpha+\xi$.

Case $\xi\in(1,\alpha)$, $\xi=l+1$ for some $l\in[1,\alpha)$.

$\Longrightarrow$) Assume $\alpha<_1\alpha+l+1$. Then by proposition 2.1, there is a strictly increasing sequence $(\xi_i)_{i\in I}\subset\mathrm{OR}$ such that $\xi_i\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and $\forall i\in I.l<\xi_i<_1\xi_i+l$. This implies, by our IH, that $\forall i\in I.\xi_i=\omega^{\omega^l\cdot s_i}$ for some $s_i\neq 0$. Moreover, note that since $(\xi_i)_{i\in I}$ is strictly increasing, then the sequence $(s_i)_{i\in I}$ has to be also strictly increasing, and therefore $\sigma:=\sup\{s_i|i\in I\}\in\mathrm{Lim}$.

On the other hand, consider $\sigma=_{\mathrm{CNF}}\omega^{S_1}a_1+\ldots+\omega^{S_m}a_m$. Then $S_m\neq 0$ (because $\sigma\in\mathrm{Lim}$) and so we can write $\sigma=\omega\cdot s$ for some $s\in\mathrm{OR}$, $s\neq 0$. From this and the previous paragraph we get that $\alpha=\sup\{\xi_i|i\in I\}=\omega^{\sup\{\omega^l\cdot s_i|i\in I\}}=\omega^{\omega^l\cdot\sup\{s_i|i\in I\}}=\omega^{\omega^l\cdot\sigma}=\omega^{\omega^l\cdot\omega\cdot s}=\omega^{\omega^{l+1}\cdot s}$ for some $s\neq 0$.

$\Longleftarrow$). Assume $\alpha=\omega^A$ with $A=\omega^{l+1}\cdot s=\omega^l\cdot(\omega s)$ for some $s\neq 0$.

Take an arbitrary finite $B\subset_{\mathrm{fin}}\alpha+l+1$. Then $B=\{a_1<\ldots<a_n<\alpha+b_1<\ldots<\alpha+b_p\}$ for some a_i, b_j. Without loss of generality (see proposition A.1 in the appendices section) we can assume that $\{a_n=\alpha,b_p=l,\alpha+l,b_1,\ldots,b_p\}\subset B$.

We want to define an $(<,<_1,+)$-isomorphism $h\colon B\longrightarrow h[B]\subset\alpha$ with $h|_{\alpha}=\mathrm{Id}_{\alpha}$. In order to achieve this, we need to do first the following observation: Let $(s_j)_{i\in J}\subset[1,\omega s)$ be a sequence such that $s_j\underset{\mathrm{cof}}{\hookrightarrow}\omega s$ (this is possible because $\omega s\in\mathrm{Lim}$). Consider the sequence $(\gamma_j)_{i\in J}$, where $\gamma_j:=\omega^{\omega^l\cdot s_j}<\omega^{\omega^l\cdot(\omega s)}=\alpha$. Then $\gamma_j\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and by our (IH), $\forall j\in J.\gamma_j<_1\gamma_j+l+1$. This shows that $\alpha\in\mathrm{Lim}\{\gamma\in\mathrm{OR}|\gamma<_1\gamma+l\}$.

On the other hand, let $C:=\{m(a)|a\in(B\cap\alpha)\wedge m(a)<\alpha\}$. Since C is finite and we know that $\alpha\in\mathrm{Lim}\{\gamma\in\mathrm{OR}|\gamma<_1\gamma+l\}$, then $\emptyset\neq(a_{n-1},\alpha)\cap(\max C,\alpha)\cap\{\gamma\in\mathrm{OR}|\gamma<_1\gamma+l\}$. Take $\rho\in(a_{n-1},\alpha)\cap(\max C,\alpha)\cap\{\gamma\in\mathrm{OR}|\gamma<_1\gamma+l\}$. Note that $\rho<_1\rho+1$ by $\leqslant_1$-connectedness and so $\rho\in\mathrm{Lim}\mathbb{P}\subset\mathbb{P}$ by proposition 1.13.

We define the function $h\colon B\longrightarrow h[B]$ as
$h(a_k):=a_k$ for any $k\in[1,n-1]$,
$h(b_k):=b_k$ for any $k\in[1,p]$,
$h(\alpha):=\rho$ and
$h(\alpha+b_k):=\rho+b_k$ for any $k\in[1,p]$.

By the definition of h, it is clear that $h|_\alpha = \mathrm{Id}_\alpha$ and that $h: B \longrightarrow h[B]$ is bijective. Now we show it is an $(<, <_1, +)$-isomorphism: The proof that h is an $(<, +)$-isomorphism is essentially the same done within the proof of proposition 1.13 (one has to check just some few more subcases).

So let's prove that h is an $<_1$-isomorphism. We need to see several cases:

1. Since $\alpha = \omega^A$ and $A = \omega^{l+1} \cdot s = \omega^l \cdot (\omega s)$, then by our SIH $\alpha <_1 \alpha + l$. On the other hand, note that $h(\alpha) <_1 h(\alpha + l) = h(\alpha) + l$ indeed holds, because $\rho \in \{\gamma \in \mathrm{OR} | \gamma <_1 \gamma + l\}$.

2. By 1. and $\leqslant_1$-connectedness, we have that $\alpha <_1 \alpha + b_k$ for any $k \in [1, p]$. But $\rho <_1 \rho + l$ and $\forall k \in [1, p]. \rho < \rho + b_k \leqslant \rho + l$ imply, by $\leqslant_1$-connectedness, that $h(\alpha) <_1 h(\alpha + b_k) = h(\alpha) + b_k$ also holds for any $k \in [1, p]$.

3. Let $\alpha + b_i, \alpha + b_j \in (B \backslash \{\alpha\})$ with $\alpha + b_i < \alpha + b_j$ be arbitrary. Then $\alpha + b_i \not<_1 \alpha + b_j$ by corollary 1.7, and because of the same reason $h(\alpha + b_i) = h(\alpha) + b_i \not<_1 h(\alpha) + b_j = h(\alpha + b_j)$.

4. Let $a_i, a_j \in B \cap \alpha$ with $a_i < a_j$ be arbitrary. Then clearly
$a_i <_1 a_j \Longleftrightarrow h(a_i) = a_i <_1 a_j = h(a_j)$.

5. Let $a_i \in B \cap \alpha$. If $a_i <_1 \alpha$, then $a_i <_1 h(\alpha)$ holds by $\leqslant_1$-connectedness (because $a_i < \rho < \alpha$). If $a_i \not<_1 \alpha$, this means $m(a_i) \in C$ and therefore $m(a_i) < \rho$, that is, $h(a_i) = a_i \not<_1 \rho = h(\alpha)$.

6. Let $a_i \in B \cap \alpha$ and $\alpha + b_j \in (B \backslash \{\alpha\})$. If $a_i <_1 \alpha + b_j$, then, using
$a_i < \rho + b_j \underset{\text{because } \alpha \in \mathbb{P}}{<} \alpha < \alpha + b_j$, we conclude that $h(a_i) = a_i <_1 \rho + b_j = h(\alpha + b_j)$ by $\leqslant_1$-connectedness. If $a_i \not<_1 \alpha + b_j$, then $a_i \not<_1 \alpha$ (because, by 2., we already know that $\alpha <_1 \alpha + b_j$; so $a_j <_1 \alpha$ would imply $a_i <_1 \alpha + b_j$ by $\leqslant_1$-transitivity). This means $m(a_i) \in C$ and therefore $m(a_i) < \rho$. Hence $h(a_i) = a_i \not<_1 \rho + b_j = h(\alpha + b_j)$.

1., 2., 3., 4., 5. and 6. show that h is also an $<_1$-isomorphism. □

Corollary 2.4. *(Wilken).* $\forall \alpha \in \mathrm{OR}. \alpha <_1 \alpha 2 \Longleftrightarrow \alpha \in \mathbb{E}$

Proof. Not hard. Left to the reader. □

Corollary 2.5. *Let* $\alpha \in \mathrm{OR}$, $\alpha \notin \mathbb{E}$ *be such that* $\alpha =_{\mathrm{CNF}} \omega^{A_1} a_1 + \ldots + \omega^{A_n} a_n$.

a) If $n \geqslant 2$ *or* $a_1 \geqslant 2$, *then* $m(\alpha) = \alpha$.

b) If $n = 1 = a_1$ *and* $A_1 =_{\mathrm{CNF}} \omega^{B_1} b_1 + \ldots + \omega^{B_s} b_s$, *then* $m(\alpha) = \alpha + B_s$. *(B_s could be zero).*

Proof. a). Suppose $n \geqslant 2$ or $a_1 \geqslant 2$. Then, by proposition 1.13, $\alpha \not<_1 \alpha + 1$; then $m(\alpha) = \alpha$.
b). Suppose $n = 1 = a_1$ and $A_1 =_{\mathrm{CNF}} \omega^{B_1} b_1 + \ldots + \omega^{B_s} b_s$.

Case $B_s = 0$. By proposition 1.13, $\alpha \not<_1 \alpha + 1$; then $m(\alpha) = \alpha = \alpha + 0 = \alpha + B_s$.

Case $B_s \neq 0$. Since by hypothesis $\alpha \notin \mathbb{E}$ and $\alpha = \omega^{\omega^{B_1} b_1 + \ldots + \omega^{B_s} b_s} \in \mathbb{P}$, then $B_s < \alpha > 1$ and therefore $B_s + 1 < \alpha > B_s$. (*)

On the other hand, let δ_i be such that $B_s + \delta_i = B_i$ for any $i \in [1, s]$. Then
$A_1 =_{\mathrm{CNF}} \omega^{B_s + \delta_1} b_1 + \ldots + \omega^{B_s + \delta_{s-1}} b_{s-1} + \omega^{B_s} b_s = \omega^{B_s} \cdot (\omega^{\delta_1} b_1 + \ldots + \omega^{\delta_{s-1}} b_{s-1} + b_s)$; moreover, $A_1 \neq \omega^{B_s + 1} \cdot D$ for any $D \in \mathrm{OR}$ (because of the uniqueness of the Cantor Normal Form: if $A_1 = \omega^{B_s + 1} \cdot D$ for some $D \in \mathrm{OR}$, then for $D =_{\mathrm{CNF}} \omega^{D_1} d_1 + \ldots + \omega^{D_k} d_k$ one gets $A_1 =_{\mathrm{CNF}} \omega^{B_s + 1 + D_1} d_1 + \ldots + \omega^{B_s + 1 + D_k} d_k$, which is different than
$\omega^{B_1} b_1 + \ldots + \omega^{B_{s-1}} b_{s-1} + \omega^{B_s} b_s$ because $B_s + 1 + D_k > B_s$). The previous and (*) imply, by theorem 2.3, that $\alpha <_1 \alpha + B_s$ and $\alpha \not<_1 \alpha + B_s + 1$. Hence $m(\alpha) = \alpha + B_s$. □

2.2 $<_1$ in the intervals $[\varepsilon_\gamma, \varepsilon_{\gamma+1})$

Our interest now is to describe the solutions of $x<_1\beta$ for $\beta>x2$. The first thing to note is that, in case we are able to find some ordinal x such that $x<_1\beta$ for some $\beta>x2$, then by $\leqslant_1$-connectedness $x<_1 x2$, and therefore, by corollary 2.4, $x\in\mathbb{E}$. This shows that the solutions of the inequalities we are now interested, in case they exist, have to be epsilon numbers. Because of this, **we define Class(1) :=** $\mathbb{E}$ and we aim at the description of epsilon numbers x such that they satisfy something of the form $x<_1\beta$, with $\beta\in[x,x^+)$ and $x^+:=\min\{e\in\mathbb{E}\mid e>x\}$; since we restrict $\beta\in[x,x^+)$, we will informally say that we are studying the relation $<_1$ in the intervals $[\varepsilon_\gamma,\varepsilon_{\gamma+1})$.

2.2.1 Substitutions

In our previous work, whenever we asserted that for certain ordinals α,ξ with $\xi\in\alpha$, it holds $\alpha<_1\alpha+\xi$, we provided, for every $B\subset_{\text{fin}}\alpha+\xi$, an $(<,<_1,+)$-isomorphism $h\colon B\longrightarrow h[B]\subset\alpha$ such that $h|_\alpha=\mathrm{Id}_\alpha$. The important aspect we want to stress is that the isomorphism we constructed had the following peculiarity: we looked for an "adequate" $\rho\in\alpha$, we defined $h(\alpha):=\rho$ (we can always consider that $\alpha\in B$ by proposition A.1 in the appendices section), $h(a):=a$ for any $a\in B\cap\alpha$, and for any $\alpha+l\in B$, we defined $h(\alpha+l):=\rho+l$. So h just "substituted α by ρ and leave the rest as it was". This suggests to study these kind of substitutions as witnesses of the $<_1$-relation; in particular, this will play an essential role for our study of $<_1$ in the intervals $[\varepsilon_\gamma,\varepsilon_{\gamma+1})$.

Definition 2.6. *For $x\in\mathrm{OR}$, let $\mathrm{Ep}(x)$ be the (finite) set of epsilon numbers appearing in the Cantor Normal Form of x, that is,*

$$\mathrm{Ep}(x):=\begin{cases}\{x\} & \text{if } x\in\mathbb{E}\\ \mathrm{Ep}(L_1)\cup\ldots\cup\mathrm{Ep}(L_n) & \text{if } x\notin\mathbb{E}\wedge x=_{\mathrm{CNF}}L_1l_1+\ldots+L_nl_n\wedge(n\geqslant 2\vee l_1\geqslant 2)\\ \mathrm{Ep}(L) & \text{if } x\notin\mathbb{E}\wedge x=_{\mathrm{CNF}}\omega^L\end{cases}$$

Definition 2.7. *Let α, $e\in\mathbb{E}$ and $x\in\mathrm{OR}$. We define the substitution of α by e in the Cantor Normal Form of x (and we denote it as $x[\alpha:=e]$) as:*

$$x[\alpha:=e]:=\begin{cases}x & \text{if } x\in\mathbb{E}\wedge x\neq\alpha\\ e & \text{if } x=\alpha\\ L_1[\alpha:=e]l_1+\ldots+L_n[\alpha:=e]l_n & \text{if } x\notin\mathbb{E}\wedge x=_{\mathrm{CNF}}L_1l_1+\ldots+L_nl_n\wedge(n\geqslant 2\vee l_1\geqslant 2)\\ \omega^{L[\alpha:=e]} & \text{if } x\notin\mathbb{E}\wedge x=_{\mathrm{CNF}}\omega^L\end{cases}$$

As the reader can see, the substitution $x[\alpha:=e]$ makes sense for any $\alpha,e\in\mathbb{E}$ and $x\in\mathrm{OR}$. We will require later the conditions $x\in\alpha^+$ and $\mathrm{Ep}(x)\cap\alpha\subset e$ in order to guarantee that $x[\alpha:=e]$ is a Cantor Normal Form already: the one obtained by simply exchanging in the Cantor Normal Form of x the epsilon number α by the epsilon number e.

Proposition 2.8. *Let $\alpha,e\in\mathbb{E}$.*

a) $\mathrm{Ep}(x)$ *is finite for any* $x\in\mathrm{OR}$.

b) $0<x[\alpha:=e]$ *for any* $x\in\mathrm{OR}\backslash\{0\}$.

c) $x[\alpha:=e]=x$ *for any* $x\in\alpha$.

Proof. Easy. □

Proposition 2.9. *Let $\alpha, e \in \mathbb{E}$ and $q, s \in \alpha^+$. Suppose that $\mathrm{Ep}(q) \cap \alpha \subset e \supset \mathrm{Ep}(s) \cap \alpha$. Then $q < s \Longleftrightarrow q[\alpha := e] < s[\alpha := e]$.*

Proof. Not hard. □

Proposition 2.10. *Let $\alpha, e \in \mathbb{E}$ and $s \in \alpha^+$.*

1. *If $\mathrm{Ep}(s) \cap \alpha \subset e$ then $s[\alpha := e] \in e^+$, $\mathrm{Ep}(s[\alpha := e]) \cap e = \mathrm{Ep}(s) \cap \alpha$ and the ordinal $s[\alpha := e]$ is already in Cantor Normal Form.*
2. *$\mathrm{Ep}(s) \cap \alpha \subset e \Longleftrightarrow \mathrm{Ep}(\omega^s) \cap \alpha \subset e$.*
3. *If $\mathrm{Ep}(s) \cap \alpha \subset e$ then $\omega^s[\alpha := e] = \omega^{s[\alpha := e]}$.*
4. *If $s =_{\mathrm{CNF}} A_1 a_1 + \ldots + A_m a_m$ then $\mathrm{Ep}(s) \cap \alpha \subset e \Longleftrightarrow (\bigcup_{1 \leqslant i \leqslant m} \mathrm{Ep}(A_i) \cap \alpha) \subset e$.*

Proof. Not hard. □

Proposition 2.11. *Let $\alpha, e \in \mathbb{E}$ and $q, s \in \alpha^+$. Suppose $\mathrm{Ep}(q) \cap \alpha \subset e \supset \mathrm{Ep}(s) \cap \alpha$. Then*

a) $\mathrm{Ep}(q + s) \cap \alpha \subset e$ *and* $(q + s)[\alpha := e] = q[\alpha := e] + s[\alpha := e]$.

b) $\mathrm{Ep}(q \cdot s) \cap \subset e$ *and* $(q \cdot s)[\alpha := e] = q[\alpha := e] \cdot s[\alpha := e]$.

c) $s[\alpha := e][e := \alpha] = s$

d) *If $s = a + c$ for some $a, c \in \mathrm{OR}$, then $\mathrm{Ep}(c) \cap \alpha \subset e$.*

e) *If $s = a \cdot b$ for some $a, b \in \mathrm{OR}$, then $\mathrm{Ep}(b) \cap \alpha \subset e$.*

Proof. Not hard. □

Definition 2.12. *For $\alpha, e \in \mathbb{E}$ we define $M(\alpha, e) := \{q \in \alpha^+ | \mathrm{Ep}(q) \cap \alpha \subset e\}$.*

We can summarize our previous results in the following two corollaries:

Corollary 2.13. *Let $\alpha, e \in \mathbb{E}$. Then:*

1. *$M(\alpha, e)$ is closed under the operations $+, \cdot, \lambda x.\omega^x$.*
2. *$M(\alpha, e) \cap [\alpha, \alpha^+)$ is closed under the operations $+, \cdot, \lambda x.\omega^x$.*

Proof. Left to the reader. □

Corollary 2.14. *Let $\alpha, e \in \mathbb{E}$. Then*

1. *The function* $\begin{array}{l} f\colon M(\alpha, e) \longrightarrow f[M(\alpha, e)] \subset \mathrm{OR} \\ \quad q \quad \longmapsto q[\alpha := e] \end{array}$ *is an $(<, +, \cdot, \lambda x.\omega^x)$-isomorphism.*
2. *If $e \leqslant \alpha$ then $M(e, \alpha) \cap [e, e^+) = [e, e^+)$ and the functions* $\begin{array}{l} h\colon [\alpha, \alpha^+) \cap M(\alpha, e) \longrightarrow [e, e^+) \\ \quad q \quad \longmapsto q[\alpha := e] \end{array}$ *and* $\begin{array}{l} k\colon [e, e^+) \longrightarrow [\alpha, \alpha^+) \cap M(\alpha, e) \\ \quad q \quad \longmapsto q[e := \alpha] \end{array}$ *are $(<, +, \cdot, \lambda x.\omega^x)$-isomorphisms with $h^{-1} = k$.*

Proof.

1.
Proposition 2.9 guarantees that f preserves the relation $<$ (this, subsequently, implies that f is injective, and therefore $f\colon M(\alpha, e) \longrightarrow f[M(\alpha, e)]$ is a bijection). Moreover, propositions 2.10 and 2.11 guarantee that f preserves the operations $\lambda x.\omega^x, +, \cdot$ too. Finally, since by corollary 2.13 $M(\alpha, e)$ is $(+, \cdot, \lambda x.\omega^x)$-closed, then we do not have to worry about f preserving the $(+, \cdot, \lambda x.\omega^x)$-closure of it's domain $M(\alpha, e)$ (that is, f is an $(<, +, \cdot, \lambda x.\omega^x)$-isomorphism in the usual sense).
2.
Left to the reader. □

Corollary 2.15. *Let $\alpha, e \in \mathbb{E}$ and $B \subset M(\alpha, e)$. Then the function $h\colon B \longrightarrow h[B]$, $h(x) := x[\alpha := e]$ is an $(<, +, \cdot, \lambda x.\omega^x)$- isomorphism.*

Proof. By previous corollary 2.14, we already know that h preserves $<, +, \cdot$ and $\lambda x.\omega^x$. Moreover, the fact that h preserves $<$ implies that h is an injection, and therefore h is a bijection from it's domain to it's image. So it only remains to show that h preserves the $(+, \cdot, \lambda x.\omega^x)$-closure of B. This is not hard: Let $\beta, \gamma \in B$. Let's denote as $\beta \Box \gamma$ to any of $\beta + \gamma$, $\beta \cdot \gamma$ or ω^γ.

Suppose $\beta \Box \gamma \in B$. Then $\beta \Box \gamma = \delta$ for some $\delta \in B$. Then $h(\beta) \Box h(\gamma) = h(\beta \Box \gamma) = h(\delta) \in h[B]$.

Suppose $h(\beta) \Box h(\gamma) \in h[B]$. Then $\beta[\alpha := e] \Box \gamma[\alpha := e] = h(\beta) \Box h(\gamma) = h(\delta) = \delta[\alpha := e]$ for some $\delta \in B \subset M(\alpha, e)$. (*). On the other hand, since $\beta, \gamma \in M(\alpha, e)$, then $\beta \Box \gamma \in M(\alpha, e)$ and $\beta[\alpha := e] \Box \gamma[\alpha := e] = (\beta \Box \gamma)[\alpha := e]$. (**). From (*) and (**) follow $(\beta \Box \gamma)[\alpha := e] = \delta[\alpha := e]$, and since the function $x \longmapsto x[\alpha := e]$ is a bijection in $M(\alpha, e)$, then $\beta \Box \gamma = \delta \in B$. □

2.2.1.1 Substitutions and $<_1$ in intervals $(\varepsilon_\gamma, \varepsilon_{\gamma+1})$.

The next two results are the main reason why we are caring so much about our substitutions $x \longmapsto x[\alpha := e]$.

Proposition 2.16. *Let $\alpha, e \in \mathbb{E}$ and $A := (\alpha, \alpha^+) \cap M(\alpha, e)$. Then A is closed under the operations $+, \cdot, \lambda x.\omega^x$ and m.*

Proof. Left to the reader. □

Corollary 2.17. *Let $\alpha, e \in \mathbb{E}$ and $A := (\alpha, \alpha^+) \cap M(\alpha, e)$. Then*

1. *The function* $\begin{array}{c} h\colon A \longrightarrow h[A] \subset (e, e^+) \\ q \longmapsto q[\alpha := e] \end{array}$ *is an $(<, +, \cdot, \lambda x.\omega^x, m)$- isomorphism.*
2. *If $\alpha \leqslant e$ then $A = (\alpha, \alpha^+)$ and then the function* $\begin{array}{rl} h\colon (\alpha, \alpha^+) & \longrightarrow h[(\alpha, \alpha^+)] \subset (e, e^+) \\ q & \longmapsto q[\alpha := e] \end{array}$ *is an $(<, +, \cdot, \lambda x.\omega^x, m)$- isomorphism.*

Proof. Clearly 2. follows from 1.

We prove 1. Let $q \in A$ be arbitrary.

$\alpha < q \underset{\text{proposition 2.9}}{\Longrightarrow} e = \alpha[\alpha := e] < q[\alpha := e] \underset{\text{proposition 2.10}}{<} e^+$. This shows that $h[A] \subset (e, e^+)$.

On the other hand, proposition 2.9 guarantees that h preserves the relation $<$. Moreover, propositions 2.10 and 2.11 guarantee that h preserves the operations $\lambda x.\omega^x, +, \cdot$ too.

Now we prove that h preserves m too.

Since $q \in (\alpha, \alpha^+)$, then $q \notin \mathbb{E}$. Then we have the following cases:

Case $q =_{\mathrm{CNF}} B_1 b_1 + \ldots + B_n b_n$ with $n \geqslant 2 \vee b_1 \geqslant 2$. Then $m(q) = q$ by corollary 2.5. On the other hand $q[\alpha := e] =_{\mathrm{CNF}} B_1[\alpha := e] b_1 + \ldots + B_n[\alpha := e] b_n$ with $n \geqslant 2 \vee b_1 \geqslant 2$; so, by corollary 2.5, $m(q[\alpha := e]) = q[\alpha := e] = m(q)[\alpha := e]$.

Case $q =_{\mathrm{CNF}} \omega^B$ with $B =_{\mathrm{CNF}} \omega^{B_1} b_1 + \ldots + \omega^{B_{n-1}} b_{n-1} + \omega^Z b_n$. Then $\boldsymbol{m(q) = q + Z}$ by corollary 2.5; moreover, $\mathrm{Ep}(Z) \cap \alpha \subset e$ by the proof of previous proposition 2.16. (*).

On the other hand $q[\alpha := e] =_{\mathrm{CNF}} \omega^{B[\alpha := e]}$ with $B[\alpha := e] =_{\mathrm{CNF}} \omega^{B_1[\alpha := e]} b_1 + \ldots + \omega^{Z[\alpha := e]} b_n$; thus $m(q[\alpha := e]) \underset{\text{corollary 2.5}}{=} q[\alpha := e] + Z[\alpha := e] \underset{\text{by (*) and proposition 2.11}}{=} (q + Z)[\alpha := e] = m(q)[\alpha := e]$.

All the previous shows that h preserves m.

Finally, since by proposition 2.16 A is $(+, \cdot, \lambda x.\omega^x, m)$-closed, then we do not have to worry about h preserving the $(+, \cdot, \lambda x.\omega^x, m)$-closure of it's domain A (that is, h is an $(<, +, \cdot, \lambda x.\omega^x, m)$-isomorphism in the usual sense). □

Remark 2.18. The function $h: A \longrightarrow h[A]$ of previous corollary 2.17 is an $<_1$-isomorphism too: For any $\beta, \gamma \in A$, the ordinals $m(\beta), m(\gamma) \in A$ (because by proposition 2.16 A is m-closed) and we have that $\beta \leqslant_1 \gamma \underset{\leqslant_1\text{-connectedness}}{\Longleftrightarrow} \beta \leqslant \gamma \leqslant m(\beta) \underset{\text{corollary 2.17}}{\Longleftrightarrow}$ $h(\beta) \leqslant h(\gamma) \leqslant h(m(\beta)) = m(h(\beta)) \underset{\leqslant_1\text{-connectedness}}{\Longleftrightarrow} h(\beta) \leqslant_1 h(\gamma)$.

2.2.2 The relation $<^1$.

With the purpose of extending our understanding between the substitutions $x \longmapsto [\alpha := e]$ and the $<_1$-relation, we introduce the following

Definition 2.19. *For $\alpha, \beta \in \mathrm{OR}$, $\alpha <^1 \beta$ means $\alpha < \beta$ and $\forall Z \subset_{\mathrm{fin}} \beta \exists \tilde{Z} \subset_{\mathrm{fin}} \alpha \exists h$ such that*

(i) $h: (Z, <, <_1, +, \lambda x.\omega^x) \rightarrow (\tilde{Z}, <, <_1, +, \lambda x.\omega^x)$ is an isomorphism.

(ii) $h|_{Z \cap \alpha} = \mathrm{Id}|_{Z \cap \alpha}$, where $\mathrm{Id}|_{Z \cap \alpha}: Z \cap \alpha \longrightarrow Z \cap \alpha$ is the identity function.

By $\alpha \leqslant^1 \beta$ we mean that $\alpha <_1 \beta$ or $\alpha = \beta$. We abbreviate $h|_{Z \cap \alpha} = \mathrm{Id}|_{Z \cap \alpha}$ as $h|_\alpha = \mathrm{Id}|_\alpha$.

Proposition 2.20. *Let $\alpha, \beta, \gamma \in \mathrm{OR}$ and $(\xi_i)_{i \in I} \subset \mathrm{OR}$. Then*

1. *$\alpha \leqslant^1 \beta \Longrightarrow \alpha \leqslant_1 \beta$.*
2. *If $\alpha \leqslant \beta \leqslant \gamma \wedge \alpha \leqslant^1 \gamma$ then $\alpha \leqslant^1 \beta$.* *($\leqslant^1$-connectedness)*
3. *If $\forall i \in I. \alpha \leqslant^1 \xi_i \wedge \xi_i \underset{\mathrm{cof}}{\longrightarrow} \beta$ then $\alpha \leqslant^1 \beta$.* *($\leqslant^1$-continuity)*

Proof. 1. follows direct from the definition of $\leqslant^1$. The proofs of $\leqslant^1$-connectedness and $\leqslant^1$-continuity are as easy as the proofs of $\leqslant_1$-connectedness and $\leqslant_1$-continuity. □

Now we show that the $<^1$-relation is closely related with the substitutions $x \longmapsto x[\alpha := e]$. We first make the following

Definition 2.21. *Let $q \in \mathrm{OR}$ with $q =_{\mathrm{CNF}} L_1 q_1 + \ldots + L_n q_n$. Let* $S_{\mathrm{CNF}}(q) := \{L_1 q_1, \ldots, L_n q_n\} \cup \{\Sigma_{i=1}^{j} L_i q_i | j \in \{1, \ldots, n\}\} \cup \bigcup_{\{A_i | i \in [1,n] \wedge L_i \notin \mathbb{E} \wedge L_i =_{\mathrm{CNF}} \omega^{A_i}\}} S_{\mathrm{CNF}}(A_i)$.

Proposition 2.22. *Let $\alpha \in \mathbb{E}$ be an arbitrary epsilon number.*

1. *Let $t \in [\alpha, \alpha^+)$ and $B(t) := S_{\mathrm{CNF}}(t) \cup \{Lj | Lq \in S_{\mathrm{CNF}}(t) \wedge L \in \mathbb{P} \wedge q \in [1, \omega) \wedge j \in \{1, \ldots, q\}\}$. Note $t \in B(t) \subset_{\mathrm{fin}} t + 1$. Then any $h: B(t) \longrightarrow h[B(t)] \subset \alpha$ that is an $(<, <_1, +, \lambda x.\omega^x)$ isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$ satisfies $h(\alpha) \in \mathbb{E} \cap \alpha$ and $\forall s \in B(t).\mathrm{Ep}(s) \cap \alpha \subset h(\alpha) \wedge h(s) = s[\alpha := h(\alpha)]$.*
2. *Let $t \in (\alpha, \alpha^+)$ and suppose $\alpha <^1 t$. Let $B \subset_{\mathrm{fin}} t$. Then there exists $\gamma \in \mathbb{E} \cap \alpha$ such that $\forall s \in B.\mathrm{Ep}(s) \cap \alpha \subset \gamma$ and the function $h: B \longrightarrow h[B] \subset \alpha$, $s \longmapsto s[\alpha := \gamma]$ is an $(<, <_1, +, \lambda x.\omega^x)$ isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$.*

Proof. We prove 2. first.

Suppose $t \in (\alpha, \alpha^+)$, $\alpha <^1 t$ and $B \subset_{\mathrm{fin}} t$. Consider the set $C := \bigcup_{s \in (B \cup \{\alpha\}) \cap [\alpha, \alpha^+)} B(s) \subset_{\mathrm{fin}} t$, where $B(s)$ is the set defined in 1. Now, since $\alpha <^1 t$, then there exists an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism $H: C \longrightarrow H[C] \subset \alpha$ with $H|_\alpha = \mathrm{Id}_\alpha$. Note that $\alpha \in B(\alpha) \subset C$ and therefore, by 1., $H(\alpha) \in \mathbb{E} \cap \alpha$. Let $\gamma := H(\alpha)$.

Let's show that $\forall s \in B.\mathrm{Ep}(s) \cap \alpha \subset \gamma$. Let $s \in B$. If $s \in B \cap \alpha$, then $s = H(s) < H(\alpha) = \gamma$ because the relation $<$ is preserved by H and so $\mathrm{Ep}(s) \cap \alpha \subset H(\alpha) = \gamma$. If $s \in B \cap [\alpha, t)$, then $s \in B(s) \subset C$ and then, by 1., $\mathrm{Ep}(s) \cap \alpha \subset H(\alpha) = \gamma$.

Finally, to show that the function $h: B \longrightarrow h[B]$, $h(s) := s[\alpha := \gamma]$ is an $(<, <_1, +, \lambda x.\omega^x)$ isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$ it is enough to show that $h = H|_B$ (since $H|_B: B \longrightarrow H|_B[B] \subset \alpha$ is already an $(<, <_1, +, \lambda x.\omega^x)$ isomorphism with $(H|_B)|_\alpha = \mathrm{Id}_\alpha$ by proposition A.1 in the appendices). So let $s \in B$. If $s < \alpha$, then $s < \gamma = H(\alpha)$ and so $h(s) = s[\alpha := \gamma] = s = H|_B(s)$. If $s \geqslant \alpha$, then $s \in B(s)$ and then by 1. we have that $H|_B(s) = H(s) = s[\alpha := H(\alpha)] = s[\alpha := \gamma] = h(s)$.

We prove 1.

Let $t \in [\alpha, \alpha)$, $B(t)$ and h as in our hypothesis. Then $h(\alpha) = h(\omega^\alpha) = \omega^{h(\alpha)}$. So $h(\alpha) \in \mathbb{E}$. Moreover, from the definition of $B(t)$ and using that h preserves the $<$ relation, it follows that $\forall s \in B(t).\mathrm{Ep}(s) \cap \alpha \subset B(t)$ and $\forall l \in \mathrm{Ep}(s) \cap \alpha. l = h(l) < h(\alpha)$; that is, $\forall s \in B(t).\mathrm{Ep}(s) \cap \alpha \subset h(\alpha)$.

We now show $\forall s \in B(t).h(s) = s[\alpha := h(\alpha)]$ by induction on the set $B(t)$ (with the usual order $<$ on the ordinals):

Let $s \in B$ with $s =_{\mathrm{CNF}} \omega^{A_1} a_1 + \ldots + \omega^{A_u} a_u$.

Suppose $\forall y \in s \cap B(t).h(y) = y[\alpha := h(\alpha)]$. **(IH)**.

If $u \geqslant 2$, then by IH $h(\omega^{A_1} a_1) = \omega^{A_1} a_1[\alpha := h(\alpha)]$, ...,$h(\omega^{A_u} a_u) = \omega^{A_1} a_u[\alpha := h(\alpha)]$ and therefore $h(s) = h(\omega^{A_1} a_1) + \ldots + h(\omega^{A_u} a_u) = \omega^{A_1} a_1[\alpha := h(\alpha)] + \ldots + \omega^{A_u} a_u[\alpha := h(\alpha)] =$ $(\omega^{A_1} a_1 + \ldots + \omega^{A_u} a_u)[\alpha := h(\alpha)] = s[\alpha := h(\alpha)]$.

If $u = 1$ and $a_1 \geqslant 2$, then by IH $h(\omega^{A_1}(a_1 - 1)) = \omega^{A_1}(a_1 - 1)[\alpha := h(\alpha)]$ and $h(\omega^{A_1}) = \omega^{A_1}[\alpha := h(\alpha)]$. Then $h(s) = h(\omega^{A_1}(a_1 - 1)) + h(\omega^{A_1}) =$
$= \omega^{A_1}(a_1 - 1)[\alpha := h(\alpha)] + \omega^{A_1}[\alpha := h(\alpha)] = \omega^{A_1} a_1[\alpha := h(\alpha)] = s[\alpha := h(\alpha)]$.

If $u = 1$ and $a_1 = 1$ (that is, $s =_{\mathrm{CNF}} \omega^{A_1}$) we have two subcases:

- $A_1 < s$. Then by IH $h(A_1) = A_1[\alpha := h(\alpha)]$ and so $h(s) = \omega^{h(A_1)} = \omega^{A_1[\alpha := h(\alpha)]} =$ $= \omega^{A_1}[\alpha := h(\alpha)] = s[\alpha := h(\alpha)]$.
- $A_1 = s$. Then $s \in \mathbb{E}$. If $s < \alpha$, then $h(s) = s = s[\alpha := h(\alpha)]$ because $h|_\alpha = \mathrm{Id}_\alpha$. If $s \geqslant \alpha$, then $s = \alpha$ (because $\alpha \leqslant s < t < \alpha^+$). So $h(s) = h(\alpha) = \alpha[\alpha := h(\alpha)] = s[\alpha := h(\alpha)]$. □

2.2.2.1 Cofinality properties of $<^1$.

What are the ordinals α such that $\alpha <^1 \alpha + 1$?. Well, one can prove the following:

Exercise 2.1. $\forall\alpha\in\mathrm{OR}.\alpha<^1\alpha+1\Longleftrightarrow\alpha\in\mathrm{Lim}\,\mathbb{E}$.

Exercise 2.2. $\forall\alpha\in\mathrm{OR}.\alpha\in\mathrm{Lim}\,\mathbb{E}\Longrightarrow\forall\xi\in(0,\alpha).\alpha<^1\alpha+\xi$.

The consideration of the previous exercises revels two very important properties of the relation $<^1$ which we prove now:

Proposition 2.23. *(First fundamental cofinality property of $<^1$). Let $\alpha\in\mathbb{E}$ and suppose $\alpha<^1 s$ for some $s\in(\alpha,\alpha^+)$. Then for any $t\in[\alpha,s)$ there exists a sequence $(c_\xi)_{\xi\in X}\subset\alpha\cap\mathbb{E}$ such that $\mathrm{Ep}(t)\cap\alpha\subset c_\xi$, $c_\xi\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and $c_\xi<_1 t[\alpha:=c_\xi]$.*

Proof. Let $\alpha\in\mathbb{E}$ and suppose $\alpha<^1 s$ for some $s\in(\alpha,\alpha^+)$. Let $t\in[\alpha,s)$. We define $M:=\max(\mathrm{Ep}(t)\cap\alpha)$. Let $\delta\in[M+1,\alpha)$ be arbitrary.

Consider the set $B_\delta:=S_{\mathrm{CNF}}(t)\cup\{Lj|Lq\in S_{\mathrm{CNF}}(t)\wedge L\in\mathbb{P}\wedge q\in[1,\omega)\wedge j\in\{1,\ldots,q\}\}\cup\{\delta\}\subset_{\mathrm{fin}}t+1\leqslant s$. By hypothesis, there exists an $(<,<_1,+,\lambda x.\omega^x)$ isomorphism $h_\delta\colon B_\delta\longrightarrow h[B_\delta]\subset\alpha$ with $h_\delta|_\alpha=\mathrm{Id}_\alpha$. Moreover, by proposition 2.22, $h_\delta(\alpha)\in\mathbb{E}$, $\mathrm{Ep}(t)\cap\alpha\subset h_\delta(\alpha)$ and $\delta=h_\delta(\delta)<h_\delta(\alpha)<_1 h_\delta(t)=t[\alpha:=h_\delta(\alpha)]$. Therefore, the set $P(t):=\{h_\delta(\alpha)|\delta\in[M+1,\alpha)\}\subset\alpha$ is confinal in α and it satisfies $\forall c\in P(t).\mathrm{Ep}(t)\cap\alpha\subset c\wedge c<_1 t[\alpha:=c]$. From this follows the claim of this proposition. $\square$

The next result is a more general version of lemma 3.11 appearing in [18]. It's proof uses the main argument used in Wilken's proof. There is, however, one point that we want to stress (something that may be overlooked by the reader): For a class of ordinals $\emptyset\neq X\subset\mathrm{OR}$, we have defined $\mathrm{Lim}(X):=\{\alpha\in\mathrm{OR}|\sup(X\cap\alpha)=\alpha\}$; that is, in general, $\mathrm{Lim}(X)\not\subset X$. This notion is very important in the whole of our work (and particularly, in the next proposition).

Proposition 2.24. *(Second fundamental cofinality property of $<^1$)*
Let $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)$. Assume $\alpha\in\mathrm{Lim}\{\gamma\in\mathbb{E}|\mathrm{Ep}(t)\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 t[\alpha:=\gamma]\}$.
Then $\forall s\in[\alpha,t+1].\alpha\leqslant^1 s$.

Proof. Let $\alpha\in\mathbb{E}$, $t\in[\alpha,\alpha^+)$ and assume $\alpha\in\mathrm{Lim}\{\gamma\in\mathbb{E}|\mathrm{Ep}(t)\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 t[\alpha:=\gamma]\}$.

We prove by induction: $\forall s\in[\alpha,t+1].\alpha\leqslant^1 s$.

For $s=\alpha$ it is clear the claim holds. So, from now on, suppose $s>\alpha$.

Case $s\in\mathrm{Lim}\cap[\alpha,t+1]$. Our induction hypothesis is $\alpha\leqslant^1\beta$ for all $\beta\in[\alpha,t+1]\cap s$. Thus $\alpha\leqslant^1 s$ by $\leqslant^1$-continuity.

Suppose $s=l+1\in[\alpha,t+1]$. Our induction hypothesis is $\alpha\leqslant^1 l$. **(IH)**

Let $B\subset_{\mathrm{fin}}s=l+1$. Without loss of generality, suppose $\alpha,l\in B$ and write $B=X\cup Y$ where $X:=B\cap\alpha$, $Y:=B\cap[\alpha,l]$, $Y:=\{y_1,\ldots,y_m|\alpha=y_1<y_2<\ldots<y_m=l\}$.

Note $l\in[\alpha,t]\subset[\alpha,\alpha^+)\ni t$ implies that $\forall e\in\mathrm{Ep}(l)\cup\mathrm{Ep}(t).e\leqslant\alpha$; moreover, since $\mathrm{Ep}(l)\cup\mathrm{Ep}(t)$ is finite and $\alpha\in\mathrm{Lim}\{\gamma\in\mathbb{E}|\mathrm{Ep}(t)\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 t[\alpha:=\gamma]\}$, then actually $\alpha\in\mathrm{Lim}\{\gamma\in\mathbb{E}|(\mathrm{Ep}(l)\cup\mathrm{Ep}(t))\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 t[\alpha:=\gamma]\}$. (*).
But for any $\gamma\in\mathbb{E}$ such that $(\mathrm{Ep}(l)\cup\mathrm{Ep}(t))\cap\alpha\subset\gamma$ we have $\gamma\leqslant l[\alpha:=\gamma]\leqslant t[\alpha:=\gamma]$; therefore, by $\leqslant_1$-connectedness and (*) we conclude $\alpha\in\mathrm{Lim}\{\gamma\in\mathbb{E}|\mathrm{Ep}(l)\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 l[\alpha:=\gamma]\}$.

Let $p:=\max\bigcup_{i\in\{1,\ldots,m\}}(\mathrm{Ep}(y_i)\cap\alpha)$ and consider the set $M:=\{\gamma\in\alpha\cap\mathbb{E}|p<\gamma\supset X\wedge\gamma\leqslant_1 l[\alpha:=\gamma]\}$. Let $C:=\{m(a)|a\in(B\cap\alpha)\wedge m(a)<\alpha\}$. Since $C\subset_{\mathrm{fin}}\alpha$ and by our previous observations M is confinal in α, then $(\max C,\alpha)\cap M\neq\emptyset$.
Let $\gamma:=\min(M\cap(\max C,\alpha))\in M$. We define the function $h\colon B\longrightarrow h[B]\subset\alpha$ as $h(x):=x[\alpha:=\gamma]$ for all $x\in B$.

Let's see that h is an $(<,<_1,+,\lambda x.\omega^x)$-isomorphism.

That h preserves is an $(<, +, \lambda x.\omega^x)$-isomorphism follows directly from the fact that $X \cup \bigcup_{i \in \{1, \ldots, m\}} (\mathrm{Ep}(y_i) \cap \alpha) \subset \gamma$ and corollary 2.15.

Let's see that h also preserves $<_1$.

- First observe that by IH $\alpha \leqslant^1 l$ and so $\alpha \leqslant_1 l$; subsequently, by $\leqslant_1$-connectedness it follows $\alpha \leqslant_1 y_i$ for any $y_i \in Y$. So we need to show $h(\alpha) \leqslant_1 h(y_i)$ for any $y_i \in Y$. But this is easy because $h(\alpha) = \gamma \leqslant_1 l[\alpha := \gamma]$ by the way we took γ, and since $\forall y_i \in Y. h(\alpha) \leqslant h(y_i) \leqslant h(l) = l[\alpha := \gamma]$, then $y_i \in Y. h(\alpha) \leqslant_1 h(y_i)$ by $\leqslant_1$-connectedness.
- Clearly $x_1 \leqslant_1 x_2 \Longleftrightarrow h(x_1) = x_1 \leqslant_1 x_2 = h(x_2)$ for any $x_1, x_2 \in X$.
- For $x \in X$ and $y_i \in Y$, $x \leqslant_1 y_i \Longrightarrow h(x) = x \leqslant_1 h(y_i)$ by $\leqslant_1$-connectedness (because $x = h(x) \leqslant h(y_i) \leqslant y_i$ for any $i \in \{1, \ldots, m\}$).
- For $x \in X$, and $y_i \in Y$, $x \nleqslant_1 y_i \Longrightarrow x \nleqslant_1 \alpha$ (otherwise, using the fact that we know $\alpha \leqslant_1 y_i$ for all $i \in \{1, \ldots, m\}$, we would have $x \leqslant_1 y_i$ by $\leqslant_1$-transitivity). So $m(x) \in C$ and then $x < m(x) < \gamma \leqslant h(y_i)$; therefore $h(x) = x \nleqslant_1 h(y_i)$.
- For $y_i, y_j \in Y \cap (\alpha, \alpha^+)$, $y_i \leqslant_1 y_j \Longleftrightarrow y_i \leqslant y_j \leqslant m(y_i) \underset{\text{corollary 2.17}}{\Longleftrightarrow}$ $h(y_i) = y_i[\alpha := \gamma] \leqslant y_j[\alpha := \gamma] = h(y_j) \leqslant m(y_i)[\alpha := \gamma] = m(y_i[\alpha := \gamma]) = m(h(y_i)) \Longleftrightarrow$ $h(y_i) \leqslant_1 h(y_j)$.

All the previous cases show that h preserves $<_1$ too and from all our work we have that h is indeed an $(<, =, <_1, +, \lambda x.\omega^x)$-isomorphism. This shows $\alpha \leqslant^1 l + 1$. □

2.3 Covering theorem

From the definition of $<^1$ it is very easy to see that $\alpha <^1 \beta \Longrightarrow \alpha <_1 \beta$. But, what about the implication $\alpha <^1 \beta \Longleftarrow \alpha <_1 \beta$?. From exercise 2.1 (or by use of the first fundamental cofinality property of $<^1$) it follows that this implication does not hold in general. The motivation for the whole of this section is the study of such implication: The main result is lemma 2.30 (covering lemma), which has two important corollaries: The proof of the minimality of the substitutions as witnesses of $\alpha <_1 \beta$ for β which are closed under the cover construction and the solution to the question when $\alpha <^1 \beta \Longleftarrow \alpha <_1 \beta$.

We introduce the following definitions as a preparation for the covering lemma.

Definition 2.25. *The following functions will be used in the main lemma of this section. For an ordinal $t =_{\mathrm{CNF}} \omega^{T_1} t_1 + \ldots + \omega^{T_n} t_n$ we define the ordinals*

$$dq := \begin{cases} 0 & \text{iff } q \notin \mathbb{P} \\ Q_m & \text{iff } q = \omega^Q \text{ with } Q =_{\mathrm{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m \end{cases},$$

$\pi t := \omega^{T_1}$, *and*

$\eta t := \max\{t, \pi t + d\pi t\}$.

Proposition 2.26. *Let $\alpha, t, s \in \mathrm{OR}$. Then*

1. $\pi(t+1) = \pi t$*; moreover, if* $t =_{\mathrm{CNF}} \omega^{T_1} t_1 + \ldots + \omega^{T_n} t_n$, *then* $d\pi t \leqslant T_1 \leqslant \omega^{T_1} = \pi t$.
2. *Suppose* $t \leqslant s$. *Then* $\pi t \leqslant \pi s$, $\pi t + d\pi t \leqslant \pi s + d\pi s$ *and therefore* $\eta t \leqslant \eta s$.
3. *If* $t \geqslant \alpha \in \mathbb{E}$ *then* $\alpha 2 \leqslant \eta t$
4. $\pi(\pi t) = \pi t$, $\pi(\pi t + d\pi t) = \pi t$ *and so* $\eta(\eta t) = \eta t$

Proof. Left to the reader. □

Proposition 2.27. *(Properties of ηt and $<_1$). Let $\alpha \in \mathbb{E}$ and $t \in (\alpha, \alpha^+)$. Then*

1. $t \notin \mathbb{P} \Longrightarrow m(t) = t$; *moreover,* $t \in \mathbb{P} \Longrightarrow m(t) = \pi t + d\pi t = \max\{t, \pi t + d\pi t\} = \eta t$. *Particularly,* $m(t) \leqslant \eta t$.
2. $\forall u \in (\alpha, t].m(u) \leqslant \eta t$. *Therefore,* $\forall s \in [\alpha, \alpha^+).\eta s = \begin{cases} \max\{m(u) | u \in (\alpha, s]\} & \text{iff} \quad s > \alpha 2 \\ \alpha 2 & \text{iff} \quad s \leqslant \alpha 2 \end{cases}$.
3. $\alpha <_1 t \Longleftrightarrow \alpha <_1 \eta t$
4. *If* $l \in [\alpha, t]$, *then* $\eta l \leqslant \eta t$
5. *It indeed happens that* $m(t) < \eta t$.

Proof. Left to the reader. □

Definition 2.28. *For any $L \in \mathbb{P}$, let*

$$F(L) := \begin{cases} \{\omega^{\omega^{V_1} v_1 + \omega^{V_2} v_2 + \ldots + \omega^{V_g} \cdot j} | g \in [1, t], j \in [1, v_g]\} \cup & \text{if} \quad L = \omega^Z \notin \mathbb{E} \wedge \\ \{\omega^{\omega^{V_1} v_1 + \omega^{V_2} v_2 + \ldots + \omega^{V_g} \cdot j} + V_g | g \in [1, t], j \in [1, v_g]\} & \quad\quad Z =_{\mathrm{CNF}} \sum_{j=1}^{t} \omega^{V_j} v_j \\ \\ \{L\} & \text{if} \quad L \in \mathbb{E} \cup \{1\} \end{cases}$$

Now, for any $\delta \in \mathrm{OR}$ with $\delta =_{\mathrm{CNF}} L_1 l_1 + \ldots + L_n l_n$, let

$C_1(\delta) := \bigcup_{L_i \notin \mathbb{E}} F(L_i)$ *and*

$C_2(\delta) := \{L_i j | i \in [1, n], j \in [1, l_i]\} \cup \{\Sigma_{i=1}^{j} L_i l_i | j \in [1, n]\}$.

Finally, for any $\delta \in \mathrm{OR}$ with $\delta =_{\mathrm{CNF}} L_1 l_1 + \ldots + L_n l_n$, we define (by recursion on $(\mathrm{OR}, <)$) the set $C(\delta)$ as

$C(\delta) := C_1(\delta) \cup \bigcup_{\sigma \in C_1(\delta)} C_2(\sigma) \cup C_2(\delta) \cup \bigcup_{V \in Y(\delta)} C(V)$, *where*
$Y(\delta) := \{V_{ij} | \exists L_i \notin \mathbb{E}. L_i = \omega^Z \wedge Z =_{\mathrm{CNF}} \sum_{j=1}^{t(i)} \omega^{V_{ij}} v_{ij}\}$ *(observe $Y \subset \delta$).*

Proposition 2.29. *Let $\delta \in \mathrm{OR}$. Then $\forall \rho \in C(\delta).C(\rho) \subset C(\delta)$.*

Proof. By induction on the ordinals one shows $\forall \delta \in \mathrm{OR}.\forall \rho \in C(\delta).C(\rho) \subset C(\delta)$. It is necessary to check the ways how ρ may be in $C(\delta)$. The details are left to the reader. □

We prove now the covering lemma.

Lemma 2.30. *(Cover for one ordinal). Let $\alpha \in \mathbb{E}$ and $\delta \in \alpha^+$ with $\delta =_{\mathrm{CNF}} L_1 l_1 + \ldots + L_n l_n$. Let $D(\alpha, \delta) := C(\delta) \cup \{\alpha, \alpha 2\}$. Then*

i. *$C(\delta)$ is a finite set.*
ii. • $\{\delta, L_1 l_1, \ldots, L_n l_n\} \subset C(\delta) \subset \max\{\delta + 1, L_1 + d(L_1) + 1\} = \max\{\delta, L_1 + d(L_1)\} + 1 = \eta\delta + 1$
  • *If $\delta \geqslant \alpha$ then* $\eta\delta \in D(\alpha, \delta) \subset \max\{\delta + 1, L_1 + d(L_1) + 1\} = \max\{\delta, L_1 + d(L_1)\} + 1 = \eta\delta + 1$
iii. *Suppose $\delta \in [\alpha, \alpha^+)$ and $h: D(\alpha, \delta) \longrightarrow h[D(\alpha, \delta)]$ is an $(<, <_1, +)$-isomorphism such that $h|_\alpha = \mathrm{Id}_\alpha$. Then $h(\alpha) \in \mathbb{E}$ and $\forall x \in D(\alpha, \delta).(\mathrm{Ep}(x) \cap \alpha) \subset h(\alpha) \wedge x[\alpha := h(\alpha)] \leqslant h(x)$.*

Proof.

$i.$
By induction on δ. Suppose $\forall r<\delta$. $C(r)$ is finite. **(IH1)**
$|C(\delta)| \leqslant |C_2(\delta)|+|C_1(\delta)|+|\bigcup_{R\in Y(\delta)} C(R)|+|\bigcup_{\sigma\in C_1(\delta)} C_2(\sigma)| \leqslant$
$|\{L_i j | i\in\{1,...,n\}, j\in\{1,...,l_i\}\}|+|\{\Sigma_i^j L_i l_i | j\in\{1,...,n\}\}|+$
$|\bigcup_{L_i\notin\mathbb{E}} F(L_i)|+|\bigcup_{R\in Y(\delta)} C(R)|+|\bigcup_{\sigma\in C_1(\delta)} C_2(\sigma)| \leqslant$
$l_1+l_2+...+l_n+\sum_{j=1}^k j+\sum_{L_i\notin\mathbb{E}} |F(L_i)|+\sum_{R\in Y(\delta)} |C(R)|+|\bigcup_{\sigma\in C_1(\delta)} C_2(\sigma)|<\omega$,
where the last inequality holds because:

(1). For any $V\in Y(\delta)$, $V<\delta$, and so $C(V)$ is finite by our (IH1); moreover, the set $Y(\delta)=\{V_{ij} | \exists L_i\notin\mathbb{E}. L_i=\omega^Z \wedge Z=_{\mathrm{CNF}} \sum_{j=1}^{t(i)} \omega^{V_{ij}} v_{ij}\}$ is finite too, since there are only a finite number of L_i's, and for each one of the $L_i\notin\mathbb{E}$ with $L_i=\omega^Z$ and $Z=_{\mathrm{CNF}} \sum_{j=1}^{t(i)} \omega^{V_{ij}} v_{ij}$, there are only a finite number of V_{ij}. Thus $\sum_{R\in Y(\delta)} |C(R)|<\omega$.

(2). For any $L_i\notin\mathbb{E}$, it is easy to see that $F(L_i)$ is finite too; moreover, as we already said, there are only a finite number of L_i's. So $|C_1(\delta)| \leqslant \sum_{L_i\notin\mathbb{E}} |F(L_i)|<\omega$.

(3). $C_2(\sigma)$ is finite for any $\sigma\in C_1(\delta)$ (exactly by the same reason why $C_2(\delta)$ is finite) and $C_1(\delta)$ is finite too (as argued in previous subcase (2)); therefore $|\bigcup_{\sigma\in C_1(\delta)} C_2(\sigma)|<\omega$.

$ii.$
- We show that $\{\delta, L_1 l_1, ..., L_n l_n\} \subset C(\delta) \subset \max\{\delta+1, L_1+d(L_1)+1\}$.

Clearly $\{\delta, L_1 l_1, ..., L_n l_n\} \subset C(\delta)$.

Let's prove by induction $\forall\delta. C(\delta) \subset \max\{\delta+1, L_1+d(L_1)+1\}$.

Suppose $\forall r<\delta. C(r) \subset \max\{r+1, \pi(r)+d(\pi(r))+1\}$. **(IH2)**

Clearly $\{L_i j | i\in\{1,...,n\}, j\in\{1,...,l_i\}\} \cup \{\Sigma_i^j L_i l_i | j\in\{1,...,n\}\} \subset \delta+1$. **(ii1*)**

Now, take $V\in Y(\delta)$. By definition it means there exist $i,j\in\omega$, where $L_i\notin\mathbb{E}$ is an additive principal number in the Cantor normal form of δ, $L_i=\omega^Z$, $Z=_{\mathrm{CNF}} \sum_{j=1}^{t(i)} \omega^{V_{ij}} v_{ij}$ and $V=V_{ij}$. Observe that $d(\pi(V)) \leqslant \pi(V) \leqslant V<L_i$ ($V<L_i\leqslant\delta$ holds because equality would imply $L_i\in\mathbb{E}$ and we know that is not the case), and since $L_i\in\mathbb{P}$, then $\pi(V)+d(\pi(V))<L_i$. So both $\pi(V)+d(\pi(V))+1, V+1\leqslant L_i\leqslant L_1\leqslant\delta<\delta+1$. Since the previous holds for any $V\in Y$, then $\bigcup_{R\in Y(\delta)} C(V) \underset{\text{by our (IH2)}}{\subset} \bigcup_{R\in Y(\delta)} \max\{V+1, \pi(V)+d(\pi(V))+1\} \subset \delta+1$. **(ii2*)**

We now check what happens with $C_1(\delta)=\bigcup_{L_i\notin\mathbb{E}} F(L_i)$. By definition, for any $L_i\notin\mathbb{E}$ with $L_i=\omega^Z$ and $Z=_{\mathrm{CNF}} \sum_{j=1}^{t(i)} \omega^{V_{ij}} v_{ij}$
$F(L_i)=\{\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j} | g\in\{1,...,t(i)\}, j\in\{1,...,v_{ig}\}\} \cup$
$\{\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j}+V_{ig} | g\in\{1,...,t(i)\}, j\in\{1,...,v_{ig}\}\}$.

Clearly $\{\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j} | g\in\{1,...,t(i)\}, j\in\{1,...,v_{ig}\}\} \subset$
$L_i+1\leqslant L_1+1\leqslant\delta+1$. **(ii3*)**

On the other hand for any $g\in\{1,...,t(i)-1\}$, $j\in\{1,...,v_{ig}\}$,
$\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j}+V_{ig} \leqslant \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j}+\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j} = \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j} 2 < \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j}\omega = \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j+1} \leqslant \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}} v_{ig}+1} \leqslant \omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{i(t(i)-1)}} v_{i(t(i)-1)}+1} \leqslant$
$\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{i(t(i)-1)}} v_{i(t(i)-1)}+\omega^{V_{it(i)}} v_{it(i)}}=L_i\leqslant L_1\leqslant\delta<\delta+1$. **(ii4*)**

For the case $g=t(i)$, $j\in\{1,...,v_{ig}\}$, $\omega^{\omega^{V_{i1}} v_{i1}+\omega^{V_{i2}} v_{i2}+...+\omega^{V_{ig}}\cdot j}+V_{ig}\leqslant L_i+V_{ig}=L_i+d(L_i)\leqslant$
$\begin{cases} L_i 2<L_{i-1}<L_1+d(L_1)+1 & \text{if } i\geqslant 2 \\ L_1+d(L_1)<L_1+d(L_1)+1 & \text{if } i=1 \end{cases}$. **(ii5*)**

So, by (ii3*), (ii4*) and (ii5*), we conclude $C_1(\delta) \subset \max\{\delta+1, L_1+d(L_1)+1\}$. **(ii6*)**

We now show that $\bigcup_{\sigma \in C_1(\delta)} C_2(\sigma) \subset \max\{\delta+1, L_1+d(L_1)+1\}$ too. By the same argument used in (ii1*), $\forall \beta \in \bigcup_{\sigma \in C_1(\delta)} C_2(\sigma).\beta \leqslant \max\{\sigma \mid \sigma \in C_1(\delta)\} \underset{\text{by (ii6*)}}{\leqslant} \max\{\delta, L_1+d(L_1)\}$.
Hence $\bigcup_{\sigma \in C_1(\delta)} C_2(\sigma) \subset \max\{\delta+1, L_1+d(L_1)+1\}$. **(ii7*)**

From (ii1*), (ii2*), (ii6*) and (ii7*) we conclude $C(\delta) \subset \max\{\delta+1, L_1+d(L_1)+1\}$.

• Suppose $\delta \geqslant \alpha$.
Then $D(\alpha,\delta) \subset \max\{\delta+1, L_1+d(L_1)+1\} = \max\{\delta, L_1+d(L_1)\}+1$ holds because $\alpha 2 \leqslant \max\{\delta, L_1+d(L_1)\}$ by proposition 2.26.

Let's prove that $\eta\delta \in D(\alpha,\delta)$. If $\delta = \eta\delta$, then (we just proved that) $\eta\delta = \delta \in C(\delta) \subset D(\alpha,\delta)$. So suppose $\delta \neq \eta\delta = \max\{\delta, \pi\delta + d\pi\delta\}$. If $\delta \in [\alpha, \alpha 2)$, then $\eta\delta = \alpha 2 \in D(\alpha,\delta)$. Suppose $\delta > \alpha 2$. Consider $\delta =_{\mathrm{CNF}} L_1 l_1 + \ldots + L_n l_n$. Note $l_1 = 1$, (otherwise $\pi d + d\pi d \leqslant L_1 + L_1 = L_1 2 \leqslant L_1 l_1 \leqslant \delta$ and then we would have that $\delta = \eta\delta$); moreover, $L_1 \notin \mathbb{E}$ (otherwise $L_1 = \alpha$ and then $\delta < \alpha 2$). This way, $L_1 \in \mathbb{P} \backslash \mathbb{E}$ and $L_1 =_{\mathrm{CNF}} \omega^Z$ for some $Z \in \mathrm{OR}$, where $Z =_{\mathrm{CNF}} \omega^{R_1} r_1 + \ldots + \omega^{R_u} r_u$ for some ordinals $R_i \in \mathrm{OR}$ and $r_i \in [1, \omega)$. Therefore, $\eta\delta = \pi\delta + d\pi\delta = L_1 + R_u \in F(L_1) \subset C(\delta) \subset D(\alpha,\delta)$.

iii.
Suppose $\delta \in [\alpha, \alpha^+)$ and $h: D(\alpha,\delta) \longrightarrow h[D(\alpha,\delta)]$ is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$. Notice from $\alpha <_1 \alpha 2$ follows $h(\alpha) <_1 h(\alpha) 2$, which is equivalent to $h(\alpha) \in \mathbb{E}$.

We now prove the claim $\forall x \in D(\alpha,\delta).\mathrm{Ep}(x) \cap \alpha \subset h(\alpha) \wedge x[\alpha := h(\alpha)] \leqslant h(x)$ by induction on the well order $(D(\alpha,\delta), <)$.

Let $x \in D(\alpha,\delta)$. Our induction hypothesis is
$\forall y \in x \cap D(\alpha,\delta).\mathrm{Ep}(y) \cap \alpha \subset h(\alpha) \wedge y[\alpha := h(\alpha)] \leqslant h(y)$. **(IH)**

If $x =_{\mathrm{CNF}} T_1 t_1 + \ldots + T_m t_m$, with $m \geqslant 2$, then $x \in C(\delta)$ and then by our (IH) and prop. 2.29 we have that $\mathrm{Ep}(T_i) \cap \alpha \subset h(\alpha)$ and $h(T_i) \geqslant T_i[\alpha := h(\alpha)]$ for all $i \in \{1, \ldots, m\}$; therefore
$\mathrm{Ep}(x) \cap \alpha \subset h(\alpha)$ and $h(x) = h(T_1) t_1 + \ldots + h(T_m) t_m \geqslant$
$T_1[\alpha := h(\alpha)] t_1 + \ldots + T_m[\alpha := h(\alpha)] t_m = x[\alpha := h(\alpha)]$.

If $x =_{\mathrm{CNF}} T_1 t_1$ with $t_1 \geqslant 2$ then, $x = \alpha 2$ or $x \in C(\delta)$. In any case, proceeding similarly as in the previous case, $\mathrm{Ep}(x) \cap \alpha \subset h(\alpha)$ and $h(x) \geqslant x[\alpha := h(\alpha)]$.

So suppose $x =_{\mathrm{CNF}} T_1$.

If $T_1 \in \mathbb{E}$ then $T_1 = \alpha$ or $T_1 \in \alpha \cap \mathbb{E}$ (because $x \leqslant \max\{\delta, L_1 + d(L_1)\} + 1 < \alpha^+$). If $T_1 = \alpha$, then $\mathrm{Ep}(x) \cap \alpha = \emptyset \subset h(\alpha) \in \mathbb{E}$ and $h(x) = h(\alpha) = x[\alpha := h(\alpha)]$. If $T_1 \in \alpha \cap \mathbb{E}$, then $x = h(x) < h(\alpha)$, and so $\mathrm{Ep}(x) \cap \alpha \subset h(\alpha)$; moreover $h(x) = x = x[\alpha := h(\alpha)]$.

So suppose $T_1 \notin \mathbb{E}$. Then $T_1 = \omega^Z$, with $Z =_{\mathrm{CNF}} \omega^{R_1} r_1 + \ldots + \omega^{R_k} r_k$. Notice that since
$\forall i \in \{1, \ldots, k\}.R_i \in D(\alpha,\delta) \wedge R_i \leqslant Z < T_1 = x$, then by (IH) $\bigcup_{1 \leqslant i \leqslant k} \mathrm{Ep}(R_i) \cap \alpha \subset h(\alpha)$ and therefore $\mathrm{Ep}(Z) \cap \alpha \subset h(\alpha)$. Thus $\mathrm{Ep}(x) \cap \alpha \subset h(\alpha)$. So it only rest to show that the inequality holds. For the case $T_1 < \alpha$, we have $h(x) = x = x[\alpha := h(\alpha)]$. So the interesting case is $\alpha < T_1 = \omega^Z \notin \mathbb{E}$.

We have the sets of inequalities (I0) and (I1):

$\omega^Z > R_1 > R_2 > \ldots > R_k$; (I0)
$\omega^{R_1} r_1 > \omega^{R_1}(r_1 - 1) \ldots > \omega^{R_1} 3 > \omega^{R_1} 2 > \omega^{R_1} \geqslant R_1$

$\omega^{R_1} r_1 + \omega^{R_2} r_2 > \omega^{R_2} r_2 > \omega^{R_2}(r_2 - 1) \ldots > \omega^{R_2} 3 > \omega^{R_2} 2 > \omega^{R_2} \geqslant R_2$ (I1)
...

$$\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k>\omega^{R_k}r_k>\omega^{R_r}(r_k-1)\ldots>\omega^{R_k}3>\omega^{R_k}2>\omega^{R_k}\geqslant R_k$$

On the other hand, from the inequalities
$R_1\leqslant\omega^{\omega^{R_1}}<\omega^{\omega^{R_1}2}<\ldots<\omega^{\omega^{R_1}r_1}$
$R_2<\omega^{\omega^{R_1}r_1+\omega^{R_2}}<\ldots<\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2}$
$R_{k-1}<\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-2}}r_{k-2}+\omega^{R_{k-1}}}<\ldots<\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-2}}r_{k-2}+\omega^{R_{k-1}}r_{k-1}}$
$R_k<\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}+\omega^{R_k}}$
and theorem 2.3 we get the inequalities (I2):

$$\begin{cases} R_1=\alpha<\omega^{\omega^{R_1}2} \quad (\text{because } \alpha<T_1=\omega^{\omega^{R_1}r_1+\ldots+\omega^{R_k}r_k}) & \text{if } R_1=\omega^{\omega^{R_1}} \\ \omega^{\omega^{R_1}}<_1\omega^{\omega^{R_1}}+R_1\leqslant\omega^{\omega^{R_1}}+\omega^{\omega^{R_1}}<\omega^{\omega^{R_1}}\omega=\omega^{\omega^{R_1}+1}\leqslant\omega^{\omega^{R_1}2} & \text{if } R_1<\omega^{\omega^{R_1}} \end{cases}$$

$\omega^{\omega^{R_1}2}<_1\omega^{\omega^{R_1}2}+R_1\leqslant\omega^{\omega^{R_1}2}+\omega^{\omega^{R_1}2}<\omega^{\omega^{R_1}2}\omega=\omega^{\omega^{R_1}2+1}\leqslant$
$\omega^{\omega^{R_1}3}<_1\omega^{\omega^{R_1}3}+R_1\leqslant\omega^{\omega^{R_1}3}+\omega^{\omega^{R_1}3}<\omega^{\omega^{R_1}3}\omega=\omega^{\omega^{R_1}3+1}\leqslant$
$\omega^{\omega^{R_1}(r_1-1)}<_1\omega^{\omega^{R_1}(r_1-1)}+R_1\leqslant\omega^{\omega^{R_1}(r_1-1)}+\omega^{\omega^{R_1}(r_1-1)}<\omega^{\omega^{R_1}(r_1-1)}\omega=\omega^{\omega^{R_1}(r_1-1)+1}\leqslant$
$\omega^{\omega^{R_1}r_1}<_1\omega^{\omega^{R_1}r_1}+R_1\leqslant\omega^{\omega^{R_1}r_1}+R_1\leqslant\omega^{\omega^{R_1}r_1}+\omega^{\omega^{R_1}r_1}<\omega^{\omega^{R_1}r_1}\omega=\omega^{\omega^{R_1}r_1+1}\leqslant$
$\omega^{\omega^{R_1}r_1+\omega^{R_2}}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}}+R_2\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}+1}\leqslant$
$\omega^{\omega^{R_1}r_1+\omega^{R_2}2}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}2}+R_2\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}2+1}\leqslant\ldots\leqslant$
$\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2}+R_2\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+1}\leqslant$
...
$\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}}+R_{k-1}\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}+1}\leqslant$
$\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}2}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}2}+R_{k-1}\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}2+1}\leqslant\ldots\leqslant$
$\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}}+R_{k-1}\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}+1}\leqslant$

$$\begin{cases} \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}}+R_k\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}+1}\leqslant & \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2}+R_k\leqslant\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2+1}\leqslant\ldots\leqslant & \text{if } R_k\neq 0 \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k}<_1\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k}+R_k=\omega^Z+d(\omega^Z) & \\ & \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}}+\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}}=\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}} & \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}}+\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2}=\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2} & \text{if } R_k=0 \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2}+\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}3}=\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}3},\ldots & \\ \omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}(r_k-1)}+\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k}=\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k}=\omega^Z+d(\omega^Z) & \end{cases}$$

Therefore, from (I1) and (I2) we get the inequalities:
$\forall i\in\{1,\ldots,k-1\}\forall j\in\{1,\ldots,r_i\}$.
$h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_i}\cdot j})<_1 h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_i}\cdot j})+h(R_i)$; (J1)

Remark: In (J1), the case $i=1,\ j=1$ is $h(\omega^{\omega^{R_1}})<_1 h(\omega^{\omega^{R_1}})+h(R_1)$ and it holds for two different reasons: If $R_1=\omega^{\omega^{R_1}}$, then $R_1=\alpha$ (because $\alpha<x\leqslant\max\{\delta,L_1+d(L_1)\}+1<\alpha^+$) and so $h(\alpha)=h(\omega^{\omega^{R_1}})=\omega^{\omega^{h(R_1)}}<_1\omega^{\omega^{h(R_1)}}+h(R_1)=h(\alpha)2$ holds because we know $\alpha<_1\alpha 2$. If $R_1<\omega^{\omega^{R_1}}$, then $h(\omega^{\omega^{R_1}})<_1 h(\omega^{\omega^{R_1}})+h(R_1)$ holds because $\omega^{\omega^{R_1}}<_1\omega^{\omega^{R_1}}+R_1$.

Moreover, from (I1) and (I2) we get the inequalities and equations (J3):
For $1\leqslant j\leqslant r_k$,

$$\begin{cases} h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}\cdot j})<_1 h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}\cdot j})+h(R_k), & \text{if } R_k\neq 0 \\ \textbf{(Observe here we use: } \boldsymbol{\omega^Z+d(\omega^Z)\in\mathrm{Dom}\,h}\textbf{)} & \\ & \\ h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}r_{k-1}})+h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}})=h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}}) & \\ h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}})+h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2})=h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2}) & \text{if } R_k=0 \\ h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}2})+h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}3})=h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}3}),\ldots & \\ h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_{k-1}}(r_k-1)})+h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k})=h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\ldots+\omega^{R_k}r_k}) & \end{cases}$$

Now, from (J1) and theorem 2.3 we get
For $i \in \{1, ..., k-1\}$,
$h(\omega^{\omega^{R_1}r_1+...+\omega^{R_{i-1}}r_{i-1}+\omega^{R_i}\cdot j}) = \omega^{\omega^{h(R_i)}S(r_1,...,r_{i-1},j)}$ for some $S(r_1, ..., r_{i-1}, j) \neq 0$.

On the other hand, from $\omega^{\omega^{R_1}} < \omega^{\omega^{R_1}2} ... < \omega^{\omega^{R_1}r_1}$ it follows $h(\omega^{\omega^{R_1}}) < h(\omega^{\omega^{R_1}2}) < ... < h(\omega^{\omega^{R_1}r_1})$, which gives us, using the equality in the previous paragraph, $\omega^{\omega^{h(R_1)}S(1)} < \omega^{\omega^{h(R_1)}S(2)} < ... < \omega^{\omega^{h(R_1)}S(r_1-1)} < \omega^{\omega^{h(R_1)}S(r_1)}$. This implies $S(1) < S(2) < ... < S(r_1)$, which subsequently implies $r_1 \leqslant S(r_1)$. Now, notice the following inductive argument: For any $j \in \{1, ..., r_2\}$, $h(\omega^{\omega^{R_1}r_1}) = \omega^{\omega^{h(R_1)}S(r_1)} < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}j}) = \omega^{\omega^{h(R_2)}S(r_1,j)}$; this way, $\omega^{h(R_1)}S(r_1) < \omega^{h(R_2)}S(r_1, j)$ and $h(R_1) > h(R_2)$ (by (I0)). Thus $S(r_1, j) > \omega^{-h(R_2)+h(R_1)}S(r_1)$ (otherwise $\omega^{h(R_1)}S(r_1) \geqslant \omega^{h(R_2)}S(r_1, j)$) and therefore it exists $q(r_1, j) \in \mathrm{OR}$ such that $S(r_1, j) = \omega^{-h(R_2)+h(R_1)}S(r_1) + q(r_1, j)$. Then $h(\omega^{\omega^{R_1}r_1+\omega^{R_2}j}) = \omega^{\omega^{h(R_2)}S(r_1,j)} = \omega^{\omega^{h(R_2)}\omega^{-h(R_2)+h(R_1)}S(r_1)+q(r_1,j)} = \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,j)}$. Moreover, observe the chain of inequalities $h(\omega^{\omega^{R_1}r_1+\omega^{R_2}}) < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}2}) < ... < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2})$ implies $q(r_1, r_2) \geqslant r_2$. But for any $j \in \{1, ..., r_3\}$, $h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2}) < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\omega^{R_3}j}) = \omega^{\omega^{h(R_3)}S(r_1,r_2,j)}$, and so $\omega^{h(R_1)}S(r_1) + \omega^{h(R_2)}q(r_1, r_2) < \omega^{h(R_3)}S(r_1, r_2, j)$; since $h(R_3) < h(R_2) < h(R_1)$, then $S(r_1, r_2, j) > \omega^{-h(R_3)+h(R_1)}S(r_1) + \omega^{-h(R_3)+h(R_2)}q(r_1, r_2)$ and so it exists $q(r_1, r_2, j) \in \mathrm{OR}$ such that $S(r_1, r_2, j) = \omega^{-h(R_3)+h(R_1)}S(r_1) + \omega^{-h(R_3)+h(R_2)}q(r_1, r_2) + q(r_1, r_2, j)$. Then $h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\omega^{R_3}j}) = \omega^{\omega^{h(R_3)}S(r_1,r_2,j)} = \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+\omega^{h(R_3)}q(r_1,r_2,j)}$. Moreover, the chain of inequalities $h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\omega^{R_3}}) < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\omega^{R_3}2}) < ... < h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+\omega^{R_3}r_3})$ implies $q(r_1, r_2, r_3) \geqslant r_3$. Inductively, we obtain
$h(\omega^{\omega^{R_1}r_1+...+\omega^{R_{k-1}}r_{k-1}}) = \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+...+\omega^{h(R_{k-1})}q(r_1,...,r_{k-1})}$ with
$S(r_1) \geqslant r_1,\ q(r_1, r_2) \geqslant r_2, q(r_1, r_2, r_3) \geqslant r_3, ..., q(r_1, ..., r_{k-1}) \geqslant r_{k-1}$.

For the case $R_k \neq 0$, doing once more the previous procedure with the equalities $h(\omega^{\omega^{R_1}r_1+...+\omega^{R_k}\cdot j}) = \omega^{\omega^{h(R_k)}S(r_1,...,r_{k-1},j)}$, we obtain:
$h(\omega^{\omega^{R_1}r_1+...+\omega^{R_k}r_k}) = \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+...+\omega^{h(R_k)}q(r_1,...,r_k)}$ with
$S(r_1) \geqslant r_1,\ q(r_1, r_2) \geqslant r_2, q(r_1, r_2, r_3) \geqslant r_3, ..., q(r_1, ..., r_k) \geqslant r_k$; therefore
$h(\omega^{\omega^{R_1}r_1+...+\omega^{R_k}r_k}) \geqslant \omega^{\omega^{h(R_1)}r_1+\omega^{h(R_2)}r_2+...+\omega^{h(R_k)}r_k}$. **(**1**)**

For the case $R_k = 0$ the additions in (J3) imply:
$h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+...+\omega^{R_k}}) \geqslant \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+...+\omega^{h(R_{k-1})}q(r_1,...,r_{k-1})+1}$
$h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+...+\omega^{R_k}2}) \geqslant \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+...+\omega^{h(R_{k-1})}q(r_1,...,r_{k-1})+2}$
...
$h(\omega^{\omega^{R_1}r_1+\omega^{R_2}r_2+...+\omega^{R_k}r_k}) \geqslant \omega^{\omega^{h(R_1)}S(r_1)+\omega^{h(R_2)}q(r_1,r_2)+...+\omega^{h(R_{k-1})}q(r_1,...,r_{k-1})+r_k}$
and therefore $h(\omega^{\omega^{R_1}r_1+...+\omega^{R_k}r_k}) \geqslant \omega^{\omega^{h(R_1)}r_1+\omega^{h(R_2)}r_2+...+\omega^{h(R_k)}r_k}$ **(**2**)**

Finally, to conclude, in any case $R_k \neq 0$ or $R_k = 0$, from (**1**) and (**2**) we have
$h(x) = (\omega^Z) = h(\omega^{\omega^{R_1}r_1+...+\omega^{R_k}r_k}) \geqslant \omega^{\omega^{h(R_1)}r_1+\omega^{h(R_2)}r_2+...+\omega^{h(R_k)}r_k} \overset{\mathrm{IH}}{\geqslant}$
$\omega^{\omega^{R_1[\alpha:=h(\alpha)]}r_1+\omega^{R_2[\alpha:=h(\alpha)]}r_2+...+\omega^{R_k[\alpha:=h(\alpha)]}r_k} = \omega^{Z[\alpha:=h(\alpha)]} = \omega^Z[\alpha := h(\alpha)] = x[\alpha := h(\alpha)]$.

This finishes the proof of this lemma. □

2.3.1 Cover of a finite set B.

Now we extend the construction of the covering for a finite set.

Definition 2.31. *(Cover of a finite set). Let* $\alpha\in\mathbb{E}$ *and* $B\subset_{\text{fin}}\alpha^+$*. We define* $\Delta(\alpha,B):=B\cup\bigcup_{\delta\in B\cap[\alpha,\alpha^+)}D(\alpha,\delta)$*, where* $D(\alpha,\delta)$ *is the set defined in previous lemma 2.30.*

Proposition 2.32. *Let* $\alpha\in\mathbb{E}$ *and* $B\subset_{\text{fin}}\alpha^+$*. If* $B\cap[\alpha,\alpha^+)=\emptyset$*, then* $\Delta(\alpha,B)=B\subset\alpha$*. If* $B\cap[\alpha,\alpha^+)\neq\emptyset$*, then for* $t:=\max B$*,* $\Delta(\alpha,B)\subset_{\text{fin}}\eta t+1\subset\alpha^+$*. In any case,* $\Delta(\alpha,B)$ *is finite.*

Proof. That $B\cap[\alpha,\alpha^+)=\emptyset$ implies $\Delta(\alpha,B)=B\subset\alpha$ is clear.

Suppose $B\cap[\alpha,\alpha^+)\neq\emptyset$ and let $t:=\max B$. Let $\delta\in B\cap[\alpha,\alpha^+)$ be arbitrary. If $\delta=\alpha$, then $D(\alpha,\delta)=\{\alpha,\alpha2\}\underset{\text{by proposition 2.26, claim 3.}}{\subset}\eta t+1$. If $\delta>\alpha$, then $\delta\leqslant t$ and so $D(\alpha,\delta)\underset{\text{by prop. 2.26 claim 3, and by prop. 2.29}}{\subset}\eta\delta+1\underset{\text{by proposition 2.27 claim 4}}{\leqslant}\eta t+1<\alpha^+$.

Finally, $\Delta(\alpha,B)$ is finite because it is finite union of finite sets. □

Theorem 2.33. *(Covering theorem). Let* $\alpha\in\mathbb{E}$ *and* $B\subset_{\text{fin}}\alpha^+$ *be such that* $B\cap[\alpha,\alpha^+)\neq\emptyset$*. Consider* $FB:=\{h\colon\Delta(\alpha,B)\longrightarrow h[\Delta(\alpha,B)]\subset\alpha\,|\,h$ *is an* $(<,<_1,+)$*-isomorphism with* $h|_\alpha=\mathrm{Id}_\alpha\}$*. Then for any* $h\in FB$ *the ordinal* $h(\alpha)\in\alpha\cap\mathbb{E}$ *and*

a) $\forall x\in\Delta(\alpha,B).\mathrm{Ep}(x)\cap\alpha\subset h(\alpha)\wedge x[\alpha:=h(\alpha)]\leqslant h(x)$*.*

b) If $\alpha\leqslant_1\max\Delta(\alpha,B)$*, then the function* $H\colon\Delta(\alpha,B)\longrightarrow H[\Delta(\alpha,B)]$*,* $H(x):=x[\alpha:=h(\alpha)]$ *is an* $(<,<_1,+,\lambda x.\omega^x)$*-isomorphism with* $H|_\alpha=\mathrm{Id}_\alpha$*.*

Proof. Let α and B as stated. Let $h\in FB$.

First note that $\alpha,\ \alpha2\in\Delta(\alpha,B)$ (because $B\cap[\alpha,\alpha^+)\neq\emptyset$) and since $\alpha<_1\alpha2$, then $h(\alpha)<_1h(\alpha)2$. This implies that $h(\alpha)\in\mathbb{E}$.

Now we show *a*).

Let $x\in\Delta(\alpha,B)$.

If $x<\alpha$, then $x=h(x)<h(\alpha)$ because h is an $<$-isomorphism such that $h|_\alpha=\mathrm{id}_\alpha$. Therefore $\mathrm{Ep}(x)\cap\alpha\subset h(\alpha)$ and $x=x[\alpha:=h(\alpha)]$.

If $x=\alpha$, then clearly $\mathrm{Ep}(x)\cap\alpha\subset h(\alpha)$ and $x[\alpha:=h(\alpha)]=h(x)$.

Case $x>\alpha$. Then $x\in D(\alpha,x)\subset\Delta(\alpha,B)$ and $h|_{D(\alpha,x)}\colon D(\alpha,x)\longrightarrow h[D(\alpha,x)]\subset\alpha$ is an $(<,<_1,+)$-isomorphism with $h|_{D(\alpha,x)}|_\alpha=\mathrm{Id}_\alpha$ by proposition A.1 in the appendices section. Therefore, by lemma 2.30, $\mathrm{Ep}(x)\cap\alpha\subset h|_{D(\alpha,x)}(\alpha)=h(\alpha)$ and $x[\alpha:=h(\alpha)]\leqslant h|_{D(\alpha,x)}(x)=h(x)$.

The previous shows *a*).

We show *b*).

Suppose $\alpha\leqslant_1\max\Delta(\alpha,B)$.

By *a*) we know $\forall x\in\Delta(\alpha,B).\mathrm{Ep}(x)\cap\alpha\subset h(\alpha)$; so, by corollary 2.15, the function H is an $(<,+,\cdot,\lambda x.\omega^x)$- isomorphism. Moreover, it is also clear that $H|_\alpha=\mathrm{Id}_\alpha$. So we just need to prove that H preserves the relation $<_1$ too. Let $\Delta(\alpha,B)\cap\alpha=\{a_1,\dots,a_N\}$ and $\Delta(\alpha,B)\cap[\alpha,\alpha^+)=\{\alpha=b_1,\dots,b_M\}$. Then:

- Note $\alpha<_1\max\Delta(\alpha,B)$ and $\leqslant_1$-connectedness imply that $\alpha<_1b_j$ for any $b_j\neq\alpha$. So we need to show $H(\alpha)<_1H(b_j)$ for any $b_j\neq\alpha$. But by *a*) we know $H(b_j)=b_j[\alpha:=h(\alpha)]\leqslant h(b_j)$; moreover, we know $h(\alpha)=H(\alpha)<H(b_j)$ and $h(\alpha)<_1h(b_j)$ for any $b_j\neq\alpha$. Thus by $\leqslant_1$-connectedness, $H(\alpha)<_1H(b_j)$ for any $b_j\neq\alpha$.
- $a_i<_1a_j\Longleftrightarrow a_i=H(a_i)<_1H(a_j)=a_j$.
- $a_i<_1\alpha\Longleftrightarrow H(a_i)=a_i=h(a_i)<_1H(\alpha)=h(\alpha)$ because h is an $<_1$ isomorphism.
- If $a_i<_1b_j$, then $H(a_i)=a_i<_1H(b_j)<\alpha<b_j$ by $<_1$-connectedness.
- If $H(a_i)<_1H(b_j)$, then $H(a_i)<_1H(\alpha)$ by $<_1$-connectedness. But $h(a_i)=H(a_i)<_1H(\alpha)=h(\alpha)\Longleftrightarrow a_i<_1\alpha$ (because h is an $<_1$ isomorphism) and since $\alpha\leqslant_1b_j$, then $a_i<_1b_j$ follows by $<_1$-transitivity.
- For $b_i\neq\alpha\neq b_j$, $b_i<_1b_j\underset{\text{corollary 2.17}}{\Longleftrightarrow}H(b_i)<_1H(b_j)$.

The previous shows b). $\square$

2.3.1.1 Consequences of the covering theorem.

Consider a finite set of ordinals $L\subset_{\text{fin}}\text{OR}$ and $FL\subset\{k|k\colon L\longrightarrow\text{OR}\}$ a class of functionals. Then FL is well ordered under the lexicographic order $<_{FL,\text{lex}}$; that is, for $h,\ k\in FL$, $h<_{\text{FL,lex}}k\colon\Longleftrightarrow\ \exists y\in L.h(y)\neq k(y)$ and for $m:=\min\{x\in L|h(x)\neq k(x)\}$ it holds $h(m)<k(m)$. Moreover, in case $FL\neq\emptyset$, we can consider $\min(FL)$, the minimum element in FL with respect to $<_{FL,\text{lex}}$. The next corollary uses this concepts.

Corollary 2.34. *Let $\alpha\in\mathbb{E}$ and $\beta\in(\alpha,\alpha^+)$ be with $\alpha<_1\beta$. Suppose $B\subset_{\text{fin}}\beta$ is such that $\Delta(\alpha,B)\subset\beta$. Consider $FB:=\{h\colon\Delta(\alpha,B)\longrightarrow h[\Delta(\alpha,B)]\subset\alpha|h$ is an $(<,<_1,+)$-isomorphism with $h|_\alpha=\text{Id}_\alpha\}$. Then $\mu:=\min(FB)$ exists, $\mu(\alpha)\in\alpha\cap\mathbb{E}$ and μ is the substitution $x\longmapsto x[\alpha:=\mu(\alpha)]$.*

Proof. Since $\alpha<_1\beta$ and $\Delta(\alpha,B)\subset_{\text{fin}}\beta$, then $FB\neq\emptyset$ and so $\mu:=\min FB$ exists. Now, by previous theorem 2.33, $\mu(\alpha)\in\mathbb{E}\cap\alpha$ and the function $H\colon\Delta(\alpha,B)\longrightarrow H[\Delta(\alpha,B)]$, $H(x)\colon=x[\alpha\colon=\mu(\alpha)]$ is well defined and satisfies the following two things: $H\in FB$ and $\forall x\in\Delta(\alpha,B).H(x)\leqslant\mu(x)$. Thus, from the minimality of the function μ, it follows $H=\mu$. $\square$

The following is the main result that relates $<_1$ with $<^1$.

Corollary 2.35. *Let $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)$. Then $\alpha<_1\eta t+1\Longleftrightarrow\alpha<^1\eta t+1$.*

Proof. The implication $\Longleftarrow$) is already known.
Let's show $\Longrightarrow$).

Let $B\subset_{\text{fin}}\eta t+1$. If $B\subset\alpha$, then $I\colon B\longrightarrow B$, $I(x):=x$ is an $(<,<_1,+,\lambda x.\omega^x)$- isomorphism with $I|_\alpha=\text{Id}_\alpha$. So suppose $B\cap[\alpha,\alpha^+)\neq\emptyset$. Let $l:=\max B\geqslant\alpha$. Proposition 2.32 guarantees that $\Delta(\alpha,B)\subset_{\text{fin}}\eta l+1$; but $\eta l\underset{\text{prop. 2.27}}{\leqslant}\eta\eta t\underset{\text{prop. 2.26}}{=}\eta t$, so $\Delta(\alpha,B)\subset_{\text{fin}}\eta l+1\leqslant\eta t+1$. Moreover, since by hypothesis $\alpha<_1\eta t+1$, then there exists $h\colon\Delta(\alpha,B)\longrightarrow h[\Delta(\alpha,B)]\subset\alpha$ an $(<,<_1,+)$-isomorphism with $h|_\alpha=\text{Id}_\alpha$. Therefore, by theorem 2.33, the function $H\colon\Delta(\alpha,B)\longrightarrow H[\Delta(\alpha,B)]\subset\alpha$ defined as $H(x):=x[\alpha:=h(\alpha)]$ is an $(<,<_1,+,\lambda x.\omega^x)$-isomorphism with $H|_\alpha=\text{Id}_\alpha$. Then, by proposition A.1 in the appendices section, $H|_B\colon B\longrightarrow H|_B[B]$ is an $(<,<_1,+,\lambda x.\omega^x)$-isomorphism with $H|_B|_\alpha=\text{Id}_\alpha$. $\square$

Corollary 2.36. $\forall\alpha\in\mathbb{E}.\alpha<_1\alpha^+\Longleftrightarrow\alpha<^1\alpha^+$

Proof. Easy. Left to the reader. $\square$

Corollary 2.37. $\forall\alpha\in\mathbb{E}.\alpha<_1\alpha^+\Longleftrightarrow$
$\alpha\in\{\beta\in\mathbb{E}|\forall t\in[\beta,\beta^+)\exists(c_\xi)_{\xi\in X}\subset\mathbb{E}\cap\beta.\text{Ep}(t)\cap\beta\subset c_\xi\wedge c_\xi<_1t[\beta:=c_\xi]\wedge c_\xi\underset{\text{cof}}{\hookrightarrow}\beta\}$.

Proof. Not hard. Left to the reader. $\square$

We want to conclude this section with a characterization of the case $\alpha<^1t+1$ for ordinals $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)$. For this (and also for our work on the next section), it will be convenient to prove following

Proposition 2.38. *Let $\alpha,\beta,t\in\mathrm{OR}$ such that $\alpha,\beta\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)\wedge\mathrm{Ep}(t)\cap\alpha\subset\beta$. Then*

a) $(\mathrm{Ep}(\pi t)\cap\alpha)\cup(\mathrm{Ep}(d\pi t)\cap\alpha)\cup(\mathrm{Ep}(\eta t)\cap\alpha)\subset\beta$

b) $\pi(t[\alpha:=\beta])=(\pi t)[\alpha:=\beta]$

c) $d\pi(t[\alpha:=\beta])=(d\pi t)[\alpha:=\beta]$

d) $\pi(t[\alpha:=\beta])+d\pi(t[\alpha:=\beta])=(\pi t+d\pi t)[\alpha:=\beta]$

e) $\eta(t[\alpha:=\beta])=(\eta t)[\alpha:=\beta]$.

Proof. Not hard. Left to the reader. $\square$

Note 2.39. Because of the previous proposition, whenever we have such hypothesis, we will simply write $\pi t[\alpha:=\beta]$, $d\pi t[\alpha:=\beta]$ and $\eta t[\alpha:=\beta]$ to the ordinals $\pi(t[\alpha:=\beta])$, $d\pi(t[\alpha:=\beta])$ and $\eta(t[\alpha:=\beta])$ respectively.

Corollary 2.40. *Let $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)$. The following are equivalent*

a) $\alpha<^1 t+1$

b) $\alpha\in\mathrm{Lim}\{\xi\in\mathbb{E}|\mathrm{Ep}(t)\cap\alpha\subset\xi\wedge\xi\leqslant_1 t[\alpha:=\xi]\}$

c) $\alpha<^1\eta t+1$

d) $\alpha<_1\eta t+1$

Proof. Let $\alpha\in\mathbb{E}$, $t\in[\alpha,\alpha^+)$.
$a)\Longleftrightarrow b)$ holds because of propositions 2.24 and 2.23.
$c)\Longleftrightarrow d)$ is corollary 2.35.
$c)\Longrightarrow a)$ holds because of $<^1$-connectedness, and so $c)\Longrightarrow b)$ (because $a)\Longleftrightarrow b)$).
So it suffices to prove $b)\Longrightarrow c)$.

Suppose $\alpha\in\mathrm{Lim}\{\xi\in\mathbb{E}|\mathrm{Ep}(t)\cap\alpha\subset\xi\wedge\xi\leqslant_1 t[\alpha:=\xi]\}$. Let $(c_j)_{j\in J}\subset\alpha\cap\mathbb{E}$ be a sequence such that $\mathrm{Ep}(t)\cap\alpha\subset c_j\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and $\forall j\in J.c_j\leqslant_1 t[\alpha:=c_j]$. Note $\forall j\in J.t[\alpha:=c_j]\in(c_j,c_j^+)$, and then, by proposition 2.27 claim 4., $\forall j\in J.c_j\leqslant_1\eta(t[\alpha:=c_j])=(\eta t)[\alpha:=c_j])$, where the last equality holds because, by proposition 2.38, $\mathrm{Ep}(\eta t)\cap\alpha\subset c_j$ and $\eta(t[\alpha:=c_j])=(\eta t)[\alpha:=c_j])$ for any $j\in J$. So, summarizing, $(c_j)_{j\in J}\subset\alpha\cap\mathbb{E}$ is a sequence of epsilon numbers such that for $\eta t\in[\alpha,\alpha^+)$, $\mathrm{Ep}(\eta t)\cap\alpha\subset c_j\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and $\forall j\in J.\mathbb{E}\ni c_j\leqslant_1(\eta t)[\alpha:=c_j]$; therefore, by proposition 2.24, $\alpha<^1\eta t+1$. $\square$

Corollary 2.41. *Let $\alpha\in\mathbb{E}$.*

a) $\forall e\in\alpha\cap\mathbb{E}.m(e)\in[e,e^+)\Longrightarrow\exists t\in[\alpha,\alpha^+).\eta t=t\wedge m(e)=t[\alpha:=e]$.

b) Suppose $m(\alpha)\in[\alpha,\alpha^+)$. Then
$\forall t\in[\alpha,m(\alpha)).t=\eta t\Longrightarrow\{\delta\in\mathbb{E}\cap\alpha|\mathrm{Ep}(t)\cap\alpha\subset\delta\wedge m(\delta)=t[\alpha:=\delta]\}$ *is confinal in α.*

Proof. Let $\alpha\in\mathbb{E}$.
a). Let $e\in\alpha\cap\mathbb{E}$ and suppose $m(e)\in[e,e^+)$. Then $\eta(m(e))\not>m(e)$ (otherwise by proposition 2.27 claim 4, $e<_1\eta(m(e))\geqslant m(e)+1$ which is impossible); but by definition $\eta(m(e))=\max\{m(e),\pi(m(e))+d\pi(m(e))\}\geqslant m(e)$, thus $\eta(m(e))=m(e)$. This way, for $t:=m(e)[e:=\alpha]$, $\eta t=\eta(m(e)[e:=\alpha])\underset{\text{proposition 2.38}}{=}\eta(m(e))[e:=\alpha]=m(e)[e:=\alpha]=t$ and clearly $m(e)=t[\alpha:=e]$.

$b)$. Suppose $m(\alpha) \in [\alpha, \alpha(+^1))$ and let $t \in [\alpha, m(\alpha))$ be such that $t = \eta t$.

Take $\gamma \in \alpha$ arbitrary.

First note that $\alpha < \eta t + 1 \leqslant m(\alpha)$, implies, by $\leqslant_1$-connectedness, $\alpha <_1 \eta t + 1$. Subsequently, by previous corollary 2.40, $\alpha \in \mathrm{Lim}\{\xi \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 t[\alpha := \xi]\} =$
$\mathrm{Lim}\{\xi \in \mathbb{E} | \mathrm{Ep}(\eta t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 (\eta t)[\alpha := \xi]\}$.
Let $e := \min(\gamma, \alpha] \cap \{\xi \in \mathbb{E} | \mathrm{Ep}(\eta t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 (\eta t)[\alpha := \xi]\}$. Then
$\gamma < e \leqslant_1 (\eta t)[\alpha := e] \underset{\text{proposition 2.38}}{=} \eta t[\alpha := e]$. We assure $e \not<_1 \eta t[\alpha := e] + 1$. Suppose the opposite $e <_1 \eta t[\alpha := e] + 1$. Then by previous corollary 2.40,
$e \in \mathrm{Lim}\{\xi \in \mathbb{E} | \mathrm{Ep}(t[\alpha := e]) \cap e \subset \xi \wedge \xi \leqslant_1 t[\alpha := e][e := \xi]\} \underset{\text{proposition 2.10}}{=}$

$\mathrm{Lim}\{\xi \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 t[\alpha := \xi]\} =$

$\mathrm{Lim}\{\xi \in \mathbb{E} | \mathrm{Ep}(\eta t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 (\eta t)[\alpha := \xi]\}$.

The latter implies that there exist some ordinal $\varphi < e$ with
$\varphi \in (\gamma, \alpha] \cap \{\xi \in \mathbb{E} | \mathrm{Ep}(\eta t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 (\eta t)[\alpha := \xi]\}$. But this is impossible since by definition $e = \min(\gamma, \alpha] \cap \{\xi \in \mathbb{E} | \mathrm{Ep}(\eta t) \cap \alpha \subset \xi \wedge \xi \leqslant_1 (\eta t)[\alpha := \xi]\}$. Contradiction.
Thus $e \not<_1 \eta t[\alpha := e] + 1$. Thus $m(e) = \eta t[\alpha := e]$.

Our previous work has provided, given an arbitrary ordinal $\gamma \in \alpha$, an ordinal $e \in \mathbb{E}$ such that $\gamma < e \in \mathbb{E} \wedge \mathrm{Ep}(t) \cap \alpha \subset e \wedge m(e) = \eta t[\alpha := e] = t[\alpha := e]$. Hence, we have shown that
$\{\delta \in \mathbb{E} \cap \alpha | \mathrm{Ep}(t) \cap \alpha \subset \delta \wedge m(\delta) = t[\alpha := \delta]\}$ is confinal in α. □

2.4 A hierarchy induced by $<_1$ and the intervals $[\varepsilon_\gamma, \varepsilon_{\gamma+1})$

In this section we will provide our theorem linking "the solutions of the $<_1$-inequality $x <_1 t$ with $t \in [x, x^+)$" with a hierarchy of ordinals obtained by a thinning procedure.

For the main theorem, we will need the following

Lemma 2.42. *Let $\alpha, t \in \mathrm{OR}$, $\alpha \in \mathbb{E}$ and $t \in (\alpha, \alpha^+) \cap \mathrm{Lim}$. Then there exists a sequence $(l_j)_{j \in I}$ with $(I \cup \{0\}) \in \mathrm{OR}$, $(I \cup \{0\}) \leqslant \alpha$ such that*

(1) For all $j \in I$, $l_j \in (\alpha, \alpha^+)$, $l_j \underset{\mathrm{cof}}{\hookrightarrow} t$ and $(l_j)_{j \in I}$ is strictly monotonous increasing.

(2) For any $\beta \in \mathbb{E} \cap \alpha \cup \{\alpha\}$ with $\mathrm{Ep}(t) \cap \alpha \subset \beta$, $\forall j \in I \cap \beta . \mathrm{Ep}(l_j) \cap \alpha \subset \beta$; moreover, the sequence $(l_j[\alpha := \beta])_{j \in I \cap \beta}$ is cofinal in $t[\alpha := \beta]$.

(3) $\forall \beta \in (\mathbb{E} \cap \alpha \cup \{\alpha\}) \forall j \in I \cap \beta$.

- *$\eta l_j[\alpha := \beta] \leqslant \eta t[\alpha := \beta]$*
- *$\eta l_j[\alpha := \beta] < \eta t[\alpha := \beta]$ if $t > \pi t + d\pi t$.*

Proof. Let α and t be as stated. Below we give only the sequence. The proof that such sequence satisfies what is stated, is long and boring and it is left to the reader.

Consider $t =_{\mathrm{CNF}} \omega^{T_1} t_1 + \ldots + \omega^{T_n} t_n$ and $T_1 =_{\mathrm{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m$. Suppose that for any $a \in (\alpha, \alpha^+) \cap \mathrm{Lim} \cap t$ we have been able to define a sequence $(l'_j)_{i \in I'}$ satisfying what the theorem state with respect to a.

Then we have cases:

So $t = \omega^{T_1} t_1$ and $T_1 =_{\mathrm{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m$.

If $t_1 = 1$ then $t = \omega^{T_1}$.

If $Q_m = 0$, then $m \geqslant 2$ and $Q_1 \geqslant \alpha$ (otherwise $t < \alpha$) and $t = \omega^{T_1} = \omega^{\omega^{Q_1} q_1 + \ldots + \omega^0 q_m}$.

Let $l_j := \omega^{\omega^{Q_1} q_1 + \ldots + \omega^0 (q_m - 1)} j$, with $j \in I := \omega \backslash \{0\}$.

If $Q_m \neq 0$. Then $\omega^{Q_m} \leqslant \omega^{T_1} = t$. Moreover, we assure $\omega^{Q_m} < \omega^{T_1} = t$. This is because

$\omega^{Q_m} = \omega^{T_1}$ implies $T_1 \leqslant \omega^{T_1} = \omega^{Q_m} \leqslant T_1$ and then $\alpha = T_1$ (since $T_1 \in [\alpha, \alpha^+)$); moreover, since $T_1 =_{\text{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m$, then $m = 1$, $q_1 = 1$ and $Q_1 = \alpha$. That is, we have the $t = \omega^\alpha = \alpha$ which is contradictory because from the beginning we picked $t \in (\alpha, \alpha^+)$. The previous showed $\omega^{Q_m} < \omega^{T_1}$.

If $Q_m < \alpha$, then $m \geqslant 2$ (otherwise $t = \omega^{T_1} = \omega^{\omega^{Q_1} q_1} < \alpha$).

If Q_m is a successor ordinal.

Let $l_j := \omega^{\omega^{Q_1} q_1 + \ldots + \omega^{Q(m-1)} q_{(m-1)} + \omega^{Q_m}(q_m - 1) + \omega^{Q_m - 1} j}$ with $j \in I := \omega \backslash \{0\}$.

If Q_m is a limit ordinal.

Let $l_j := \omega^{\omega^{Q_1} q_1 + \ldots + \omega^{Q(m-1)} q_{(m-1)} + \omega^{Q_m}(q_m - 1) + \omega^j}$ with $j \in I := Q_m \backslash \{0\}$.

If $Q_m = \alpha$. Since $T_1 =_{\text{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m \in [\alpha, \alpha^+)$, then $m = 1$ and so $t = \omega^{T_1} = \omega^{\omega^{Q_1} q_1} = \omega^{\omega^\alpha q_1}$.

Let $l_j := \omega^{\omega^\alpha (q_1 - 1) + \omega^j}$ with $j \in I := Q_1 \backslash \{0\} = \alpha \backslash \{0\}$.

If $Q_m > \alpha$. Then $\omega^{Q_m} \in (\alpha, \alpha^+) \cap \text{Lim}$ and moreover, we already know $\omega^{Q_m} < \omega^{T_1} = t$. Then by our induction hypothesis applied to ω^{Q_m} there exists a sequence $(\xi_j)_{j \in I}$ with $I \cup \{0\} \in \text{OR}$ and $I \cup \{0\} \leqslant \alpha$, such that (1), (2), and (3) hold with respect to the sequence $(\xi_j)_{j \in I}$ and ω^{Q_m}.

Let $l_j := \omega^{\omega^{Q_1} q_1 + \ldots + \omega^{Q(m-1)} q_{(m-1)} + \omega^{Q_m}(q_m - 1) + \xi_j}$ with $j \in I$.

Case $t_1 \geqslant 2$.

If $T_1 = \alpha$, then $t_1 = \omega^\alpha t_1$.

Let $l_j := \omega^{T_1}(t_1 - 1) + j$ with $j \in I := T_1 \backslash \{0\} = \alpha \backslash \{0\}$.

If $T_1 > \alpha$, then $t = \omega^{T_1} t_1 > \omega^{T_1} \in (\alpha, \alpha^+)$; so by our induction hypothesis applied to ω^{T_1} there exists a sequence $(\xi_j)_{j \in I}$ with $I \cup \{0\} \in \text{OR}$ and $I \cup \{0\} \leqslant \alpha$, such that (1), (2), and (3) hold with respect to the sequence $(\xi_j)_{j \in I}$ and ω^{T_1}.

Let $l_j := \omega^{T_1}(t_1 - 1) + \xi_j$ with $j \in I$.

Case $n \geqslant 2$.

So $t = \omega^{T_1} t_1 + \omega^{T_2} t_2 + \ldots + \omega^{T_n} t_n$, $T_1 =_{\text{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m$ and $T_1 > T_n \neq 0$ (because $t \in \text{Lim}$). Then $\omega^{T_n} < t$.

If $T_n < \alpha$.

Let $l_j := \omega^{T_1} t_1 + \omega^{T_2} t_2 + \ldots + \omega^{T(n-1)} t_{(n-1)} + \omega^{T_n}(t_n - 1) + j$ with $j \in I := \omega^{T_n} \backslash \{0\}$. So clearly $I \cup \{0\} < \alpha$.

If $T_n = \alpha$. Then the argument is almost the same as in the previous subcase:

Let $l_j := \omega^{T_1} t_1 + \omega^{T_2} t_2 + \ldots + \omega^{T_n}(t_n - 1) + j$ with $j \in I := T_n \backslash \{0\} = \alpha \backslash \{0\}$.

If $T_n > \alpha$. Then $\omega^{T_n} \in (\alpha, \alpha^+) \cap \text{Lim}$ and moreover, we already know $\omega^{T_n} < \omega^{T_1} \leqslant t$. Then by our induction hypothesis applied to ω^{T_1} there exists a sequence $(\xi_j)_{j \in I}$ with $I \cup \{0\} \in \text{OR}$ and $I \cup \{0\} \leqslant \alpha$, such that (1), (2), and (3) hold with respect to the sequence $(\xi_j)_{j \in I}$ and ω^{T_n}.

Let $l_j := \omega^{T_1} t_1 + \ldots + \omega^{T_n}(t_n - 1) + \xi_j$ with $j \in I$. □

The following will be also needed in the main theorem of this section (theorem 2.45).

Proposition 2.43. *Let* $\beta \in \text{OR}. \beta <^1 \beta 2 + 1 \Longleftrightarrow \beta \in \text{Lim}\,\mathbb{E}$

Proof. Not hard. Left to the reader. □

Definition 2.44. *Let* $A: [\varepsilon_\omega, \infty) \longrightarrow \text{Subclases(OR)}$ *be defined recursively as:*

For $l + 1 \in [\varepsilon_\omega, \infty)$,

$$A(l+1) := \begin{cases} A(l) & \text{if } \; l < \pi l + d\pi l \\ \text{Lim} A(l) & \text{otherwise} \end{cases}$$

For $t \in [\varepsilon_\omega, \infty) \cap \mathrm{Lim}$,

$$A(t) := \begin{cases} (\mathrm{Lim}\mathbb{E}) \cap (M, \alpha+1) & \textit{iff} \quad t \in [\alpha, \alpha 2] \\ \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in I \cap r} A(l_j)\} & \textit{iff} \quad t > \pi t + d\pi t \wedge t \in (\alpha 2, \alpha^+) \\ \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in S \cap r} A(e_j)\} & \textit{iff} \quad t \leqslant \pi t + d\pi t \wedge t \in (\alpha 2, \alpha^+) \end{cases},$$

where $\alpha \in \mathbb{E}$ *is such that* $t \in [\alpha, \alpha^+)$; $(l_j)_{j \in I}$ *is obtained by lemma 2.42 applied to* t *and* α; $(e_j)_{j \in S}$ *is obtained by lemma 2.42 applied to* πt *and* α; *and* $M := \begin{cases} \max(\mathrm{Ep}(t) \cap \alpha) & \textit{iff} \ \mathrm{Ep}(t) \cap \alpha \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$.

On the other hand, we define G: $[\varepsilon_\omega, \infty) \longrightarrow \mathrm{Subclases(OR)}$ *in the following way: Consider* $t \in [\varepsilon_\omega, \infty)$ *and* $\alpha \in \mathbb{E}$ *such that* $t \in [\alpha, \alpha^+)$. *Let*

$$\begin{aligned} G(t) &:= \{\beta \in \mathrm{OR} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \ \textit{and} \ \beta \leqslant^1 (\eta t)[\alpha := \beta] + 1\} \underset{\textit{theorem 2.45}}{=} \\ &= \{\beta \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \ \textit{and} \ \beta \leqslant^1 (\eta t)[\alpha := \beta] + 1\} \underset{\textit{proposition 2.38 and corollary 2.35}}{=} \\ &= \{\beta \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \ \textit{and} \ \beta \leqslant_1 (\eta t)[\alpha := \beta] + 1\}. \end{aligned}$$

Theorem 2.45.

1. $\forall t \in [\varepsilon_\omega, \infty). G(t) \subset \mathrm{Lim}\mathbb{E}$
2. $\forall t \in [\varepsilon_\omega, \infty). G(t) = A(t)$

Proof.
1.
First observe the following. Let $t \in [\varepsilon_\omega, \infty)$ and $\alpha \in \mathbb{E}$ such that $t \in [\alpha, \alpha^+)$ and take $\beta \in G(t)$; so $\beta \leqslant^1 (\eta t)[\alpha := \beta] + 1$. Since by proposition 2.26 $\alpha 2 \leqslant \eta t$, then $\beta < \beta 2 \leqslant (\eta t)[\alpha := \beta]$ and so $\beta \leqslant^1 \beta 2 + 1$ by $\leqslant^1$-connectedness. So $\beta \in \mathrm{Lim}\mathbb{E}$ by proposition 2.43.

2.
We first prove the following easy case: Let $\alpha \in [\varepsilon_\omega, \infty) \cap \mathbb{E}$. Consider $t \in [\alpha, \alpha 2]$. Then $\pi t + d\pi t = \alpha 2$ and so $\eta t = \max\{t, \alpha 2\} = \alpha 2$. Then
$G(t) = \{\beta \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^1 \eta t[\alpha := \beta] + 1\} =$
$= \{\beta \in \mathbb{E} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^1 \alpha 2[\alpha := \beta] + 1\} =$
$= \{\beta \in \mathbb{E} | \beta \leqslant^1 \beta 2 + 1\} \cap (\max(\mathrm{Ep}(t) \cap \alpha), \alpha + 1) \underset{\text{prop. 2.43}}{=} \mathrm{Lim}\mathbb{E} \cap (\max(\mathrm{Ep}(t) \cap \alpha), \alpha + 1)$.

On the other hand, we prove by induction that for $t \in [\alpha, \alpha 2]$, $A(t) = \mathrm{Lim}\mathbb{E} \cap (\max(\mathrm{Ep}(t) \cap \alpha), \alpha + 1)$. For $t \in \mathrm{Lim}$ it is clear. So suppose $t = l + 1$ is a successor. Then $l < l + 1 < \alpha 2 = \pi l + d\pi l$, and so $A(t) = A(l + 1) = A(l) = \mathrm{Lim}\mathbb{E} \cap (\max(\mathrm{Ep}(t) \cap \alpha), \alpha + 1)$, where the last equality holds because of the induction hypothesis.

Hence we have shown that $G(t) = A(t) = \mathrm{Lim}\mathbb{E} \cap (\max(\mathrm{Ep}(t) \cap \alpha), \alpha + 1)$ for all $t \in [\alpha, \alpha 2]$, with $\alpha \in [\varepsilon_\omega, \infty) \cap \mathbb{E}$.

Now we proceed to prove that $G(t) = A(t)$ for arbitrary $t \in [\varepsilon_\omega, \infty)$. We proceed by induction on the class $[\varepsilon_\omega, \infty)$.

So let $t \in [\varepsilon_\omega, \infty]$ and $\alpha \in \mathbb{E}$ be such that $t \in [\alpha, \alpha^+)$.

Suppose $\forall l \in [\varepsilon_\omega, \infty) \cap t. A(l) = G(l)$. **(IH)**

Successor case.
Suppose $t = l+1$.

Subcase $l < \pi l + d\pi l$. Then $\eta(l+1) = \max\{l+1, \pi(l+1) + d\pi(l+1)\} =$, proposition 2.26,
$= \max\{l+1, \pi l + d\pi l\} = \pi l + d\pi l = \max\{l, \pi l + d\pi l\} = \eta l$.
Thus $G(t) = G(l+1) = \{\beta \in \mathbb{E} | \mathrm{Ep}(l+1) \cap \alpha \subset \beta \wedge \beta \leqslant^1 (\eta(l+1))[\alpha := \beta] + 1\} =$
$= \{\beta \in \mathbb{E} | \mathrm{Ep}(l) \cap \alpha \subset \beta \wedge \beta \leqslant^1 (\eta l)[\alpha := \beta] + 1\} = G(l) \underset{\mathrm{IH}}{=} A(l) = A(l+1) = A(t)$.

Subcase $l \geqslant \pi l + d\pi l$.
Let's see $G(t) = G(l+1) \subset A(l+1) = A(t)$.
Take $\beta \in G(t)$; so $\alpha \geqslant \beta \leqslant^1 (\eta(l+1))[\alpha := \beta] + 1$ and $(\eta(l+1))[\alpha := \beta] \in [\beta, \beta^+)$. So, by proposition 2.23, there is a sequence $(c_\xi)_{\xi \in X}$, such that $c_\xi \in \mathbb{E}$, $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta \leqslant \alpha$,
$\mathrm{Ep}((\eta(l+1))[\alpha := \beta]) \cap \beta \subset c_\xi$ and $c_\xi \leqslant_1 (\eta(l+1))[\alpha := \beta][\beta := c_\xi] = (\eta(l+1))[\alpha := c_\xi] =$
$(\eta l + 1)[\alpha := c_\xi] = (\eta l)[\alpha := c_\xi] + 1$ where the last two equalities hold because
$\mathrm{Ep}(\eta(l+1)) \cap \alpha = \mathrm{Ep}((\eta(l+1))[\alpha := \beta]) \cap \beta \subset c_\xi$ (and then $(\mathrm{Ep}(l) \cap \alpha) \cup (\mathrm{Ep}(\eta l) \cap \alpha) \subset c_\xi$) and because $\eta(l+1) = \max\{l+1, \pi(l+1) + d\pi(l+1)\} =$, by proposition 2.26,
$= \max\{l+1, \pi l + d\pi l\} = l+1 = \eta l + 1$.
Now, by proposition 2.38, $(\eta l)[\alpha := c_\xi] = \eta(l[\alpha := c_\xi])$ so we have $c_\xi \leqslant_1 \eta(l[\alpha := c_\xi]) + 1$; moreover, this holds iff, by corollary 2.35, $c_\xi \leqslant^1 \eta(l[\alpha := c_\xi]) + 1 = (\eta l)[\alpha := c_\xi]) + 1$. This way, we have actually shown that $c_\xi \in G(l)$ for all $\xi \in X$; therefore $\beta \in \mathrm{Lim}\, G(l) \underset{\mathrm{IH}}{=} \mathrm{Lim}\, A(l) = A(l+1) = A(t)$. This shows $G(t) \subset A(t)$.
Let's see $G(t) = G(l+1) \supset A(l+1) = A(t)$.
Let $\beta \in A(t) = A(l+1) = \mathrm{LimA}(l) \underset{\mathrm{by\,IH}}{=} \mathrm{LimG}(l)$. Then there exists a sequence $(c_\xi)_{\xi \in X}$, with $c_\xi \in G(l)$ and $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$; i.e., for all $\xi \in X$ it also holds $\mathrm{Ep}(l) \cap \alpha \subset c_\xi \leqslant \alpha$, $c_\xi \in \mathbb{E}$ and
$c_\xi \leqslant^1 (\eta l)[\alpha := c_\xi] + 1 = (\eta l + 1)[\alpha := c_\xi] = (\eta l + 1)[\alpha := \beta][\beta := c_\xi]$. It is easy to see that
$(\mathrm{Ep}(l+1) \cap \alpha) \cup (\mathrm{Ep}(\eta l + 1) \cap \alpha) \subset \beta$ and that the last equality hold; the reason to introduce them is the following: from all the previous we have $\beta \in \mathbb{E}$, $(\eta l + 1)[\alpha := \beta] \in [\beta, \beta^+)$, $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta \leqslant \alpha$, $\forall \xi \in X. c_\xi \in \mathbb{E} \wedge \mathrm{Ep}((\eta l + 1)[\alpha := \beta]) \cap \beta \subset c_\xi$ and $c_\xi \leqslant_1 (\eta l + 1)[\alpha := \beta][\beta := c_\xi]$. Therefore, applying proposition 2.24, $\beta \leqslant_1 = (\eta l + 1)[\alpha := \beta] + 1 = (\eta(l+1))[\alpha := \beta] + 1 = \eta((l+1)[\alpha := \beta]) + 1$, where the last equalities hold because of proposition 2.38. From this, and corollary 2.35 we get $\beta \leqslant^1 \eta((l+1)[\alpha := \beta]) + 1 = (\eta(l+1))[\alpha := \beta] + 1$. So we have shown $\beta \in G(l+1) = G(t)$.

Limit case.
Suppose $t \in \mathrm{Lim}$. Moreover, since we have already proved what happens for $t \in [\alpha, \alpha 2]$, then suppose $t \in (\alpha 2, \alpha^+)$.

Subcase $t > \pi t + d\pi t$.

To show $G(t) \subset A(t)$.
Let $\beta \in G(t)$. So $\alpha \geqslant \beta \leqslant^1 (\eta t)[\alpha := \beta] + 1 = t[\alpha := \beta] + 1$. Then by proposition 2.23 there exists a sequence $(c_\xi)_{\xi \in X}$ such that $\mathrm{Ep}(t) \cap \alpha = \mathrm{Ep}(t[\alpha := \beta]) \cap \beta \subset c_\xi$ (so $c_\xi > 1$), $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$ and $c_\xi \leqslant_1 t[\alpha := \beta][\beta := c_\xi] = t[\alpha := c_\xi]$.
On the other hand, by lemma 2.42 we know that for the sequence $(l_j)_{j \in I}$, it holds:
- $I \cup \{0\} \leqslant \alpha$
- $(l_j[\alpha := c_\xi])_{j \in I \cap c_\xi}$ is cofinal in $t[\alpha := c_\xi]$ and
- For any $j \in I \cap c_\xi$, $\eta l_j[\alpha := c_\xi] < \eta t[\alpha := c_\xi]$.

Therefore, for any $\xi \in X$ and for any $j \in I \cap c_\xi$, $\alpha \geqslant c_\xi \leqslant l_j[\alpha := c_\xi] + 1 \leqslant \eta l_j[\alpha := c_\xi] + 1 \leqslant$
$\eta t[\alpha := c_\xi] = t[\alpha := c_\xi]$, which implies, by $\leqslant_1$-connectedness, $\forall j \in I \cap c_\xi$, $c_\xi \leqslant_1 \eta l_j[\alpha := c_\xi] + 1$. Then, by corollary 2.35 we obtain $\forall j \in I \cap c_\xi$, $c_\xi \leqslant^1 \eta l_j[\alpha := c_\xi] + 1$.
The previous shows $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$, and $\mathrm{Ep}(t) \cap \alpha \subset c_\xi \in$ $\bigcap_{j \in I \cap c_\xi} G(l_j) \underset{\mathrm{IH}}{=} \bigcap_{j \in I \cap c_\xi} A(l_j)$. Thus $\beta \in \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in I \cap r} A(l_j)\} = A(t)$.

To show $G(t) \supset A(t)$.

Let $\beta \in A(t) = \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in I \cap r} A(l_j)\}$. Since we know $l_j < t$ for any $j \in I$, then $A(l_j) = G(l_j)$ for any $j \in I \cap r$ by induction hypothesis. This way $\beta \in \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in I \cap r} G(l_j)\}$, which means there exists a sequence $(c_\xi)_{\xi \in X}$, such that $\alpha \geqslant c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$ and $\mathrm{Ep}(t) \cap \alpha \subset c_\xi \in \bigcap_{j \in I \cap c_\xi} G(l_j)$; i.e., $\forall j \in I \cap c_\xi$, $c_\xi \leqslant^1 \eta l_j[\alpha := c_\xi] + 1$. This way, for any $\xi \in X$ and any $j \in I \cap c_\xi$, $c_\xi \leqslant^1 l_j[\alpha := c_\xi]$ by $<^1$-connectedness (because from $\alpha < l_j \leqslant \eta l_j$ follows $c_\xi < l_j[\alpha := c_\xi] \leqslant \eta l_j[\alpha := c_\xi] < \eta l_j[\alpha := c_\xi] + 1$). But by lemma 2.42, $(l_j[\alpha := c_\xi])_{j \in I \cap c_\xi}$ is cofinal in $t[\alpha := c_\xi]$; therefore $\forall \xi \in X. c_\xi \leqslant^1 t[\alpha := c_\xi]$ by $\leqslant^1$-continuity. (*)

On the other hand, $\forall \xi \in X. c_\xi \in \mathbb{E}$ (because $c_\xi \in G(l_1) \underset{\text{by 1.}}{\subset} \mathrm{Lim}\mathbb{E}$), and since $\mathrm{Ep}(t) \cap \alpha \subset c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$, then $\mathrm{Ep}(t) \cap \alpha \subset \beta \in \mathbb{E}$. So $t[\alpha := \beta] \in [\beta, \beta^+)$ and $\mathrm{Ep}(t) \cap \alpha = \mathrm{Ep}(t[\alpha := \beta]) \cap \beta$. From all this and the fact that (*) implies $\forall \xi \in X. c_\xi \leqslant_1 t[\alpha := c_\xi] = t[\alpha := \beta][\beta := c_\xi]$, we conclude $\beta \in \mathrm{Lim}\{\gamma \in \mathbb{E} | \mathrm{Ep}(t[\alpha := \beta]) \cap \beta \subset \gamma \wedge \gamma \leqslant_1 t[\alpha := \beta][\beta := \gamma]\}$. This implies, by proposition 2.24, $\beta \leqslant_1 t[\alpha := \beta] + 1 = \eta t[\alpha := \beta] + 1$ and subsequently, by corollary 2.35, $\beta \leqslant^1 \eta t[\alpha := \beta] + 1$. So $\beta \in G(t)$.

All the previous shows $G(t) = A(t)$ for the subcase $t > \pi t + d\pi t$.

Subcase $t \leqslant \pi t + d\pi t$.

Write $t =_{\mathrm{CNF}} \omega^{T_1} t_1 + \ldots + \omega^{T_n} t_n$ and $T_1 =_{\mathrm{CNF}} \omega^{Q_1} q_1 + \ldots + \omega^{Q_m} q_m$. Note $Q_m < T_1$ and then $\omega^{Q_m} < \omega^{T_1}$ (otherwise $T_1 = Q_m \leqslant \omega^{Q_m} \leqslant T_1$ and then $T_1 = Q_m \in \mathbb{E}$; from this and the fact that $t \in (\alpha 2, \alpha^+)$ follows that $T_1 = \alpha$, but then $t \leqslant \omega^\alpha + \alpha = \alpha 2$, which is contradictory with our supposition $t \in (\alpha 2, \alpha^+)$). The previous also shows $T_1 > \alpha$. This way, the inequalities $t \leqslant \pi t + d\pi t = \omega^{T_1} + Q_m$ and $Q_m < T_1$ imply that t looks like $t =_{\mathrm{CNF}} \omega^{T_1} + \omega^{T_2} t_2 \ldots + \omega^{T_n} t_n$ with $\omega^{T_2} t_2 \ldots + \omega^{T_n} t_n \leqslant Q_m$, and $T_1 > \alpha$.

Lets show now $G(t) \subset A(t)$.

Let $\beta \in G(t)$. So $\mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant^1 (\eta t)[\alpha := \beta] + 1$ and $\alpha \geqslant \beta \in \mathbb{E}$. Then, by proposition 2.23, there is a sequence $(c_\xi)_{\xi \in X}$, such that $c_\xi \in \mathbb{E}$, $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$, $\mathrm{Ep}(\eta t) \cap \alpha = \mathrm{Ep}(\eta t[\alpha := \beta]) \cap \beta \subset c_\xi$, and $c_\xi \leqslant_1 \eta t[\alpha := \beta][\beta := c_\xi] = \eta t[\alpha := c_\xi] = \eta \pi t[\alpha := c_\xi]$.

We now need to remember how the sequence $(e_j)_{j \in S}$ is defined. Consider the ordinal ω^{Q_m}. If $Q_m > \alpha$, let $(a_j)_{j \in K}$, with $K \leqslant \alpha$ be the sequence obtained by lemma 2.42 applied to ω^{Q_m}. If $0 \neq Q_m \leqslant \alpha$ let $K := \omega^{Q_m} \backslash \{0\}$ and $a_j := j$ for any $j \in K$. Then

$$e_j = \begin{cases} \omega^{\omega^{Q_1} q_1 + \ldots + \omega^{Q(m-1)} q_{(m-1)} + q_m - 1} j, \text{ with } j \in S := \omega \backslash \{0\} & \text{iff } Q_m = 0 \\ \omega^{\omega^{Q_1} q_1 + \ldots + \omega^{Q(m-1)} q_{(m-1)} + \omega^{Q_m} (q_m - 1) + a_j}, \text{ with } j \in S := K & \text{iff } Q_m \neq 0 \end{cases}.$$

As we know, $(e_j)_{j \in S}$ is cofinal in $\omega^{T_1} = \pi t$. Besides, since $\forall \xi \in X. \mathrm{Ep}(\eta t) \cap \alpha \subset c_\xi$, then $\forall \xi \in X. \mathrm{Ep}(\pi t) \cap \alpha \subset c_\xi$; this way, for any $\xi \in X$,

- for any $j \in S \cap c_\xi$, $\mathrm{Ep}(e_j) \cap \alpha \subset c_\xi$ and
- $(e_j[\alpha := c_\xi])_{j \in S \cap c_\xi}$ is cofinal in $\omega^{T_1}[\alpha := c_\xi]$.

Moreover, notice $\forall j \in S. \eta e_j < \eta t = \eta \pi t$; so $\forall j \in S \cap c_\xi. \eta e_j[\alpha := c_\xi] < \eta \pi t[\alpha := c_\xi]$.

From all our previous work we obtain: $\forall \xi \in X. \forall j \in S \cap c_\xi. c_\xi \leqslant \eta e_j[\alpha := c_\xi] + 1 \leqslant \eta \pi t[\alpha := c_\xi]$, which implies, by $\leqslant_1$-connectedness, $\forall \xi \in X. \forall j \in S \cap c_\xi. c_\xi \leqslant_1 \eta e_j[\alpha := c_\xi] + 1$, which in turn is equivalent (by corollary 2.35) to $\forall \xi \in X. \forall j \in S \cap c_\xi. c_\xi \leqslant^1 \eta e_j[\alpha := c_\xi] + 1$. Finally, since $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta \supset \mathrm{Ep}(t) \cap \alpha$ and $\mathrm{Ep}(t) \cap \alpha$ is a finite set, then there exists $y \subset X$ such that $(c_\xi)_{\xi \in (X \backslash y)} \underset{\mathrm{cof}}{\hookrightarrow} \beta$ and $\forall \xi \in (X \backslash y). \forall j \in S \cap c_\xi. \mathrm{Ep}(t) \cap \alpha \subset c_\xi \leqslant^1 \eta e_j[\alpha := c_\xi] + 1$.

The previous paragraph shows $\forall \xi \in X \backslash y. M < c_\xi \in \bigcap_{j \in S \cap c_\xi} G(e_j) \underset{\mathrm{IH}}{=} \bigcap_{j \in S \cap c_\xi} A(e_j)$ and $(c_\xi)_{\xi \in (X \backslash y)} \underset{\mathrm{cof}}{\hookrightarrow} \beta$; i.e., it shows $\beta \in \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in S \cap r} A(e_j)\} = A(t)$.

To show $G(t) \supset A(t)$.

Let $\beta \in A(t) = \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in S \cap r} A(e_j)\} \underset{\mathrm{IH}}{=} \mathrm{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{j \in S \cap r} G(e_j)\}$. Then there exists a sequence $(c_\xi)_{\xi \in X}$ such that $\alpha \geqslant c_\xi \leqslant^1 \eta e_j[\alpha := c_\xi] + 1$ for all $j \in S \cap c_\xi$ and $M < c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$. Of course, the last inequality means $\mathrm{Ep}(t) \cap \alpha \subset c_\xi$, which implies $\mathrm{Ep}(\pi t) \cap \alpha \subset c_\xi$.

Now, noting that $\forall j \in S \cap c_\xi . c_\xi \leqslant e_j[\alpha := c_\xi] < e_j[\alpha := c_\xi] + 1 \leqslant \eta e_j[\alpha := c_\xi] + 1$, we obtain by $\leqslant^1$-connectedness $c_\xi \leqslant^1 e_j[\alpha := c_\xi]$ for all $j \in S \cap c_\xi$. But the fact that $\forall \xi \in X . \mathrm{Ep}(\pi t) \cap \alpha \subset c_\xi$ implies (by lemma 2.42) that the sequence $(e_j[\alpha := c_\xi])_{j \in S \cap c_\xi}$ is cofinal in $\pi t[\alpha := c_\xi]$ for any $\xi \in X$, and so we conclude $\forall \xi \in X . c_\xi \leqslant^1 \pi t[\alpha := c_\xi]$ by $\leqslant^1$-continuity.

From the work done in the previous paragraph follows immediately that $\forall \xi \in X . c_\xi \leqslant_1 \pi t[\alpha := c_\xi]$; but $\pi t[\alpha := c_\xi] \leqslant_1 \pi t[\alpha := c_\xi] + d\pi t[\alpha := c_\xi]$ by theorem 2.3; thus by $\leqslant_1$ transitivity we conclude $\forall \xi \in X . c_\xi \leqslant_1 \pi t[\alpha := c_\xi] + d\pi t[\alpha := c_\xi] = (\pi t + d\pi t)[\alpha := c_\xi] = \eta t[\alpha := c_\xi]$, where the last two equalities hold by proposition 2.38. Finally applying proposition 2.24 to $\forall \xi \in X . c_\xi \leqslant_1 \eta t[\alpha := c_\xi]$ and to $M < c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \beta$, and using the fact that $\forall \xi \in X . c_\xi \leqslant \alpha$, we conclude $\mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha$ and $\beta \leqslant_1 \eta t[\alpha := \beta] + 1$. Observe the latter is equivalent (by corollary 2.35) to $\beta \leqslant^1 \eta t[\alpha := \beta] + 1$. Thus $\beta \in G(t)$. ☐

2.4.1 Uncountable regular ordinals and the $A(t)$ sets

Up to this moment we have shown that the sets $A(t)$ consists of the ordinals that are "solutions of certain $<_1$-inequalities of the form $x <_1 t$ with $t \in [x, x^+)$", but we still do not know whether these solutions indeed exist. We address this problem now: our purpose is to study closer the $A(t)$ sets and, very specifically, by the introduction of an uncountable regular ordinal κ, show that for $t \in [\kappa, \kappa^+)$, the $A(t)$ sets have to have elements.

In the following we will use the next two propositions.

Proposition 2.46. *Let κ be an uncountable regular ordinal and let X be a class of ordinals that are club in κ. Then* $\mathrm{Lim}\, X$ *is club in κ.*

Proof. Known result about club classes. ☐

Proposition 2.47. *Let κ be an uncountable regular ordinal and let $(X_i)_{i<I}$ be a sequence of classes of ordinals that are club in κ.*

- *If $|I| < \kappa$, then $\bigcap\limits_{i<I} X_i$ is club in κ.*
- *Suppose $I = \kappa$. Then $\{\xi < \kappa | \xi \in \bigcap\limits_{i<\xi} X_i\}$ is club in κ.*

Proof. Known result about club classes. ☐

Proposition 2.48. *Let κ be an uncountable regular ordinal. Then $\forall t \in [\kappa, \kappa^+)$, $A(t)$ is club in κ.*

Proof. We prove the claim by induction on the interval $[\kappa, \kappa^+)$.

Case $t = \kappa$.

Then $A(t) = (\mathrm{Lim}\mathbb{E}) \cap (0, \kappa + 1)$ is club in κ because $\mathbb{E}$ is club in κ and by proposition 2.46.

Our induction hypothesis is $\forall s < t . A(s)$ is club in κ. **(IH)**

Case $t = l + 1 \in [\kappa, \kappa^+)$.

Then $A(t) = A(l+1) = \begin{cases} A(l) & \text{if } l < \pi l + d\pi l \\ \mathrm{Lim} A(l) & \text{otherwise} \end{cases}$; this way, by our IH and proposition 2.46, $A(t)$ is club in κ in any case.

Case $t \in [\kappa, \kappa^+) \cap \mathrm{Lim}$.

Let $(l_j)_{j\in I}$ be the sequence obtained by the application of lemma 2.42 to t and κ and $(e_j)_{j\in S}$ the sequence obtained by the application of lemma 2.42 to πt and κ. Moreover, in case $\mathrm{Ep}(t) \cap \alpha \neq \emptyset$, let $M := \max \mathrm{Ep}(t) \cap \alpha$; in case $\mathrm{Ep}(t) \cap \alpha = \emptyset$, let $M := 0$. Then by definition

$$A(t) = \begin{cases} (\mathrm{Lim}\mathbb{E}) \cap (M, \kappa+1) & \text{iff} \quad t \in [\kappa, \kappa 2] \\ \mathrm{Lim}\{r \leqslant \kappa | M < r \in \bigcap_{j \in I \cap r} A(l_j)\} & \text{iff } t > \pi t + d\pi t \wedge t \in (\kappa 2, \kappa^+) \\ \mathrm{Lim}\{r \leqslant \kappa | M < r \in \bigcap_{j \in S \cap r} A(e_j)\} & \text{iff} \quad t \leqslant \pi t + d\pi t \wedge t \in (\kappa 2, \kappa^+) \end{cases}$$

and we have some subcases:

If $t \in [\kappa, \kappa 2]$, then $A(t) = (\mathrm{Lim}\mathbb{E}) \cap (M, \kappa+1)$ is club in κ because $\mathbb{E}$ is club in κ and because of proposition 2.46.

Subcase $t > \pi t + d\pi t \wedge t \in (\kappa 2, \kappa^+)$.

Note it is enough to show that $Y := \{r \leqslant \kappa | M < r \in \bigcap_{j \in I \cap r} A(l_j)\}$ is club in κ because knowing this we conclude $\mathrm{Lim} Y$ is club in κ by proposition 2.46. In order to see that Y is club in κ, we define for any $i \in \kappa$, $X_i := \begin{cases} A(l_i) & \text{iff } i \in I \\ A(l_1) & \text{iff } i \notin I \end{cases}$. Since by lemma 2.42, $l_j < t$ for any $j \in I$, then by our IH we have that X_i is club in κ for any $i \in \kappa$; consequently, by proposition 2.47, the set
$X := \{\xi < \kappa | \xi \in \bigcap_{i<\xi} X_i\}$ is club in κ.

We now show $Y \cap \kappa = X \backslash (M+1)$.

$''\supset''$. Let $r \in X \backslash (M+1)$. Then $M + 1 \leqslant r < \kappa$ and $r \in \bigcap_{i<r} X_i \subset \bigcap_{i \in I \cap r} X_i = \bigcap_{i \in I \cap r} A(l_i)$. This shows $r \in Y \cap \kappa$.

$''\subset''$. Let $r \in Y \cap \kappa$. Then $M < r < \kappa$ and $r \in \bigcap_{i \in I \cap r} A(l_i) = \bigcap_{i \in I \cap r} X_i = \bigcap_{i \in I \cap r} X_i \cap X_1 =$
$= \bigcap_{i \in I \cap r} X_i \cap \bigcap_{i \in (r \backslash I)} X_i = \bigcap_{i<r} X_i$. So $r \in X \backslash (M+1)$.

Hence, since $Y \cap \kappa = X \backslash (M+1)$ and X is club in κ, then $Y = \begin{cases} Y \cap \kappa \cup \{\kappa\} & \text{iff } \kappa \in Y \\ Y \cap \kappa & \text{otherwise} \end{cases}$ is also club in κ.

Subcase $t \leqslant \pi t + d\pi t \wedge t \in (\kappa 2, \kappa^+)$.

It is enough to show that $Z := \{r \leqslant \kappa | M < r \in \bigcap_{j \in S \cap r} A(e_j)\}$ is club in κ because of the same reasons of the previous subcase. For any $i \in \kappa$, let $W_i := \begin{cases} A(e_i) & \text{iff } i \in S \\ A(e_1) & \text{iff } i \notin S \end{cases}$. Since by lemma 2.42, $e_j < \pi t \leqslant t$ for any $j \in S$, then by our IH we have that W_i is club in κ for any $i \in \kappa$; consequently, by proposition 2.47, the set $W := \{\xi < \kappa | \xi \in \bigcap_{i<\xi} W_i\}$ is club in κ.

We show $Z \cap \kappa = W \backslash (M+1)$.

$''\supset''$. Let $r \in W \backslash (M+1)$. Then $M + 1 \leqslant r < \kappa$ and $r \in \bigcap_{i<r} W_i \subset \bigcap_{i \in S \cap r} W_i = \bigcap_{i \in S \cap r} A(e_i)$. From this we conclude $r \in Z \cap \kappa$.

$''\subset''$. Let $r \in Z \cap \kappa$. Then $M < r < \kappa$ and $r \in \bigcap_{i \in S \cap r} A(e_i) = \bigcap_{i \in S \cap r} W_i = \bigcap_{i \in S \cap r} W_i \cap W_1 =$
$= \bigcap_{i \in S \cap r} W_i \cap \bigcap_{i \in (r \backslash S)} W_i = \bigcap_{i<r} W_i$. So $r \in W \backslash (M+1)$.

Therefore, since $Z \cap \kappa = W \backslash (M+1)$ and W is club in κ, then $Z = \begin{cases} Z \cap \kappa \cup \{\kappa\} & \text{iff } \kappa \in Z \\ Z \cap \kappa & \text{otherwise} \end{cases}$ is club in κ. □

Consider an epsilon number $\alpha \in \mathbb{E}$, $t \in [\alpha, \alpha^+)$ and a non-countable regular ordinal $\kappa > \alpha$. The "solutions to the $<_1$-inequality $x <_1 \eta t[\alpha := x] + 1$ in interval $[0, \kappa]$" are the same as the "solutions to the $<_1$-inequality $x <_1 \eta t[\alpha := \kappa][\kappa := x] + 1$ in interval $[0, \kappa]$", which are, of course, the elements of the set $G(t[\alpha := \kappa]) \underset{\text{theorem 2.45}}{=} A(t[\alpha := \kappa])$. This way, proposition 2.48 shows us that such solutions are indeed many: $G(t[\alpha := \kappa])$ is club in κ. So our hierarchy $A(l)_{l \in [\kappa, \kappa^+)}$ captures all these solutions (in interval $[0, \kappa]$) and such solutions do exist. Now we just want to make explicit that we get a similar result for arbitrary " $<_1$-inequality $x <_1 t[\alpha := x]$".

Proposition 2.49. *Let κ be an uncountable regular ordinal and $\alpha \in \mathbb{E} \cap \kappa$. Then for any $t \in [\alpha 2, \alpha^+)$, there are $\gamma \in \mathbb{E} \cap \kappa$ and $s \in [\gamma 2, \gamma^+)$ such that*

$$\{\beta \in \mathbb{E} \mid \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \kappa \wedge \beta \leqslant_1 t[\alpha := \beta]\} = \begin{cases} \mathbb{E} \cap (\kappa + 1) & \textit{iff } t = \alpha 2 \\ [\gamma, \kappa] \cap \bigcap_{\xi \in [\gamma 2, s)} A(\xi[\gamma := \kappa]) & \textit{iff } t \in (\alpha 2, \alpha^+) \end{cases}$$

and the set $\{\beta \in \mathbb{E} \mid \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \kappa \wedge \beta \leqslant_1 t[\alpha := \beta]\}$ is club in κ.

Proof. Let κ and $\alpha \in \mathbb{E} \cap \kappa$ be as stated. Take $t \in [\alpha 2, \alpha^+)$ and consider $\gamma := \min\{e \in \mathbb{E} \cap \kappa \mid \mathrm{Ep}(t) \cap \alpha \subset e\}$ (γ exists because $\mathrm{Ep}(t) \cap \alpha \subset \alpha < \kappa$). Then $t[\alpha := \gamma] \in [\gamma 2, \gamma^+)$ and $C := \{\beta \in \mathbb{E} \mid \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \kappa \wedge \beta \leqslant_1 t[\alpha := \beta]\} =$
$= \{\beta \in \mathbb{E} \mid \mathrm{Ep}(t[\alpha := \gamma]) \cap \gamma \subset \beta \leqslant \kappa \wedge \beta \leqslant_1 t[\alpha := \gamma][\gamma := \beta]\}$.

We have two cases:

- Case $t[\alpha := \gamma] = \gamma 2$. Then $C = \mathbb{E} \cap (\kappa + 1)$ is (clearly) club in κ.
- Case $t[\alpha := \gamma] \in (\gamma 2, \gamma^+)$. Let $s := \min\{z \in [\gamma 2, t[\alpha := \gamma]] \mid m(z) \geqslant t[\alpha := \gamma]\}$ (of course s exists because $m(t[\alpha := \gamma]) \geqslant t[\alpha := \gamma]$). Let $Z := \max(\mathrm{Ep}(s) \cap \gamma)$. We show that

$$C = [\gamma, \kappa] \cap \bigcap_{\xi \in [\gamma 2, s)} A(\xi[\gamma := \kappa]) = [\gamma, \kappa] \cap \bigcap_{\xi \in [\gamma 2, s)} G(\xi[\gamma := \kappa]). \qquad (\circ)$$

"$\subset$". Take $\beta \in C$. Then $\beta \geqslant \gamma$ (because $\mathrm{Ep}(t) \cap \alpha \subset \beta$) and
$\mathrm{Ep}(t[\alpha := \gamma]) \cap \gamma \subset \beta \in \mathbb{E} \cap (\kappa + 1) \wedge \beta \leqslant_1 t[\alpha := \gamma][\gamma := \beta]$. **(*)**
On the other hand, let $\xi \in [\gamma 2, s)$ be arbitrary. Then $\mathrm{Ep}(\eta \xi + 1) \cap \gamma \subset \gamma \leqslant \beta$, and then, using that $\eta \xi + 1 \underset{\text{proposition 2.27}}{=} \max\{m(\alpha) \mid a \in (\alpha, \xi]\} + 1 \leqslant t[\alpha := \gamma]$ we get
$\beta \leqslant (\eta \xi + 1)[\gamma := \beta] \leqslant t[\alpha := \gamma][\gamma := \beta]$. This, (*) and $\leqslant_1$-connectedness imply that
$\beta \leqslant_1 (\eta \xi + 1)[\gamma := \beta] = \eta \xi[\gamma := \beta] + 1$.

Our previous work has shown that $\beta \geqslant \gamma$ and that
$\beta \in \{e \in \mathbb{E} | \mathrm{Ep}(\xi) \cap \gamma \subset e \leqslant \kappa \text{ and } e \leqslant_1 \eta \xi[\gamma := e] + 1\} =$
$\{e \in \mathbb{E} | \mathrm{Ep}(\xi[\gamma := \kappa]) \cap \kappa \subset e \leqslant \kappa \text{ and } e \leqslant_1 (\eta \xi[\gamma := \kappa])[\kappa := e] + 1\} \underset{\text{theorem 2.45}}{=} A(\xi[\gamma := \kappa])$.
Since this was done for arbitrary $\xi \in [\gamma 2, s)$, we have shown $\beta \in [\gamma, \kappa] \cap \bigcap_{\xi \in [\gamma 2, s)} A(\xi[\gamma := \kappa])$.

"$\supset$". Take $\beta \in [\gamma, \kappa] \cap \bigcap_{\xi \in [\gamma 2, s)} A(\xi[\gamma := \kappa])$. Then for any $\xi \in [\gamma 2, s)$
$\mathrm{Ep}(\xi[\gamma := \alpha]) \cap \alpha = \mathrm{Ep}(\xi) \cap \gamma = \mathrm{Ep}(\xi[\gamma := \kappa]) \cap \kappa \subset \beta \leqslant \kappa$ and
$\beta \leqslant_1 (\eta \xi[\gamma := \kappa])[\kappa := \beta] + 1 = \eta \xi[\gamma := \beta] + 1 = \eta \xi[\gamma := \alpha][\alpha := \beta] + 1$. **(**)**

Subcase $s \in \mathrm{Lim}$. Since by (**) we have that $\forall [\gamma 2, s). \beta \leqslant_1 \eta \xi[\gamma := \beta] + 1$, then by $\leqslant_1$-connectedness $\forall \xi \in [\gamma 2, s). \beta \leqslant_1 \xi[\gamma := \beta]$; but $(\xi[\gamma := \beta])_{[\gamma 2, s)} \underset{\text{cof}}{\hookrightarrow} s[\gamma := \beta]$, thus, by $\leqslant_1$-continuity, $\beta \leqslant_1 s[\gamma := \beta]$. **(***)**. On the other hand, the inequalities $s[\gamma := \beta] \leqslant t[\alpha := \gamma][\gamma := \beta] \leqslant m(s)[\gamma := \beta] \underset{\text{corollary 2.17}}{=} m(s[\gamma := \beta])$ imply, by $\leqslant_1$-connectedness, that $s[\gamma := \beta] \leqslant_1 t[\alpha := \gamma][\gamma := \beta]$; from this, (***) and $\leqslant_1$-transitivity we conclude $\beta \leqslant_1 t[\alpha := \gamma][\gamma := \beta] = t[\alpha := \beta]$. Hence $\beta \in C$.

Subcase $s=l+1$ for some $l\in\mathrm{OR}$. Then $t[\alpha:=\gamma]\geqslant s=m(s)\geqslant t[\alpha:=\gamma]$, that is, $l+1=s=t[\alpha:=\gamma]$. On the other hand, $l\leqslant \eta l=\max\{m(a)\mid a\in(\alpha,l]\}<m(s)=s$, so $\eta l=l$ and from all this we conclude $t[\alpha:=\gamma]=s=\eta l+1$. But by hypothesis $\beta\in A(l[\gamma:=\kappa])$, then $\beta\leqslant_1\eta l[\gamma:=\kappa][\kappa:=\beta]+1=\eta l[\gamma:=\beta]+1=(\eta l+1)[\gamma:=\beta]=s[\gamma:=\beta]=t[\alpha:=\gamma][\gamma:=\beta]$. Hence $\beta\in C$.

The previous concludes the proof of ($\circ$).

Finally, since $|[\gamma 2,s)|\leqslant|s|<\kappa$ then $\bigcap\limits_{\xi\in[\gamma 2,s)} A(\xi[\gamma:=\kappa])$ is club in κ (by proposition 2.47) and therefore $C \underset{\text{by }(\circ)}{=} [\gamma,\kappa]\cap\bigcap\limits_{\xi\in[\gamma 2,s)} A(\xi[\gamma:=\kappa])$ is club in κ too. □

2.4.2 Epsilon numbers α satisfying $\alpha<_1\alpha^+$. Class(2).

We comment on this subsection after the next

Corollary 2.50. *Let κ be an uncountable regular ordinal. Then*

a) $\kappa<^1\kappa^+$

b) $\kappa\in\bigcap\limits_{t\in[\kappa,\kappa^+)} A(t)$

Proof.
a).
By proposition 2.48, for any $t\in[\kappa,\kappa^+)$, $A(t)$ is club in κ. This means there exist a sequence $(c_\xi)_{\xi\in X}$ such that $c_\xi\in A(t)$ and $c_\xi \underset{\text{cof}}{\hookrightarrow} \kappa$. But by theorem 2.45, $A(t)=G(t)=$ $\{\beta\in\mathbb{E}|\mathrm{Ep}(t)\cap\kappa\subset\beta\leqslant\kappa\in\mathbb{E}$ and $\beta\leqslant^1\eta t[\kappa:=\beta]+1\}$ which implies $\forall\xi\in X.c_\xi\leqslant_1\eta t[\kappa:=c_\xi]$. Now, from all this and proposition 2.24 we obtain $\kappa\leqslant^1\eta t+1$. The previous shows that $\forall t\in[\kappa,\kappa^+).\kappa\leqslant^1\eta t+1$, and since the sequence $(\eta t+1)_{\kappa\leqslant t<\kappa^+}$ is cofinal in κ^+, then $\kappa\leqslant^1\kappa^+$ by $<^1$-continuity.

b).
$\kappa<^1\kappa^+$ is equivalent to $\kappa\in\bigcap\limits_{t\in[\kappa,\kappa^+)} A(t)$ by next proposition 2.51. □

We had seen previously that for $\alpha\in\mathbb{E}$ and $t\in[\alpha,\alpha^+)$ arbitrary, the "solutions of the $<_1$-inequality $x<_1 t[\alpha:=x]$ in interval $[0,\kappa]$" can always be given in terms of our hierarchy $A(l)_{l\in[\kappa,\kappa^+)}$. But we can tell even more: Consider $B:=\min\{\beta\in\mathbb{E}|\beta<_1\beta^+\}$ (previous corollary 2.50 guarantees the existence of B). Then corollaries 2.41 and 2.52 provide the big picture of what happens in $B\cap\mathbb{E}$ (indeed, they provide the following characterization of B):

I. For any ordinal $\alpha\in B\cap\mathbb{E}$, $m(\alpha)\in[\alpha,\alpha^+)$ and therefore $m(\alpha)=s[B:=\alpha]$ for some $s\in[B,B^+)$ with $s=\eta s$.

II. For every $s=\eta s\in[B,B^+)$, there are cofinal many ordinals in B with $m(\alpha)=s[B:=\alpha]$.

III. B is the only one ordinal such that I and II hold.

This way, for $\alpha\in B\cap\mathbb{E}$ (note $B\leqslant\kappa$ by previous corollary 2.50) we have:

- Case $m(\alpha)=\alpha 2$. Then $\alpha\in\mathbb{E}\backslash A(m(\alpha)[\alpha:=\kappa])$;
- Case $\alpha 2<m(\alpha)\wedge\exists s\in[\alpha 2,m(\alpha)).\eta s+1\geqslant m(\alpha)$. Let $z:=\min\{s\in[\alpha 2,m(\alpha))\mid\eta s+1\geqslant m(\alpha)\}$. Then $\eta z+1=m(\alpha)$ (otherwise the inequalities $\alpha<z<m(\alpha)<\eta z+1\geqslant m(\alpha)+1$ would imply, by $\leqslant_1$-connectedness and proposition 2.27, that $\alpha<_1 m(\alpha)+1$, which is contradictory). Therefore $\alpha\in A(\eta z[\alpha:=\kappa])\backslash A((\eta z+1)[\alpha:=\kappa])=A(\eta z[\alpha:=\kappa])\backslash A(m(\alpha)[\alpha:=\kappa])$.

- Case $\alpha 2 < m(\alpha) \wedge \forall s \in [\alpha 2, m(\alpha)).\eta s + 1 < m(\alpha)$. Then
  $\alpha \in [\bigcap_{s \in [\alpha 2, m(\alpha))} A(s[\alpha := \kappa])] \backslash A(m(\alpha)[\alpha := \kappa])$.

So our theorems explain us quite well what happens in the segment $[0, B)$, but what about ordinals bigger that B?. Corollary 2.50 showed us, for the first time, that the class of ordinals **Class(2)** $:= \{\alpha \in \mathrm{E} \mid \alpha <_1 \alpha^+\}$ is nonempty. We now focus our attention on those ordinals. Our goals are propositions 2.58 and 2.59 which relate Class(2) with our hierarchies $(A(t))_{t \in [\kappa, \kappa^+)}$, for κ an uncountable regular ordinal.

Proposition 2.51. $\forall \alpha \in \mathrm{OR}.\alpha <_1 \alpha^+ \Longleftrightarrow \alpha <^1 \alpha^+ \Longleftrightarrow \alpha \in \bigcap_{t \in [\alpha, \alpha^+)} A(t)$

Proof. Let $\alpha \in \mathrm{OR}$. We already know $\alpha <_1 \alpha^+ \Longleftrightarrow \alpha <^1 \alpha^+$. We now show $\alpha <^1 \alpha^+ \Longleftrightarrow \alpha \in \bigcap_{t \in [\alpha, \alpha^+)} A(t)$.
$\Longrightarrow$).
Suppose $\alpha <^1 \alpha^+$. Let $t \in [\alpha, \alpha^+)$. Then $\alpha <^1 \eta t[\alpha := \alpha] + 1 = \eta t + 1$ by $<^1$-connectedness. So $\alpha \in G(t) = \{\beta \in \mathrm{OR} | \mathrm{Ep}(t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^1 \eta t[\alpha := \beta] + 1\} \underset{\text{theorem 2.45}}{=} A(t)$. Since this holds for an arbitrary $t \in [\alpha, \alpha^+)$, we have then actually shown $\alpha \in \bigcap_{t \in [\alpha, \alpha^+)} A(t)$.
$\Longleftarrow$).
Suppose $\alpha \in \bigcap_{t \in [\alpha, \alpha^+)} A(t) \underset{\text{theorem 2.45}}{=} \bigcap_{t \in [\alpha, \alpha^+)} G(t)$. Then for any $t \in [\alpha, \alpha^+)$, $\alpha <^1 \eta t[\alpha := \alpha] + 1 = \eta t + 1$, and since $(\eta t + 1)_{\alpha \leqslant t < \alpha^+}$ is cofinal in α^+, then $\alpha \leqslant^1 \alpha^+$ by $<^1$-continuity. □

Corollary 2.52. *Let ρ be an epsilon number such that $\rho <_1 \rho^+$. Then* $\forall t \in [\rho, \rho^+).t = \eta t \Longrightarrow \{\alpha \in \rho \cap \mathbb{E} | \mathrm{Ep}(t) \cap \rho \subset \alpha \wedge m(\alpha) = t[\rho := \alpha]\}$ *is confinal in ρ.*

Proof. Not hard. Left to the reader. □

Proposition 2.53. *Let $\alpha, \beta \in \mathbb{E}$, $t \in (\alpha, \alpha^+) \cap \mathrm{Lim}$, $\beta < \alpha$ and $\mathrm{Ep}(t) \cap \alpha \subset \beta$.
Let $(l_j)_{j \in I}$ be obtained by lemma 2.42 applied to t and α. Since $\mathrm{Ep}(t) \cap \alpha \subset \beta$ and $t \in (\alpha, \alpha^+) \cap \mathrm{Lim}$, then $t[\alpha := \beta] \in (\beta, \beta^+) \cap \mathrm{Lim}$. Then $(l_j[\alpha := \beta])_{j \in I \cap (\beta+1)}$ is the sequence obtained by lemma 2.42 applied to $t[\alpha := \beta]$ and β.*

Proof. Long and boring. Left to the reader. □

Proposition 2.54. *Let $A \subset \mathrm{OR} \ni \alpha$. Then* $\mathrm{Lim}(A) \cap (\alpha + 1) = \mathrm{Lim}(A \cap (\alpha + 1))$

Proof. $''\subset''$. Let $r \in \mathrm{Lim}(A) \cap (\alpha + 1)$. Then there exist a sequence $(c_i)_{i \in I} \subset A$ such that $c_i \underset{\text{cof}}{\hookrightarrow} r \leqslant \alpha$. Then $(c_i)_{i \in I} \subset (\alpha + 1)$. All this means $(c_i)_{i \in I} \subset A \cap (\alpha + 1)$ and $c_i \underset{\text{cof}}{\hookrightarrow} r$, i.e., $r \in \mathrm{Lim}(A \cap (\alpha + 1))$.
$''\supset''$. Let $r \in \mathrm{Lim}(A \cap (\alpha + 1))$. Then there exist a sequence $(c_i)_{i \in I} \subset A \cap (\alpha + 1)$ such that $c_i \underset{\text{cof}}{\hookrightarrow} r$; note that since $(c_i)_{i \in I} \subset (\alpha + 1)$, then $r \leqslant \alpha$. So $r \in \mathrm{Lim}(A) \cap (\alpha + 1)$. □

The next proposition is (much) easier to prove using theorem 2.45 and the properties we already know about the substitutions $t \longmapsto t[\beta := \alpha]$. The reader can do that as an easy exercise. We provide here our original proof.

Proposition 2.55. *Let $\beta \in \mathbb{E}$, $\alpha \in \mathbb{E} \cap \beta$ and $t \in [\beta, \beta^+)$ be such that $\mathrm{Ep}(t) \cap \beta \subset \alpha$. So $t[\beta := \alpha] \in [\alpha, \alpha^+)$. Then $A(t) \cap (\alpha + 1) = A(t[\beta := \alpha])$.*

Proof. Let β and $\alpha \in \mathbb{E} \cap \beta$ be as stated. We will prove by induction on $[\beta, \beta^+)$ the statement $\forall t \in [\beta, \beta^+).\mathrm{Ep}(t) \cap \beta \subset \alpha \Longrightarrow A(t) \cap (\alpha+1) = A(t[\beta := \alpha])$.

Let $t \in [\beta, \beta^+)$. Our induction hypothesis is
$\forall r \in [\beta, \beta^+) \cap t.\mathrm{Ep}(r) \cap \beta \subset \alpha \Longrightarrow A(r) \cap (\alpha+1) = A(r[\beta := \alpha])$ **(IH)**

Suppose $\mathrm{Ep}(t) \cap \beta \subset \alpha$. Let $M := \max(\mathrm{Ep}(t) \cap \alpha)$.

Case $t \in [\beta, \beta 2] \cap \mathbf{Lim}$. Then $A(t) = (\mathrm{Lim}\mathbb{E}) \cap (M, \beta+1)$ and so
$A(t) \cap (\alpha+1) = (\mathrm{Lim}\mathbb{E}) \cap (M, \beta+1) \cap (\alpha+1) = (\mathrm{Lim}\mathbb{E}) \cap (M, \alpha+1) = A(t[\beta := \alpha])$, where the last equality is because $t[\beta := \alpha] \in [\alpha, \alpha 2] \cap \mathrm{Lim}$.

Case $t = l+1 \in [\beta, \beta^+)$. Then clearly $\mathrm{Ep}(l) \cap \beta \subset \alpha$ and so

$$A(l+1) \cap (\alpha+1) = \begin{cases} A(l) \cap (\alpha+1) & \text{iff } l < \pi l + d\pi l \\ (\mathrm{Lim}A(l)) \cap (\alpha+1) & \text{otherwise} \end{cases}$$
$$\underset{\text{prop.2.54}}{=} \begin{cases} A(l) \cap (\alpha+1) & \text{iff } l < \pi l + d\pi l \\ \mathrm{Lim}(A(l) \cap (\alpha+1)) & \text{otherwise} \end{cases}$$
$=$, by our IH and because of proposition 2.54,
$$= \begin{cases} A(l[\beta := \alpha]) & \text{iff } l[\beta := \alpha] < \pi(l[\beta := \alpha]) + d\pi(l[\beta := \alpha]) \\ (\mathrm{Lim}\, A(l[\beta := \alpha])) & \text{otherwise} \end{cases}$$
$$= A(l[\beta := \alpha]+1) = A((l+1)[\beta := \alpha]) = A(t[\beta := \alpha]).$$

Case $t \in (\beta 2, \beta^+) \cap \mathbf{Lim}$. Let $(l_j)_{j \in I}$ be obtained by lemma 2.42 applied to t and β; moreover, let $(e_j)_{j \in S}$ be obtained by lemma 2.42 applied to πt and β.

Subcase $t > \pi t + d\pi t$.

$$A(t) \cap (\alpha+1) = (\mathrm{Lim}\{r \leqslant \beta | M < r \in \textstyle\bigcap_{j \in I \cap r} A(l_j)\}) \cap (\alpha+1) \underset{\text{prop. 2.54}}{=}$$
$$= \mathrm{Lim}(\{r \leqslant \beta | M < r \in \textstyle\bigcap_{j \in I \cap r} A(l_j)\} \cap (\alpha+1)) = \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in I \cap r} A(l_j)\} =$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in I \cap r} (A(l_j) \cap (\alpha+1))\} \underset{\text{IH}}{=}$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in I \cap r} A(l_j[\beta := \alpha])\} =$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in (I \cap (\alpha+1)) \cap r} A(l_j[\beta := \alpha])\} = A(t[\beta := \alpha]),$$

where the last equality holds by proposition 2.53 and because $\mathrm{Ep}(t) \cap \beta \subset \alpha$ implies $\mathrm{Ep}(\pi t + d\pi t) \cap \beta \subset \alpha$ and so $t > \pi t + d\pi t \Longleftrightarrow t[\beta := \alpha] > (\pi t + d\pi t)[\beta := \alpha] = \pi(t[\beta := \alpha]) + d\pi(t[\beta := \alpha])$.

Subcase $t \leqslant \pi t + d\pi t$.

$$A(t) \cap (\alpha+1) = (\mathrm{Lim}\{r \leqslant \beta | M < r \in \textstyle\bigcap_{j \in S \cap r} A(e_j)\}) \cap (\alpha+1) \underset{\text{prop. 2.54}}{=}$$
$$= \mathrm{Lim}(\{r \leqslant \beta | M < r \in \textstyle\bigcap_{j \in S \cap r} A(e_j)\} \cap (\alpha+1)) = \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in S \cap r} A(e_j)\} =$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in S \cap r} (A(e_j) \cap (\alpha+1))\} \underset{\text{IH}}{=}$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in S \cap r} A(e_j[\beta := \alpha])\} =$$
$$= \mathrm{Lim}\{r \leqslant \alpha | M < r \in \textstyle\bigcap_{j \in (S \cap (\alpha+1)) \cap r} A(e_j[\beta := \alpha])\} = A(t[\beta := \alpha]),$$

where the last equality holds by proposition 2.53 (more precisely, since $(e_j)_{j \in S}$ approximates $\pi t \in [\beta, \beta^+)$, then $(e_j[\beta := \alpha])_{S \cap (\alpha+1)}$ approximates $(\pi t)[\beta := \alpha] = \pi(t[\beta := \alpha]) \in [\alpha, \alpha^+)$)), and because $t \leqslant \pi t + d\pi t \Longleftrightarrow t[\beta := \alpha] \leqslant (\pi t + d\pi t)[\beta := \alpha] = \pi(t[\beta := \alpha]) + d\pi(t[\beta := \alpha])$. □

Proposition 2.56. *Let $\beta \in \mathbb{E}$ and $\alpha \in \mathbb{E} \cap \beta$. Consider the set of ordinals* $M(\beta, \alpha) = \{q \in \beta^+ | \mathrm{Ep}(q) \cap \beta \subset \alpha\}$. *Then* $\bigcap_{t \in [\alpha, \alpha^+)} A(t) = (\alpha+1) \cap \bigcap_{t \in M(\beta,\alpha) \cap [\beta, \beta^+)} A(t)$.

Proof. Not hard. Left to the reader. □

Proposition 2.57. *Let $\beta \in \mathbb{E}$, $\alpha \in \mathbb{E} \cap \beta$ and $M(\beta,\alpha) = \{q \in \beta^+ | \mathrm{Ep}(q) \cap \beta \subset \alpha\}$. Then*

a) $\forall \gamma \in \bigcap_{t \in M(\beta,\alpha) \cap [\beta,\beta^+)} A(t).\gamma \geqslant \alpha$.

b) $\bigcap_{t \in [\alpha,\alpha^+)} A(t) = (\alpha+1) \cap \bigcap_{t \in M(\beta,\alpha) \cap [\beta,\beta^+)} A(t)$*; moreover, if* $\bigcap_{t \in [\alpha,\alpha^+)} A(t) \neq \emptyset$ *then* $\bigcap_{t \in [\alpha,\alpha^+)} A(t) = \{\alpha\}$.

Proof. Not hard. Left to the reader. □

Proposition 2.58. *Let $\kappa \in \mathbb{E}$ be an uncountable regular ordinal and $\alpha \in \mathbb{E} \cap \kappa$.*
Suppose $\gamma \in \bigcap_{t \in M(\kappa,\alpha) \cap [\kappa,\kappa^+)} A(t)$. Then $\gamma <^1 \gamma^+$.

Proof. Let $a_0 := \kappa$, $a_{n+1} := \kappa^{a_n}$.
Observe $\forall e \in \mathbb{E}.\{a_n | n \in \omega\} \subset [\kappa,\kappa^+) \cap M(\kappa,e)$. This way, since $\gamma \in \bigcap_{t \in M(\kappa,\alpha) \cap [\kappa,\kappa^+)} A(t)$, then $\gamma \in A(a_n) = G(a_n) = \{\beta \in \mathbb{E} | \mathrm{Ep}(a_n) \cap \kappa \subset \beta \leqslant \kappa \wedge \beta \leqslant^1 \eta a_n[\kappa := \beta] + 1\}$ for any $n \in \omega$. Therefore for any $n \in \omega$, $\gamma \leqslant^1 \eta a_n[\kappa := \gamma] + 1$ and since $\gamma \leqslant a_n[\kappa := \gamma] \leqslant \eta a_n[\kappa := \gamma] + 1$, then we conclude by $\leqslant^1$-connectedness $\forall n \in \omega.\gamma \leqslant^1 a_n[\kappa := \gamma])]$. But the sequence $(a_n[\kappa := \gamma])_{n \in \omega}$ is confinal in γ^+, therefore by $\leqslant^1$-continuity, $\gamma \leqslant^1 \gamma^+$. □

Proposition 2.59. *Let $\kappa \in \mathrm{OR}$ be an uncountable regular ordinal and $\sigma \in \kappa \cap [\varepsilon_\omega, \infty] \cap \mathbb{E}$. Then*

i. $\bigcap_{t \in M(\kappa,\sigma) \cap [\kappa,\kappa^+)} A(t)$ *is club in κ.*

ii. $\mathrm{Class}(2) := \{\alpha \in \mathbb{E} \mid \alpha <_1 \alpha^+\}$ *is club in κ.*

Proof.
i.
Take $\kappa \in \mathrm{OR}$ as stated and $\sigma \in \kappa \cap [\varepsilon_\omega, \infty] \cap \mathbb{E}$. Then, directly from corollary 2.14, the function
$$\begin{aligned} f: M(\kappa,\sigma) \cap [\kappa,\kappa^+) &\longrightarrow [\sigma,\sigma^+) \\ t &\longmapsto t[\kappa := \sigma] \end{aligned}$$
is an $(<, +, \cdot, \lambda x.\omega^x)$-isomorphism and therefore a bijection.
This way, $|M(\kappa,\sigma) \cap [\kappa,\kappa^+)| = |[\sigma,\sigma^+)| < \kappa$. (1)

On the other hand, by proposition 2.48, for all $t \in M(\kappa,\sigma) \cap [\kappa,\kappa^+)$, $A(t)$ is club in κ. (2)

From (1), (2) and proposition 2.47, we conclude $\bigcap_{t \in M(\kappa,\sigma) \cap [\kappa,\kappa^+)} A(t)$ is club in κ.

ii.
Direct from previous proposition 2.58 and *i*. □

Part II

The upper classes

Chapter 3

Upper classes of ordinals induced by $<_1$

3.0.3 Class(n)

We want to generalize the results of previous chapter. Our main guide of how to do this is the following: Departing from $\mathbb{E} = \text{Class}(1)$, we have been able to obtain ordinals $\alpha \in \text{Class}(1)$ such that $\alpha <_1 \alpha^+$, where $\alpha^+ = \min\{\beta \in \text{Class}(1) | \alpha < \beta\}$, and we have called Class(2) to the collection of such ordinals. The idea is to iterate this process.

Definition 3.1. *We define by recursion on* ω
$\text{Class}(1) := \mathbb{E}$*;*
$\text{Class}(n+1) := \{\alpha \in \text{OR} | \alpha \in \text{Class}(n) \wedge \alpha(+^n) \in \text{Class}(n) \wedge \alpha <_1 \alpha(+^n)\}$,
where for $\alpha \in \text{Class}(n)$ *we define* $\alpha(+^n) := \begin{cases} \min\{\beta \in \text{Class}(n) | \alpha < \beta\} & \textit{iff } \{\beta \in \text{Class}(n) | \alpha < \beta\} \neq \emptyset \\ \infty & \textit{otherwise} \end{cases}$,
and we make the conventions $\infty \notin \text{OR}$ *and* $\forall \gamma \in \text{OR}. \gamma < \infty$

In the upper definition of $(+^n)$, which we can call "the successor functional of Class(n)", we needed to consider the case when, for $\alpha \in \text{Class}(n)$, such "successor of α in Class(n)" may not exist. We want to tell the reader that this is just a merely formal necessity: one of our purposes is to show that such successor always exists and that Class(n) "behaves like the class $\mathbb{E}$" in the sense of being a closed unbounded class of ordinals. This is one of the important results we want to generalize, although it's proof will take some effort.

Let's see now some basic properties of the elements of Class(n).

Proposition 3.2.

1. $\forall n, i \in [1, \omega). i \leqslant n \Longrightarrow \text{Class}(n) \subset \text{Class}(i)$
2. *For any* $n \in [1, \omega)$ *and any* $\alpha \in \text{Class}(n)$ *define recursively on* $[0, n-1]$
   $\alpha_n := \alpha$, $\alpha_{n-(k+1)} := \alpha_{n-k}(+^{n-(k+1)})$.
   Then $\forall i \in [1, n]. \alpha_i \in \text{Class}(i)$ *and* $\alpha = \alpha_n <_1 \alpha_{n-1} <_1 \ldots <_1 \alpha_2 <_1 \alpha_1 <_1 \alpha_1 2$.
3. *For any* $n \in [1, \omega)$ *and any* $\alpha \in \text{Class}(n)$ *consider the sequence defined in 2.*
   If $\alpha <_1 \alpha_1 2 + 1$ *then* $\alpha \in \text{Lim}\,\text{Class}(n)$.
4. $\forall n \in [2, \omega). \text{Class}(n) \subset \text{Lim}\,\text{Class}(n-1)$.
5. $\forall n, m \in [1, \omega) \forall \alpha. (m < n \wedge \alpha \in \text{Class}(n)) \Longrightarrow \alpha \in \text{Class}(m) \wedge \alpha(+^m) < \alpha(+^n)$.

Proof. 1, 2 and 5 are left to the reader.
3.
Let $\rho \in \alpha$ be arbitrary. Define $B_\rho := \{\rho\} \cup \{\alpha_i, \alpha_i 2 | n \in \{1, \ldots, n\}\}$. Since $\alpha <_1 \alpha_1 2 + 1$ and $B_\rho \subset_{\text{fin}} \alpha_1 2 + 1$, then there exists an $(<, <_1, +)$-isomorphism $h_\rho : B_\rho \longrightarrow h[B_\rho] \subset \alpha$ with
$h_\rho|_\alpha = \text{Id}_\alpha$. Note this implies the following facts in the following order (the order is important since the later facts use the previous to assert their conclusion):

(1') $\forall i\in[1,n].\alpha_i<_1\alpha_i 2\Longleftrightarrow h_\rho(\alpha_i)<_1 h_\rho(\alpha_i)2$. So $h_\rho(\alpha_i)\in\mathbb{E}=\mathrm{Class}(1)$.
(2') $\forall i\in[2,n].\alpha_i<_1\alpha_1\Longleftrightarrow h_\rho(\alpha_i)<_1 h_\rho(\alpha_1)$.
So by (1') and $<_1$-connectedness $\forall i\in[2,n].h_\rho(\alpha_i)\in\mathrm{Class}(2)$.
(3') $\forall i\in[3,n].\alpha_i<_1\alpha_2\Longleftrightarrow h_\rho(\alpha_i)<_1 h_\rho(\alpha_2)$.
So by (1') and (2') and $<_1$-connectedness $\forall i\in[3,n-1].h_\rho(\alpha_i)\in\mathrm{Class}(3)$.
... (inductively)
(n') $\alpha_n<_1\alpha_{n-1}\Longleftrightarrow h_\rho(\alpha_n)<_1 h_\rho(\alpha_{n-1})$.
So by (1'), (2'),.., (n-1') and $<_1$-connectedness $h_\rho(\alpha_n)\in\mathrm{Class}(n)$.

The previous shows that the set (remember that $\alpha_n=\alpha$)
$A:=\{h_\rho(\alpha)|\rho\in\alpha\wedge h_\rho\colon B_\rho\longrightarrow\alpha$ is an $(<,<_1,+)$-iso such that $h_\rho|_\alpha=\mathrm{Id}_\alpha\}\subset\mathrm{Class}(n)$ contains for any $\rho\in\alpha$ an element $h_\rho(\alpha)=h_\rho(\alpha_n)\in\mathrm{Class}(n)$; moreover, since $\rho<\alpha=\alpha_n$ implies $\rho=h_\rho(\rho)<h_\rho(\alpha_n)=h_\rho(\alpha)<\alpha$, then A is confinal in α. Hence $\alpha\in\mathrm{Lim}\,\mathrm{Class}(n)$.

4.
We proceed by induction on $[2,\omega)$.
Take $n\in[2,\omega)$.
Suppose $\forall l\in n\cap[2,\omega).\mathrm{Class}(l)\subset\mathrm{Lim}\,\mathrm{Class}(l-1)$. **(cIH)**

If $\mathrm{Class}(n)=\emptyset$ then we are done. So suppose $\mathrm{Class}(n)\neq\emptyset$ and take $\alpha\in\mathrm{Class}(n)$. By definition this means $\alpha\in\mathrm{Class}(n-1)\ni\alpha(+^{n-1})$ and $\alpha<_1\alpha(+^{n-1})$. **(*3)**

Case $n-1=1$.
Then by (*3), the inequality $\alpha<\alpha 2+1<\alpha(+^1)$ and $<_1$-connectedness we get $\alpha<_1\alpha 2+1$. Then by corollary 2.40, $\alpha\in\mathrm{Lim}\,\mathbb{E}=\mathrm{Lim}\,\mathrm{Class}(1)$.

Case $n-1\geqslant 2$.
By (*3) we know $\alpha\in\mathrm{Class}(n-1)$. Then, by 2., the on $[0,n-2]$ recursively defined sequence of ordinals $\beta_{n-1}:=\alpha$, $\beta_{n-1-(k+1)}:=\beta_{n-1-k}(+^{n-1-(k+1)})$ satisfies
$\beta_{n-1}<_1\beta_{n-2}<_1...<_1\beta_2<_1\beta_1<_1\beta_1 2$ and $\forall i\in[1,n-1].\beta_i\in\mathrm{Class}(i)$. **(*4)**

Let $\gamma:=\alpha(+^{n-1})\in\mathrm{Class}(n-1)$.
We show now by a side induction that $\forall u\in[2,n-1].\beta_{n-u}<\gamma$. **(**3**)**

Let $u\in[2,n-1]$.
Suppose for $l\in u\cap[2,n-1].\beta_{n-l}<\gamma$. **(SIH)**
Since $\gamma\in\mathrm{Class}(n-1)$, then by (3*) (in case $u=2$), by our SIH (in case $u>2$) and 1. we have that $\beta_{n-(u-1)}<\gamma\in\mathrm{Class}(n-(u-1))$, that is, $\gamma\in\{e\in\mathrm{Class}(n-(u-1))|\beta_{n-(u-1)}<_1 e\}$. But $\beta_{n-u}=\beta_{n-(u-1)}(+^{n-u})=\min\{e\in\mathrm{Class}(n-u)|\beta_{n-(u-1)}<e\}$. From all this follows $\beta_{n-u}\leqslant\gamma$. (*5)
On the other hand, our cIH applied to $\gamma\in\mathrm{Class}(n-(u-1))$ implies $\gamma\in\mathrm{Lim}\,\mathrm{Class}(n-u)$; however, since $\beta_{n-u}=\min\{e\in\mathrm{Class}(n-u)|\beta_{n-(u-1)}<e\}$, then $\beta_{n-u}\notin\mathrm{Lim}\,\mathrm{Class}(n-u)$. From this and (*5) follows $\beta_{n-u}<\gamma$. This shows (**3**).

From the fact that $\gamma\in\mathbb{E}$, (**3**) and (*4) we have
$\alpha=\beta_{n-1}<_1\beta_{n-2}<_1...<_1\beta_2<_1\beta_1<_1\beta_1 2<\beta_1 2+1<\gamma=\alpha(+^{n-1})$; moreover, from this, (*3) and $<_1$- connectedness we obtain $\beta_{n-1}<_1\beta_1 2+1$. This way, by 3., it follows that $\alpha=\beta_{n-1}\in\mathrm{Lim}\,\mathrm{Class}(n-1)$ as we needed to show. □

Proposition 3.3. *Let $j\in[2,\omega)$ and $c\in\mathrm{Class}(j)$. Then for any $d\in[1,j)$ there exists a sequence $(c_\xi)_{\xi\in X}\subset\mathrm{Class}(d)$ such that $c_\xi\underset{cof}{\hookrightarrow}c$.*

Proof. Left to the reader. □

Corollary 3.4. *For any* $n\in[1,\omega)$ *and any* $\alpha\in\mathrm{Class}(n)$ *define by recursion on* $[0,n-1]$ *the ordinals* $\alpha_n:=\alpha$, $\alpha_{n-(k+1)}:=\alpha_{n-k}(+^{n-(k+1)})$. *Then*

a) $\alpha=\alpha_n\in\mathrm{Class}(n)$.

b) $\forall j\in[1,n-1].\alpha_j\in\mathrm{Class}(j)\backslash\mathrm{Class}(j+1)$.

c) $\alpha=\alpha_n<_1\alpha_{n-1}<_1\alpha_{n-2}<_1\alpha_{n-3}<_1\ldots<_1\alpha_2<_1\alpha_1<_1\alpha_1 2<\alpha(+^n)$.

d) $\forall j\in[1,n-1].m(\alpha_j)=\alpha_1 2$.

Proof. Left to the reader. □

Proposition 3.5.

1. $\forall j\in[1,\omega)\forall(a_1,\ldots,a_j)\in\mathrm{OR}^j.a_j<_1 a_{j-1}<_1\ldots<_1 a_1<_1 a_1 2\Longrightarrow a_j\in\mathrm{Class}(j)$

2. $\forall j\in[1,\omega)\forall a\in\mathrm{OR}.a\in\mathrm{Class}(j)\Longleftrightarrow\exists(a_1,\ldots,a_j)\in\mathrm{OR}^j.a=a_j<_1 a_{j-1}<_1\ldots<_1 a_2<a_1<a_1 2$

Proof.
1.
By induction on $[1,\omega)$.
For $j=1$ it is clear.

Suppose for $j\in[1,\omega)$ the claim holds, that is
$\forall(a_1,\ldots,a_j)\in\mathrm{OR}^j.a_j<_1 a_{j-1}<_1\ldots<_1 a_1<_1 a_1 2\Longrightarrow a_j\in\mathrm{Class}(j)$. **(IH)**

We show that the claim holds for $j+1$.

Let $(a_1,\ldots,a_{j+1})\in\mathrm{OR}^{j+1}$ be such that $a_{j+1}<_1 a_j<_1\ldots<_1 a_1<_1 a_1 2$. **(*)**
By our (IH) follows $a_j\in\mathrm{Class}(j)$ and therefore, $\forall s\in[1,j].a_j\in\mathrm{Class}(s)$. **(**)**
Now, observe the following argument:
$a_{j+1}<2a_{j+1}\underset{\text{because } a_{j+1}<a_j\in\mathbb{E}\subset\mathbb{P}}{<}a_j$ and then $a_{j+1}<_1 a_{j+1}2$ by (*) and $<_1$-connectedness; that is, $a_{j+1}\in\mathbb{E}=\mathrm{Class}(1)$. But then, $a_{j+1}<a_{j+1}(+^1)\underset{\text{because } a_{j+1}<a_j\in\mathbb{E}}{\leqslant}a_j$ and then $a_{j+1}<_1 a_{j+1}(+^1)$ by (*) and $<_1$-connectedness; that is, $a_{j+1}\in\mathrm{Class}(2)$. But then $a_{j+1}<a_{j+1}(+^2)\underset{\text{because } a_{j+1}<a_j\in\mathrm{Class}(2)}{\leqslant}a_j$ and then $a_{j+1}<_1 a_{j+1}(+^2)$ by (*) and $<_1$-connectedness; that is, $a_{j+1}\in\mathrm{Class}(3)$. Inductively, we get $a_{j+1}\in\mathrm{Class}(j)$, $a_{j+1}<a_{j+1}(+^j)\underset{\text{because } a_{j+1}<a_j\in\mathrm{Class}(j)}{\leqslant}a_j$ and then, by (*) and $<_1$-connectedness, $a_{j+1}<_1 a_{j+1}(+^j)$; that is, $a_{j+1}\in\mathrm{Class}(j+1)$.

2.
Left to the reader. □

Proposition 3.6. $\forall k\in[1,\omega).\forall\alpha,\beta\in\mathrm{OR}.\alpha<_1\beta\in\mathrm{Class}(k)\Longrightarrow\alpha\in\mathrm{Class}(k+1)$.

Proof.
By induction on $[1,\omega)$.
Let $k=1$ and $\alpha,\beta\in\mathrm{OR}$ be such that $\alpha<_1\beta\in\mathrm{Class}(1)$. Then $\alpha<_1\alpha 2$ by $\leqslant_1$-connectedness (because $\alpha<\alpha 2<\beta$), which, as we know, means $\alpha\in\mathbb{E}$. This way $\alpha<\alpha(+^1)\leqslant\beta$, and then, by $\leqslant_1$-connectedness again, $\alpha<_1\alpha(+^1)$, that is, $\alpha\in\mathrm{Class}(2)$.
Suppose the claim holds for $k\in[1,\omega)$. **(IH)**

Let $\alpha, \beta \in \mathrm{OR}$ be such that $\alpha <_1 \beta \in \mathrm{Class}(k+1)$. Then $\beta \in \mathrm{Class}(k)$ by proposition 3.2. So $\alpha <_1 \beta \in \mathrm{Class}(k)$, and our (IH) implies $\alpha \in \mathrm{Class}(k+1)$. But then $\alpha < \alpha(+^{k+1}) \leqslant \beta$, which implies by $\leqslant_1$-connectedness that $\alpha <_1 \alpha(+^{k+1})$. Thus $\alpha \in \mathrm{Class}(k+2)$. □

Proposition 3.7. *Let $i \in [1, \omega)$ and $\alpha \in \mathrm{Class}(i)$. Then*

1. $\forall z \in [1, i).m(\alpha(+^{i-1})(+^{i-2})...(+^{z})) = \alpha(+^{i-1})(+^{i-2})...(+^{z})(+^{z-1})...(+^{1})2$.
2. $\forall t \in (\alpha, \alpha(+^{i-1})(+^{i-2})...(+^{2})(+^{1})2].m(t) \leqslant \alpha(+^{i-1})(+^{i-2})...(+^{2})(+^{1})2$.

Proof. Left to the reader. □

3.0.4 More general substitutions

For our subsequent work we need to introduce a more general notion of substitutions than the one used in previous chapter.

Definition 3.8. *Let $x \in \mathrm{OR}$ and $f: \mathrm{Dom} f \subset \mathbb{E} \longrightarrow \mathbb{E}$ a strictly increasing function such that $\mathrm{Ep}(x) \subset \mathrm{Dom} f$. We define $x[f]$, the "simultaneous substitution of all the epsilon numbers $\mathrm{Ep}(x)$ of the Cantor Normal Form of x by the values of f on them", as*

$$x[f] = \begin{cases} f(x) & \text{if } x \in \mathbb{E} \\ \sum_{i=1}^{n} T_i[f]t_i & \text{if } x \notin \mathbb{E} \text{ and } x =_{\mathrm{CNF}} \sum_{i=1}^{n} T_i t_i \text{ and } (t_1 \geqslant 2 \vee n \geqslant 2) \\ \omega^{Z[f]} & \text{if } x \notin \mathbb{E} \text{ and } x = \omega^{Z} \text{ for some } Z \in \mathrm{OR} \\ x & \text{if } x < \varepsilon_0 \end{cases}$$

Moreover, for $\mathrm{Ep}(x) = \{e_1 > ... > e_k\}$ and a set $Y := \{\sigma_1 > ... > \sigma_k\} \subset \mathbb{E}$ of epsilon numbers, we may also write $x[\mathrm{Ep}(x) := Y]$ instead of $x[h]$, where $h: \mathrm{Ep}(x) \longrightarrow Y$ is the function $h(e_i) := \sigma_i$.

Definition 3.9. *Let $S \subset \mathrm{OR}$ and $f_1, f_2: S \longrightarrow \mathrm{OR}$. We will denote as usual:*

- $f_1 \leqslant f_2 \;:\Longleftrightarrow\; \forall e \in S. f_1(e) \leqslant f_2(e)$.
- $f_1 < f_2 \;:\Longleftrightarrow\; f_1 \leqslant f_2 \wedge \exists e \in S. f_1(e) < f_2(e)$.

Now we enunciate the properties about these kind of substitutions that are of our interest.

Proposition 3.10. *Let $x, y \in \mathrm{OR}$. Let $f: S \subset \mathbb{E} \longrightarrow \mathbb{E}$ be a strictly increasing function with $\mathrm{Ep}(x) \cup \mathrm{Ep}(y) \subset S$. Then*

1. $y \in \mathbb{P} \Longleftrightarrow y[f] \in \mathbb{P}$.
2. $x < y \Longleftrightarrow x[f] < y[f]$.
3. $y \in \mathbb{E} \Longleftrightarrow y[f] \in \mathbb{E}$ *and* $y \in \mathbb{P} \backslash \mathbb{E} \Longleftrightarrow y[f] \in \mathbb{P} \backslash \mathbb{E}$

Proof. Left to the reader. □

Proposition 3.11. *Let $f: \mathrm{Dom} f \subset \mathbb{E} \longrightarrow \mathbb{E}$ be a strictly increasing function. Let $A := \{x \in \mathrm{OR} | \mathrm{Ep}(x) \subset \mathrm{Dom} f\}$. Then the assignation $\varphi: A \longrightarrow \mathrm{OR}$ defined as $\varphi(x) := x[f]$ is a function with respect to the equality in the ordinals, that is, $\forall x, y \in A. x = y \Longrightarrow \varphi(x) = \varphi(y)$.*

Proof. Left to the reader. □

Proposition 3.12. *Let $x \in \mathrm{OR}$ and $f: S \subset \mathbb{E} \longrightarrow \mathbb{E}$ be a strictly increasing function, where* $\mathrm{Ep}(x) \subset S$. *Then*

1. *$x[f]$ is already in Cantor Normal Form.*
2. $\mathrm{Ep}(x[f]) = f[\mathrm{Ep}(x)] \subset \mathrm{Im} f$.
3. *It exists $f^{-1}: \mathrm{Im} f \longrightarrow S$, f^{-1} is strictly increasing and $(x[f])[f^{-1}] = x$.*
4. *Let $\alpha \in \mathbb{E}$. Then $x \in [\alpha, \alpha(+^1)) \Longleftrightarrow \alpha \in S \wedge x[f] \in [f(\alpha), f(\alpha)(+^1))$.*

Proof. Left to the reader. □

Proposition 3.13. *Let $f, g: S \subset \mathbb{E} \longrightarrow \mathbb{E}$ be strictly increasing functions. Let $D := \{e \in S | f(e) < g(e)\}$. Then*

1. $f \leqslant g \Longleftrightarrow \forall x \in \mathrm{OR}. \mathrm{Ep}(x) \subset S \Longrightarrow x[f] \leqslant x[g])$.
2. $f < g \Longrightarrow \forall x \in \mathrm{OR}. (\mathrm{Ep}(x) \subset S \wedge \mathrm{Ep}(x) \cap D \neq \emptyset) \Longrightarrow x[f] < x[g])$.

Proof. Left to the reader. □

Proposition 3.14. *Let x, $y \in \mathrm{OR}$. Let f: $S \subset \mathbb{E} \longrightarrow \mathbb{E}$ be a strictly increasing function with* $\mathrm{Ep}(x) \cup \mathrm{Ep}(y) \subset S$. *Then*

1. $\mathrm{Ep}(x + y) \cup \mathrm{Ep}(\omega^x) \cup \mathrm{Ep}(x \cdot y) \subset S$
2. $(x + y)[f] = x[f] + y[f]$
3. $\omega^x[f] = \omega^{x[f]}$
4. $(x \cdot y)[f] = x[f] \cdot y[f]$

Proof. Left to the reader. □

Proposition 3.15. *Let g: $S \subset \mathbb{E} \longrightarrow Z \subset \mathbb{E}$ and f: $Z \subset \mathbb{E} \longrightarrow \mathbb{E}$ be strictly increasing functions. Then $f \circ g: S \subset \mathbb{E} \longrightarrow \mathbb{E}$ is strictly increasing and for any $t \in \mathrm{OR}$ with $\mathrm{Ep}(t) \subset S$, $t[f \circ g] = t[g][f]$.*

Proof. Left to the reader. □

3.1 The main theorem.

Now we introduce certain notions that are necessary to enunciate the main theorem.

Definition 3.16. *For $k \in [1, \omega)$, $\alpha \in \mathrm{Class}(k)$ and $t \in [\alpha, \alpha(+^k))$, the ordinal $\eta(k, \alpha, t)$ is defined as*

$$\eta(k,\alpha,t) := \begin{cases} \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2 & \textit{iff} \quad t \in [\alpha, \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2] \\ \max\{m(e) | e \in (\alpha, t]\} & \textit{iff} \quad t > \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2 \end{cases}$$

Our next proposition 3.17 shows that $\eta(k, \alpha, t)$ is well defined.

Proposition 3.17. *Let $k \in [1, \omega)$, $\alpha \in \mathrm{Class}(k)$, $t \in (\alpha, \alpha(+^k))$ and $P := \{r \in (\alpha, t] | m(r) \geqslant t\}$. Then*

(0). P is finite; more specifically $1 \leqslant |P| \leqslant k + 1$.
(1). $\max\{m(e) | e \in P\}$ exists and $\max\{m(e) | e \in P\} = \max\{m(e) | e \in (\alpha, t]\}$.
(2). $\eta(k, \alpha, t)$ is well defined.
(3). $\eta(k, \alpha, t) \geqslant m(t) \geqslant t$.

Proof. Consider i, α and t as stated.

(0)

Clearly $t \in P$. So $|P| \geqslant 1$. We prove the other inequality by contradiction. Suppose $|P| \geqslant k + 2$. Then there exist $k + 1$ ordinals $E_0, E_1, ..., E_{k-1}, E_k \in P$ such that $E_k < E_{k-1} < ... < E_1 < E_0 < t$; that is, $\forall l \in [0, k]. E_k < ... < E_l < ... < E_0 < t \leqslant m(E_l)$, and therefore, by $\leqslant_1$-connectedness, we get:

a. $E_0 <_1 E_0 + 1 \leqslant t$, that is, $E_0 \in \mathrm{Lim}\,\mathbb{P} \subset \mathbb{P}$.
b. $E_1 < 2E_1 < E_0 < t \leqslant m(E_1)$; then, by $\leqslant_1$-connectedness $E_1 <_1 2E_1$, that is, $E_1 \in \mathbb{E}$.
c. $\alpha < E_k <_1 E_{k-1} <_1 ... <_1 E_1 <_1 E_1 2 < E_0 < t < \alpha(+^k)$

This way, from c. and proposition 3.5 follows $E_k \in \mathrm{Class}(k) \cap (\alpha, \alpha(+^k))$. Contradiction.

Therefore $|P| \leqslant k + 1$.

(1)

Since by (0) $P \neq \emptyset$ is finite, then $\{m(e) | e \in P\}$ is finite too and thus $\mu := \max\{m(e) | e \in P\}$ exists. Then:

(I). $\mu \geqslant m(t) \geqslant t$ because $t \in P$ (and because $m(\beta) \geqslant \beta$ for any ordinal).
(II). Since $P \subset (\alpha, t]$, then $\mu \in \{m(e) | e \in (\alpha, t]\}$.

On the other hand, let $e \in (\alpha, t]$ be arbitrary. If $m(e) < t$, then $m(e) < \mu$ because of (I). If $m(e) \geqslant t$, then $e \in P$ and then $m(e) \leqslant \mu$. This shows that $\forall e \in (\alpha, t]. m(e) \leqslant \mu$ and since by (II) $\mu \in \{m(e) | e \in (\alpha, t]\}$, we have shown $\mu = \max\{m(e) | e \in (\alpha, t]\}$.

(2)

If $t \in [\alpha, \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2]$, then it is clear that $\eta(k, \alpha, t)$ is well defined. So suppose $t \in (\alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2, \alpha(+^k))$. By (1) $\max\{m(e) | e \in P\}$ exists and $\max\{m(e) | e \in P\} = \max\{m(e) | e \in (\alpha, t]\} \underset{\text{by definition}}{=} \eta(k, \alpha, t)$. That is, $\eta(k, \alpha, t)$ exists.

(3)

For $t > \alpha(+^{k-1})...(+^2)(+^1)2$ the assertion is clear. For $t \leqslant \alpha(+^{k-1})...(+^2)(+^1)2$, we get by proposition 3.7, $t \leqslant m(t) \leqslant \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2 = \eta(k, \alpha, t)$. □

Remark 3.18. The ordinal $\eta(i, \alpha, t)$ is meant to play in Class(i) the analogous role that the ordinal ηt played in $\mathrm{Class}(1) = \mathbb{E}$. Particularly, for $i = 1, \alpha \in \mathrm{Class}(1) = \mathbb{E}$ and $t \in [\alpha, \alpha(+^1))$, $\eta(1, \alpha, t) \underset{\text{proposition 2.27}}{=} \eta t$.

Proposition 3.19. $\forall i\in[1,\omega)\forall\alpha\in\mathrm{Class}(i)\forall t\in[\alpha,\alpha(+^i)).\eta(i,\alpha,t)\in(\alpha,\alpha(+^i))$.

Proof. Left to the reader. □

Definition 3.20. *Let $i\in[1,\omega)$, $\alpha\in\mathrm{Class}(i)$ and $t\in[\alpha,\alpha(+^i))$. We define*

$$l(i,\alpha,t):=\begin{cases}\alpha(+^{i-1})(+^{i-2})...(+^2)(+^1)2 & \text{iff } t\in[\alpha,\alpha(+^{i-1})(+^{i-2})...(+^2)(+^1)2]\\ \min\{r\in(\alpha,t]\mid m(r)=\eta(i,\alpha,t)\} & \text{iff } t>\alpha(+^{i-1})(+^{i-2})...(+^2)(+^1)2\end{cases}.$$

Proposition 3.21. *Let $i\in[1,\omega)$, $\alpha\in\mathrm{Class}(i)$ and $t\in(\alpha(+^{i-1})(+^{i-2})...(+^2)(+^1)2,\alpha(+^i))$. Then*

1. $l(i,\alpha,t)>\alpha(+^{i-1})...(+^1)2$
2. $\eta(i,\alpha,l(i,\alpha,t))=\max\{m(e)|e\in(\alpha,l(i,\alpha,t)]\}=m(l(i,\alpha,t))=\eta(i,\alpha,t)$.

Proof. Left to the reader. □

Remark 3.22. With respect to definition 3.20, consider the case $i=1$. Let $t\in(\alpha2,\alpha^+)$ and suppose $l(1,\alpha,t)\in(\alpha,t)$. The inequalities $l(1,\alpha,t)\leqslant t<\eta t=m(l(1,\alpha,t))$ and the fact that $l(1,\alpha,t)\notin\mathbb{E}$ imply, by theorem 2.3 and corollary 2.4, that $\mathbb{P}\ni l(1,\alpha,t)\wedge m(l(1,\alpha,t))<l(1,\alpha,t)2$. Therefore $\mathbb{P}\ni l(1,\alpha,t)\leqslant t<m(l(1,\alpha,t))<l(1,\alpha,t)2$, which subsequently implies (by considering the cantor normal form of t) that $\pi t=l(1,\alpha,t)$. From this we conclude:

For any $s\in[\alpha,\alpha^+)$, $l(1,\alpha,s)=\begin{cases}\alpha2 & \text{iff } s\in[\alpha,\alpha2]\\ \pi s & \text{iff } s\in(\alpha2,\alpha^+)\wedge l(1,\alpha,s)<s\\ s & \text{iff } s\in(\alpha2,\alpha^+)\wedge l(1,\alpha,s)\not< s\end{cases}.$

Definition 3.23. *Let $i\in[1,\omega)$, $\alpha\in\mathrm{Class}(i)$, $t\in[\alpha,\alpha(+^i)$ and $j\in[1,i]$. We define $\lambda(j,t)$ as the only one ordinal δ satisfying $\delta\in\mathrm{Class}(j)$ and $t\in[\delta,\delta(+^j))$ in case such ordinal exists, and $-\infty$ otherwise.*

Remark 3.24. For $i\in[1,\omega)$, $\alpha\in\mathrm{Class}(i)$, $t\in[\alpha,\alpha(+^i))$ and $j\in[1,i]$, (i.e., all the conditions above) $\lambda(j,t)$ will always be an ordinal. Again, the reason to give the definition this way is just because the existence of $\lambda(j,t)$ is not completely obvious (we will see that later).

We can now present the theorem stating the generalization of the main results we have up to now.

Theorem 3.25. *For any $n\in[1,\omega)$,*
(1). Class(n) *is κ-club for any non-countable regular ordinal κ.*

There exist a binary relational $\leqslant^n\subset\mathrm{Class}(n)\times\mathrm{OR}$ such that:
For $\alpha,c\in\mathrm{Class}(n)$ and any $t\in\alpha(+^n)$ there exist
- A finite set $T(n,\alpha,t)\subset\mathbb{E}\cap\alpha(+^n)$,
- A strictly increasing function $g(n,\alpha,c)\colon\mathrm{Dom}\,g(n,\alpha,c)\subset\mathbb{E}\cap\alpha(+^n)\longrightarrow\mathbb{E}\cap c(+^n)$
such that:

(2) The function $H\colon(\mathrm{Dom}\,g(n,\alpha,c))\cap(\alpha,\alpha(+^n))\longrightarrow(c,c(+^n))$, $H(e):=e[g(n,\alpha,c)]$ is an $(<,+,\cdot,<_1,\lambda x.\omega^x,(+^1),(+^2),...,(+^{n-1}))$ isomorphism.

(3) The relation $\leqslant^n$ *satisfies* $\leqslant^n$*-connectedness,* $\leqslant^n$*-continuity and is such that* $(t\in[\alpha,\alpha(+^n)]\wedge\alpha\leqslant^n t)\Longrightarrow\alpha\leqslant_1 t$.

(4) (First fundamental cofinality property of $\leqslant^n$*).*
If $t\in[\alpha,\alpha(+^n))\wedge\alpha\leqslant^n t+1$*, then there exists a sequence* $(c_\xi)_{\xi\in X}\subset\alpha\cap\mathrm{Class}(n)$ *such that* $c_\xi \underset{\mathrm{cof}}{\hookrightarrow}\alpha$*,* $\forall\xi\in X.T(n,\alpha,t)\cap\alpha\subset c_\xi$ *and* $c_\xi\leqslant_1 t[g(n,\alpha,c_\xi)]$.

(5). (Second fundamental cofinality property of $\leqslant^n$*).*
Suppose $t\in[\alpha,\alpha(+^n))\wedge\alpha\in\mathrm{Lim}\{\gamma\in\mathrm{Class}(n)|T(n,\alpha,t)\cap\alpha\subset\gamma\wedge\gamma\leqslant_1 t[g(n,\alpha,\gamma)]\}$*. Then*
(5.1) $\forall s\in[\alpha,t+1].\alpha\leqslant^n s$*, and therefore*
(5.2) $\alpha\leqslant_1 t+1$

(6). $(t\in[\alpha,\alpha(+^n))\wedge\alpha<_1\eta(n,\alpha,t)+1)\Longrightarrow\alpha\leqslant^n\eta(n,\alpha,t)+1$.

Theorem 3.25 states the general result we are striving for. But the proof of theorem 3.25 is a very long journey: we need to overcome many technical difficulties not stated in it; because of that, we restate it in a more technical way: theorem 3.26. It is exactly the statement of theorem 3.26 that we will be proving along this and the next 3 chapters.

Theorem 3.26. *For any* $n\in[1,\omega)$
(0). $\mathrm{Class}(n)$ *is* κ*-club for any non-countable regular ordinal* κ.

(1). For any $\alpha\in\mathrm{Class}(n)$ *the functions*

- $S(n,\alpha)\colon\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))\longrightarrow\mathrm{Subsets}(\mathrm{Class}(n-1)\cap(\gamma,\gamma(+^n)))$
  $S(n,\alpha)(\delta):=\{e\in\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))\cap\delta\mid m(e)[g(n-1,e,\delta)]\geqslant m(\delta)\}$
- $f(n,\alpha)\colon\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))\longrightarrow\mathrm{Subsets}(\mathrm{OR})$

$$f(n,\alpha)(\delta):=\begin{cases}\{\delta\} & \textit{iff}\quad S(n,\alpha)(\delta)=\emptyset\\ \\ f(n,\alpha)(s)\cup\{\delta\} & \textit{iff}\quad S(n,\alpha)(\delta)\neq\emptyset\wedge s:=\sup(S(n,\alpha)(\delta))\end{cases}$$

are well defined and are such that
(1.1) If $S(n,\alpha)(\delta)\neq\emptyset$ *then* $\sup(S(n,\alpha)(\delta))\in S(n,\alpha)(\delta)\subset\mathrm{Class}(n-1)\cap\delta$.
(1.2) $\forall\delta\in\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n)).\delta\in f(n,\alpha)(\delta)\subset(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)$
and $f(n,\alpha)(\delta)$ *is finite.*
(1.3) $\forall q\in[1,\omega).\forall\sigma\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)$*. If* $f(n,\alpha)(\sigma)=\{\sigma_1>\ldots>\sigma_q\}$ *for some* $\sigma_1,\ldots,\sigma_q\in\mathrm{OR}$ *then*
(1.3.1) $\sigma_1=\sigma$,
(1.3.2) $q\geqslant 2\Longrightarrow\forall j\in\{1,\ldots,q-1\}.m(\sigma_j)\leqslant m(\sigma_{j+1})[g(n-1,\sigma_{j+1},\sigma_j)]$ *and*
(1.3.3) $\sigma_q=\min\{e\in(\alpha,\sigma_q]\cap\mathrm{Class}(n-1)\mid m(e)[g(n-1,e,\sigma_q)]\geqslant m(\sigma_q)\}$.
(1.3.4) $m(\sigma)=m(\sigma_1)\leqslant m(\sigma_2)[g(n-1,\sigma_2,\sigma)]\leqslant\ldots\leqslant m(\sigma_q)[g(n-1,\sigma_q,\sigma)]$.
(1.3.5)
$$\begin{aligned}\sigma_q&=\min\{e\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)\mid e\leqslant\sigma_q\wedge m(e)[g(n-1,e,\sigma_q)]\geqslant m(\sigma_q)\}\\&=\min\{e\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)\mid e\leqslant\sigma_q\wedge m(e)[g(n-1,e,\sigma_q)]=m(\sigma_q)\}.\end{aligned}$$
(1.3.6) For any $j\in\{1,\ldots,q-1\}$,
$$\begin{aligned}\sigma_j&=\min\{e\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)\mid\\&\qquad\sigma_{j+1}<e\leqslant\sigma_j\wedge m(e)[g(n-1,e,\sigma_j)]\geqslant m(\sigma_j)\}\\&=\min\{e\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)\mid\\&\qquad\sigma_{j+1}<e\leqslant\sigma_j\wedge m(e)[g(n-1,e,\sigma_j)]=m(\sigma_j)\}\end{aligned}$$
Note: $S(1,\alpha)=\emptyset=f(1,\alpha)$*. These functions are interesting for* $n\geqslant 2$.

(2).
(2.1) For any $\alpha\in\mathrm{Class}(n)$ *and any* $t\in\alpha(+^n)$ *consider the set* $T(n,\alpha,t)$ *defined as:*
$T(n,\alpha,t):=\bigcup_{E\in\mathrm{Ep}(t)}T(n,\alpha,E)$ *if* $t\notin\mathbb{E}$*;*
$T(n,\alpha,t):=\{t\}$ *if* $t\in\mathbb{E}\cap(\alpha+1)$*;*

$T(n,\alpha,t):=\bigcup_{i\in\omega} O(i,t)$, *if* $t\in(\alpha,\alpha(+^n))\cap\mathbb{E}$,
where for $E\in(\alpha,\alpha(+^n))\cap\mathbb{E}$:
we define $E_1:=\lambda(1,m(E)), E_2:=\lambda(2,E_1),\ldots,E_n:=\lambda(n,E_{n-1})$,
(note $\alpha=E_n\leqslant\ldots E_3\leqslant E_2<E_1$*) and*

$$O(0,E):=\bigcup_{\substack{\delta\in W(0,k,E)\\ k=1,\ldots,n-1}} f(k+1,\lambda(k+1,\delta))(\delta)\cup\mathrm{Ep}(m(\delta))\cup\{\lambda(k+1,\delta)\};$$

$$W(0,k,E):=(\alpha,\alpha(+^n))\cap\{E_1>E_2\geqslant E_3\geqslant\ldots\geqslant E_n=\alpha\}\cap(\mathrm{Class}(k)\backslash\mathrm{Class}(k+1));$$

$$O(l+1,E):=\bigcup_{\substack{\delta\in W(l,k,E)\\ k=1,\ldots,n-1}} f(k+1,\lambda(k+1,\delta))(\delta)\cup\mathrm{Ep}(m(\delta))\cup\{\lambda(k+1,\delta)\};$$

$$W(l,k,E):=(\alpha,\alpha(+^n))\cap O(l,E)\cap(\mathrm{Class}(k)\backslash\mathrm{Class}(k+1)).$$

Then $T(n,\alpha,t)\subset\mathbb{E}\cap\alpha(+^n)$ *is such that:*

(2.1.1) $\mathrm{Ep}(t)\subset T(n,\alpha,t)$ *and* $T(n,\alpha,t)$ *is finite.*
(2.1.2) $T(n,\alpha,t+1)=T(n,\alpha,t)$
(2.1.3) $\alpha(+^{n-1})(+^{n-2})\ldots(+^2)(+^1)2\leqslant t\Longrightarrow T(n,\alpha,\eta(n,\alpha,t))\cap\alpha\subset T(n,\alpha,t)\cap\alpha$
(2.1.4) $\alpha(+^{n-1})(+^{n-2})\ldots(+^2)(+^1)2\leqslant t\Longrightarrow T(n,\alpha,l(n,\alpha,t))\subset T(n,\alpha,t)$

(2.2) For any $\alpha,c\in\mathrm{Class}(n)$ *there exist a function*
$g(n,\alpha,c)$: $\mathrm{Dom}\,g(n,\alpha,c)\subset\mathbb{E}\cap\alpha(+^n)\longrightarrow\mathbb{E}\cap c(+^n)$ *such that*
(2.2.1) $g(n,\alpha,c)|_{c\cap\alpha\cap(\mathrm{Dom}\,g(n,\alpha,c))}$ *and* $g(n,\alpha,\alpha)$ *are the identity functions in their respective domain.*
(2.2.2) $g(n,\alpha,c)$ *is strictly increasing.*
(2.2.3) $\forall t\in\alpha(+^n).T(n,\alpha,t)\cap\alpha\subset c\Longleftrightarrow\mathrm{Ep}(t)\subset\mathrm{Dom}\,g(n,\alpha,c)$
(2.2.4) $\forall t\in\alpha(+^n).\mathrm{Ep}(t)\subset\mathrm{Dom}\,g(n,\alpha,c)\Longrightarrow T(n,c,t[g(n,\alpha,c)])\cap c=T(n,\alpha,t)\cap\alpha$
(2.2.5) For any $t\in[\alpha,\alpha(+^n))$ *with* $\mathrm{Ep}(t)\subset\mathrm{Dom}\,g(n,\alpha,c)$, $\mathrm{Ep}(\eta(n,\alpha,t))\subset\mathrm{Dom}\,g(n,\alpha,c)$ *and* $\eta(n,\alpha,t)[g(n,\alpha,c)]=\eta(n,c,t[g(n,\alpha,c)])$.

By (2.2.2), $g(n,\alpha,c)$ *is bijective in its image. Let's denote* $g^{-1}(n,\alpha,c)$ *to the inverse function of* $g(n,\alpha,c)$.

(2.3) For (2.3.1), (2.3.2) and (2.3.3) we suppose $c\leqslant\alpha$. *Then*
(2.3.1) $\mathrm{Dom}\,g(n,c,\alpha)=\mathbb{E}\cap c(+^n)$
(2.3.2) $g(n,\alpha,c)=g^{-1}(n,c,\alpha)$
(2.3.3) $g(n,\alpha,c)[\mathrm{Dom}\,g(n,\alpha,c)]=\mathbb{E}\cap c(+^n)$

(2.4) $g(n,\alpha,c)$ *has the following homomorphism-like properties:*
(2.4.1) $g(n,\alpha,c)(\alpha)=c$
(2.4.2) For any $i\in[1,n]$ *and any* $e\in(\mathrm{Dom}\,g(n,\alpha,c))\cap[\alpha,\alpha(+^n))$,
$e\in\mathrm{Class}(i)\Longleftrightarrow g(n,\alpha,c)(e)\in\mathrm{Class}(i)$
(2.4.3) The function $e\longmapsto e[g(n,\alpha,c)]$ *with domain* $(\mathrm{Dom}\,g(n,\alpha,c))\cap(\alpha,\alpha(+^n))$ *is an* $(<,+,\cdot,<_1,\lambda x.\omega^x,(+^1),(+^2),\ldots,(+^{n-1}))$ *isomorphism*
(2.4.4) $\forall e\in(\mathrm{Dom}\,g(n,\alpha,c))\cap(\alpha,\alpha(+^n)).m(g(n,\alpha,c)(e))=m(e)[g(n,\alpha,c)]$.
(2.4.5) Suppose $n\geqslant 2$. *Then*
$\forall i\in[2,n]$.
$\forall e\in\mathrm{Class}(i)\cap(\mathrm{Dom}\,g(n,\alpha,c))\cap[\alpha,\alpha(+^n))$.
$\forall E\in(e,e(+^i))\cap\mathrm{Class}(i-1)$.
$f(i,e)(E)=\{E_1>\ldots>E_q\}\Longleftrightarrow$
$f(i,g(n,\alpha,c)(e))(g(n,\alpha,c)(E))=\{g(n,\alpha,c)(E_1)>\ldots>g(n,\alpha,c)(E_q)\}$

(2.4.6) Suppose $n \geqslant 2$. *Then*
$\forall i \in [2, n]. \forall s \in \mathrm{Class}(i-1) \cap [\alpha, \alpha(+^n)).$
$g(n, \alpha, c)(\lambda(i, s)) = \lambda(i, g(n, \alpha, c)(s))$

(2.5) For (2.5.1), (2.5.2) and (2.5.3) we suppose $c \leqslant \alpha$. *Then for all* $d \in \mathrm{Class}(n) \cap [c, \alpha]$,
(2.5.1) $\mathrm{Dom}\, g(n, \alpha, c) \subset \mathrm{Dom}\, g(n, \alpha, d)$
(2.5.2) $g(n, \alpha, d)[\mathrm{Dom}\, g(n, \alpha, c)] \subset \mathrm{Dom}\, g(n, d, c)$
(2.5.3) $g(n, \alpha, c) = g(n, d, c) \circ g(n, \alpha, d)|_{\mathrm{Dom}\, g(n, \alpha, c)}$ *and therefore*
$g^{-1}(n, \alpha, d) \circ g^{-1}(n, d, c) = g^{-1}(n, \alpha, c) \colon \mathbb{E} \cap c(+^n) \longrightarrow \mathrm{Dom}\, g(n, \alpha, c).$

(3). There exists a binary relational $\leqslant^n \subset \mathrm{Class}(n) \times \mathrm{OR}$ *satisfying* $\leqslant^n$*-connectedness and* $\leqslant^n$*-continuity such that* $\forall \alpha \in \mathrm{Class}(n). \forall t \in [\alpha, \alpha(+^n)]. \alpha \leqslant^n t \Longrightarrow \alpha \leqslant_1 t$*; moreover:*

(4) (First fundamental cofinality property of $\leqslant^n$*).*
Let $\alpha \in \mathrm{Class}(n)$ *and* $t \in [\alpha, \alpha(+^n))$ *be arbitrary. If* $\alpha \leqslant^n t+1$*, then there exists a sequence* $(c_\xi)_{\xi \in X} \subset \alpha \cap \mathrm{Class}(n)$ *such that* $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \alpha$, $\forall \xi \in X. T(n, \alpha, t) \cap \alpha \subset c_\xi$ *and* $c_\xi \leqslant_1 t[g(n, \alpha, c_\xi)]$.

(5). (Second fundamental cofinality property of $\leqslant^n$*).*
Let $\alpha \in \mathrm{Class}(n)$ *and* $t \in [\alpha, \alpha(+^n))$.
Suppose $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | T(n, \alpha, t) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$. *Then*
(5.1) $\forall s \in [\alpha, t+1]. \alpha \leqslant^n s$*, and therefore*
(5.2) $\alpha \leqslant_1 t+1$

(6). For $\alpha \in \mathrm{Class}(n)$ *and* $t \in [\alpha, \alpha(+^n))$, $\alpha <_1 \eta(n, \alpha, t) + 1 \Longrightarrow \alpha \leqslant^n \eta(n, \alpha, t) + 1$

The proof of the previous theorem 3.26 will be carried out by induction on $([1, \omega), <)$, and one proves simultaneously (0), (1), (2), (3), (4), (5) and (6). Indeed, such proof is now our current goal.

3.1.1 The case $n = 1$ of theorem 3.26

Proposition 3.27. *Theorem 3.26 holds for* $n = 1$.

Proof.
(0).
$\mathbb{E}$ is κ-club for any non-countable regular ordinal κ.

(1).
Let $\alpha \in \mathbb{E} = \mathrm{Class}(1)$. We define $S(1, \alpha) := \emptyset$ and $f(1, \alpha) := \emptyset$. Then clearly $S(1, \alpha)$ and $f(1, \alpha)$ satisfy the properties stated.

(2).
(2.1)
Let $\alpha, c \in \mathbb{E} = \mathrm{Class}(1)$ with $c \leqslant \alpha$. Let $t \in \alpha(+^1)$. Note $T(1, \alpha, c) = \mathrm{Ep}(t) \subset \mathbb{E}$ is well defined and clearly (2.1.1) and (2.1.2) hold. Now, suppose $t \geqslant \alpha 2$. Since $\eta(1, \alpha, t) = \eta t = \max\{t, \pi t + d\pi t\}$, then (2.1.3) holds. Finally, by remark 3.22, $l(1, \alpha, t) \in \{\alpha 2, \pi t, t\}$ and therefore (2.1.4) holds too.
(2.2)
Let $\alpha, c \in \mathbb{E} = \mathrm{Class}(1)$. Consider $\mathrm{Dom}\, g(1, \alpha, c) := (\mathbb{E} \cap c \cap \alpha) \cup \{\alpha\}$ and $g(1, \alpha, c) \colon \mathrm{Dom}\, g(1, \alpha, c) \longrightarrow \mathbb{E} \cap c(+^1)$ be the function defined as

$$\begin{array}{lll} e \longmapsto e & \text{iff} & e \in \mathbb{E} \cap c \cap \alpha \\ \alpha \longmapsto c & & \end{array}$$

Then it is easy to see that $g(1, \alpha, c)$ satisfies (2.2.1), (2.2.2) and (2.2.3). Besides, $g(1, \alpha, c)$ also satisfies (2.2.4): Take $t \in \alpha^+$ with $\mathrm{Ep}(t) \subset \mathrm{Dom}\, g(1, \alpha, c) = (\mathbb{E} \cap c \cap \alpha) \cup \{\alpha\}$. Then $\mathrm{Ep}(t) \cap \alpha \subset c$ and $t[g(1, \alpha, c)] = t[\alpha := c]$ and therefore $T(1, c, t[g(1, \alpha, c)]) \cap c = T(1, c, t[\alpha := c]) \cap c = \mathrm{Ep}(t[\alpha := c]) \cap c \underset{\text{proposition 2.10}}{=} \mathrm{Ep}(t) \cap \alpha = T(1, \alpha, t)$. Finally, we show that $g(1, \alpha, c)$ satisfies (2.2.5): Take $t \in \alpha^+$ with $\mathrm{Ep}(t) \subset \mathrm{Dom}\, g(1, \alpha, c)$. Then $\mathrm{Ep}(t) \cap \alpha \subset c$ and so $\mathrm{Ep}(\eta(1, \alpha, t)) \cap \alpha \underset{\text{remark 3.18}}{=} \mathrm{Ep}(\eta t) \cap \alpha \underset{\text{proposition 2.38}}{\subset} c$, which means $\mathrm{Ep}(\eta(1, \alpha, t)) \subset \mathrm{Dom}\, g(1, \alpha, c)$. Moreover, $\eta(1, \alpha, t)[g(1, \alpha, c)] = (\eta t)[\alpha := c] \underset{\text{proposition 2.38}}{=} \eta(t[\alpha := c]) = \eta(1, c, t[\alpha := c]) = \eta(1, c, t[g(1, \alpha, c)])$.

(2.3)

Considering $\alpha, c \in \mathbb{E}$ and $g(1, \alpha, c)$ as in (2.2) with the extra assumption $c \leqslant \alpha$ it is immediate that (2.3.1), (2.3.2) and (2.3.3) hold.

(2.4)

Given α, $c \in \mathbb{E}$ and $g(1, \alpha, c)$ as in (2.2), it is clear that (2.4.1), (2.4.2), (2.4.5) and (2.4.6) hold. Moreover, (2.4.3) and (2.4.4) are corollary 2.17 and remark 2.18.

(2.5)

Take $\alpha, d, c \in \mathbb{E}$ with $c \leqslant d \leqslant \alpha$. Then $\mathrm{Dom}\, g(1, \alpha, c) = (\mathbb{E} \cap c \cap \alpha) \cup \{\alpha\} \subset (\mathbb{E} \cap d \cap \alpha) \cup \{\alpha\} = \mathrm{Dom}\, g(1, \alpha, d)$, that is, (2.5.1) holds. Moreover, $g(1, \alpha, d)[\mathrm{Dom}\, g(1, \alpha, c)] = \{g(1, \alpha, d)(e) \mid e \in (\mathbb{E} \cap c \cap \alpha) \cup \{\alpha\}\} = (\mathbb{E} \cap c \cap \alpha) \cup \{d\} \subset (\mathbb{E} \cap c \cap d) \cup \{d\} = \mathrm{Dom}\, g(1, d, c)$, i.e., (2.5.2) holds. Let's show that (2.5.3) also holds: For $e \in \mathrm{Dom}\, g(1, \alpha, c) = (\mathbb{E} \cap c \cap \alpha) \cup \{\alpha\}$, $e \underset{g(1,\alpha,d)}{\longmapsto} \begin{cases} e \underset{g(1,d,c)}{\longmapsto} c = g(1, \alpha, c)(e) & \text{iff } e = \alpha \\ \\ e \underset{g(1,d,c)}{\longmapsto} e = g(1, \alpha, c)(e) & \text{iff } e \neq \alpha \end{cases}$, that is, $g(1, \alpha, c) = g(1, d, c) \circ g(1, \alpha, d)|_{\mathrm{Dom}\, g(n, \alpha, c)}$; finally, direct from the previous equality follows that $g^{-1}(n, \alpha, c) = g^{-1}(n, \alpha, d) \circ g^{-1}(n, d, c)$ because $g(1, \alpha, c)$, $g(1, d, c)$ and $g(1, \alpha, d)|_{\mathrm{Dom}\, g(n, \alpha, c)}$ are invertible functions, and since by (2.3.2) $g^{-1}(1, \alpha, c) = g(1, c, \alpha)$, then $g^{-1}(1, \alpha, c) = g(1, c, \alpha) \colon (\mathbb{E} \cap \alpha \cap c) \cup \{c\} = \mathbb{E} \cap c^+ \longrightarrow (\mathbb{E} \cap \alpha \cap c) \cup \{\alpha\} = \mathrm{Dom}\, g(1, \alpha, c)$.

(3).

Of course, the relation $\leqslant^1$ worked in the first chapters satisfies $\forall \alpha \in \mathrm{Class}(1). \forall t \in [\alpha, \alpha(+^1)]. \alpha \leqslant^1 t \Longrightarrow \alpha \leqslant_1 t$ and moreover:

(4) holds because of proposition 2.23;

(5) holds too because of proposition 2.24;

(6) holds because of corollary 2.35. □

Working on case $n > 1$ of theorem 3.26

It is in this moment that the hard work starts. As we have already said, we prove theorem 3.26 by induction on $[1, \omega)$, and since we have already seen that it holds for $n = 1$, then for the next 3 Chapters (that is, until we complete the whole proof of theorem 3.26) we consider a fixed $n \in [2, \omega)$ and our induction hypothesis is that theorem 3.26 holds for any $i \in [1, n)$. We name **GenThmIH** to this induction hypothesis.

Chapter 4

Clause (0) of theorem 3.26

We want to show that clause (0) of theorem 3.26 holds. In order to do this, our first goal is to provide a generalized version of the hierarchy theorem done for the intervals $[\varepsilon_\gamma, \varepsilon_{\gamma+1})$. We first prove certain propositions that will be necessary later.

Proposition 4.1. *Let $i \in [1, n-1]$. Let κ be an uncountable regular ordinal. Then $\kappa \in \mathrm{Class}(i)$.*

Proof. Take i, κ as stated. Let ρ be an uncountable regular ordinal, $\rho > \kappa$ (ρ exists because the class of regular ordinals is unbounded in the class of ordinals). Since $\mathrm{Class}(i) \cap \kappa$ is bounded in ρ and $\mathrm{Class}(i)$ is club in ρ by GenThmIH, then $\sup\,(\mathrm{Class}(i) \cap \kappa) \in \mathrm{Class}(i)$. But $\mathrm{Class}(i) \cap \kappa$ is unbounded in κ (by GenThmIH) and therefore $\sup\,(\mathrm{Class}(i) \cap \kappa) = \kappa$. These two observations prove $\kappa = \sup\,(\mathrm{Class}(i) \cap \kappa) \in \mathrm{Class}(i)$. $\square$

Proposition 4.2. *For any $i \in [1, n]$, $\mathrm{Class}(n)$ is closed.*

Proof. For $i \leqslant n-1$ the claim is clear by GenThmIH. So suppose $i = n$.

Let $\alpha \in \mathrm{Lim}\,\mathrm{Class}(n)$. Then there exists a sequence $(c_\xi)_{\xi \in X} \subset \mathrm{Class}(n) \cap \alpha$ with $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \alpha$. So $(c_\xi)_{\xi \in X} \subset \mathrm{Class}(n-1)$ and since by (0) of GenThmIH $\mathrm{Class}(n-1)$ is club in any non-countable regular ordinal κ, then $\alpha \in \mathrm{Class}(n-1)$.

Now we want to show that $\forall t \in (\alpha, \alpha(+^{n-1})).\alpha <_1 t$. **(*)**

Let $t \in (\alpha, \alpha(+^n))$. Since $T(n-1, \alpha, t)$ is finite and $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} \alpha$, then we can assume without loss of generality that $\forall \xi \in X.T(n-1, \alpha, t) \cap \alpha \subset c_\xi$. This way, for all $\xi \in X$, the ordinal $t[g(n-1, \alpha, c_\xi)] \in (c_\xi, c_\xi(+^{n-1}))$ and since by hypothesis $c_\xi \in \mathrm{Class}(n)$, (i.e., $c_\xi <_1 c_\xi(+^{n-1})$), then $c_\xi <_1 t[g(n-1, \alpha, c_\xi)]$ by $<_1$-connectedness. This shows $\forall \xi \in X.c_\xi <_1 t[g(n-1, \alpha, c_\xi)]$.

From our work in the previous paragraph follows that $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n-1, \alpha, \gamma)]\}$, and therefore, by use of GenThmIH (5) (Second fundamental cofinality property of $\leqslant^{n-1}$), follows $\alpha \leqslant_1 t$.

The previous shows (*).

Finally, for the sequence $(d_\xi)_{\xi \in (\alpha, \alpha(+^n))}$ defined as $d_\xi := \xi$, it follows from (*) that $\alpha <_1 d_\xi \underset{\mathrm{cof}}{\hookrightarrow} \alpha(+^{n-1})$; therefore, by $\leqslant_1$-continuity, $\alpha <_1 \alpha(+^{n-1})$, that is, $\alpha \in \mathrm{Class}(n)$. $\square$

Remark 4.3. Consider $i \in [1, n]$, $\alpha \in \mathrm{Class}(i)$ and $t \in [\alpha, \alpha(+^i))$. Let $j \in [1, i]$. Then $\lambda(j, e)$ was defined as the only one ordinal δ satisfying $\delta \in \mathrm{Class}(j) \wedge e \in [\delta, \delta(+^j))$ or $-\infty$ in case such ordinal does not exist. We want **to show that $\boldsymbol{\lambda(j, e)}$ is indeed an ordinal**:

Let $U := (e+1) \cap \mathrm{Class}(j)$. Then $\beta \in U \neq \emptyset$ because $j \leqslant i$ implies $\mathrm{Class}(i) \subset \mathrm{Class}(j)$ by proposition 3.2. Let $u := \sup U$. Then, by previous proposition 4.2, $u \in \mathrm{Class}(j) \cap (e+1)$. Moreover, $e \in [u, u(+^j))$. This shows that $\lambda(e, j) = u \in \mathrm{OR}$.

Proposition 4.4. *Let* $k<n$ *and* $\beta\in\mathrm{Class}(k)$. *Then* $\beta\leqslant_1\beta(+^{k-1})...(+^2)(+^1)2+1\Longleftrightarrow\beta\leqslant^k\beta(+^{k-1})...(+^2)(+^1)2+1\Longleftrightarrow\beta\in\mathrm{Lim}(\mathrm{Class}(k))$.

Proof. Let $k<n$ and $\beta\in\mathrm{Class}(k)$.

Note $\beta\leqslant_1\beta(+^{k-1})...(+^2)(+^1)2+1\Longleftrightarrow\beta\leqslant^k\beta(+^{k-1})...(+^2)(+^1)2+1$ holds because $\eta(k,\beta,\beta(+^{k-1})...(+^1)2)=\beta(+^{k-1})...(+^1)2$ and because of (3) and (6) of GenThmIH. Moreover, $\beta\leqslant^k\beta(+^{k-1})...(+^1)2+1\Longrightarrow\beta\in\mathrm{Lim}(\mathrm{Class}(k))$ holds because of (4) of GenThmIH.

It only remains to show that $\beta\leqslant^k\beta(+^{k-1})...(+^1)2+1\Longleftarrow\beta\in\mathrm{Lim}(\mathrm{Class}(k))$. Take $\beta\in\mathrm{Lim}(\mathrm{Class}(k))$. Then there is a sequence $(c_\xi)_{\xi\in X}\subset\mathrm{Class}(k)$ with $c_\xi\underset{\mathrm{cof}}{\hookrightarrow}\beta$. Now, by (2.1.1) of GenThmIH, $T(k,\beta,\beta(+^{k-1})...(+^1)2)$ is finite, and so $T(k,\beta,\beta(+^{k-1})...(+^1)2)\cap\beta$ is finite too. This way, there is a subsequence $(d_j)_{j\in J}$ of $(c_\xi)_{\xi\in X}$ such that $\forall j\in J.T(k,\beta,\beta(+^{k-1})...(+^1)2)\cap\beta\subset d_j$ and $d_j\underset{\mathrm{cof}}{\hookrightarrow}\beta$.

From the previous paragraph we get that $\forall j\in J.T(k,\beta,\beta(+^{k-1})...(+^1)2)\cap\beta\subset d_j$, $d_j\underset{\mathrm{cof}}{\hookrightarrow}\beta$ and $\forall j\in J.d_j\underset{\text{by proposition 3.2}}{\leqslant_1}d_j(+^{k-1})...(+^1)2\underset{\text{by (2.4.3) and (2.4.1) of GenThmIH}}{=}$
$=(\beta(+^{k-1})...(+^1)2)[g(k,\beta,d_j)]$. That is, we have shown
$\beta\in\mathrm{Lim}\{\gamma\in\mathrm{Class}(k)|T(k,\beta,\beta(+^{k-1})...(+^1)2)\cap\beta\subset\gamma\wedge$
$\gamma\leqslant_1(\beta(+^{k-1})...(+^1)2)[g(k,\beta,\gamma)]\}$. Therefore, by (5) of GenThmIH, we conclude $\beta\leqslant^k\beta(+^{k-1})...(+^1)2+1$. □

Definition 4.5. *Let* $i\in[1,n)$, $\alpha\in\mathrm{Class}(i)$ *and* $t\in(\alpha,\alpha(+^i))$. *For any ordinal* $r\in\mathrm{OR}$, *let* $S(i,\alpha,r,t):=\{q\in(\alpha,l(i,\alpha,t))\mid T(i,\alpha,q)\cap\alpha\subset r\}$.

Remark 4.6. With respect to our previous definition, note $S(i,\alpha,r,t)\subset l(i,\alpha,t)\leqslant t$. Moreover, since $i\in[1,n)$, then by (2.2.3) of GenThmIH,
$r\in\mathrm{Class}(i)\Longrightarrow S(i,\alpha,r,t)=\{q\in(\alpha,l(i,\alpha,t))\mid\mathrm{Ep}(q)\subset(\mathrm{Dom}\,g(i,\alpha,r))\}$.

4.1 The Generalized Hierarchy Theorem

Now we are ready to define a hierarchy of sets $A^{n-1}(t)$ which generalizes the hierarchy of the sets $A(t)$ worked in the first part of this thesis.

Definition 4.7. *Let* $C^{n-1}:\mathrm{OR}\longrightarrow\mathrm{Class}(n-1)$ *be the counting functional of* $\mathrm{Class}(n-1)$, *(by GenThmIH follows* $\mathrm{Class}(n-1)$ *is a closed unbounded class of ordinals) and for* $j\in\mathrm{OR}$, *let's write* C^{n-1}_j *for* $C^{n-1}(j)$.

We define by recursion on the interval $[C^{n-1}_\omega,\infty)$ *the functional* $A^{n-1}:[C^{n-1}_\omega,\infty)\longrightarrow\mathrm{Subclasses(OR)}$ *as:*

For $t\in[C^{n-1}_\omega,\infty)$, *let* $\alpha\in\mathrm{Class}(n-1)$ *be such that* $t\in[\alpha,\alpha(+^{n-1}))$.

Let $M:=\begin{cases}\max(T(n-1,\alpha,t)\cap\alpha) & \textit{iff}\quad T(n-1,\alpha,t)\cap\alpha\neq\emptyset\\ -\infty & \textit{otherwise}\end{cases}$.

Case $t = l + 1$.

$$A^{n-1}(l+1) := \begin{cases} A^{n-1}(l) & \textit{iff } \ l < \eta(n-1, \alpha, l) \\ \operatorname{Lim} A^{n-1}(l) & \textit{otherwise; that is, } l = \eta(n-1, \alpha, l) \end{cases}$$

Case $t \in \operatorname{Lim}$.

$$A^{n-1}(t) := \begin{cases} (\operatorname{Lim} \operatorname{Class}(n-1)) \cap (M, \alpha+1) & \textit{iff} \quad t \in [\alpha, \alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2] \\ \operatorname{Lim}\{r \leqslant \alpha | M < r \in \bigcap_{s \in S(n-1,\alpha,r,t)} A^{n-1}(s)\} & \textit{otherwise} \end{cases}.$$

On the other hand, we define the functional $G^{n-1}: [C_\omega^{n-1}, \infty) \longrightarrow \operatorname{Subclasses(OR)}$ *in the following way:*

For $t \in [C_\omega^{n-1}, \infty)$, *let* $\alpha \in \operatorname{Class}(n-1)$ *be such that* $t \in [\alpha, \alpha(+^{n-1}))$ *and let*
$G^{n-1}(t) := \{\beta \in \operatorname{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^{n-1} \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1\}$
$=$, *by GenThmIH (3) and (6),*
$= \{\beta \in \operatorname{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant_1 \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1\}$.

Remark 4.8. Notice that $G^{n-1}(t)$ is well defined because for $\beta \in \operatorname{Class}(n-1)$ satisfying $T(n-1, \alpha, t) \cap \alpha \subset \beta$, (2.2.3) and (2.2.4) of GenThmIH imply $T(n-1, \alpha, \eta(n-1, \alpha, t)) \cap \alpha \subset \beta$; therefore, again by (2.2.3) of GenThmIH, $\operatorname{Ep}(\eta(n-1, \alpha, t)) \subset \operatorname{Dom}(g(n-1, \alpha, \beta))$.

Proposition 4.9. *Let* $\alpha \in \operatorname{Class}(n-1) \cap [C_\omega^{n-1}, \infty)$. *Then*
$\forall t \in [\alpha, \alpha(+^{n-2})...(+^1)2].A^{n-1}(t) = (\operatorname{Lim} \operatorname{Class}(n-1)) \cap (\max(T(n-1, \alpha, t) \cap \alpha), \alpha+1)$

Proof. Left to the reader. □

Theorem 4.10. $\forall t \in [C_\omega^{n-1}, \infty).G^{n-1}(t) = A^{n-1}(t)$

Proof. We proceed by induction on the class $[C_\omega^{n-1}, \infty)$.

Let $t \in [C_\omega^{n-1}, \infty)$ and $\alpha \in \operatorname{Class}(n-1)$ be with $t \in [\alpha, \alpha(+^{n-1}))$.

Suppose $\forall s \in [C_\omega^{n-1}, \infty) \cap t.G^{n-1}(s) = A^{n-1}(s)$. **(cIH)**

Case $t \in [\alpha, \alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2]$.
Then $\eta(n-1, \alpha, t) = \alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2$ and so

$$\begin{aligned} G^{n-1}(t) &= \{\beta \in \operatorname{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \\ &\quad \beta \leqslant^{n-1} \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1\} = \\ &= \{\beta \in \operatorname{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \\ &\quad \beta \leqslant^{n-1} \alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2[g(n-1, \alpha, \beta)] + 1\} = \\ &= \{\beta \in \operatorname{Class}(n-1) | T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant \alpha \wedge \\ &\quad \beta \leqslant^{n-1} \beta(+^{n-2})(+^{n-3})...(+^2)(+^1)2 + 1\} = \\ &\underset{\text{proposition 4.4}}{=} (\operatorname{Lim} \operatorname{Class}(n-1)) \cap (\max(T(n-1, \alpha, t) \cap \alpha), \alpha+1) = \\ &\underset{\text{by proposition 4.9}}{=} A^{n-1}(t). \end{aligned}$$

The previous shows the theorem holds in interval $[\alpha, \alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2]$. So, **from now on, we suppose** $\boldsymbol{t \in (\alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2, \alpha(+^{n-1}))}$. **(A0)**

Successor subcase. Suppose $t=s+1$ for some $s\in[\alpha(+^{n-2})(+^{n-3})...(+^2)(+^1)2,t)$.

First note

$$\eta(n,\alpha,s+1)=\max\{m(e)|e\in(\alpha,s+1]\}=\max\{\max\{m(e)|e\in(\alpha,s]\},m(s+1)=s+1\}=$$
$$=\begin{cases}\max\{\max\{m(e)|e\in(\alpha,s]\},s+1\} & \text{iff} \quad s=\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\\ \max\{\max\{m(e)|e\in(\alpha,s]\},s+1\} & \text{iff} \quad s>\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\end{cases}$$
$$=\begin{cases}\max\{\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2,s+1\} & \text{iff} \quad s=\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\\ \max\{\eta(n-1,\alpha,s),s+1\} & \text{iff} \quad s>\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\end{cases}$$
$$=\begin{cases}\max\{\eta(n-1,\alpha,s),s+1\} & \text{iff} \quad s=\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\\ \max\{\eta(n-1,\alpha,s),s+1\} & \text{iff} \quad s>\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\end{cases}$$
$$=\max\{\eta(n-1,\alpha,s),s+1\}. \qquad \mathbf{(A1)}$$

Subsubcase $s<\eta(n-1,\alpha,s)$.

Then, using (A1), $\eta(n-1,\alpha,s+1)=\eta(n-1,\alpha,s)$. Therefore,

$$G^{n-1}(t)=G^{n-1}(s+1)=\{\beta\in\text{Class}(n-1)|T(n-1,\alpha,s+1)\cap\alpha\subset\beta\leqslant\alpha\wedge$$
$$\beta\leqslant^{n-1}\eta(n-1,\alpha,s+1)[g(n-1,\alpha,\beta)]+1\}=$$
$$\underset{\text{by (2.1.2) of GenThmIH}}{=}\{\beta\in\text{Class}(n-1)|T(n-1,\alpha,s)\cap\alpha\subset\beta\leqslant\alpha\wedge$$
$$\beta\leqslant^{n-1}\eta(n-1,\alpha,s)[g(n-1,\alpha,\beta)]+1\}=$$
$$=G^{n-1}(s)\underset{\text{by cIH}}{=}A^{n-1}(s)\underset{\text{because } s<\eta(n-1,\alpha,s)}{=}A^{n-1}(s+1).$$

Subsubcase $s=\eta(n-1,\alpha,s)$.

So, from (A1), $\eta(n-1,\alpha,s+1)=s+1=\eta(n-1,\alpha,s)+1$. **(A2)**.

To show $G^{n-1}(t)\subset A^{n-1}(t)$. (A3)

Let $\beta\in G^{n-1}(t)=G^{n-1}(s+1)$. Then $\beta\in\text{Class}(n-1)$, $T(n-1,\alpha,s+1)\cap\alpha\subset\beta\leqslant\alpha$ and $\beta\leqslant^{n-1}\eta(n-1,\alpha,s+1)[g(n-1,\alpha,\beta)]+1\underset{\text{by (A2)}}{=}(\eta(n-1,\alpha,s)+1)[g(n-1,\alpha,\beta)]+1$; from this and (4) of GenThmIH follows the existence of a sequence $(c_\xi)_{\xi\in X}\subset\text{Class}(n-1)\cap\beta$, $c_\xi\underset{\text{cof}}{\hookrightarrow}\beta$ such that for all $\xi\in X$,
$T(n-1,\beta,(\eta(n-1,\alpha,s)+1)[g(n-1,\alpha,\beta)])\cap\beta\subset c_\xi$ and
$c_\xi\leqslant_1\boldsymbol{(\eta(n-1,\alpha,s)+1)}[g(n-1,\alpha,\beta)]g[(n-1,\beta,c_\xi)]$. **(A4)**

On the other hand, for any $\xi\in X$, $c_\xi\supset T(n-1,\beta,(\eta(n-1,\alpha,s)+1)[g(n-1,\alpha,\beta)])\cap\beta=$ $T(n-1,\beta,(s+1)[g(n-1,\alpha,\beta)])\cap\beta\underset{\text{by (2.2.4) of GenThmIH}}{=}T(n-1,\alpha,s+1)\cap\alpha=$ $T(n-1,\alpha,\eta(n-1,\alpha,s)+1)\cap\alpha$. **(A5)**

Now, note that for any $\xi\in X$, by (A5) and (2.2.3) of GenThmIH, we have that $\text{Ep}(\eta(n-1,\alpha,s)+1)\subset\text{Dom}(g(n-1,\alpha,c_\xi))$. Then
$(\eta(n-1,\alpha,s)+1)[g(n-1,\alpha,\beta)][g(n-1,\beta,c_\xi)]=$
$(\eta(n-1,\alpha,s)+1)[g(n-1,\beta,c_\xi)\circ g(n-1,\alpha,\beta)]\underset{\text{by (2.5.3) of GenThmIH}}{=}$
$(\eta(n-1,\alpha,s)+1)[g(n-1,\alpha,c_\xi)]=\eta(n-1,\alpha,s)[g(n-1,\alpha,c_\xi)]+1\underset{\text{by (2.2.5) of GenThmIH}}{=}$
$\eta(n-1,c_\xi,s[g(n-1,\alpha,c_\xi)])+1$. **(A6)**

Done the previous work, from (A4), (A5) (and (2.1.2) of GenThmIH) and (A6) follows $\forall\xi\in X.T(n-1,\alpha,s)\cap\alpha\subset c_\xi\leqslant\alpha\wedge c_\xi\leqslant_1\eta(n-1,\alpha,s[g(n-1,\alpha,c_\xi)])+1$ and therefore, by (6) of GenThmIH, $\forall\xi\in X.T(n-1,\alpha,s)\cap\alpha\subset c_\xi\leqslant\alpha\wedge c_\xi\leqslant^{n-1}\eta(n-1,\alpha,s[g(n-1,\alpha,c_\xi)])+1$. This shows that $(c_\xi)_{\xi\in X}\subset G^{n-1}(s)\underset{\text{by our cIH}}{=}A^{n-1}(s)$, and since $c_\xi\underset{\text{cof}}{\hookrightarrow}\beta$, then we have that $\beta\in\text{Lim}\,A^{n-1}(s)=A(s+1)=A(t)$. This proves **(A3)**.

We now show $G^{n-1}(t)\supset A^{n-1}(t)$. (B1)

Let $\beta \in A^{n-1}(t) = A^{n-1}(s+1) = \operatorname{Lim} A^{n-1}(s) \underset{\text{by our cIH}}{=} \operatorname{Lim} G^{n-1}(s)$. So there is a sequence $(c_\xi)_{\xi \in X} \subset G^{n-1}(s)$ such that $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$. So for all $\xi \in X$,
$T(n-1, \alpha, s) \cap \alpha \subset c_\xi \in \operatorname{Class}(n-1) \cap \beta \subset \alpha$ and
$c_\xi \leqslant^{n-1} \eta(n-1, \alpha, s)[g(n-1, \alpha, c_\xi)] + 1 = (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, c_\xi)]$. **(B2)**

We will argue similarly as in the proof of (A3). Let $\xi_0 \in X$. Then
$T(n-1, \alpha, \eta(n-1, \alpha, s)+1) \cap \alpha = T(n-1, \alpha, \eta(n-1, \alpha, s)) \cap \alpha = T(n-1, \alpha, s) \cap \alpha \subset c_{\xi_0} < \beta$, so $\boldsymbol{T(n-1, \alpha, s) \cap \alpha \subset \beta}$ and the ordinal $(\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)] \in [\beta, \beta(+^{n-1}))$ is well defined. Now, for any $\xi \in X$, $T(n-1, \beta, (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)]) \cap \beta =$
$T(n-1, \alpha, (\eta(n-1, \alpha, s)+1)) \cap \alpha = T(n-1, \alpha, s+1) \cap \alpha = T(n-1, \alpha, s) \cap \alpha \subset c_\xi$, and so the ordinal $(\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)][g(n-1, \beta, c_\xi)] \in [c_\xi, c_\xi(+^{n-1}))$ is well defined too; moreover, using this and (2.5.3) of GenThmIH, we get
$(\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)][g(n-1, \beta, c_\xi)] =$
$(\eta(n-1, \alpha, s)+1)[g(n-1, \beta, c_\xi) \circ g(n-1, \alpha, \beta)] = (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, c_\xi)]$. But from this and (B2) we get
$\forall \xi \in X. T(n-1, \alpha, (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)]) \cap \beta \subset c_\xi < \beta \wedge$
$c_\xi \leqslant_1 (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, c_\xi)] = (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)][g(n-1, \beta, c_\xi)]$;
note these previous two lines and the fact that $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$ means
$\beta \in \operatorname{Lim}\{\gamma \in \operatorname{Class}(n-1) | T(n-1, \beta, (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)]) \cap \beta \subset \gamma \wedge$
$\gamma \leqslant_1 (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)][g(n-1, \beta, c_\xi)]\}$. Thus, from all of the above and using (5.1) of GenThmIH, we conclude
$T(n-1, \alpha, s) \cap \alpha \subset \beta \leqslant \alpha \wedge$
$\beta \leqslant^{n-1} (\eta(n-1, \alpha, s)+1)[g(n-1, \alpha, \beta)] + 1 \underset{\text{by (A2)}}{=} \eta(n-1, \alpha, s+1)[g(n-1, \alpha, \beta)] + 1 =$
$= \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1.$ **(B3)**.

(B3) shows $\beta \in G^{n-1}(t)$. Hence we have shown $G^{n-1}(t) \supset A^{n-1}(t)$.

All the previous work shows that for t a successor ordinal the theorem holds. Now we have to see what happens when t is a limit ordinal.

Subcase $t \in \operatorname{Lim}$. We remind the reader that, by (A0), we also know that $t \in (\alpha(+^{n-2})(+^{n-3}) \ldots (+^2)(+^1)2, \alpha(+^{n-1}))$.

To show $\boldsymbol{G^{n-1}(t) \subset A^{n-1}(t)}$. **(B4)**

Let $\beta \in G^{n-1}(t)$. So $T(n-1, \alpha, t) \cap \alpha \subset \beta \leqslant^{n-1} \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1$ and $\alpha \geqslant \beta \in \operatorname{Class}(n-1)$. Then, by (4) of GenThmIH there exists a sequence $(c_\xi)_{\xi \in X} \subset \operatorname{Class}(n-1) \cap \beta$, $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$ such that for all $\xi \in X$,
$T(n-1, \beta, \eta(n-1, \alpha, t))[g(n-1, \alpha, \beta)]) \cap \beta \subset c_\xi$ and
$c_\xi \leqslant_1 \boldsymbol{\eta(n-1, \alpha, t)}[g(n-1, \alpha, \beta)]g[(n-1, \beta, c_\xi)]$. **(B5)**

On the other hand, since $T(n-1, \alpha, t) \cap \alpha \subset \beta$, $T(n-1, \alpha, t)$ is finite (by (2.1.1) of GenThmIH) and $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$, then $T(n-1, \alpha, t) \cap \alpha$ is also finite and therefore we can assume without loss of generality that $\forall \xi \in X. T(n-1, \alpha, t) \cap \alpha \subset c_\xi$. **(B6)**

Now, notice for any $\xi \in X$,
$T(n-1, \alpha, \eta(n-1, \alpha, t)) \cap \alpha \underset{\text{by (2.1.3) of GenThmIH}}{\subset} T(n-1, \alpha, t) \cap \alpha \subset c_\xi$ and therefore, by (B6) and (2.2.3) of GenThmIH,
$\operatorname{Ep}(\eta(n-1, \alpha, t)) \subset \operatorname{Dom}(g(n-1, \alpha, c_\xi))$. This way,
$\eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)][g(n-1, \beta, c_\xi)] =$
$\eta(n-1, \alpha, t)[g(n-1, \beta, c_\xi) \circ g(n-1, \alpha, \beta)] \underset{\text{(2.5.3) of GenThmIH}}{=}$
$\eta(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$. From this, (B5) and (B6) we obtain
$\forall \xi \in X. T(n-1, \alpha, t) \cap \alpha \subset c_\xi \leqslant \alpha \wedge$
$c_\xi \leqslant_1 \eta(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)] = \eta(n-1, \alpha, l(n-1, \alpha, t))[g(n-1, \alpha, c_\xi)]$. **(C1)**

Let's see now that $\forall \xi \in X. c_\xi \in \bigcap_{s \in S(n-1,\alpha,c_\xi,t)} A^{n-1}(s)$. **(C2)**

Let $\xi \in X$ be arbitrary. Take $s \in S(n-1, \alpha, c_\xi, t)$. Then $s \in (\alpha, l(n-1, \alpha, t))$ and then, by the definition of $l(n-1, \alpha, t)$, it follows that $\eta(n-1, \alpha, s) < \eta(n-1, \alpha, l(n-1, \alpha, t))$. On the other hand, since $T(n-1, \alpha, s) \cap \alpha \subset c_\xi$ then $T(n-1, \alpha, \eta(n-1, \alpha, s)) \cap \alpha \subset c_\xi$ (by (2.1.3) of GenThmIH); moreover, we know $T(n-1, \alpha, t) \cap \alpha \subset c_\xi$, so by (2.1.4) of GenThmIH, $T(n-1, \alpha, l(n-1, \alpha, t)) \cap \alpha \subset c_\xi$.

From the previous paragraph follows that, for any $\xi \in X$, the ordinals $\eta(n-1, \alpha, s)[g(n-1, \alpha, c_\xi)], l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)] \in (c_\xi, c_\xi(+^{n-1})) \subset \beta < \alpha$ are well defined and that $c_\xi < \eta(n-1, \alpha, s)[g(n-1, \alpha, c_\xi)] + 1 \leqslant l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$. This last inequalities imply, by (C1) and $\leqslant_1$-connectedness, that
$c_\xi <_1 \eta(n-1, \alpha, s)[g(n-1, \alpha, c_\xi)] + 1 \underset{\text{by (2.2.5) of GenThmIH}}{=}$
$\eta(n-1, c_\xi, s[g(n-1, \alpha, c_\xi)]) + 1$, and then (by (6) of GenThmIH)
$c_\xi <^{n-1} \eta(n-1, c_\xi, s[g(n-1, \alpha, c_\xi)]) + 1 \underset{\text{by (2.2.5) of GenThmIH}}{=} \eta(n-1, \alpha, s)[g(n-1, \alpha, c_\xi)] + 1$.

The previous shows that for all $\xi \in X$ and all $s \in S(n-1, \alpha, c_\xi, t)$, $c_\xi \in G^{n-1}(s) \underset{\text{cIH}}{=} A^{n-1}(s)$, that is, we have shown (C2). From (C2) and the fact that $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$ we conclude that $\beta \in \mathrm{Lim}\{r \leqslant \alpha \mid M < r \in \bigcap_{s \in S(n-1,\alpha,r,t)} A^{n-1}(s)\} = A^{n-1}(t)$. This shows (B4).

Now we show $G^{n-1}(t) \supset A^{n-1}(t)$. **(C3)**

Let $\beta \in A^{n-1}(t) = \mathrm{Lim}\{r \leqslant \alpha \mid M < r \in \bigcap_{s \in S(n-1,\alpha,r,t)} A^{n-1}(s)\} \underset{\text{cIH}}{=}$
$= \mathrm{Lim}\{r \leqslant \alpha \mid M < r \in \bigcap_{s \in S(n-1,\alpha,r,t)} G^{n-1}(s)\}$. Then there is a sequence $(c_\xi)_{\xi \in X}$ such that $M < c_\xi \underset{\text{cof}}{\longrightarrow} \beta$ and $\forall \xi \in X. c_\xi \in \bigcap_{s \in S(n-1,\alpha,c_\xi,t)} G^{n-1}(s)$. **(C4)**

Note that since $\forall \xi \in X. c_\xi \in \bigcap_{s \in S(n-1,\alpha,c_\xi,t)} G^{n-1}(s) \subset \mathrm{Class}(n-1)$ and $(c_\xi)_{\xi \in X}$ is cofinal in β, then, by proposition 4.2, $\beta \in \mathrm{Class}(n-1)$.

Now, for any $\xi \in X$, we know $\max(T(n-1, \alpha, t) \cap \alpha) = M < c_\xi < \beta$; therefore, by (2.2.3) and (2.1.4) of GenThmIH, we have hat $t[g(n-1, \alpha, \beta)], l(n-1, \alpha, t)[g(n-1, \alpha, \beta)] \in (\beta, \beta(+^{n-1}))$ and $t[g(n-1, \alpha, c_\xi)], l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)] \in (c_\xi, c_\xi(+^{n-1}))$ are well defined. **(C5)**

Our next aim is to show that $\forall \xi \in X. c_\xi \leqslant_1 l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$. **(C6)**

Let $\xi \in X$ be arbitrary. First note that, since $t \in \mathrm{Lim}$, then $l(n-1, \alpha, t) \in \mathrm{Lim}$ (because $l(n-1, \alpha, t) = t \in \mathrm{Lim}$ or $l(n-1, \alpha, t) < l(n-1, \alpha, t) + 1 \leqslant t \leqslant m(t) = m(l(n-1, \alpha, t))$; the latter case implies $l(n-1, \alpha, t) <_1 l(n-1, \alpha, t) + 1$ by $\leqslant_1$-connectedness and so $l(n-1, \alpha, t) \in \mathbb{P}$) and then $l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)] \in \mathrm{Lim}$ (simply because $l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$ is the result of substituting epsilon numbers by other epsilon numbers in the cantor normal form of $l(n-1, \alpha, t)$). Now, let $q \in (c_\xi, l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]) \underset{\text{by (2.2) of GenThmIH}}{\subset} (c_\xi, c_\xi(+^n))$ be arbitrary. Then by (2.3.1) of GenThmIH, $\mathrm{Ep}(q) \subset \mathrm{Dom}(g(n-1, c_\xi, \alpha))$ and then
$q[g(n-1, c_\xi, \alpha)] \underset{\text{by (2.4.3) of GenThmIH}}{\in} (\alpha, l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)][g(n-1, c_\xi, \alpha)]) =$
$\underset{\text{by (2.3.2) of GenThmIH}}{=} (\alpha, l(n-1, \alpha, t))$. This shows
$q[g(n-1, c_\xi, \alpha)] \in \mathrm{Im}(g(n-1, c_\xi, \alpha)) \cap (\alpha, l(n-1, \alpha, t)) \underset{\text{by (2.3.2) of GenThmIH}}{=}$
$(\mathrm{Dom}(g(n-1, \alpha, c_\xi))) \cap (\alpha, l(n-1, \alpha, t)) \underset{\text{remark 4.6}}{=} S(n-1, \alpha, c_\xi, t)$, and so by (C4), $c_\xi \in G^{n-1}(q[g(n-1, c_\xi, \alpha)])$. Finally, observe the latter implies that
$c_\xi \leqslant^{n-1} \eta(n-1, \alpha, q[g(n-1, c_\xi, \alpha)])[g(n-1, \alpha, c_\xi)] + 1 \underset{\text{(2.2.5) of GenThmIH}}{=}$
$= \eta(n-1, c_\xi, q[g(n-1, c_\xi, \alpha)][g(n-1, \alpha, c_\xi)]) + 1 = \eta(n-1, c_\xi, q) + 1$, which subsequently implies, (using $c_\xi < q \leqslant \eta(n-1, c_\xi, q)$ and $\leqslant_1$-connectedness) that $c_\xi \leqslant_1 q$.

Last paragraph proves that, for $\xi \in X$, the sequence $(d_q)_{q\in Y}$ defined as $d_q := q$ and $Y := (c_\xi, l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)])$, satisfies $\forall q \in Y. c_\xi \leqslant_1 q$; but this and the fact that $d_q \underset{\text{cof}}{\hookrightarrow} l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$ (we already showed $l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)] \in \mathrm{Lim}$) imply $c_\xi \leqslant_1 l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$ by $\leqslant_1$-continuity. Since the previous was done for arbitrary $\xi \in X$, we conclude $\forall \xi \in X. c_\xi \leqslant_1 l(n-1, \alpha, t)[g(n-1, \alpha, c_\xi)]$. This proves (C6).

We continue with the proof of (C3).

Let $\xi \in X$. Using (C6) we get

$$c_\xi \leqslant_1 l(n-1,\alpha,t)[g(n-1,\alpha,c_\xi)] \leqslant_1 \eta(n-1, c_\xi, l(n-1,\alpha,t)[g(n-1,\alpha,c_\xi)]) = \underset{\text{by (2.5.3) of GenThmIH}}{=} \eta(n-1, c_\xi, l(n-1,\alpha,t)[g(n-1,\beta,c_\xi)\circ g(n-1,\alpha,\beta)]) = \eta(n-1, c_\xi, l(n-1,\alpha,t)[g(n-1,\alpha,\beta)][g(n-1,\beta,c_\xi)]) = \underset{\text{by (2.2.5) of GenThmIH}}{=} \eta(n-1,\beta, l(n-1,\alpha,t)[g(n-1,\alpha,\beta)])[g(n-1,\beta,c_\xi)];$$

therefore, by $\leqslant_1$-transitivity, $c_\xi \leqslant_1 \eta(n-1, \beta, l(n-1, \alpha, t)[g(n-1, \alpha, \beta)])[g(n-1, \beta, c_\xi)]$. But since this was done for arbitrary $\xi \in X$, we have proved

$\forall \xi \in X. c_\xi \leqslant_1 \eta(n-1, \beta, l(n-1,\alpha,t)[g(n-1,\alpha,\beta)])[g(n-1,\beta,c_\xi)]$. **(C7)**

Finally, from (C7), the fact that $c_\xi \underset{\text{cof}}{\hookrightarrow} \beta$ and (5) of GenThmIH follow that

$$\beta \leqslant^{n-1} \eta(n-1,\beta, l(n-1,\alpha,t)[g(n-1,\alpha,\beta)]) + 1 \underset{\text{by (2.2.5) of GenThmIH}}{=} \eta(n-1,\alpha, l(n-1,\alpha,t))[g(n-1,\alpha,\beta)] + 1 \underset{\text{by proposition 3.21}}{=}$$

$= \eta(n-1, \alpha, t)[g(n-1, \alpha, \beta)] + 1$. This and (C5) show that $\beta \in G^{n-1}(t)$. But the previous we have done for arbitrary $\beta \in A^{n-1}(t)$, so we have proved $A^{n-1}(t) \subset G^{n-1}(t)$, i.e., we have proved (C3). □

4.2 Uncountable regular ordinals and the $A^{n-1}(t)$ sets

Proposition 4.11. *Let κ be an uncountable regular ordinal ($\kappa \in \mathrm{Class}(n-1)$ by proposition 4.1). Then $\forall t \in [\kappa, \kappa(+^{n-1}))$, $A^{n-1}(t)$ is club in κ.*

Proof. We prove the claim by induction on the interval $[\kappa, \kappa(+^{n-1}))$.

Case $t = \kappa$.

Then $T(n-1, \kappa, t) \cap \kappa \underset{\text{definition of } T(n-1,\kappa,\kappa)}{=} \emptyset$. So

$A^{n-1}(t) = (\mathrm{Lim}\,\mathrm{Class}(n-1)) \cap (-\infty, \kappa+1) = \mathrm{Lim}\,\mathrm{Class}(n-1)$ is club in κ because $\mathrm{Class}(n-1)$ is club in κ (by GenThmIH (0)) and because of proposition 2.46.

Our induction hypothesis is

$\forall s \in [\kappa, \kappa(+^{n-1})) \cap t. A^{n-1}(s)$ is club in κ. **(IH)**

Case $t = l + 1 \in [\kappa, \kappa(+^{n-1}))$.

Then $A^{n-1}(t) = A^{n-1}(l+1) = \begin{cases} A^{n-1}(l) & \text{if } l < \eta(n-1,\kappa,l) \\ \mathrm{Lim} A^{n-1}(l) & \text{otherwise} \end{cases}$; this way, by our (IH) and proposition 2.46, $A^{n-1}(t)$ is club in κ.

Case $t \in [\kappa, \kappa(+^{n-1})) \cap \mathrm{Lim}$.

By definition

$$M := \begin{cases} \max(T(n-1,\kappa,t)\cap\kappa) & \text{iff} \quad T(n-1,\kappa,t)\cap\kappa \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

and

$$A^{n-1}(t) = \begin{cases} (\mathrm{Lim\,Class}(n-1)) \cap (M,\kappa+1) & \text{iff} \quad t \in [\kappa, \kappa(+^{n-2})...(+^2)(+^1)2] \\ \mathrm{Lim}\{r \leqslant \kappa \mid M < r \in \bigcap_{s\in S(n-1,\kappa,r,t)} A^{n-1}(s)\} & \text{otherwise} \end{cases}$$

If $t \in [\kappa, \kappa(+^{n-2})...(+^2)(+^1)2]$, then $A^{n-1}(t) = (\mathrm{Lim\ Class}(n-1)) \cap (M, \kappa+1)$ is club in κ because of exactly the same reasons as in the case $t = \kappa$.

So from now on we suppose $t \in (\kappa(+^{n-2})...(+^2)(+^1)2, \kappa(+^{n-1}))$.

First we make the following four observations:

- It is enough to show that $Y := \{r \leqslant \kappa \mid M < r \in \bigcap_{s\in S(n-1,\kappa,r,t)} A^{n-1}(s)\}$ is club in κ because, knowing this, we conclude $\mathrm{Lim}Y = A^{n-1}(t)$ is club in κ by proposition 2.46. Moreover, note that as a consequence of theorem 4.10, $\forall z \in \mathrm{Dom}\ A^{n-1}.A^{n-1}(z) \subset \mathrm{Class}(n-1)$ and therefore
$Y = \{r \in \mathrm{Class}(n-1) \cap (\kappa+1) \mid M < r \in \bigcap_{s\in S(n-1,\kappa,r,t)} A^{n-1}(s)\}$. **(0*)**

- For $r \in \mathrm{Class}(n-1) \cap \kappa$,
$\{q \in (\kappa, l(n-1,\kappa,t)) \mid \mathrm{Ep}(q) \subset (\mathrm{Im}\, g(n-1,r,\kappa))\} \underset{\text{by (2.3.2) of GenThmIH}}{=}$
$\{q \in (\kappa, l(n-1,\kappa,t)) \mid \mathrm{Ep}(q) \subset (\mathrm{Dom}\, g(n-1,\kappa,r))\} \underset{\text{by remark 4.6}}{=}$
$S(n-1,\kappa,r,t) \underset{\text{by remark 4.6}}{\subset} l(n-1,\kappa,t) \underset{\text{by remark 4.6}}{\leqslant} t.$ **(1*)**

- By (1*) and our (IH), $\forall r \in \mathrm{Class}(n-1)\cap\kappa \forall s \in S(n-1,\kappa,r,t)$, $A^{n-1}(s)$ is club in κ. **(2*)**

- Let $r \in \mathrm{Class}(n-1) \cap \kappa$. By (0) of GenThmIH, $\mathrm{Class}(n-1)$ is club in κ and consequently $r(+^{n-1}) \in \mathrm{Class}(n-1) \cap \kappa$; moreover, by proposition 4.1, $\kappa \in \mathrm{Class}(n-1)$ and subsequently, $r < r(+^{n-1}) < \kappa < \kappa(+^{n-1})$. Consider the function P_r: $r(+^{n-1}) \longrightarrow \kappa(+^{n-1})$ defined as $P_r(x) := x[g(n-1,r,\kappa)]$. P_r is well defined because of (2.3.1) of GenThmIH. We now show that $S(n-1,\kappa,r,t) \subset \mathrm{Im}\, P_r$. This is easy: Take $q \in S(n-1,\kappa,r,t)$. Then, by (1*), $\mathrm{Ep}(q) \subset \mathrm{Dom}(g(n-1,\kappa,r))$ and therefore $q[g(n-1,\kappa,r)]$ is well defined; but then, by (2.3.3) and (2.3.2) of GenThmIH, $q[g(n-1,\kappa,r)] \in r(+^{n-1})$ and $q = q[g(n-1,\kappa,r)][g(n-1,r,\kappa)] = P_r(q[g(n-1,\kappa,r)])$. This shows $S(n-1,\kappa,r,t) \subset \mathrm{Im}\ P_r$ as we assured. Finally, since P_r is a strictly increasing function (so it is injective), then
$|S(n-1,\kappa,r,t)| \leqslant |\mathrm{Im}\, P_r| = |r(+^{n-1})| \underset{\text{because } \kappa \text{ is a cardinal}}{<} \kappa.$ **(3*)**

After the previous observations, we continue with the proof of the theorem, that is, as already said in (0*), we want to show that Y is club in κ.

We show first that Y is κ-closed.

Let $(r'_i)_{i\in I'} \subset Y \cap \kappa$ be such that $|I'| < \kappa$ and $r'_i \underset{\mathrm{cof}}{\longrightarrow} \rho$ for some $\rho < \kappa$. To show that $\rho \in Y$.

Since $Y \subset \mathrm{Class}(n-1)$ and by (0) of GenThmIH $\mathrm{Class}(n-1)$ is club in κ, then $\rho \in \mathrm{Class}(n-1)$. Now consider $s \in S(n-1,\kappa,\rho,t) =$
$\{d \in (\kappa, l(n-1,\kappa,t)) \subset (\kappa, \kappa(+^{n-1})) \mid T(n-1,\kappa,d) \cap \kappa \subset \rho\}$. Since by (2.1.1) of GenThmIH $T(n-1,\kappa,s) \cap \kappa$ is finite and $r'_i \underset{\mathrm{cof}}{\longrightarrow} \rho$, then there exists a subsequence $(r_i)_{i\in I}$ of the sequence $(r'_i)_{i\in I'}$, such that $r_i \underset{\mathrm{cof}}{\longrightarrow} \rho$, $\forall i \in I.T(n-1,\kappa,s) \cap \kappa \subset r_i$ and $|I| \leqslant |I'| < \kappa$; that is,
$\forall i \in I.s \in S(n-1,\kappa,r_i,t)$. This and the fact that $(r_i)_{i\in I} \subset Y$ means $\forall i \in I.r_i \in A^{n-1}(s)$. But by (2*) $A^{n-1}(s)$ is club in κ, so $\rho = \sup\{r_i \mid i \in I\} \in A^{n-1}(s)$. Our previous work shows that, for arbitrary $s \in S(n-1,\kappa,\rho,t)$, $\rho \in A^{n-1}(s)$, i.e., $\rho \in \bigcap_{s\in S(n-1,\kappa,\rho,t)} A^{n-1}(s)$. From this it follows that $\rho \in Y$. Hence Y is κ-closed.

Now our aim is to to prove that Y is unbounded in κ. **(b0)**
We do first the following:

Let $R := \mathrm{Class}(n-1) \cap \kappa$ and $B_r := \bigcap_{s \in S(n-1,\kappa,r,t)} A^{n-1}(s)$ for any $r \in R$.

Let's show first that $\forall \xi \in \mathrm{Lim}\, R \cap \kappa. \bigcap_{r \in R \cap \xi} B_r = B_\xi$. **(b1)**
Proof of (b1):
Let $\xi \in \mathrm{Lim}\, R \cap \kappa$.

We show (b1) contention $'' \subset ''$.
Let $x \in \bigcap_{r \in R \cap \xi} B_r = \bigcap_{r \in R \cap \xi} (\bigcap_{s \in S(n-1,\kappa,r,t)} A^{n-1}(s))$ be arbitrary. This means $\forall r \in R \cap \xi. \forall s \in S(n-1,\kappa,r,t). x \in A^{n-1}(s)$. **(b2)**.
On the other hand, let $z \in S(n-1,\kappa,\xi,t)$ be arbitrary. By definition of $S(n-1,\kappa,\xi,t)$, this means $z \in (\kappa, l(n-1,\kappa,t))$ and $T(n-1,\kappa,z) \cap \kappa \subset \xi$. But $T(n-1,\kappa,d) \cap \kappa$ is a finite set (by (2.1.1) of GenThmIH), so, since $\xi \in \mathrm{Lim}\, R$, there exists $r \in R$ such that $T(n-1,\kappa,z) \cap \kappa \subset r < \xi$. This means $z \in S(n-1,\kappa,r,t)$, and then, by (b2), $x \in A^{n-1}(z)$. Note the previous shows $\forall z \in S(n-1,\kappa,\xi,t). x \in A^{n-1}(z)$, i.e., $x \in \bigcap_{s \in S(n-1,\kappa,\xi,t)} A^{n-1}(s) = B_\xi$. Finally, since this was done for arbitrary $x \in \bigcap_{r \in R \cap \xi} B_r$, then we have actually shown that $\bigcap_{r \in R \cap \xi} B_r \subset B_\xi$.

Now we show (b1) contention $'' \supset ''$.
Let $x \in B_\xi = \bigcap_{s \in S(n-1,\kappa,\xi,t)} A^{n-1}(s)$ be arbitrary. This means $\forall s \in S(n-1,\kappa,\xi,t). x \in A^{n-1}(s)$. **(b4)**

On the other hand, let $r \in R \cap \xi$ be arbitrary. Take $z \in S(n-1,\kappa,r,t)$. By definition, this means $z \in (\kappa, l(n-1,\kappa,t))$ and $T(n-1,\kappa,z) \cap \kappa \subset r$. But since $r < \xi$, this implies that actually $z \in S(n-1,\kappa,\xi,t)$, which, together with (b4), implies $x \in A^{n-1}(z)$. Note we have shown $\forall z \in S(n-1,\kappa,r,t). x \in A^{n-1}(z)$, i.e, $x \in \bigcap_{z \in S(n-1,\kappa,r,t)} A^{n-1}(z) = B_r$; moreover, we have shown this for arbitrary $r \in R \cap \xi$, i.e., we have shown $x \in \bigcap_{r \in R \cap \xi} B_r$. Finally, since this was done for arbitrary $x \in B_\xi$, we have shown $\bigcap_{r \in R \cap \xi} B_r \supset B_\xi$.
This concludes the proof of (b1).

Now we show that $X := \{r \in R \mid M < r \in B_r\}$ is unbounded in κ. **(c0)**.

By (2*), (3*) and proposition 2.47 we have that for any $r \in R$, B_r is club in κ. **(c1)**

Let $\delta \in \kappa$ be arbitrary. Moreover, let $a := \min R$. We define by recursion the function $r: \omega \longrightarrow R$ as:
$r(0) := \min\{s \in \kappa \cap B_a \mid \delta < s > M\}$. Note $r(0)$ exists because of (c1).
Suppose we have defined $r(l) \in R = \mathrm{Class}(n-1) \cap \kappa$, for $l \in \omega$. **(rIH)**
Note that $|R \cap r(l)| \leqslant r(l) \underset{\text{by (rIH)}}{<} \kappa$, and then, by (c1) and proposition 2.47 it follows that $\bigcap\limits_{z \in R \cap r(l)} B_z$ is club in κ. So we define $r(l+1) := \min\{s \in \kappa \cap \bigcap\limits_{z \in R \cap r(l)} B_z \mid r(l) < s\}$.

Consider $\rho := \sup\{r(l) \mid l \in \omega\}$.
First note that, by construction, $(r(l))_{l \in \omega}$ is a strictly increasing sequence of ordinals in R (because any B_s is club in κ and $B_s \subset \mathrm{Class}(n-1)$) and so $\rho \in \mathrm{Class}(n-1) \cap (\mathrm{Lim}\, R)$. Moreover, since κ is an uncountable regular ordinal and $r: \omega \longrightarrow \kappa$, then $\rho < \kappa$. Summarizing all these observations: $\rho \in R \cap (\mathrm{Lim}\, R)$ **(c2)**

Now we show $M < \rho \wedge \delta < \rho \in B_\rho$. **(c3)**

That $\delta < \rho > M$ is clear from the definition of the function r. Now, let $\gamma \in R \cap \rho$ be arbitrary. Then there exists $l \in \omega$ such that $r(l) > \gamma$. Now, by the definition of our function r, $r(l+1) \in \bigcap_{z \in R \cap r(l)} B_z$; but this implies that the sequence $(r(s))_{s \in [l+1,\omega)} \subset B_\gamma$, and since $\rho = \sup\{r(s) \mid s \in [l+1,\omega)\}$ and B_γ is club in κ, then $\rho \in B_\gamma$. Finally, since this was done for arbitrary $\gamma \in R \cap \rho$, then we have actually shown that $\rho \in \bigcap_{\gamma \in R \cap \rho} B_\gamma \underset{\text{by (b1) and (c2)}}{=} B_\rho$. This concludes the proof of (c3).

Finally, observe that (c2) and (c3) have actually shown that $\forall \delta \in \kappa \exists \rho \in R. \delta < \rho \in X \subset R \subset \kappa$. Therefore (c0) holds. But $X \underset{\text{by (0*)}}{=} Y \cap \kappa \subset Y$. So Y is unbounded in κ. This has proven (b0). □

Proposition 4.12.
$\forall \alpha \in \text{Class}(n-1). \alpha <_1 \alpha(+^{n-1}) \Longleftrightarrow \alpha <^{n-1} \alpha(+^{n-1}) \Longleftrightarrow \alpha \in \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} A^{n-1}(t)$.

Proof. Let $\alpha \in \text{Class}(n-1)$.

To show $\alpha <^{n-1} \alpha(+^{n-1}) \Longrightarrow \alpha \in \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} A^{n-1}(t)$. **(a)**

Suppose $\alpha <^{n-1} \alpha(+^{n-1})$. Let $t \in [\alpha, \alpha(+^{n-1}))$ be arbitrary.
Then $\alpha \leqslant^{n-1} \eta(n-1,\alpha,t)[g(n-1,\alpha,\alpha)] + 1 = \eta(n-1,\alpha,t) + 1$ by $<^{n-1}$-connectedness. So $\alpha \in \{\beta \in \text{Class}(n-1) | T(n-1,\alpha,t) \cap \alpha \subset \beta \leqslant \alpha \wedge \beta \leqslant^{n-1} \eta(n-1,\alpha,t)[g(n-1,\alpha,\beta)] + 1\} = G^{n-1}(t) \underset{\text{theorem 4.10}}{=} A^{n-1}(t)$. Since this holds for an arbitrary $t \in [\alpha, \alpha(+^{n-1}))$, we have shown $\alpha \in \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} A^{n-1}(t)$. This shows (a).

To show $\alpha <^{n-1} \alpha(+^{n-1}) \Longleftarrow \alpha \in \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} A^{n-1}(t)$. **(b)**

Suppose $\alpha \in \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} A^{n-1}(t) \underset{\text{theorem 4.10}}{=} \bigcap_{t \in [\alpha, \alpha(+^{n-1}))} G^{n-1}(t)$. Then for any $t \in [\alpha, \alpha(+^{n-1}))$, $\alpha \leqslant^{n-1} \eta(n-1,\alpha,t)[g(n-1,\alpha,\alpha)] + 1 = \eta(n-1,\alpha,t) + 1$; thus, by (3) of GenThmIH (that is, by $\leqslant^{n-1}$-continuity), $\alpha \leqslant^{n-1} \alpha(+^{n-1})$. This shows (b).

To show $\alpha <_1 \alpha(+^{n-1}) \Longrightarrow \alpha <^{n-1} \alpha(+^{n-1})$. **(c)**

Suppose $\alpha <_1 \alpha(+^{n-1})$. Then for any $t \in [\alpha, \alpha(+^{n-1}))$, $\eta(n-1,\alpha,t)) + 1 \in (\alpha, \alpha(+^{n-1}))$ and so, by $\leqslant_1$-connectedness, $\alpha \leqslant_1 \eta(n-1,\alpha,t)) + 1$. Subsequently, by (6) of GenThmIH, $\alpha \leqslant^{n-1} \eta(n-1,\alpha,t)) + 1$. The previous shows that $\forall t \in [\alpha, \alpha(+^{n-1})). \alpha \leqslant^{n-1} \eta(n-1,\alpha,t)) + 1$, and since the sequence $\{\eta(n-1,\alpha,t)) + 1 \mid t \in [\alpha, \alpha(+^{n}))\}$ is confinal in $\alpha(+^{n-1})$, then by (3) of GenThmIH (that is, $\leqslant^{n-1}$-continuity), $\alpha <^{n-1} \alpha(+^{n-1})$. This shows (c).

Finally, $\alpha <_1 \alpha(+^{n-1}) \Longleftarrow \alpha <^{n-1} \alpha(+^{n-1})$ clearly holds by (3) of GenThmIH. □

Corollary 4.13. *Let κ be an uncountable regular ordinal ($\kappa \in \text{Class}(n-1)$ by proposition 4.1). Then*

a) $\kappa <^{n-1} \kappa(+^{n-1})$ *and therefore* $\kappa \in \text{Class}(n)$.

b) $\kappa \in \bigcap_{s \in [\kappa, \kappa(+^{n-1}))} A^{n-1}(s)$.

Proof. Left to the reader. □

Corollary 4.14.

1. $\text{Class}(n) \neq \emptyset$

2. For any $\alpha \in \text{Class}(n)$, $\alpha(+^n) < \infty$; that is, $\alpha(+^n)$ is an ordinal.

Proof. Left to the reader. $\square$

Lemma 4.15. *Let $k \in [1, n)$, $q \in \text{Class}(k)$, $t = \eta(k, q, t) \in [q, q(+^k))$ and $q <_1 t + 1$. Then there is a sequence $(\xi_j)_{j \in J} \subset \text{Class}(k)$ such that $\xi_j \underset{\text{cof}}{\hookrightarrow} q$ and such that for all $j \in J$, $T(k, q, t) \cap q \subset \xi_j$ and $m(\xi_j) = t[g(k, q, \xi_j)]$.*

Proof. Let k, q and t as stated. Then by (6) and (4) of GenThmIH, there exists a sequence $(l_i)_{i \in I} \in q \cap \text{Class}(k)$, $l_i \underset{\text{cof}}{\hookrightarrow} q$ such that for all $i \in I$,
$T(k, q, t) \cap q \subset l_i$ and $m(l_i) \geqslant t[g(k, q, l_i)]$. (*)

We have now two cases:
(a). For some subsequence $(l_d)_{d \in D} \subset (l_i)_{i \in I}$ it occurs $\forall d \in D. m(l_d) = t[g(k, q, l_d)]$.
Then $(l_d)_{d \in D}$ is the sequence we are looking for.

(b). For every subsequence $(l_d)_{d \in D} \subset (l_i)_{i \in I}$ $\exists d \in D. m(l_d) \neq t[g(k, q, l_d)]$.
Choose an arbitrary $e < q$ and let
$l := \min \{r \in q \cap \text{Class}(k) |\ T(k, q, t) \cap q \subset r > e \wedge m(r) > t[g(k, q, r)]\}$. Observe l exists because of (b) and (*). Then
$e < l <_1 t[g(k, q, l)] + 1 = \eta(k, q, t)[g(k, q, l)] + 1 \underset{\text{by (2.2.3) and (2.2.5) of GenThmIH}}{=} \eta(k, l, t[g(k, q, l)]) + 1$, which implies, by (6) and (4) of GenThmIH, the existence of a sequence $(s_u)_{u \in U}$, $s_u \underset{\text{cof}}{\hookrightarrow} l$ such that for all $u \in U$,

$$T(k, q, \eta(k, q, t)) \cap q \underset{\text{by (2.2.3) and (2.2.4) of GenThmIH}}{=} T(k, l, \eta(k, q, t)[g(k, q, l)]) \cap l \underset{\text{by (2.2.3) and (2.2.5) of GenThmIH}}{=} T(k, l, \eta(k, l, t[g(k, q, l)])) \cap l \subset s_u \qquad \textbf{(1*)}$$

and

$$s_u \leqslant_1 \eta(k, l, t[g(k, q, l)])[g(k, l, s_u)] \underset{\text{by (2.2.3) and (2.2.5) of GenThmIH}}{=} \eta(k, q, t)[g(k, q, l)][g(k, l, s_u)] = \eta(k, q, t)[g(k, l, s_u) \circ g(k, q, l)] \underset{\text{by (2.5.3) of GenThmIH}}{=} \eta(k, q, t)[g(k, q, s_u)] = t[g(k, q, s_u)]. \qquad \textbf{(2*)}$$

Now, note that (1*) and (2*) assert $\forall u \in U. T(k, q, \eta(k, q, t)) \cap q \subset s_u \wedge m(s_u) \geqslant t[g(k, q, s_u)]$. Therefore, since $s_u \underset{\text{cof}}{\hookrightarrow} l$, there is some $a \in U$ such that $e < s_a < l$,
$T(k, q, t) \cap q \subset s_a$ and $m(s_a) \geqslant t[g(k, q, s_a)]$; moreover, by the definition of l,
$m(s_a) \ngtr t[g(k, q, s_a)]$ and then $m(s_a) = t[g(k, q, s_a)]$. We define $\xi_e := s_a$. Then, the sequence $(\xi_e)_{e \in q}$ is the sequence we are looking for. $\square$

4.3 Canonical sequence of an ordinal $e(+^i)$

Reminder: For $e \in \mathbb{E}$, we denote by $(\omega_k(e))_{k \in \omega}$ to the recursively defined sequence $\omega_0(e) := e + 1$, $\omega_{k+1}(e) := \omega^{\omega_k(e)}$.

We want now to define, for $e \in \text{Class}(i)$, a (canonical) sequence cofinal in $e(+^i)$.

Definition 4.16. *(Canonical sequence of an ordinal $e(+^i)$)*
For $i \in [1, n)$, $e \in \text{Class}(i)$, and $k \in [1, \omega)$ we define the set $X_k(i, e)$ and the ordinals $x_k(i, e)$ and $\gamma_k(i, e)$ simultaneously by recursion on $([1, n), <)$ as follows:

Let $i=1$, $e\in\mathrm{Class}(1)$ *and* $k\in[1,\omega)$. *Let it be*
$X_k(1,e):=\{\omega_k(e)\}$,
$x_k(1,e):=\omega_k(e)=\min X_k(1,e)$ *and*
$\gamma_k(1,e):=m(x_k(1,e))=m(\omega_k(e))=\pi(\omega_k(e))+d\pi(\omega_k(e))=\omega_k(e)+d(\omega_k(e))=\eta(\omega_k(e))=$
$\underset{\text{properties of }\eta}{=}\eta(\eta(\omega_k(e)))=\eta(\gamma_k(1,e))=\eta(1,e,\gamma_k(1,e))$.

Suppose $i+1\in[2,n)$ *and that for* $i\in[1,n)$, $X_k(i,E)$, $x_k(i,E)$ *and* $\gamma_k(i,E)$ *have already been defined for arbitrary* $E\in\mathrm{Class}(i)$ *and* $k\in[1,\omega)$.

Let $e\in\mathrm{Class}(i+1)$ *and* $k\in[1,\omega)$. *We define*
$X_k(i+1,e):=\{r\in(e,e(+^{i+1}))\cap\mathrm{Class}(i)\mid m(r)=\gamma_k(i,r)\}$,
$x_k(i+1,e):=\min X_k(i+1,e)$ *and*
$\gamma_k(i+1,e):=m(x_k(i+1,e))\in(x_k(i+1,e),x_k(i+1,e)(+^i))\subset(e,e(+^{i+1}))$.

For $e\in\mathrm{Class}(i)$, *we call* $(\gamma_k(i,e))_{k\in[1,\omega)}$ *the canonical sequence of* $e(+^i)$.

To assure that our previous definition 4.16 is correct we need to show that min $X_k(i,\ e)$ exists. This is one of the reasons for our next

Proposition 4.17. $\forall i\in[1,n)\forall e\in\mathrm{Class}(i)$.

1. *For any* $k\in[1,\omega)$, $X_k(i,e)\neq\emptyset$ *and therefore* $\min X_k(i,e)$ *exists.*
2. $(\gamma_j(i,e))_{j\in[1,\omega)}\subset(e,e(+^i))$
3. $\forall j\in[1,\omega).\gamma_j(i,e)=\eta(i,e,\gamma_j(i,e))$
4. $(\gamma_j(i,e))_{j\in[1,\omega)}$ *and cofinal in* $e(+^i)$.
5. *If* $i=1$, *then* $\forall z\in[1,\omega).T(i,e,\gamma_z(i,e))=\{\lambda(1,\gamma_z(1,e))=e\}$.
   Case $i\geqslant 2$. *Then for any* $z\in[1,\omega)$,
   - $m(x_z(i,e))=m(x_z(i-1,x_z(i,e)))=\ldots=m(x_z(2,x_z(3,\ldots x_z(i-1,x_z(i,e))\ldots)))$;
   - $T(i,e,\gamma_z(i,e))=\{o_1>o_2>\ldots>o_{i-1}>o_i=e\}$, *where*
     $o_1:=\lambda(1,\gamma_z(i,e))$,
     $o_2:=\lambda(2,\gamma_z(i,e)),\ldots$,
     $o_{i-1}:=\lambda(i-1,\gamma_z(i,e))$,
     $o_i:=\lambda(i,\gamma_z(i,e))=e$;
   - *Moreover,*
     $x_z(i,e)=o_{i-1}$,$x_z(i-1,x_z(i,e))=o_{i-2},\ldots$,$x_z(2,x_z(3,\ldots x_z(i-1,x_z(i,e))\ldots))=o_1$
     and
     $o_2=\lambda(2,o_1),o_3=\lambda(3,o_2),\ldots,o_i=\lambda(i,o_{i-1}),o_{i+1}=\lambda(i+1,o_i)$.
6. $\forall j\in[1,\omega).\forall a\in(T(i,e,\gamma_j(i,e))\backslash\{e\}).m(a)=\gamma_j(i,e)$
7. $\forall\alpha\in\mathrm{Class}(i).\forall j\in[1,\omega).\emptyset=T(i,e,\gamma_j(i,e))\cap e\subset\alpha\wedge\gamma_j(i,e)[g(i,e,\alpha)]=\gamma_j(i,\alpha)$

Proof. We prove simultaneously 1, 2, 3, 4, 5, 6 and 7 by induction on $[1,n)$.

Case $i=1$ and $e\in\mathrm{Class}(1)$.

It follows immediately from definition 4.16 (and the equalities explicitly given there) that 1, 2 and 3 hold. Moreover, it is also clear that 4. holds.

Now, let $j\in[1,\omega)$ be arbitrary. Then, by the definition (see statement of theorem 3.26), $T(1,\ e,\ \gamma_j(1,e))=\ \bigcup_{E\in\mathrm{Ep}(\gamma_j(1,e))}T(1,\ e,\ E)=\mathrm{Ep}(\gamma_j(1,\ e))=\{e=\lambda(1,\ \gamma_j(1,\ e))\}$. So 5. holds. Moreover, by the equality $T(1,e,\gamma_j(1,e))=\{e\}$ it is clear that 6. holds too.

Finally, let $\alpha \in \text{Class}(1)$ and $j \in [1, \omega)$ be arbitrary. Then by 5. $\emptyset = T(i, e, \gamma_j(i, e)) \cap e \subset \alpha$. Moreover, by definition of $\gamma_j(1, e)$ and the usual properties of the substitution $x \longmapsto x[e := \alpha]$, we have $\gamma_j(1, e)[g(1, e, \alpha)] = \gamma_j(1, e)[e := \alpha] = \gamma_j(1, \alpha)$, that is, 7. holds.

Let $i+1 \in [2, n)$ and suppose the claim holds for i. **(IH)**

Let $e \in \text{Class}(i+1)$.

To show that 1. holds.

Let $k \in [1, \omega)$.

Since $e(+^{i+1}) \in \text{Class}(i+1) \subset \text{Class}(i)$, then by our (IH), $\eta(i, e(+^{i+1}), \gamma_k(i, e(+^{i+1}))) = \gamma_k(i, e(+^{i+1})) \in (e(+^{i+1}), e(+^{i+1})(+^i))$; from this, the fact that $e(+^{i+1}) <_1 e(+^{i+1})(+^i)$ and $<_1$-connectedness follows $e(+^{i+1}) <_1 \eta(i, e(+^{i+1}), \gamma_k(i, e(+^{i+1}))) + 1$. Thus, by lemma 4.15, there is a sequence $(\xi_j)_{j \in J} \subset \text{Class}(i)$ such that $\xi_j \underset{\text{cof}}{\hookrightarrow} e(+^{i+1})$ and such that for all $j \in J$, $T(i, e(+^{i+1}), \gamma_k(i, e(+^{i+1}))) \cap e(+^{i+1}) \subset \xi_j$ and $m(\xi_j) = \gamma_k(i, e(+^{i+1}))[g(i, e(+^{i+1}), \xi_j)] \underset{\text{by 7. of our (IH)}}{=} \gamma_k(i, \xi_j)$.

From the previous follows that $X_k(i+1, e) = \{r \in (e, e(+^{i+1})) \cap \text{Class}(i) \mid m(r) = \gamma_k(i, r)\} \neq \emptyset$. Hence 1. holds.

2. holds.

This is clear from the definition of $(\gamma_j(i+1, e))_{j \in [1, \omega)}$ (the fact that $X_k(i+1, e) \neq \emptyset$ implies that $(\gamma_j(i+1, e))_{j \in [1, \omega)}$ is well defined).

To show that $[\gamma_k(i+1, e))_{k \in [1, \omega)}$ satisfies 3.

Let $k \in [1, \omega)$.

Since $x_k(i+1, e) \in (e, e(+^{i+1})) \cap \text{Class}(i)$, then $x_k(i+1, e) \geqslant e(+^i)$ and so $m(x_k(i+1, e)) \geqslant e(+^i)(+^{i-1})...(+^1)2$. (1*)

On the other hand, for any $t \in (e, x_k(i+1, e))$ proposition 3.6 implies $m(t) < x_k(i+1, e) \leqslant m(x_k(i+1, e))$. Moreover, notice for any $t \in [x_k(i+1, e), m(x_k(i+1, e))]$, $m(t) \not> m(x_k(i+1, e))$: Assume the opposite. Then the inequalities $x_k(i+1, e) \leqslant t \leqslant m(x_k(i+1, e)) < m(x_k(i+1, e)) + 1 \leqslant m(t)$ imply by $\leqslant_1$-connectedness that $x_k(i+1, e) \leqslant_1 t <_1 m(x_k(i+1, e)) + 1$ and then, by $\leqslant_1$-transitivity, $x_k(i+1, e) <_1 m(x_k(i+1, e)) + 1$. Contradiction. Hence, from all this we conclude $\forall t \in (e, m(x_k(i+1, e))].m(t) \leqslant m(x_k(i+1, e))$. (2*)

Finally,

$$\begin{aligned}\eta(i+1, e, \gamma_k(i+1, e)) = \eta(i+1, e, m(x_k(i+1, e))) &\underset{\text{by (1*)}}{=} \\ = \max\{m(\beta) \mid \beta \in (e, m(x_k(i+1, e))]\} &\underset{\text{by (2*)}}{=} \\ = m(x_k(i+1, e)) = \gamma_k(i+1, e).&\end{aligned}$$

Thus 3. holds.

To show 4., that is, $(\gamma_k(i+1, e))_{k \in [1, \omega)}$ is cofinal in $e(+^{i+1})$.

First note that since $e(+^{i+1})(+^{i-1})...(+^2)(+^1)2+1 \in (e(+^{i+1}), e(+^{i+1})(+^i))$, then $e(+^{i+1}) <_1 e(+^{i+1})(+^{i-1})...(+^2)(+^1)2+1 = \eta(i, e(+^{i+1}), e(+^{i+1})(+^{i-1})...(+^2)(+^1)2) + 1$ by $<_1$-connectedness; then, by (6) and (4) of GenThmIH, there exist a sequence of elements in Class(i) that is cofinal in $e(+^{i+1})$. So, to show that $(\gamma_k(i+1, e))_{k \in [1, \omega)}$ is cofinal in $e(+^{i+1})$ it is enough to show $\forall \sigma \in (e, e(+^{i+1})) \cap \text{Class}(i). \exists s \in [1, \omega). \gamma_s(i+1, e) > \sigma$. **(b1)**

To show (b1).

Let $\sigma \in \text{Class}(i) \cap (e, e(+^{i+1}))$. Then by (1.3), (1.3.1), (1.3.5), (1.3.6) and (1.3.4) of GenThmIH $f(i+1,e)(\sigma)=\{\sigma=\sigma_1>...>\sigma_q\}$ for some $q\in[1,\omega)$, **(c0)**
where
$\sigma_q=\min\{d\in(e,\sigma_q]\cap\text{Class}(i)\mid m(d)[g(i,d,\sigma_q)]\geqslant m(\sigma_g)\}$, **(c1)**
$\forall l\in[1,q-1].\sigma_l=\min\{d\in(\sigma_{l+1},\sigma_l]\cap\text{Class}(i)\mid m(d)[g(i,d,\sigma_l)]\geqslant m(\sigma_l)\}$ **(c2)**
and
$m(\sigma)=m(\sigma_1)\leqslant m(\sigma_2)[g(i,\sigma_2,\sigma)]\leqslant m(\sigma_3)[g(i,\sigma_3,\sigma)]\leqslant...\leqslant m(\sigma_q)[g(i,\sigma_q,\sigma)]$. **(c3)**

On the other hand, by (IH) $(\gamma_j(i,\sigma_q))_{j\in[1,\omega)}$ is cofinal in $\sigma_q(+^i)$, so there exists $z\in[1,\omega)$ such that $\gamma_z(i,\sigma_q)\in(m(\sigma_q),\sigma_q(+^i))$. **(c4)**
But by (c1),
$\forall d\in(e,\sigma_q]\cap\text{Class}(i).m(d)[g(i,d,\sigma_q)]\leqslant m(\sigma_q)<\gamma_z(i,\sigma_q)$, particularly,
$\forall d\in(e,\sigma_q]\cap\text{Class}(i).m(d)[g(i,d,\sigma_q)]<\gamma_z(i,\sigma_q)$. From this and using (2.3.2) of GenThmIH and (7) of our (IH), we get
$\forall d\in(e,\sigma_q]\cap\text{Class}(i)$.
$m(d)=m(d)[g(i,d,\sigma_q)][g(i,\sigma_q,d)]<\gamma_z(i,\sigma_q)[g(i,\sigma_q,d)]\underset{\text{by (7) of our (IH)}}{=}\gamma_z(i,d)$ **(*)**

Now let $l\in[1,q-1]$ and $d\in(\sigma_{l+1},\sigma_l]\cap\text{Class}(i)$. By (c2), $m(d)[g(i,d,\sigma_l)]\leqslant m(\sigma_l)$; this inequality, (c0) and (2.5.3) and (2.3.1) of GenThmIH imply,
$m(d)[g(i,d,\sigma)]=m(d)[g(i,d,\sigma_l)][g(i,\sigma_l,\sigma)]\leqslant m(\sigma_l)[g(i,\sigma_l,\sigma)]\leqslant$
$\underset{\text{by (c3)}}{\leqslant}m(\sigma_q)[g(i,\sigma_q,\sigma)]\underset{\text{using (c4)}}{<}\gamma_z(i,\sigma_q)[g(i,\sigma_q,\sigma)]\underset{\text{(7) of our (IH)}}{=}\gamma_z(i,\sigma)$. From this, by (2.3.2) and (2.5.3) of GenThmIH,
$m(d)=m(d)[g(i,d,\sigma)][g(i,\sigma,d)]<\gamma_z(i,\sigma)[g(i,\sigma,d)]\underset{\text{(7) of our (IH)}}{=}\gamma_z(i,d)$. The previous shows
$\forall l\in[1,q-1]\forall d\in(\sigma_{l+1},\sigma_l]\cap\text{Class}(i).m(d)<\gamma_z(i,d)$. **(**)**

From (*) and (**) follows $\forall d\in(e,\sigma]\cap\text{Class}(i).m(d)<\gamma_z(i,d)$, and therefore
$\forall d\in(e,\sigma]\cap\text{Class}(i).d<\min X_z(i+1,e)=x_z(i+1,e)\leqslant m(x_z(i+1,e))=\gamma_z(i+1,e)$.

This shows (b1). Hence 4. holds.

<u>To show that $(\gamma_k(i+1,e))_{k\in[1,\omega)}$ satisfies 5.</u>

First note that for arbitrary $k,j\in[1,\omega)$ and $c\in\text{Class}(j+1)$
$x_k(j+1,c)=\min\{r\in(c,c(+^{j+1}))\cap\text{Class}(j)\mid m(r)=\gamma_k(j,r)\}$. **(J0)**
So
$m(x_k(j+1,c))=\gamma_k(j,x_z(j+1,c))=m(x_k(j,x_z(j+1,c)))$; **(J1)**
$x_k(j+1,c)\in\text{Class}(j)\backslash\text{Class}(j+1)$; **(J2)**
$\gamma_k(j+1,c)=m(x_k(j+1,c))\in(x_k(j+1,c),x_k(j+1,c)(+^j))$; **(J3)**
$\lambda(j,m(x_k(j+1,c))=\lambda(j,\gamma_k(j+1,c))=x_k(j+1,c)$. **(J4)**

Let $z\in[1,\omega)$.

We show now

$$\gamma_z(i+1,e)=m(x_z(i+1,e))=m(x_z(i,x_z(i+1,e)))=...=$$
$$=m(x_z(2,x_z(3,...x_z(i,x_z(i+1,e))...)))\qquad\textbf{(J5)}$$

This is easy:

$$\gamma_z(i+1,e)=m(x_z(i+1,e))\underset{\text{by (J1)}}{=}m(x_z(i,x_z(i+1,e)))\underset{\text{by (J1)}}{=}$$
$$=m(x_z(i-1,x_z(i,x_z(i+1,e))))\underset{\text{by (J1)}}{=}....\underset{\text{by (J1)}}{=}$$
$$=m(x_z(2,x_z(3,...x_z(i,x_z(i+1,e))...))).$$

This shows (J5).

Let's abbreviate
$o_1 := \lambda(1, \gamma_z(i+1, e))$,
$o_2 := \lambda(2, \gamma_z(i+1, e)), \ldots,$
$o_i := \lambda(i, \gamma_z(i+1, e))$,
$o_{i+1} := \lambda(i+1, \gamma_z(i+1, e))$. **(d1)**

To show $o_{i+1} = e$,
$x_z(i+1, e) = o_i$,$x_z(i, x_z(i+1, e)) = o_{i-1}, \ldots$,$x_z(2, x_z(3, \ldots x_z(i, x_z(i+1, e))\ldots)) = o_1$ and
$o_2 = \lambda(2, o_1), o_3 = \lambda(3, o_2), \ldots, o_i = \lambda(i, o_{i-1}), o_{i+1} = \lambda(i+1, o_i)$. **(J6)**

First let's see $o_{i+1} = e$. **(J6.1)**
Note $\gamma_z(i+1, e) \underset{\text{By (J3)}}{\in} (x_z(i+1, e), x_z(i+1, e)(+^i)) \underset{\text{By (J0)}}{\subset} (e, e(+^{i+1}))$. Then, since $(e, e(+^{i+1})) \cap \mathrm{Class}(i) = \emptyset$, we get $o_{i+1} = \lambda(i+1, \gamma_z(i+1, e)) = e$. So (J6.1) holds.

Now let's show $x_z(i+1, e) = o_i, \ldots$,$x_z(2, x_z(3, \ldots x_z(i, x_z(i+1, e))\ldots)) = o_1$. **(J6.2)**
This is also easy:
$x_z(i+1, e) \underset{\text{by (J4)}}{=} \lambda(i, \gamma_z(i+1, e)) = o_i$,
$x_z(i, x_z(i+1, e)) \underset{\text{by (J4)}}{=} \lambda(i-1, m(x_z(i, x_z(i+1, e)))) \underset{\text{by (J5)}}{=} \lambda(i-1, \gamma_z(i+1, e)) = o_{i-1}$,
...
$x_z(2, x_z(3, \ldots x_z(i, x_z(i+1, e))\ldots)) \underset{\text{by (J4)}}{=} \lambda(1, m(x_z(2, x_z(3, \ldots x_z(i, x_z(i+1, e))\ldots)))) \underset{\text{by (J5)}}{=}$
$= \lambda(1, \gamma_z(i+1, e)) = o_1$.
So (J6.2) holds.

Let's see that $o_2 = \lambda(2, o_1), o_3 = \lambda(3, o_2), \ldots, o_i = \lambda(i, o_{i-1}), o_{i+1} = \lambda(i+1, o_i)$. **(J6.3)**
Note for any $k \in [1, i]$
$o_k \underset{\text{by (J6.2) and (J6.1)}}{=} x_z(k+1, o_{k+1}) \underset{\text{by (J0)}}{\in} (o_{k+1}, o_{k+1}(+^{k+1})) \cap \mathrm{Class}(k)$, so $\lambda(k+1, o_k) = o_{k+1}$.
So (J6.3) holds.

Hence (J6) holds because of the proofs of (J6.1), (J6.2), (J6.3).

To show $T(i+1, e, \gamma_z(i+1, e)) = \{o_1 > o_2 > \ldots > o_{i-1} > o_{i+1}\}$ **(J7)**

Since $\gamma_z(i+1, e) \underset{\text{by (J5) and (J6)}}{=} m(o_1)$, then
$T(i+1, e, \gamma_z(i+1, e)) = T(i+1, e, m(o_1)) = \bigcup_{d \in \mathrm{Ep}(m(o_1))} T(i+1, e, d) \underset{\mathrm{Ep}(m(o_1)) = T(1, o_1, m(o_1))}{=}$ $\bigcup_{d \in T(1, o_1, m(o_1))} T(i+1, e, d) \underset{\text{by our (IH)}}{=} \bigcup_{d \in \{o_1\}} T(i+1, e, d) = T(i+1, e, o_1) = \bigcup_{k \in \omega} O(k, o_1)$,

where by definition
$E_1 = \lambda(1, m(o_1)) = o_1$,$E_2 = \lambda(2, E_1) \underset{\text{by (d1)}}{=} o_2, \ldots$,$E_{i+1} = \lambda(i+1, E_i) \underset{\text{by (d1)}}{=} o_{i+1}$ and

$$O(0, o_1) := \bigcup_{\substack{\delta \in W(0, k, o_1) \\ k = 1, \ldots, i}} f(k+1, \lambda(k+1, \delta))(\delta) \cup \mathrm{Ep}(m(\delta)) \cup \{\lambda(k+1, \delta)\};$$

$W(0, k, o_1) := (e, e(+^{i+1})) \cap \{E_1 > E_2 \geqslant E_3 \geqslant \ldots \geqslant E_{i+1} = e\} \cap (\mathrm{Class}(k) \backslash \mathrm{Class}(k+1))$;

$$O(l+1, o_1) := \bigcup_{\substack{\delta \in W(l, k, o_1) \\ k = 1, \ldots, i}} f(k+1, \lambda(k+1, \delta))(\delta) \cup \mathrm{Ep}(m(\delta)) \cup \{\lambda(k+1, \delta)\};$$

$W(l, k, o_1) := (e, e(+^{i+1})) \cap O(l, o_1) \cap (\mathrm{Class}(k) \backslash \mathrm{Class}(k+1))$.

But note for any $k\in[1,i]$,
$W(0,k,o_1)=\{o_k\}$,
$\lambda(k+1,o_k)=o_{k+1}$,
$f(k+1,\lambda(k+1,o_k))(o_k)=f(k+1,o_{k+1})(o_k)=\{o_k\}$,
$\mathrm{Ep}(m(o_k))\underset{\text{by (J5) and (J6)}}{=}\mathrm{Ep}(m(o_1))=\{o_1\}$.
Therefore $f(k+1,\lambda(k+1,o_k))(o_k)\cup\mathrm{Ep}(m(o_k))\cup\{\lambda(k+1,o_k)\}=\{o_k,o_1,o_{k+1}\}$. This way $O(0,o_1)=\{o_1>o_2\geqslant o_3\geqslant\ldots\geqslant o_{i+1}=e\}$, and moreover, exactly because of the same reasoning, $\forall l\in\omega.O(l+1,o_1)=\{o_1>o_2\geqslant o_3\geqslant\ldots\geqslant o_{i+1}=e\}$. Thus, we conclude
$T(i+1,e,\gamma_z(i+1,e))=\{o_1>o_2\geqslant o_3\geqslant\ldots\geqslant o_{i+1}=e\}$. Finally, we just make the reader aware that actually $o_1>o_2>o_3>\ldots>o_{i+1}$ holds because of (J6) and (J2). So we have shown (J7). This concludes the proof of 5.

<u>To show that $(\gamma_k(i+1,e))_{k\in[1,\omega)}$ satisfies 6.</u>

From 5. we get $T(i,e,\gamma_z(i,e))\backslash\{e\}=\{o_1>o_2>\ldots>o_{i-1}\}$ with $m(o_{i-1})=\ldots=m(o_1)$.

<u>To show $(\gamma_k(i+1,e))_{k\in[1,\omega)}$ satisfies 7.</u>

Let $\alpha\in\mathrm{Class}(i+1)$ and $z\in[1,\omega)$.
Let $o_1,\ldots,o_{i+1}$ as in 5. (that is, for $k\in[1,i+1]$, $o_k:=\lambda(k,\gamma_z(i+1,e))$). By 5., we know that $T(i+1,e,\gamma_z(i+1,e))=\{o_1>o_2>\ldots>o_{i-1}>o_{i+1}=e\}$. So $T(i+1,e,\gamma_z(i+1,e))\cap e=\emptyset$. Now, for any $k\in[1,i+1]$, $T(i+1,e,o_k)\underset{\text{definition of }T(i+1,e,o_k)}{\subset}T(i+1,e,\gamma_z(i+1,e))$, which means $\forall k\in[1,i+1].\emptyset=T(i+1,e,o_k)\cap e\subset\alpha$. The latter expression implies, by (2.2.3) of GenThmIH that $\forall k\in[1,i+1].\mathrm{Ep}(o_k)\subset\mathrm{Dom}[g(i+1,e,\alpha)]$. So for $k\in[1,i+1]$, let $\boldsymbol{u_k:=o_k[g(i+1,e,\alpha)]}$.

We will need the following observations (K1), (K2), (K3) and (W):

Since by 5. we know $\forall k\in[1,i].o_{k+1}=\lambda(k+1,o_k)$, then by (2.4.6) of GenThmIH, this implies $\forall k\in[1,i].u_{k+1}=o_{k+1}[g(i+1,e,\alpha)]=\lambda(k+1,o_k[g(i+1,e,\alpha)])=\lambda(k+1,u_k)$. **(K1)**

Note $o_1\underset{\text{by 5.}}{=}x_z(2,o_2)\in X_z(2,o_2):=\{r\in(o_2,o_2(+^2))\cap\mathrm{Class}(1)\mid m(r)=\gamma_z(1,r)\}$. This implies $m(o_1)=\gamma_z(1,o_1)$. **(K2)**

Moreover, observe that
$o_i<_1 o_{i-1}<_1\ldots<_1 o_1<_1\gamma_z(i+1,e)=m(o_1)\underset{\text{by (K2)}}{=}\gamma_z(1,o_1)\underset{\text{by definition}}{=}m(\omega_z(o_1))$ and
$\forall j\in[1,i].o_j\not\leqslant_1 m(\omega_z(o_1))+1$ imply, by (2.4.3) of GenThmIH, that
$u_i<_1\ldots<_1 u_1<_1 m(\omega_z(o_1))[g(i+1,e,\alpha)]\underset{\text{by (2.4.4) of GenThmIH}}{=}m((\omega_z(o_1))[g(i+1,e,\alpha)])=$
$=m(\omega_z(u_1))$
and
$\forall j\in[1,i].u_j\not\leqslant_1(m(\omega_z(o_1))+1)[g(i+1,e,\alpha)]\underset{\text{by (2.4.4) of GenThmIH}}{=}m((\omega_z(o_1))[g(i+1,e,\alpha)])+1$
$=m(\omega_z(u_1))+1$.
From this follows $\forall j\in[1,i].m(u_j)=m(\omega_z(u_1))\underset{\text{by definition}}{=}\gamma_z(1,u_1)$. **(K3)**

Now we show that $\forall j\in[1,i].u_j=x_z(j+1,u_{j+1})$ **(W)**

We prove (W) by a (side)induction on $([1,i],<)$.

Let $j\in[1,i]$.
Suppose $\forall l\in j\cap[1,i].u_l=x_z(l+1,u_{l+1})$. **(WIH)**

Note $m(u_j) \underset{\text{by (K3)}}{=} \gamma_z(1,u_1) \underset{\text{if } j\geqslant 2}{=} m(u_{j-1}) \underset{\text{by (WIH)}}{=} m(x_z(j,u_j)) = \gamma_z(j,u_j)$. This shows that, in any case, $m(u_j) = \gamma_z(j,u_j)$ **(M1)**.
This way,
$u_j \underset{\text{by (M1)}}{\in} \{r \in (\lambda(j+1,u_j), \lambda(j+1,u_j)(+^{j+1})) \cap \text{Class}(j) | m(r) = \gamma_z(j,r)\} =$
$= X_z(j+1, \lambda(j+1,u_j)) \underset{\text{by (K1)}}{=} X_z(j+1,u_{j+1})$. Moreover, since
$f(j+1,o_{j+1})(o_j) = \{o_j\} \underset{\text{by (2.4.5) of GenThmIH}}{\Longleftrightarrow} f(j+1,u_{j+1})(u_j) = \{u_j\}$, then by (1.3.5) of GenThmIH,
$u_j = \min\{s \in (u_{j+1}, u_{j+1}(+^{j+1})) \cap \text{Class}(j) \mid s \leqslant u_j \wedge m(s)[g(j,s,u_j)] = m(u_j)\} =$
$= \min\{s \in (u_{j+1}, u_{j+1}(+^{j+1})) \cap \text{Class}(j) \mid s \leqslant u_j \wedge m(s)[g(j,s,u_j)] \underset{\text{by (M1)}}{=} \gamma_z(j,u_j)\} =$
$=$, since by (IH) 7., applied to $j \leqslant i$, $u_j, s \in \text{Class}(j)$, we get $T(j,u_j,\gamma_z(j,u_j)) \cap u_j = \emptyset \subset s$,
$= \min\{s \in (u_{j+1}, u_{j+1}(+^{j+1})) \cap \text{Class}(j) \mid s \leqslant u_j \wedge m(s) = \gamma_z(j,u_j)[g(j,u_j,s)]\}$
$=$, by (IH) 7. applied to $j \leqslant i$, $u_j, s \in \text{Class}(j)$ and $z \in [1,\omega)$,
$= \min\{s \in (u_{j+1}, u_{j+1}(+^{j+1})) \cap \text{Class}(j) \mid s \leqslant u_j \wedge m(s) = \gamma_z(j,s)\} =$
$= \min\{s \in (u_{j+1}, u_{j+1}(+^{j+1})) \cap \text{Class}(j) \mid m(s) = \gamma_z(j,s)\} =$
$= \min X_z(j+1,u_{j+1}) = x_z(j+1,u_{j+1})$.
This shows that (W) holds.

Finally,
$\gamma_z(i+1,e)[g(i+1,e,\alpha)] = m(x_z(i+1,e))[g(i+1,e,\alpha)] \underset{\text{by 5.}}{=} m(o_i)[g(i+1,e,\alpha)] =$
$= m(o_i[g(i+1,e,\alpha)]) = m(u_i) \underset{\text{by (W)}}{=} m(x_z(i+1,u_{i+1})) =$
$= m(x_z(i+1,\alpha)) = \gamma_z(i+1,\alpha)$.
Since this last equality was done for arbitrary $\alpha \in \text{Class}(i+1)$ and $z \in [1.\omega)$, then 7. holds. □

Remark 4.18. For $i \in [1,n)$ and $e \in \text{Class}(i)$, it is not hard to see that the sequences $(x_k(i,e))_{k\in[1,\omega)}$ and $(\gamma_k(i,e))_{k\in[1,\omega)}$ are strictly increasing.

Moreover, for any $k \in [1,\omega)$, $\eta(i,e,x_k(i,e)) = m(x_k(i,e))$. This equality holds because $x_k(i,e) \leqslant m(x_k(i,e))$, implies
$m(x_k(i,e)) \leqslant \eta(i,e,x_k(i,e)) \leqslant \eta(i,e,m(x_k(i,e)) \underset{\text{by 3. of previous proposition 4.17}}{=} m(x_k(i,e))$.

4.4 Class(n) is κ-club

Proposition 4.19. *Let κ be an uncountable regular ordinal and $r \in \text{Class}(n-1) \cap \kappa$ arbitrary. Let $M^{n-1}(r,\kappa) := \{q \in [\kappa, \kappa(+^{n-1})) \mid T(n-1,\kappa,q) \cap \kappa \subset r\}$. Then $\bigcap_{s\in M^{n-1}(r,\kappa)} A^{n-1}(s) \subset \text{Class}(n)$.*

Proof. Let $\alpha \in \bigcap_{s\in M^{n-1}(r,\kappa)} A^{n-1}(s)$

Consider $(\gamma_j(n-1,\kappa))_{j\in[1,\omega)}$, the canonical sequence of $\kappa(+^{n-1}) \in \text{Class}(n-1)$. Then $\forall j \in [1,\omega).\gamma_j(n-1,\kappa) \in M^{n-1}(r,\kappa)$. Therefore for any $j \in [1,\omega)$,
$\alpha \in A^{n-1}(\gamma_j(n-1,\kappa)) \underset{\text{theorem 4.10}}{=} G^{n-1}(\gamma_j(n-1,\kappa))$.
$= \{\beta \in \text{Class}(n-1) \mid T(n-1,\kappa,\gamma_j(n-1,\kappa)) \cap \kappa \subset \beta \leqslant \kappa \wedge$
$\beta \leqslant^{n-1} \eta(n-1,\kappa,\gamma_j(n-1,\kappa))[g(n-1,\kappa,\beta)]+1\}$.

The previous means, for any $j \in [1,\omega)$,
$\alpha \leqslant^{n-1} \eta(n-1,\kappa,\gamma_j(n-1,\kappa))[g(n-1,\kappa,\alpha)]+1 = \gamma_j(n-1,\kappa)[g(n-1,\kappa,\alpha)])+1 =$
$= \gamma_j(n-1,\alpha)+1$. But, by previous proposition 4.17, $(\gamma_j(n-1,\alpha))_{j\in[1,\omega)}$ is cofinal in $\alpha(+^{n-1})$; therefore, by $\leqslant^{n-1}$-continuity follows $\alpha \leqslant^n \alpha(+^{n-1})$. Hence $\alpha \in \text{Class}(n)$. □

Proposition 4.20. *Let κ be an uncountable regular ordinal. Then* Class(n) *is club in κ.*

Proof. We already know that Class(n) is closed in κ. So we only need to show that Class(n) is unbounded in κ.

Let $\beta\in\kappa$.
Since we know Class$(n-1)$ is club in κ, take $r, r(+^{n-1})\in\mathrm{Class}(n-1)\cap\kappa\neq\emptyset$.

Consider $M^{n-1}(r,\kappa):=\{q\in[\kappa,\kappa(+^{n-1}))\,|\,T(n-1,\kappa,q)\cap\kappa\subset r\}$.

Consider $R\colon[r,r(+^{n-1}))\longrightarrow R[r(+^{n-1})]\subset[\kappa,\kappa(+^{n-1}))$, $R(t):=t[g(n-1,r,\kappa)]$. Then R is a bijection. We assure that $R[[r,r(+^{n-1}))]=M^{n-1}(r,\kappa)$. **(a)**

To show $R[[r,r(+^{n-1}))]\subset M^{n-1}(r,\kappa)$. **(a1)**
Take $t\in[r,r(+^{n-1}))$. Then by (2.3.1) of GenThmIH, $\mathrm{Ep}(t)\in\mathrm{Dom}(g(n-1,r,\kappa))$ and then, by (2.2.4) of GenThmIH, $T(n-1,\kappa,t[g(n-1,r,\kappa)])\cap\kappa=T(n-1,r,t)\cap r\subset r$. Moreover, it is clear also from GenThmIH that $t[g(n-1,r,\kappa)]\in[\kappa,\kappa(+^{n-1}))$. This shows that $R(t)=t[g(n-1,r,\kappa)]\in M^{n-1}(r,t)$, and since this was done for $t\in r(+^{n-1})$ arbitrary, then (a1) holds.

To show $R[[r,r(+^{n-1}))]\supset M^{n-1}(r,\kappa)$. **(a2)**
Let $s\in M^{n-1}(r,\kappa)$. By (2.2.3) of GenThmIH we have that $M^{n-1}(r,\kappa)=\{t\in[\kappa,\kappa(+^{n-1}))|\mathrm{Ep}(t)\subset\mathrm{Dom}\ g(n-1,\kappa,r)\}$. Therefore, easily from GenThmIH we get that $s[g(n-1,\kappa,r)]\in[r,r(+^{n-1}))$. But then $R(s[g(n-1,\kappa,r)])=s[g(n-1,\kappa,r)][g(n-1,r,\kappa)]\underset{\text{by (2.3.2) of GenThmIH}}{=}s$. This shows that $s\in R[[r,r(+^{n-1}))]$, and since this was done for arbitrary $s\in M^{n-1}(r,\kappa)$, then (a2) holds.

(a1) and (a2) show (a).

By (a) and (2.3.2) of GenThmIH, the function $H:=R^{-1}\colon M^{n-1}(r,\kappa)\longrightarrow[r,r(+^{n-1}))$, $H(s):=s[g(n-1,\kappa,r)]$ is a bijection. **(b)**

On the other hand, since $r\in\mathrm{Class}(n-1)\subset\mathbb{E}\subset[\omega,\infty)$ (because $n-1\geqslant 1$), then there exists $\delta\in\mathrm{OR}$ such that $\aleph_\delta=|r|$. Then $\aleph_\delta\leqslant r<\aleph_{\delta+1}\leqslant\kappa$. But $\aleph_{\delta+1}$ is a regular uncountable ordinal (because it is a successor cardinal), and then, by (0) of GenThmIH, Class$(n-1)$ is club in $\aleph_{\delta+1}$. Hence, $\aleph_\delta\leqslant r<r(+^{n-1})<\aleph_{\delta+1}\leqslant\kappa$ and subsequently $|[r,r(+^{n-1}))|\leqslant|r(+^{n-1})|=|r|<\kappa$. **(c)**

Finally, from (c), (b), proposition 4.11 and proposition 2.47 follows that the set $\bigcap_{s\in M^{n-1}(r,\kappa)}A^{n-1}(s)$ is club in κ. So there exists $\gamma\in\bigcap_{s\in M^{n-1}(r,\kappa)}A^{n-1}(s)$, with $\gamma>\beta$. But by previous proposition 4.19, $\gamma\in\mathrm{Class}(n)$. Since the previous was done for an arbitrary $\beta\in\kappa$, then we have shown that Class(n) is unbounded in κ. □

Chapter 5

Clauses (1) and (2) of theorem 3.26

5.1 Clause (1) of theorem 3.26

The reason of Clause (1) of theorem 3.26 is the following: For $\alpha, c \in \mathrm{Class}(j)$, we would like to have a function $g(j, \alpha, c)$ as stated in (2) of theorem 3.26. However, to prove the existence of such a function is not easy, and it turns out that, for $i \in [1, j)$ and $e \in \mathrm{Class}(i)$, we can use the functions m and $f(i,\ e)$ to provide a "local description" of the elements in an interval $[\alpha, \alpha(+^j)) \cap \mathbb{E}$, and later, based on these ideas, prove the existence of $g(j, \alpha, c)$.

Proposition 5.1. *For any $\alpha \in \mathrm{Class}(n)$, the functions*

- $S(n,\alpha)\colon \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)) \longrightarrow \mathrm{Subsets}(\mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)))$
  $S(n,\alpha)(\delta) := \{e \in \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)) \cap \delta \mid m(e)[g(n-1, e, \delta)] \geqslant m(\delta)\}$
- $f(n,\alpha)\colon \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)) \longrightarrow \mathrm{Subsets}(\mathrm{OR})$
  $$f(n,\alpha)(\delta) := \begin{cases} \{\delta\} & \textit{iff} \quad S(n,\alpha)(\delta) = \emptyset \\ \\ f(n,\alpha)(s) \cup \{\delta\} & \textit{iff} \quad S(n,\alpha)(\delta) \neq \emptyset \wedge s := \sup(S(n,\alpha)(\delta)) \end{cases}$$

are well defined and satisfy (1.1), (1.2), (1.3.1), (1.3.2) and (1.3.3) of theorem 3.26, that is,

(1.1) If $S(n,\alpha)(\delta) \neq \emptyset$ *then* $\sup(S(n,\alpha)(\delta)) \in S(n,\alpha)(\delta) \subset \mathrm{Class}(n-1) \cap \delta$.

(1.2) $\forall \delta \in \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n)).\delta \in f(n,\alpha)(\delta) \subset (\alpha, \alpha(+^n)) \cap \mathrm{Class}(n-1)$ *and $f(n,\alpha)(\delta)$ is finite.*

(1.3) $\forall q \in [1, \omega).\forall \sigma \in (\alpha, \alpha(+^n)) \cap \mathrm{Class}(n-1)$*. If* $f(n,\alpha)(\sigma) = \{\sigma_1 > \ldots > \sigma_q\}$ *for some* $\sigma_1, \ldots, \sigma_q \in \mathrm{OR}$ *then*

(1.3.1) $\sigma_1 = \sigma$,

(1.3.2) $q \geqslant 2 \Longrightarrow \forall j \in \{1, \ldots, q-1\}.m(\sigma_j) \leqslant m(\sigma_{j+1})[g(n-1, \sigma_{j+1}, \sigma_j)]$

(1.3.3) $\sigma_q = \min\{e \in (\alpha, \sigma_q] \cap \mathrm{Class}(n-1) \mid m(e)[g(n-1, e, \sigma_q)] \geqslant m(\sigma_q)\}$

Proof. Let $\alpha \in \mathrm{Class}(n)$ be arbitrary.

Clearly $S(n,\alpha)$ is well defined. (The fact that $f(n,\alpha)$ is well defined follows from (1.1)).

We prove (1.1).
Let $\delta \in \mathrm{Class}(n-1) \cap (\alpha, \alpha(+^n))$.

We assure $\mathrm{Lim}\, S(n,\alpha)(\delta) \subset S(n,\alpha)(\delta)$. (**S)

Let's use (**S) to prove (1.1) and after that we prove (**S). Suppose $S(n,\ \alpha)(\delta) \neq \emptyset$. Let $\Delta = \sup(S(n,\alpha)(\delta))$. If $\Delta \in \mathrm{Lim}\, S(n,\alpha)(\delta) \underset{(**S)}{\subset} S(n,\alpha)(\delta)$, then $\Delta \in S(n,\alpha)(\delta) \subset \mathrm{Class}(n-1) \cap \delta$. If $\Delta \notin \mathrm{Lim}\, S(n,\ \alpha)(\delta)$, then $\exists s \in \Delta.\forall l \in [s,\ \Delta).l \notin S(n,\ \alpha)(\delta)$; but since $\Delta = \sup S(n,\ \alpha)(\delta)$, then $\Delta \in S(n,\alpha)(\delta)$, that is, $\Delta = \max S(n,\alpha)(\delta) \in S(n,\alpha)(\delta) \subset \mathrm{Class}(n-1) \cap \delta$.

We show now (**S).

If $\mathrm{Lim}\, S(n,\alpha)(\delta)=\emptyset$, then clearly $\mathrm{Lim}\, S(n,\alpha)(\delta)\subset S(n,\alpha)(\delta)$, so suppose $\mathrm{Lim}\, S(n,\alpha)(\delta)\neq\emptyset$. Let $\mu\in\mathrm{Lim}\, S(n,\alpha)(\delta)$. Since $S(n,\alpha)(\delta)\subset\mathrm{Class}(n-1)$, then by (0) of our GenThmIH, $\mu\in\mathrm{Class}(n-1)$. Let $(e_i)_{i\in I}\subset S(n,\alpha)(\delta)$ such that $e_i\underset{\mathrm{cof}}{\hookrightarrow}\mu$. We assure $\exists i\in I.m(e_i)[g(n-1,e_i,\mu)]\leqslant m(\mu)$. (***S)

Suppose the opposite: $\forall i\in I.m(e_i)[g(n-1,e_i,\mu)]>m(\mu)$. Since $e_i\underset{\mathrm{cof}}{\hookrightarrow}\mu$ and $T(n-1,\mu,m(\mu))$ is finite, we can assume without loss of generality that $\forall i\in I.T(n-1,\mu,m(\mu))\cap\mu\subset e_i$, that is, by (2.2.3) of our GenThmIH, that $\forall i\in I.\mathrm{Ep}(m(\mu))\subset\mathrm{Dom}\, g(n-1,\mu,e_i)$. Then, by applying $g(n-1,\mu,e_i)$ to both sides of the inequality $m(e_i)[g(n-1,e_i,\mu)]>m(\mu)$ we get $\forall i\in I.m(e_i)=m(e_i)[g(n-1,e_i,\mu)][g(n-1,\mu,e_i)]>m(\mu)[g(n-1,\mu,e_i)]$. This and the fact that $e_i\underset{\mathrm{cof}}{\hookrightarrow}\mu$ imply, by (5) of GenThmIH, that $\mu\leqslant_1 m(\mu)+1$. Contradiction. Thus (***S) holds. Now, by (***S), let $i_0\in I$ such that $m(\mu)\geqslant m(e_{i_0})[g(n-1,e_{i_0},\mu)]$. From this inequality follows $m(\mu)[g(n-1,\mu,\delta)]\geqslant m(e_{i_0})[g(n-1,e_{i_0},\mu)][g(n-1,\mu,\delta)]=$ $m(e_{i_0})[g(n-1,\mu,\delta)\circ g(n-1,e_{i_0},\mu)]=m(e_{i_0})[g(n-1,e_{i_0},\delta)]\geqslant m(\delta)$, where the last inequality holds simply because $e_{i_0}\in S(n,\alpha)(\delta)$. This shows $m(\mu)[g(n-1,\mu,\delta)]\geqslant m(\delta)$. Finally, we need to show that $\mu<\delta$. Since by definition $S(n,\alpha)(\delta)\subset\delta$, then $\mu\not>\delta$. So it only rest to show that $\mu\neq\delta$. Suppose $\mu=\delta$. Since $e_i\underset{\mathrm{cof}}{\hookrightarrow}\mu=\delta$, $\forall i\in I.m(e_i)[g(n-1,e_i,\delta)]\geqslant m(\delta)$ (because $(e_i)_{i\in I}\subset S(n,\alpha)(\delta))$ and $T(n-1,\delta,m(\delta))$ is finite, then there is a subsequence $(e_j)_{j\in J}$ of $(e_i)_{i\in I}$ such that $(e_j)_{j\in J}$ is cofinal in $\mu=\delta$ and such that $\forall i\in J.T(n-1,\delta,m(\delta))\cap\delta\subset e_j$ (once more, this last condition means by (2.2.3) of GenThmIH, that $\forall i\in J.\mathrm{Ep}(m(\delta))\subset\mathrm{Dom}\, g(n-1,\delta,e_j)$); therefore, using now (2.3.2) of GenThmIH, $\forall j\in J.m(e_j)=m(e_j)[g(n-1,e_j,\delta)][g(n-1,\delta,e_j)]\geqslant m(\delta)[g(n-1,\delta,e_j)]$, and then, by (5) of GenThmIH, follows $\delta\leqslant_1 m(\delta)+1$. Contradiction. So $\mu\neq\delta$.

All the previous shows (**S).

Remark: Observe that (1.1) implies that $f(n,\alpha)$ is well defined (by recursion).

We prove now (1.2).

Let $\delta\in\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))$.

We proceed by induction on $\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))$.
If $\delta=\alpha(+^{n-1})$, then clearly $\delta\in f(n,\alpha)(\delta)=\{\alpha(+^{n-1})\}\subset\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^{n-1}))$ and $f(n,\alpha)(\delta)$ is finite.

Suppose $e\in f(n,\alpha)(e)\subset\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^{n-1}))$ and $f(n,\alpha)(e)$ is finite for any $e\in\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))\cap\delta$. **(cIH)**

Then $\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^2))\supset f(n,\alpha)(\delta)=\{\delta\}\ni\delta$ or $\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^2))\supset f(n,\alpha)(\delta)=f(n,\alpha)(\sup S(n,\alpha)(\delta))\cup\{\delta\}\ni\delta$; this way, in any case and using our cIH, $\delta\in f(n,\alpha)(\delta)\subset\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^2))$ and $f(n,\alpha)(\delta)$ is finite.

Now we prove (1.3).

We proceed by induction on $[1,\omega)$ and show (1.3.1), (1.3.2) and (1.3.3) simultaneously.

Suppose $q=1$.

So let $\sigma\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(n-1)$ such that $f(n,\alpha)(\sigma)=\{\sigma_1\}$. Then $\sigma_1=\sigma$ by (1.2) and so (1.3.1) holds. On the other hand, (1.3.2) clearly holds. Finally, observe $f(n,\alpha)(\sigma)=\{\sigma\}$ means $S(n,\alpha)(\sigma)=\{e\in\mathrm{Class}(n-1)\cap(\alpha,\alpha(+^n))\cap\sigma\mid m(e)[g(n-1,e,\sigma)]\geqslant m(\sigma)\}=\emptyset$ and therefore (1.3.3) holds.

Now suppose (1.3.1), (1.3.2) and (1.3.3) hold for $m\in[1,\omega)$. **(ccIH)**

Lets show (1.3.1), (1.3.2) and (1.3.3) for $q=m+1\geqslant 2$.

Let $\sigma \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1)$ and assume $f(n,\alpha)(\sigma) = \{\sigma_1 > \ldots > \sigma_{m+1}\}$ for some $\sigma_1, \ldots, \sigma_{m+1} \in \text{OR}$. So $f(n,\alpha)(\sigma) = \{\sigma\} \cup f(n,\alpha)(s)$, with $s = \sup(S(n,\alpha)(\sigma)) < \sigma$ (where this last inequality is due to (1.1)). This means $f(n,\alpha)(s) = \{\sigma_2 > \ldots > \sigma_{m+1}\}$ and since by ccIH, (1.3.1), (1.3.2) and (1.3.3) hold for $f(n,\alpha)(s)$, we have $\sigma_2 = s < \sigma \in f(n,\alpha)(\sigma) = \{\sigma_1 > \ldots > \sigma_{m+1}\}$. So $\sigma = \sigma_1$ and we have shown (1.3.1).

Let's show (1.3.2).

Since by ccIH (1.3.2) holds for $f(n,\alpha)(s) = \{s = \sigma_2 > \ldots > \sigma_{m+1}\}$, then we just need to see that $m(\sigma) \leqslant m(s)[g(n-1,s,\sigma)]$; but this is clear, since by (1.1), $s \in S(n,\alpha)(\sigma)$.

We prove (1.3.3).

By ccIH, $\sigma_q = \min\{e \in (\alpha, \sigma_q] \cap \text{Class}(n-1) \mid m(e)[g(n-1,e,\sigma_q)] \geqslant m(\sigma_q)\}$.

This concludes the proof of the whole proposition. □

Corollary 5.2. *Let $\alpha \in \text{Class}(n)$, $\sigma \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1)$, $q \in [1,\omega)$ and $f(n,\alpha)(\sigma) = \{\sigma_1 > \ldots > \sigma_q\}$. Then (1.3.4), (1.3.5) and (1.3.6) of theorem 3.26 hold, that is:*

(1.3.4). $m(\sigma) = m(\sigma_1) \leqslant m(\sigma_2)[g(n-1,\sigma_2,\sigma)] \leqslant \ldots \leqslant m(\sigma_q)[g(n-1,\sigma_q,\sigma)]$.

(1.3.5). For $\gamma := \sigma_q$,

$$\gamma = \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid e \leqslant \gamma \wedge m(e)[g(n-1,e,\gamma)] \geqslant m(\gamma)\}$$
$$= \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid e \leqslant \gamma \wedge m(e)[g(n-1,e,\gamma)] = m(\gamma)\}.$$

(1.3.6). For any $j \in \{1, \ldots, q-1\}$,

$$\sigma_j = \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid \sigma_{j+1} < e \leqslant \sigma_j \wedge m(e)[g(n-1,e,\sigma_j)] \geqslant m(\sigma_j)\}$$
$$= \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid \sigma_{j+1} < e \leqslant \sigma_j \wedge m(e)[g(n-1,e,\sigma_j)] = m(\sigma_j)\}.$$

Proof.

(1.3.4).

We prove this claim in the form $\forall i \in [1, q-1].m(\sigma_i)[g(n-1,\sigma_i,\sigma)] \leqslant m(\sigma_{i+1})[g(n-1,\sigma_{i+1},\sigma)]$.

Case $i = 1$. Then $m(\sigma) = m(\sigma_1)[g(n-1,\sigma_1,\sigma)] \leqslant m(\sigma_2)[g(n-1,\sigma_2,\sigma)]$ holds by previous proposition 5.1 and because $g(n-1,\sigma,\sigma)$ is the identity function (by (2.2.1) of GenThmIH).

Case $i \in (1, q-1]$.

Since by proposition 5.1, $m(\sigma_i) \leqslant m(\sigma_{i+1})[g(n-1,\sigma_{i+1},\sigma_i)]$, then $m(\sigma_i)[g(n-1,\sigma_i,\sigma)] \leqslant m(\sigma_{i+1})[g(n-1,\sigma_{i+1},\sigma_i)][g(n-1,\sigma_i,\sigma)] = m(\sigma_{i+1})[g(n-1,\sigma_i,\sigma) \circ g(n-1,\sigma_{i+1},\sigma_i)] = m(\sigma_{i+1})[g(n-1,\sigma_{i+1},\sigma)]$, where the last equality holds by (2.5.3) of GenThmIH.

(1.3.5).

Easy. The proof is essentially the same that we carry out for (1.3.6).

(1.3.6).

Let $j \in [1, q-1)$.

Let's show $\sigma_j = \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid \sigma_{j+1} < e \leqslant \sigma_j \wedge m(e)[g(n-1,e,\sigma_j)] \geqslant m(\sigma_j)\}$.

Let $X_j := \{e \in (\sigma_{j+1}, \sigma_j] \cap \text{Class}(n-1) \mid m(e)[g(n-1,e,\sigma_j)] \geqslant m(\sigma_j)\}$.

Clearly $\sigma_j \in X_j$ and we need to show $\sigma_j = \min X_j$. For this, it is enough to show $\forall c \in (\sigma_{j+1}, \sigma_j).c \notin X_j$. This easy: Observe $X_j \cap (\sigma_{j+1}, \sigma_j) \subset S(n,\alpha)(\sigma_j) = \{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \cap \sigma_j \mid m(e)[g(n-1,e,\sigma_j)] \geqslant m(\sigma_j)\}$. Since by definition $\sigma_{j+1} = \sup S(n,\alpha)(\sigma_j)$, then for any $c \in (\sigma_{j+1}, \sigma_j)$, $c \notin S(n,\alpha)(\sigma_j)$ and therefore $\forall c \in (\sigma_{j+1}, \sigma_j).c \notin X_j$.

Thus $\sigma_j = \min X_j$.

Let's show $\sigma_j = \min\{e \in (\alpha, \alpha(+^n)) \cap \text{Class}(n-1) \mid \sigma_{j+1} < e \leqslant \sigma_j \wedge m(e)[g(n-1,e,\sigma_j)] = m(\sigma_j)\}$.

Let $Y_j := \{e \in (\sigma_{j+1}, \sigma_j] \cap \text{Class}(n-1) \mid m(e)[g(n-1,e,\sigma_j)] = m(\sigma_j)\}$. Clearly $\sigma_j \in Y_j$ and since $Y_j \subset X_j$, then $\sigma_j = \min X_j \leqslant \min Y_j \leqslant \sigma_j$. So $\sigma_j = \min Y_j$. □

Concluding, by previous proposition 5.1 and corollary 5.2, clause (1) of theorem 3.26 holds.

5.2 The $T(n,\alpha,t)$ sets and Clause (2.1) of theorem 3.26

Remark 5.3. For $k\in[1,n]$, $\alpha\in\mathrm{Class}(k)$ and $t\in\alpha(+^k)$, the set $T(k,\alpha,t)\subset\mathbb{E}\cap\alpha(+^k)$ is inductively defined by the clauses

a) $\mathrm{Ep}(t)\subset T(k,\alpha,t)$

b) $\xi\in T(k,\alpha,t)\cap(\alpha,\alpha(+^k))\cap\mathbb{E}\Longrightarrow \mathrm{Ep}(m(\xi))\subset T(k,\alpha,t)$

c) $l\in[1,k)\wedge\xi\in T(k,\alpha,t)\cap(\alpha,\alpha(+^k))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))\Longrightarrow\lambda(l+1,\xi)\in T(k,\alpha,t)$

d) $l\in[1,k)\wedge\xi\in T(k,\alpha,t)\cap(\alpha,\alpha(+^k))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))\Longrightarrow$
$f(l+1,\lambda(l+1,\xi))(\xi)\subset T(k,\alpha,t)$

Proposition 5.4. *Let $k\in[1,n]$, $\alpha\in\mathrm{Class}(k)$.*

1. $\forall t\in\alpha(+^k)\forall s\in T(k,\alpha,t).T(k,\alpha,s)\subset T(k,\alpha,t)$.
2. $\forall l\in[1,k].\forall e\in[\alpha,\alpha(+^k))\cap\mathrm{Class}(l).\forall t\in[e,e(+^l)).T(l,e,t)\subset T(k,\alpha,t)$
3. $\forall l\in[1,k].\forall t\in[\alpha,\alpha(+^l)).T(l,\alpha,t)=T(k,\alpha,t)$

Proof. Left to the reader. □

Proposition 5.5. *Let $\alpha\in\mathrm{Class}(n)$ and $t\in\alpha(+^n)$. Then $T(n,\alpha,t)$ is finite.*

Proof. If $t\leqslant\alpha$, then $T(n,\alpha,t)=\mathrm{Ep}(t)$ is clearly finite. So suppose $t\in(\alpha,\alpha(+^n))$. Since for the case $t\notin\mathbb{E}$, $T(n,\alpha,t)=\bigcup_{E\in\mathrm{Ep}(t)}T(n,\alpha,E)$ and $\mathrm{Ep}(t)$ is finite, then it is enough to show that for any $E\in(\alpha,\alpha(+^n))\cap\mathbb{E}$, $T(n,\alpha,E)$ is finite.

So let $E\in(\alpha,\alpha(+^n))\cap\mathbb{E}$. The set $T(n,\alpha,E)$ can be thought as the tree with root E, and such that for any node $\xi\in T(n,\alpha,E)$ with $\xi\in(\alpha,\alpha(+^n))\cap\mathrm{Class}(l)\backslash\mathrm{Class}(l+1)$ and $l\in[1,n-1]$, the children of ξ are the elements of the finite set $\mathrm{Ep}(m(\xi))\cup\{\lambda(l+1,\xi)\}\cup f(l+1,\lambda(l+1,\xi))(\xi)$ that "have not been generated previously". We formalize this idea in what follows.

Let $F_1,...,F_{2n}:\mathrm{OR}\longrightarrow\mathrm{Subsets(OR)}$ be the functions
$F_1(x):=\emptyset$,
For $i\in[2,n]$, $F_i(x):=\begin{cases}\emptyset & \text{iff } x\notin\mathrm{Class}(i-1)\backslash\mathrm{Class}(i)\\ \{\lambda(i,x)\} & \text{otherwise}\end{cases}$,
$F_{n+1}(x):=\begin{cases}\emptyset & \text{otherwise}\\ \mathrm{Ep}(m(x)) & \text{iff } m(x)\neq\infty\end{cases}$,
For $i\in[2,n]$, $F_{n+i}(x):=\begin{cases}\emptyset & \text{iff } x\notin\mathrm{Class}(i-1)\cap(F_i(x),F_i(x)(+^i))\\ f(i,F_i(x))(x) & \text{otherwise}\end{cases}$.

Moreover, we define the sets W_i (with $i\in\omega$) recursively on ω as
$W_0:=\{E\}$;
$W_{j+1}:=\bigcup_{e\in W_j}R_j(e)$, where R_j is defined on W_j (by recursion on $(W_j,<)$) as
$R_j(e):=(\bigcup_{i\in[1,2n]}F_i(e))\backslash(\bigcup_{d\in[0,j]}W_d\cup\bigcup_{s\in e\cap W_j}R_j(s))$.

To show $\forall i\in\omega.W_i\subset T(n,\alpha,E)$. **(w1)**

We proceed by induction on ω.

Clearly $W_0=\{E\}\subset T(n,\alpha,E)$.

Take $i+1\in\omega$ and suppose $W_i\subset T(n,\alpha,E)$. **(cIH)**

Then $W_{i+1}=\bigcup_{e\in W_i} R_i(e)\subset\bigcup_{e\in W_i}\bigcup_{j\in[1,2n]} F_j(e)\underset{\text{by (cIH) and the definition of } T(n,\alpha,E)}{\subset} T(n,\alpha,E)$.

To show $\bigcup_{i\in\omega} W_i\supset T(n,\alpha,E)$. **(w2)**

We proceed by induction on the definition of $T(n,\alpha,E)$.

- Clearly $\mathrm{Ep}(E)=\{E\}=W_0\subset\bigcup_{i\in\omega} W_i$.
- Let $\xi\in\bigcup_{i\in\omega} W_i\cap(\alpha,\alpha(+^n))\cap\mathbb{E}$; that is, for some $i\in[1,\omega)$ $\xi\in W_i\cap(\alpha,\alpha(+^n))\cap\mathbb{E}$. Then $m(\xi)<\infty$ and $\mathrm{Ep}(m(\xi))=F_{n+1}(\xi)\subset R_i(\xi)\cup(\bigcup_{\delta\in[0,i]} W_\delta\cup\bigcup_{s\in\xi\cap W_i} R_i(s))\subset(\bigcup_{\delta\in[0,i]} W_\delta)\cup W_{i+1}\subset\bigcup_{\delta\in\omega} W_\delta$.
- Take $l\in[1,n)\wedge\xi\in(\bigcup_{i\in\omega} W_i)\cap(\alpha,\alpha(+^n))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))$; so $\xi\in W_i\cap(\alpha,\alpha(+^n))\cap$ for some $i\in[1,\omega)$ and then $\{\lambda(l+1,\xi)\}=F_{l+1}(\xi)\subset R_i(\xi)\cup(\bigcup_{\delta\in[0,i]} W_\delta\cup\bigcup_{s\in\xi\cap W_i} R_i(s))\subset(\bigcup_{\delta\in[0,i]} W_\delta)\cup W_{i+1}\subset\bigcup_{\delta\in\omega} W_\delta$.
- Take $l\in[1,n)\wedge\xi\in(\bigcup_{i\in\omega} W_i)\cap(\alpha,\alpha(+^n))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))$. Then, just as in the previous case, it follows $f(l+1,\xi)(\xi)=f(l+1,F_{l+1}(\xi))(\xi)=F_{n+l+1}(\xi)\subset\bigcup_{\delta\in\omega} W_\delta$.

The previous show that $\bigcup_{i\in\omega} W_i$ is closed under clauses a), b), c) and d) of the inductive definition of $T(n,\alpha,E)$. So $T(n,\alpha,E)\subset\bigcup_{i\in\omega} W_i$ and we have shown (w2).

Concluding, from (w1) and (w2) we get that $\bigcup_{i\in\omega} W_i=T(n,\alpha,E)$. **(w3)**

Done the previous, for $e,\beta\in T(n,\alpha,E)$, we define the binary relation $\prec$ as $e\prec\beta$ if and only if for some $i\in\omega$, $e\in W_i\wedge\beta\in R_i(e)$. Moreover, let $\sqsubset$ be the transitive closure of $\prec$. Note that $\sqsubset$ is irreflexive and transitive. From now on, we work on the tree $(T(n,\alpha,E),\sqsubset)$.

It is very easy to see (by induction on ω), that $\forall i\in\omega\forall e\in W_i.|W_i|<\omega>|R_i(e)|$; in particular, it follows that $(T(n,\alpha,E),\sqsubset)$ is finitely branching. So, by Königs lemma, to see that $T(n,\alpha,E)$ is finite, it suffices to show that every branch B of $(T(n,\alpha,E),\sqsubset)$ is finite. **(w4)**

To show (w4).

Let B be an arbitrary branch of $(T(n,\alpha,E),\sqsubset)$. Clearly $E\in B\cap(\alpha,\alpha(+^n))$ (because E is the root of our tree) and so we define, $\xi:=\min(B\cap(\alpha,\alpha(+^n)))$ (minimum with respect to the usual order $<$ in the ordinals). Moreover, by the very definition of $\sqsubset$, it follows that $\xi\in W_i\backslash W_{i+1}$ for some $i\in\omega$. Let $B(\xi):=B\cap\bigcup_{j\in[i+1,\omega)} W_i$. Note that $\xi\in B$ implies that $B(\xi)\subset T(n,\alpha,\xi)$, **(w5)**

because $\xi\in T(n,\alpha,\xi)$ and $T(n,\alpha,\xi)$ is closed under the operations $F_1,\ldots,F_{2n}$.

Let $\mu:=\lambda(n-1,\xi)\in T(n,\alpha,\xi)\cap\mathrm{Class}(n-1)\cap[\alpha,\alpha(+^n))$.

Clearly $\xi\in[\mu,\mu(+^{n-1}))$. **(w6)**

We have two cases:

Case $\mu=\alpha$.

Then $\xi\in(\alpha,\alpha(+^{n-1}))$ and therefore $T(n-1,\alpha,\xi)\underset{\text{by proposition 5.4}}{=}T(n,\alpha,\xi)$. But by GenThmIH $T(n-1,\alpha,\xi)$ is finite. So $B(\xi)$ is finite, and subsequently, B is finite.

Case $\mu>\alpha$.

Let $C:=\{\sigma\in\mu(+^{n-1})|\sigma\in[\mu,\mu(+^{n-1}))\Longrightarrow\sigma\in T(n-1,\mu,m(\mu))\cup T(n-1,\mu,\xi)\}$. We will need to show that $T(n,\alpha,\xi)\subset C$. **(w7)**

In order to see that (w7) holds, we prove that C is closed under clauses a), b), c) and d) of the inductive definition of $T(n, \alpha, \xi)$.
- Clearly $\mathrm{Ep}(\xi) = \{\xi\} \subset T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)$, and so $\mathrm{Ep}(\xi) \subset C$.
- Suppose $\sigma \in C \cap (\alpha, \alpha(+^n)) \cap \mathbb{E}$. We want to show that $\mathrm{Ep}(m(\sigma)) \subset C$.
Case $\sigma \notin [\mu, \mu(+^{n-1}))$. Then $\sigma < \mu$ and then, by proposition 3.6, $m(\sigma) < \mu$. Therefore $\mathrm{Ep}(m(\sigma)) \subset \mu$ and so $\mathrm{Ep}(m(\sigma)) \subset C$.
Case $\sigma \in [\mu, \mu(+^{n-1}))$. Then $\sigma \in T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi) \cap \mathbb{E}$. Now, if $\sigma = \mu$, then clearly $\mathrm{Ep}(m(\sigma)) \subset T(n-1, \mu, m(\mu)) \subset T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)$ and so $\mathrm{Ep}(m(\sigma)) \subset C$. However, if $\sigma \neq \mu$, then $\sigma \in (T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)) \cap (\mu, \mu(+^{n-1}))$ which imply, by the definitions of $T(n-1, \mu, m(\mu))$ and $T(n-1, \mu, \xi)$, that
$\mathrm{Ep}(m(\sigma)) \subset T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)$; subsequently $\mathrm{Ep}(m(\sigma)) \subset C$ too.
- Suppose $l \in [1, n) \wedge \sigma \in C \cap (\alpha, \alpha(+^n)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$. To show $\lambda(l+1, \sigma) \in C$.
Case $\sigma \notin [\mu, \mu(+^{n-1}))$. Then $\sigma < \mu$ and then $\lambda(l+1, \sigma) < \mu$. So $\lambda(l+1, \sigma) \in C$.
Case $\sigma \in [\mu, \mu(+^{n-1}))$. If $l = n-1$, then $\sigma = \mu$ and $\lambda(l+1, \sigma) = \alpha < \mu$; so $\lambda(l+1, \sigma) \in C$. If $l \neq n-1$, then $l \in [1, n-1)$ and $\sigma \in (\mu, \mu(+^{n-1})) \cap (T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi))$. Note this means, by the definitions of $T(n-1, \mu, m(\mu))$ and $T(n-1, \mu, \xi)$, that
$\lambda(l+1, \sigma) \in (T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi))$; subsequently, $\lambda(l+1, \sigma) \in C$.
- Suppose $l \in [1, n) \wedge \sigma \in C \cap (\alpha, \alpha(+^n)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$.
To show $f(l+1, \lambda(l+1, \sigma))(\sigma) \subset C$.
Case $\sigma \notin [\mu, \mu(+^{n-1}))$. Then $\sigma < \mu$ and then $f(l+1, \lambda(l+1, \sigma))(\sigma) \subset \sigma + 1 < \mu$. Therefore $f(l+1, \lambda(l+1, \sigma))(\sigma) \subset C$.
Case $\sigma \in [\mu, \mu(+^{n-1}))$. If $l = n-1$, then $\sigma = \mu$ and $\mu = \max f(l+1, \lambda(l+1, \sigma))(\sigma)$; therefore $f(l+1, \lambda(l+1, \sigma))(\sigma) \subset C$. If $l \neq n-1$, then $l \in [1, n-1)$ and
$\sigma \in (\mu, \mu(+^{n-1})) \cap (T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi))$. Then, by the definitions of
$T(n-1, \mu, m(\mu))$ and $T(n-1, \mu, \xi)$,
$f(l+1, \lambda(l+1, \sigma))(\sigma) \subset (T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi))$; subsequently,
$f(l+1, \lambda(l+1, \sigma))(\sigma) \in C$.

The previous shows that C is closed under clauses a), b), c) and d) of the inductive definition of $T(n, \alpha, \xi)$. Therefore (w7) holds.

Done the previous work, note
(a1). $B(\xi) \cap [0, \alpha]$ contains at most 1 element, because of (w5) and because every $\beta \in T(n, \alpha, \xi) \cap [0, \alpha]$ is a leaf of $T(n, \alpha, \xi)$.
(a2). $(B(\xi) \cap (\alpha, \mu)) = \emptyset$, because $\mu \leqslant \xi = \min(B \cap (\alpha, \alpha(+^n)))$.
(a3). $T(n, \alpha, \xi) \cap [\mu, \mu(+^{n-1})) \subset T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)$, because of (w7).
This way, $B(\xi) = (B(\xi) \cap [0, \alpha]) \cup (B(\xi) \cap (\alpha, \mu)) \cup (B(\xi) \cap [\mu, \mu(+^{n-1}))) \underset{\text{by (a2) and (w5)}}{\subset}$
$(B(\xi) \cap [0, \alpha]) \cup (T(n, \alpha, \xi) \cap [\mu, \mu(+^{n-1}))) \underset{\text{by (a3)}}{\subset}$
$(B(\xi) \cap [0, \alpha]) \cup T(n-1, \mu, m(\mu)) \cup T(n-1, \mu, \xi)$. But by (a1) and GenThmIH, the sets $B(\xi) \cap [0, \alpha]$, $T(n-1, \mu, m(\mu))$ and $T(n-1, \mu, \xi)$ are finite. So $B(\xi)$ is finite, and therefore B is finite. Finally, since this was shown for an arbitrary branch B of $(T(n, \alpha, E), \sqsubset)$, then we have shown (w4). □

Proposition 5.6. *Let* $k \in [1, n]$, $\alpha \in \mathrm{Class}(k)$ *and* $t \in (\alpha, \alpha(+^k))$.
Then $\forall s \in [1, k]. \lambda(s, t) \in T(k, \alpha, t) \cap [\alpha, \alpha(+^k))$.

Proof. Left to the reader. □

Proposition 5.7. *Let* $k \in [1, n]$, $\alpha \in \mathrm{Class}(k)$ *and* $t \in (\alpha, \alpha(+^k))$.

- *If* $t \notin \mathrm{Lim}\mathbb{P}$, *then* $T(k, \alpha, m(t)) = T(k, \alpha, t)$
- *If* $t \in (\mathrm{Lim}\mathbb{P}) \backslash \mathbb{E}$, *then* $T(k, \alpha, m(t)) \subset T(k, \alpha, t)$
- *If* $t \in \mathbb{E}$, *then* $T(k, \alpha, m(t)) = T(k, \alpha, t)$

Proof. Left to the reader. □

Proposition 5.8. *Let* $k\in[1,n]$*,* $\alpha\in\mathrm{Class}(n)$ *and* $t\in[\alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2,\alpha(+^k))$*. Then*

1. $l(k,\alpha,t)\in t\backslash\mathbb{E}\Longrightarrow l(k,\alpha,t)=\pi t$.
2. $l(k,\alpha,t)\in t\cap\mathbb{E}\Longrightarrow l(k,\alpha,t)\in\{\lambda(1,t),...,\lambda(k-1,t)\}$.

Proof. Let α,t as stated.
1.
Suppose $l(k,\alpha,t)\in t\backslash\mathbb{E}$. Then $l(k,\alpha,t)<t\leqslant m(t)=m(l(k,\alpha,t))\underset{\text{by corollary 2.4}}{<}l(k,\alpha,t)2$. **(1*)**
This implies, by theorem 2.3, $l(k,\alpha,t)\in\mathbb{P}$. **(2*)**
Now, note it is impossible that $l(k,\alpha,t)<\pi t$, otherwise $l(k,\alpha,t)2\underset{\text{because }\pi t\in\mathbb{P}}{<}\pi t\leqslant t$, which contradicts (1*). So $\pi t\leqslant l(k,\alpha,t)$. Moreover, $\pi t\not<l(k,\alpha,t)$, otherwise, since $l(k,\alpha,t)\in\mathbb{P}$ by (2*), then we would have $t<l(k,\alpha,t)$, which contradicts (1*). Thus $\pi t=l(k,\alpha,t)$.

2.
Suppose $l(k,\alpha,t)\in t\cap\mathbb{E}$. Then there is $j\in[1,k)$ such that $l(k,\alpha,t)\in\mathrm{Class}(j)\backslash\mathrm{Class}(j+1)$. Consider $e:=\lambda(j,t)$. Since both $l(k,\alpha,t),e\in\mathrm{Class}(j)$ satisfy $e\leqslant t\geqslant l(k,\alpha,t)$, then $e\geqslant l(k,\alpha,t)$. Now, suppose $e>l(k,\alpha,t)$. Then we have $l(k,\alpha,t)<e\leqslant t\leqslant m(t)=m(l(k,\alpha,t))$ which implies, by proposition 3.6, $l(k,\alpha,t)\in\mathrm{Class}(j+1)$. Contradiction. So $e\not>l(k,\alpha,t)$. All the previous shows $e=l(k,\alpha,t)$. □

Proposition 5.9. *Let* $\alpha\in\mathrm{Class}(n)$ *and* $t\in\alpha(+^n)$*. Then (2.1.1), (2.1.2), (2.1.3) and (2.1.4) of theorem 3.26 hold, that is:*
(2.1.1) $\mathrm{Ep}(t)\subset T(n,\alpha,t)$ *and* $T(n,\alpha,t)$ *is finite.*
(2.1.2) $T(n,\alpha,t+1)=T(n,\alpha,t)$.
(2.1.3) $\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\leqslant t\Longrightarrow T(n,\alpha,\eta(n,\alpha,t))\cap\alpha\subset T(n,\alpha,t)\cap\alpha$.
(2.1.4) $\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2\leqslant t\Longrightarrow T(n,\alpha,l(n,\alpha,t))\subset T(n,\alpha,t)$.

Proof.
(2.1.1)
Clear from remark 5.3 and proposition 5.5.

(2.1.2)
Note $T(n,\alpha,t+1)$ and $T(n,\alpha,t)$ are (both) the closure of $\mathrm{Ep}(t+1)=\mathrm{Ep}(t)$ under clauses b), c) and d) of remark 5.3; therefore $T(n,\alpha,t+1)=T(n,\alpha,t)$.

We show first (2.1.4)
Suppose $t\geqslant\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2$. If $t=\alpha(+^{n-1})...(+^1)2$, then
$l(n,\alpha,t)=\alpha(+^{n-1})...(+^1)2=t$ and clearly $T(n,\alpha,l(n,\alpha,t))\subset T(n,\alpha,t)$.

So suppose $t>\alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2$.

If $l(n,\alpha,t)=t$, then clearly $T(n,\alpha,l(n,\alpha,t))\subset T(n,\alpha,t)$.

So assume $l(n,\alpha,t)<t$.

Case $l(n,\alpha,t)\notin\mathbb{E}$.
Then, by proposition 5.8, $l(n,\alpha,t)=\pi t$. But $\mathrm{Ep}(\pi t)\subset\mathrm{Ep}(t)$ and then, from the definition of $T(n,\alpha,t)$, it follows that $T(n,\alpha,t)$ is closed under clauses a), b), c) and d) of the definition of $T(n,\alpha,l(n,\alpha,t))$. Thus $T(n,\alpha,l(n,\alpha,t))\subset T(n,\alpha,t)$.

Case $l(n,\alpha,t)\in\mathbb{E}$.

Then, by proposition 5.8, $l(n, \alpha, t) = \lambda(j, t)$ for some $j \in [1, n)$. This and proposition 5.6 imply that $l(n, \alpha, t) \in T(n, \alpha, t)$, which, subsequently, implies $T(n, \alpha, l(n, \alpha, t)) \subset T(n, \alpha, t)$ by proposition 5.4.

(2.1.3)
Suppose $t \geqslant \alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2$. If $t = \alpha(+^{n-1})...(+^1)2$, then $\eta(n, \alpha, t) = l(n, \alpha, t) = \alpha(+^{n-1})...(+^1)2 = t$ and clearly $T(n, \alpha, \eta(n, \alpha, t)) \cap \alpha \subset T(n, \alpha, t) \cap \alpha$. So suppose $t > \alpha(+^{n-1})(+^{n-2})...(+^2)(+^1)2$. Then $T(n, \alpha, \eta(n, \alpha, t)) \underset{\text{by proposition 3.21}}{=} T(n, \alpha, m(l(n, \alpha, t))) \underset{\text{by proposition 5.7}}{\subset} T(n, \alpha, l(n, \alpha, t))$; thus $T(n, \alpha, \eta(n, \alpha, t)) \cap \alpha \subset T(n, \alpha, l(n, \alpha, t)) \cap \alpha \underset{\text{by previous (2.1.4)}}{\subset} T(n, \alpha, t) \cap \alpha$ as we wanted to show. □

5.3 Clause (2.2) of theorem 3.26

5.3.1 The Extension theorem

It is now time to provide the functions $g(n, \alpha, c)$. This is a task that takes considerable effort, and in fact, in our way to achieve this, we provide what is maybe the most important theorem in this thesis: the extension theorem.

Theorem 5.10. *(Extension theorem)*
$\forall j \forall \varepsilon \forall \sigma \forall p.$
if $j \in [1, n] \wedge \varepsilon, \sigma \in \mathrm{Class}(j) \wedge \varepsilon \leqslant \sigma \wedge p : \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E} \wedge p$ *is a strictly increasing function, then there exists a unique* $F : \varepsilon(+^j) \cap \mathbb{E} \longrightarrow p[\varepsilon \cap \mathbb{E}] \cup ([\sigma, \sigma(+^j)) \cap \mathbb{E})$ *such that*

1. *F is strictly increasing*
2. $F(\varepsilon) = \sigma$
3. $F|_{\varepsilon \cap \mathbb{E}} = p$
4. *The function* $H_F : (\varepsilon, \varepsilon(+^j)) \longrightarrow H_F[(\varepsilon, \varepsilon(+^j))] \subset (\sigma, \sigma(+^j))$, $t \longmapsto t[F]$ *is an* $(<, +, \cdot, <_1, \lambda x.\omega^x)$ *isomorphism.*
5. $\forall i \in [1, j]. \forall e \in [\varepsilon, \varepsilon(+^j)) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow F(e) \in \mathrm{Class}(i)$.
6. *H_F is also an* $(+^1), (+^2), ..., (+^{j-1})$ *isomorphism.*
7. *If* $j \geqslant 2$, *then* $\forall i \in [2, j]. \forall e \in \mathrm{Class}(i) \cap [\varepsilon, \varepsilon(+^j)). \forall E \in (e, e(+^i)) \cap \mathrm{Class}(i-1).$
   $f(i, e)(E) = \{E_1 > ... > E_q\} \Longleftrightarrow f(i, F(e))(F(E)) = \{F(E_1) > ... > F(E_q)\}$
8. *If* $j \geqslant 2$, *then* $\forall i \in [2, j]. \forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, \varepsilon(+^j)). F(\lambda(i, s)) = \lambda(i, F(s))$

Proof. We proceed by induction on $[1, n]$.

Base case $j = 1$. Let $\varepsilon, \sigma \in \mathrm{Class}(1) \wedge \varepsilon \leqslant \sigma \wedge p : \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E} \wedge p$ strictly increasing.

We define $F : \varepsilon(+^1) \cap \mathbb{E} \longrightarrow p[\varepsilon] \cup ([\sigma, \sigma(+^1)) \cap \mathbb{E})$ as $F|_\varepsilon := p$, $F(\varepsilon) := \sigma$.

Then clearly F satisfies 1, 2, 3, 5, 6, 7 and 8, and it is the only one function with domain $\varepsilon(+^1) \cap \mathbb{E}$ satisfying 3 and 2. So it only rests to prove 4.

So consider the function $H_F : (e, e(+^1)) \longrightarrow (\sigma, \sigma(+^1))$, $H_F(t) := t[F]$.

Then H_F has such codomain and preserves $<, +, \cdot, \lambda x.\omega^x$ according to the theorems we know about general substitutions (propositions 3.12, 3.10 and 3.14).

So we only need to show that H_F preserves the $<_1$ relation:

Let $a, b \in (\varepsilon, \varepsilon(+^1))$.

Then $a, b \notin \mathbb{E}$. Moreover, by propositions 3.12 and 3.10, $a[F], b[F] \in (F(\varepsilon), F(\varepsilon)(+^1))$, and therefore $a[F], b[F] \notin \mathbb{E}$ either. (*1)

We now show $a <_1 b \Longrightarrow a[F] <_1 b[F]$. Suppose $a <_1 b$. Then $b < a2$ (otherwise $a <_1 a2$ and then $a \in \mathbb{E}$, which is contradictory with (*1)). So let $\xi \in [1, \alpha 2)$ be such that $b = a + \xi$. So we have $a <_1 b = a + \xi$, and this holds if and only if (by theorem 2.3) $a = \omega^{\omega^{\xi} \cdot \beta}$ for some $\beta \in (1, a)$. This implies, by proposition 3.14, $a[F] = \omega^{\omega^{\xi} \cdot \beta}[F] = \omega^{\omega^{\xi[F]} \cdot (\beta[F])}$ with $\beta[F] \in (1, a[F])$, and therefore, by theorem 2.3 again, $a[F] <_1 a[F] + \xi[F] = (a + \xi)[F] = b[F]$.

We now show $a[F] <_1 b[F] \Longrightarrow a <_1 b$. Suppose $a[F] <_1 b[F]$. Then $b[F] < a[F]2$ (otherwise $a[F] <_1 a[F]2$ and then $a[F] \in \mathbb{E}$, which is contradictory with (*1)). So let $\delta \in [1, a[F]2)$ be such that $b[F] = a[F] + \delta$ (note this equality and the fact that $\mathrm{Ep}(b) \subset \mathrm{Dom}\, F$ implies $\mathrm{Ep}(\delta) \subset \mathrm{Im}\, F$). So we have $a[F] <_1 b[F] = a[F] + \delta$, and this holds if and only if, by theorem 2.3, $a[F] = \omega^{\omega^{\delta} \cdot \gamma}$ for some $\gamma \in (1, a[F])$ (again, note this equality and the fact that $\mathrm{Ep}(a) \subset \mathrm{Dom}\, F$ implies $\mathrm{Ep}(\gamma) \subset \mathrm{Im}\, F$). This way, by proposition 3.12, $a = a[F][F^{-1}] = \omega^{\omega^{\delta} \cdot \gamma}[F^{-1}] = \omega^{\omega^{\delta[F^{-1}]} \cdot (\gamma[F^{-1}])}$ with $\gamma[F^{-1}] \in (1, a[F][F^{-1}]) = (1, a)$, and therefore, once more by theorem 2.3, $a <_1 a + \delta[F^{-1}] = a[F][F^{-1}] + \delta[F^{-1}] = (a[F] + \delta)[F^{-1}] = b[F][F^{-1}] = b$.

The previous shows the theorem holds for $j = 1$.

Now, let $j \in (1, n]$ and suppose the theorem holds for any $m \in [1, j)$. **(IH)**

Let $\varepsilon, \sigma \in \mathrm{Class}(j) \wedge \varepsilon \leqslant \sigma \wedge p: \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E} \wedge p$ strictly increasing. We first show the following

Claim1:

For any $E \in [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1)$ there exists a unique pair (F_E, U_E) such that

i. $U_E \in \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^j))$

ii. $F_E: E(+^{j-1}) \cap \mathbb{E} \longrightarrow U_E(+^{j-1}) \cap \mathbb{E}$ is an strictly increasing function

iii. $F_E(\varepsilon) = \sigma$ and $F_E(E) = U_E$

iv. $F_E|_{\varepsilon \cap \mathbb{E}} = p$

v. The function $R_{F_E}: (\varepsilon, E(+^{j-1})) \longrightarrow R_{F_E}[(\varepsilon, E(+^{j-1}))] \subset (\sigma, U_E(+^{j-1}))$, $t \longmapsto t[F_E]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism.

vi. $\forall i \in [1, j]. \forall e \in [\varepsilon, E(+^{j-1})) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow F_E(e) \in \mathrm{Class}(i)$.

vii. R_{F_E} is an $(+^1), \ldots, (+^{j-1})$ isomorphism

viii. If $j \geqslant 2$ then
$\forall i \in [2, j]. \forall e \in \mathrm{Class}(i) \cap [\varepsilon, E(+^{j-1})). \forall s \in (e, e(+^i)) \cap E(+^{j-1}) \cap \mathrm{Class}(i-1)$.
$f(i, e)(s) = \{s_1 > \ldots > s_k\} \Longleftrightarrow f(i, F_E(e))(F_E(s)) = \{F_E(s_1) > \ldots > F_E(s_k)\}$

ix. If $j \geqslant 2$, then $\forall i \in [2, j]. \forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, E(+^{j-1})). F_E(\lambda(i, s)) = \lambda(i, F_E(s))$

We warn the reader that the proof of Claim1 is **very long** and it will require that we prove **twelve assertions.**

We proceed by a Side Induction on the well order $([\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1), <)$.

Case $E = \varepsilon \in [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1)$. Let $U_E := \sigma$. Our (IH) implies the existence of a function F_E such that 1, 2, 3, 4, 5, 6, 7 and 8 hold with respect to $E, U_E \in \mathrm{Class}(j-1)$; therefore i, ii, iii, iv, v, vi, vii, viii and ix hold for the pair (F_E, U_E). Now, suppose (G_E, V_E) is another pair such that i, ii, iii, iv, v, vi, vii, viii and ix hold. Then, by iii, $V_E = G_E(E) = \sigma = F_E(E) = U_E$, and $G_E: E(+^{j-1}) \cap \mathbb{E} \longrightarrow U_E(+^{j-1}) \cap \mathbb{E}$ is a function satisfying 1, 2, 3, 4, 5, 6, 7 and 8 with respect to $E, U_E \in \mathrm{Class}(j-1)$; thus, since by our (IH) there is only one such a function, $G_E = F_E$.

Let's prove now the general case of Claim1. Let $E \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1)$ and suppose Claim1 holds for $[\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap E$. **(SIH)**

Consider $\boldsymbol{A} := [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap E$ and $\boldsymbol{G} := \bigcup_{S \in A} F_S$.

Assertion0: $G: E \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is a function.

Proof of Assertion0:

Notice $G \subset \mathrm{OR} \times \mathrm{OR}$ is a binary relation with $\mathrm{Dom}\, G = \{a \in \mathrm{OR} | \exists b.(a, b) \in G\} = \bigcup_{S \in A} \mathrm{Dom}\, F_S = \bigcup_{S \in A} S(+^{j-1}) \cap \mathbb{E}) = E \cap \mathbb{E}$. Moreover, $\mathrm{Range}\, G = \{b \in \mathrm{OR} | \exists a.(a, b) \in G\} = \bigcup_{S \in A} \mathrm{Range}\, F_S = \bigcup_{S \in A} F_S[\mathrm{Dom}\, F_S] = \bigcup_{S \in A} F_S[S(+^{j-1}) \cap \mathbb{E}] \subset \bigcup_{S \in A} U_S(+^{j-1}) \cap \mathbb{E} \subset \sigma(+^j) \cap \mathbb{E}$.

The previous paragraph shows that G is a binary relation with $G: E \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$. Let's see now that G is a function. Note we just need to show that for any $a, b \in A$, F_a and F_b are compatible, that is, for any $x \in \mathrm{Dom}\, F_a \cap \mathrm{Dom}\, F_b$, $F_a(x) = F_b(x)$.

So let $a, b \in A$, and $x \in \mathrm{Dom}\, F_a \cap \mathrm{Dom}\, F_b$ be arbitrary. If $a = b$ then $F_a = F_b$ because by our (SIH), the pair (F_a, U_a) is unique. So without loss of generality, suppose $a < b$. Notice then the following:

•1. $a(+^{j-1}) \leqslant b$ and so $a(+^{j-1}) \in b(+^{j-1}) \cap \mathbb{E} = \mathrm{Dom}\, F_b$.

•2. $F_b(a), F_b(a(+^{j-1})) \in \mathrm{Class}(j-1) \cap \mathrm{Im}\, F_b \subset \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^j))$.

•3. $F_b(a(+^{j-1})) = F_b(a)(+^{j-1})$.

•4. $\forall q \in a(+^{j-1}) \cap \mathbb{E}. F_b(q) \in \mathbb{E} \cap F_b(a(+^{j-1})) = \mathbb{E} \cap (F_b(a)(+^{j-1}))$. This holds because $\forall x \in b(+^{j-1}) \cap \mathbb{E}. \mathbb{E} \ni F_b(x) = x[F_b] = H_{F_b}(x)$, because of •3. and because H_{F_b} is an $<$-isomorphism.

•5. By •4., $F_b|_{a(+^{j-1}) \cap \mathbb{E}}: a(+^{j-1}) \cap \mathbb{E} \longrightarrow F_b(a)(+^{j-1}) \cap \mathbb{E}$; moreover, it is clear this function is strictly increasing.

•6. $F_b|_{a(+^{j-1}) \cap \mathbb{E}}(\varepsilon) = \sigma$ and $F_b|_{a(+^{j-1}) \cap \mathbb{E}}(a) = F_b(a)$

•7. $F_b|_{a(+^{j-1}) \cap \mathbb{E}}|_{\varepsilon \cap \mathbb{E}} = F_b|_{\varepsilon \cap \mathbb{E}} = p$

•8. Because of our (SIH),

$\forall i \in [1, j]. \forall e \in [\varepsilon, a(+^{j-1})) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow F_b|_{a(+^{j-1}) \cap \mathbb{E}}(e) \in \mathrm{Class}(i)$ and the function $R_{F_b|_{a(+^{j-1}) \cap \mathbb{E}}}: (e, a(+^{j-1})) \longrightarrow R_{F_b|_{a(+^{j-1}) \cap \mathbb{E}}}[(e, a(+^{j-1}))] \subset (\sigma, F_b(a)(+^{j-1}))$, $t \longmapsto t[F_b|_{a(+^{j-1}) \cap \mathbb{E}}]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x, (+^1), \ldots, (+^{j-1}))$ isomorphism.

•9. If $j \geqslant 2$, then

$$\forall i \in [2, j]. \forall e \in \mathrm{Class}(i) \cap [\varepsilon, a(+^{j-1})). \forall s \in (e, e(+^i)) \cap a(+^{j-1}) \cap \mathrm{Class}(i-1).$$
$$f(i, e)(s) = \{s_1 > \ldots > s_k\} \Longleftrightarrow f(i, F_b(e))(F_b(s)) = \{F_b(s_1) > \ldots > F_b(s_k)\}$$

•10. If $j \geqslant 2. \forall i \in [2, j]. \forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, a(+^{j-1})). F_b(\lambda(i, s)) = \lambda(i, F_b(s))$

So, for $a \in [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1)$ and due to •1, •2, •3, •4, •5, •6, •7, •8, •9 and •10, the pairs (F_a, U_a) and $(F_b|_{a(+^{j-1}) \cap \mathbb{E}}, F_b(a))$ are two witnesses of i, ii, ii, iv, v, vi, vii, viii and ix of Claim1. Therefore, since by our (SIH) such pairs are unique, $F_a = F_b|_{a(+^{j-1}) \cap \mathbb{E}}$ and $U_a = F_b(a)$. From these equalities follows immediately that F_a and F_b are compatible.

Hence $G: E \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is a function. This proves Assertion0.

Assertion1: For any $a \in A = [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap E$ there exists $y \in \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^j))$ such that $G|_{a(+^{j-1}) \cap \mathbb{E}}: a(+^{j-1}) \cap \mathbb{E} \longrightarrow y \cap \mathbb{E}$ satisfies i, ii, iii, iv, v, vi, vii, viii and ix.

Proof of Assertion1:

For $a \in A$, $G|_{a(+^{j-1}) \cap \mathbb{E}} = F_a$ and $y = U_a(+^{j-1})$; so by our (SIH), F_a satisfies i, ii, iii, iv, v, vi, vii, viii and ix. This proves Assertion1.

Consider now the following: Let $\boldsymbol{\varphi := \sigma(+^j)}$. By Assertion0 and Assertion1 follows $G: E \cap \mathbb{E} \longrightarrow \varphi \cap \mathbb{E}$ is a strictly increasing function and since $E \in \mathrm{Class}(j-1) \ni \varphi$ and $E < \varphi$, then by our (IH) there exists a unique extension $\Theta: E(+^{j-1}) \cap \mathbb{E} \longrightarrow \varphi(+^{j-1}) \cap \mathbb{E}$ of G such that 1, 2, 3, 4, 5, 6, 7 and 8 hold with respect to E and φ.

This way, for $f(j, \varepsilon)(E) = \{E = E_1 > E_2 \ldots > E_q\}$, we define

$J_1 := \begin{cases} E_2 & \text{if } q \geqslant 2 \\ \varepsilon & \text{otherwise} \end{cases}$,

$Q := \{s \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) | G(J_1) < s \wedge m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\}$ and $\rho := \min Q$.

Of course, for ρ to make sense, we need to see that $Q \neq \emptyset$. Let's see this. Since $\varphi \leqslant m(E)[\Theta] + 1 \leqslant \eta(j-1, \varphi, m(E)[\Theta]) + 1 < \varphi(+^{j-1}) < m(\varphi)$, then, by $\leqslant_1$-connectedness and (6) of GenThmIH we get $\varphi \leqslant^{j-1} \eta(j-1, \varphi, m(E)[\Theta]) + 1$; but this implies, by $\leqslant^{j-1}$-connectedness (that is, by (3) of GenThmIH) $\varphi \leqslant^{j-1} m(E)[\Theta] + 1$. Therefore, by (4) of GenThmIH , there exists a sequence $(\varphi_i)_{i \in I} \subset \mathrm{Class}(j-1) \cap \varphi$ such that $\forall i \in I. T(j-1, \varphi, m(E)[\Theta]) \cap \varphi \subset \varphi_i$, $\varphi_i \underset{\mathrm{cof}}{\hookrightarrow} \varphi$ and $\varphi_i \leqslant_1 m(E)[\Theta][g(j-1, \varphi, \varphi_i)]$. Since the latter relations means $\forall i \in I. m(E)[\Theta][g(j-1, \varphi, \varphi_i)] \leqslant m(\varphi_i)$, then, using (2.3.2) of GenThmIH, we conclude $\forall i \in I. T(j-1, \varphi, m(E)[\Theta]) \cap \varphi \subset \varphi_i$, $\varphi_i \underset{\mathrm{cof}}{\hookrightarrow} \varphi$ and $m(E)[\Theta] \leqslant m(\varphi_i)[g(j-1, \varphi_i, \varphi)]$. From the previous line follows the existence of some $i_0 \in I$ such that $\varphi_{i_0} \in Q$. So $Q \neq \emptyset$.

As a final remark before our next assertion, we remind the reader that we know that $E = \min\{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | J_1 < e \leqslant E \wedge m(e)[g(j-1, e, E)] \geqslant m(E)\}$ by (1.3.6) of GenThmIH (for the case $j \leqslant n-1$) and by corollary 5.2 (for the case $j = n$).

Assertion2: Let $a \in \mathrm{Class}(j-1)$, $a \leqslant E$ and $b \in (\sigma, \varphi) \cap \mathrm{Class}(j-1)$ be arbitrary and suppose $G|_{a \cap \mathbb{E}}: a \cap \mathbb{E} \longrightarrow b \cap \mathbb{E}$. Let $\Phi: a(+^{j-1}) \cap \mathbb{E} \longrightarrow b(+^{j-1}) \cap \mathbb{E}$ be the only one extension of $G|_{a \cap \mathbb{E}}$ satisfying 1, 2, 3, 4, 5, 6, 7 and 8 with respect to a and b and which is obtained by our (IH) applied to a, b and $G|_{a \cap \mathbb{E}}$. Then $g(j-1, b, \varphi) \circ \Phi = \Theta \circ g(j-1, a, E)$.

Proof of Assertion2:

Let $P_1 := g(j-1, b, \varphi) \circ \Phi$ and $P_2 := \Theta \circ g(j-1, a, E)$. Then $P_1, P_2: a(+^{j-1}) \cap \mathbb{E} \longrightarrow \varphi(+^{j-1}) \cap \mathbb{E}$ are functions satisfying:

1*. P_1 and P_2 are strictly increasing (they are composition of strictly increasing functions);

2*. $P_1(a) = g(j-1, b, \varphi)(\Phi(a)) = g(j-1, b, \varphi)(b) = \varphi = \Theta(E) = \Theta(g(j-1, a, E)(a)) = P_2(a)$;

3*. $\forall e \in a \cap \mathbb{E}. P_1(e) = g(j-1, b, \varphi)(\Phi(e)) = g(j-1, b, \varphi)(G(e)) = G(e)$ and $\forall e \in a \cap \mathbb{E}. P_2(e) = \Theta(g(j-1, a, E)(e)) = \Theta(e) = G(e)$; that is, $P_1|_{a \cap \mathbb{E}} = G|_{a \cap \mathbb{E}} = P_2|_{a \cap \mathbb{E}}$

4*. $H_{P_1}: (a, a(+^{j-1})) \longrightarrow H_{P_1}[(a, a(+^{j-1}))] \subset (\varphi, \varphi(+^{j-1}))$, $t \longmapsto t[P_1]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism. This is because for $t \in (a, a(+^{j-1}))$, $t[P_1] = t[g(j-1, b, \varphi) \circ \Phi] = t[\Phi][g(j-1, b, \varphi)]$, that is, $H_{P_1} = H_{g(j-1, b, \varphi)} \circ H_\Phi$ and since H_Φ and $H_{g(j-1, b, \varphi)}$ are $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphisms, then H_{P_1} is $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism.

Analogously, $H_{P_2}: (a, a(+^{j-1})) \longrightarrow H_{P_2}[(a, a(+^{j-1}))] \subset (\varphi, \varphi(+^{j-1}))$, $t \longmapsto t[P_2]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism, because $H_{P_2} = H_\Theta \circ H_{g(j-1, a, E)}$.

5*. $\forall i \in [1, j-1]. \forall e \in [a, a(+^{j-1})) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow P_1(e) \in \mathrm{Class}(i) \ni P_2(e)$.

6*. H_{P_1} and H_{P_2} are also $(+^1), \ldots, (+^{j-2})$ isomorphisms, because $H_{P_1} = H_{g(j-1, b, \varphi)} \circ H_\Phi$ and $H_{P_2} = H_\Theta \circ H_{g(j-1, a, e)}$ and because $H_{g(j-1, b, \varphi)}$, H_Φ, H_Θ and $H_{g(j-1, a, e)}$ are $(+^1), \ldots, (+^{j-2})$ isomorphisms.

7*. $\forall i \in [2, j-1]. \forall s \in \mathrm{Class}(i) \cap [a, a(+^{j-1})). \forall Z \in (s, s(+^i)) \cap \mathrm{Class}(i-1)$.
$f(i, s)(Z) = \{Z_1 > \ldots > Z_d\} \Longleftrightarrow f(i, \Phi(s))(\Phi(Z)) = \{\Phi(Z_1) > \ldots > \Phi(Z_d)\} \Longleftrightarrow$
$f(i, g(j-1, b, \varphi)(\Phi(s)))(g(j-1, b, \varphi)(\Phi(Z))) =$
$\{g(j-1, b, \varphi)(\Phi(Z_1)) > \ldots > g(j-1, b, \varphi)(\Phi(Z_d))\} \Longleftrightarrow$
$f(i, P_1(s))(P_1(Z)) = \{P_1(Z_1) > \ldots > P_1(Z_d)\}$.

Analogously, $f(i, s)(Z) = \{Z_1 > \ldots > Z_d\} \Longleftrightarrow f(i, P_2(s))(P_2(Z)) = \{P_2(Z_1) > \ldots > P_2(Z_d)\}$.

8*. $\forall i \in [2, j-1]. \forall s \in \mathrm{Class}(i-1) \cap [a, a(+^{j-1}))$.
$P_1(\lambda(i, s)) = g(j-1, b, \varphi)(\Phi(\lambda(i, s))) = g(j-1, b, \varphi)(\lambda(i, \Phi(s))) = \lambda(i, g(j-1, b, \varphi)(\Phi(s))) =$
$\lambda(i, P_1(s))$.

Analogously $P_2(\lambda(i, s)) = \lambda(i, P_2(s))$.

Now, according to our IH applied to $a \in \mathrm{Class}(j-1) \ni \varphi$ and $G|_{a \cap \mathbb{E}}: a \cap \mathbb{E} \longrightarrow \varphi \cap \mathbb{E}$, there is exactly one extension of $G|_{a \cap \mathbb{E}}$ to $a(+^{j-1}) \cap \mathbb{E} \longrightarrow \varphi(+^{j-1}) \cap \mathbb{E}$ satisfying 1*, 2*, 3*, 4*, 5*, 6*, 7* and 8*. Thus $P_1 = P_2$.

So we have shown Assertion2.

Assertion3.

Let $a\in(\varepsilon,\varepsilon(+^j))\cap E\cap\mathrm{Class}(j-1)$ with $f(j,\varepsilon)(a)=\{a=a_1>a_2>\ldots>a_m\}$. We define $J_2(a):=\begin{cases} G(a_2) & \text{if } m\geqslant 2\\ \sigma & \text{otherwise}\end{cases}$,
$K_1(a):=\{e\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)|J_2(a)<e\leqslant G(a)\wedge m(e)[g(j-1,e,G(a))]\geqslant m(G(a))\}$, and
$K_2(a):=\{e\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)|J_2(a)<e\wedge m(e)[g(j-1,e,\varphi)]\geqslant m(a)[g(j-1,a,E)][\Theta]\}$.

Then $G(a)=\min K_1(a)=\min K_2(a)$.

Proof of Assertion3:

Since $f(j,\varepsilon)(a)=\{a=a_1>a_2>\ldots>a_m\}$, then $f(j,G(\varepsilon))(G(a))\underset{\text{Assertion1}}{=}$ $\{G(a)=G(a_1)>G(a_2)>\ldots>G(a_m)\}$. **(A0*)**.
Therefore, $G(a)=\min K_1(a)$ holds because of (1.3.6) of GenThmIH (for the case $j\leqslant n-1$) and because of corollary 5.2 (for the case $j=n$).

On the other hand, using Assertion2 with a, $b:=G(a)$ and
$\Phi:=G|_{a(+^{j-1})\cap\mathbb{E}}: a(+^{j-1})\cap\mathbb{E}\longrightarrow G(a)(+^{j-1})\cap\mathbb{E}$ (the only one extension of $G|_{a\cap\mathbb{E}}$ satisfying 1, 2, 3, 4, 5, 6, 7 and 8 with respect to a and $G(a)$ and which is obtained by our (IH) applied to a, $G(a)$ and $G|_{a\cap\mathbb{E}}$), we obtain the equality
$g(j-1,G(a),\varphi)\circ G|_{a(+^{j-1})\cap\mathbb{E}}=\Theta\circ g(j-1,a,E)$. **(A1*)**. This will be used below.

Now let's see that $G(a)\in K_2(a)$. By (A0*) we know $J_2(a)<G(a)$. Moreover, note
$m(G(a))[g(j-1,G(a),\varphi)]\underset{\text{Assertion1}}{=}$
$=m(a)[G|_{a(+^{j-1})\cap\mathbb{E}}][g(j-1,G(a),\varphi)]=$
$=m(a)[g(j-1,G(a),\varphi)\circ G|_{a(+^{j-1})\cap\mathbb{E}}]\underset{\text{by (A1*)}}{=}$
$=m(a)[\Theta\circ g(j-1,a,E)]=m(a)[g(j-1,a,E)][\Theta]$. This shows $G(a)\in K_2(a)$.

Now, to prove that $G(a)=\min K_2(a)$ it suffices to show that $\neg\exists\xi\in G(a)\cap K_2(t)$.

Suppose $\exists\xi\in G(a)\cap K_2(t)$. Then $J_2(a)<\xi<G(a)$ and
$m(\xi)[g(j-1,\xi,\varphi)]\geqslant m(a)[g(j-1,a,E)][\Theta]=[\Theta\circ g(j-1,a,E)]$. **(A3*)**.
But $m(\xi)[g(j-1,\xi,\varphi)]=m(\xi)[g(j-1,G(a),\varphi)\circ g(j-1,\xi,G(a))]=$
$=m(\xi)[g(j-1,\xi,G(a))][g(j-1,G(a),\varphi)]$ and
$\Theta\circ g(j-1,a,E)\underset{\text{by (A1*)}}{=}g(j-1,G(a),\varphi)\circ G|_{a(+^{j-1})\cap\mathbb{E}}$; thus, from this equalities and (A3*) we get
$m(\xi)[g(j-1,\xi,G(a))][g(j-1,G(a),\varphi)]\geqslant m(a)[g(j-1,G(a),\varphi)\circ G|_{a(+^{j-1})\cap\mathbb{E}}]=$
$m(a)[G|_{a(+^{j-1})\cap\mathbb{E}}][g(j-1,G(a),\varphi)]\underset{\text{Assertion1}}{=}m(G(a))[g(j-1,G(a),\varphi)]$.
But this means $m(\xi)[g(j-1,\xi,G(a))]\geqslant m(G(a))$, that is, $\xi\in G(a)\cap K_1(a)$. Contradiction. Hence $\neg\exists\xi\in G(a)\cap K_2(t)$, and therefore $G(a)=\min K_2(a)$.

This concludes the proof of Assertion3.

Assertion4: $\forall a\in A=[\varepsilon,\varepsilon(+^j))\cap\mathrm{Class}(j-1)\cap E.G(a)<\rho$.

Proof of Assertion4:

We proceed by induction on $(A,<)$.

Take $a\in A$ and suppose $\forall b\in A\cap a.G(b)<\rho$. **(IHAssertion4)**

If $a\leqslant J_1$, then $G(a)\leqslant G(J_1)<\rho$. So suppose $a\in(J_1,E)\cap\mathrm{Class}(j-1)$.
Take $f(j,\varepsilon)=\{a=a_1>\ldots>a_m\}$. Then, by Assertion3, $G(a)=\min K_2(a)$, **(A4*)**
where
$J_2(a)=\begin{cases} G(a_2) & \text{if } m\geqslant 2\\ G(\varepsilon)=\sigma & \text{otherwise}\end{cases}$, and
$K_2(a)=\{e\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)|J_2(a)<e\wedge m(e)[g(j-1,e,\varphi)]\geqslant m(a)[g(j-1,a,E)][\Theta]\}$.

Let $\Phi: a(+^{j-1}) \cap \mathbb{E} \longrightarrow \rho(+^{j-1}) \cap \mathbb{E}$ be the only one extension of $G|_{a\cap\mathbb{E}}$ satisfying 1, 2, 3, 4, 5, 6, 7 and 8 with respect to a and ρ and which is obtained by our (IH) applied to a, ρ and $G|_{a\cap\mathbb{E}}$.

We assure $m(a)[\Phi] < m(\rho)$. **(A5*)**

Suppose the opposite, that $m(a)[\Phi] \geqslant m(\rho)$. Then
$m(a)[\Phi] \geqslant m(\rho) \Longleftrightarrow$
$m(a)[g(j-1,\rho,\varphi) \circ \Phi] = m(a)[\Phi][g(j-1,\rho,\varphi)] \geqslant m(\rho)[g(j-1,\rho,\varphi)]$. **(A6*)**.
But $m(\rho)[g(j-1,\rho,\varphi)] \underset{\text{by definition of } \rho}{\geqslant} m(E)[\Theta]$ and $g(j-1,\rho,\varphi) \circ \Phi \underset{\text{by Assertion2}}{=}$ $\Theta \circ g(j-1,a,E)$, so these observations and (A6*) imply that
$m(a)[g(j-1,a,E)][\Theta] = m(a)[\Theta \circ g(j-1,a,E)] \geqslant m(E)[\Theta]$. **(A7*)**

On the other hand, since $a \in (J_1, E) \cap \mathrm{Class}(j-1)$ and
$E = \min\{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | J_1 < e \leqslant E \wedge m(e)[g(j-1,e,E)] \geqslant m(E)\}$, then
$m(a)[g(j-1, a, E)] < m(E)$, and so $m(a)[g(j-1, a, E)][\Theta] < m(E)[\Theta]$. This is a clear contradiction with (A7*). Therefore (A5*) holds.

To show that $\eta(j-1,\rho,m(a)[\Phi]) = m(a)[\Phi]$. **(A8*)**

Let $v_1 := \max \mathrm{Ep}(m(a))$, $v_0 := v_1(+^1)$ and for
$k \in [1, j-2]$, $v_{k+1} := \lambda(k+1, v_k)$. Then $v_{j-1} = a \leqslant v_{j-2} \leqslant \ldots \leqslant v_2 \leqslant v_1 < m(a) < v_0$,
$\forall i \in [1, j-1].v_i \in \mathrm{Class}(i)$ and $\forall i \in [1, j-1].m(v_i) \leqslant m(a)$; this implies
$\Phi(v_{j-1}) = \rho \leqslant \Phi(v_{j-2}) \leqslant \ldots \leqslant \Phi(v_2) \leqslant \Phi(v_1) < m(a)[\Phi] < \Phi(v_0)$, $\forall i \in [1,\, j-1].\Phi(v_i) \in \mathrm{Class}(i)$ and $\forall i \in [1, j-1].v_i \neq a \Longrightarrow m(\Phi(v_i)) = m(v_i)[\Phi] \leqslant m(a)[\Phi]$.

Let $\xi \in (\rho, m(a)[\Phi]]$. If $\xi \in \{\Phi(v_{j-1}), \ldots, \Phi(v_1)\}$, then $m(\xi) \leqslant m(a)[\Phi]$. Moreover, if $\xi \notin \{\Phi(v_{j-1}), \ldots, \Phi(v_1)\}$, then there exist some $l \in [1,\, j-1]$ such that $\xi \in (\Phi(v_l), \Phi(v_{(l-1)}))$; note this implies $\xi \notin \mathrm{Class}(l)$ and therefore $m(\xi) < \Phi(v_{(l-1)})$ (since $m(\xi) \geqslant \Phi(v_{(l-1)})$ would imply $\xi \in \mathrm{Class}(l)$). Now we have two cases: Case $l \geqslant 2$. Then
$m(\xi) < \Phi(v_{(l-1)})) \leqslant m(v_{(l-1)})[\Phi] \leqslant m(a)[\Phi]$. Case $l = 1$. If $\xi \notin \mathbb{P}$, then $m(\xi) = \xi \leqslant m(a)[\Phi]$. If $\xi \in \mathbb{P}$, then $\xi =_{\mathrm{CNF}} \omega^R$ for some R, $m(a) =_{\mathrm{CNF}} \omega^{Y_1} y_1 + \ldots + \omega^{Y_l} y_l$ for some Y_i, y_k and so $m(a)[\Phi] =_{\mathrm{CNF}} \omega^{Y_1[\Phi]} y_1 + \ldots + \omega^{Y_l[\Phi]} y_l$. If $R < Y_1[\Phi]$, then
$m(\xi) = \xi + d\xi < \omega^R + \omega^R < \omega^{Y_1[\Phi]} \leqslant \omega^{Y_1[\Phi]} y_1 + \ldots + \omega^{Y_l[\Phi]} y_l = m(a)[\Phi]$. If $R = Y_1[\Phi]$, then
$\xi = \omega^{Y_1}[\Phi]$; but this and the facts that $m(\omega^{Y_1}) = \omega^{Y_1} + d\omega^{Y_1} \leqslant m(a)$ and $d(\omega^{Y_1[\Phi]}) = (d\omega^{Y_1})[\Phi]$ (the latter holds because for $Y_1 =_{\mathrm{CNF}} \omega^{\omega^{K_1} k_1 + \ldots + \omega^{K_d} k_d}$, $d\omega^{Y_1} = K_d$ and so $(d\omega^{Y_1})[\Phi] = K_d[\Phi] = d(\omega^{\omega^{K_1[\Phi]} k_1 + \ldots + \omega^{K_d[\Phi]} k_d}) = d(\omega^{(\omega^{K_1} k_1 + \ldots + \omega^{K_d} k_d)[\Phi]}) = d\omega^{Y_1[\Phi]})$ imply that
$m(\xi) = \xi + d\xi = \omega^{Y_1[\Phi]} + d\omega^{Y_1[\Phi]} = \omega^{Y_1}[\Phi] + (d\omega^{Y_1})[\Phi] = (\omega^{Y_1} + d\omega^{Y_1})[\Phi] \leqslant m(a)[\Phi]$.

The previous shows $\forall \xi \in (\rho, m(a)[\Phi]].m(\xi) \leqslant m(a)[\Phi]$. So $\eta(j-1, \rho, m(a)[\Phi]) = m(a)[\Phi]$ and (A8*) holds.

We continue with the proof of Assertion4. From (A8*) and (A5*) we have that
$\eta(j-1,\rho,m(a)[\Phi]) + 1 = m(a)[\Phi] + 1 \leqslant m(\rho)$; then, by $\leqslant_1$-connectedness
$\rho \leqslant_1 \eta(j-1,\rho,m(a)[\Phi]) + 1$ which, by (6) of GenThmIH, implies
$\rho \leqslant^{j-1} \eta(j-1,\, \rho,\, m(a)[\Phi]) + 1 = m(a)[\Phi] + 1$. Hence, by use of (4) of GenThmIH we obtain a sequence $(\rho_i)_{i\in I} \subset \rho \cap \mathrm{Class}(j-1)$ such that $\rho_i \underset{\mathrm{cof}}{\hookrightarrow} \rho$ and $\forall i \in I.m(\rho_i)[g(j-1,\rho_i,\rho)] \geqslant m(a)[\Phi]$;
this implies $\forall i \in I.m(\rho_i)[g(j-1,\rho_i,\rho)][g(j-1,\rho,\varphi)] \geqslant m(a)[\Phi][g(j-1,\rho,\varphi)]$ **(A9*)**.
But note that for any $i \in I$, $m(\rho_i)[g(j-1,\, \rho_i,\, \varphi)] = m(\rho_i)[g(j-1,\, \rho_i,\, \rho)][g(j-1,\, \rho,\, \varphi)]$ and
$m(a)[\Phi][g(j-1,\rho,\varphi)] = m(a)[g(j-1,\rho,\varphi) \circ \Phi] \underset{\text{by Assertion2}}{=} m(a)[\Theta \circ g(j-1,a,E)] =$
$m(a)[g(j-1,a,E)][\Theta]$; this way, (A9*) can be restated as: there is a sequence
$(\rho_i)_{i\in I} \subset \rho \cap \mathrm{Class}(j-1)$ such that $\rho_i \underset{\mathrm{cof}}{\hookrightarrow} \rho$ and
$\forall i \in I.m(\rho_i)[g(j-1,\rho_i,\varphi)] \geqslant m(a)[g(j-1,a,E)][\Theta]$. **(A10*)**.

Finally, since $\varepsilon \in A \cap a$ and for the case $f(j,\varepsilon)(a) = \{a = a_1 > \ldots > a_m\}$ with $m \geqslant 2$ it also holds $a_2 \in A \cap a$, then by our (IHAssertion4), $J_2(a) < \rho$. This and (A10*) imply that the set $\rho \cap K_2(a)$ is confinal in ρ. Therefore, since by (A4*), $G(a) = \min K_2(a)$, then
$G(a) \in \rho \cap K_2(a)$. Hence $G(a) < \rho$.

This proves Assertion4.

Assertion5: $G\colon E\cap\mathbb{E}\longrightarrow\rho\cap\mathbb{E}$

Proof of Assertion5:

Case $\boldsymbol{E\notin}$ Lim(Class($\boldsymbol{j-1}$)).

Then $\neg\forall l\in[\varepsilon,E)\cap\mathrm{Class}(j-1)\exists r\in\mathrm{Class}(j-1)\cap E.l<r$ and so $l:=\max[\varepsilon,E)\cap\mathrm{Class}(j-1)$ exists. This way, $l(\,+^{j-1}\,)=E$ and $G=\bigcup_{S\in A}F_S=F_l$. From this and our (SIH) we get that (F_l,U_l) is such that $U_l\in[\sigma,\sigma(\,+^{j}\,))\cap\mathrm{Class}(j-1)$ and $F_l\colon E\cap\mathbb{E}\longrightarrow U_l(\,+^{j-1}\,)\cap\mathbb{E}\subset\rho\cap\mathbb{E}$, where $U_l(\,+^{j-1}\,)\cap\mathbb{E}\subset\rho\cap\mathbb{E}$ holds because $\rho\underset{\text{by Assertion4}}{>}G(l)=F_l(l)=U_l$ and so $U_l(\,+^{j-1}\,)\leqslant\rho$.

Case $\boldsymbol{E\in}$ Lim(Class($\boldsymbol{j-1}$)).

Let $e\in E\cap\mathbb{E}$. Since $E\in\mathrm{Lim}(\mathrm{Class}(j-1))$, then there exists $a\in E\cap(\mathrm{Class}(j-1))$ such that $e<a$. Since we already know that G is increasing, then $G(e)<G(a)<\rho$.

This concludes the proof of Assertion5.

We continue with the proof of Claim1. Let $U_E:=\rho$ and $F_E\colon E(\,+^{j-1}\,)\cap\mathbb{E}\longrightarrow U_E(\,+^{j-1}\,)\cap\mathbb{E}$ be the only one function that is extension of the function $G\colon E\cap\mathbb{E}\longrightarrow U_E\cap\mathbb{E}$ which is obtained by our (IH) applied to $E,U_E\in\mathrm{Class}(j-1)$, $E<U_E$ and G. According to our (IH), F_E satisfies 1, 2, 3, 4, 5, 6, 7 and 8 with respect to $E\in\mathrm{Class}(j-1)$ and $U_E\in\mathrm{Class}(j-1)$. **(B1*)**

Assertion6: $m(E)[F_E]=m(\rho)=m(F_E(E))$

Proof of Assertion6:

The right hand side equality is clear, because $F_E(E)=\rho$. So we only need to prove left hand side equality.

First we show $m(E)[F_E]\not>m(\rho)$. Suppose $m(E)[F_E]>m(\rho)$.

Then $m(E)[g(j-1,\rho,\varphi)\circ F_E]=m(E)[F_E][g(j-1,\rho,\varphi)]>m(\rho)[g(j-1,\rho,\varphi)]$, but since by Assertion2 $g(j-1,\rho,\varphi)\circ F_E=\Theta\circ g(j-1,E,E)=\Theta$, then $m(E)[\Theta]>m(\rho)[g(j-1,\rho,\varphi)]$. Contradiction, because by definition $\rho=\min Q\in Q$.

To show that $\eta(j-1,\rho,m(E)[F_E])=m(E)[F_E]$. **(B2*)**

The proof is essentially the same as the proof of (A8*):
Let $v_1:=\max\mathrm{Ep}(m(E))$, $v_0:=v_1(\,+^{1}\,)$ and for $k\in[1,j-2]$, $v_{k+1}:=\lambda(k+1,v_k)$. Then $v_{j-1}=E\leqslant v_{j-2}\leqslant\ldots\leqslant v_2\leqslant v_1<m(E)<v_0$, $\forall i\in[1,j-1].v_i\in\mathrm{Class}(i)$ and $\forall i\in[1,j-1].m(v_i)\leqslant m(E)$; this implies $F_E(v_{j-1})=\rho\leqslant F_E(v_{j-2})\leqslant\ldots\leqslant F_E(v_1)<m(E)[F_E]<F_E(v_0)$, $\forall i\in[1,j-1].F_E(v_i)\in\mathrm{Class}(i)$ and $\forall i\in[1,j-1].v_i\neq E\Longrightarrow m(F_E(v_i))=m(v_i)[F_E]\leqslant m(E)[F_E]$.

Let $\xi\in(\rho,m(E)[F_E]]$. If $\xi\in\{F_E(v_{j-1}),\ldots,F_E(v_1)\}$, then $m(\xi)\leqslant m(E)[F_E]$. Moreover, if $\xi\notin\{F_E(v_{j-1}),\ldots,F_E(v_1)\}$, then there exist some $l\in[1,j-1]$ such that $\xi\in(F_E(v_l),\ F_E(v_{(l-1)}))$; this implies $\xi\notin\mathrm{Class}(l)$ and therefore $m(\xi)<F_E(v_{(l-1)})$ (since $m(\xi)\geqslant F_E(v_{(l-1)})$ would imply $\xi\in\mathrm{Class}(l)$). Now we have two cases: Case $l\geqslant 2$. Then $m(\xi)<F_E(v_{(l-1)}))\leqslant m(v_{(l-1)})[F_E]\leqslant m(E)[F_E]$. Case $l=1$. If $\xi\notin\mathbb{P}$, then $m(\xi)=\xi\leqslant m(E)[F_E]$. If $\xi\in\mathbb{P}$, then $\xi=_{\mathrm{CNF}}\omega^R$ for some R, $m(E)=_{\mathrm{CNF}}\omega^{Y_1}y_1+\ldots+\omega^{Y_l}y_l$ for some Y_i,y_k and so $m(E)[F_E]=_{\mathrm{CNF}}\omega^{Y_1[F_E]}y_1+\ldots+\omega^{Y_l[F_E]}y_l$. If $R<Y_1[F_E]$, then $m(\xi)=\xi+d\xi<\omega^R+\omega^R<\omega^{Y_1[F_E]}\leqslant\omega^{Y_1[F_E]}y_1+\ldots+\omega^{Y_l[F_E]}y_l=m(E)[F_E]$. If $R=Y_1[F_E]$, then $\xi=\omega^{Y_1}[F_E]$; this and the facts that $m(\omega^{Y_1})=\omega^{Y_1}+d\omega^{Y_1}\leqslant m(E)$ and $d(\omega^{Y_1[F_E]})=(d\omega^{Y_1})[F_E]$ imply that $m(\xi)=\xi+d\xi=\omega^{Y_1[F_E]}+d\omega^{Y_1[F_E]}=(\omega^{Y_1}+d\omega^{Y_1})[F_E]\leqslant m(E)[F_E]$.

The previous shows $\forall\xi\in(\rho,m(E)[F_E].m(\xi)\leqslant m(E)[F_E]$, from which follows $\eta(j-1,\rho,m(E)[F_E])=m(E)[F_E]$. Hence (B2*) holds.

Now we show $m(E)[F_E] \not< m(\rho)$. Suppose $m(E)[F_E] < m(\rho)$. This implies, using (B2*), $\leqslant_1$ -connectedness and (6) of GenThmIH, that $\rho \leqslant^{j-1} m(E)[F_E] + 1$. But then, by (4) of GenThmIH, there exists a sequence $(\rho_i)_{i \in I} \subset \mathrm{Class}(j-1) \cap \rho$ such that
$\forall i \in I.T(j-1, \rho, m(E)[F_E]) \cap \rho \subset \rho_i$, $\rho_i \underset{\mathrm{cof}}{\hookrightarrow} \rho$ and $\rho_i \leqslant m(E)[F_E][g(j-1, \rho, \rho_i)] \leqslant m(\rho_i)$.
Hence, there exists $i_0 \in I$ such that $\rho_{i_0} > G(J_1)$. Let $\psi := \rho_{i_0}$. Note then that
$m(E)[F_E][g(j-1, \rho, \psi)] \leqslant m(\psi)$ implies
$m(E)[F_E] = m(E)[F_E][g(j-1, \rho, \psi)][g(j-1, \psi, \rho)] \leqslant m(\psi)[g(j-1, \psi, \rho)]$, which subsequently implies
$m(E)[F_E][g(j-1, \rho, \varphi)] \leqslant m(\psi)[g(j-1, \psi, \rho)][g(j-1, \rho, \varphi)] = m(\psi)[g(j-1, \psi, \varphi)]$. But since by Assertion2 $g(j-1, \rho, \varphi) \circ F_E = \Theta \circ g(j-1, E, E) = \Theta$, then the previous is
$m(E)[\Theta] \leqslant m(\psi)[g(j-1, \psi, \varphi)]$. This shows $\psi \in Q \cap \rho$. Contradiction because $\rho = \min Q$. Thus $m(E)[F_E] \not< m(\rho)$.

Hence, from $m(E)[F_E] \not> m(\rho)$ and $m(E)[F_E] \not< m(\rho)$ we conclude $m(E)[F_E] = m(\rho)$.

This proves Assertion6.

Assertion7: (F_E, U_E) satisfies i, ii, iii, iv, vi and ix of Claim1.

Proof of Assertion7:

i*. $U_E \in \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^j))$

ii*. $F_E: E(+^{j-1}) \cap \mathbb{E} \longrightarrow U_E(+^{j-1}) \cap \mathbb{E}$ is strictly increasing (because of (B1*)).

iii*. $F_E(\varepsilon) = G(\varepsilon) = \sigma$ and $F_E(E) = U_E$ (this last equality holds because of (B1*)).

iv*. $F_E|_{\varepsilon \cap \mathbb{E}} = G|_{\varepsilon \cap \mathbb{E}} = p$

vi*. $\forall i \in [1, j]. \forall e \in [\varepsilon, E(+^{j-1})) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow F_E(e) \in \mathrm{Class}(i)$. It is easy to see this holds: Let $i \in [1, j]$ and $e \in [\varepsilon, E(+^{j-1})) \cap \mathbb{E}$.
Case $e \in [\varepsilon, E) \cap \mathbb{E}$. Then $e \in \mathrm{Class}(i) \Longleftrightarrow G(e) \in \mathrm{Class}(i) \Longleftrightarrow F_E(e) \in \mathrm{Class}(i)$.
Case $e \in [E, E(+^{j-1})) \cap \mathbb{E}$. Then $e \in \mathrm{Class}(i) \Longleftrightarrow e \in \mathrm{Class}(i) \wedge i \in [1, j-1] \underset{\text{by (B1*)}}{\Longleftrightarrow}$
$F_E(e) \in \mathrm{Class}(i)$.

ix*. If $j \geqslant 2$, then $\forall i \in [2, j]. \forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, E(+^{j-1})). F_E(\lambda(i, s)) = \lambda(i, F_E(s))$. It is easy to see this holds: Suppose $j \geqslant 2$ and let $i \in [2, j]$ and $s \in \mathrm{Class}(i-1) \cap [\varepsilon, E(+^{j-1}))$ be arbitrary.
Case $s \in [\varepsilon, E) \cap \mathbb{E}$. Then $\lambda(i, s) \in [\varepsilon, E) \cap \mathbb{E}$ and
$F_E(\lambda(i, s)) = G(\lambda(i, s)) = \lambda(i, G(s)) = \lambda(i, F_E(s))$.
Case $s \in [E, E(+^{j-1})) \cap \mathbb{E}$. Then $F_E(\lambda(i, s)) = \lambda(i, F_E(s))$ because of (B1*).

The previous proves Assertion7.

Assertion8: (F_E, U_E) satisfies v of Claim1.

Proof of Assertion8:

v*. We assure the function $R_{F_E}: (\varepsilon, E(+^{j-1})) \longrightarrow R_{F_E}[(\varepsilon, E(+^{j-1}))] \subset (\sigma, U_E(+^{j-1}))$, $t \longmapsto t[F_E]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism.

Note R_{F_E} preserves $<, +, \cdot, \lambda x.\omega^x$ because $F_E: E(+^{j-1}) \cap \mathbb{E} \longrightarrow U_E(+^{j-1}) \cap \mathbb{E}$ is an strictly increasing function and because of the general properties we know about substitutions (propositions 3.12, 3.10 and 3.14). So we only have to see that R_{F_E} preserves $<_1$ too: Let $x, y \in (\varepsilon, E(+^{j-1}))$ with $x < y$.

First, we assure $x <_1 y \Longleftrightarrow x <_1 y < E \vee E \leqslant x <_1 y$. **(B3*)**.

The reason of (B3*) is that $x <_1 y$ with $x < E \leqslant y$ is impossible: Assume $x <_1 y \wedge x < E \leqslant y$. Then $x \in \mathrm{Class}(j)$ by proposition 3.6. But this is a contradiction, because
$x \in (\varepsilon, E(+^{j-1})) \subset (\varepsilon, \varepsilon(+^j))$ and $(\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j) = \emptyset$. Therefore (B3*) holds.

Now, suppose $x <_1 y$. By (B3*) we have three cases:

Case $x <_1 y < E$. Then there exists $a \in \mathrm{Class}(j-1) \cap [\varepsilon, E)$ such that $x, y \in (\varepsilon, a(+^{j-1}))$; moreover, by Assertion1, the function $G|_{a(+^{j-1}) \cap \mathbb{E}} = F_E|_{a(+^{j-1}) \cap \mathbb{E}}$ is such that
$R_{F_E|_{a(+^{j-1}) \cap \mathbb{E}}}: (\varepsilon, a(+^{j-1})) \longrightarrow R_{F_E|_{a(+^{j-1}) \cap \mathbb{E}}}[(\varepsilon, a(+^{j-1}))]$, $t \longmapsto t[F_E|_{a(+^{j-1}) \cap \mathbb{E}}]$ is an
$(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism. So $x[F_E|_{a(+^{j-1}) \cap \mathbb{E}}] <_1 y[F_E|_{a(+^{j-1}) \cap \mathbb{E}}] \Longleftrightarrow x[F_E] <_1 y[F_E]$.

Case $E < x <_1 y$. Then $x, y \in (E, E(+^{j-1}))$, and since by (B1*) the function H_{F_E}: $(E, E(+^{j-1})) \longrightarrow H_{F_E}[(E, E(+^{j-1}))]$, $t \longmapsto t[F_E]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism, then $x[F_E] <_1 y[F_E]$.

Case $E = x <_1 y$. Then $E < y \leqslant m(E)$, and then $F_E(E) = E[F_E] < y[F_E] \leqslant m(E)[F_E] \underset{\text{by Assertion6}}{=} m(F_E(E))$, that is, $x[F_E] <_1 y[F_E]$.

Suppose $x \not<_1 y$. We have four cases:

Case $x < y < E$. Then there exists $a \in \mathrm{Class}(j-1) \cap [\varepsilon, E)$ such that $x, y \in (\varepsilon, a(+^{j-1}))$; moreover, by Assertion1, the function $G|_{a(+^{j-1}) \cap \mathbb{E}} = F_E|_{a(+^{j-1}) \cap \mathbb{E}}$ is such that $R_{F_E|_{a(+^{j-1}) \cap \mathbb{E}}}$: $(\varepsilon, a(+^{j-1})) \longrightarrow R_{F_E|_{a(+^{j-1}) \cap \mathbb{E}}}[(\varepsilon, a(+^{j-1}))]$, $t \longmapsto t[F_E|_{a(+^{j-1}) \cap \mathbb{E}}]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism. So $x[F_E|_{a(+^{j-1}) \cap \mathbb{E}}] \not<_1 y[F_E|_{a(+^{j-1}) \cap \mathbb{E}}] \Longleftrightarrow x[F_E] \not<_1 y[F_E]$.

Case $E < x < y$. Then $x, y \in (E, E(+^{j-1}))$, and since by (B1*) the function H_{F_E}: $(E, E(+^{j-1})) \longrightarrow H_{F_E}[(E, E(+^{j-1}))]$, $t \longmapsto t[F_E]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism, then $x[F_E] \not<_1 y[F_E]$.

Case $x < E \leqslant y$. Then $x[F_E] < E[F_E] = F_E(E) = \rho \leqslant y[F_E]$; that is, $x[F_E] \in (\sigma, \rho) \subset (\sigma, \sigma(+^j))$ and $y[F_E] \in [\rho, \rho(+^{j-1}))$. So $x[F_E] \notin \mathrm{Class}(j)$ and from all this, just as in the proof of (B3*), it follows $x[F_E] \not<_1 y[F_E]$.

Case $E = x < y$. This means $E < m(E) < y$, and then $F_E(E) = E[F_E] < m(E)[F_E] < y[F_E]$. But by Assertion6 $m(E)[F_E] = m(F_E(E))$, so the previous inequality is $F_E(E) = E[F_E] < m(F_E(E)) < y[F_E]$, which means $x[F_E] \not<_1 y[F_E]$.

This proves Assertion8.

Assertion9: (F_E, U_E) satisfies vii of Claim1.

vii*. The function R_{F_E}: $(\varepsilon, E(+^{j-1})) \longrightarrow R_{F_E}[(\varepsilon, E(+^{j-1}))] \subset (\sigma, U_E(+^{j-1}))$, $t \longmapsto t[F_E]$ is an $(+^1), \ldots, (+^{j-1})$ isomorphism.

Proof of Assertion9:

Let $i \in [1, j-1]$ and $e \in [\varepsilon, E(+^{j-1})) \cap \mathrm{Class}(i)$ such that $e(+^i) \in (\varepsilon, E(+^{j-1}))$. We have some cases:

Case $e(+^i) \in (\varepsilon, E)$. Then there exists $a \in \mathrm{Class}(j-1) \cap (\varepsilon, E)$ such that $e(+^i) \in a(+^{j-1}) \cap \mathbb{E}$ and then
$(e(+^i))[F_E] = (e(+^i))[G|_{a(+^{j-1}) \cap \mathbb{E}}] \underset{\text{by Assertion1}}{=} (e[G|_{a(+^{j-1}) \cap \mathbb{E}}])(+^i) = (e[F_E])(+^i)$.

Case $e(+^i) \in (E, E(+^{j-1}))$. Then $(e(+^i))[F_E] \underset{\text{by (B1*)}}{=} (e[F_E])(+^i)$.

Case $e(+^i) = E$. Then $i = j-1$, because $E \in \mathrm{Class}(j-1)$ implies $\forall l \in [1, j-2].E \in \mathrm{Lim}\,\mathrm{Class}(l)$. But we know $E = e(+^{j-1})$ implies $m(E) = E(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2$ and then, since the function Θ: $E(+^{j-1}) \cap \mathbb{E} \longrightarrow \varphi(+^{j-1}) \cap \mathbb{E}$ is an extension of G that satisfies 1, 2, 3, 4, 5, 6, 7 and 8 with respect to E and φ, we have
$m(E)[\Theta] = (E(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2)[\Theta] = (E[\Theta])(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2 =$
$\varphi(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2$. **(B4*)**.

On the other hand, we know
$E = \min\{s \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | J_1 < s \leqslant E \wedge m(s)[g(j-1, s, E)] \geqslant m(E)\} =$
$= \min\{s \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | J_1 < s \leqslant E \wedge$
$m(s)[g(j-1, s, E)] \geqslant E(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2\}$.

But $\delta := J_1(+^{j-1}) \in (J_1, E] \cap \mathrm{Class}(j-1)$ is such that $m(\delta) = \delta(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2$ and $m(\delta)[g(j-1, \delta, E)] = (\delta(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2)[g(j-1, \delta, E)] =$
$(\delta[g(j-1, \delta, E)])(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2 = E(+^{j-2})(+^{j-3})\ldots(+^2)(+^1)2$. From this follows $E = \delta = J_1(+^{j-1})$ and $J_1 = e$. **(B5*)**

Now, by definition,

$Q = \{s \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) | G(J_1) < s \wedge m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\} \underset{\text{by (B4*)}}{=}$
$= \{s \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) | G(J_1) < s \wedge$
$m(s)[g(j-1, s, \varphi)] \geqslant \varphi(+^{j-2})(+^{j-3})...(+^2)(+^1)2\}$.
But $\xi := G(J_1)(+^{j-1}) \in (G(J_1), \sigma(+^j)) \cap \mathrm{Class}(j-1)$ is such that
$m(\xi) = \xi(+^{j-2})(+^{j-3})...(+^2)(+^1)2$ and
$m(\xi)[g(j-1, \xi, \varphi)] = (\xi(+^{j-2})(+^{j-3})...(+^2)(+^1)2)[g(j-1, \xi, \varphi)] =$
$(\xi[g(j-1, \xi, \varphi)])(+^{j-2})(+^{j-3})...(+^2)(+^1)2 = \varphi(+^{j-2})(+^{j-3})...(+^2)(+^1)2$. From this follows $G(J_1)(+^{j-1}) = \xi = \min Q = \rho$. **(B6*)**

Finally, from (B5*) and (B6*) we have that
$e(+^{j-1})[F_E] = J_1(+^{j-1})[F_E] = E[F_E] = \rho = G(J_1)(+^{j-1}) =$, because F_E extends G,
$= F_E(J_1)(+^{j-1}) = (J_1[F_E])(+^{j-1}) = (e[F_E])(+^{j-1})$.

All our previous work shows Assertion9.

Assertion10: (F_E, U_E) satisfies viii of Claim1, that is:
viii*. If $j \geqslant 2$, then
$\forall i \in [2, j]. \forall e \in \mathrm{Class}(i) \cap [\varepsilon, E(+^{j-1})). \forall s \in (e, e(+^i)) \cap E(+^{j-1}) \cap \mathrm{Class}(i-1)$.
$f(i, e)(s) = \{s_1 > ... > s_k\} \Longleftrightarrow f(i, F_E(e))(F_E(s)) = \{F_E(s_1) > ... > F_E(s_k)\}$.

Proof of Assertion10:
Suppose $j \geqslant 2$, $i \in [2, j]$, $e \in \mathrm{Class}(i) \cap [\varepsilon, E(+^{j-1}))$ and $s \in (e, e(+^i)) \cap \mathrm{Class}(i-1)$.

(10a). Case $s \in [\varepsilon, E)$.
Then $e < E$ and there exists $a \in [\varepsilon, e] \cap \mathrm{Class}(j-1)$ such that $s \in (e, e(+^i)) \cap a(+^{j-1}) \cap \mathbb{E}$. This way, $f(i, e)(s) = \{s_1 > ... > s_k\} \underset{\text{because of Assertion1}}{\Longleftrightarrow}$
$f(i, G|_{a(+^{j-1}) \cap \mathbb{E}}(e))(G|_{a(+^{j-1}) \cap \mathbb{E}}(s)) = \{G|_{a(+^{j-1}) \cap \mathbb{E}}(s_1) > ... > G|_{a(+^{j-1}) \cap \mathbb{E}}(s_k)\}$
$\underset{\text{because } F_E|_E = G}{\Longleftrightarrow}$
$f(i, F_E(e))(F_E(s)) = \{F_E(s_1) > ... > F_E(s_k)\}$.

(10b). Case $s \in (E, E(+^{j-1}))$.
Then $E \leqslant e$ and because of (B1*),
$f(i, e)(s) = \{s_1 > ... > s_k\} \Longleftrightarrow f(i, F_E(e))(F_E(s)) = \{F_E(s_1) > ... > F_E(s_k)\}$.

(10c). Case $s = E$.
Then $i = j$, $e = \varepsilon$ and $f(j, \varepsilon)(E) = \{E = E_1 > ... > E_q\}$.
So we need to show $f(j, F_E(\varepsilon))(F_E(E)) = \{F_E(E_1) > ... > F_E(E_q)\}$, i.e.,
$f(j, \sigma)(\rho) = \{F_E(E_1) > ... > F_E(E_q)\}$. **(B7*)**

Subcase $q = 1$, i.e., $f(j, \varepsilon)(E) = \{E\}$.
We assure that $S(j, \sigma)(\rho) = \emptyset$.
Suppose the opposite.
Let $Z \in S(j, \sigma)(\rho) = \{s \in (\sigma, \sigma(+^{j-1})) \cap \mathrm{Class}(j-1) \cap \rho \mid m(s)[g(j-1, s, \rho)] \geqslant m(\rho)\}$. Then $m(Z)[g(j-1, Z, \varphi)] = m(Z)[g(j-1, \rho, \varphi) \circ g(j-1, Z, \rho)] = m(Z)[g(j-1, Z, \rho)][g(j-1, \rho, \varphi)] \geqslant m(\rho)[g(j-1, \rho, \varphi)] \geqslant m(E)[\Theta]$. This shows that
$Z \in \rho \cap \{s \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) | m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\} \underset{\text{by definition of } J_1}{=}$
$\rho \cap \{s \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) | G(J_1) < s \wedge m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\} = \rho \cap Q$.
This is a contradiction to the fact that $\rho = \min Q$. Therefore $S(j, \sigma)(\rho) = \emptyset$.
Finally, notice $S(j, \sigma)(\rho) = \emptyset$ means, by definition of the function $f(j, \sigma)$, that
$f(j, \sigma)(\rho) = \{\rho\} = \{F_E(E)\}$. This shows (B7*) for the case $q = 1$.

Subcase $q \geqslant 2$.

Note $f(j,\varepsilon)(E)=\{E=E_1>\ldots>E_q\}=\{E\}\cup f(j,\varepsilon)(E_2)$, where by definition of $S(j,\varepsilon)$ and of $f(j,\varepsilon)$, $E_2=\sup S(j,\varepsilon)(E)\in(\varepsilon,E)$ and $f(j,\varepsilon)(E_2)=\{E_2>\ldots>E_q\}$. **(B8*)**
Then, by previous case (10a), we have that $f(j,F_E(\varepsilon))(F_E(E_2))=f(j,G(\varepsilon))(G(E_2))=$
$f(j,\sigma)(G(E_2))=\{G(E_2)>\ldots>G(E_q)\}=\{F_E(E_2)>\ldots>F_E(E_q)\}$. **(B9*)**
But by definition $f(j,\ \sigma)(\rho)\ =\ \begin{cases}\{\rho\}\cup f(j,\sigma)(\sup S(j,\sigma)(\rho)) & \text{iff } S(j,\sigma)(\rho)\neq\emptyset\\ \{\rho\} & \text{otherwise}\end{cases}$, this way, by (B9*) follows that to prove (B7*) it is enough to prove $G(E_2)=\sup S(j,\sigma)(\rho)$. **(B10*)**

Proof of (B10*):
By (B8*), $E_2=\sup S(j,\varepsilon)(E)=\{s\in(\sigma,\sigma(+^{j-1}))\cap\mathrm{Class}(j-1)\cap E|m(s)[g(j-1,s,E)]\geqslant m(E)\}$.
So $m(E)\leqslant m(E_2)[g(j-1,E_2,E)]$, which implies

$$\begin{aligned}m(E)[\Theta]\leqslant m(E_2)[g(j-1,E_2,E)][\Theta]&=m(E_2)[\Theta\circ g(j-1,E_2,E)]\underset{\text{by Assertion2}}{=}\\&=m(E_2)[g(j-1,G(E_2),\varphi)\circ G|_{E_2(+^{j-1})\cap\mathbb{E}}]=\\&=m(E_2)[G|_{E_2(+^{j-1})\cap\mathbb{E}}][g(j-1,G(E_2),\varphi)]=\\&=m(G(E_2))[g(j-1,G(E_2),\varphi)].\qquad\textbf{(B11*)}\end{aligned}$$

On the other hand, $\rho=\min Q$ with

$$\begin{aligned}Q&=\{s\in(\sigma,\sigma(+^{j}))\cap\mathrm{Class}(j-1)|G(J_1)<s\wedge m(s)[g(j-1,s,\varphi)]\geqslant m(E)[\Theta]\}\underset{\text{by definition of }J_1}{=}\\&=\{s\in(\sigma,\sigma(+^{j}))\cap\mathrm{Class}(j-1)|G(E_2)<s\wedge m(s)[g(j-1,s,\varphi)]\geqslant m(E)[\Theta]\}.\end{aligned}$$

So clearly $\rho>G(E_2)$.

We assure $m(G(E_2))[g(j-1,G(E_2),\rho)]\geqslant m(\rho)$.
Suppose the opposite, that $m(G(E_2))[g(j-1,G(E_2),\rho)]<m(\rho)$. Then

$$\begin{aligned}m(G(E_2))[g(j-1,G(E_2),\varphi)]&=m(G(E_2))[g(j-1,\rho,\varphi)\circ g(j-1,G(E_2),\rho)]=\\&=m(G(E_2))[g(j-1,G(E_2),\rho)][g(j-1,\rho,\varphi)]<\\&<m(\rho)[g(j-1,\rho,\varphi)]=m(F_E(E))[g(j-1,\rho,\varphi)]=\\&=m(F_E(E))[g(j-1,\rho,\varphi)]=m(E)[F_E][g(j-1,\rho,\varphi)]=\\&=m(E)[g(j-1,\rho,\varphi)\circ F_E]\underset{\text{by Assertion 2}}{=}\\&=m(E)[\Theta\circ g(j-1,E,E)]=m(E)[\Theta].\end{aligned}$$

But this is a contradiction with (B11*). So $m(G(E_2))[g(j-1,G(E_2),\rho)]\geqslant m(\rho)$ and from all this work we conclude that

$$\begin{aligned}G(E_2)\in S(j,\sigma)(\rho)=\{s\in(\sigma,\sigma(+^{j-1}))\cap\mathrm{Class}(j-1)\cap\rho\ |\\ m(s)[g(j-1,s,\rho)]\geqslant m(\rho)\}.\qquad\textbf{(B12*)}\end{aligned}$$

Now, we will show that $\neg\exists Z\in(G(E_2),\rho).Z\in S(j,\sigma)(\rho)$. **(B13*)**
Suppose the opposite. Let $Z\in(G(E_2),\rho).Z\in S(j,\sigma)(\rho)$.
Then $G(E_2)<Z$ and $m(Z)[g(j-1,Z,\rho)]\geqslant m(\rho)$, which implies

$$\begin{aligned}m(Z)[g(j-1,Z,\varphi)]&=m(Z)[g(j-1,\rho,\varphi)\circ g(j-1,Z,\rho)]=\\&=m(Z)[g(j-1,Z,\rho)][g(j-1,\rho,\varphi)]\geqslant m(\rho)[g(j-1,\rho,\varphi)]\geqslant m(E)[\Theta].\end{aligned}$$

But then $Z\in Q\cap\rho$. Contradiction because $\rho=\min Q$. Hence (B13*) holds.

Finally, from (B12*) and B(13*) follows $G(E_2)=\sup S(j,\sigma)(\rho)$, i.e., we have proven (B10*).
This concludes the proof of Assertion10.

We continue with the proof of Claim1. Up to know we have shown that the pair (F_E,U_E) defined in (B1*) satisfies i, ii, iii, iv, v, vi, vii, viii and ix of Claim1. So it only remains to prove that such pair is unique.

Assertion11: Suppose that (F'_E,U'_E) is a pair such that $U'_E\in\mathrm{Class}(j-1)\cap[\sigma,\sigma(+^{j-1}))$ and $F'_E\colon E(+^{j-1})\cap\mathbb{E}\longrightarrow U'_E(+^{j-1})\cap\mathbb{E}$ is an extension of $p\colon\varepsilon\cap\mathbb{E}\longrightarrow\sigma\cap\mathbb{E}$ such that i, ii, iii, iv, v, vi, vii, viii, and ix of Claim1 hold. Then $F_E=F'_E$ and $U_E=U'_E$.

Proof of Assertion11:

Let $a \in A = [\varepsilon, \varepsilon(+^j)) \cap \text{Class}(j-1) \cap E$ be arbitrary. Then the pair $(F'_E|_{a(+^{j-1})}, F'_E(a))$ is such that $F'_E|_{a(+^{j-1})}$: $a(+^{j-1}) \cap \mathbb{E} \longrightarrow F'_E(a)(+^{j-1}) \cap \mathbb{E}$ is an extension of p: $\varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ such that i, ii, iii, iv, v, vi, vii, viii and ix of Claim1 hold with respect to a. Therefore, by our Side Induction Hypothesis $F'_E|_{a(+^{j-1})} = F_a = G|_{a(+^{j-1})}$ and $F'_E(a) = U_a = F_a(a) = G(a)$. Note that since this was done for arbitrary $a \in A$, it follows $F'_E|_E = G = F_E|_E$.

From the previous paragraph (and the fact that F'_E is an extension of p: $\varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ such that i, ii, iii, iv, v, vi, vii, viii and ix of Claim1 hold) follows that the function F'_E: $E(+^{j-1}) \cap \mathbb{E} \longrightarrow U'_E(+^{j-1}) \cap \mathbb{E}$ is an extension of G: $E \cap \mathbb{E} \longrightarrow F'_E(E) \cap \mathbb{E} = U'_E \cap \mathbb{E}$ which satisfies 1, 2, 3, 4, 5, 6, 7 and 8 with respect to E and U'_E (of course $E, U'_E \in \text{Class}(j-1)$); so, by our (IH), F'_E is the only one extension of G: $E \cap \mathbb{E} \longrightarrow U'_E \cap \mathbb{E}$ such that 1, 2, 3, 4, 5, 6, 7 and 8 hold with respect to E and U'_E. **(B14*)**

We now want to show that $F'_E(E) \in Q$. **(B15*)**

Consider $f(j, \varepsilon)(E) = \{E = E_1 > E_2 > \ldots > E_q\}$.

Then $f(j, \sigma)(F'_E(E)) = f(j, F'_E(\varepsilon))(F'_E(E)) = \{F'_E(E_1) > F'_E(E_2) > \ldots > F'_E(E_q)\} \underset{\text{by (B14*)}}{=}$

$= \{F'_E(E_1) > G(E_2) > \ldots > G(E_q)\}$, which implies, because of (1.3.6) of GenThmIH (for the case $j \leqslant n-1$) and because of corollary 5.2 (for the case $j = n$), that

$F'_E(E) = \min\{s \in (\sigma, \sigma(+^j)) \cap \text{Class}(j-1) \mid G(J_1) < s \leqslant F'_E(E) \wedge$
$m(s)[g(j-1, s, F'_E(E))] \geqslant m(F'_E(E))\}$. **(B16*)**

On the other hand, note

$$\begin{aligned} m(F'_E(E))[g(j-1, F'_E(E), \varphi)] &= m(E)[F'_E][g(j-1, F'_E(E), \varphi)] = \\ &= m(E)[g(j-1, F'_E(E), \varphi) \circ F'_E] \underset{\text{by (B14*)}}{} \underset{\text{and Assertion2}}{=} \\ &= m(E)[\Theta \circ g(j-1, E, E)] = m(E)[\Theta]. \qquad \textbf{(B17*)} \end{aligned}$$

This way, by (B16*) and (B17*) we have that

$F'_E(E) \in Q = \{s \in (\sigma, \sigma(+^j)) \cap \text{Class}(j-1) | G(J_1) < s \wedge m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\}$, that is, (B15*) holds.

Now we show $\forall s \in (\sigma, \sigma(+^j)) \cap \text{Class}(j-1). s < F'_E(E) \Longrightarrow s \notin Q$. **(B18*)**

Suppose $s \in (\sigma, \sigma(+^j)) \cap \text{Class}(j-1) \cap F'_E(E)$. If $s \leqslant G(J_1)$ then clearly $s \notin Q$. So suppose $G(J_1) < s$. Then by (B16*) $m(s)[g(j-1, s, F'_E(E))] < m(F'_E(E))$, and so

$$\begin{aligned} m(s)[g(j-1, s, \varphi)] &= m(s)[g(j-1, F'_E(E), \varphi) \circ g(j-1, s, F'_E(E))] = \\ &= m(s)[g(j-1, s, F'_E(E))][g(j-1, F'_E(E), \varphi)] < \\ &< m(F'_E(E))[g(j-1, F'_E(E), \varphi)] \underset{\text{by (B17*)}}{=} m(E)[\Theta]. \end{aligned}$$

This way, $s \notin Q = \{s \in (\sigma, \sigma(+^j)) \cap \text{Class}(j-1) | G(J_1) < s \wedge m(s)[g(j-1, s, \varphi)] \geqslant m(E)[\Theta]\}$. This shows (B18*).

Finally, from (B15*) and (B18*) follows $U'_E = F'_E(E) = \min Q = \rho = U_E$. Note this means, by (B14*) and the definition of F_E, that F'_E and F_E are two extensions of G: $E \cap \mathbb{E} \longrightarrow \rho \cap \mathbb{E}$ such that 1, 2, 3, 4, 5, 6, 7 and 8 hold with respect to E and ρ; therefore, by our (IH), $F'_E = F_E$.

This concludes the proof of Assertion11.

This concludes the proof of Claim1.

Now we continue with the proof of the theorem.

Consider $D := [\varepsilon, \varepsilon(+^j)) \cap \text{Class}(j-1)$ and $F := \bigcup_{E \in D} F_E$, where for $E \in [\varepsilon, \varepsilon(+^j)) \cap \text{Class}(j-1)$, (F_E, U_E) is the pair obtained by Claim1.

Claim2. F: $\varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is a function

Proof of Claim2:

Notice $F \subset \mathrm{OR} \times \mathrm{OR}$ is a binary relation with
$\mathrm{Dom}\ F = \{a \in \mathrm{OR} | \exists b.(a, b) \in F\} = \bigcup_{E \in D} \mathrm{Dom}\ F_E = \bigcup_{E \in D} E(+^{j-1}) \cap \mathbb{E}) = \varepsilon(+^j) \cap \mathbb{E}$.
Moreover, $\mathrm{Range}\ F = \{b \in \mathrm{OR} | \exists a.(a, b) \in F\} = \bigcup_{E \in D} \mathrm{Range}\ F_E \subset (\bigcup_{S \in E} U_E(+^{j-1}) \cap \mathbb{E}) \subset \sigma(+^j) \cap \mathbb{E}$.
The previous shows F is a binary relation such that $F: \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$. Let's see now that F is a function. We just need to show that for any $a, b \in D$, F_a and F_b are compatible, that is, for any $x \in \mathrm{Dom}\, F_a \cap \mathrm{Dom}\, F_b$, $F_a(x) = F_b(x)$.

Let $a, b \in D$ and $x \in \mathrm{Dom}\, F_a \cap \mathrm{Dom}\, F_b$ be arbitrary. If $a = b$ then $F_a = F_b$ because by Claim1 the pair (F_a, U_a) is unique. So without loss of generality, suppose $a < b$. Notice then the following:

□1. $a(+^{j-1}) \leqslant b$ and so $a(+^{j-1}) \in b(+^{j-1}) \cap \mathbb{E} = \mathrm{Dom}\, F_b$.

□2. $F_b(a), F_b(a(+^{j-1})) \in \mathrm{Class}(j-1) \cap \mathrm{Im}\, F_b \subset \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^j))$.

□3. $F_b(a(+^{j-1})) = F_b(a)(+^{j-1})$.

□4. $\forall q \in a(+^{j-1}) \cap \mathbb{E}.F_b(q) \in \mathbb{E} \cap F_b(a(+^{j-1})) = \mathbb{E} \cap (F_b(a)(+^{j-1}))$. This holds by □3., because $\forall x \in b(+^{j-1}) \cap \mathbb{E}.\mathbb{E} \ni F_b(x) = x[F_b] = H_{F_b}(x)$ and because H_{F_b} is an $<$-isomorphism.

□5. By □4., $F_b|_{a(+^{j-1}) \cap \mathbb{E}}$: $a(+^{j-1}) \cap \mathbb{E} \longrightarrow F_b(a)(+^{j-1}) \cap \mathbb{E}$; moreover, it is clear this function is strictly increasing.

□6. $F_b|_{a(+^{j-1}) \cap \mathbb{E}}(\varepsilon) = \sigma$ and $F_b|_{a(+^{j-1}) \cap \mathbb{E}}(a) = F_b(a)$

□7. $F_b|_{a(+^{j-1}) \cap \mathbb{E}}|_{\varepsilon \cap \mathbb{E}} = F_b|_{\varepsilon \cap \mathbb{E}} = p$

□8. Because of Claim1,
$\forall i \in [1, j].\forall e \in [\varepsilon, a(+^{j-1})) \cap \mathbb{E}.e \in \mathrm{Class}(i) \Longleftrightarrow F_b|_{a(+^{j-1}) \cap \mathbb{E}}(e) \in \mathrm{Class}(i)$ and the function $R_{F_b|_{a(+^{j-1}) \cap \mathbb{E}}}: (e, a(+^{j-1})) \longrightarrow R_{F_b|_{a(+^{j-1}) \cap \mathbb{E}}}[(e, a(+^{j-1}))] \subset (\sigma, F_b(a)(+^{j-1}))$,
$t \longmapsto t[F_b|_{a(+^{j-1}) \cap \mathbb{E}}]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x, (+^1), ..., (+^{j-1}))$ isomorphism.

□9. If $j \geqslant 2$, then
$\forall i \in [2, j].\forall e \in \mathrm{Class}(i) \cap [\varepsilon, a(+^{j-1})).\forall s \in (e, e(+^i)) \cap a(+^{j-1}) \cap \mathrm{Class}(i-1)$.

$f(i, e)(s) = \{s_1 > ... > s_k\} \Longleftrightarrow f(i, F_b(e))(F_b(s)) = \{F_b(s_1) > ... > F_b(s_k)\}$

□10. If $j \geqslant 2.\forall i \in [2, j].\forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, a(+^{j-1})).F_b(\lambda(i, s)) = \lambda(i, F_b(s))$

So, for $a \in [\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1)$ and due to □1, □2, □3, □4, □5, □6, □7, □8, □9 and □10, the pairs (F_a, U_a) and $(F_b|_{a(+^{j-1}) \cap \mathbb{E}}, F_b(a))$ are two witnesses of i, ii, ii, iv, v, vi, vii, viii and ix of Claim1; therefore, by the uniqueness of such pairs, $F_a = F_b|_{a(+^{j-1}) \cap \mathbb{E}}$ and $U_a = F_b(a)$. From these equalities follows that F_a and F_b are compatible.

Hence $F: \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is a function.

This concludes the proof of Claim2.

We will make use of the following observation later:
$\forall e \in \varepsilon(+^j) \cap \mathbb{E}.\exists a \in \mathrm{Class}(j-1) \cap \varepsilon(+^j).e \in a(+^{j-1})$. **(C1*)**
Note (C1*) holds because $\mathrm{Class}(j-1)$ is confinal in $\varepsilon(+^j) \in \mathrm{Class}(j)$.

Claim3. $F: \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ satisfies 1, 2, 3, 4, 5, 6, 7 and 8.

Proof of Claim3:

1*. F is strictly increasing. This is easy: For any $e, l \in \varepsilon(+^j) \cap \mathbb{E}$, by (C1*) there exists $a \in \mathrm{Class}(j-1)$ such that $e, l \in a(+^{j-1}) \cap \mathbb{E}$. So $e < l \Longleftrightarrow F(e) = F_a(e) < F_b(l) = F(b)$.

2*. $F(\varepsilon) = F_\varepsilon(\varepsilon) = \sigma$.

3*. $F|_{\varepsilon \cap \mathbb{E}} = F_\varepsilon|_{\varepsilon \cap \mathbb{E}} = p$.

4*. The function $H_F: (\varepsilon, \varepsilon(+^j)) \longrightarrow H_F[(\varepsilon, \varepsilon(+^j))] \subset (\sigma, \sigma(+^j))$, $t \longmapsto t[F]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism. The proof of this fact is not hard: We know that H_F preserves $(<, +, \cdot, \lambda x.\omega^x)$ by propositions 3.12, 3.10 and 3.14. So we only have to see that H_F preserves the $<_1$ relation too. Let $x, y \in (\varepsilon, \varepsilon(+^j))$. Then, by (C1*), there exists $a \in \mathrm{Class}(j-1)$ such that $e, l \in (\varepsilon, a(+^{j-1}))$. So $x <_1 y \Longleftrightarrow x[F] = x[F_a] <_1 y[F_a] = x[F]$.

5*. $\forall i \in [1, j].\forall e \in [\varepsilon, \varepsilon(+^j)) \cap \mathbb{E}.e \in \mathrm{Class}(i) \Longleftrightarrow F(e) \in \mathrm{Class}(i)$. The proof is easy: Let $i \in [1, j]$, $e \in [\varepsilon, \varepsilon(+^j)) \cap \mathbb{E}$. By (C1*), let $a \in \mathrm{Class}(j-1)$ such that $e \in [\varepsilon, a(+^{j-1})) \cap \mathbb{E}$. Then $e \in \mathrm{Class}(i) \Longleftrightarrow F(e) = F_a(e) \in \mathrm{Class}(i)$.

6*. H_F is also an $(+^1), (+^2), \ldots, (+^{j-1})$ isomorphism. Proof: Let $i \in [1, j-1]$ and $e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(i)$. Then $e(+^i) \in (\varepsilon, \varepsilon(+^j))$ and by (C1*) there exists $a \in \mathrm{Class}(j-1)$ such that $e(+^i) \in (\varepsilon, a(+^{j-1}))$. This way, $F(e(+^i)) = F_a(e(+^i)) = F_a(e)(+^i)) = F(e)(+^i))$.

7*. If $j \geqslant 2$, then

$$\forall i \in [2, j].\forall e \in \mathrm{Class}(i) \cap [\varepsilon, \varepsilon(+^j)).\forall E \in (e, e(+^i)) \cap \mathrm{Class}(i-1).$$
$$f(i,e)(E) = \{E_1 > \ldots > E_q\} \Longleftrightarrow f(i, F(e))(F(E)) = \{F(E_1) > \ldots > F(E_q)\}.$$

Proof: Suppose $j \geqslant 2$, $i \in [2, j]$, $e \in \mathrm{Class}(i) \cap [\varepsilon, \varepsilon(+^j))$ and $E \in (e, e(+^i)) \cap \mathrm{Class}(i-1)$. Then, by (C1*), there exists $a \in \mathrm{Class}(j-1)$ such that $e < E \in a(+^{j-1}))$; that is, we have that $E \in (e, e(+^i)) \cap a(+^{j-1}) \cap \mathrm{Class}(i-1)$. Then

$$f(i,e)(E) = \{E_1 > \ldots > E_k\} \Longleftrightarrow f(i, F_a(e))(F_a(E)) = \{F_a(E_1) > \ldots > F_a(E_k)\} \Longleftrightarrow$$
$$\Longleftrightarrow f(i, F(e))(F(E)) = \{F(E_1) > \ldots > F(E_k)\}.$$

8*. If $j \geqslant 2$, then $\forall i \in [2, j].\forall s \in \mathrm{Class}(i-1) \cap [\varepsilon, \varepsilon(+^j)).F(\lambda(i, s)) = \lambda(i, F(s))$.

Proof: Suppose $j \geqslant 2$, $i \in [2, j]$ and $s \in \mathrm{Class}(i-1) \cap [\varepsilon, \varepsilon(+^j))$. Then, by (C1*), there exists $a \in \mathrm{Class}(j-1)$ such that $\lambda(i, s) \leqslant s \in [\varepsilon, a(+^{j-1}))$. Then $F(\lambda(i, s)) = F_a(\lambda(i, s)) = \lambda(i, F_a(s)) = \lambda(i, F(s))$.

This proves Claim3.

Claim4. The function $F\colon \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is the only one function satisfying 1, 2, 3, 4, 5, 6, 7 and 8.

Proof of Claim4.

Suppose $F'\colon \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ is a function satisfying 1, 2, 3, 4, 5, 6, 7 and 8. Let $e \in \varepsilon(+^j) \cap \mathbb{E}$. Then, by (C1*), there exists $a \in \mathrm{Class}(j-1)$ such that $e \in a(+^{j-1}) \cap \mathbb{E}$. Then note that the pair $(F'|_{a(+^{j-1}) \cap \mathbb{E}}, F'(a))$ is such that $F'(a) \in \mathrm{Class}(j-1) \cap [\sigma, \sigma(+^{j-1}))$, $F'|_{a(+^{j-1}) \cap \mathbb{E}}\colon a(+^{j-1}) \cap \mathbb{E} \longrightarrow F'(a)(+^{j-1}) \cap \mathbb{E}$ is an strictly increasing function such that i, ii, iii, iv, v, vi, vii, viii and ix of Claim1 hold with respect to a and $F'(a)$ and p. Therefore, by Claim1, $F'|_{a(+^{j-1}) \cap \mathbb{E}} = F_a$ and $F'(a) = U_a$, and so $F'(e) = F'|_{a(+^{j-1}) \cap \mathbb{E}}(e) = F_a(e) = F(e)$. Since this was shown for an arbitrary $e \in \varepsilon(+^j) \cap \mathbb{E}$, it follows $F' = F$.

So the theorem holds because of Claim1, Claim2, Claim3 and Claim4. □

5.3.2 The functions $g(n, \alpha, c)$

Definition 5.11. *Let $\alpha, c \in \mathrm{Class}(n)$ with $\alpha \leqslant c$. We define the function $g(n, \alpha, c)\colon \alpha(+^n) \cap \mathbb{E} \longrightarrow c(+^n) \cap \mathbb{E}$ as the only one function obtained by the application of theorem 5.10 to n, α, c and the identity function $\mathrm{Id}\colon \alpha \cap \mathbb{E} \longrightarrow c \cap \mathbb{E}$, $\mathrm{Id}(e) := e$. Moreover, since $g(n, \alpha, c)$ is injective, (because it is strictly increasing) we define $g(n, c, \alpha)$ as the inverse function of $g(n, \alpha, c)$, i.e., $g(n, c, \alpha) := g^{-1}(n, \alpha, c)$.*

Remark 5.12. For any $k \in [1, n]$ and any $\alpha, c \in \mathrm{Class}(k)$ with $\alpha \leqslant c$, by previous definition 5.11 and by GenThmIH, we have that $g(k, \alpha, c)\colon \alpha(+^k) \cap \mathbb{E} \longrightarrow \alpha \cap \mathbb{E} \cup ([c, c(+^k)) \cap \mathbb{E})$ satisfy

1. $g(k, \alpha, c)$ is strictly increasing.
2. $g(k, \alpha, c)(\alpha) = c$.
3. $g(k, \alpha, c)|_{\alpha \cap \mathbb{E}} = \mathrm{Id}_\alpha$, where $\mathrm{Id}_\alpha\colon \alpha \cap \mathbb{E} \longrightarrow c \cap \mathbb{E}$, $\mathrm{Id}_\alpha(e) := e$.

4. The function $H_{g(k,\alpha,c)}: (\alpha, \alpha(+^k)) \longrightarrow H_{g(k,\alpha,c)}[(\alpha, \alpha(+^k))] \subset (c, c(+^k)),\ t \longmapsto t[g(k,\alpha,c)]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism.
5. $\forall i \in [1,k]. \forall e \in [\alpha, \alpha(+^k)) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow g(k,\alpha,c)(e) \in \mathrm{Class}(i)$.
6. $H_{g(k,\alpha,c)}$ is also an $(+^1), (+^2), \ldots, (+^{j-1})$ isomorphism.
7. If $k \geqslant 2$, then $\forall i \in [2,k]. \forall e \in \mathrm{Class}(i) \cap [\alpha, \alpha(+^k)). \forall E \in (e, e(+^i)) \cap \mathrm{Class}(i-1)$.
 $f(i,e)(E) = \{E_1 > \ldots > E_q\} \Longleftrightarrow$
 $f(i, g(k,\alpha,c)(e))(g(k,\alpha,c)(E)) = \{g(k,\alpha,c)(E_1) > \ldots > g(k,\alpha,c)(E_q)\}$.
8. If $k \geqslant 2$, then $\forall i \in [2,k]. \forall s \in \mathrm{Class}(i-1) \cap [\alpha, \alpha(+^k))$.
 $g(k,\alpha,c)(\lambda(i,s)) = \lambda(i, g(k,\alpha,c)(s))$.

But by theorem 5.10 there exist only one such function. Therefore $g(k, \alpha, c)$ is the only one extension of the identity $\mathrm{Id}_\alpha: \alpha \cap \mathbb{E} \longrightarrow c \cap \mathbb{E}$, $\mathrm{Id}_\alpha(e) := e$ in interval $[\alpha, \alpha(+^k)) \cap \mathbb{E}$ provided by theorem 5.10. Besides, $g(n, c, \alpha)$ is the inverse of $g(n, \alpha, c)$.

5.4 The functions $\Phi(j, \varepsilon, \sigma, p)$

Notation 5.13. *Let $j \in [1, n]$ and $\varepsilon, \sigma \in \mathrm{Class}(j)$ be with $\varepsilon \leqslant \sigma$. Let $p: \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ be a strictly increasing function. We will denote as $\Phi(j, \varepsilon, \sigma, p)$ to the function $\Phi(j, \varepsilon, \sigma, p): \varepsilon(+^j) \cap \mathbb{E} \longrightarrow p[\varepsilon \cap \mathbb{E}] \cup ([\sigma, \sigma(+^j)) \cap \mathbb{E})$ obtained by theorem 5.10 applied to j, ε, σ and p.*

Remark 5.14. By the proof of theorem 5.10, for $j \in [1, n]$, the function $\Phi(j, \varepsilon, \sigma, p)$ is defined in the following recursive way:

Case $j = 1 \in [1, n]$.
For arbitrary $\varepsilon, \sigma \in \mathrm{Class}(1)$ with $\varepsilon < \sigma$ and $p: \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ a strictly increasing function,
$\Phi(1, \varepsilon, \sigma, p) := \begin{cases} e \longmapsto p(e) & \text{iff } \ e \in \varepsilon \cap \mathbb{E} \\ \varepsilon \longmapsto \sigma & \end{cases}$,
$\Phi(1, \sigma, \varepsilon, p) := (\Phi(1, \varepsilon, \sigma, p))^{-1}$.

Case $j + 1 \in [1, n]$.
By induction hypothesis $\Phi(j, \varepsilon', \sigma', p')$ and $\Phi(j, \sigma', \varepsilon', p')$ are already defined for arbitrary $\varepsilon', \sigma' \in \mathrm{Class}(j)$ with $\varepsilon' < \sigma'$ and $p': \varepsilon' \cap \mathbb{E} \longrightarrow \sigma' \cap \mathbb{E}$ a strictly increasing function.

Now, for any $\varepsilon, \sigma \in \mathrm{Class}(j+1)$ with $\varepsilon < \sigma$ and $p: \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ a strictly increasing function, $\Phi(j+1, \varepsilon, \sigma, p): \varepsilon(+^{j+1}) \cap \mathbb{E} \longrightarrow \sigma(+^{j+1}) \cap \mathbb{E}$ is given by a (side)-recursion on the well order $(\varepsilon(+^{j+1}) \cap \mathbb{E}, <)$ as:
$\Phi(j+1, \varepsilon, \sigma, p)(e) := p(e)$ if and only if $e \in \varepsilon \cap \mathbb{E}$;
$\Phi(j+1, \varepsilon, \sigma, p)(\varepsilon) := \sigma$;
$\Phi(j+1, \varepsilon, \sigma, p)(e) := \Phi(j, \xi, \Phi(j+1, \varepsilon, \sigma, p)(\xi), \Phi(j+1, \varepsilon, \sigma, p)|_\xi)(e)$ if and only if
$e \in (\xi, \xi(+^j)) \cap \mathbb{E} \wedge \xi \in [\varepsilon, \varepsilon(+^{j+1})) \cap \mathrm{Class}(j)$;
$\Phi(j+1, \varepsilon, \sigma, p)(\xi) := \min\{\delta \in (\sigma, \sigma(+^{j+1})) \cap \mathrm{Class}(j) \mid \Phi(j+1, \varepsilon, \sigma, p)(J) < \delta \wedge$
$m(\delta)[g(j, \delta, \sigma(+^{j+1}))] \geqslant m(\xi)[\Phi(j, \xi, \sigma(+^{j+1}), \Phi(j+1, \varepsilon, \sigma, p)|_\xi)]\}$
where $\xi \in \mathrm{Class}(j) \cap (\varepsilon, \varepsilon(+^{j+1}))$, $f(j+1, \varepsilon)(\xi) = \{\xi = \xi_1 > \ldots > \xi_k\}$ and
$J := \begin{cases} \xi_2 & \text{iff } k \geqslant 2 \\ \varepsilon & \text{otherwise} \end{cases}$.

Proposition 5.15. *Let $j\in[1,n]$ and $\varepsilon,\sigma\in\mathrm{Class}(j)$ be with $\varepsilon\leqslant\sigma$. Let $p\colon\varepsilon\cap\mathbb{E}\longrightarrow\sigma\cap\mathbb{E}$ be strictly increasing. Suppose $a_1,a_2\in[0,\varepsilon(+^j)]\cap\mathrm{Class}(j-1)$ and $b_1,b_2\in\mathrm{Class}(j-1)$ are such that*

- $a_1<a_2\leqslant b_1<b_2$,
- $\Phi(j,\varepsilon,\sigma,p)|_{a_1}\colon a_1\cap\mathbb{E}\longrightarrow b_1\cap\mathbb{E}$ *and*
- $\Phi(j,\varepsilon,\sigma,p)|_{a_2}\colon a_2\cap\mathbb{E}\longrightarrow b_2\cap\mathbb{E}$

Then
$g(j-1,b_1,b_2)\circ\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})=\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})\circ g(j-1,a_1,a_2)$

Proof. Let $j,\varepsilon,\sigma,p,a_1,a_2,b_1$ and b_2 as stated. Let
$P_1:=g(j-1,b_1,b_2)\circ\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})$ and
$P_2:=\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})\circ g(j-1,a_1,a_2)$. Then
$P_1,P_2\colon a_1(+^{j-1})\cap\mathbb{E}\longrightarrow b_2(+^{j-1})\cap\mathbb{E}$ are functions satisfying:

1*. P_1 and P_2 are strictly increasing (they are composition of strictly increasing functions);

2*. $P_1(a_1)=g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(a_1))=g(j-1,b_1,b_2)(b_1)=b_2=$
$\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})(a_2)=\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})(g(j-1,a_1,a_2)(a_1))=$
$P_2(a_1)$;

3*. $\forall e\in a_1\cap\mathbb{E}.P_1(e)=g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(e))=$
$g(j-1,b_1,b_2)(\Phi(j,\varepsilon,\sigma,p)|_{a_1}(e))=\Phi(j,\varepsilon,\sigma,p)|_{a_1}(e)=\Phi(j,\varepsilon,\sigma,p)(e)$ and
$\forall e\in a_1\cap\mathbb{E}.P_2(e)=\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})(g(j-1,a_1,a_2)(e))=$
$=\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})(e)=\Phi(j,\varepsilon,\sigma,p)|_{a_2}(e)=\Phi(j,\varepsilon,\sigma,p)(e)$. That is,
$P_1|_{a_1\cap\mathbb{E}}=\Phi(j,\varepsilon,\sigma,p)|_{a_1}=P_2|_{a_1\cap\mathbb{E}}$.

4*. $H_{P_1}\colon(a_1,a_1(+^{j-1}))\longrightarrow H_{P_1}[(a_1,a_1(+^{j-1}))]\subset(b_2,b_2(+^{j-1}))$, $t\longmapsto t[P_1]$ is an $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism. This is because for $t\in(a_1,a_1(+^{j-1}))$,
$t[\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})]\in(b_1,b_1(+^{j-1}))$ and then
$t[\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})][g(j-1,b_1,b_2)]\in(b_2,b_2(+^{j-1}))$; but
$t[P_1]=t[g(j-1,b_1,b_2)\circ\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})]=$
$t[\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})][g(j-1,b_1,b_2)]$, that is,
$H_{P_1}=H_{g(j-1,b_1,b_2)}\circ H_{\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})}$ and since $H_{\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})}$ and $H_{g(j-1,b_1,b_2)}$ are $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphisms, then H_{P_1} is $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism.

Analogously, $H_{P_2}\colon(a_1,a_1(+^{j-1}))\longrightarrow H_{P_2}[(a_1,a_1(+^{j-1}))]\subset(b_2,b_2(+^{j-1}))$, $t\longmapsto t[P_2]$ is an $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism, because $H_{P_2}=H_{\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})}\circ H_{g(j-1,a_1,a_2)}$.

5*. $\forall i\in[1,j-1].\forall e\in[a_1,a_1(+^{j-1}))\cap\mathbb{E}.e\in\mathrm{Class}(i)\Longleftrightarrow P_1(e)\in\mathrm{Class}(i)\ni P_2(e)$.

6*. H_{P_1} and H_{P_2} are also $(+^1),\dots,(+^{j-2})$ isomorphisms, because
$H_{P_1}=H_{g(j-1,b_1,b_2)}\circ H_{\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})}$, $H_{P_2}=H_{\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})}\circ H_{g(j-1,a_1,a_2)}$, and $H_{g(j-1,b_1,b_2)}, H_{\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})}, H_{\Phi(j-1,a_2,b_2,\Phi(j,\varepsilon,\sigma,p)|_{a_2})}$ and $H_{g(j-1,a_1,a_2)}$ are $(+^1),\dots,(+^{j-2})$ isomorphisms.

7*. $\forall i\in[2,j-1].\forall s\in\mathrm{Class}(i)\cap[a_1,a_1(+^{j-1})).\forall Z\in(s,s(+^{i-1}))\cap\mathrm{Class}(i-1)$.
$f(i,s)(Z)=\{Z_1>\dots>Z_d\}\Longleftrightarrow$
$f(i,\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(s))(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z))=$
$\{\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z_1)>\dots>\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z_d)\}\Longleftrightarrow$
$f(i,g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(s)))(g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z)))=$
$\{g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z_d))>\dots>g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(Z_d))\}$
$\Longleftrightarrow$
$f(i,P_1(s))(P_1(Z))=\{P_1(Z_1)>\dots>P_1(Z_d)\}$.

Analogously, $f(i,s)(Z)=\{Z_1>\dots>Z_d\}\Longleftrightarrow f(i,P_2(s))(P_2(Z))=\{P_2(Z_1)>\dots>P_2(Z_d)\}$.

8*. $\forall i\in[2,j-1].\forall s\in\mathrm{Class}(i-1)\cap[a_1,a_1(+^{j-1}))$.
$P_1(\lambda(i,s))=g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(\lambda(i,s)))=$
$g(j-1,b_1,b_2)(\lambda(i,\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(s)))=$
$\lambda(i,g(j-1,b_1,b_2)(\Phi(j-1,a_1,b_1,\Phi(j,\varepsilon,\sigma,p)|_{a_1})(s)))=\lambda(i,P_1(s))$.

Analogously $\forall i\in[2,j-1].\forall s\in\mathrm{Class}(i-1)\cap[a_1,a_1(+^{j-1})).P_2(\lambda(i,s))=\lambda(i,P_2(s))$.

Now, by theorem 5.10, applied to $j-1$, $a_1, b_2 \in \mathrm{Class}(j-1)$ with $a_1 \leqslant b_2$ and $\Phi(j,\ \varepsilon,\ \sigma,\ p)|_{a_1}$: $a_1 \cap \mathbb{E} \longrightarrow b_2 \cap \mathbb{E}$, there is exactly one extension of $\Phi(j,\ \varepsilon,\ \sigma,\ p)|_{a_1}$ to $a_1(+^{j-1}) \cap \mathbb{E} \longrightarrow b_2(+^{j-1}) \cap \mathbb{E}$ satisfying 1*, 2*, 3*, 4*, 5*, 6*, 7* and 8*. Thus $P_1 = P_2$. □

Proposition 5.16. *Let $j \in [1,n]$ and $\varepsilon, a_2, \sigma \in \mathrm{Class}(j)$ be with $\varepsilon \leqslant a_2 \leqslant \sigma$. Then $g(j,\varepsilon,\sigma) = g(j,a_2,\sigma) \circ g(j,\varepsilon,a_2)$.*

Proof. Let $\mathrm{Id}{:}\, \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ be the identity function. Consider $P_1 := g(j,\varepsilon,\sigma)$, $P_2 := g(j,a_2,\sigma) \circ g(j,\varepsilon,a_2)$. Then $P_1, P_2{:}\, \varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ are functions satisfying:

1*. P_1 and P_2 are strictly increasing (they are composition of strictly increasing functions);
2*. $P_1(\varepsilon) = g(j,\varepsilon,\sigma)(\varepsilon) = \sigma = g(j,a_2,\sigma)(a_2) = g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(\varepsilon)) = P_2(a_1)$;
3*. $\forall e \in \varepsilon \cap \mathbb{E}. P_1(e) = g(j,\varepsilon,\sigma)(e) = e = g(j,a_2,\sigma)(e) = g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(e)) = P_2(e)$.
That is, $P_1|_{\varepsilon\cap\mathbb{E}} = \mathrm{Id} = P_2|_{\varepsilon\cap\mathbb{E}}$.
4*. We know $H_{P_1}{:}\, (\varepsilon, \varepsilon(+^j)) \longrightarrow H_{P_1}[(\varepsilon,\varepsilon(+^j))] \subset (\sigma,\sigma(+^j))$, $t \longmapsto t[P_1]$ is an $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism.
On the other hand, $H_{P_2}{:}\, (\varepsilon,\varepsilon(+^j)) \longrightarrow H_{P_2}[(\varepsilon,\varepsilon(+^j))] \subset (\sigma,\sigma(+^j))$ is also an $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism. This is the case because for $t \in (\varepsilon,\varepsilon(+^j))$, $t[g(j,\varepsilon,a_2)] \in (a_2, a_2(+^j))$ and then $t[g(j,\varepsilon,a_2)][g(j,a_2,\sigma)] \in (\sigma,\sigma(+^j))$; but $t[P_2] = t[g(j,a_2,\sigma) \circ g(j,\varepsilon,a_2)] = t[g(j,\varepsilon,a_2)][g(j,a_2,\sigma)]$, that is, $H_{P_2} = H_{g(j,a_2,\sigma)} \circ H_{g(j,\varepsilon,a_2)}$ and since $H_{g(j,\varepsilon,a_2)}$ and $H_{g(j,a_2,\sigma)}$ are $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphisms, then H_{P_1} is $(<,+,\cdot,<_1,\lambda x.\omega^x)$ isomorphism.
5*. $\forall i \in [1,j]. \forall e \in [\varepsilon,\varepsilon(+^j)) \cap \mathbb{E}. e \in \mathrm{Class}(i) \Longleftrightarrow P_1(e) \in \mathrm{Class}(i) \ni P_2(e)$.
6*. H_{P_1} and H_{P_2} are also $(+^1),\ldots,(+^{j-1})$ isomorphisms, because $H_{P_1} = H_{g(j,\varepsilon,\sigma)}$, and $H_{P_2} = H_{g(j,a_2,\sigma)} \circ H_{g(j,\varepsilon,a_2)}$, and we know $H_{g(j,\varepsilon,\sigma)}$, $H_{g(j,\varepsilon,a_2)}$ and $H_{g(j,a_2,\sigma)}$ are $(+^1),\ldots,(+^{j-1})$ isomorphisms.
7*. $\forall i \in [2,j]. \forall s \in \mathrm{Class}(i) \cap [\varepsilon,\varepsilon(+^j)). \forall Z \in (s, s(+^i)) \cap \mathrm{Class}(i-1)$.
$f(i,s)(Z) = \{Z_1 > \ldots > Z_d\} \Longleftrightarrow$
$f(i, g(j,\varepsilon,a_2)(s))(g(j,\varepsilon,a_2)(Z)) = \{g(j,\varepsilon,a_2)(Z_1) > \ldots > g(j,\varepsilon,a_2)(Z_d)\} \Longleftrightarrow$
$f(i, g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(s)))(g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(Z))) =$
$\{g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(Z_1)) > \ldots > g(j,a_2,\sigma)(g(j,\varepsilon,a_2)(Z_d))\} \Longleftrightarrow$
$f(i, P_2(s))(P_2(Z)) = \{P_2(Z_1) > \ldots > P_2(Z_d)\}$.
On the other hand, we know
$f(i,s)(Z) = \{Z_1 > \ldots > Z_d\} \Longleftrightarrow$
$f(i, g(j,\varepsilon,\sigma)(s))(g(j,\varepsilon,\sigma)(Z)) = \{g(j,\varepsilon,\sigma)(Z_1) > \ldots > g(j,\varepsilon,\sigma)(Z_d)\} \Longleftrightarrow$
$f(i, P_1(s))(P_1(Z)) = \{P_1(Z_1) > \ldots > P_1(Z_d)\}$.
8*. $\forall i \in [2,j]. \forall s \in \mathrm{Class}(i-1) \cap [\varepsilon,\varepsilon(+^j))$.
$P_2(\lambda(i,s)) = g(j,a_2,\sigma) \circ g(j,\varepsilon,a_2)(\lambda(i,s)) =$
$g(j,\sigma,a_2)(\lambda(i, g(j,\varepsilon,a_2)(s))) = \lambda(i, g(j,\sigma,a_2)(g(j,\varepsilon,a_2)(s))) = \lambda(i, P_2(s))$.
On the other hand, we know $P_1(\lambda(i,s)) = g(j,\varepsilon,\sigma)(\lambda(i,s)) = \lambda(i, g(j,\varepsilon,\sigma)(s)) = \lambda(i, P_1(s))$.

Hence, by theorem 5.10 applied to j, $\varepsilon, \sigma \in \mathrm{Class}(j)$ with $\varepsilon \leqslant \sigma$ and $\mathrm{Id}{:}\, \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$, there is exactly one extension of Id to $\varepsilon(+^j) \cap \mathbb{E} \longrightarrow \sigma(+^j) \cap \mathbb{E}$ satisfying 1*, 2*, 3*, 4*, 5*, 6*, 7* and 8*. Thus $P_1 = P_2$. □

Proposition 5.17. *Let $j \in [1,n]$ and $\varepsilon, a, \sigma \in \mathrm{Class}(j)$. Suppose $e \in (\mathrm{Dom}\, g(j,\varepsilon,\sigma)) \cap (\mathrm{Dom}\, g(j,\varepsilon,a))$ and $g(j,\varepsilon,a)(e) \in (\mathrm{Dom}\, g(j,a,\sigma))$. Then $g(j,\varepsilon,\sigma)(e) = (g(j,a,\sigma) \circ g(j,\varepsilon,a))(e)$.*

Proof. Take j, ε, a and σ as stated. Moreover, let $e \in (\mathrm{Dom}\, g(j,\varepsilon,\sigma)) \cap (\mathrm{Dom}\, g(j,\varepsilon,a))$ be such that $g(j,\varepsilon,a)(e) \in (\mathrm{Dom}\, g(j,a,\sigma))$.

Case $\varepsilon \leqslant \sigma$.

Subcase $\varepsilon \leqslant a \leqslant \sigma$. Then the result holds by proposition 5.16.

Subcase $a < \varepsilon \leqslant \sigma$. Then
$g(j, \varepsilon, \sigma)(e) \underset{\text{by definition of } g(j,a,\varepsilon)}{=} (g(j, \varepsilon, \sigma) \circ g(j, a, \varepsilon))(g(j, \varepsilon, a)(e)) \underset{\text{by proposition 5.16}}{=}$
$g(j, a, \sigma)(g(j, \varepsilon, a)(e)) = (g(j, a, \sigma) \circ g(j, \varepsilon, a))(e)$. So the claim also holds in this case.

Subcase $\varepsilon \leqslant \sigma < a$. Then
$g(j, \varepsilon, \sigma)(e) \underset{\text{by definition of } g(j,a,\sigma)}{=} g(j, a, \sigma)(g(j, \sigma, a)(g(j, \varepsilon, \sigma))(e))) =$
$g(j, a, \sigma)((g(j, \sigma, a) \circ g(j, \varepsilon, \sigma))(e)) \underset{\text{by proposition 5.16}}{=} g(j, a, \sigma)(g(j, \varepsilon, a)(e)) =$
$(g(j, a, \sigma) \circ g(j, \varepsilon, a))(e)$. Thus the claim holds in this case too.

Case $\sigma < \varepsilon$.

Subcase $\sigma \leqslant a \leqslant \varepsilon$.
By proposition 5.16 we have that $g(j, \sigma, \varepsilon) = g(j, a, \varepsilon) \circ g(j, \sigma, a)$; therefore:
$\mathrm{Dom}\, g^{-1}(j, \sigma, \varepsilon) \subset \mathrm{Dom}\, g^{-1}(j, a, \varepsilon)$,
$g^{-1}(j, a, \varepsilon)[\mathrm{Dom}\, g^{-1}(j, \sigma, \varepsilon)] \subset \mathrm{Im}\, g(j, \sigma, a) = \mathrm{Dom}\, g^{-1}(j, \sigma, a)$ and
$\forall s \in \mathrm{Dom}\, g^{-1}(j, \sigma, \varepsilon). g^{-1}(j, \sigma, \varepsilon)(s) = (g^{-1}(j, \sigma, a) \circ g^{-1}(j, a, \varepsilon))(s)$.
Note the latter is $\forall s \in \mathrm{Dom}\, g(j, \varepsilon, \sigma). g(j, \varepsilon, \sigma)(s) = g(j, a, \sigma) \circ g(j, \varepsilon, a)(s)$, which in particular means $g(j, \varepsilon, \sigma)(e) = (g(j, a, \sigma) \circ g(j, \varepsilon, a))(e)$. So the claim holds for this case.

Subcase $a < \sigma < \varepsilon$.
By proposition 5.16 we know that $g(j, a, \varepsilon) = g(j, \sigma, \varepsilon) \circ g(j, a, \sigma)$ and therefore (analogous as in subcase $\sigma \leqslant a \leqslant \varepsilon$),
$\mathrm{Dom}\, g(j, \varepsilon, a) \subset \mathrm{Dom}\, g(j, \varepsilon, \sigma)$,
$g(j, \varepsilon, \sigma)[\mathrm{Dom}\, g(j, \varepsilon, a)] \subset \mathrm{Dom}\, g(j, \sigma, a)$ and
$\forall s \in \mathrm{Dom}\, g(j, \varepsilon, a). g(j, \varepsilon, a)(s) = g(j, \sigma, a) \circ g(j, \varepsilon, \sigma)(s)$ **(*)**
This way,
$g(j, \varepsilon, \sigma))(e) \underset{\text{by definition of } g(j,a,\sigma)}{=} g(j, a, \sigma)((g(j, \sigma, a) \circ g(j, \varepsilon, \sigma))(e)) \underset{\text{by (*)}}{=}$
$g(j, a, \sigma)(g(j, \varepsilon, a)(e)) = (g(j, a, \sigma) \circ g(j, \varepsilon, a))(e)$. Thus the claim holds for this case.

Subcase $\sigma < \varepsilon < a$.
By proposition 5.16 we know that $g(j, \sigma, a) = g(j, \varepsilon, a) \circ g(j, \sigma, \varepsilon)$. Thus, in the same way we have done before,
$\mathrm{Dom}\, g(j, a, \sigma) \subset \mathrm{Dom}\, g(j, a, \varepsilon)$,
$g(j, a, \varepsilon)[\mathrm{Dom}\, g(j, a, \sigma)] \subset \mathrm{Dom}\, g(j, \varepsilon, \sigma)$ and
$\forall s \in \mathrm{Dom}\, g(j, a, \sigma). g(j, a, \sigma)(s) = g(j, \varepsilon, \sigma) \circ g(j, a, \varepsilon)(s)$ **(**)**
So
$g(j, \varepsilon, \sigma)(e) \underset{\text{by definition of } g(j,a,\varepsilon)}{=} (g(j, \varepsilon, \sigma) \circ g(j, a, \varepsilon))(g(j, \varepsilon, a)(e)) \underset{\text{by (**)}}{=}$
$g(j, a, \sigma)(g(j, \varepsilon, a)(e)) = (g(j, a, \sigma) \circ g(j, \varepsilon, a))(e)$. Thus the claim holds in this case too. □

Proposition 5.18. *Let $j \in [1, n]$ and $\varepsilon, \sigma \in \mathrm{Class}(j)$ be with $\varepsilon \leqslant \sigma$. Let $p: \varepsilon \cap \mathbb{E} \longrightarrow \sigma \cap \mathbb{E}$ be a strictly increasing function.*
Then, for any $t \in \varepsilon(+^j)$, $\Phi(j, \varepsilon, \sigma, p)[T(j, \varepsilon, t)] = T(j, \sigma, t[\Phi(j, \varepsilon, \sigma, p)])$.

Proof. In order to facilitate our notation, let's abbreviate $\Phi(j, \varepsilon, \sigma, p)$ as Φ. Take $t \in \varepsilon(+^j)$. Let $C := \{e \in \mathrm{OR} | \Phi(e) \in T(j, \sigma, t[\Phi])\}$.

To show $T(j,\varepsilon,t) \subset C$. **(1*)**

To obtain (1*) we show that C is closed under clauses a), b), c) and d) of the definition of $T(j,\varepsilon,t)$ (see remark 5.3).

- Note $\Phi[\mathrm{Ep}(t)] = \mathrm{Ep}(t[\Phi]) \subset T(j,\sigma,t[\Phi])$ by clause a) of the definition of $T(j,\sigma,t[\Phi])$. So $\mathrm{Ep}(t) \subset C$.
- Suppose $\xi \in C \cap (\varepsilon, \varepsilon(+^j)) \cap \mathbb{E}$. Then $\Phi(\xi) \in T(j,\sigma,t[\Phi])$. But by theorem 5.10, we know that $\Phi(\xi) \in (\sigma, \sigma(+^j)) \cap \mathbb{E}$, so, by clause b) of the definition of $T(j,\sigma,t[\Phi])$, we have that $T(j,\sigma,t[\Phi]) \supset \mathrm{Ep}(m(\Phi(\xi))) \underset{\text{theorem 5.10}}{=} \mathrm{Ep}(m(\xi)[\Phi]) = \Phi[\mathrm{Ep}(m(\xi))]$. Thus $\mathrm{Ep}(m(\xi)) \subset C$.
- Suppose $\xi \in C \cap (\varepsilon, \varepsilon(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$ for some $l \in [1,j)$. Then $\Phi(\xi) \in T(j,\sigma,t[\Phi])$; moreover, by theorem 5.10, we know that $\Phi(\xi) \in (\sigma, \sigma(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$. Therefore, by clause c) of the definition of $T(j,\sigma,t[\Phi])$, $\Phi(\lambda(l+1,\xi)) \underset{\text{theorem 5.10}}{=} \lambda(l+1,\Phi(\xi)) \in T(j,\sigma,t[\Phi])$. Thus $\lambda(l+1,\xi) \in C$.
- Suppose $\xi \in C \cap (\varepsilon, \varepsilon(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$ for some $l \in [1,j)$. Then, just as in the previous case, $\Phi(\xi) \in T(j,\sigma,t[\Phi]) \cap (\sigma, \sigma(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$ and then, by clause d) of the definition of $T(j,\sigma,t[\Phi])$, $T(j,\sigma,t[F]) \supset f(l+1, \lambda(l+1,\Phi(\xi)))(\Phi(\xi)) \underset{\text{by 8. of theorem 5.10}}{=} f(l+1, \Phi(\lambda(l+1,\xi))(\Phi(\xi)) \underset{\text{by 7. of theorem 5.10}}{=} \Phi[f(l+1,\lambda(l+1,\xi))(\xi)]$. Therefore $f(l+1,\lambda(l+1,\xi))(\xi) \subset C$.

The previous concludes the proof of (1*).

Note (1*) actually proves that $\Phi[T(j,\varepsilon,t)] \subset T(j,\sigma,t[\Phi]$. **(2*)**

So it only remains to show $\Phi[T(j,\varepsilon,t)] \supset T(j,\sigma,t[\Phi])$. **(3*)**

To prove (3*) we show that $\Phi[T(j,\varepsilon,t)]$ is closed under clauses a), b), c) and d) of the definition of $T(j,\sigma,t[\Phi])$.

- $\mathrm{Ep}(t[\Phi]) = \Phi[\mathrm{Ep}(t)] \subset \Phi[T(j,\varepsilon,t)]$, since $\mathrm{Ep}(t) \subset T(j,\varepsilon,t)$.
- Suppose $\xi \in \Phi[T(j,\varepsilon,t)] \cap (\sigma,\sigma(+^j)) \cap \mathbb{E}$. Then, by theorem 5.10, $\xi = \Phi(e)$ for some $e \in T(j,\varepsilon,t) \cap (\varepsilon, \varepsilon(+^j)) \cap \mathbb{E}$. So, by clause b) of $T(j,\varepsilon,t)$ definition, $\mathrm{Ep}(m(e)) \subset T(j,\varepsilon,t)$. Thus $\mathrm{Ep}(m(\xi)) = \mathrm{Ep}(m(\Phi(e))) \underset{\text{by theorem 5.10}}{=} \mathrm{Ep}(m(e)[\Phi]) = \Phi[\mathrm{Ep}(m(e))] \subset \Phi[T(j,\varepsilon,t)]$.
- Suppose $\xi \in \Phi[T(j,\varepsilon,t)] \cap (\sigma,\sigma(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$ for some $l \in [1,j)$. Then, by theorem 5.10, $\xi = \Phi(e)$ for some $e \in T(j,\varepsilon,t) \cap (\varepsilon,\varepsilon(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$. So, by clause c) of the definition of $T(j,\varepsilon,t)$, $\lambda(l+1,e) \in T(j,\varepsilon,t)$ and therefore $\lambda(l+1,\xi) = \lambda(l+1,\Phi(e)) \underset{\text{theorem 5.10}}{=} \Phi(\lambda(l+1,e)) \in \Phi[T(j,\varepsilon,t)]$.
- Suppose $\xi \in \Phi[T(j,\varepsilon,t)] \cap (\sigma,\sigma(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$ for some $l \in [1,j)$. Then, by theorem 5.10, $\xi = \Phi(e)$ for some $e \in T(j,\varepsilon,t) \cap (\varepsilon,\varepsilon(+^j)) \cap (\mathrm{Class}(l) \backslash \mathrm{Class}(l+1))$. This way, by clause d) of the definition of $T(j,\varepsilon,t)$, $T(j,\varepsilon,t) \supset f(l+1,\lambda(l+1,e))(e)$; this implies $\Phi[T(j,\varepsilon,t)] \supset \Phi[f(l+1,\lambda(l+1,e))(e)] \underset{\text{by 7. of theorem 5.10}}{=} f(l+1,\Phi(\lambda(l+1,e)))(\Phi(e)) \underset{\text{by 8. of theorem 5.10}}{=} f(l+1,\lambda(l+1,\Phi(e)))(\Phi(e)) = f(l+1,\lambda(l+1,\xi))(\xi)$.

The previous shows (3*). □

Proposition 5.19. *Let $j \in [1,n]$ and $\varepsilon, \sigma \in \mathrm{Class}(j)$ be with $\varepsilon \leqslant \sigma$. Then, for any $t \in \varepsilon(+^j)$, $T(j,\varepsilon,t) \cap \varepsilon = T(j,\sigma,t[g(j,\varepsilon,\sigma)]) \cap \sigma$.*

Proof. Consider j, ε, σ and t as stated.

To show $T(j,\varepsilon,t) \cap \varepsilon \subset T(j,\sigma,t[g(j,\varepsilon,\sigma)]) \cap \sigma$. **(1*)**

Let $e \in T(j,\varepsilon,t) \cap \varepsilon$. Then $e = g(j,\varepsilon,\sigma)(e) < g(j,\varepsilon,\sigma)(\varepsilon) = \sigma$ and $e = g(j,\varepsilon,\sigma)(e) \underset{\text{by proposition 5.18}}{\in} T(j,\sigma,t[g(j,\varepsilon,\sigma)]) \cap \sigma$. This shows $e \in T(j,\sigma,t[g(j,\varepsilon,\sigma)]) \cap \sigma$, and since this was done for arbitrary $e \in T(j,\varepsilon,t) \cap \varepsilon$, then we have shown **(1*)**.

To show $T(j,\varepsilon,t)\cap\varepsilon\supset T(j,\sigma,t[g(j,\varepsilon,\sigma)])\cap\sigma$. **(2*)**

Let $e\in T(j,\sigma,t[g(j,\varepsilon,\sigma)])\cap\sigma$. Then, by proposition 5.18, there exists $d\in T(j,\varepsilon,t)$ with $g(j,\varepsilon,\sigma)(d)=e$; but $d\not\geqslant\varepsilon$, otherwise $g(j,\varepsilon,\sigma)(d)\geqslant\sigma$. So $T(j,\varepsilon,t)\cap\varepsilon\ni d=g(j,\varepsilon,\sigma)(d)=e$. This shows $e\in T(j,\varepsilon,t)\cap\varepsilon$, and since this was done for arbitrary $e\in T(j,\sigma,t[g(j,\varepsilon,\sigma)])\cap\sigma$, then we have shown **(2*)**. □

Proposition 5.20. $\forall j,\varepsilon,\sigma,p$.
if $j\in[1,n]\wedge\varepsilon,\sigma\in\mathrm{Class}(j)\wedge\varepsilon<\sigma\wedge p{:}\,\varepsilon\cap\mathbb{E}\longrightarrow\sigma\cap\mathbb{E}\wedge p$ *is a strictly increasing function, then* $\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)=L(j,\varepsilon,\sigma,p)$*, where* $L(j,\varepsilon,\sigma,p):=\{s\in\sigma(+^j)\cap\mathbb{E}|T(j,\sigma,s)\cap\sigma\subset\mathrm{Im}\,p\}$.

Proof. By induction on $([1,n],<)$.

Case $j=1$.

Take $\varepsilon,\sigma\in\mathrm{Class}(1)$, $\varepsilon<\sigma$ and $p{:}\,\varepsilon\cap\mathbb{E}\longrightarrow\sigma\cap\mathbb{E}$ a strictly increasing function.
Then $\Phi(1,\varepsilon,\sigma,p){:}\,\varepsilon(+^1)\cap\mathbb{E}\longrightarrow p[\varepsilon\cap\mathbb{E}]\cup([\sigma,\sigma(+^1))\cap\mathbb{E})$ is the function

$$\Phi(1,\varepsilon,\sigma,p)(e):=\begin{cases}p(e) & \text{iff } e\in\varepsilon\cap\mathbb{E}\\ \sigma & \text{iff } e=\varepsilon\end{cases}\quad\text{and } \forall s\in\sigma(+^1)\cap\mathbb{E}.T(1,\sigma,s)=\{s\}.\qquad\textbf{(1*)}$$

Therefore $\mathrm{Im}\,\Phi(1,\varepsilon,\sigma,p)=\{\Phi(1,\varepsilon,\sigma,p)(e)\mid e\in\varepsilon(+^1)\cap\mathbb{E}\}\underset{\text{by (1*)}}{=}\{p(e)\mid e\in\varepsilon\cap\mathbb{E}\}\cup\{\sigma\}=$ $\{s\in\sigma(+^1)\cap\mathbb{E}\mid s\in\mathrm{Im}\,p\}\cup\{\sigma\}=\{s\in\sigma(+^1)\cap\mathbb{E}\mid\{s\}\subset\mathrm{Im}\,p\}\cup\{\sigma\}\underset{\text{by (1*)}}{=}$ $\{s\in\sigma(+^1)\cap\mathbb{E}|T(1,\sigma,s)\subset\mathrm{Im}\,p\}\cup\{\sigma\}\underset{\text{by (1*)}}{=}\{s\in\sigma(+^1)\cap\mathbb{E}|T(1,\sigma,s)\cap\sigma\subset\mathrm{Im}\,p\}=L(1,\varepsilon,\sigma,p)$.

This proves the proposition for the case $j=1$.

So let $j\in(1,n]$.
Suppose the assertion of the theorem holds for any $i\in[1,n]\cap j$. **(IH)**.

Take $\varepsilon,\sigma\in\mathrm{Class}(j)$ with $\varepsilon<\sigma$ and $p{:}\,\varepsilon\cap\mathbb{E}\longrightarrow\sigma\cap\mathbb{E}$ a strictly increasing function. Let $\Phi(j,\varepsilon,\sigma,p){:}\,\varepsilon(+^j)\cap\mathbb{E}\longrightarrow p[\varepsilon\cap\mathbb{E}]\cup([\sigma,\sigma(+^j))\cap\mathbb{E})$ be the function obtained by theorem 5.10 applied to j,ε,σ and p.

To show $\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)\subset L(j,\varepsilon,\sigma,p)$. **(D0)**

Let $s\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$ be arbitrary. Then there is $e\in\varepsilon(+^j)\cap\mathbb{E}$ with $\Phi(j,\varepsilon,\sigma,p)(e)=s$. Now take any $Z\in T(j,\sigma,s)\cap\sigma$. By proposition 5.18 there is $E\in T(j,\varepsilon,e)$ such that $\Phi(j,\varepsilon,\sigma,p)(E)=Z$. This means $\Phi(j,\varepsilon,\sigma,p)(E)=Z<\sigma=\Phi(j,\varepsilon,\sigma,p)(\varepsilon)$, which, by theorem 5.10 implies $E<\varepsilon$. But $\Phi(j,\varepsilon,\sigma,p)|_{\varepsilon\cap\mathbb{E}}=p$ by theorem 5.10, so $p(E)=\Phi(j,\varepsilon,\sigma,p)(E)=Z$. This shows that $Z\in\mathrm{Im}\,p$, and since this was done for arbitrary $Z\in T(j,\sigma,s)\cap\sigma$, it follows $T(j,\sigma,s)\cap\sigma\subset\mathrm{Im}\,p$. Moreover, since this was done for arbitrary $s\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$, then we have shown (D0).

To show $\Phi(j,\varepsilon,\sigma,p)\supset L(j,\varepsilon,\sigma,p)$.
We show $\forall S\in L(j,\varepsilon,\sigma,p).S\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. **(E0)**
by a side induction on the well order $(L(j,\varepsilon,\sigma,p),<)$.

Let $S\in L(j,\varepsilon,\sigma,p)=\{s\in\sigma(+^j)\cap\mathbb{E}|T(j,\sigma,s)\cap\sigma\subset\mathrm{Im}\,p\}$ and suppose $\forall e\in S\cap L(j,\varepsilon,\sigma,p).e\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. **(SIH)**

Case $S<\sigma$.
Then $\{S\}=T(j,\sigma,S)=T(j,\sigma,S)\cap\sigma\subset\mathrm{Im}\,p\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$, that is, $S\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$.

Case $S=\sigma$.

Then $\Phi(j,\varepsilon,\sigma,p)(\varepsilon)=\sigma$. So $S\in \mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$.

Case $S\in(\sigma,\sigma(+^j))$.

Subcase $S\in \mathrm{Class}(i)\backslash\mathrm{Class}(i+1)$ for some $i\in[1,j-2]$.

Let $q:=\lambda(j-1,S)\in[\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)$. Note $S\in(q,q(+^{j-1}))\subset[\sigma,\sigma(+^j))$ and $q\in T(j,\sigma,q)\underset{\text{by proposition 5.4}}{\subset}T(j,\sigma,S)$, which implies $T(j,\sigma,q)\cap\sigma\subset T(j,\sigma,S)\cap\sigma\subset \mathrm{Im}\,p$; that is, $q\in S\cap L(j,\varepsilon,\sigma,p)$. This way, by our (SIH), $q\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. Let $r:=\Phi^{-1}(j,\varepsilon,\sigma,p)(q)$. Note $\Phi(j,\varepsilon,\sigma,p)|_r: r\cap\mathbb{E}\longrightarrow q\cap\mathbb{E}$ and then, by our (IH) applied to $j-1,r,q$ and $\Phi(j,\varepsilon,\sigma,p)|_r$, we have that $\mathrm{Im}\,\Phi(j-1,r,q,\Phi(j,\varepsilon,\sigma,p)|_r)=L(j-1,r,q,\Phi(j,\varepsilon,\sigma,p)|_r)$. Therefore
$\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)|_{r(+^{j-1})}\underset{\text{by definition of }\Phi(j,\varepsilon,\sigma,p)}{=}$
$\mathrm{Im}\,\Phi(j-1,r,q,\Phi(j,\varepsilon,\sigma,p)|_r)=L(j-1,r,q,\Phi(j,\varepsilon,\sigma,p)|_r)=$
$\{\xi\in q(+^{j-1})\cap\mathbb{E}|T(j-1,q,\xi)\cap q\subset\Phi(j,\varepsilon,\sigma,p)|_r\}$.
Because of these equalities, to show that $S\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)|_{r(+^{j-1})}\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$ it is enough to show that $T(j-1,q,S)\cap q\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)|_r$. **(E1)**

We prove (E1):
Let $e\in T(j-1,q,S)\cap q$. Then
$e\in T(j-1,q,e)\underset{\text{by proposition 5.4}}{\subset}T(j-1,q,S)\underset{\text{by proposition 5.4}}{\subset}T(j,\sigma,S)$; this and proposition 5.4 imply $T(j,\sigma,e)\subset T(j,\sigma,S)$, and therefore $T(j,\sigma,e)\cap\sigma\subset T(j,\sigma,S)\cap\sigma\subset\mathrm{Im}\,p$. Moreover, since $e<q<S$, we have that $e\in S\cap L(j,\varepsilon,\sigma,p)$. Thus, using our (SIH),
$e\in q\cap\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)=\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)|_r$. Since this was done for arbitrary $e\in T(j-1,q,S)\cap q$, then we have shown (E1).
So (E0) holds for this subcase.

Subcase $S\in\mathrm{Class}(j-1)$.

Subsubcase $S\notin\mathrm{Lim}(\mathrm{Class}(j-1))$.
Then there exists $\mu\in S\cap[\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)$ with $\mu(+^{j-1})=S$.
Moreover, $m(S)=S(+^{j-2})(+^{j-3})...(+^2)(+^1)2$, which implies
$f(j,\sigma)(S)=\begin{cases}\{S\} & \text{iff } \mu=\sigma\\ \{S=S_1>\mu=S_2>S_3>...>S_d\}\text{ for some } d\in[2,\omega) & \text{otherwise}\end{cases}$; so $\mu\in S\cap T(j,\sigma,S)$.
From this follows $T(j,\sigma,\mu)\subset T(j,\sigma,S)$ and so $T(j,\sigma,\mu)\cap\sigma\subset T(j,\sigma,S)\cap\sigma\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. All this means $\mu\in S\cap L(j,\varepsilon,\sigma,p)$, which by our (SIH) implies $\mu\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. This way, let $z:=(\Phi^{-1}(j,\varepsilon,\sigma,p)(\mu)$. Then note that
$S=\mu(+^{j-1})=\Phi(j,\varepsilon,\sigma,p)(z)(+^{j-1})\underset{\text{theorem 5.10}}{=}\Phi(j,\varepsilon,\sigma,p)(z(+^{j-1}))\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$.

Subsubcase $S\in\mathrm{Lim}(\mathrm{Class}(j-1))$. **(E2)**
Let $\mu':=\max T(j,\sigma,S)\cap S\cap\mathrm{Class}(j-1)\geqslant\sigma$ and $\mu:=\mu'(+^{j-1})\underset{\text{by (E2)}}{<}S$. Note μ' is well defined because we know that $T(j,\sigma,S)$ is finite (by GenThmIH for the case $j\in[1,n-1]$, and by proposition 5.5 for the case $j=n$). But the fact that $\mu'\in T(j,\sigma,S)$, implies that
$T(j,\sigma,\mu')\cap\sigma\subset T(j,\sigma,S)\cap\sigma\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$, which, together with the fact that $\mu'<S$, means $\mu'\in S\cap L(j,\varepsilon,\sigma,p)$. Subsequently, by our (SIH),
$\mu'\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. Let $\boldsymbol{r':=\Phi^{-1}(j,\varepsilon,\sigma,p)(\mu')}$ and $\boldsymbol{r:=r'(+^{j-1})}$. Note
$\mu=(\Phi(j,\varepsilon,\sigma,p)(r'))(+^{j-1})=\Phi(j,\varepsilon,\sigma,p)(r'(+^{j-1}))=$
$\Phi(j,\varepsilon,\sigma,p)(r)\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. **(E3)**

On the other hand, by 2. of proposition 5.4,
$T(j-1,S,m(S))\cap S\subset T(j,\sigma,m(S))\cap S\underset{\text{by proposition 5.7}}{=}T(j,\sigma,S)\cap S\subset\mu$, and then by our (IH), $\mathrm{Ep}(m(S))\subset\mathrm{Im}\,\Phi(j-1,\mu,S,\mathrm{Identity}:\mu\cap\mathbb{E}\longrightarrow S\cap\mathbb{E})\underset{\text{by remark 5.12}}{=}\mathrm{Im}\,g(j-1,\mu,S)$. So $m(S)[g(j-1,S,\mu)]\in(\mu,\mu(+^{j-1}))$. **(E4)**

Assertion0. Let $D:=\{\delta\in\mathrm{OR}|\delta<\mu\Longrightarrow\delta\in T(j,\sigma,m(S))$ and
$\delta\geqslant\mu\Longrightarrow\delta\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])\}$.
Then $T(j,\sigma,m(S)[g(j-1,S,\mu)])\subset D$.

Proof of Assertion0:

We will show that D is closed under clauses a), b), c) and d) of the inductive definition of $T(j,\sigma,m(S)[g(j-1,S,\mu)])$.

- Let $e\in\mathrm{Ep}(m(S)[g(j-1,S,\mu)])$ be arbitrary. If $e<\mu$, then $e\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])\cap\mu\underset{\text{by proposition 5.19}}{=}T(j-1,S,m(S))\cap S\underset{\text{by prop. 5.4}}{\subset}T(j,\sigma,m(S))$; that is, $e\in D$ in this case. If $e\geqslant\mu$, then clearly $e\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])$; so $e\in D$ in this case too. Since the previous was done for arbitrary $e\in\mathrm{Ep}(m(S)[g(j-1,S,\mu)])$, then we have shown $\mathrm{Ep}(m(S)[g(j-1,S,\mu)])\subset D$.
- Take $\xi\in D\cap(\sigma,\sigma(+^j))\cap\mathbb{E}$.

If $\xi<\mu$, then $\xi\in T(j,\sigma,m(S))$ (because $\xi\in D$) and $m(\xi)<\mu$ (otherwise $\xi<\mu\leqslant m(\xi)$ would imply $\xi\in\mathrm{Class}(j)$, which is impossible). This way, $\mathrm{Ep}(m(\xi))\subset\mu$ and $\mathrm{Ep}(m(\xi))\subset T(j,\sigma,m(S))$. Thus $\mathrm{Ep}(m(\xi))\subset D$ in this case.

If $\xi\geqslant\mu$, then $\xi\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])$ (because $\xi\in D$). Let $e\in\mathrm{Ep}(m(\xi))$ be arbitrary. If $e<\mu$, then $e\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])\cap\mu\underset{\text{by proposition 5.19}}{=}$ $T(j-1,S,m(S))\cap S\underset{\text{by prop. 5.4}}{\subset}T(j,\sigma,m(S))$; that is, $e\in D$. If $e\geqslant\mu$, then simply $e\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])$, that is, $e\in D$ too. The previous shows $\mathrm{Ep}(m(\xi))\subset D$.
- Take $\xi\in D\cap(\sigma,\sigma(+^j))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))$ for some $l\in[1,j)$.

If $\xi<\mu$, then $\xi\in T(j,\sigma,m(S))$ and $\mu>\xi>\lambda(l+1,\xi)\in T(j,\sigma,m(S))$. So $\lambda(l+1,\xi)\in D$.

If $\xi\geqslant\mu$, then $\xi\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])\subset\mu(+^{j-1})$ and we have two subcases:

If $l=j-1$, then $\xi=\mu$ and $\lambda(l+1,\xi)=\sigma\in\mu\cap T(j,\sigma,m(S))$; that is, $\lambda(l+1,\xi)\in D$.

If $l\neq j-1$, then $\xi\in(\mu,\mu(+^{j-1}))$ and so $\mu\leqslant\lambda(l+1,\xi)\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])$; that is, $\lambda(l+1,\xi)\in D$ in this case too.
- Take $\xi\in D\cap(\sigma,\sigma(+^j))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))$ for some $l\in[1,j)$.

If $\xi<\mu$, then $\xi\in T(j,\sigma,m(S))$ and so $\mu>\xi+1\supset f(l+1,\lambda(l+1,\xi))(\xi)\subset T(j,\sigma,m(S))$. This shows $f(l+1,\lambda(l+1,\xi))(\xi)\subset D$ in this case.

If $\xi\geqslant\mu$, then $\xi\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])\subset\mu(+^{j-1})$ and we have two subcases:

If $l\neq j-1$, then $\xi\in(\mu,\mu(+^{j-1}))\cap(\mathrm{Class}(l)\backslash\mathrm{Class}(l+1))$. This way, $f(l+1,\lambda(l+1,\xi))(\xi)\subset T(j-1,\mu,m(S)[g(j-1,S,\mu)])$ and $\forall e\in f(l+1,\lambda(l+1,\xi))(\xi).e>\lambda(l+1,\xi)\geqslant\mu$. Therefore $f(l+1,\lambda(l+1,\xi))(\xi)\subset D$.

If $l=j-1$, then $\xi=\mu$ and $\lambda(l+1,\xi)=\sigma$. Moreover, note $\mu'=\sup S(j,\sigma)(\mu)=\sup\{e\in\mathrm{Class}(j-1)\cap(\sigma,\sigma(+^j))\cap\mu\,|\,m(e)[g(j-1,e,\mu)]\geqslant m(\mu)\}$; therefore, $f(l+1,\lambda(l+1,\xi))(\xi)=f(j,\sigma)(\mu)=\{\mu\}\cup f(j,\sigma)(\mu')$ **(E5)**. Now, let $e\in f(j,\sigma)(\mu)$ be arbitrary. If $e\geqslant\mu$, then $e\underset{\text{by (E5)}}{=}\mu\in T(j-1,\mu,m(S)[g(j-1,S,\mu)])$; that is, $e\in D$. If $e<\mu$, then $e\underset{\text{by (E5)}}{\in}f(j,\sigma)(\mu')\underset{\text{because }\mu'\in T(j,\sigma,S)}{\subset}T(j,\sigma,S)\underset{\text{by proposition 5.7}}{=}T(j,\sigma,m(S))$; that is, $e\in D$ too.

This concludes the proof of Assertion0.

Now, consider the following:
$T(j,\sigma,m(S)[g(j-1,S,\mu)])\cap\sigma\underset{\text{because }\sigma<\mu\text{ and Assertion0}}{\subset}T(j,\sigma,m(S))\cap\sigma\subset\mathrm{Im}\,p$ and $\mathrm{Ep}(m(S)[g(j-1,S,\mu)])\subset S$; this means $\mathrm{Ep}(m(S)[g(j-1,S,\mu)])\subset S\cap L(j,\varepsilon,\sigma,p)$, and therefore by our (SIH), $\mathrm{Ep}(m(S)[g(j-1,S,\mu)])\subset\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. **(E6)**
This way, we define:
$t:=m(S)[g(j-1,S,\mu)][\Phi^{-1}(j,\varepsilon,\sigma,p)]\in(r,r(+^{j-1}))$ and
$Z:=\min\{e\in(\varepsilon,\varepsilon(+^j))\cap\mathrm{Class}(j-1)|r\leqslant e\wedge m(e)\geqslant t[g(j-1,r,e)]\}$.

Claim1. $\emptyset \neq \{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1, r, e)]\}$ and therefore Z is well defined.

Proof of Claim1:

Since $t[g(j-1, r, \varepsilon(+^j))] \in (\varepsilon(+^j), \varepsilon(+^j)(+^{j-1}))$, then $\eta(j-1, \varepsilon(+^j), t[g(j-1, r, \varepsilon(+^j))]) < \varepsilon(+^j)(+^{j-1}) < m(\varepsilon(+^j))$; this way, by $\leqslant_1$-connectedness $\varepsilon <_1 \eta(j-1, \varepsilon(+^j), t[g(j-1, r, \varepsilon(+^j))]) + 1$. From this, (6) of GenThmIH and $\leqslant_1$-connectedness we get that $\varepsilon <^{j-1} t[g(j-1, r, \varepsilon(+^j))] + 1$, which in turn implies, by (4) of GenThmIH, the existence of a sequence $(\delta_i)_{i \in I} \subset \mathrm{Class}(j-1) \cap \varepsilon(+^j)$ such that $\delta_i \underset{\mathrm{cof}}{\hookrightarrow} \varepsilon(+^j)$, $\forall i \in I. T(j-1, \delta_i, t[g(j-1, r, \varepsilon(+^j))]) \cap \delta_i \subset \delta_i$ and $m(\delta_i) \geqslant t[g(j-1, r, \varepsilon(+^j))][g(j-1, \varepsilon(+^j), \delta_i)] \underset{\text{proposition 5.17}}{=} t[g(j-1, r, \delta_i)]$. So for some $i \in I$, $\delta_i \in \{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1, r, e)]\} \neq \emptyset$.

This shows Claim1.

Assertion1. $Z \in \mathrm{Lim}\,\mathrm{Class}(j-1)$

Proof of Assertion1:

Since $S \in \mathrm{Lim}\,\mathrm{Class}(j-1)$, then $m(S) \geqslant S(+^{j-2})...(+^1)2+1$. This way,
$m(Z) \underset{\text{by definition}}{\geqslant} t[g(j-1, r, Z)] = m(S)[g(j-1, S, \mu)][\Phi^{-1}(j, \varepsilon, \sigma, p)][g(j-1, r, Z)] \geqslant$
$(S(+^{j-2})...(+^1)2+1)[g(j-1, S, \mu)][\Phi^{-1}(j, \varepsilon, \sigma, p)][g(j-1, r, Z)] =$
$(\mu(+^{j-2})...(+^1)2+1)[\Phi^{-1}(j, \varepsilon, \sigma, p)][g(j-1, r, Z)] =$
$(r(+^{j-2})...(+^1)2+1)[g(j-1, r, Z)] = Z(+^{j-2})...(+^1)2+1$.

But $m(Z) \geqslant Z(+^{j-2})...(+^1)2+1$ implies, by proposition 3.2, that $Z \in \mathrm{Lim}\,\mathrm{Class}(j-1)$.

This proves Assertion1.

Assertion2. $(\forall e' \in (r, Z) \cap \mathrm{Class}(j-1). \Phi(j, \varepsilon, \sigma, p)(e') < S) \Longrightarrow \forall e \in Z \cap \mathbb{E}. \Phi(j, \varepsilon, \sigma, p)(e) < S$.

Proof of Assertion2:

Suppose $\forall e' \in (r, Z) \cap \mathrm{Class}(j-1). \Phi(j, \varepsilon, \sigma, p)(e') < S$. **(As0*)**

Let $e \in Z \cap \mathbb{E}$ be arbitrary. If $e \leqslant r$, then $\Phi(j, \varepsilon, \sigma, p)(e) \leqslant \Phi(j, \varepsilon, \sigma, p)(r) = \mu < S$. So suppose $e \in (r, Z) \cap \mathbb{E}$. Then $r \leqslant \lambda(j-1, e) \leqslant e < \lambda(j-1, e)(+^{j-1}) \underset{\text{by Assertion1}}{\in} Z$. Therefore
$\Phi(j, \varepsilon, \sigma, p)(e) < \Phi(j, \varepsilon, \sigma, p)(\lambda(j-1, e)(+^{j-1})) \underset{\text{by (As0*)}}{<} S$.

This proves Assertion2.

Assertion3. $\forall e \in Z \cap \mathbb{E}. \Phi(j, \varepsilon, \sigma, p)(e) \in S \cap \mathbb{E}$

Proof of Assertion3:

By Assertion2, it is enough to show that $\forall e' \in (r, Z) \cap \mathrm{Class}(j-1). \Phi(j, \varepsilon, \sigma, p)(e') < S$. We show the latter by contradiction:

Suppose $\exists e' \in (r, Z) \cap \mathrm{Class}(j-1). \Phi(j, \varepsilon, \sigma, p)(e') \geqslant S$. **(As1*)**

Let $e := \min\{e' \in (r, Z) \cap \mathrm{Class}(j-1) \mid \Phi(j, \varepsilon, \sigma, p)(e') \geqslant S\}$. Note that $e \in (r, Z)$ and $Z = \min\{d \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \mid r \leqslant d \wedge m(d) \geqslant t[g(j-1, r, d)]\}$ imply $m(e) < t[g(j-1, r, e)]$. Therefore:
$m(\Phi(j, \varepsilon, \sigma, p)(e)) = m(e)[\Phi(j, \varepsilon, \sigma, p)] < t[g(j-1, r, e)][\Phi(j, \varepsilon, \sigma, p)] =$
$t[g(j-1, r, e)][\Phi(j-1, e, \Phi(j, \varepsilon, \sigma, p)(e), \Phi(j, \varepsilon, \sigma, p)|_e)] =$
$t[\Phi(j-1, e, \Phi(j, \varepsilon, \sigma, p)(e), \Phi(j, \varepsilon, \sigma, p)|_e) \circ g(j-1, r, e)] =$, by proposition 5.15,
$t[g(j-1, \mu, \Phi(j, \varepsilon, \sigma, p)(e)) \circ \Phi(j-1, r, \mu, \Phi(j, \varepsilon, \sigma, p)|_r)] =$
$t[\Phi(j-1, r, \mu, \Phi(j, \varepsilon, \sigma, p)|_r)][g(j-1, \mu, \Phi(j, \varepsilon, \sigma, p)(e))] =$
$m(S)[g(j-1, S, \mu)][\Phi^{-1}(j, \varepsilon, \sigma, p)][\Phi(j-1, r, \mu, \Phi(j, \varepsilon, \sigma, p)|_r)][g(j-1, \mu, \Phi(j, \varepsilon, \sigma, p)(e))] =$
$m(S)[g(j-1, S, \mu)][g(j-1, \mu, \Phi(j, \varepsilon, \sigma, p)(e))] =$
$m(S)[g(j-1, S, \Phi(j, \varepsilon, \sigma, p)(e))]$. **(As2*)**

Now, note
$S \underset{\text{by (As2*)}}{\in} S(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(e)) = \{d \in (\sigma, \sigma(+^j)) \cap \mathrm{Class}(j-1) \cap \Phi(j,\varepsilon,\sigma,p)(e) \mid$
$m(d)[g(j-1, d, \Phi(j,\varepsilon,\sigma,p)(e))] \geqslant m(\Phi(j,\varepsilon,\sigma,p)(e))\}$;
therefore, by definition of $f(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(e))$,
$f(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(e)) = \{\Phi(j,\varepsilon,\sigma,p)(e) > d_2 > \ldots > d_u\}$, for some $u \in [2,\omega)$ and where
$d_2 = \sup S(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(e))) \underset{\text{because } S \in S(j,\sigma)\Phi(j,\varepsilon,\sigma,p)(e))}{\geqslant} S.$ **(As3*)**
But by theorem 5.10,
$f(j,\varepsilon)(e) = \{e > e_2 \ldots > e_x\} \Longleftrightarrow$
$f(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(e)) = \{\Phi(j,\varepsilon,\sigma,p)(e) > \Phi(j,\varepsilon,\sigma,p)(e_2) \ldots > \Phi(j,\varepsilon,\sigma,p)(e_x)\}$. So $x = u$ and there are $e_2, \ldots, e_u \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap e$ such that $\forall i \in [2,u].\Phi(j,\varepsilon,\sigma,p)(e_i) = d_i$.
Finally, note that $\Phi(j,\varepsilon,\sigma,p)(e_2) = d_2 \geqslant S$ and
$e_2 < e = \min\{e' \in (r,Z) \cap \mathrm{Class}(j-1) \mid \Phi(j,\varepsilon,\sigma,p)(e') \geqslant S\}$, imply that $e_2 \leqslant r$; so
$S \leqslant d_2 = \Phi(j,\varepsilon,\sigma,p)(e_2) \leqslant \Phi(j,\varepsilon,\sigma,p)(r) = \mu < S$. Contradiction.

Hence (As1*) does not hold and we have shown Assertion3.

Claim2. $m(Z) = t[g(j-1,r,Z)]$

Proof of Claim2:

Clearly $m(Z) \geqslant t[g(j-1,r,Z)]$. We assure $m(Z) \not> t[g(j-1,r,Z)]$. **(F0)**

Assume $m(Z) > t[g(j-1,r,Z)]$. **(F1)**

We assure $\eta(j-1,Z,t[g(j-1,r,Z)]) \not> t[g(j-1,r,Z)]$. **(F2)**

Assume the negation of (F2). So there exists $l \in (Z, t[g(j-1,r,Z)]$ with
$m(l) = \eta(j-1,Z,t[g(j-1,r,Z)]) > t[g(j-1,r,Z)]$; but by Assertion3
$\Phi(j,\varepsilon,\sigma,p)|_Z \colon Z \cap \mathbb{E} \longrightarrow S \cap \mathbb{E}$, so:
$m(\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l)) =$
$m(l)[\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)] >$
$t[g(j-1,r,Z)][\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)] =$, by prop. 5.15 and $g(j-1,Z,r) = g^{-1}(j-1,r,Z)$,
$t[g(j-1,r,Z)][g(j-1,\mu,S) \circ \Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r) \circ g(j-1,Z,r)] =$
$t[g(j-1,r,Z)][g(j-1,Z,r)][\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,S)] =$
$t[\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,S)] =$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j,\varepsilon,\sigma,p)][\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,S)] =$
$m(S)[g(j-1,S,\mu)][g(j-1,\mu,S)] = m(S)$ **(F3)**;
moreover,
$\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l) = l[\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)] <$
$t[g(j-1,r,Z)][\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)] \underset{\text{equalities in (F3)}}{=} m(S)$ **(F4)**.

Note (F3) and (F4) lead us to the $<$-inequalities
$S < \Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l) < m(S) < m(S)+1 \leqslant m(\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l))$,
which together with the $<_1$-inequalities
$S <_1 m(S)$ and $\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l) <_1 m(\Phi(j-1,Z,S,\Phi(j,\varepsilon,\sigma,p)|_Z)(l))$ and the use of $<_1$-connectedness and $<_1$-transitivity imply that $S <_1 m(S)+1$. Contradiction.

Therefore (F2) holds.

Now, from (F2) follows $\eta(j-1, Z, t[g(j-1, r, Z)]) = t[g(j-1, r, Z)]$, which subsequently implies, by (F1) and $\leqslant_1$-connectedness, $Z <_1 \eta(j-1, Z, t[g(j-1, r, Z)]) + 1$; by this and (6) and (4) of GenThmIH we get a sequence $(\xi_i)_{i \in I} \subset Z \cap \mathrm{Class}(j-1)$ such that $\xi_i \underset{\mathrm{cof}}{\hookrightarrow} Z$, $T(j-1, Z, t[g(j-1, r, Z)]) \cap Z \subset \xi_i$ and $\xi_i <_1 t[g(j-1, r, Z)][g(j-1, Z, \xi_i)]$. Therefore, there exists $i_0 \in I$ such that $r \leqslant \xi_{i_0} < Z$ and $\xi_{i_0} <_1 t[g(j-1, r, Z)][g(j-1, Z, \xi_{i_0})] = t[g(j-1, r, \xi_{i_0})] \leqslant m(\xi_{i_0})$. But this implies that $\xi_{i_0} \in Z \cap \{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1, r, e)]\}$ which is impossible, because $Z = \min\{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1, r, e)]\}$. Contradiction.

Thus (F0) holds and this concludes the proof of Claim2.

Claim3. If $S(j,\sigma)(S) \neq \emptyset$ then $\sup S(j,\sigma)(S) \in \mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$ and $\Phi^{-1}(j,\varepsilon,\sigma,p)(\sup S(j,\sigma)(S)) \in S(j,\varepsilon)(Z)$.

Proof of Claim3:

Suppose $S(j,\sigma)(S) \neq \emptyset$. Then $\sup S(j,\sigma)(S) \in S(j,\sigma)(S)$, $f(j,\sigma)(S) = \{S_1 > S_2 > \ldots > S_d\}$ for some $d \geqslant 2$ and $S_2 = \sup S(j,\sigma)(S)$. But $S_2 \in T(j,\sigma,S)$, so $T(j,\sigma,S_2) \cap \sigma \subset T(j,\sigma,S) \cap \sigma \subset \mathrm{Im}\, p$. Thus $S_2 \in S \cap L(j,\varepsilon,\sigma,p)$, which by our (SIH) implies $\sup S(j,\sigma)(S) = S_2 \in \mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$. **(H0)**.

On the other hand, let $\boldsymbol{Z_2 := \Phi^{-1}(j,\varepsilon,\sigma,p)(S_2)}$. By theorem 5.10, $f(j,\varepsilon)(Z_2) = \{Z_2 > Z_3 \ldots > Z_{d'}\}$ is such that $f(j,\sigma)(S_2) = f(j,\sigma)(\Phi(j,\varepsilon,\sigma,p)(Z_2)) = \{\Phi(j,\varepsilon,\sigma,p)(Z_2) > \ldots > \Phi(j,\varepsilon,\sigma,p)(Z_{d'})\}$. From this follows that $d = d'$, $\forall i \in [2,d]. Z_i \leqslant r' < r$ and $\forall i \in [2,d]. \Phi(j,\varepsilon,\sigma,p)(Z_i) = S_i < \mu$. **(H1)**

Now we show $Z_2 \in S(j,\varepsilon)(Z) = \{e \in (\varepsilon,\varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap Z | m(e) g(j-1,e,Z) \geqslant m(Z)\}$. By (H1) we know $Z_2 < r \leqslant Z$ and $Z_2 \in (\varepsilon,\varepsilon(+^j)) \cap \mathrm{Class}(j-1)$; so we only need to show that $m(Z_2)[g(j-1,Z_2,Z)] \geqslant m(Z)$. **(H4)**

Proof of (H4):

$m(Z_2)[g(j-1,Z_2,r)][\Phi(j,\varepsilon,\sigma,p)][g(j-1,\mu,S)] =$
$m(Z_2)[g(j-1,Z_2,r)][\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,S)] =$
$m(Z_2)[\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r) \circ g(j-1,Z_2,r)][g(j-1,\mu,S)] =$, by proposition 5.15,
$m(Z_2)[g(j-1,S_2,\mu) \circ \Phi(j-1,Z_2,S_2,\Phi(j,\varepsilon,\sigma,p)|_{Z_2})][g(j-1,\mu,S)] =$
$m(Z_2)[\Phi(j-1,Z_2,S_2,\Phi(j,\varepsilon,\sigma,p)|_{Z_2})][g(j-1,S_2,\mu)][g(j-1,\mu,S)] =$
$m(Z_2)[\Phi(j,\varepsilon,\sigma,p)][g(j-1,S_2,\mu)][g(j-1,\mu,S)] =$
$m(S_2)[g(j-1,S_2,\mu)][g(j-1,\mu,S)] \underset{\text{proposition 5.17}}{=}$
$m(S_2)[g(j-1,S_2,S)] \underset{\text{because } S_2 \in \sup S(j,\sigma)(S)}{\geqslant} m(S)$.

So, from the previous inequalities,
$m(Z_2)[g(j-1,Z_2,r)][\Phi(j,\varepsilon,\sigma,p)] \geqslant m(S)[g(j-1,S,\mu)]$, and then
$m(Z_2)[g(j-1,Z_2,r)] \geqslant m(S)[g(j-1,S,\mu)][\Phi^{-1}(j,\varepsilon,\sigma,p)] = t$; this implies
$m(Z_2)[g(j-1,Z_2,Z)] = m(Z_2)[g(j-1,Z_2,r)][g(j-1,r,Z)] \geqslant t[g(j-1,r,Z)] \underset{\text{by Claim2}}{=} m(Z)$.

So (H4) holds, and thus $Z_2 \in S(j,\varepsilon)(Z)$.

All the previous shows that Claim3 holds.

Claim4. If $S(j,\varepsilon)(Z) \neq \emptyset$ then $\Phi(j,\varepsilon,\sigma,p)(\sup S(j,\varepsilon)(S)) \in S(j,\sigma)(S)$.

Proof of Claim4:

Suppose $\emptyset \neq S(j,\varepsilon)(Z) = \{e \in (\varepsilon,\varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap Z | m(e) g(j-1,e,Z) \geqslant m(Z)\}$. Let $E := \sup S(j,\varepsilon)(Z) \in S(j,\varepsilon)(Z)$.
So $m(E)[g(j-1,E,Z)] \geqslant m(Z) = t[g(j-1,r,Z)]$ and $E < Z$. **(G1)**

We assure $E < r$. **(G2)**

Assume the opposite, that $E \geqslant r$. **(G3)**
Note $\mathrm{Ep}(t[g(j-1,r,Z)]) \subset \mathrm{Im}\, g(j-1,r,Z) \underset{\text{by (IH)}}{=}$
$\{\xi \in Z(+^{j-1}) \cap \mathbb{E} | T(j-1,Z,\xi) \cap Z \subset \mathrm{Im\, Identity}\colon r \cap \mathbb{E} \longrightarrow Z \cap \mathbb{E}\} =$
$\{\xi \in Z(+^{j-1}) \cap \mathbb{E} | T(j-1,Z,\xi) \cap Z \subset r\}$; therefore
$T(j-1,Z,t[g(j-1,r,Z)]) \cap Z \subset r \underset{\text{by G3}}{\subset} E$, and again by our (IH), this means
$\mathrm{Ep}(t[g(j-1,r,Z)]) \subset \mathrm{Im}\, g(j-1,E,Z) = \mathrm{Dom}\, g(j-1,Z,E)$. **(G4)**

Now, (G1) and (G4) imply:
$m(E) = m(E)[g(j-1,E,Z)][g(j-1,Z,E)] \geqslant$
$t[g(j-1,r,Z)][g(j-1,Z,E)] \underset{\text{proposition 5.17}}{=} t[g(j-1,r,E)]$. **(G5)**

This way, from (G1), (G3) and (G5) we have that
$E \in Z \cap \{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1,r,e)]\}$, which is impossible, since $Z = \min\{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) | r \leqslant e \wedge m(e) \geqslant t[g(j-1,r,e)]\}$. Contradiction.
So (G3) does not hold, that is, (G2) holds.

With the help of (G1) and (G2) we do now the following:
$m(E)[g(j-1,E,r)][g(j-1,r,Z)] \underset{\text{proposition 5.17}}{=} m(E)[g(j-1,E,Z)] \geqslant t[g(j-1,r,Z)]$, which implies
$m(E)[g(j-1,E,r)] \geqslant t$; this in turn implies
$m(E)[g(j-1,E,r)][\Phi(j,\varepsilon,\sigma,p)][g(j-1,\mu,S)] \geqslant$
$t[\Phi(j,\varepsilon,\sigma,p)][g(j-1,\mu,S)] = m(S)$. **(G6)**
But
$m(E)[g(j-1,E,r)][\Phi(j,\varepsilon,\sigma,p)][g(j-1,\mu,S)] =$
$m(E)[g(j-1,E,r)][\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,S)] =$
$m(E)[\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r) \circ g(j-1,E,r)][g(j-1,\mu,S)] \underset{\text{proposition 5.15}}{=}$
$m(E)[g(j-1,\Phi(j,\varepsilon,\sigma,p)(E),\mu) \circ \Phi(j-1,E,\Phi(j,\varepsilon,\sigma,p)(E),\Phi(j,\varepsilon,\sigma,p)|_E)][g(j-1,\mu,S)] =$
$m(E)[\Phi(j-1,E,\Phi(j,\varepsilon,\sigma,p)(E),\Phi(j,\varepsilon,\sigma,p)|_E)][g(j-1,\Phi(j,\varepsilon,\sigma,p)(E),\mu)][g(j-1,\mu,S)] =$
$m(\Phi(j,\varepsilon,\sigma,p)(E))[g(j-1,\Phi(j,\varepsilon,\sigma,p)(E),\mu)][g(j-1,\mu,S)] \underset{\text{proposition 5.17}}{=}$
$m(\Phi(j,\varepsilon,\sigma,p)(E))[g(j-1,\Phi(j,\varepsilon,\sigma,p)(E),S)]$. **(G7)**

Notice (G6) and (G7) together show $m(\Phi(j,\varepsilon,\sigma,p)(E))[g(j-1,\Phi(j,\varepsilon,\sigma,p)(E),S)] \geqslant m(S)$, and since by (G2) $E < r$, then $\Phi(j,\varepsilon,\sigma,p)(E) < \Phi(j,\varepsilon,\sigma,p)(r) = \mu < S$. Therefore
$\Phi(j,\varepsilon,\sigma,p)(E) \in \{e \in (\varepsilon, \varepsilon(+^j)) \cap \mathrm{Class}(j-1) \cap S | m(e) g(j-1,e,Z) \geqslant m(Z)\} = S(j,\sigma)(S)$.
The previous concludes the proof of Claim4.

Claim5. $S(j,\sigma)(S) \neq \emptyset \Longleftrightarrow S(j,\varepsilon)(Z) \neq \emptyset$. Moreover, if $S(j,\varepsilon)(Z) \neq \emptyset$ then
$\Phi(j,\varepsilon,\sigma,p)(\sup S(j,\varepsilon)(Z)) = \sup S(j,\sigma)(S)$.

Proof of Claim5:
$S(j,\sigma)(S) \neq \emptyset \Longleftrightarrow S(j,\varepsilon)(Z) \neq \emptyset$ is now very easy to prove:
$S(j,\sigma)(S) \neq \emptyset \Longleftrightarrow \sup S(j,\sigma)(S) \in S(j,\sigma)(S) \underset{\text{Claim 3}}{\Longrightarrow} \Phi^{-1}(\sup S(j,\sigma)(S)) \in S(j,\varepsilon)(Z) \neq \emptyset$.
$S(j,\varepsilon)(Z) \neq \emptyset \Longleftrightarrow \sup S(j,\varepsilon)(Z) \in S(j,\varepsilon)(Z) \underset{\text{Claim 4}}{\Longrightarrow} \Phi(\sup S(j,\varepsilon)(Z)) \in S(j,\sigma)(S) \neq \emptyset$.

Let's show $S(j,\varepsilon)(Z) \neq \emptyset \Longrightarrow \Phi(j,\varepsilon,\sigma,p)(\sup S(j,\varepsilon)(Z)) = \sup S(j,\sigma)(S)$.
Suppose $S(j,\varepsilon)(Z) \neq \emptyset$. Then $S(j,\sigma)(S) \neq \emptyset$ as we just proved. Now, by Claim3
$\Phi^{-1}(j,\varepsilon,\sigma,p)(\sup S(j,\sigma)(S)) \in S(j,\varepsilon)(Z)$ and so
$\Phi^{-1}(j,\varepsilon,\sigma,p)(\sup S(j,\sigma)(S)) \leqslant \sup S(j,\varepsilon)(Z) \in S(j,\varepsilon)(Z)$; this implies
$\sup S(j,\sigma)(S) \leqslant \Phi(j,\sigma,\varepsilon,p)(\sup S(j,\varepsilon)(Z))$. **(G8)**
On the other hand, by Claim4, $\Phi(j,\varepsilon,\sigma,p)(\sup S(j,\varepsilon)(Z)) \in S(j,\sigma)(S)$, which implies
$\Phi(j,\varepsilon,\sigma,p)(\sup S(j,\varepsilon)(Z)) \leqslant \sup S(j,\sigma)(S)$. From this last inequality and (G8) we get
$\sup S(j,\sigma)(S) = \Phi(j,\sigma,\varepsilon,p)(\sup S(j,\varepsilon)(Z))$.
This concludes the proof of Claim5.

Claim6. $\Phi(j,\varepsilon,\sigma,p)(Z)=S$.

Proof of Claim6:

Consider $f(j,\sigma)(S)=\{S=S_1>S_2>\ldots>S_d\}$. By Claim5 (in case $d\geqslant 2$), $\Phi^{-1}(j,\varepsilon,\sigma,p)(S_2)=\sup S(j,\varepsilon)(Z)$, so $f(j,\varepsilon)(Z)=\{Z\}\cup f(j,\varepsilon)(\Phi^{-1}(j,\varepsilon,\sigma,p)(S_2))\underset{\text{Claim3 and (H1)}}{=}\{Z=Z_1>Z_2>\ldots>Z_d\}$, where by definition $Z_2=\Phi^{-1}(j,\varepsilon,\sigma,p)(S_2)$ and, as shown in (H1), it holds $f(j,\varepsilon)(Z_2)=\{Z_2>\ldots>Z_d\}$ and $\forall i\in[2,d].\Phi(j,\varepsilon,\sigma,p)(Z_i)=S_i$. **(N0)**

Let

$$J_2:=\begin{cases} S_2=\sup S(j,\sigma)(S) & \text{iff } \ d\geqslant 2 \ \ (\Longleftrightarrow S(j,\sigma)(S)\neq\emptyset)\\ \sigma & \text{otherwise}\end{cases}$$

and

$$J_1:=\begin{cases} Z_2=\sup S(j,Z) & \text{iff } \ d\geqslant 2 \ \ (\Longleftrightarrow S(j,\varepsilon)(Z)\neq\emptyset)\\ \varepsilon & \text{otherwise}\end{cases}.$$

Note $J_2\leqslant\mu'<\mu$, $J_1\leqslant r'<r$ and by (N0), $J_1=\Phi^{-1}(j,\varepsilon,\sigma,p)(J_2)$. **(N2)**

Besides, by the proof of theorem 5.10, $\Phi(j,\varepsilon,\sigma,p)(Z)=\min Q$, where $\varphi:=\sigma(+^j)$ and

$$Q:=\{\xi\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)\mid \Phi(j,\varepsilon,\sigma,p)|_Z(J_1)<\xi\wedge m(\xi)[g(j-1,\xi,\varphi)]\geqslant m(Z)[\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]\}=$$
$$\underset{\text{by (N2)}}{=}\{\xi\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)\mid J_2<\xi\wedge m(\xi)[g(j-1,\xi,\varphi)]\geqslant m(Z)[\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]\}.$$

Now, observe the following:

$m(S)[g(j-1,S,\varphi)]=m(S)[g(j-1,S,\mu)][g(j-1,\mu,S)][g(j-1,S,\varphi)]=$
$m(S)[g(j-1,S,\mu)][g(j-1,S,\varphi)\circ g(j-1,\mu,S)]\underset{\text{proposition 5.17}}{=}$
$m(S)[g(j-1,S,\mu)][g(j-1,\mu,\varphi)]\underset{\text{using (E6)}}{=}$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j,\varepsilon,\sigma,p)][\Phi(j,\varepsilon,\sigma,p)][g(j-1,\mu,\varphi)]=$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,\varphi)]=$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,\mu,\varphi)\circ\Phi(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)]=$
$=$, by proposition 5.15,
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)\circ g(j-1,r,Z)]=$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j-1,r,\mu,\Phi(j,\varepsilon,\sigma,p)|_r)][g(j-1,r,Z)][\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]=$
$m(S)[g(j-1,S,\mu)][\Phi^{-1}(j,\varepsilon,\sigma,p)][g(j-1,r,Z)][\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]=$
$t[g(j-1,r,Z)][\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]\underset{\text{Claim2}}{=}$
$m(Z)[\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]$. **(N3)**

Notice from (N3) and the fact that $J_2<S$ follows $S\in Q$. **(N4)**

Now let's see that $S=\min Q$.

Take $x\in(J_2,S)$. Then $m(x)[g(j-1,x,S)]<m(S)$ (because in case $S(j,\sigma)(S)\neq\emptyset$, $J_2=\sup S(j,\sigma)(S)=\sup\{e\in(\sigma,\sigma(+^j))\cap\mathrm{Class}(j-1)\cap S|m(e)g(j-1,e,S)\geqslant m(S)\}$). Note $m(x)[g(j-1,x,S)]<m(S)$ implies
$m(x)[g(j-1,x,\varphi)]\underset{\text{proposition 5.17}}{=}m(x)[g(j-1,x,S)][g(j-1,S,\varphi)]<$
$m(S)[g(j-1,S,\varphi)]\underset{\text{by (N3)}}{=}m(Z)[\Phi(j-1,Z,\varphi,\Phi(j,\varepsilon,\sigma,p)|_Z)]$.
So $x\notin Q$. **(N5)**

Finally, from (N4) and (N5) we conclude $S=\min Q=\Phi(j,\varepsilon,\sigma,p)(Z)$.
This concludes the proof of Claim6.

All our previous work shows that $S\in\mathrm{Im}\,\Phi(j,\varepsilon,\sigma,p)$, i.e., we have shown (E0).

This concludes the proof of the whole proposition. □

5.5 Clause (2.2) of theorem 3.26

Corollary 5.21. *Clauses (2.2), (2.2.1), (2.2.2), (2.2.3) and (2.2.4) of theorem 3.26 also hold for n, that is,*

(2.2) For any $\alpha, c \in \mathrm{Class}(n)$ there exist a function

$g(n,\alpha,c)\colon \mathrm{Dom}\, g(n,\alpha,c) \subset \mathbb{E} \cap \alpha(+^n) \longrightarrow \mathbb{E} \cap c(+^n)$ *such that*

(2.2.1) $g(n,\alpha,c)|_{c\cap\alpha\cap(\mathrm{Dom}\, g(n,\alpha,c))}$ and $g(n,\alpha,\alpha)$ are the identity functions in their respective domain.

(2.2.2) $g(n,\alpha,c)$ is strictly increasing.

(2.2.3) $\forall t \in \alpha(+^n).T(n,\alpha,t) \cap \alpha \subset c \Longleftrightarrow \mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n,\alpha,c)$

(2.2.4) $\forall t \in \alpha(+^n).\mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n,\alpha,c) \Longrightarrow T(n,c,t[g(n,\alpha,c)]) \cap c = T(n,\alpha,t) \cap \alpha$

Proof. Let $\alpha, c \in \mathrm{Class}(n)$. Definition 5.11 gives us a function $g(n,\alpha,c)\colon \mathrm{Dom}\, g(n,\alpha,c) \subset \mathbb{E} \cap \alpha(+^n) \longrightarrow \mathbb{E} \cap c(+^n)$ such that $g(n,\alpha,c)|_{c\cap\alpha\cap(\mathrm{Dom}\, g(n,\alpha,c))}$ is the identity.

On the other hand, note that $\mathrm{Id}\colon \mathbb{E} \cap \alpha(+^n) \longrightarrow \mathbb{E} \cap \alpha(+^n)$, $\mathrm{Id}(e) = e$ is such that

1. Id is strictly increasing.
2. $\mathrm{Id}(\alpha) = \alpha$.
3. $\mathrm{Id}|_{\alpha\cap\alpha} = \mathrm{Id}_\alpha$, where $\mathrm{Id}_\alpha\colon \alpha \cap \mathbb{E} \longrightarrow \alpha \cap \mathbb{E}$, $\mathrm{Id}_\alpha(e) := e$.
4. The function $H_{\mathrm{Id}}\colon (\alpha, \alpha(+^n)) \longrightarrow H_{\mathrm{Id}}[(\alpha, \alpha(+^n))] \subset (\alpha, \alpha(+^n))$, $t \longmapsto t[\mathrm{Id}]$ is an $(<, +, \cdot, <_1, \lambda x.\omega^x)$ isomorphism.
5. $\forall i \in [1,k].\forall e \in [\alpha, \alpha(+^k)) \cap \mathbb{E}.e \in \mathrm{Class}(i) \Longleftrightarrow \mathrm{Id}(e) \in \mathrm{Class}(i)$.
6. H_{Id} is also an $(+^1), (+^2), \ldots, (+^{j-1})$ isomorphism.
7. If $k \geqslant 2$, then $\forall i \in [2,k].\forall e \in \mathrm{Class}(i) \cap [\alpha, \alpha(+^k)).\forall E \in (e, e(+^i)) \cap \mathrm{Class}(i-1)$.
   $f(i,e)(E) = \{E_1 > \ldots > E_q\} \Longleftrightarrow f(i,\mathrm{Id}(e))(\mathrm{Id}(E)) = \{\mathrm{Id}(E_1) > \ldots > \mathrm{Id}(E_q)\}$.
8. If $k \geqslant 2$, then $\forall i \in [2,k].\forall s \in \mathrm{Class}(i-1) \cap [\alpha, \alpha(+^k))$.
   $\mathrm{Id}(\lambda(i,s)) = \lambda(i, \mathrm{Id}(s))$.

The previous shows that Id is the only one function obtained by the application of theorem 5.10 to n, α, α and the identity function $\mathrm{Id}_\alpha\colon \alpha \cap \mathbb{E} \longrightarrow \alpha \cap \mathbb{E}$, $\mathrm{Id}_\alpha(e) = e$; since by definition such function is $g(n,\alpha,\alpha)$, then we have that $\mathrm{Id} = g(n,\alpha,\alpha)$.

All the previous shows that (2.2) and (2.2.1) hold. Moreover, by theorem 5.10 we know that $g(n, \alpha, c)$ is strictly increasing, that is, (2.2.2) holds too. So it only remains to show (2.2.3), (2.2.4) and (2.2.5).

Proof of (2.2.3):

Case $\alpha \leqslant c$.
Let $t \in \alpha(+^n)$. Then $T(n,\alpha,t) \cap \alpha \subset \alpha \subset c$ and $\mathrm{Ep}(t) \subset \alpha(+^n) \cap \mathbb{E} = \mathrm{Dom}\, g(n,\alpha,c)$. So (2.2.3) holds in this case.

Case $\alpha > c$.
Let $\mathrm{Id}_c\colon c \cap \mathbb{E} \longrightarrow c \cap \mathbb{E}$ be the identity $\mathrm{Id}_c(e) := e$.
Let $t \in \alpha(+^n)$ be arbitrary.

We show that $T(n,\alpha,t)\cap\alpha\subset c\Longrightarrow \mathrm{Ep}(t)\subset \mathrm{Dom}\, g(n,\alpha,c)$. **(1*)**

Suppose $T(n,\alpha,t)\cap\alpha\subset c$. Consider $e\in\mathrm{Ep}(t)$. Then $T(n,\alpha,e)\cap\alpha \underset{\text{by 1. of proposition 5.4}}{\subset} T(n,\alpha,t)\cap\alpha\subset \mathrm{Im}\,\mathrm{Id}_c$ and then, by proposition 5.20, $e\in \mathrm{Im}\ g(n,c,\alpha)=\mathrm{Dom}\ g(n,\alpha,c)$. Since this was done for arbitrary $e\in\mathrm{Ep}(t)$, then we have shown that $\mathrm{Ep}(t)\subset\mathrm{Dom}\, g(n,\alpha,c)$. This proves (1*).

We show that $T(n,\alpha,t)\cap\alpha\subset c\Longleftarrow \mathrm{Ep}(t)\subset \mathrm{Dom}\, g(n,\alpha,c)$. **(2*)**

Suppose $\mathrm{Ep}(t)\subset\mathrm{Dom}\, g(n,\alpha,c)=\mathrm{Im}\, g(n,c,\alpha)$. Then, by proposition 5.20, $\mathrm{Ep}(t)\subset\{s\in\alpha(+^n)\cap\mathbb{E}|T(n,\alpha,s)\cap\alpha\subset\mathrm{Im}\,\mathrm{Id}_c\}$; therefore $T(n,\alpha,t)\cap\alpha=(\bigcup_{s\in\mathrm{Ep}(t)}T(n,\alpha,s))\cap\alpha=\bigcup_{s\in\mathrm{Ep}(t)}(T(n,\alpha,s)\cap\alpha)=\subset\mathrm{Im}\,\mathrm{Id}_c\subset c$. So (2*) holds.

This concludes the proof of (2.2.3).

Proof of (2.2.4):

Let $t\in\alpha(+^n)$ be arbitrary and suppose $\mathrm{Ep}(t)\subset\mathrm{Dom}\, g(n,\alpha,c)$.

Case $\alpha\leqslant c$.
Then $T(n,c,t[g(n,\alpha,c)])\cap c=T(n,\alpha,t)\cap\alpha$ by proposition 5.19.

Case $\alpha>c$.
Since $\mathrm{Ep}(t)\subset\mathrm{Dom}\, g(n,\alpha,c)$, then consider $t[g(n,\alpha,c)]\in c(+^n)$. Then $T(n,\alpha,t)\cap\alpha=T(n,\alpha,t[g(n,\alpha,c)][g(n,c,\alpha)])\cap\alpha \underset{\text{by proposition 5.19}}{=} T(n,c,t[g(n,\alpha,c)])\cap c$.

Since the previous was done for $t\in\alpha(+^n)$ arbitrary, then we have shown (2.2.4). □

Proposition 5.22. *Let $k\in[1,n]$, $\alpha\in\mathrm{Class}(k)$ and $t\in(\alpha(+^{k-1})(+^{k-2})...(+^1)2,\alpha(+^k)]$. Suppose $s\in(\alpha,t]$ is such that $t\leqslant m(s)$. Then $l(k,\alpha,t)\leqslant s$.*

Proof. Let k,α and t be as stated. Let $s\in(\alpha,t]$ be such that $t\leqslant m(s)$. We proceed by contradiction: Suppose $s<l(k,\alpha,t)$. **(*)**
Then $s<l(k,\alpha,t)\leqslant t\leqslant m(s)\leqslant\max\{m(e)|e\in(\alpha,t]\}=\eta(k,\alpha,t)=m(l(k,\alpha,t))$ and therefore, by $\leqslant_1$-connectedness,$s<_1 l(k,\alpha,t)\leqslant_1\eta(k,\alpha,t)$, which subsequently implies, by $\leqslant_1$-transitivity, $s\leqslant_1\eta(k,\alpha,t)$. So $m(s)\geqslant\eta(k,\alpha,t)\geqslant m(s)$, i.e., $m(s)=\eta(k,\alpha,t)$. This shows that $s\in l(k,\alpha,t)\cap\{e\in(\alpha,t]\mid m(e)=\eta(k,\alpha,t)\}$ which is impossible, since $l(k,\alpha,t)=\min\{e\in(\alpha,t]\mid m(e)=\eta(k,\alpha,t)\}$. Contradiction.
Thus $s\not<l(k,\alpha,t)$, that is, $l(k,\alpha,t)\leqslant s$. □

Proposition 5.23. *Let $k\in[1,n]$, $\alpha,c\in\mathrm{Class}(k)$ and $t\in[\alpha,\alpha(+^k)]$ be with $\mathrm{Ep}(t)\subset\mathrm{Dom}\, g(k,\alpha,c)$. Then $\mathrm{Ep}(l(k,\alpha,t))\subset\mathrm{Dom}\, g(k,\alpha,c)$ and $l(k,\alpha,t)[g(k,\alpha,c)]=l(k,c,t[g(k,\alpha,c)])$*

Proof. Consider k,α,c and t as stated.

Since $\mathrm{Ep}(t)\subset\mathrm{Dom}\ g(k,\alpha,c)$, then, by (2.2.3) of GenThmIH (for the case $k\in[1,n)$) and by (2.2.3) of corollary 5.21 (for $k=n$), we have that $T(n,\alpha,t)\cap\alpha\subset c$. But $T(n,\alpha,l(n,\alpha,t))\cap\alpha \underset{\text{by prop. 5.9}}{\subset} T(n,\alpha,t)\cap\alpha\subset c$. So, again by both (2.2.3) of GenThmIH and (2.2.3) of corollary 5.21, $\mathrm{Ep}(l(k,\alpha,t))\subset\mathrm{Dom}\, g(k,\alpha,c)$.

So it only remains to show the equality $l(k,\alpha,t)[g(k,\alpha,c)]=l(k,c,t[g(k,\alpha,c)])$. We have several cases:

Case $t \in [\alpha, \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2]$.
Then $t[g(k,\alpha,c)] \in [c, c(+^{k-1})(+^{k-2})...(+^2)(+^1)2]$ **(1*)**
and $l(k,\alpha,t)[g(k,\alpha,c)] = \alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2[g(k,\alpha,c)] =$
$\alpha[g(k,\alpha,c)](+^{k-1})(+^{k-2})...(+^2)(+^1)2 = c(+^{k-1})(+^{k-2})...(+^2)(+^1) \underset{\text{by (1*)}}{=}$
$l(k,c,t[g(k,\alpha,c)])$.

Case $t \in (\alpha(+^{k-1})(+^{k-2})...(+^2)(+^1)2, \alpha(+^k)]$.

Subcase $l(k,\alpha,t) = t$.
We proceed by contradiction. Suppose $l(k, \alpha, t[g(k, \alpha, c)]) \neq l(k, \alpha, t)[g(k, \alpha, c)] = t[g(k, \alpha, c)]$. Then $l(k,\alpha,t[g(k,\alpha,c)]) < t[g(k,\alpha,c)]$ and we have some subcases

- $l(k,\alpha,t[g(k,\alpha,c)]) \notin \mathbb{E}$. Then $\pi t[g(k,\alpha,c)] = \pi(t[g(k,\alpha,c)]) \underset{\text{by proposition 5.8}}{=}$ $l(k,\alpha,t[g(k,\alpha,c)]) < t[g(k,\alpha,c)]$, which implies $\pi t < t$ **(2*)**.
  On the other hand, $(\pi t + d\pi t)[g(k,\alpha,c)] = \pi t[g(k,\alpha,c)] + d\pi t[g(k,\alpha,c)] =$
  $\pi(t[g(k,\alpha,c)]) + d(\pi t[g(k,\alpha,c)]) = m(l(k,\alpha,t[g(k,\alpha,c)])) \geqslant m(t[g(k,\alpha,c)]) = m(t)[g(k,\alpha,c)]$;
  from these last inequalities we conclude $m(\pi t) = \pi t + d\pi t \geqslant m(t)$. But this, (2*) and proposition 5.22 imply $l(k,\alpha,t) \leqslant \pi t < t$. Contradiction.
- $l(k,\alpha,t[g(k,\alpha,c)]) \in \mathbb{E}$. Then, by proposition 5.8, for some $r \in [1,k)$,
  $\lambda(r,t)[g(k,\alpha,c)] = \lambda(r,t[g(k,\alpha,c)]) \underset{\text{prop. 5.8}}{=} l(k,\alpha,t[g(k,\alpha,c)]) < t[g(k,\alpha,c)]$, which implies $\lambda(r,t) < t$ **(3*)**.
  On the other hand, $m(t)[g(k,\alpha,c)] = m(t[g(k,\alpha,c)]) \leqslant m(l(k,\alpha,t[g(k,\alpha,c)])) =$
  $m(\lambda(r,t[g(k,\alpha,c)])) = m(\lambda(r,t)[g(k,\alpha,c)]) = m(\lambda(r,t))[g(k,\alpha,c)]$ and therefore
  $m(t) \leqslant m(\lambda(r,t))$. This, (3*) and proposition 5.22 imply $l(k,\alpha,t) \leqslant \lambda(r,t) < t$. Contradiction.

Hence, from the previous we conclude $l(k, \alpha, t[g(k, \alpha, c)]) = l(k, \alpha, t)[g(k, \alpha, c)]$ for the subcase $l(k,\alpha,t) = t$.

Subcase $l(k,\alpha,t) \in t \backslash \mathbb{E}$.
Then $l(k,\alpha,t) \underset{\text{by proposition 5.8}}{=} \pi t < t$ and $m(\pi t) = \pi t + d\pi t \geqslant m(t)$. **(4*)**
Note these inequalities imply $\pi t[g(k,\alpha,c)] < t[g(k,\alpha,c)]$ and
$m(\pi t[g(k,\alpha,c)]) = m(\pi t)[g(k,\alpha,c)] = (\pi t + d\pi t)[g(k,\alpha,c)] \geqslant m(t)[g(k,\alpha,c)]$; i.e.,
$l(n,c,t[g(k,\alpha,c)]) \leqslant \pi t[g(k,\alpha,c)]$. **(5*)**
Now, suppose $l(n,c,t[g(k,\alpha,c)]) < \pi t[g(k,\alpha,c)]$. **(6*)**
Then $l(n,c,t[g(k,\alpha,c)]) \in \mathbb{E} \cap t[g(k,\alpha,c)]$, which, by proposition 5.8, implies
$l(n,c,t[g(k,\alpha,c)]) = \lambda(r,t[g(k,\alpha,c)]) = \lambda(r,t)[g(k,\alpha,c)]$ for some $r \in [1,k)$. **(7*)**
So, $m(t)[g(k,\alpha,c)] = m(t[g(k,\alpha,c)]) \leqslant m(l(n,c,t[g(k,\alpha,c)])) \underset{\text{by (7*)}}{=} m(\lambda(r,t)[g(k,\alpha,c)] =$
$m(\lambda(r,t))[g(k,\alpha,c)]$; observe this means, $m(t) \leqslant m(\lambda(r,t))$. From this and proposition 5.22 we
conclude $l(k,\alpha,t) \leqslant \lambda(r,t) < \pi t$, which is contradictory with **(4*)**.
Hence (6*) does not hold, which, by (5*), means
$l(n,c,t[g(k,\alpha,c)]) = \pi t[g(k,\alpha,c)] \underset{\text{by (4*)}}{=} l(k,\alpha,t)[g(k,\alpha,c)]$.

Subcase $l(k,\alpha,t) \in t \cap \mathbb{E}$.
Then, by proposition 5.8, $l(n, c, t) = \lambda(r, t) < t$ for some $r \in [1, k)$. **(8*)**. Note this implies $m(t[g(k,\alpha,c)]) = m(t)[g(k,\alpha,c)] \leqslant m(l(n,c,t))[g(k,\alpha,c)] = m(\lambda(r,t))[g(k,\alpha,c)] =$
$m(\lambda(r,t)[g(k,\alpha,c)]) = m(\lambda(r,t[g(k,\alpha,c)]))$ and
$\lambda(r,t[g(k,\alpha,c)]) = \lambda(r,t)[g(k,\alpha,c)] < t[g(k,\alpha,c)]$. So, from all this follows
$l(k,c,t[g(k,\alpha,c)]) \leqslant \lambda(r,t[g(k,\alpha,c)])$. **(9*)**
Suppose $l(k,c,t[g(k,\alpha,c)]) < \lambda(r,t[g(k,\alpha,c)])$. **(10*)**
Then, by proposition 5.8, (10*) and the fact that $\lambda(r, t[g(k, \alpha, c)]) < \pi t[g(k, \alpha, c)$, we have that $l(k,c,t[g(k,\alpha,c)]) = \lambda(s,t[g(k,\alpha,c)])$ for some $s \in [r+1,k)$. This way, we obtain
$m(\lambda(s,t))[g(k,\alpha,c)] = m(\lambda(s,t)[g(k,\alpha,c)]) = m(\lambda(s,t[g(k,\alpha,c)])) = m(l(k,c,t[g(k,\alpha,c)])) \geqslant$

$m(t[g(k,\alpha,c)]) = m(t)[g(k,\alpha,c)]$, which imply $m(\lambda(s,t)) \geqslant m(t)$. From this and proposition 5.22 we conclude $l(s,\alpha,t) \leqslant \lambda(s,t) \underset{\text{because } s\geqslant r+1}{<} \lambda(r,t)$ which contradicts (8*).

Thus (10*) doesn't hold, which, together with (9*) means
$l(k,c,t[g(k,\alpha,c)]) = \lambda(r,t[g(k,\alpha,c)]) = \lambda(r,t)[g(k,\alpha,c)] = l(k,\alpha,t)[g(k,\alpha,c)]$. □

Corollary 5.24. *Clause (2.2.5) of theorem 3.26 also hold for n; that is, given* $\alpha, c \in \mathrm{Class}(n)$*,*
(2.2.5) For any $t \in [\alpha, \alpha(+^n))$ *with* $\mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n,\alpha,c)$*,* $\mathrm{Ep}(\eta(n,\alpha,t)) \subset \mathrm{Dom}\, g(n,\alpha,c)$ *and* $\eta(n,\alpha,t)[g(n,\alpha,c)] = \eta(n,c,t[g(n,\alpha,c)])$*.*

Proof. Let $\alpha, c \in \mathrm{Class}(n)$ and $t \in [\alpha, \alpha(+^n))$.
Suppose $\mathrm{Ep}(t) \subset \mathrm{Dom}\, g(n,\alpha,c)$. **(3*)**

Case $t \in [\alpha, \alpha(+^{n-1})(+^{n-1})...(+^1)2]$.
Then $\eta(n,\alpha,t) = \alpha(+^{n-1})(+^{n-2})...(+^1)2$ and
$T(n,\alpha,\eta(n,\alpha,t)) = \{\alpha, \alpha(+^{n-1}), ..., \alpha(+^{n-1})(+^{n-2})...(+^1)\}$. Therefore
$T(n,\alpha,\eta(n,\alpha,t)) \cap \alpha = \emptyset \subset c$, which, by (2.2.3) of previous corollary 5.21, implies
$\mathrm{Ep}(\eta(n,\alpha,t)) \subset \mathrm{Dom}\, g(n,\alpha,c)$.

On the other hand, since $t \in [\alpha, \alpha(+^{n-1})(+^{n-1})...(+^1)2)$, then
$t[g(n,\alpha,c)] \in [c, c(+^{n-1})(+^{n-1})...(+^1)2)$. **(4*)**
This way, $\eta(n,\alpha,t)[g(n,\alpha,c)] = (\alpha(+^{n-1})(+^{n-2})...(+^1)2)[g(n,\alpha,c)] =$
$c(+^{n-1})(+^{n-2})...(+^1)2 \underset{\text{by } (4^*)}{=} \eta(n,c,t[g(n,\alpha,c)])$.

The previous shows that (2.2.5) holds for the case $t \in [\alpha, \alpha(+^{n-1})(+^{n-1})...(+^1)2]$.

Case $t \in (\alpha(+^{n-1})(+^{n-1})...(+^1)2, \alpha(+^n))$.

Then $T(n,\alpha,\eta(n,\alpha,t)) \cap \alpha = T(n,\alpha,m(l(n,\alpha,t))) \cap \alpha \underset{\text{by proposition 5.7}}{\subset}$
$T(n, \alpha, l(n, \alpha, t)) \cap \alpha \underset{\text{by prop. 5.9}}{\subset} T(n, \alpha, t) \cap \alpha \underset{\text{by (3*) and (2.2.3) of corollary 5.21}}{\subset} c$. So, from the previous and (2.2.3) of corollary 5.21 again, we conclude $\mathrm{Ep}(\eta(n,\alpha,t)) \cup \mathrm{Ep}(l(n,\alpha,t)) \subset \mathrm{Dom}\, g(n, \alpha, c)$. This way, note $\eta(n, \alpha, t)[g(n, \alpha, c)] = m(l(n, \alpha, t))[g(n, \alpha, c)] = m(l(n, \alpha, t)[g(n, \alpha, c)]) \underset{\text{by proposition 5.23}}{=}$
$m(l(n,c,t[g(n,\alpha,c)])) = \eta(n,c,t[g(n,\alpha,c))$. □

5.6 Clauses (2.3), (2.4) and (2.5) of theorem 3.26

Corollary 5.25. *Clauses (2.3.1), (2.3.2) and (2.3.3) of theorem 3.26 also hold for n; that is, given* $\alpha, c \in \mathrm{Class}(n)$ *with* $c \leqslant \alpha$*,*
(2.3.1) $\mathrm{Dom}\, g(n,c,\alpha) = \mathbb{E} \cap c(+^n)$
(2.3.2) $g(n,\alpha,c) = g^{-1}(n,c,\alpha)$
(2.3.3) $g(n,\alpha,c)[\mathrm{Dom}\, g(n,\alpha,c)] = \mathbb{E} \cap c(+^n)$

Proof. Direct from the definition of $g(n,c,\alpha)$ and $g(n,\alpha,c)$. □

Corollary 5.26. *Clauses (2.4.1), (2.4.2), (2.4.3), (2.4.4), (2.4.5) and (2.4.6) of theorem 3.26 also hold for n; that is, given* $\alpha, c \in \mathrm{Class}(n)$*,*

(2.4.1) $g(n,\alpha,c)(\alpha)=c$
(2.4.2) For any $i\in[1,n]$ *and any* $e\in(\mathrm{Dom}\, g(n,\alpha,c))\cap[\alpha,\alpha(+^n))$,
$e\in\mathrm{Class}(i)\Longleftrightarrow g(n,\alpha,c)(e)\in\mathrm{Class}(i)$
(2.4.3) The function $e\longmapsto e[g(n,\alpha,c)]$ *with domain* $(\mathrm{Dom}\, g(n,\alpha,c))\cap(\alpha,\alpha(+^n))$ *is an* $(<,+,\cdot,<_1,\lambda x.\omega^x,(+^1),(+^2),\ldots,(+^{n-1}))$ *isomorphism*
(2.4.4) $\forall e\in(\mathrm{Dom}\, g(n,\alpha,c))\cap(\alpha,\alpha(+^n)).m(g(n,\alpha,c)(e))=m(e)[g(n,\alpha,c)]$.
(2.4.5) Suppose $n\geqslant 2$. *Then*
$\forall i\in[2,n]$.
$\forall e\in\mathrm{Class}(i)\cap(\mathrm{Dom}\, g(n,\alpha,c))\cap[\alpha,\alpha(+^n))$.
$\forall E\in(e,e(+^i))\cap\mathrm{Class}(i-1)$.
$f(i,e)(E)=\{E_1>\ldots>E_q\}\Longleftrightarrow$
$f(i,g(n,\alpha,c)(e))(g(n,\alpha,c)(E))=\{g(n,\alpha,c)(E_1)>\ldots>g(n,\alpha,c)(E_q)\}$
(2.4.6) Suppose $n\geqslant 2$. *Then*
$\forall i\in[2,n].\forall s\in\mathrm{Class}(i-1)\cap[\alpha,\alpha(+^n))$.
$g(n,\alpha,c)(\lambda(i,s))=\lambda(i,g(n,\alpha,c)(s))$

Proof. Direct from theorem 5.10 and the definitions of $g(n,c,\alpha)$ and $g(n,\alpha,c)$. □

Corollary 5.27. *Clauses (2.5.1), (2.5.2) and (2.5.3) of theorem 3.26 also hold for n; that is, given* $\alpha,c\in\mathrm{Class}(n)$ *with* $c\leqslant\alpha$, *then for all* $d\in\mathrm{Class}(n)\cap[c,\alpha]$,
(2.5.1) $\mathrm{Dom}\, g(n,\alpha,c)\subset\mathrm{Dom}\, g(n,\alpha,d)$
(2.5.2) $g(n,\alpha,d)[\mathrm{Dom}\, g(n,\alpha,c)]\subset\mathrm{Dom}\, g(n,d,c)$
(2.5.3) $g(n,\alpha,c)=g(n,d,c)\circ g(n,\alpha,d)|_{\mathrm{Dom}\, g(n,\alpha,c)}$ *and therefore*
$g^{-1}(n,\alpha,d)\circ g^{-1}(n,d,c)=g^{-1}(n,\alpha,c)\colon\mathbb{E}\cap c(+^n)\longrightarrow\mathrm{Dom}\, g(n,\alpha,c)$.

Proof. Consider $c,d,\alpha\in\mathrm{Class}(n)$ as stated. By proposition 5.16 and the definitions of $g(n,\alpha,c)$ and of $g(n,c,\alpha)$, follow $g^{-1}(n,\alpha,d)\circ g^{-1}(n,d,c)=g(n,d,\alpha)\circ g(n,c,d)=$
$=g(n,c,\alpha)=g^{-1}(n,\alpha,c)\colon\mathbb{E}\cap c(+^n)\longrightarrow\mathrm{Dom}\, g(n,\alpha,c)$. But this implies
$g(n,\alpha,c)=g(n,d,c)\circ g(n,\alpha,d)|_{\mathrm{Dom}\, g(n,\alpha,c)}$, $g(n,\alpha,d)[\mathrm{Dom}\, g(n,\alpha,c)]\subset\mathrm{Dom}\, g(n,d,c)$ and $\mathrm{Dom}\, g(n,\alpha,c)\subset\mathrm{Dom}\, g(n,\alpha,d)$. Hence (2.5.3), (2.5.2) and (2.5.1) hold. □

Chapter 6

Clauses (3),(4),(5),(6) of theorem 3.26

6.1 The $\leqslant^n$-relation

Definition 6.1. *Let $\alpha \in \mathrm{Class}(n)$, $t \in [\alpha, \alpha(+^n)]$. By $\alpha <^n t$ we mean*

1. $\alpha < t$
2. $\forall B \subset_{\mathrm{fin}} t. \exists \delta \in \mathrm{Class}(n) \cap \alpha$ *such that*

   i. $(\bigcup_{x \in B} T(n, \alpha, x)) \cap \alpha \subset \delta$

   ii. The function $h: B \longrightarrow h[B]$ *defined as* $h(x) := x[g(n, \alpha, \delta)]$ *is an* $(<, <_1, +, \lambda x.\omega^x)$*-isomorphism with* $h|_\alpha = \mathrm{Id}_\alpha$.

As usual, $\alpha \leqslant^n t$ *just means* $\alpha <^n t$ *or* $\alpha = t$

Proposition 6.2. *Let $\alpha \in \mathrm{Class}(n)$ and $(c_\xi)_{\xi \in I} \subset [\alpha, \alpha(+^n)] \ni c$.*

1. *Let $b \in [\alpha, c]$. If $\alpha <^n c$ then $\alpha \leqslant^n b$.* *($<^n$-connectedness)*
2. *Suppose $\forall \xi \in I. \alpha <^n c_\xi$ and $c_\xi \underset{cof}{\hookrightarrow} c$. Then $\alpha <^n c$.* *($<^n$-continuity)*

Proof.
1.
Assume α, b, c as stated in our proposition.
If $b = \alpha$ then clearly $\alpha \leqslant^n b$. So suppose $\alpha < b < c \leqslant \alpha(+^n)$. Let $B \subset_{\mathrm{fin}} b$ be arbitrary. Then $B \subset_{\mathrm{fin}} c$ and then, since $\alpha <^n c$, there exists $\delta \in \mathrm{Class}(n) \cap \alpha$ such that $(\bigcup_{x \in B} T(n, \alpha, x)) \cap \alpha \subset \delta$ and such that the function $h: B \longrightarrow h[B]$, $x \longmapsto x[g(n, \alpha, \delta)]$ is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$. Since the previous was done for arbitrary $B \subset_{\mathrm{fin}} b$, we have actually shown that $\alpha <^n b$.

2.
Assume $\alpha, (c_\xi)_{\xi \in I}, c$ are as stated in our proposition.
Let $B \subset_{\mathrm{fin}} c$ be arbitrary. Since B is finite and $c_\xi \underset{\mathrm{cof}}{\hookrightarrow} c$, then there exists $\xi \in I$ such that $B \subset_{\mathrm{fin}} c_\xi$. From this and the fact that $\alpha <^n c_\xi$ we conclude that there exists $\delta \in \mathrm{Class}(n) \cap \alpha$ such that $(\bigcup_{x \in B} T(n, \alpha, x)) \cap \alpha \subset \delta$ and such that the function $h: B \longrightarrow h[B]$, $x \longmapsto x[g(n, \alpha, \delta)]$ is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$. Since the previous was done for arbitrary $B \subset_{\mathrm{fin}} c$, we have shown that $\alpha <^n c$. $\square$

Remark 6.3. (3) of theorem 3.26 holds for n, that is, the binary relation $\leqslant^n \subset \mathrm{Class}(n) \times \mathrm{OR}$ given in definition 6.1 satisfies $\leqslant^n$-connectedness and $\leqslant^n$-continuity (by proposition 6.2). Moreover, it is clear from the definition of $\leqslant^n$ that $\forall \alpha \in \mathrm{Class}(n). \forall t \in [\alpha, \alpha(+^n)]. \alpha \leqslant^n t \Longrightarrow \alpha \leqslant_1 t$.

Remark 6.4. $<^n$-transitivity does not make sense in general.

6.2 Clauses (4) and (5) of theorem 3.26

Proposition 6.5. *(First cofinality property of $\leqslant^n$). (4) of theorem 3.26 holds for n. Explicitly:*
Let $\alpha \in \mathrm{Class}(n)$ and $s \in (\alpha, \alpha(+^n)]$ be such that $\alpha <^n s$. Then, for any $t \in [\alpha, s)$ there is a sequence $(c_\xi)_{\xi \in X} \subset \alpha \cap \mathrm{Class}(n)$ such that $T(n, \alpha, t) \cap \alpha \subset c_\xi$, $c_\xi \underset{cof}{\hookrightarrow} \alpha$ and $c_\xi \leqslant_1 t[g(n, \alpha, c_\xi)]$.

Proof. Let $\alpha \in \mathrm{Class}(n)$, $s \in (\alpha, \alpha(+^n)]$ and suppose $\alpha <^n s$.

Take $t \in [\alpha, s)$. Moreover, take $\gamma \in \alpha$ arbitrary.

Consider the set $B_\gamma := \{\gamma, \alpha, t\} \subset_{\mathrm{fin}} t+1 \leqslant s$. By hypothesis there exists $\delta_\gamma \in \mathrm{Class}(n) \cap \alpha$ such that $(\bigcup_{x \in B_\gamma} T(n, \alpha, x)) \cap \alpha \subset \delta_\gamma$ and the function $h_\gamma \colon B_\gamma \longrightarrow h[B_\gamma]$, $h_\gamma(x) := x[g(n, \alpha, \delta_\gamma)] \subset \alpha$ is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism with $h_\gamma|_\alpha = \mathrm{Id}_\alpha$. Therefore, since $\gamma < \alpha \leqslant_1 t$, then $\gamma = h_\gamma(\gamma) < h_\gamma(\alpha) \leqslant_1 h_\gamma(t) = t[g(n, \alpha, \delta_\gamma)]$; besides, $h_\gamma(\alpha) = \alpha[g(n, \alpha, \delta_\gamma)] = \delta_\gamma \in \mathrm{Class}(n)$. Hence, by defining $c_\gamma := h_\gamma(\alpha)$ for any $\gamma \in \alpha$, we have that the sequence $(c_\gamma)_{\gamma \in \alpha}$ satisfies all what is stated. $\square$

Proposition 6.6. *(Second cofinality property of $\leqslant^n$). (5) of theorem 3.26 holds for n. Explicitly:*
Let $\alpha \in \mathrm{Class}(n)$ and $t \in [\alpha, \alpha(+^n))$ be arbitrary.
Suppose $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | T(n, \alpha, t) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$. Then $\forall s \in [\alpha, t+1]. \alpha \leqslant^n s$ and therefore $\alpha \leqslant_1 t+1$.

Proof. Let $\alpha \in \mathrm{Class}(n)$, $t \in [\alpha, \alpha(+^n))$ and assume
$\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | T(n, \alpha, t) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$.

We prove by induction on $([\alpha, t+1], <)$ that $\forall s \in [\alpha, t+1]. \alpha \leqslant^n s$.

For $s = \alpha$ it is clear. So, from now on, suppose $s > \alpha$.

Case $s \in \mathrm{Lim} \cap [\alpha, t+1]$. Our induction hypothesis is $\alpha \leqslant^n \beta$ for all $\beta \in [\alpha, t+1] \cap s$. Thus $\alpha \leqslant^n s$ by $\leqslant^n$-continuity.

Suppose $s = l+1 \in [\alpha, t+1]$.
Our induction hypothesis is $\alpha \leqslant^n l$. **(IH)**

Let $B \subset_{\mathrm{fin}} s = l+1$. Without loss of generality, suppose $\alpha, l \in B$ and write $B = X \cup Y$ where $X := B \cap \alpha$, $Y := B \cap [\alpha, l]$, $Y := \{y_1, ..., y_m | \alpha = y_1 < y_2 < ... < y_m = l\}$.
Note $l \in [\alpha, t] \subset [\alpha, \alpha(+^n)) \ni t$. Moreover, since $T(n, \alpha, l) \cup T(n, \alpha, t)$ is finite and $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | T(n, \alpha, t) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$, then actually $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | (T(n, \alpha, l) \cup T(n, \alpha, t)) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 t[g(n, \alpha, \gamma)]\}$. **(*)**. But for any $\gamma \in \mathrm{Class}(n)$ such that $(T(n, \alpha, l) \cup T(n, \alpha, t)) \cap \alpha \subset \gamma$ we have $\gamma \leqslant l[g(n, \alpha, \gamma)] \leqslant t[g(n, \alpha, \gamma)]$; therefore, by $\leqslant_1$-connectedness and (*) we conclude $\alpha \in \mathrm{Lim}\{\gamma \in \mathrm{Class}(n) | T(n, \alpha, l) \cap \alpha \subset \gamma \wedge \gamma \leqslant_1 l[g(n, \alpha, \gamma)]\}$.

Let $p := \max \bigcup_{i\in\{1,\ldots,m\}} (T(n,\alpha,y_i)\cap\alpha)$ and consider the set $M := \{\gamma \in \alpha \cap \mathrm{Class}(n) | p < \gamma \supset X \wedge \gamma \leqslant_1 l[g(n,\alpha,\gamma)]\}$. By our previous observations M is confinal in α. Let $X' := \{x \in X | x \not\leqslant_1 \alpha\}$. Then for any $x \in X'$ there exists $\gamma_x \in M$ such that $x \not\leqslant_1 \gamma_x$ (otherwise, by $\leqslant_1$-continuity $x \leqslant_1 \alpha$). Let $\gamma := \max(\{\gamma_x | x \in X'\} \cup \{\min M\})$. Clearly $\gamma \in M$.

We define the function $h: B \longrightarrow h[B] \subset \alpha$ as $h(x) := x[g(n,\alpha,\gamma)]$ for all $x \in B$ (particularly note that $h(\alpha) = \gamma$). Let's see that h is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism.

That h preserves $<$, $+$ and $\lambda x.\omega^x$ follows directly from the fact that $X \cup \bigcup_{i\in\{1,\ldots,m\}} (\mathrm{Ep}(y_i)\cap\alpha) \subset \gamma$ and the theorems we know about substitutions.

Since $h|_\alpha = \mathrm{Id}_\alpha$, then clearly h preserves $<_1$ in $B\cap\alpha$; moreover, from the properties of the function $g(n,\alpha,\gamma)$, we know h preserves $<_1$ in $B \cap (\alpha, \alpha(+^n))$ too. Therefore it only remains to see that h preserves $<_1$ in the cases of the form $\alpha \leqslant_1 y_i$ or $x \leqslant_1 y_i$ for $y_i \in Y$. Let's see this:

- By (IH) $\alpha \leqslant^n l$ and so $\alpha \leqslant_1 l$; subsequently, by $\leqslant_1$-connectedness it follows $\alpha \leqslant_1 y_i$ for any $y_i \in Y$. So we need to show $h(\alpha) \leqslant_1 h(y_i)$ for any $y_i \in Y$. But this is easy because $h(\alpha) = \gamma \leqslant_1 l[g(n,\alpha,\gamma)]$ by the way we took γ, and since $\forall y_i \in Y. h(\alpha) \leqslant h(y_i) \leqslant h(l) = l[g(n,\alpha,\gamma)]$, then by $\leqslant_1$-connectedness $\forall y_i \in Y. h(\alpha) \leqslant_1 h(y_i)$.
- Suppose $x \in X$ and $y_i \in Y$ satisfy $x \leqslant_1 y_i$. Then $h(x) = x \leqslant_1 h(y_i)$ by $\leqslant_1$-connectedness (because $x = h(x) \leqslant h(y_i) \leqslant y_i$ for any $i \in \{1,\ldots,m\}$).
- Suppose $x \in X$ and $y_i \in Y$ satisfy $x \not\leqslant_1 y_i$. Then $x \not\leqslant_1 \alpha$ (otherwise, using the fact that we know $\alpha \leqslant_1 y_i$ for all $i \in \{1,\ldots,m\}$, we would have $x \leqslant_1 y_i$ by $\leqslant_1$-transitivity). So $x \in X'$ and then $x \not\leqslant_1 \gamma_x \leqslant \gamma \leqslant h(y_i)$; therefore $h(x) = x \not\leqslant_1 h(y_i)$.

All the previous cases show that h preserves $<_1$ too and from all our work we have that h is indeed an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism. This shows $\alpha \leqslant^n l + 1$

Our precedent work shows $\forall s \in [\alpha, t+1]. \alpha \leqslant^n s$, which clearly implies $\alpha \leqslant_1 t+1$. □

6.3 Clause (6) of theorem 3.26

6.3.1 Generalized covering of a finite set.

Definition 6.7. *Let $i \in [1,n]$, $\alpha \in \mathrm{Class}(i)$, $B \subset_{\mathrm{fin}} \alpha(+^i)$. In the following, we use the definitions of $T(i,\alpha,x)$ and of $D(\alpha,\delta)$ (see lemma 2.30 and definition 2.28).*
Let it be

$$Q := B \cup \bigcup_{x\in B} T(i,\alpha,x),$$

$$W := Q \cup \bigcup_{\substack{\xi \in Q\cap \mathrm{Class}(j)\cap[\alpha,\alpha(+^i)) \\ j\in[2,i]}} \{\xi(+^{j-1}), \ldots, \xi(+^{j-1})(+^{j-2})\ldots(+^1), \xi(+^{j-1})(+^{j-2})\ldots(+^1)2\},$$

and

$$Z := W \cup \bigcup_{\gamma\in\mathbb{E}\wedge x\in W\cap\mathbb{E}\cap(\alpha,\alpha(+^i))\wedge m(x)\in(\gamma,\gamma(+^1))} D(\gamma, m(x)) \cup \bigcup_{\gamma\in\mathbb{E}\wedge x\in W\cap(\alpha,\alpha(+^i))\cap(\gamma,\gamma(+^1))} D(\gamma, x)$$

We define the set $\Delta(i,\alpha,B)$ *as* $\Delta(i,\alpha,B):=Z\cup\{\gamma 2\mid\gamma\in\mathbb{E}\cap Z\}$.

Moreover, for an ordinal $\beta\in\alpha(+^i)$*, we will write* $\Delta(i,\alpha,\beta)$ *instead of* $\Delta(i,\alpha,\{\beta\})$.

Proposition 6.8. *Let* $i\in[1,n]$, $\alpha\in\mathrm{Class}(i)$.

1. $\forall t_1,t_2\in[\alpha,\alpha(+^i)).t_1\leqslant t_2\Longrightarrow\eta(i,\alpha,t_1)\leqslant\eta(i,\alpha,t_2)$.
2. $\forall t\in[\alpha,\alpha(+^i)).\eta(i,\alpha,\eta(i,\alpha,t))=\eta(i,\alpha,t)$.

Proof.
1.
Let $t_1,t_2\in[\alpha,\alpha(+^i))$ be such that $t_1\leqslant t_2$.

Case $t_2\leqslant\alpha(+^{i-1})...(+^1)2$.
Then $\eta(i,\alpha,t_1)=\alpha(+^{i-1})...(+^1)2=\eta(i,\alpha,t_2)$.

Case $t_1\leqslant\alpha(+^{i-1})...(+^1)2<t_2$.
Then $\eta(i,\alpha,t_1)=\alpha(+^{i-1})...(+^1)2<t_2\leqslant\eta(i,\alpha,t_2)$.

Case $\alpha(+^{i-1})...(+^1)2<t_1\leqslant t_2$.
Then $\eta(i,\alpha,t_2)=\max\{m(e)\mid e\in(\alpha,t_2]\}\geqslant\max\{m(e)\mid e\in(\alpha,t_1]\}=\eta(i,\alpha,t_1)$.

2.
Let $t\in[\alpha,\alpha(+^i))$ be arbitrary.

Case $t\in[\alpha,\alpha(+^{i-1})...(+^1)2]$.
Then $\eta(i,\alpha,t)=\alpha(+^{i-1})...(+^1)2=\eta(i,\alpha,\alpha(+^{i-1})...(+^1)2)=\eta(i,\alpha,\eta(i,\alpha,t))$.

Case $t\in(\alpha(+^{i-1})...(+^1)2,\alpha(+^i))$.

To show that $\forall s\in(\alpha,\eta(i,\alpha,t)].m(s)\leqslant\eta(i,\alpha,t)$. **(*)**

Proof of (*):

Let $s\in(\alpha,\eta(i,\alpha,t)]$ be arbitrary.

Case $s\in(\alpha,t]$. Then $m(s)\leqslant\max\{m(e)\mid e\in(\alpha,t]\}=\eta(i,\alpha,t)$.

Case $s\in(t,\eta(i,\alpha,t)]$. We proceed by contradiction. Suppose $\eta(i,\alpha,t)<m(s)$. **(**)**.
Then note $l(i,\alpha,t)\leqslant t<s\leqslant\eta(i,\alpha,t)=m(l(i,\alpha,t))<m(l(i,\alpha,t))+1\leqslant m(s)$, which implies, by $\leqslant_1$-connectedness, $l(i,\alpha,t)\leqslant_1 s\leqslant_1 m(l(i,\alpha,t))+1$, and then by $\leqslant_1$-transitivity $l(i,\alpha,t)\leqslant_1 m(l(i,\alpha,t))+1$. Contradiction. Thus (**) does not hold, i.e., $m(s)\leqslant\eta(i,\alpha,t)$.

This concludes the proof of (*).

Finally, since $t\in(\alpha(+^{i-1})...(+^1)2,\alpha(+^i))$, then $\eta(i,\alpha,t)\in(\alpha(+^{i-1})...(+^1)2,\alpha(+^i))$ and therefore $\eta(i,\alpha,\eta(i,\alpha,t))=\max\{m(s)\mid s\in(\alpha,\eta(i,\alpha,t)]\}\underset{\text{by }(*)}{=}\eta(i,\alpha,t)$. □

Proposition 6.9. *Let* $i\in[1,n]$, $\alpha\in\mathrm{Class}(i)$ *and* $t\in\alpha(+^i)$. *Then* $\forall s\in T(i,\alpha,t).s\leqslant\eta(i,\alpha,t)$.

Proof. If $t=\alpha$, then $T(i,\alpha,t)=\{\alpha\}\subset\alpha(+^{i-1})(+^{i-2})...(+^1)2=\eta(i,\alpha,t)$. So from now on, suppose $t\in(\alpha,\alpha(+^i))$.

Let $C:=\{x\in\alpha(+^i)\mid x\leqslant\eta(i,\alpha,t)\}$. Then C is closed under clauses a), b), c) and d) of the definition of $T(i,\alpha,t)$. The details of the proof are left to the reader. □

Proposition 6.10. *Let* $i\in[1,n]$, $\alpha\in\mathrm{Class}(i)$, $B\subset_{\text{fin}}\alpha(+^i)$. *Then, using the same notation as in previous definition 6.7, the following holds:*

i. $\Delta(i,\alpha,B)$ *is finite.*

ii. $Q \subset \Delta(i, \alpha, B) \subset \alpha(+^i)$*; moreover,* $B \cap [\alpha, \alpha(+^i)) \neq \emptyset \Longrightarrow \alpha \in Q \subset \Delta(i, \alpha, B)$.

iii. $\forall x \in \Delta(i, \alpha, B). x \in \mathbb{E} \Longrightarrow x \in W$.

iv. Suppose $B \cap [\alpha, \alpha(+^i)) \neq \emptyset$*. Let* $t := \max B$*. Then* $\alpha \in \Delta(i, \alpha, B) \subset_{\text{fin}} \eta(i, \alpha, t) + 1 < \alpha(+^i)$.

Proof. Along this proof, we use the same notation as in definition 6.7.

i.
Q is finite because B is finite and because every $T(i, \alpha, x)$ is finite; therefore W is finite too. This way, Z is finite too, because W is finite and because by lemma 2.30, the sets $D(\gamma, m(x))$ and $D(\gamma, x)$ are finite. Finally, $\Delta(i, \alpha, B)$ is finite since it has, at most, twice the number of elements of Z.

ii.
Clearly $Q \subset \Delta(i, \alpha, B)$; moreover, if $B \cap [\alpha, \alpha(+^i)) \neq \emptyset$, then take for $x \in B \cap [\alpha, \alpha(+^i))$, we have that $\alpha \in T(i, \alpha, x) \subset Q \subset \Delta(i, \alpha, B)$. So it only remains to show that $\Delta(i, \alpha, B) \subset \alpha(+^i)$. Let's do this: $Q \subset \alpha(+^i)$ because $B \subset \alpha(+^i)$ and because of the definition of $T(i, \alpha, x)$; this implies that $W \subset \alpha(+^i)$, because for $\xi \in \alpha(+^i) \cap \text{Class}(r)$ with $r < i$, $\xi(+^r)(+^{r-1})...(+^1)2 < \alpha(+^i)$. Consequently, $Z \subset \alpha(+^i)$ because $W \subset \alpha(+^i)$ and because of lemma 2.30. Concluding: $\Delta(i, \alpha, B) \subset \alpha(+^i)$ because all of the previous and because $\alpha(+^i)$ is an additive principal number.

iii.
Take $x \in \Delta(i, \alpha, B) \cap \mathbb{E}$. Then $x \in \Delta(i, \alpha, B) \backslash \{\delta 2 \mid \delta \in Z \cap \mathbb{E}\}$. Now, if $x \in W$ then we are done, so suppose for some $\gamma \in \mathbb{E}$ and $y \in W \cap (\alpha, \alpha(+^i))$,
$x \in (D(\gamma, m(y)) \wedge y \in W \cap \mathbb{E} \cap (\alpha, \alpha(+^i)) \wedge m(y) \in (\gamma, \gamma(+^1))$ **(a1*)** or
$x \in D(\gamma, y) \wedge y \in W \cap (\alpha, \alpha(+^i)) \cap (\gamma, \gamma(+^1))$. **(a2*)**
By the definition of the sets $D(\beta, \delta)$, (see lemma 2.30 and definition 2.28), for the cases (a1*) and (a2*), it holds
$\{\gamma, x\} \subset D(\gamma, m(y)) \cap \mathbb{E} = \text{Ep}(m(y))$ and $\{\gamma, x\} \subset D(\gamma, y) \cap \mathbb{E} = \text{Ep}(y)$ respectively. **(0*)**.
Let's consider the cases (a1*) and (a2*) more carefully:

Case (a1*).
We now see the ways in which $y \in W$.
Subcase $y \in \{\xi(+^{j-1}), ..., \xi(+^{j-1})(+^{j-2})...(+^1)\}$, for some $\xi \in Q \cap \text{Class}(j) \cap [\alpha, \alpha(+^i))$ with $j \in [2, 1)$. Then $m(y) = \xi(+^{j-1})(+^{j-2})...(+^1)2$; thus $\text{Ep}(m(y)) = \{\xi(+^{j-1})(+^{j-2})...(+^1)\}$ and by (0*) we have $x = \gamma = \xi(+^{j-1})(+^{j-2})...(+^1) \in W$.
Subcase $y \in Q$.
If $y \in B$, then $\{\gamma, x\} \underset{\text{by (0*)}}{\subset} \text{Ep}(m(y)) \subset T(i, \alpha, y) \subset W$.
If $y \in T(i, \alpha, s)$ for some $s \in B$, then $\{\gamma, x\} \underset{\text{by (0*)}}{\subset} \text{Ep}(m(y)) \subset T(i, \alpha, y) \underset{\text{by prop. 5.4}}{\subset} T(i, \alpha, s) \subset W$.

Case (a2*).
We see again the ways in which $y \in W$.
Subcase $y = \xi(+^{j-1})(+^{j-2})...(+^1)2$, for some $\xi \in Q \cap \text{Class}(j) \cap [\alpha, \alpha(+^i))$ with $j \in [2, 1)$. Then $\text{Ep}(y) = \{\xi(+^{j-1})(+^{j-2})...(+^1)\}$ and thus, by (0*), $x = \gamma = \xi(+^{j-1})(+^{j-2})...(+^1) \in W$.
Subcase $y \in Q$.
Then $y \in B$ (because the sets $T(i, \alpha, s) \subset \mathbb{E}$). So $\{x, \gamma\} \underset{\text{by (0*)}}{\subset} \text{Ep}(y) \subset T(i, \alpha, y) \subset W$.

iv.
By ii. it is clear that $\alpha \in \Delta(i, \alpha, B)$; moreover, we already know $\eta(i, \alpha, t) + 1 < \alpha(+^i)$ and that $\Delta(i, \alpha, B)$ is finite. So we only have to show that $\Delta(i, \alpha, B) \subset \eta(i, \alpha, t) + 1$. Let's do this:

Note by proposition 6.9,
$Q \subset \max\{\{t+1\} \cup \{\eta(i,\alpha,x)+1 \mid x \in B\}\} \underset{\text{using proposition 6.8 and } t \leqslant \eta(i,\alpha,t)}{=} \eta(i,\alpha,t)+1.$
(a3*).

On the other hand, for $j \in [2,i]$ and $\xi \in Q \cap \mathrm{Class}(j) \cap [\alpha, \alpha(+^i))$ arbitrary, we have that, by **(a3*)**, $\xi \leqslant \eta(i,\alpha,t)$; therefore $\eta(i,\alpha,\xi) \underset{\text{by prop. 6.8}}{\leqslant} \eta(i,\alpha,\eta(i,\alpha,t)) \underset{\text{by prop. 6.8}}{=} \eta(i,\alpha,t)$. **(a4*)**.
So $\{\xi(+^{j-1}),...,\xi(+^{j-1})...(+^1),\xi(+^{j-1})...(+^1)2\} \subset m(\xi)+1 \leqslant \eta(i,\alpha,\xi)+1 \underset{\text{by (a4*)}}{\leqslant} \eta(i,\alpha,t)+1$.
From our work in this paragraph follows $W \subset \eta(i,\alpha,t)+1$. **(a5*)**.

<u>Now we show that $Z \subset \eta(i,\alpha,t)+1$</u>.

Consider $D(\gamma, m(x))$, with $\gamma \in \mathbb{E} \wedge x \in W \cap \mathbb{E} \cap (\alpha, \alpha(+^i)) \wedge m(x) \in (\gamma, \gamma(+^1))$. Note this means $x \leqslant \gamma < m(x)$ and therefore it is not possible that $m(\gamma) > m(x)$, otherwise the inequalities $x \leqslant \gamma < m(x) < m(x)+1 \leqslant m(\gamma)$ imply, by $\leqslant_1$-connectedness and $\leqslant_1$-transitivity, $x \leqslant_1 m(x)+1$ which is contradictory. Thus $x \leqslant \gamma \leqslant m(\gamma) \leqslant m(x) < \gamma(+^1)$ **(a6*)**.

On the other hand, since $x \in W$, then $x \leqslant \eta(i,\alpha,t)$ and then by proposition 6.8,
$m(x) \leqslant \eta(i,\alpha,x) \leqslant \eta(i,\alpha,\eta(i,\alpha,t)) = \eta(i,\alpha,t)$ **(a7*)**; but then, again by proposition 6.8,
$\eta(i,\alpha,m(x)) \leqslant \eta(i,\alpha,\eta(i,\alpha,x)) = \eta(i,\alpha,x) \leqslant \eta(i,\alpha,t)$ which together with the inequalities $m(\pi(m(x))) \leqslant \eta(i,\alpha,\pi(m(x))) \underset{\text{because } \pi(m(x)) \leqslant m(x)}{\leqslant} \eta(i,\alpha,m(x))$ yields
$m(\pi(m(x))) \leqslant \eta(i,\alpha,t)$. **(a8*)**.

Finally, by lemma 2.30, $D(\gamma, m(x)) \subset \max\{m(x), \pi(m(x)) + d\pi(m(x))\}+1 =$

$$\left\{\begin{array}{lll} \max\{m(x), m(\pi(m(x)))\}+1 & \text{iff} & \pi(m(x)) \notin \mathbb{E} \\ \max\{m(x), \gamma 2\}+1 & \text{iff} & \pi(m(x)) \in \mathbb{E} \end{array}\right\} \underset{\text{by (a6*) and because } m(\gamma) \geqslant \gamma 2}{\leqslant}$$

$$\left\{\begin{array}{lll} \max\{m(x), m(\pi(m(x)))\}+1 & \text{iff} & \pi(m(x)) \notin \mathbb{E} \\ m(x)+1 & \text{iff} & \pi(m(x)) \in \mathbb{E} \end{array}\right\} \leqslant$$

$$\max\{m(x), m(\pi(m(x)))\}+1 \underset{\text{by (a7*) and (a8*)}}{\leqslant} \eta(i,\alpha,t)+1.$$

Now the final case: consider $D(\gamma, x)$ with $\gamma \in \mathbb{E} \wedge x \in W \cap \mathbb{E} \cap (\alpha, \alpha(+^i)) \wedge x \in (\gamma, \gamma(+^1))$. Since $x \in W$, then $\pi x \leqslant x \leqslant \eta(i,\alpha,t)$; from these inequalities and using proposition 6.8, we get
$m(x) \leqslant \eta(i,\alpha,x) \leqslant \eta(i,\alpha,\eta(i,\alpha,t)) = \eta(i,\alpha,t)$ **(b1*)**
and
$m(\pi x) \leqslant \eta(i,\alpha,\pi x) \leqslant \eta(i,\alpha,\eta(i,\alpha,t)) = \eta(i,\alpha,t)$. **(b2*)**

Finally, by lemma 2.30,
$D(\gamma, x) \subset \max\{x, \pi x + d\pi x\}+1 =$, using that $\pi x \in \mathbb{E} \Longleftrightarrow \pi x = \gamma$,

$$\left\{\begin{array}{lll} \max\{m(x), m(\pi x)\}+1 & \text{iff} & \pi x \notin \mathbb{E} \\ \max\{m(x), \gamma 2\}+1 & \text{iff} & \pi x \in \mathbb{E} \end{array}\right\} \leqslant, \text{ using } m(\gamma) \geqslant 2\gamma \text{ and } \pi x \in \mathbb{E} \Longleftrightarrow \pi x = \gamma,$$

$$\left\{\begin{array}{lll} \max\{m(x), m(\pi x)\}+1 & \text{iff} & \pi x \notin \mathbb{E} \\ \max\{m(x), m(\pi x)\}+1 & \text{iff} & \pi x \in \mathbb{E} \end{array}\right\} \underset{\text{by (b1*) and (b2*)}}{\leqslant} \eta(i,\alpha,t)+1.$$

The previous shows $Z \subset \eta(i,\alpha,t)+1$.

<u>To show $\Delta(i,\alpha,B) \subset \eta(i,\alpha,t)+1$</u>.

It only remains to show that $\{\gamma 2 \mid \gamma \in \mathbb{E} \cap Z\} \subset \eta(i,\alpha,t)+1$. So let $\gamma \in \mathbb{E} \cap Z$. Since we already know that $Z \subset \eta(i,\alpha,t)+1$, then $\gamma \leqslant \eta(i,\alpha,t)$; but this implies, by proposition 6.8, that $\eta(i,\alpha,\gamma) \leqslant \eta(i,\alpha,\eta(i,\alpha,t)) = \eta(i,\alpha,t)$. **(00*)**. Now, if $\gamma \leqslant \alpha$, then clearly
$\gamma 2 \leqslant \alpha 2 < \alpha(+^{i-1})...(+^1)2 \leqslant \eta(i,\alpha,t)$. So suppose $\alpha < \gamma$. Then
$\gamma 2 \underset{\text{because } \gamma \in \mathbb{E}}{\leqslant} m(\gamma) \underset{\text{by proposition 3.17}}{\leqslant} \eta(i,\alpha,\gamma) \underset{\text{by (00*)}}{\leqslant} \eta(i,\alpha,t)$.

Hence $\Delta(i,\alpha,B)\subset\eta(i,\alpha,t)+1$. □

Proposition 6.11. *Let* $i\in[1,n]$, $\alpha\in\mathrm{Class}(i)$, $B\subset_{\mathrm{fin}}\alpha(+^i)$.
If $y\in\Delta(i,\alpha,B)$, *then* $T(i,\alpha,y)\subset\Delta(i,\alpha,B)$.

Proof. Along this proof, we use the same notation as in definition 6.7.

Take $y\in\Delta(i,\alpha,B)$.

Case $y\in\mathbb{E}$.

Subcase 1.1. $y\in Q$.
If $y\in B$, then $T(i,\alpha,y)\subset\Delta(i,\alpha,B)$. If $y\notin B$ then $y\in\bigcup_{x\in B}T(i,\alpha,x)$, that is, $y\in T(i,\alpha,x_0)$ for some $x_0\in B$. Thus $T(i,\alpha,y)\underset{\text{by proposition 5.4}}{\subset}T(i,\alpha,x_0)\subset\Delta(i,\alpha,B)$.

Subcase 1.2.
$y\in W\backslash Q=\bigcup_{\substack{\xi\in Q\cap\mathrm{Class}(j)\\ \xi\in[\alpha,\alpha(+^i))\\ j\in[2,i]}}\{\xi(+^{j-1}),...,\xi(+^{j-1})(+^{j-2})...(+^1),\xi(+^{j-1})(+^{j-2})...(+^1)2\}$. Then,
$y=\xi(+^{j-1})(+^{j-2})...(+^{j-l})$ for some $j\in[2,i]$, $l\in[1,j)$ and
$\xi\in Q\cap\mathrm{Class}(j)\cap[\alpha,\alpha(+^i))$. So $m(y)=\xi(+^{j-1})(+^{j-2})...(+^1)2$ and then
$T(i,\alpha,y)\underset{\text{easy}}{=}\{\xi(+^{j-1}),...,\xi(+^{j-1})...(+^1),\xi(+^{j-1})...(+^1)2\}\cup T(i,\alpha,\xi)\underset{\text{by Subcase 1.1}}{\subset}\Delta(i,\alpha,B)$.

Finally, we make the reader aware that 1.1 and 1.2 are the only possible subcases because by proposition 6.10 *iii.*, $\Delta(i,\alpha,B)\backslash W$ has no epsilon numbers.

Case $y\notin\mathbb{E}$.
Note that in this case $m(y)=y\vee m(y)=\pi y+d\pi y$.

Subcase 2.1. $y\in Q$. Then $y\in B$ because $(Q\backslash B)\subset\mathbb{E}$; so $T(i,\alpha,y)\subset Q\subset\Delta(i,\alpha,B)$.

Subcase 2.2. $y\in W\backslash Q$. Then $y=\xi(+^{j-1})(+^{j-2})...(+^1)2$ for some $j\in[2,i]$ and
$\xi\in Q\cap\mathrm{Class}(j)\cap[\alpha,\alpha(+^i))$. So
$T(i,\alpha,y)\underset{\text{easy}}{=}\{\xi(+^{j-1}),...,\xi(+^{j-1})...(+^1),\xi(+^{j-1})...(+^1)2\}\cup T(i,\alpha,\xi)\underset{\text{by Subcase 1.1}}{\subset}\Delta(i,\alpha,B)$.

Subcase 2.3. $y\in Z\backslash W$.

Subsubcase $y\in\bigcup_{\gamma\in\mathbb{E}\wedge x\in W\cap\mathbb{E}\cap(\alpha,\alpha(+^i))\wedge m(x)\in(\gamma,\gamma(+^1))}D(\gamma,m(x))$.
Then $y\in D(\gamma,m(x_0))$ for some $\gamma\in\mathbb{E}$ and $x_0\in W\cap\mathbb{E}\cap(\alpha,\alpha(+^i))$ with
$m(x_0)\in(\gamma,\gamma(+^1))$. First note that since $\gamma\in D(\gamma,m(x_0))\cap\mathbb{E}\subset\Delta(i,\alpha,B)\cap\mathbb{E}\underset{\text{by prop. 6.10}}{\subset}W$, then by the subcases 1.1 and 1.2, $T(\alpha,i,\gamma)\subset\Delta(i,\alpha,B)$. **(1*)**

Now, in case $y=\gamma 2$, then $T(i,\alpha,y)\underset{\text{easy}}{=}T(i,\alpha,\gamma)\underset{\text{by (1*)}}{\subset}\Delta(i,\alpha,B)$. In case $y\neq\gamma 2$, then, by the definition of the set $D(\gamma,m(x_0))$, y is a cantor normal form constructed only using epsilon numbers appearing in $\mathrm{Ep}(m(x_0))$, and therefore
$T(i,\alpha,y)=\bigcup_{e\in\mathrm{Ep}(y)}T(i,\alpha,e)\subset\bigcup_{e\in\mathrm{Ep}(m(x_0))}T(i,\alpha,e)=T(i,\alpha,m(x_0))\underset{\text{by proposition 5.7}}{\subset}$
$T(i,\alpha,x_0)\underset{\text{because }x_0\in W\text{ and subcases 1.1. and 1.2}}{\subset}\Delta(i,\alpha,B)$

Subsubcase $y \in \bigcup_{\gamma \in \mathbb{E} \wedge x \in W \cap (\alpha, \alpha(+^i)) \cap (\gamma, \gamma(+^1))} D(\gamma, x)$.
Then $y \in D(\gamma, x_0)$ for some $\gamma \in \mathbb{E}$ and $x_0 \in W \cap (\alpha, \alpha(+^i)) \cap (\gamma, \gamma(+^1))$ (note $x_0 \notin \mathbb{E}$ because $x_0 \in (\gamma, \gamma(+^1))$). But $\gamma \in D(\gamma, x_0) \cap \mathbb{E} \subset \Delta(i, \alpha, B) \cap \mathbb{E} \underset{\text{by prop. 6.10}}{\subset} W$, which implies, by the subcases 1.1 and 1.2, $T(\alpha, i, \gamma) \subset \Delta(i, \alpha, B)$. **(2*)**.

Finally, we have two cases: If $y = \gamma 2$, then $T(i, \alpha, y) \underset{\text{easy}}{=} T(i, \alpha, \gamma) \underset{\text{by (2*)}}{\subset} \Delta(i, \alpha, B)$. If $y \neq \gamma 2$, then by definition of $D(\gamma, x_0)$, y is a cantor normal form constructed only using epsilon numbers in $\mathrm{Ep}(x_0)$; this way, $T(i, \alpha, y) = \bigcup_{e \in \mathrm{Ep}(y)} T(i, \alpha, e) \subset \bigcup_{e \in \mathrm{Ep}(x_0)} T(i, \alpha, e) =$
$T(i, \alpha, x_0) \underset{\text{because } x_0 \in W \backslash \mathbb{E} \text{ and subcases 2.1. and 2.2}}{\subset} \Delta(i, \alpha, B)$.

Subcase 2.4. $y \in \Delta(i, \alpha, B) \backslash Z$. Then $y = \gamma 2$ for some $\gamma \in \mathbb{E} \cap Z$. Then
$T(i, \alpha, y) \underset{\text{easy}}{=} T(i, \alpha, \gamma) \underset{\text{by subcases 1.1 and 1.2}}{\subset} \Delta(i, \alpha, B)$. □

Proposition 6.12. *Let* $i \in [1, n]$, $\alpha \in \mathrm{Class}(i)$, $B \subset_{\text{fin}} \alpha(+^i)$. *Then*

a) $B \subset \Delta(i, \alpha, B)$

b) $\forall y \in \Delta(i, \alpha, B) \cap (\alpha, \alpha(+^i))$.
If $y \in \mathbb{E}$ *then* $\mathrm{Ep}(m(y)) \subset \Delta(i, \alpha, B)$;
If $y \notin \mathbb{E}$ *then* $\mathrm{Ep}(m(y)) \subset \mathrm{Ep}(y) \subset \Delta(i, \alpha, B)$.

Proof. In the following proof, we will use the same notation of definition 6.7.
a). Clear.
b). Take $y \in \Delta(i, \alpha, B) \cap (\alpha, \alpha(+^i))$.

Case $y \in \mathbb{E}$.
Then $y \in T(i, \alpha, y)$ and then $\mathrm{Ep}(m(y)) \underset{\text{by definition of } T(i,\alpha,y)}{\subset} T(i, \alpha, y) \underset{\text{prop. 6.11}}{\subset} \Delta(i, \alpha, B)$.

Case $y \notin \mathbb{E}$.
Then $m(y) = y \vee m(y) = \pi y + d\pi y$. So $\mathrm{Ep}(m(y)) \subset \mathrm{Ep}(y) \subset T(i, \alpha, y) \underset{\text{prop. 6.11}}{\subset} \Delta(i, \alpha, B)$. □

Proposition 6.13. *Let* $i \in [1, n]$, $\alpha \in \mathrm{Class}(i)$, $B \subset_{\text{fin}} \alpha(+^i)$.
If $\gamma \in [\alpha, \alpha(+^i)) \cap \mathbb{E}$ *and* $x \in \Delta(i, \alpha, B) \cap (\gamma, \gamma(+^1))$, *then* $D(\gamma, x) \subset \Delta(i, \alpha, B)$.

Proof. Along this proof, we use the same notation as in definition 6.7.

Take i, α and B as stated. Let $\gamma \in [\alpha, \alpha(+^i)) \cap \mathbb{E}$ and $x \in \Delta(i, \alpha, B) \cap (\gamma, \gamma(+^1))$. Then $x \notin \mathbb{E}$ and this leaves us the following alternatives:

- $x \in B \subset W$. Then $D(\gamma, x) \subset Z = \Delta(i, \alpha, B)$ by the definition of Z.

- $x = \xi(+^{j-1})(+^{j-2})...(+^1)2$, where $\{\xi(+^{j-1}), ..., \xi(+^{j-1})(+^{j-2})...(+^1), \xi(+^{j-1})(+^{j-2})...(+^1)2\} \subset W$ for some $\xi \in Q \cap \mathrm{Class}(j) \cap [\alpha, \alpha(+^i))$ with $j \in [2, i]$. Then $\gamma = \xi(+^{j-1})(+^{j-2})...(+^1)$ and $D(\gamma, x) = \{\xi(+^{j-1})...(+^1), \xi(+^{j-1})...(+^1)2\} \subset W \subset \Delta(i, \alpha, B)$.

- $x \in D(\delta, m(y))$ for some $\delta \in \mathbb{E} \wedge y \in W \cap \mathbb{E} \cap (\alpha, \alpha(+^i)) \wedge m(y) \in (\delta, \delta(+^1))$. If $x = \delta 2$, then $\gamma = \delta$ and $D(\gamma, x) = \{\gamma, \gamma 2\} = \{\delta, \delta 2\} \subset D(\delta, m(y)) \subset \Delta(i, \alpha, B)$. In case $x \neq \delta 2$, then $x \in C(m(y))$ ($x \neq \delta$ because $x \notin \mathbb{E}$) and then:

- $C(x) \underset{\text{prop. 2.29}}{\subset} C(m(y)) \subset D(\delta, m(y)) \subset \Delta(i, \alpha, B)$; **(0*)**

• By the definition of $C(m(y))$, x is just a cantor normal form constructed only using epsilon numbers appearing in $\mathrm{Ep}(m(y))$. This means $\gamma \in \mathrm{Ep}(x) \subset \mathrm{Ep}(m(y)) \underset{\text{prop. 6.12}}{\subset} \Delta(i,\alpha,B)$; note this actually means that $\gamma \in Z$, since $\Delta(i,\alpha,B) = Z \cup \{\rho 2 \mid \rho \in \mathbb{E} \cap Z\}$. Therefore, by the definition of $\Delta(i,\alpha,B)$, $\{\gamma,\gamma(2)\} \subset \Delta(i,\alpha,B)$. **(1*)**

Finally, from (0*) and (1*) we conclude $D(\gamma,x) = \{\gamma,\gamma 2\} \cup C(x) \subset \Delta(i,\alpha,B)$.

- $x \in D(\delta, y)$ for some $\delta \in \mathbb{E} \wedge y \in W \cap (\alpha, \alpha(+^i)) \cap (\delta, \delta(+^1))$. Then arguing exactly as in the previous case we get $D(\gamma,x) \subset \Delta(i,\alpha,B)$.

- $x = \gamma 2$ for $\gamma \in Z \cap \mathbb{E}$. Then $D(\gamma,x) = \{\gamma,\gamma 2\} \subset Z \cup \{\gamma 2\} \subset \Delta(i,\alpha,B)$. $\square$

Proposition 6.14. *Let it be $k, i \in [1,n]$, $k \leqslant i$, $\alpha \in \mathrm{Class}(i)$ and $y \in [\alpha, \alpha(+^i)) \cap \mathrm{Class}(k)$. Suppose $A \subset_{\mathrm{fin}} y(+^k)$ and $B \subset_{\mathrm{fin}} \alpha(+^i)$ are two finite sets such that $A \subset \Delta(i,\alpha,B)$. Then $\Delta(k,y,A) \subset \Delta(i,\alpha,B)$.*

Proof. Using the same notation as in definition 6.7, $\Delta(i,\alpha,B) = Z \cup \{\gamma 2 \mid \gamma \in \mathbb{E} \cap Z\}$, where

$$Q = B \cup \bigcup_{x \in B} T(i,\alpha,x),$$

$$W = Q \cup \bigcup_{\substack{\xi \in Q \cap \mathrm{Class}(j) \cap [\alpha,\alpha(+^i)) \\ j \in [2,i]}} \{\xi(+^{j-1}), \ldots, \xi(+^{j-1})(+^{j-2})\cdots(+^1), \xi(+^{j-1})(+^{j-2})\cdots(+^1)2\},$$

and

$$Z = W \cup \bigcup_{\gamma \in \mathbb{E} \wedge x \in W \cap \mathbb{E} \cap (\alpha,\alpha(+^i)) \wedge m(x) \in (\gamma,\gamma(+^1))} D(\gamma, m(x)) \cup \bigcup_{\gamma \in \mathbb{E} \wedge x \in W \cap (\alpha,\alpha(+^i)) \cap (\gamma,\gamma(+^1))} D(\gamma,x)$$

Similarly, $\Delta(k,y,A) := Z_2 \cup \{\gamma 2 \mid \gamma \in \mathbb{E} \cap Z_2\}$, where we define

$$Q_2 := A \cup \bigcup_{x \in A} T(k,y,x),$$

$$W_2 := Q_2 \cup \bigcup_{\substack{\xi \in Q_2 \cap \mathrm{Class}(j) \cap [y,y(+^k)) \\ j \in [2,k]}} \{\xi(+^{j-1}), \ldots, \xi(+^{j-1})\cdots(+^1), \xi(+^{j-1})(+^{j-2})\cdots(+^1)2\}$$

and

$$Z_2 := W_2 \cup \bigcup_{\gamma \in \mathbb{E} \wedge x \in W_2 \cap \mathbb{E} \cap (y,y(+^k)) \wedge m(x) \in (\gamma,\gamma(+^1))} D(\gamma, m(x)) \cup \bigcup_{\gamma \in \mathbb{E} \wedge x \in W_2 \cap (y,y(+^k)) \cap (\gamma,\gamma(+^1))} D(\gamma,x)$$

First note $Q_2 = A \cup \bigcup_{x \in A} T(k, y, x) \underset{\text{clause 2 of prop. 5.4}}{\subset} A \cup \bigcup_{x \in A} T(i, \alpha, x) \underset{\text{prop. 6.11}}{\subset} \Delta(i, \alpha, B)$. **(1*)**

Now we show that $W_2 \subset \Delta(i,\alpha,B)$. **(2*)**

Let $\beta \in W_2$ be arbitrary.

If $\beta \in Q_2$, then by (1*) we know $\beta \in \Delta(i,\alpha,B)$. So suppose $\beta \in W_2 \backslash Q_2$. Then $\beta \in \{\xi(+^{j-1}), \ldots, \xi(+^{j-1})(+^{j-2})\cdots(+^1), \xi(+^{j-1})(+^{j-2})\cdots(+^1)2\}$ for some $\xi \in Q_2 \cap \mathrm{Class}(j) \cap [y, y(+^k))$ and some $j \in [2, k]$. This way, by (1*) and clause *iii* of proposition 6.10, we have that $\xi \in W$ and subsequently we need to consider two subcases:

Subcase $\xi \in Q$. Then $\beta \in \{\xi(+^{j-1}), \ldots, \xi(+^{j-1})\cdots(+^1), \xi(+^{j-1})\cdots(+^1)2\} \subset W \subset \Delta(i,\alpha,B)$.

Subcase $\xi\notin Q$. Then $\xi\in\{\rho(+^{l-1}),...,\rho(+^{l-1})(+^{l-2})\cdots(+^1),\rho(+^{l-1})(+^{l-2})\cdots(+^1)2\}$ for some $\rho\in Q\cap\mathrm{Class}(l)\cap[\alpha,\alpha(+^i))$ and some $l\in[2,i]$, and therefore
$\beta\in\{\xi(+^{j-1}),...,\xi(+^{j-1})(+^{j-2})\cdots(+^1),\xi(+^{j-1})(+^{j-2})\cdots(+^1)2\}\subset$
$\{\rho(+^{l-1}),...,\rho(+^{l-1})(+^{l-2})\cdots(+^1),\rho(+^{l-1})(+^{l-2})\cdots(+^1)2\}\subset W\subset\Delta(i,\alpha,B)$.

The previous shows that, in any case, $\beta\in\Delta(i,\alpha,B)$ and since we proved this for arbitrary $\beta\in W_2$, then we have shown (2*).

To show $Z_2\subset\Delta(i,\alpha,B)$. **(3*)**

First note that for $\gamma,x\in\mathrm{OR}$, if $\gamma\in\mathbb{E}\wedge x\in W_2\cap\mathbb{E}\cap(y,y(+^k))\wedge m(x)\in(\gamma,\gamma(+^1))$, then by (2*) and clause *iii* of proposition 6.10, $\gamma\in\mathbb{E}\wedge x\in W\cap\mathbb{E}\cap(\alpha,\alpha(+^i))\wedge m(x)\in(\gamma,\gamma(+^1))$. From this follows $\bigcup_{\gamma\in\mathbb{E}\wedge x\in W_2\cap\mathbb{E}\cap(y,y(+^k))\wedge m(x)\in(\gamma,\gamma(+^1))} D(\gamma,m(x))\subset Z\subset\Delta(i,\alpha,B)$. **(4*)**

On the other hand, for any $\gamma,x\in\mathrm{OR}$, if $\gamma\in\mathbb{E}\wedge x\in W_2\cap(y,y(+^k))\cap(\gamma,\gamma(+^1))$, then using (2*) we have that $\gamma\in[\alpha,\alpha(+^i))\cap\mathbb{E}\wedge x\in\Delta(i,\alpha,B)\cap(\gamma,\gamma(+^1))$ and then, by proposition 6.13, we have that $D(\gamma,x)\subset\Delta(i,\alpha,B)$. From this follows
$\bigcup_{\gamma\in\mathbb{E}\wedge x\in W_2\cap(y,y(+^k))\cap(\gamma,\gamma(+^1))} D(\gamma,x)\subset\Delta(i,\alpha,B)$ **(5*)**.

Hence, (2*), (4*) and (5*) prove (3*).

Concluding the proof of our theorem: $\Delta(k,y,A)=Z_2\cup\{\gamma 2\mid\gamma\in\mathbb{E}\cap Z_2\}\underset{\text{by (3*)}}{\subset}$
$\Delta(i,\alpha,B)\cup\{\gamma 2\mid\gamma\in\Delta(i,\alpha,B)\cap\mathbb{E}\}\underset{\text{by clause }iii\text{ of prop. 6.10}}{=}$
$\Delta(i,\alpha,B)\cup\{\gamma 2\mid\gamma\in W\cap\mathbb{E}\}\subset\Delta(i,\alpha,B)\cup\{\gamma 2\mid\gamma\in Z\cap\mathbb{E}\}\subset\Delta(i,\alpha,B)$. □

Proposition 6.15. *Let $i\in[1,n]$ and $\alpha,\beta\in\mathrm{Class}(i)$.*

a) $\forall\delta\in\mathrm{OR}.C(\delta)\cap\mathbb{E}\subset\mathrm{Dom}\,g(i,\alpha,\beta)\Longrightarrow C(\delta[g(i,\alpha,\beta)])=\{z[g(i,\alpha,\beta)]\mid z\in C(\delta)\}$.

b) $\forall\gamma\in\mathbb{E}.\forall\delta\in[\gamma,\gamma(+^1)).D(\gamma,\delta)\cap\mathbb{E}\subset\mathrm{Dom}\,g(i,\alpha,\beta)\Longrightarrow$
$D(\gamma[g(i,\alpha,\beta)],\delta[g(i,\alpha,\beta)])=\{z[g(i,\alpha,\beta)]\mid z\in D(\gamma,\delta)\}$.

Proof. In this proof we use the same notation as in definition 2.28.

Let i,α,β be as stated.

a)

By induction on $(\mathrm{OR},<)$ we will show
$\forall\delta\in\mathrm{OR}.C(\delta)\cap\mathbb{E}\subset\mathrm{Dom}\,g(i,\alpha,\beta)\Longrightarrow C(\delta[g(i,\alpha,\beta)])=\{z[g(i,\alpha,\beta)]|z\in C(\delta)\}$ **(*)**

Take $\delta\in\mathrm{OR}$ with $C(\delta)\cap\mathbb{E}\subset\mathrm{Dom}\,g(i,\alpha,\beta)$.
Assume (*) holds for any $\sigma\in\delta$. **(IH)**

Consider $\delta=_{\mathrm{CNF}}L_1l_1+...+L_ml_m$.

Note $\delta[g(i,\alpha,\beta)]=_{\mathrm{CNF}}L_1[g(i,\alpha,\beta)]l_1+...+L_m[g(i,\alpha,\beta)]l_m$ **(0*)** and therefore
$Y(\delta)=\{V_{kj}|\exists L_k\notin\mathbb{E}.L_k=\omega^Z\wedge Z=_{\mathrm{CNF}}\sum_{j=1}^{t(k)}\omega^{V_{kj}}v_{kj}\}\Longleftrightarrow$
$Y(\delta[g(i,\alpha,\beta)])=\{V'_{kj}|\exists L_k[g(i,\alpha,\beta)]\notin\mathbb{E}.L_k[g(i,\alpha,\beta)]=\omega^{Z'}\wedge Z'=_{\mathrm{CNF}}\sum_{j=1}^{t(k)}\omega^{V_{kj}}v_{kj}\}=$
$\{V_{kj}[g(i,\alpha,\beta)]|\exists L_k\notin\mathbb{E}.L_k=\omega^Z\wedge Z=_{\mathrm{CNF}}\sum_{j=1}^{t(k)}\omega^{V_{kj}}v_{kj}\}=$
$\{V[g(i,\alpha,\beta)]\mid V\in Y(\delta)\}$. **(1*)**

From the fact that $Y(\delta)\subset\delta$, our (IH) and (1*) we get
$\bigcup_{V'\in Y(\delta[g(i,\alpha,\beta)])}C(V')=\bigcup_{V\in Y(\delta)}C(V[g(i,\alpha,\beta)])=\bigcup_{V\in Y(\delta)}\{z[g(i,\alpha,\beta)]\,|z\in C(V)\}=$

$$= \{z[g(i,\alpha,\beta)] \mid z \in \bigcup_{V \in Y(\delta)} C(V)\} = . \qquad \textbf{(a*)}$$

On the other hand, for any L_k, $L_k \notin \mathbb{E} \Longleftrightarrow L_k[g(i,\alpha,\beta)] \notin \mathbb{E}$; **(2*)**
moreover, in case $L_k \notin \mathbb{E}$ and $L_k = \omega^Z \wedge Z =_{\mathrm{CNF}} \sum_{j=1}^{t(k)} \omega^{V_{kj}} v_{kj}$,
$F(L_k) = \{\omega^{\omega^{V_{k1}} v_{k1} + \omega^{V_{k2}} v_{k2} + \ldots + \omega^{V_{ks}} \cdot j} \mid s \in [1,t(k)], j \in [1, v_{ks}]\} \cup$
$\{\omega^{\omega^{V_{k1}} v_{k1} + \omega^{V_{k2}} v_{k2} + \ldots + \omega^{V_{ks}} \cdot j} + V_{ks} \mid s \in [1,t(k)], j \in [1, v_{ks}]\}$
$\Longleftrightarrow$
$F(L_k[g(i,\alpha,\beta)]) = \{\omega^{\omega^{V_{k1}[g(i,\alpha,\beta)]} v_{k1} + \omega^{V_{k2}[g(i,\alpha,\beta)]} v_{k2} + \ldots + \omega^{V_{ks}[g(i,\alpha,\beta)]} \cdot j} \mid s \in [1,t(k)], j \in [1, v_{ks}]\} \cup$
$\{\omega^{\omega^{V_{k1}[g(i,\alpha,\beta)]} v_{k1} + \ldots + \omega^{V_{ks}[g(i,\alpha,\beta)]} \cdot j} + V_{ks}[g(i,\alpha,\beta)] \mid s \in [1,t(k)], j \in [1, v_{ks}]\} =$
$= \{z[g(i,\alpha,\beta)] \mid z \in F(L_k)\}$. **(3*)**

From (0*), (2*) and (3*) we get
$C_1(\delta[g(i,\alpha,\beta)]) = \bigcup_{L_k[g(i,\alpha,\beta)] \notin \mathbb{E}} F(L_k[g(i,\alpha,\beta)]) = \bigcup_{L_k \notin \mathbb{E}} \{z[g(i,\alpha,\beta)] \mid z \in F(L_k)\} =$

$= \{z[g(i,\alpha,\beta)] \mid z \in \bigcup_{L_k \notin \mathbb{E}} F(L_k)\} = \{z[g(i,\alpha,\beta)] \mid z \in C_1(\delta)\}$. **(b*)**.
Moreover, from (0*) we also get
$C_2(\delta[g(i,\alpha,\beta)]) = \{L_k[g(i,\alpha,\beta)] j \mid k \in [1,m], j \in [1,l_k]\} \cup \{\Sigma_{k=1}^{j} L_k[g(i,\alpha,\beta)] l_k \mid j \in [1,m]\} =$
$= \{z[g(i,\alpha,\beta)] \mid z \in \{L_k j \mid k \in [1,m], j \in [1,l_k]\} \cup \{\Sigma_{k=1}^{j} L_k l_k \mid j \in [1,m]\}\} =$
$= \{z[g(i,\alpha,\beta)] \mid z \in C_2(\delta)\}$. **(c*)**

On the other hand,
$\bigcup_{\sigma \in C_1(\delta[g(i,\alpha,\beta)])} C_2(\sigma) \underset{\text{by (b*)}}{=} \bigcup_{\sigma \in C_1(\delta)} C_2(\sigma[g(i,\alpha,\beta)]) \underset{\text{same reasoning as in (c*)}}{=}$

$$= \bigcup_{\sigma \in C_1(\delta)} \{z[g(i,\alpha,\beta)] \mid z \in C_2(\sigma)\} =$$

$$= \{z[g(i,\alpha,\beta)] \mid \bigcup_{\sigma \in C_1(\delta)} z \in C_2(\sigma)\}. \qquad \textbf{(d*)}$$

Finally, to conclude our proof, from (b*), (d*), (c*) and (a*)
$C(\delta[g(i,\alpha,\beta)]) =$
$C_1(\delta[g(i,\alpha,\beta)]) \cup \bigcup_{\sigma \in C_1(\delta[g(i,\alpha,\beta)])} C_2(\sigma) \cup C_2(\delta[g(i,\alpha,\beta)]) \cup \bigcup_{V \in Y(\delta[g(i,\alpha,\beta)])} C(V) =$
$\{z[g(i,\alpha,\beta)] \mid z \in C_1(\delta)\} \cup \{z[g(i,\alpha,\beta)] \mid \bigcup_{\sigma \in C_1(\delta)} z \in C_2(\sigma)\} \cup \{z[g(i,\alpha,\beta)] \mid z \in C_2(\delta)\} \cup$
$\{z[g(i,\alpha,\beta)] \mid z \in \bigcup_{V \in Y(\delta)} C(V)\} =$
$\{z[g(i,\alpha,\beta)] \mid z \in C_1(\delta) \cup \bigcup_{\sigma \in C_1(\delta)} z \in C_2(\sigma) \cup C_2(\delta) \cup \bigcup_{V \in Y(\delta)} C(V)\} = \{z[g(i,\alpha,\beta)] \mid z \in C(\delta)\}$.

b)
Let $\gamma \in \mathbb{E}$, $\delta \in [\gamma, \gamma(+^1))$ and suppose $D(\gamma,\delta) \cap \mathbb{E} \subset \mathrm{Dom}\, g(i,\alpha,\beta)$. Then
$\gamma[g(i,\alpha,\beta)] \in \mathbb{E}$, $\delta[g(i,\alpha,\beta)] \in [\gamma[g(i,\alpha,\beta)], \gamma[g(i,\alpha,\beta)](+^1))$ and
$D(\gamma[g(i,\alpha,\beta)], \delta[g(i,\alpha,\beta)]) = \{\gamma[g(i,\alpha,\beta)], \gamma[g(i,\alpha,\beta)]2\} \cup C(\delta[g(i,\alpha,\beta)]) \underset{\text{by } a)}{=}$
$\{z[g(i,\alpha,\beta)] \mid z \in \{\gamma, \gamma 2\} \cup C(\delta)\} = \{z[g(i,\alpha,\beta)] \mid D(\gamma,\delta)\}$. $\square$

6.3.2 Generalized covering lemma.

Lemma 6.16. *(Generalized covering lemma).*
$\forall i \forall \alpha \forall B \forall h$.
If $i \in [1,n] \wedge \alpha \in \mathrm{Class}(i) \wedge B \subset_{\mathrm{fin}} \alpha(+^i) \wedge B \cap [\alpha, \alpha(+^i)) \neq \emptyset$, *then*

if

- $h: \Delta(i, \alpha, B) \longrightarrow h[\Delta(i, \alpha, B)] \subset \alpha$ *is a function that is an* $(<, +)$*-isomorphism,*
- $h|_{\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))}$ *is an* $<_1$*-isomorphism onto* $h[\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))] \subset \alpha$*,*
- $\alpha <_1 \alpha(+^{i-1}) <_1 ... <_1 \alpha(+^{i-1})...(+^1) <_1 \alpha(+^{i-1})...(+^1)2 \Longleftrightarrow$ $h(\alpha) <_1 h(\alpha(+^{i-1})) <_1 ... <_1 h(\alpha(+^{i-1})...(+^1)) <_1 h(\alpha(+^{i-1})...(+^1)2)$*, and*
- $h|_\alpha \geqslant \mathrm{Id}_\alpha$

then:

1. $\forall l \in [1, i].\forall \sigma \in \Delta(i, \alpha, B) \cap \mathrm{Class}(l) \cap [\alpha, \alpha(+^i)).h(\sigma) \in \mathrm{Class}(l) \cap \alpha$. *In particular,* $h(\alpha) \in \mathrm{Class}(i) \cap \alpha$*;*
2. $\forall x \in \Delta(i, \alpha, B).T(i, \alpha, x) \cap \alpha \subset h(\alpha) \wedge x[g(i, \alpha, h(\alpha))] \leqslant h(x)$.

Proof. By induction on $[1, n]$.

Let $i \in [1, n]$.
Suppose the claim holds for any $l \in [1, i)$. **(IH)**

Let $\alpha \in \mathrm{Class}(i)$, $B \subset_{\text{fin}} \alpha(+^i)$ with $B \cap [\alpha, \alpha(+^i)) \neq \emptyset$ and $h: \Delta(i, \alpha, B) \longrightarrow h[\Delta(i, \alpha, B)] \subset \alpha$ be a function accomplishing the hypothesis of the lemma.

1.
Let $l \in [1, i]$ and $\sigma \in \Delta(i, \alpha, B) \cap \mathrm{Class}(l) \cap [\alpha, \alpha(+^i))$ be arbitrary. With the abbreviations
$r(\sigma, l) := \sigma$,
$r(\sigma, l-1) := \sigma(+^{l-1})$,
$r(\sigma, l-2) := \sigma(+^{l-1})(+^{l-2})$,
...
$r(\sigma, 2) := \sigma(+^{l-1})(+^{l-2})...(+^2)$,
$r(\sigma, 1) := \sigma(+^{l-1})(+^{l-2})...(+^2)(+^1)$,
we have that $\sigma <_1 r(\sigma, i-1) <_1 r(\sigma, i-2) <_1 ... <_1 r(\sigma, 1) <_1 r(\sigma, 1)2$ and therefore $h(\sigma) <_1 h(r(\sigma, i-1)) <_1 h(r(\sigma, i-2)) <_1 ... <_1 h(r(\sigma, 1)) <_1 h(r(\sigma, 1)2) = h(r(\sigma, 1))2$. From this and proposition 3.5, we have $h(\sigma) \in \mathrm{Class}(l) \cap \alpha$.

2.
We first show $\forall x \in \Delta(i, \alpha, B).T(i, \alpha, x) \cap \alpha \subset h(\alpha)$. **(B1)**

Let $x \in \Delta(i, \alpha, B)$. Take $y \in T(i, \alpha, x) \cap \alpha$ arbitrary. Then by proposition 6.11, $y \in \Delta(i, \alpha, B) \cap \alpha$ and so $y \underset{\text{By hypothesis } \mathrm{Id}_\alpha \leqslant h|_\alpha}{\leqslant} h(y) < h(\alpha)$. Since this was done for arbitrary $y \in T(i, \alpha, x) \cap \alpha$, then $T(i, \alpha, x) \cap \alpha \subset h(\alpha)$.
This proves (B1).

Now we show by a (side)induction on $(\Delta(i, \alpha, B), <)$, that
$\forall x \in \Delta(i, \alpha, B).x[g(i, \alpha, h(\alpha))] \leqslant h(x)$. **(B2)**

Let $y \in \Delta(i, \alpha, B)$ and suppose $\forall x \in \Delta(i, \alpha, B) \cap y.x[g(i, \alpha, h(\alpha))] \leqslant h(x)$. **(SIH)**

Case $y < \alpha$. Then clearly $y[g(i, \alpha, h(\alpha))] = y \underset{\text{By hypothesis } \mathrm{Id}_\alpha \leqslant h|_\alpha}{\leqslant} h(y)$.

Case $y = \alpha$. Then clearly $y[g(i, \alpha, h(\alpha))] = h(\alpha) = h(y)$.

Case $y > \alpha$.

- Subcase $y \notin \mathbb{P}$.

Then $y =_{\mathrm{CNF}} \omega^{A_1} a_1 + \ldots + \omega^{A_m} a_m$ with $m \geqslant 2$ or $a_1 \geqslant 2$.
Now, because of our definition of $\Delta(i,\alpha,B)$, we can apply h to the subterms of y as follows:
If $m \geqslant 2$, then $h(y) = h(\omega^{A_1} a_1) + \ldots + h(\omega^{A_m} a_m) \underset{(\mathrm{SIH})}{\geqslant}$

$$\geqslant \omega^{A_1} a_1[g(i,\alpha,h(\alpha))] + \ldots + \omega^{A_m} a_m[g(i,\alpha,h(\alpha))] = \\ = (\omega^{A_1} a_1 + \ldots + \omega^{A_m} a_m)[g(i,\alpha,h(\alpha))] = y[g(i,\alpha,h(\alpha))].$$

If $m = 1$ and $a_1 \geqslant 2$, then $h(y) = h(\omega^{A_1} a_1) = h(\omega^{A_1}) a_1 \underset{(\mathrm{SIH})}{\geqslant} \omega^{A_1}[g(i,\alpha,h(\alpha))] a_1 =$

$$= (\omega^{A_1} a_1)[g(i,\alpha,h(\alpha))] = y[g(i,\alpha,h(\alpha))].$$

- **Subcase** $y \in \mathbb{P} \backslash \mathbb{E}$.

Then $y = \omega^Z$ with $Z =_{\mathrm{CNF}} \omega^{R_1} r_1 + \ldots + \omega^{R_k} r_k$, $y = \omega^Z > R_1 > \ldots > R_k$ and $y \in (\beta, \beta(+^1))$ for some $\beta \in \cap [\alpha, \alpha(+^i)) \cap \mathbb{E} \cap y$. Then, by proposition 6.13, $D(\beta, y) \subset \Delta(i,\alpha,B)$ and therefore, carrying out the same procedure as in clause *iii* of lemma 2.30 with y, Z, R_1, ...R_k and β we get: $h(y) = h(\omega^Z) = h(\omega^{\omega^{R_1} r_1 + \ldots + \omega^{R_k} r_k}) \geqslant \omega^{\omega^{h(R_1)} r_1 + \ldots + \omega^{h(R_k)} r_k}$ **(B3)**; but

$$\omega^{\omega^{h(R_1)} r_1 + \ldots + \omega^{h(R_k)} r_k} \underset{\text{by our (SIH)}}{\geqslant} \omega^{\omega^{R_1[g(i,\alpha,h(\alpha))]} r_1 + \ldots + \omega^{R_k[g(i,\alpha,h(\alpha))]} r_k} = \\ = \left(\omega^{\omega^{R_1} r_1 + \ldots + \omega^{R_k} r_k}\right)[g(i,\alpha,h(\alpha))] = y[g(i,\alpha,h(\alpha))] \qquad \textbf{(B4)}$$

Thus, from (B3) and (B4) follow $h(y) \geqslant y[g(i,\alpha,h(\alpha))]$.

- **Subcase** $y \in \mathbb{E}$.

Then $y \in \mathrm{Class}(k) \backslash \mathrm{Class}(k+1)$, for some $k \in [1, i-1]$ (because $y \in (\alpha, \alpha(+^i))$) and therefore $y \in (\beta, \beta(+^{k+1}))$ for $\beta := \lambda(k+1, y) \in [\alpha, \alpha(+^i)) \cap \Delta(i,\alpha,B) \cap \mathrm{Class}(k+1) \cap y$.

Note that by 1., $h(\beta) \in \mathrm{Class}(k+1)$; besides, by our (SIH), $h(\beta) \geqslant \beta[g(i,\alpha,h(\alpha))]$. Moreover, we also now $\beta[g(i,\alpha,h(\alpha))] \in \mathrm{Class}(k+1)$. **(B5)**

From (B5) we get some cases:

Subsubcase $h(\beta) > \beta[g(i,\alpha,h(\alpha))]$.

Then $\beta[g(i,\alpha,h(\alpha))](+^{k+1}) \leqslant h(\beta)$. So
$y[g(i,\alpha,h(\alpha))] < \beta[g(i,\alpha,h(\alpha))](+^{k+1}) \leqslant h(\beta) < h(y)$.

Subsubcase $h(\beta) = \beta[g(i,\alpha,h(\alpha))]$.

Subsubsubcase $h(y) \geqslant \beta[g(i,\alpha,h(\alpha))](+^{k+1})$.

Then $y[g(i,\alpha,h(\alpha))] < \beta[g(i,\alpha,h(\alpha))](+^{k+1}) \leqslant h(y)$

Subsubsubcase $h(y) < \beta[g(i,\alpha,h(\alpha))](+^{k+1})$.

Then $\beta[g(i,\alpha,h(\alpha))] = h(\beta) < h(y) < h(\beta)(+^{k+1}) = \beta[g(i,\alpha,h(\alpha))](+^{k+1})$, which means $h(y) \in \mathrm{Class}(k) \backslash \mathrm{Class}(k+1)$ and so $m(h(y)) \in (h(y), h(y)(+^k))$. **(B6)**

Now, consider $f(k+1,\beta)(y) = \{y = y_1 > \ldots > y_q\} \subset \mathrm{Class}(k) \cap (\beta, \beta(+^{k+1})) \subset (\alpha, \alpha(+^i))$. Then $h(\beta) < h(y_q) < \ldots < h(y_2) < h(y)$, $h(y_q), \ldots, h(y_2), h(y) \underset{\text{by } 2.1}{\in} \mathrm{Class}(k)$ and by our (SIH) $h(y_2) \geqslant y_2[g(i,\alpha,h(\alpha))]$. **(B7)**

On the other hand, from the properties of the function $g(i,\alpha,h(\alpha))$, it follow
$f(k+1, \beta[g(i,\alpha,h(\alpha))])(y[g(i,\alpha,h(\alpha))]) = \{y_1[g(i,\alpha,h(\alpha))] > \ldots > y_q[g(i,\alpha,h(\alpha))]\}$ **(B8)**
and so

$$y[g(i,\alpha,h(\alpha))] = g(i,\alpha,h(\alpha))(y) = \\ = \min\{s \in (\beta[g(i,\alpha,h(\alpha))], \beta[g(i,\alpha,h(\alpha))](+^{k+1})) \cap \mathrm{Class}(k) | \\ y_2[g(i,\alpha,h(\alpha))] < s \leqslant y[g(i,\alpha,h(\alpha))] \wedge \\ m(s)[g(k,s,y[g(i,\alpha,h(\alpha))])] \geqslant m(y[g(i,\alpha,h(\alpha))])\}. \qquad \textbf{(B9)}$$

We now show that $h(y) \geqslant y[g(i,\alpha,h(\alpha))]$ by contradiction. **(B10)**.

Assume the opposite $h(y) < y[g(i,\alpha,h(\alpha))]$. **(***)**

Then by (B7) $y_2[g(i,\alpha,h(\alpha))] \leqslant h(y_2) < h(y) < y[g(i,\alpha,h(\alpha))]$, which implies by (B6) and (B9) that $m(h(y))[g(k,h(y),y[g(i,\alpha,h(\alpha))])] < m(y[g(i,\alpha,h(\alpha))])$ **(B11)**.

Now, let $P := \mathrm{Ep}(m(y[g(i,\alpha,h(\alpha))])) \subset_{\text{fin}} y[g(i,\alpha,h(\alpha))](+^k)$ and let's abbreviate $\mathrm{Int} := [y[g(i,\alpha,h(\alpha))], y[g(i,\alpha,h(\alpha))](+^k))$ and $\mathrm{Int}^\circ = (y[g(i,\alpha,h(\alpha))], y[g(i,\alpha,h(\alpha))](+^k))$. Note $P \cap \mathrm{Int} \neq \emptyset$ (and remember $k < i$). **(C1)**

On the other hand, by clause b) of proposition 6.12, $\mathrm{Ep}(m(y)) \subset \Delta(i,\alpha,B)$ and then, by proposition 6.14, $\Delta(k,y,\mathrm{Ep}(m(y))) \subset \Delta(i,\alpha,B) \underset{\text{by (B1)}}{\subset} \mathrm{Dom}\ g(i,\alpha,h(\alpha))$, that is, we can apply the transformation $x \longmapsto x[g(i,\alpha,h(\alpha))]$ on the elements of $\Delta(k,y,\mathrm{Ep}(m(y)))$ without problems.

We assure $\Delta(k,y[g(i,\alpha,h(\alpha))],P) = \{z[g(i,\alpha,h(\alpha))] \mid z \in \Delta(k,y,\mathrm{Ep}(m(y)))\}$ **(C2)**

Proof of C2:
By definition
$\Delta(k,y[g(i,\alpha,h(\alpha))],P) = \Upsilon_2 \cup \{\gamma 2 \mid \gamma \in \mathbb{E} \cap \Upsilon_2\}$, where
$\Upsilon := P \cup \bigcup_{x \in P} T(k,y[g(i,\alpha,h(\alpha))],x)$,
$$\Upsilon_1 := \Upsilon \cup \bigcup_{\substack{\xi \in \Upsilon \cap \mathrm{Class}(j) \cap \mathrm{Int} \\ j \in [2,k]}} \{\xi(+^{j-1}),\ldots,\xi(+^{j-1})(+^{j-2})\ldots(+^1),\xi(+^{j-1})(+^{j-2})\ldots(+^1)2\}$$
and
$$\Upsilon_2 := \Upsilon_1 \cup \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Upsilon_1 \cap \mathbb{E} \\ x \in (y[g(i,\alpha,h(\alpha))],y[g(i,\alpha,h(\alpha))](+^k)) \\ m(x) \in (\gamma,\gamma(+^1))}} D(\gamma,m(x)) \cup \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Upsilon_1 \cap \mathbb{E} \\ x \in (y[g(i,\alpha,h(\alpha))],y[g(i,\alpha,h(\alpha))](+^k)) \\ x \in (\gamma,\gamma(+^1))}} D(\gamma,x)$$
and

$\Delta(k,y,\mathrm{Ep}(m(y))) = \Omega_2 \cup \{\gamma 2 \mid \gamma \in \mathbb{E} \cap \Omega_2\}$, where
$\Omega := \mathrm{Ep}(m(y)) \cup \bigcup_{z \in \mathrm{Ep}(m(y))} T(k,y,z)$,
$$\Omega_1 := \Omega \cup \bigcup_{\substack{\xi \in \Omega \cap \mathrm{Class}(j) \cap [y,y(+^k)) \\ j \in [2,k]}} \{\xi(+^{j-1}),\ldots,\xi(+^{j-1})(+^{j-2})\ldots(+^1),\xi(+^{j-1})(+^{j-2})\ldots(+^1)2\}$$
and
$$\Omega_2 := \Omega_1 \cup \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Omega_1 \cap \mathbb{E} \\ x \in (y,y(+^k)) \\ m(x) \in (\gamma,\gamma(+^1))}} D(\gamma,m(x)) \cup \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Omega_1 \cap \mathbb{E} \\ x \in (y,y(+^k)) \\ x \in (\gamma,\gamma(+^1))}} D(\gamma,x)$$

First note that $P = \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\}$ **(C3)**, because $m(y[g(i,\alpha,h(\alpha))]) = m(y)[g(i,\alpha,h(\alpha))]$.

Now we show that $\Upsilon = \{z[g(i,\alpha,h(\alpha))] \mid z \in \Omega\}$. **(C4)**
Observe
$$\Upsilon = P \cup \bigcup_{x \in P} T(k,y[g(i,\alpha,h(\alpha))],x) \underset{\text{by (C3)}}{=}$$
$$= \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\} \cup \bigcup_{x \in P} T(k,y[g(i,\alpha,h(\alpha))],x) =$$

$= \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\} \cup \bigcup_{x \in \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\}} T(k, y[g(i,\alpha,h(\alpha))], x) =$
$= \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\} \cup \bigcup_{z \in \mathrm{Ep}(m(y))} T(k, y[g(i,\alpha,h(\alpha))], z[g(i,\alpha,h(\alpha))]) \underset{\text{by prop. 5.18}}{=}$
$= \{z[g(i,\alpha,h(\alpha))] \mid z \in \mathrm{Ep}(m(y))\} \cup \bigcup_{z \in \mathrm{Ep}(m(y))} \{l[g(i,\alpha,h(\alpha))] \mid l \in T(k,y,z)\} =$
$= \{z[g(i,\alpha,h(\alpha))] \mid z \in \Omega\}.$
This shows (C4).

To show that $\Upsilon_1 = \{z[g(i,\alpha,h(\alpha))] \mid z \in \Omega_1\}$. **(C5)**
Because of (C4), (C5) holds in case we show the following equality **(C5')**:

$$\bigcup_{\substack{\xi \in \Upsilon \cap \mathrm{Class}(j) \\ \xi \in \mathrm{Int} \\ j \in [2,k]}} \{\xi(+^{j-1}), ..., \xi(+^{j-1})(+^{j-2}) \cdots (+^1), \xi(+^{j-1})(+^{j-2}) \cdots (+^1)2\} =$$

$$\{z[g(i,\alpha,h(\alpha))] \mid z \in \bigcup_{\substack{\xi \in \Omega \cap \mathrm{Class}(j) \\ \xi \in [y, y(+^k)) \\ j \in [2,k]}} \{\xi(+^{j-1}), ..., \xi(+^{j-1}) \cdots (+^1), \xi(+^{j-1})(+^{j-2}) \cdots (+^1)2\}.$$

So let's prove (C5').

Let δ be an ordinal in the set in the left hand side of equality (C5'). Then there exist $j \in [2,k]$, $l \in [1,j)$ and $\xi \in \Upsilon \cap \mathrm{Class}(j) \cap \mathrm{Int}$ such that $\delta = \xi(+^{j-1})(+^{j-2})...(+^{j-l})$ or $\delta = \xi(+^{j-1})(+^{j-2})...(+^1)2$. Then, by (C4), $\xi = z[g(i, \alpha, h(\alpha))]$ for some $z \in \Omega$; moreover, by the properties of the substitution $x \longmapsto x[g(i, \alpha, h(\alpha))]$, $z \in [y, y(+^k)) \cap \mathrm{Class}(j)$. This way,
$\delta = \xi(+^{j-1})(+^{j-2})...(+^{j-l}) = z[g(i,\alpha,h(\alpha))](+^{j-1})(+^{j-2})...(+^{j-l}) =$
$= (z(+^{j-1})(+^{j-2})...(+^{j-l}))[g(i,\alpha,h(\alpha))]$ or
$\delta = \xi(+^{j-1})(+^{j-2})...(+^1)2 = z[g(i,\alpha,h(\alpha))](+^{j-1})(+^{j-2})...(+^1)2 =$
$= (z(+^{j-1})(+^{j-2})...(+^1)2)[g(i, \alpha, h(\alpha))]$. From all this follows that δ belongs to the set in the right hand side of equality (C5').

Now, let $z[g(i, \alpha, h(\alpha))]$ be an ordinal in the set in the right hand side of equality (C5'). Then there exist $j \in [2,k]$, $l \in [1,j)$ and $\varphi \in \Omega \cap \mathrm{Class}(j) \cap [y, y(+^k))$ such that
$z[g(i,\alpha,h(\alpha))] = (\varphi(+^{j-1})(+^{j-2})...(+^{j-l}))[g(i,\alpha,h(\alpha))] =$
$= (\varphi[g(i,\alpha,h(\alpha))])(+^{j-1})(+^{j-2})...(+^{j-l})$ or
$z[g(i,\alpha,h(\alpha))] = (\varphi(+^{j-1})(+^{j-2})...(+^1)2)[g(i,\alpha,h(\alpha))] =$
$= (\varphi[g(i, \alpha, h(\alpha))])(+^{j-1})(+^{j-2})...(+^1)2$. From these equalities (and the fact that by (C4) and the properties or $g(i,\alpha,h(\alpha)$ we know $\varphi[g(i,\alpha,h(\alpha))] \in \Upsilon \cap \mathrm{Int} \cap \mathrm{Class}(j))$ follow that δ belongs to the set in the left hand side of equality (C5').

The previous shows (C5') and subsequently, (C5) holds.

We now show $\Upsilon_2 = \{z[g(i,\alpha,h(\alpha))] \mid z \in \Omega_2\}$. **(C6.1)**
Because of (C5), to show (C6) it is enough to show the following two equalities:

(C6.2). $$\bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Upsilon_1 \cap \mathbb{E} \\ x \in \mathrm{Int}^\circ \\ m(x) \in (\gamma, \gamma(+^1))}} D(\gamma, m(x)) = \{z[g(i,\alpha,h(\alpha))] \mid z \in \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Omega_1 \cap \mathbb{E} \\ x \in (y, y(+^k)) \\ m(x) \in (\gamma, \gamma(+^1))}} D(\gamma, m(x))\};$$

and

(C6.3). $$\bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Upsilon_1 \\ x \in \mathrm{Int}^\circ \\ x \in (\gamma, \gamma(+^1))}} D(\gamma, x) = \{z[g(i,\alpha,h(\alpha))] \mid z \in \bigcup_{\substack{\gamma \in \mathbb{E} \wedge x \in \Omega_1 \\ x \in (y, y(+^k)) \\ x \in (\gamma, \gamma(+^1))}} D(\gamma, x)\}.$$

Let's prove (C6.2).

First notice that, for any $l \in \Upsilon_1$ and $\gamma \in \mathbb{E}$, if $l \in (\gamma, \gamma(+^1))$, then $\gamma \in \mathrm{Ep}(l)$; then, by propositions 6.10 and 6.12 b), $\gamma \in \mathrm{Ep}(l) \subset \Delta(k, y[g(i,\alpha,h(\alpha))], P) \cap \mathbb{E} \subset \Upsilon_1$. **(C7)**

Now we prove (C6.2) as follows:

$$\bigcup_{\substack{\gamma\in\mathbb{E}\wedge x\in\Upsilon_1\cap\mathbb{E}\\ x\in\mathrm{Int}^\circ\\ m(x)\in(\gamma,\gamma(+^1))}} D(\gamma,m(x)) \underset{\text{by (C7)}}{=} \bigcup_{\substack{\gamma\in\mathbb{E}\cap\Upsilon_1\wedge x\in\Upsilon_1\cap\mathbb{E}\\ x\in\mathrm{Int}^\circ\\ m(x)\in(\gamma,\gamma(+^1))}} D(\gamma,m(x)) \underset{\text{by (C5)}}{=}$$

$$\bigcup_{\substack{\gamma\in\mathbb{E}\cap\{z[g(i,\alpha,h(\alpha))]\ |\ z\in\Omega_1\}\\ x\in\{z[g(i,\alpha,h(\alpha))]\ |\ z\in\Omega_1\}\cap\mathbb{E}\\ x\in\mathrm{Int}^\circ\\ m(x)\in(\gamma,\gamma(+^1))}} D(\gamma,m(x)) = \text{, by properties of the functions } \begin{matrix} x\longmapsto x[g(i,\alpha,h(\alpha))]\\ x\longmapsto x[g(i,h(\alpha),\alpha)]\end{matrix}\ ,$$

$$\bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\\ z\in\Omega_1\cap\mathbb{E}\\ z\in(y,y(+^k))\\ m(z)\in(\rho,\rho(+^1))}} D(\rho[g(i,\alpha,h(\alpha))],m(z)[g(i,\alpha,h(\alpha))]) \underset{\text{by prop. 6.15}}{=}$$

$$\bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\\ z\in\Omega_1\cap\mathbb{E}\\ z\in(y,y(+^k))\\ m(z)\in(\rho,\rho(+^1))}} \{l[g(i,\alpha,h(\alpha))]\,|\,l\in D(\rho,m(z))\} =$$

$$\{l[g(i,\alpha,h(\alpha))]\,|\,l\in \bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\wedge z\in\Omega_1\cap\mathbb{E}\\ z\in(y,y(+^k))\\ m(z)\in(\rho,\rho(+^1))}} D(\rho,m(z))\} = \text{, by analogous argument as in (C7),}$$

$$\{l[g(i,\alpha,h(\alpha))]\,|\,l\in \bigcup_{\substack{\rho\in\mathbb{E}\wedge z\in\Omega_1\cap\mathbb{E}\\ z\in(y,y(+^k))\\ m(z)\in(\rho,\rho(+^1))}} D(\rho,m(z))\}.$$

Thus (C6.2) holds.

Now we prove (C6.3).

As we did with (C6.2),

$$\bigcup_{\substack{\gamma\in\mathbb{E}\wedge x\in\Upsilon_1\\ x\in\mathrm{Int}^\circ\\ x\in(\gamma,\gamma(+^1))}} D(\gamma,x) \underset{\text{by (C7)}}{=} \bigcup_{\substack{\gamma\in\mathbb{E}\cap\Upsilon_1\wedge x\in\Upsilon_1\\ x\in\mathrm{Int}^\circ\\ x\in(\gamma,\gamma(+^1))}} D(\gamma,x) \underset{\text{by (C5)}}{=}$$

$$\bigcup_{\substack{\gamma\in\mathbb{E}\cap\{z[g(i,\alpha,h(\alpha))]\ |\ z\in\Omega_1\}\\ x\in\{z[g(i,\alpha,h(\alpha))]\ |\ z\in\Omega_1\}\\ x\in\mathrm{Int}^\circ\\ x\in(\gamma,\gamma(+^1))}} D(\gamma,x) = \text{, by properties of the functions } \begin{matrix} x\longmapsto x[g(i,\alpha,h(\alpha))]\\ x\longmapsto x[g(i,h(\alpha),\alpha)]\end{matrix}\ ,$$

$$\bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\\ z\in\Omega_1\\ z\in(y,y(+^k))\\ z\in(\rho,\rho(+^1))}} D(\rho[g(i,\alpha,h(\alpha))],z[g(i,\alpha,h(\alpha))]) \underset{\text{by prop. 6.15}}{=}$$

$$\bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\\ z\in\Omega_1\\ z\in(y,y(+^k))\\ z\in(\rho,\rho(+^1))}} \{l[g(i,\alpha,h(\alpha))]\,|\,l\in D(\rho,z)\} =$$

$\{l[g(i,\alpha,h(\alpha))] \mid l \in \bigcup_{\substack{\rho\in\mathbb{E}\cap\Omega_1\wedge z\in\Omega_1\\ z\in(y,y(+^k))\\ z\in(\rho,\rho(+^1))}} D(\rho,z)\} =$, by analogous argument as in (C7),

$\{l[g(i,\alpha,h(\alpha))] \mid l \in \bigcup_{\substack{\rho\in\mathbb{E}\wedge z\in\Omega_1\\ z\in(y,y(+^k))\\ z\in(\rho,\rho(+^1))}} D(\rho,z)\}$.

Thus (C6.3) holds.

(C5), (C6.2) and (C6.3) prove that (C6.1) holds.

Finally,
$\Delta(k,y[g(i,\alpha,h(\alpha))],P)=\Upsilon_2\cup\{\gamma 2\mid\gamma\in\mathbb{E}\cap\Upsilon_2\} \underset{\text{by (C6.1)}}{=}$
$\{z[g(i,\alpha,h(\alpha))]\mid z\in\Omega_2\}\cup\{\gamma 2\mid\gamma\in\mathbb{E}\cap\{z[g(i,\alpha,h(\alpha))]\mid z\in\Omega_2\}\}=$
$\{z[g(i,\alpha,h(\alpha))]\mid z\in\Omega_2\}\cup\{z[g(i,\alpha,h(\alpha))]2\mid z\in\mathbb{E}\cap\Omega_2\}=$
$\{z[g(i,\alpha,h(\alpha))]\mid z\in\Delta(k,y,\mathrm{Ep}(m(y)))\}$, that is, (C2) holds.

This concludes the proof of (C2).

After having proved (C2), we continue with the proof of the lemma.

Consider the function
ϕ: $\Delta(k,\ y[g(i,\ \alpha,\ h(\alpha))],\ P) \longrightarrow \phi[\Delta(k,\ y[g(i,\ \alpha,\ h(\alpha))],\ P)] \subset h(y)(\ +^k\ ) \leqslant y[g(i,\ \alpha,\ h(\alpha))]$
defined as $\phi(z[g(i,\alpha,h(\alpha))]):=h(z)$.

We assure
a) $\phi|_{y[g(i,\alpha,h(\alpha))]}\geqslant \mathrm{Id}|_{y[g(i,\alpha,h(\alpha))]}$.
b) ϕ is an $(<,+)$- isomorphism;
c) $y[g(i,\alpha,h(\alpha))]<_1 y[g(i,\alpha,h(\alpha))](+^{k-1})<_1\ldots<_1 y[g(i,\alpha,h(\alpha))](+^{k-1})\ldots(+^1)2\Longleftrightarrow$
$\phi(y[g(i,\alpha,h(\alpha))])<_1\ldots<_1\phi(y[g(i,\alpha,h(\alpha))](+^{-1})(+^{j-1})\ldots(+^1)2)$
d) ϕ restricted to $\Delta(k,y[g(i,\alpha,h(\alpha))],P)\cap(y[g(i,\alpha,h(\alpha))],y[g(i,\alpha,h(\alpha))](+^k))$ is an $<_1$-isomorphism.

a)
Let $z'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)\cap y[g(i,\alpha,h(\alpha))]$. By (C2) and using that the substitution $x\longmapsto x[g(i,\alpha,h(\alpha))]$ for $\mathrm{Ep}(x)\subset\mathrm{Dom}\,g(i,\alpha,h(\alpha))$ preserves inequalities, it follows that $z'=z[g(i,\alpha,h(\alpha))]$ for some $z\in\Delta(i,\alpha,B)\cap y$. Then
$\phi(z[g(i,\alpha,h(\alpha))])=h(z) \underset{\text{by SIH}}{\geqslant} z[g(i,\alpha,h(\alpha))]$. This shows a).

b)
To show ϕ is an $<$-isomorphism.
Let $c',e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$. By (C2), $c'=c[g(i,\alpha,h(\alpha))]$, $e'=e[g(i,\alpha,h(\alpha))]$ for some $c,e\in\Delta(i,\alpha,B)$. Then:
$c[g(i,\alpha,h(\alpha))]=c'<e'=e[g(i,\alpha,h(\alpha))]\Longleftrightarrow c<e \underset{\text{h is }<\text{-isomorphism}}{\Longleftrightarrow}$
$h(c)=\phi(c')<\phi(e')=h(e)$. So ϕ is $<$-isomorphism.

To show ϕ is an $+$-isomorphism.
Let $c',e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$. By (C2), $c'=c[g(i,\alpha,h(\alpha))]$, $e'=e[g(i,\alpha,h(\alpha))]$ for some $c,e\in\Delta(i,\alpha,B)$.
Suppose $c'+e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$. Then there exists $d'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$ such that $c'+e'=d'$; but by C2, $d'=d[g(i,\alpha,h(\alpha))]$ for some $d\in\Delta(i,\alpha,B)$ too, so this equality is actually $c[g(i,\alpha,h(\alpha))]+e[g(i,\alpha,h(\alpha))]=d[g(i,\alpha,h(\alpha))]$ which holds if and only if $c+e=d\in\Delta(i,\alpha,B)$. Then, since h is an $+$-isomorphism,
$\phi(c')+\phi(e')=h(c)+h(e)=h(d)=\phi(d')\in\mathrm{Im}\phi$.

The previous paragraph has shown the following two things:
1b. $c'+e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)\Longrightarrow\phi(c')+\phi(e')\in\mathrm{Im}\phi$.
2b. $c'+e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)\Longrightarrow\phi(c')+\phi(e')=\phi(c'+e')$.
So, to be able to assure that ϕ is an $+$-isomorphism, it only remains to show that
3b. $c'+e'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)\Longleftarrow\phi(c')+\phi(e')\in\mathrm{Im}\phi$.

Let's show that 3b. indeed holds:
Suppose $\phi(c')+\phi(e')\in\mathrm{Im}\phi$. Then there exists $d'\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$ such that $\phi(c')+\phi(e')=\phi(d')$. Now, by (C2), $d'=d[g(i,\alpha,h(\alpha))]$ for some $d\in\Delta(k,\,y,\,\mathrm{Ep}(m(y)))\subset\Delta(i,\,\alpha,\,B)$. **(C8)** Because of all this, we can rewrite the equality $\phi(c')+\phi(e')=\phi(d')$ as $h(c)+h(e)=\phi(c[g(i,\,\alpha,\,h(\alpha))])+\phi(e[g(i,\,\alpha,\,h(\alpha))])=\phi(d[g(i,\,\alpha,\,h(\alpha))])=h(d)$. This shows that that $h(c)+h(e)=h(d)\in\mathrm{Im}\,h$, and since h is an $+$-isomorphism, then $c+e=d$. Finally, from this last equality we have that $c'+e'=c[g(i,\,\alpha,\,h(\alpha))]+e[g(i,\,\alpha,\,h(\alpha))]=d[g(i,\,\alpha,\,h(\alpha))]\underset{\text{by (C8) and (C2)}}{\in}\Delta(k,\,y[g(i,\,\alpha,\,h(\alpha))],\,P)$. This shows 3b.; moreover, this concludes the proof that ϕ is an $+$-isomorphism.

c)
This is very easy to prove:
$y[g(i,\alpha,h(\alpha))]<_1 y[g(i,\alpha,h(\alpha))](+^{k-1})<_1\ldots<_1 y[g(i,\alpha,h(\alpha))](+^{k-1})\ldots(+^1)2\Longleftrightarrow$
$y<_1 y(+^{k-1})<_1\ldots<_1 y(+^{k-1})\ldots(+^1)2\Longleftrightarrow$
$h(y)<_1 h(y(+^{k-1}))<_1\ldots<_1 h(y(+^{k-1})\ldots(+^1)2)\Longleftrightarrow$
$\phi(y[g(i,\alpha,h(\alpha))])<_1\ldots<_1\phi(y[g(i,\alpha,h(\alpha))](+^{k-1})\ldots(+^1)2)$.

d)
Let $c',\,e'\subset\Delta(k,\,y[g(i,\,\alpha,\,h(\alpha))],\,P)\cap(y[g(i,\,\alpha,\,h(\alpha))],\,y[g(i,\,\alpha,\,h(\alpha))](+^k))$ be arbitrary. Using the same argument as in a) we have that $c'=c[g(i,\,\alpha,\,h(\alpha))]$ and $e'=e[g(i,\,\alpha,\,h(\alpha))]$ for some $c,e\in\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))$. **(D1)**
Now, $c[g(i,\alpha,h(\alpha))]<_1 e[g(i,\alpha,h(\alpha))]\Longleftrightarrow c<_1 e\Longleftrightarrow$ $\phi(c')=\phi(c[g(i,\,\alpha,\,h(\alpha))])=h(c)<_1 h(e)=\phi(e[g(i,\,\alpha,\,h(\alpha))])=\phi(e')$, where the first "$\Longleftrightarrow$" holds because of (D1) and because the substitution $x\longmapsto x[g(i,\,\alpha,\,h(\alpha))]$ is an $<_1$-isomorphism in $\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))$; and where the second "$\Longleftrightarrow$" holds because h is an $<_1$-isomorphism in $\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))$ too. This shows d).

Now, using our (IH) applied to $k<i$, $y[g(i,\alpha,h(\alpha))]\in\mathrm{Class}(k)$, $P\subset_{\mathrm{fin}} y[g(i,\alpha,h(\alpha))](+^k)$ with $P\cap[y[g(i,\alpha,h(\alpha))],y[g(i,\alpha,h(\alpha))](+^k))\neq\emptyset$ and ϕ: $\Delta(k,\,y[g(i,\,\alpha,\,h(\alpha))],\,P)\longrightarrow\phi[\Delta(k,\,y[g(i,\,\alpha,\,h(\alpha))],\,P)]\subset y[g(i,\,\alpha,\,h(\alpha))]$ satisfying a), b), c) and d), we get that for any $r\in\Delta(k,y[g(i,\alpha,h(\alpha))],P)$,
$r[g(k,y[g(i,\alpha,h(\alpha))],h(y))]=r[g(k,y[g(i,\alpha,h(\alpha))],\phi(y[g(i,\alpha,h(\alpha))]))]\leqslant\phi(r)$. **(D2)**.

We now make the following observation: Since $y,m(y)\in\Delta(i,\alpha,B)\cap(\alpha,\alpha(+^i))$, $y\leqslant_1 m(y)$ and the function h: $\Delta(i,\,\alpha,\,B)\cap(\alpha,\,\alpha(+^i))\longrightarrow h[\Delta(i,\,\alpha,\,B)\cap(\alpha,\,\alpha(+^i))]\subset\alpha$ is an $<_1$-isomorphism, then $h(y)\leqslant_1 h(m(y))$; note this last $\leqslant_1$-inequality means $h(m(y))\leqslant m(h(y))$. **(D3)**

Now, from (B11) we have $m(h(y))[g(k,\,h(y),\,y[g(i,\,\alpha,\,h(\alpha))])]<m(y[g(i,\,\alpha,\,h(\alpha))])$, which implies
$m(h(y))<m(y[g(i,\alpha,h(\alpha))])[g(k,y[g(i,\alpha,h(\alpha))],h(y))]\underset{\text{by D2}}{\leqslant}$
$\phi(m(y[g(i,\alpha,h(\alpha))]))=\phi(m(y)[g(i,\alpha,h(\alpha))])=h(m(y))\underset{\text{by D3}}{\leqslant}m(h(y))$. Contradiction.

This shows (***) can not hold. Hence $h(y)\geqslant y[g(i,\alpha,h(\alpha))]$ as we needed to show.

This concludes the whole proof of our lemma. □

6.3.3 Consequences of the covering lemma.

6.3.3.1 Generalized covering theorem

Theorem 6.17. *(Generalized covering theorem). Let $i \in [1,n]$, $\alpha \in \mathrm{Class}(i)$ and $A \subset_{\mathrm{fin}} \alpha(+^i)$ be such that $A \cap [\alpha, \alpha(+^i)) \neq \emptyset$. Suppose $h: \Delta(i,\alpha,A) \longrightarrow h[\Delta(i,\alpha,A)] \subset \alpha$ is an $(<,<_1,+)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$. Then*

a) $h(\alpha) \in \alpha \cap \mathrm{Class}(i)$ and $\forall x \in \Delta(i,\alpha,A).T(i,\alpha,x) \cap \alpha \subset h(\alpha)$.

b) Suppose $B \subset \Delta(i,\alpha,A)$, $B \cap [\alpha,\alpha(+^i)) \neq \emptyset$ is such that $\alpha \leqslant_1 \max B$. Then the function $H_B: B \longrightarrow H_B[B] \subset_{\mathrm{fin}} \alpha$, $H_B(x) := x[g(i,\alpha,h(\alpha))]$ is an $(<,<_1,+,\lambda x.\omega^x)$-isomorphism with $H_B|_\alpha = \mathrm{Id}_\alpha$.

Proof. Let $i \in [1,n]$, α, A and h be as stated.

a)
Direct from lemma 6.16.

b)
Suppose B is as stated. We know that
$B \subset \Delta(i,\alpha,A) \subset \{x \in \alpha(+^i) | T(i,\alpha,x) \cap \alpha \subset h(\alpha)\} \underset{\text{corollary 5.21}}{=} \{x \in \alpha(+^i) \mid \mathrm{Ep}(x) \subset \mathrm{Dom}\, g(i,\alpha, h(\alpha))\}$; therefore, from the properties of $g(i,\alpha,h(\alpha))$ we know that $H_B|_\alpha = \mathrm{Id}_\alpha$ and H_B is an $(<,+,\lambda x.\omega^x)$-isomorphism; moreover, we also know that $H_B|_{B\cap(\alpha,\alpha(+^i))}$ preserves $<_1$. **(*)**

Now we show that H_B is, in whole B, an $<_1$-isomorphism. Let $x, y \in B$ and without loss of generality, suppose $x < y$.

Case 1. $y < \alpha$. $x <_1 y \underset{\text{because } h|_\alpha = \mathrm{Id}_\alpha}{\Longleftrightarrow} H_B(x) = x <_1 y = H_B(y)$.

Case 2. $y = \alpha$. $x <_1 y \underset{\text{because } h \text{ is } <_1\text{-iso}}{\Longleftrightarrow} H_B(x) = x = h(x) <_1 h(y) = h(\alpha) = H_B(\alpha) = H_B(y)$.

Case 3. $\alpha < x$. $x <_1 y \underset{\text{by (*)}}{\Longleftrightarrow} H_B(x) <_1 H_B(y)$.

Case 4. $\alpha = x$. By hypothesis $\alpha \leqslant_1 \max B$, and therefore, by $<_1$-connectedness, $\alpha <_1 y$. But $h: \Delta(i,\alpha,A) \longrightarrow h[\Delta(n,i,A)]$ is an $(<,+,<_1)$-iso and $B \subset \Delta(i,\alpha,A)$, so $h(\alpha) <_1 h(y)$. **(**)**

On the other hand, by lemma 6.16, $H_B(\alpha) = h(\alpha) < H_B(y) \leqslant h(y)$; from this, (**) and $<_1$-connectedness follow $H_B(x) = H_B(\alpha) <_1 H_B(y)$. So H_B preserves $<_1$ in this case too.

Case 5. $x < \alpha < y$. Then

$$x <_1 y \underset{\text{because } x<\alpha<y,\ \alpha<_1 \max B \text{ and by } <_1\text{-connectedness}}{\Longleftrightarrow} x <_1 \alpha <_1 y \underset{\text{by case 2 and case 4}}{\Longleftrightarrow}$$

$$H_B(x) <_1 H_B(\alpha) <_1 H_B(y) \underset{\substack{\text{Direction } \Longrightarrow.\ \text{By } <_1\text{-transitivity} \\ \text{Direction } \Longleftarrow.\ \text{Because } x<\alpha<y \Longleftrightarrow H_B(x)<H_B(\alpha)<H_B(y) \\ \text{and by } <_1\text{-connectedness and case 4}}}{\Longleftrightarrow} H_B(x) <_1 H_B(y). \quad \square$$

6.3.3.2 Minimal isomorphisms of the covering

Reminder: For a finite set of ordinals $L \subset_{\mathrm{fin}} \mathrm{OR}$ and $FL \subset \{k | k: L \longrightarrow \mathrm{OR}\}$ a class of functionals, FL is well ordered under the lexicographic order $<_{FL,\mathrm{lex}}$; that is, for $h, k \in FL$, $h <_{\mathrm{FL,lex}} k :\Longleftrightarrow \exists y \in L.h(y) \neq k(y)$ and for $m := \min\{x \in L | h(x) \neq k(x)\}$ it holds $h(m) < k(m)$. Moreover, in case $FL \neq \emptyset$, we can consider $\min(FL)$, the minimum element in FL with respect to $<_{FL,\mathrm{lex}}$. The next corollary uses this concepts.

Corollary 6.18. *Let $i \in [1, n]$ and $\alpha \in \mathrm{Class}(i) \backslash \mathrm{Class}(i+1)$. Consider a finite set $A \subset_{\mathrm{fin}} m(\alpha)$ with $A \cap [\alpha, \alpha(+^i)) \neq \emptyset$. Then $\max \Delta(i, \alpha, A) \leqslant m(\alpha)$.*
Moreover, if $h: \Delta(i, \alpha, A) \longrightarrow h[\Delta(i, \alpha, A)] \subset \alpha$ is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$, then the function $H: \Delta(i, \alpha, A) \longrightarrow H[\Delta(i, \alpha, A)] \subset \alpha$, $H(x) := x[g(i, \alpha, h(\alpha))]$ is well defined and it is an $(<, <_1, +)$-isomorphism with $H|_\alpha = \mathrm{Id}_\alpha$.

Proof. Note $\max \Delta(i, \alpha, A) \underset{\text{proposition 6.10}}{\leqslant} \eta(i, \alpha, \max A) \leqslant \eta(i, \alpha, m(\alpha)) \underset{\text{easy}}{=} m(\alpha)$. The rest of the claim follows directly from the previous inequality and theorem 6.17. □

Corollary 6.19. *Let $i \in [1, n]$, $\alpha \in \mathrm{Class}(i)$ and $\beta \in (\alpha, \alpha(+^i))$ be such that $\alpha <_1 \beta$. Suppose $A \subset_{\mathrm{fin}} \beta$ is such that $A \cap [\alpha, \alpha(+^i)) \neq \emptyset$ and $\Delta(i, \alpha, A) \subset \beta$. Consider*
$F\Delta(i, \alpha, A) := \{h: \Delta(i, \alpha, A) \longrightarrow h[\Delta(i, \alpha, A)] \subset \alpha \mid h$ is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha\}$.
Then $\mu := \min(F\Delta(i, \alpha, A))$ exists, $\mu(\alpha) \in \alpha \cap \mathrm{Class}(i)$, $\forall x \in \Delta(i, \alpha, A).T(i, \alpha, x) \cap \alpha \subset \mu(\alpha)$ and μ is the substitution $x \longmapsto x[g(i, \alpha, \mu(\alpha))]$.

Proof. Let i, α, A, β be as stated.

First note that, since $\alpha < \max \Delta(i, \alpha, A) < \beta$, then, by $\leqslant_1$-connectedness,
$\alpha <_1 \max \Delta(i, \alpha, A)$. **(1*)**

On the other hand, since $\alpha <_1 \beta$ and $\Delta(i, \alpha, A) \subset_{\mathrm{fin}} \beta$, then $F\Delta(i, \alpha, A) \neq \emptyset$ and
$\mu := \min(F\Delta(i, \alpha, A))$ exists. Now, by (1*) and previous theorem 6.17, it follows that
$\mu(\alpha) \in \alpha \cap \mathrm{Class}(i)$;
$\forall x \in \Delta(i, \alpha, A).T(i, \alpha, x) \cap \alpha \subset \mu(\alpha)$; and
the function $H_{\Delta(i,\alpha,A)} \in F\Delta(i, \alpha, A)$ **(2*)**,
where $H_{\Delta(i,\alpha,A)}: \Delta(i, \alpha, A) \longrightarrow H[\Delta(i, \alpha, A)]$ is defined as $H_{\Delta(i,\alpha,A)}(x) := x[g(i, \alpha, \mu(\alpha))]$.

Finally, by lemma 6.16, $\forall x \in \Delta(i, \alpha, A).H_{\Delta(i,\alpha,A)}(x) = x[g(i, \alpha, \mu(\alpha))] \leqslant \mu(x)$; therefore, by (2*) and the minimality of μ in $F\Delta(i, \alpha, A)$ it follows $H_{\Delta(i,\alpha,A)} = \mu$. □

Corollary 6.20. *(6) of theorem 3.26 holds for n, that is:*
For any $\alpha \in \mathrm{Class}(n)$ and any $t \in [\alpha, \alpha(+^n))$, $\alpha <_1 \eta(n, \alpha, t) + 1 \Longrightarrow \alpha <^n \eta(n, \alpha, t) + 1$

Proof. Let $\alpha \in \mathrm{Class}(n)$ and $t \in [\alpha, \alpha(+^n))$ and assume $\alpha <_1 \eta(n, \alpha, t) + 1$.

Clearly $\alpha < \eta(n, \alpha, t) + 1$. **(1*)**

Now, take $B \subset_{\mathrm{fin}} \eta(n, \alpha, t) + 1$ arbitrary and $A := \{\alpha\} \cup B$. Then $A \cap [\alpha, \alpha(+^n)) \neq \emptyset$, $A \subset_{\mathrm{fin}} \eta(n, \alpha, t) + 1$ and by proposition 6.10, $\Delta(n, \alpha, A) \subset_{\mathrm{fin}} \eta(n, \alpha, t) + 1$. Then all the conditions of corollary 6.19 are fulfilled by n, α, $\beta := \eta(n, \alpha, t) + 1$ and A and therefore, for $\mu := \min\{h: \Delta(n, \alpha, A) \longrightarrow h[\Delta(n, \alpha, A)] \subset \alpha \mid h$ is an $(<, <_1, +)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha\}$ we have that $\delta := \mu(\alpha) \in \mathrm{Class}(n) \cap \alpha$, $\forall x \in \Delta(n, \alpha, A).T(n, \alpha, x) \cap \alpha \subset \delta \wedge \mu(x) = x[g(n, \alpha, \delta)]$. Since the previous was done for arbitrary B, it follows easily:
$\forall B \subset_{\mathrm{fin}} \eta(n, \alpha, t) + 1.\exists \delta \in \mathrm{Class}(n) \cap \alpha$.
- $\forall x \in B.T(n, \alpha, x) \cap \alpha \subset \delta$
- $h: B \longrightarrow h[B]$, $h(x) := x[g(n, \alpha, \delta)]$ is an $(<, <_1, +, \lambda x.\omega^x)$-isomorphism with $h|_\alpha = \mathrm{Id}_\alpha$. **(2*)**

Finally, from (1*) and (2*) and according to definition 6.1, we conclude $\alpha <^n \eta(n, \alpha, t) + 1$. □

Part III

$<_1$ and the ψ_i functions

Chapter 7

Class(ω) and the ordinals O_i.

Now that we have finally finished the proof of theorem 3.26, we have a lot of results that give us a pretty good understanding of the $<_1$ relation: At our disposal are not only the assertions of theorem 3.26, but all the many results obtained on the way of the proof of such theorem. The first thing we want to do now is to introduce a new class of ordinals induced by $<_1$ and show that this class is the last (or "thinnest") class induced by the $<_1$-relation: We have arrived to the point where we can partition the whole class of ordinals in the classes of ordinals having "the same $\leqslant_1$-reach up to a replacement" (see proposition 7.6).[7.1]

7.1 Class(ω).

Definition 7.1.
$\text{Class}(\omega) := \bigcap_{n\in[1,\omega)} \text{Class}(n)$.
For any $\alpha \in \text{Class}(\omega)$, *let* $\alpha(+^{\omega}) := \min\{\beta \in \text{Class}(\omega) \mid \alpha < \beta\}$

Remark 7.2. Consider an arbitrary non-countable regular ordinal κ. Since we know that for any $n \in [1, \omega)$ Class(n) is κ-club, then Class(ω) is also κ-club (by proposition 2.47). Therefore, for any $\alpha \in \text{Class}(\omega)$, $\alpha(+^{\omega})$ is well defined.

Our current goal now is to characterize Class(ω). Corollary 7.5. provides such characterization.

Proposition 7.3. $\forall \alpha \in \text{Class}(\omega). \alpha <_1 \alpha(+^{\omega})$

Proof. Let $\alpha \in \text{Class}(\omega)$. For any $i \in [1, \omega)$, let $\xi_i := \alpha(+^i)$. Let $\xi := \sup\{\xi_n \mid n \in [1, \omega)\}$.

To show $\forall n \in [1, \omega). \xi \in \text{Class}(n)$. **(1*)**
Let $n \in [1, \omega)$.
Now, for any $i \in \omega$, let $\gamma_i := \xi_{n+i}$. Then $\forall i \in \omega. \gamma_i \in \text{Class}(n)$ and since Class(n) is a closed class we have that $\xi = \sup\{\gamma_i \mid i \in \omega\} \in \text{Class}(n)$.
This shows (1*).

To show $\alpha(+^{\omega}) \leqslant \xi \in \text{Class}(\omega)$. **(2*)**
From (1*) it is clear that $\alpha < \xi \in \text{Class}(\omega)$. Thus $\alpha(+^{\omega}) \leqslant \xi$.
This shows (2*).

7.1. The expression "the same $\leqslant_1$-reach up to a replacement" was suggested to me by Basil Karadais while I was explaining this theorem in one of our weekly meetings and I used to say "the same abstract $\leqslant_1$-reach".

Now we show that $\forall n \in [1, \omega).\alpha <_1 \xi_n$. **(3*)**
Let $n \in [1, \omega)$.
Since $\alpha \in \text{Class}(\omega)$, then $\alpha \in \text{Class}(n+1)$ and therefore $\alpha <_1 \alpha(+^n) = \xi_n$.
This shows (3*).

Finally, note that (3*) and $<_1$-continuity imply that $\alpha <_1 \xi$; therefore, using (2*) and $<_1$-connectedness we conclude $\alpha <_1 \alpha(+^\omega)$. □

Proposition 7.4. $\forall \alpha \in \text{Class}(\omega).m(\alpha) = \infty$.

Proof. We first show $\forall \beta \in \text{Class}(\omega).\alpha \leqslant \beta \Longrightarrow \alpha \leqslant_1 \beta$ by induction on $(\text{Class}(\omega), <)$. **(0*)**

Let $\beta \in \text{Class}(\omega)$.

Suppose $\forall \gamma \in \beta \cap \text{Class}(\omega).\alpha \leqslant \gamma \Longrightarrow \alpha \leqslant_1 \gamma$. **(IH)**

Assume $\beta \geqslant \alpha$. Then we have the following cases:

a) $\beta = \alpha$. Then clearly $\alpha \leqslant_1 \beta$.

b) $\alpha < \beta \notin \text{Lim}(\text{Class}(\omega))$. Then $\beta = \gamma(+^\omega)$ for some $\gamma \in [\alpha, \beta) \cap \text{Class}(\omega)$. Then $\alpha \underset{\text{by (IH)}}{\leqslant_1} \gamma \underset{\text{by proposition 7.3}}{<_1} \gamma(+^\omega)$. So $\alpha \leqslant_1 \gamma(+^\omega) = \beta$ by $\leqslant_1$-transitivity.

c) $\alpha < \beta \in \text{Lim}(\text{Class}(\omega))$. Then there exists a sequence $(\gamma_i)_{i \in I} \subset [\alpha, \beta) \cap \text{Class}(\omega)$ such that $\gamma_i \underset{\text{cof}}{\hookrightarrow} \beta$, and since $\forall i \in I.\alpha \underset{\text{by (IH)}}{\leqslant_1} \gamma_i$, then $\alpha <_1 \beta$ by $<_1$-continuity.

The previous concludes the proof of (0*).

Now we show that $m(\alpha) = \infty$, that is, we show $\forall \gamma \in \text{OR}.\alpha \leqslant \gamma \Longrightarrow \alpha \leqslant_1 \gamma$.

Let $\gamma \geqslant \alpha$. Then, since Class(ω) is κ-club for any non-countable regular ordinal κ, then there exists $\beta \in \text{Class}(\omega)$ such that $\alpha \leqslant \gamma \leqslant \beta$. From this, (0*) and $\leqslant_1$-connectedness we get $\alpha \leqslant_1 \gamma$. □

Corollary 7.5. $\text{Class}(\omega) = \{\alpha \in \text{OR} \mid m(\alpha) = \infty\}$.

Proof. $\text{Class}(\omega) \underset{\text{by prop. 7.4}}{\subset} \{\delta \in \text{OR} \mid m(\delta) = \infty\}$, so it only remains to show that $\{\delta \in \text{OR} \mid m(\delta) = \infty\} \subset \text{Class}(\omega)$. **(0*)**

Let $\alpha \in \{\delta \in \text{OR} \mid m(\delta) = \infty\}$ be arbitrary. **(*)**

To show $\forall n \in [1, \omega).\alpha \in \text{Class}(n)$. **(1*)**

We carry out the proof of (1*) by induction on $([1, \omega), <)$. Let $n \in [1, \omega)$.
Suppose $\alpha \in \text{Class}(n)$. **(IH)**
Case $n = 1$. Then $\alpha < \alpha 2$ and so, by (*), $\alpha <_1 \alpha 2$. Thus $\alpha \in \text{Class}(1)$.
Case $n = l + 1$ for some $l \in [1, \omega)$. Since by our (IH) $\alpha \in \text{Class}(l)$, then $\alpha(+^l)$ is defined and $\alpha < \alpha(+^l)$. Then, by (*), $\alpha <_1 \alpha(+^n)$, that is, $\alpha \in \text{Class}(l+1) = \text{Class}(n)$.
The previous concludes the proof of (1*).

Concluding, by (1*), $\alpha \in \bigcap_{n \in [1, \omega)} \text{Class}(n) = \text{Class}(\omega)$, and since this was done for arbitrary $\alpha \in \{\delta \in \text{OR} \mid m(\delta) = \infty\}$, then (0*) holds. □

To finish this section, we prove the following two results

Proposition 7.6. *Let $\sim$ be the following binary relation on the ordinals:*

$$\alpha \sim \beta :\Longleftrightarrow \begin{cases} m(\alpha) = \infty = m(\beta) \\ \textit{or} \\ \alpha \notin \mathbb{E} \not\ni \beta \wedge m(\alpha) = \alpha + l \wedge m(\beta) = \beta + l \textit{ for some } l \in \alpha \cap \beta. \\ \textit{or} \\ \{\alpha, \beta\} \subset \mathrm{Class}(n) \backslash \mathrm{Class}(n+1) \wedge T(n, \alpha, m(\alpha)) \cap \alpha \subset \beta \wedge m(\alpha)[g(n, \alpha, \beta)] = m(\beta) \textit{ for some } n \in [1, \omega) \end{cases}$$

Then $\sim$ is an equivalence relation.

Proof. Not hard. □

Proposition 7.7. *Every non-countable cardinal belongs to* $\mathrm{Class}(\omega)$.

Proof. We show $\forall \alpha \in \mathrm{OR}.\alpha \geqslant 1 \Longrightarrow \aleph_\alpha \in \mathrm{Class}(\omega)$ by induction on $(\mathrm{OR}, <)$.

Let $\alpha \in \mathrm{OR}$ with $\alpha \geqslant 1$.
Suppose $\forall \sigma \in \mathrm{OR} \cap \alpha.\sigma \geqslant 1 \Longrightarrow \aleph_\alpha \in \mathrm{Class}(\omega)$. **(IH)**

First note that $\aleph_{\alpha+1}$ is a regular non-countable ordinal and therefore, by remark 7.2, $\mathrm{Class}(\omega)$ is closed unbounded in $\aleph_{\alpha+1}$. **(*0)**

Case $\alpha = \beta + 1$ for some $\beta \in \mathrm{OR}$.
Then $\aleph_{\beta+1}$ is a regular non-countable ordinal and then
$\aleph_\alpha = \aleph_{\beta+1} \underset{\text{remark 7.2}}{=} \sup(\mathrm{Class}(\omega) \cap \aleph_{\beta+1}) \underset{\text{by (*0)}}{\in} \mathrm{Class}(\omega) \cap \aleph_{\beta+1+1}$.

Case $\alpha \in \mathrm{Lim}$.
Then, by our (IH), $\{\aleph_\sigma \mid \sigma \in [1, \alpha)\} \subset \mathrm{Class}(\omega)$ and therefore
$\aleph_\alpha = \sup\{\aleph_\sigma \mid \sigma \in [1, \alpha)\} \underset{\text{by (*0)}}{\in} \mathrm{Class}(\omega) \cap \aleph_{\alpha+1}$. □

7.2 The ordinals O_i.

Having available the classes $\mathrm{Class}(n)$, for $n \in [1, \omega]$, we define

Definition 7.8. *For any $i \in [1, \omega]$, let $O_i := \min \mathrm{Class}(i)$.*

A first observation relative to the ordinals O_i is

Proposition 7.9. *For any $i \in [1, \omega]$, O_i is countable.*

Proof. $\aleph_1$ is regular. Thus for any $i \in [1, \omega]$, $\mathrm{Class}(i)$ is club in $\aleph_1$. So $O_i = \min\mathrm{Class}(i) < \aleph_1$. □

7.2.1 O_ω is the core of R_1

Consider the structure $R_1 := (\mathrm{ORD}, 0, +, \leqslant, \leqslant_1)$. Carlson defines in [10] an *isominimal* substructure of R_1 as a finite substructure of R_1 which is minimal in the pointwise ordering of the collection of all finite substructures of R_1 which are isomorphic to it; moreover, he defines the core of R_1 as the set of ordinals which occur in some isominimal substructure of R_1. From our previous work and the work of Carlson follows that O_ω is the core of R_1. We state now this result as

Corollary 7.10. *O_ω is the core of R_1.*

Proof. $O_\omega \underset{\text{corollary 7.5.}}{=} \min\{\alpha \in \mathrm{OR} \mid m(\alpha) = \infty\} \underset{\text{see [10]}}{=} \text{Core of } R_1 \underset{\text{see [10]}}{=} |\Pi^1_1\text{-CA}_0|$. □

Corollary 7.11. *$(O_i)_{i \in [1,\omega)}$ is strictly increasing and cofinal in $O_\omega = |\Pi^1_1\text{-CA}_0|$.*

Proof. It is easy to see that $(O_i)_{i \in [1,\omega)}$ is strictly increasing and that $(O_i)_{i \in [1,\omega)} \subset O_\omega$. Moreover, using the same argument used in proposition 7.3, it follows $\sup\{O_i \mid i \in [1,\omega)\} \in \mathrm{Class}(\omega)$. From all this we get $\sup\{O_i \mid i \in [1,\omega)\} \leqslant O_\omega \leqslant \sup\{O_i \mid i \in [1,\omega)\}$, that is,
$\sup\{O_i \mid i \in [1,\omega)\} = O_\omega \underset{\text{corollary 7.10}}{=} |\Pi^1_1\text{-CA}_0|$. □

Remark 7.12. As a final comment on this section, we want to stress the following observation made by Prof. Buchholz: How many non-countable regular ordinals do we need to use for the proof of the existence of the ordinals O_i?.

To answer this question, let's convey to denote $1(+^i) := O_i$ for any $i \in [1,\omega]$.

Consider $\rho \in \{1\} \cup \{\text{non-countable regular ordinals}\}$. A careful reading of the proof of theorem 3.26 shows the following:

1. To show the existence of $\rho(+^2)$ we just use one non-countable regular ordinal $\kappa_1 > \rho$, $\kappa_1 \in \mathrm{Class}(1)$, together with it's "Class(1)-successor" $\kappa_1(+^1) \in \mathrm{Class}(1)$. Indeed, using the interval $[\kappa_1, \kappa_1(+^1))$ one shows that Class(2) is κ_1-club (and that $\kappa_1 \in \mathrm{Class}(2)$ too).
2. For the proof of the existence of $\rho(+^3)$, in a similar way as in the previous case, we use the pair $\kappa_1, \kappa_1(+^2) \in \mathrm{Class}(2)$ and show that Class(3) is κ_1-club and that $\kappa_1 \in \mathrm{Class}(3)$. However, for the existence of $\kappa_1(+^2)$, as we just mentioned in 1, we need to use another non-countable regular ordinal $\kappa_2 > \kappa_1$, $\kappa_2 \in \mathrm{Class}(1)$. So, actually, for the existence of $\rho(+^3)$ we have used the existence of at least two non-countable regular ordinals κ_1 and κ_2 satisfying $\rho < \kappa_1 < \kappa_2$.
3. Inductively, for $i \in [1,\omega)$, to show the existence of $\rho(+^{i+1})$ one shows that $\mathrm{Class}(i+1)$ is κ_1-club (and that $\kappa_1 \in \mathrm{Class}(i+1)$) using the pair $\kappa_1, \kappa_1(+^i) \in \mathrm{Class}(i)$, where for the proof of the existence of $\kappa_1(+^i)$ we need $(i-1)$-non-countable regular ordinals $\kappa_2 < \kappa_3 < \ldots < \kappa_i$ bigger than κ_1. That is, in total, for the existence of $\rho(+^{i+1})$ we have used i-non-countable regular ordinals $\kappa_1, \ldots, \kappa_i$ satisfying $\rho < \kappa_1 < \kappa_2 < \ldots < \kappa_i$.
4. The proof of the existence of $\rho(+^\omega)$ only requires the ordinals $\{\kappa_j \mid j \in [1,\omega)\}$, since just using these ordinals it follows that $\kappa_1 \in \bigcap_{i \in [1,\omega)} \mathrm{Class}(n) = \mathrm{Class}(\omega)$.

Summarizing, our proof of the existence of the O_i ordinals (with $i \in [1,\omega]$) requires the existence of at least ω many non-countable regular ordinals κ_j.

7.3 The ψ_v functions and the ordinals $\psi_v(\Omega_{v+m})$

For $n \in [1, \omega)$, consider the theory ID_n of n-iterated inductive definitions and it's proof theoretic ordinal $|\mathrm{ID}_n|$. In this section, for Buchhoz ψ_n functions, we show that
$\forall n \in \omega . \psi_n(\Omega_{n+2}) = \Omega_n(+^2)$ (corollary 7.44), which in particular means $|\mathrm{ID}_1| = O_2$. Moreover, an incomplete but (in the opinion of the author of this thesis) plausible proof of the statement
$\forall n \in \omega \forall m \in [1, \omega) . \psi_n(\Omega_{n+m}) \leqslant \Omega_n(+^m)$ (which in particular would mean
$\forall n \in [1, \omega) . |\mathrm{ID}_n| \leqslant O_{n+1})$ is also presented.

First let's remind the reader of the following induction principle that will be used several times later.

Theorem 7.13. *(Induction principle for monotone inductive definitions). Let U be a non-empty set. Let $\mathfrak{P}(U)$ be the power set of U and $\Gamma: \mathfrak{P}(U) \longrightarrow \mathfrak{P}(U)$ be a monotone operator. Let I_Γ be the least fixed point of Γ. Then for any $X \subset U$, if $\Gamma(X \cap I_\Gamma) \subset X$, then $I_\Gamma \subset X$.*

Proof. Not hard. See [14]. □

7.3.1 Buchhoz ψ_v functions (with $v \in [0, \omega]$)

Now we introduce Buchholz ψ_v functions (as given in [4]) and present several known properties of them necessary for our purposes.

Definition 7.14. *For $v \in [0, \omega]$, let* $\Omega_v := \begin{cases} 1 & \text{iff } v = 0 \\ \aleph_v & \text{otherwise} \end{cases}$

Convention 7.15. It will be useful for later to make the following convention:
For any $i \in [1, \omega]$, let $\Omega_0(+^i) := O_i = \min \mathrm{Class}(i)$.

Definition 7.16. *By recursion on OR, the functions ψ_u and C_u (for any $u \in [0, \omega]$) are simultaneously defined in the following way: Suppose that $C_u(\xi)$ and $\psi_u(\xi)$ are defined for all $\xi < \alpha$ and for all $u \in [0, \omega]$.*

Then, for any $v \in [0, \omega]$, $\psi_v \alpha := \min \{\gamma \in \mathrm{OR} \mid \gamma \notin C_v(\alpha)\}$, where the set $C_v(\alpha)$ is inductively defined by the following clauses:

- ***(C1).*** $\Omega_v \subset C_v(\alpha)$
- ***(C2).*** $\sigma, \delta \in C_v(\alpha) \Longrightarrow \sigma + \delta \in C_v(\alpha)$
- ***(C3).*** $n \in [0, \omega] \wedge \sigma \in \alpha \cap C_v(\alpha) \cap C_n(\sigma) \Longrightarrow \psi_n \sigma \in C_v(\alpha)$

Theorem 7.17. $\forall v \in [0, \omega) . \psi_0(\varepsilon_{\Omega_v + 1}) = |\mathrm{ID}_v|$

Proof. See [4]. □

Proposition 7.18.

a) $\psi_v 0 = \Omega_v$.

b) $\psi_v \alpha \in \mathbb{P}$.

c) $\Omega_v \leqslant \psi_v \alpha < \Omega_{v+1}$.

d) $\alpha \leqslant \beta \Longrightarrow C_v(\alpha) \subset C_v(\beta)$ *and* $\psi_v \alpha \leqslant \psi_v \beta$

e) $\gamma \in C_v(\alpha) \wedge \gamma =_{\text{CNF}} L_1 l_1 + \ldots + L_k l_k \Longleftrightarrow \{L_1, \ldots, L_k\} \subset C_v(\alpha)$

f) $\xi, \delta \in C_v(\alpha) \Longrightarrow \xi + \delta \in C_v(\alpha)$

g) $\xi + \delta \in C_v(\alpha) \Longrightarrow \delta \in C_v(\alpha)$

h) *If* $\alpha_0 < \alpha$ *and* $[\alpha_0, \alpha) \cap C_v(\alpha_0) = \emptyset$, *then* $C_v(\alpha_0) = C_v(\alpha)$

Proof. See [4]. □

Proposition 7.19.

a) *If* $\alpha < \beta$ *and* $\alpha \in C_v(\alpha)$, *then* $\psi_v \alpha < \psi_v \beta$

b) *If* $\psi_a \alpha = \psi_b \beta$ *and* $\alpha \in C_a(\alpha) \wedge \beta \in C_b(\beta)$, *then* $a = b \wedge \alpha = \beta$.

c) *If* $\Omega_l \leqslant \gamma \in \mathbb{P} \cap C_l(\alpha)$, *then* $\exists u \exists \xi \in C_u(\xi) \cap \alpha \cap C_l(\alpha). \gamma = \psi_u \xi$

d) *If* $\Omega_l \leqslant \psi_u \xi \in C_l(\alpha) \wedge \xi \in C_u(\xi)$, *then* $\xi \in \alpha \cap C_l(\alpha)$

Proof. See [4]. □

Proposition 7.20.

a) $C_v(\alpha) \cap \Omega_{v+1} = \psi_v \alpha$.

b) $\psi_v(\alpha + 1) = \begin{cases} \min \{\gamma \in \mathbb{P} \mid \psi_v \alpha < \gamma\} & \textit{if} \quad \alpha \in C_v(\alpha) \\ \psi_v \alpha & \textit{otherwise} \end{cases}$.

c) *If* $\alpha \in \mathrm{Lim}$, *then* $\psi_v \alpha = \sup \{\psi_v \xi \mid \xi < \alpha \wedge \xi \in C_v(\xi)\}$.

Proof. See [4]. □

Proposition 7.21.

a) $\alpha < \varepsilon_0 \Longrightarrow \alpha \in C_0(\alpha) \wedge \psi_0 \alpha = \omega^\alpha$.

b) $\alpha < \varepsilon_{\Omega_v + 1} \wedge v \neq 0 \Longrightarrow \alpha \in C_v(\alpha) \wedge \psi_v \alpha = \omega^{\Omega_v + \alpha}$.

c) $C_v(\alpha) \subset \varepsilon_{\Omega_\omega + 1}$.

d) $\varepsilon_{\Omega_\omega + 1} \leqslant \alpha \Longrightarrow C_v(\varepsilon_{\Omega_\omega + 1}) = C_v(\alpha)$.

Proof. See [4]. □

Proposition 7.22.

a) $(\neg \exists \xi \in C_v(\xi) \cap [\alpha, \beta)) \Longrightarrow C_v(\beta) \subset C_v(\alpha)$.

b) $(\exists \xi \in C_v(\xi) \cap [\alpha, \beta)) \Longleftrightarrow \psi_v \alpha < \psi_v \beta$.

c) $\alpha \in \beta \cap C_v(\beta) \Longrightarrow \psi_v \alpha < \psi_v \beta$.

Proof. Not hard. □

Proposition 7.23. $\forall m \in \omega. \forall \xi \in \Omega_{m+1}. \forall n \leqslant m. \xi \in C_n(\xi) \Longrightarrow C_n(\xi) \cap [\Omega_m(+^1), \Omega_{m+1}) = \emptyset$.

Proof. Let $m \in [1, \omega)$. We prove $\forall \xi \in \Omega_{m+1}. \forall n \leqslant m. \xi \in C_n(\xi) \Longrightarrow C_n(\xi) \cap [\Omega_m(+^1), \Omega_{m+1}) = \emptyset$ by induction on $(\Omega_{m+1}, <)$.

Let $\xi \in \Omega_{m+1}$.

Suppose $\forall \delta \in \xi \cap \Omega_{m+1}.\forall n \leqslant m.\delta \in C_n(\delta) \Longrightarrow C_n(\delta) \cap [\Omega_m(+^1), \Omega_{m+1}) = \emptyset$. **(IH)**

Let $n \in [0, m]$ and suppose $\xi \in C_n(\xi)$. Let $X := \{\beta \in \mathrm{OR} \mid \beta \notin [\Omega_m(+^1), \Omega_{m+1})\}$. Now we show that X is closed under the clauses of the inductive definition of $C_n(\xi)$.

- $\Omega_n \subset X$ clearly.
- Suppose $\rho, \delta \in X$. Then $\rho + \delta \begin{cases} < \Omega_m(+^1) & \text{iff } \rho, \delta < \Omega_m(+^1) \\ \geqslant \Omega_{m+1} & \text{iff } \exists x \in \{\rho, \delta\}.x \geqslant \Omega_{m+1} \end{cases}$. So $\rho + \delta \in X$.
- Suppose $\rho = \psi_q(\delta)$ for $\delta \in \xi \cap X \cap C_q(\delta)$ for some $q \in [0, \omega]$. Then we have some cases:
  + $q \geqslant m+1$. Then $\psi_q(\delta) \geqslant \Omega_q \geqslant \Omega_{m+1}$, i.e, $\rho \in X$.
  + $q < m$. Then $\psi_q(\delta) < \Omega_{q+1} \leqslant \Omega_m < \Omega_m(+^1)$, i.e, $\rho \in X$.
  + $q = m$. Then we have $\delta \in \xi \cap C_m(\delta)$ and so, by our (IH), $C_m(\delta) \cap [\Omega_m(+^1), \Omega_{m+1}) = \emptyset$;

this together with the fact $\delta \in \xi \in \Omega_{m+1}$ implies $\delta < \Omega_m(+^1)$. **(*)**
Finally, (*) and proposition 7.21 yield $\psi_m(\delta) = \omega^{\Omega_m + \delta} \underset{\text{by (*)}}{<} \Omega_m(+^1)$. So $\rho = \psi_m(\delta) \in X$.

From the previous follows $C_n(\xi) \subset X$, that is, the theorem holds. □

Proposition 7.24.

1. *$\forall m \in \omega.\forall \xi \in [\Omega_m(+^1), \Omega_{m+1})\forall n \leqslant m.\xi \notin C_n(\xi)$.*
2. *$\forall \xi \in [\varepsilon_0, \Omega_1).\xi \notin C_0(\xi)$.*

Proof.
1.
Let $m \in [1, \omega)$, $\xi \in [\Omega_m(+^1), \Omega_{m+1})$ and $n \leqslant m$. Then $\xi \in C_n(\xi)$ imply
$\xi \in C_n(\xi) \cap [\Omega_m(+^1), \Omega_{m+1}) \underset{\text{proposition 7.23}}{=} \emptyset$. Contradiction. So $\xi \notin C_n(\xi)$.

2.
Direct from 1 and the fact that $\Omega_0(+^1) = \varepsilon_0$. □

7.3.2 The type of an ordinal α.

The following definition will play a major role later.

Definition 7.25. *Let $\Omega \in \mathrm{Class}(\omega)$. For $\alpha < \Omega$ we define $\mathrm{tp}(\alpha, \Omega)$, the type of α in terms of Ω, as:*

$$\mathrm{tp}(\alpha, \Omega) := \begin{cases} \infty & \textit{iff } \alpha \in \mathrm{Class}(\omega) \\ m(\alpha)[g(n, \alpha, \Omega)] & \textit{iff } \alpha \in \mathrm{Class}(n) \backslash \mathrm{Class}(n+1) \textit{ for some } n \in [1, \omega) \\ \Omega + q & \textit{iff } m(\alpha) = \alpha + q \textit{ for some } q \in [0, \alpha) \end{cases}$$

Now we will work on certain results concerning $\mathrm{tp}(\alpha, \Omega)$ and limit procedures that will be necessary.

Proposition 7.26. *Let $n \in [1, \omega)$, $\alpha \in \mathrm{Class}(n) \backslash \mathrm{Class}(n+1)$ and*
$t \in (\alpha(+^{n-1})...(+^1)2, \alpha(+^n))$. Then

1. *$\forall s \in (\alpha, l(n, \alpha, t)).\eta(n, \alpha, s) < l(n, \alpha, t)$*
2. *If $l(n, \alpha, t) \in \mathrm{Lim}$, then $l(n, \alpha, t) = \sup \{\eta(n, \alpha, s) \mid s \in (\alpha, l(n, \alpha, t))\}$*

Proof. Let ω, α and t be as stated.

1.
Take $s \in (\alpha, l(n, \alpha, t))$ arbitrary. If $s \leqslant \alpha(+^{n-1})...(+^1)2$, then it is easy to see that $\eta(n, \alpha, s) = \alpha(+^{n-1})...(+^1)2 \underset{\text{because } t > \alpha(+^{n-1})...(+^1)2}{<} l(n, \alpha, t)$. So assume $s > \alpha(+^{n-1})...(+^1)2$.

To show $l(n, \alpha, t) \nleqslant \eta(n, \alpha, s)$. **(1*)**

Assume the opposite. Since $\eta(n, \alpha, s) = \max\{m(e) \mid e \in (\alpha, s]\}$, then there exists $e \in (\alpha, s]$ such that $e \leqslant s < l(n, \alpha, t) \leqslant m(e)$; these inequalities together with the fact that $m(l(n, \alpha, t)) = \eta(n, \alpha, t)$ imply, by $\leqslant_1$-connectedness, that $e <_1 l(n, \alpha, t) \leqslant_1 \eta(n, \alpha, t)$ and therefore, by $\leqslant_1$-transitivity $e \leqslant_1 \eta(n, \alpha, t)$, that is, $m(e) \geqslant \eta(n, \alpha, t)$. **(*)**. But then $m(e) \underset{\text{always holds}}{\leqslant} \eta(n, \alpha, e) \underset{\text{because } e \leqslant t}{\leqslant} \eta(n, \alpha, t) \underset{\text{by } (*)}{\leqslant} m(e)$, that is $m(e) = \eta(n, \alpha, t)$. From our previous work follows $e \in (\alpha, l(n, \alpha, t)) \cap \{\beta \in (\alpha, t] \mid m(\beta) = \eta(n, \alpha, t)\}$. Contradiction because $l(n, \alpha, t) = \min\{\beta \in (\alpha, t] \mid m(\beta) = \eta(n, \alpha, t)\}$.

Thus (1*) holds, that is, $\eta(n, \alpha, s) < l(n, \alpha, t)$.

2.
Note $\forall s \in (\alpha, l(n, \alpha, t)).s \leqslant \eta(n, \alpha, s) < \eta(n, \alpha, s) + 1 \underset{\text{by 1.}}{\leqslant} l(n, \alpha, t)$, therefore $l(n, \alpha, t) = \sup(\alpha, l(n, \alpha, t)) \leqslant \sup\{\eta(n, \alpha, s) \mid s \in (\alpha, l(n, \alpha, t))\} \leqslant l(n, \alpha, t)$, i.e., $l(n, \alpha, t) = \sup\{\eta(n, \alpha, s) \mid s \in (\alpha, l(n, \alpha, t))\}$. □

Proposition 7.27. *Let $n \in [1, \omega)$ and $\alpha \in \text{Class}(n) \backslash \text{Class}(n+1)$. Then*

1. $\forall t \in (\alpha, m(\alpha)].m(t) \leqslant m(\alpha)$
2. $\forall t \in (\alpha, m(\alpha)].\eta(n, \alpha, t) \leqslant m(\alpha)$.
3. $\eta(n, \alpha, m(\alpha)) = m(\alpha)$

Proof. Let n and α be as stated.

1.
Take $t \in (\alpha, m(\alpha)]$. Assume $m(t) > m(\alpha)$. Then the inequalities $\alpha \leqslant t \leqslant m(\alpha) < m(\alpha) + 1 \leqslant m(t)$ imply, by $\leqslant_1$-connectedness, that $\alpha \leqslant_1 t \leqslant_1 m(\alpha) + 1$, which subsequently implies, by $\leqslant_1$-continuity, that $\alpha \leqslant_1 m(\alpha) + 1$. Contradiction. Thus $m(t) \leqslant m(\alpha)$.

2.
Take $t \in (\alpha, m(\alpha)]$. Then:

If $t \in (\alpha, \alpha(+^{n-1})...(+^1)2]$, then $\eta(n, \alpha, t) = \alpha(+^{n-1})...(+^1)2 \leqslant m(\alpha)$.

If $t \in (\alpha(+^{n-1})...(+^1)2, m(\alpha)]$, then $\eta(n, \alpha, t) = \max\{m(e) | e \in (\alpha, t]\} \underset{\text{by 1.}}{\leqslant} m(\alpha)$.

3.
$m(\alpha) \leqslant \eta(n, \alpha, m(\alpha)) \underset{\text{by 2.}}{\leqslant} m(\alpha)$. So $\eta(n, \alpha, m(\alpha)) = m(\alpha)$. □

Convention 7.28. Consider $n \in [1, \omega)$ and $\alpha \in \text{Class}(n) \cap O_{n+1}$. We want to be able to take $f(n+1, \lambda(n+1, \alpha))(\alpha)$ as we have done in previous chapters. So we just extend in the natural way the definitions of $\lambda(n+1, \cdot)$ and $f(n+1, \cdot)$. We convey:

- $\lambda(n+1, \alpha) := 0$,
- $S(n+1, 0) \colon \text{Class}(n) \cap O_{n+1} \longrightarrow \text{Subsets}(\text{Class}(n) \cap O_{n+1})$
  $S(n+1, 0)(\alpha) := \{e \in \text{Class}(n) \cap O_{n+1} \cap \alpha \mid m(e)[g(n, e, \alpha)] \geqslant m(\alpha)\}$
- $f(n+1, 0) \colon \text{Class}(n) \cap O_{n+1} \longrightarrow \text{Subsets}(\text{OR})$
  $$f(n+1, 0)(\alpha) := \begin{cases} \{\alpha\} & \text{iff} \quad S(n+1, 0)(\alpha) = \emptyset \\ \\ f(n+1, 0)(s) \cup \{\alpha\} & \text{iff} \quad S(n+1, 0)(\alpha) \neq \emptyset \wedge s := \sup(S(n+1, 0)(\alpha)) \end{cases}$$

Proposition 7.29. *For any $n\in[1,\omega)$, propositions 5.1 and corollary 5.2 hold for $S(n+1,0)$ and $f(n+1,0)$ too.*

Proof. Clear. The proofs of propositions 5.1 and corollary 5.2 hold for these cases too. $\square$

Proposition 7.30. *Let $n\in[1,\omega)$ and $\alpha\in(\mathrm{Lim\,Class}\,(n))\backslash\mathrm{Class}(n+1)$. Consider* $\beta(\alpha):=\begin{cases}\sigma_2 & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1>\ldots>\sigma_q\}\wedge q\geqslant 2\\ \lambda(n+1,\alpha) & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1\}\end{cases}$. *Then*

1. $\{\eta(n,\alpha,s)\mid s\in(\alpha,l(n,\alpha,m(\alpha)))\}=\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}$
2. $l(n,\alpha,m(\alpha))\in\mathrm{Lim}\Longrightarrow l(n,\alpha,m(\alpha))=\sup\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}$.
3. *Suppose $l(n,\alpha,m(\alpha))\notin\mathrm{Lim}$. Then $l(n,\alpha,m(\alpha))=\gamma+1$ for some $\gamma\in[\alpha,\alpha(+^n))$ such that*
   - 3.1. $m(\alpha)=\eta(n,\alpha,l(n,\alpha,m(\alpha)))=m(l(n,\alpha,m(\alpha)))=\gamma+1=l(n,\alpha,m(\alpha))$
   - 3.2. $\eta(n,\alpha,\gamma)=\gamma$
   - 3.3. $\gamma=\sup\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}=$
     $=\max\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}$.

Proof. Let α and $\beta(\alpha)$ be as stated. To simplify our notation, in the subsequent we write β instead of $\beta(\alpha)$.
1.
Let $e\in(\alpha,l(n,\alpha,m(\alpha)))$ be arbitrary. Then
$\alpha<e\leqslant\eta(n,\alpha,e)<\eta(n,\alpha,e)+1\underset{\text{because } m(\alpha)>\alpha(+^{n-1})\ldots(+^1)2 \text{ and by proposition 7.26}}{\leqslant}l(n,\alpha,m(\alpha))\leqslant$
$m(l(n,\alpha,m(\alpha)))\underset{\text{by prop. 3.21}}{=}m(\alpha)$, which implies by $\leqslant_1$-connectedness, that
$\alpha<_1\eta(n,\alpha,e)+1$. So, by lemma 4.15, there is a sequence $(\rho_j)_{j\in D}\subset\mathrm{Class}(n)$ with $\forall j\in D.T(n,\alpha,\eta(n,\alpha,e))\cap\alpha\subset\rho_j\wedge m(\rho_j)=\eta(n,\alpha,e)[g(n,\alpha,\rho_j)]$ and $\rho_j\underset{\text{cof}}{\hookrightarrow}\alpha$. Let $k_0\in D$ be such that $\rho_{k_0}>\beta$. Then $\eta(n,\alpha,e)=m(\rho_{k_0})[g(n,\rho_{k_0},\alpha)]\in\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta,\alpha)\cap\mathrm{Class}(n)\}$. This shows
$\{\eta(n,\alpha,s)\mid s\in(\alpha,l(n,\alpha,m(\alpha)))\}\subset\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta,\alpha)\cap\mathrm{Class}(n)\}$. **(1*)**

To show $\{\eta(n,\alpha,s)\mid s\in(\alpha,l(n,\alpha,m(\alpha)))\}\supset\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta,\alpha)\cap\mathrm{Class}(n)\}$. **(2*)**

Let $q\in\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta,\alpha)\cap\mathrm{Class}(n)\}$, that is $q=m(\rho)[g(n,\rho,\alpha)]$ for some $\rho\in(\beta,\alpha)\cap\mathrm{Class}(n)$. By proposition 7.29 5.2, $m(\rho)[g(n,\rho,\alpha)]<m(\alpha)$. **(*)**

We assure $l(n,\alpha,m(\alpha))\nleqslant m(\rho)[g(n,\rho,\alpha)]$. **(3*)**
Assume the opposite, i.e., $l(n,\alpha,m(\alpha))\leqslant m(\rho)[g(n,\rho,\alpha)]$. Then $\eta(n,\alpha,m(\rho)[g(n,\rho,\alpha)])=m(\alpha)$, and subsequently $m(\rho)=\eta(n,\rho,m(\rho))=\eta(n,\rho,m(\rho)[g(n,\rho,\alpha)][g(n,\alpha,\rho)])=\eta(n,\alpha,m(\rho)[g(n,\rho,\alpha)])[g(n,\alpha,\rho)]=m(\alpha)[g(n,\alpha,\rho)]$, that is, $m(\rho)[g(n,\rho,\alpha)]=m(\alpha)$, which is contradictory with (*). Thus (3*) holds.

Finally, from (3*) we have $l(n,\alpha,m(\alpha))>q=m(\rho)[g(n,\rho,\alpha)]=\eta(n,\rho,m(\rho))[g(n,\rho,\alpha)]=\eta(n,\alpha,m(\rho)[g(n,\rho,\alpha)])$, which means
$q=\eta(n,\alpha,m(\rho)[g(n,\rho,\alpha)])\in\{\eta(n,\alpha,s)\mid s\in(\alpha,l(n,\alpha,m(\alpha)))\}$. Since this was done for arbitrary $q\in\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}$, then we have shown (2*).

So the claim holds because of (1*) and (2*).

2.
$l(n,\alpha,m(\alpha))\underset{\text{prop. 7.26}}{=}\sup\{\eta(n,\alpha,s)\mid s\in(\alpha,l(n,\alpha,m(\alpha)))\}=$

$$\underset{\text{by 1.}}{=} \sup\{m(\delta)[g(n,\delta,\alpha)] \mid \delta \in (\beta(\alpha),\alpha)\cap \mathrm{Class}(n)\}.$$

3.
Suppose $l(n,\alpha,m(\alpha))\notin \mathrm{Lim}$ and let $\gamma\in \mathrm{OR}$ be such that $l(n,\alpha,m(\alpha))=\gamma+1$. Clearly $\gamma\in[\alpha(+^{n-1})\cdots(+^1),\alpha(+^n))$.

3.1.
Direct from propositions 3.21 and 7.27.

3.2.
Since $l(n,\alpha,m(\alpha))\geqslant \alpha(+^{n-1})\cdots(+^1)2$, then $\gamma\geqslant\alpha(+^{n-1})\cdots(+^1)2$. From this follows that $\eta(n,\alpha,\gamma)=\max\{m(e)|e\in(\alpha,\gamma]\}$. **(a)**. But $\max\{m(e)|e\in(\alpha,\gamma]\}\not>\gamma$, otherwise, for some $e\in(\alpha,\gamma]$, we had $\alpha<e\leqslant\gamma<\gamma+1=m(\alpha)\leqslant m(e)$, which implies $m(\alpha)=m(e)$ and therefore $l(n,\alpha,m(\alpha))\leqslant e\leqslant\gamma<\gamma+1$. Contradiction with our assumption $l(n,\alpha,m(\alpha))=\gamma+1$. Hence $\max\{m(e)|e\in(\alpha,\gamma]\}\not>\gamma$, that is, (using (a)), $\eta(n,\alpha,\gamma)\leqslant\gamma$. From this and the fact that it always hods $\gamma\leqslant\eta(n,\alpha,\gamma)$, we get $\eta(n,\alpha,\gamma)=\gamma$ as we wanted.

3.3.
Direct from 1. and 3.2. □

Proposition 7.31. *Let $n\in[1,\omega)$ and $\alpha\in\mathrm{Class}(n)\backslash\mathrm{Class}(n+1)$ be arbitrary. Consider* $\beta(\alpha):=\begin{cases}\sigma_2 & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1>\ldots>\sigma_q\}\wedge q\geqslant 2\\ \lambda(n+1,\alpha) & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1\}\end{cases}$ *and* $\gamma:=\begin{cases}\sup\{m(\delta)[g(n,\delta,\alpha)] \mid \delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\} & \text{if } \alpha\in\mathrm{LimClass}(n)\\ -1 & \text{otherwise}\end{cases}$. *Then*

a) $m(\alpha)=\begin{cases}\alpha(+^{n-1})\cdots(+^1)2 & \text{iff } \alpha\notin\mathrm{Lim}(\mathrm{Class}(n))\\ \eta(n,\alpha,\gamma) & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\in\mathrm{Lim}\\ \gamma+1 & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\notin\mathrm{Lim}\end{cases}$

b) *For any $\Omega\in\mathrm{Class}(\omega)$ with $\Omega>\alpha$,*
$\mathrm{tp}(\alpha,\Omega)=\begin{cases}\Omega(+^{n-1})\cdots(+^1)2 & \text{iff } \alpha\notin\mathrm{Lim}(\mathrm{Class}(n))\\ \eta(n,\Omega,\gamma[g(n,\alpha,\Omega)]) & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\in\mathrm{Lim}\\ \gamma[g(n,\alpha,\Omega)]+1 & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\notin\mathrm{Lim}\end{cases}$

Proof. Let $n\in[1,\omega)$ and α be as stated.
a)
If $\alpha\notin\mathrm{Lim}(\mathrm{Class}(n))$, then (we already know) $m(\alpha)=\alpha(+^{n-1})\cdots(+^1)2$. If $\alpha\in\mathrm{Lim}(\mathrm{Class}(n))$, then we have two cases:
- $l(n,\alpha,m(\alpha))\in\mathrm{Lim}$. Then, $m(\alpha)=\eta(n,\alpha,l(n,\alpha,m(\alpha)))\underset{\text{proposition 7.30}}{=}\eta(n,\alpha,\gamma)$.
- $l(n,\alpha,m(\alpha))\notin\mathrm{Lim}$. Then $m(\alpha)=\eta(n,\alpha,l(n,\alpha,m(\alpha)))\underset{\text{proposition 7.30}}{=}\gamma+1$.

b)
Direct from *a*). □

Proposition 7.32. *Let $n\in[1,\omega)$, $\Omega\in\mathrm{Class}(\omega)$, $\alpha\in[1,\Omega)$ and $\nu\in\{0\}\cup(\alpha\cap\mathrm{Class}(n+1))$ be arbitrary. Suppose $\rho\in[\Omega,\Omega(+^n))$ is such that $\forall\delta\in(\nu,\alpha).\mathrm{tp}(\delta,\Omega)<\rho$. Then*

a) $\alpha\notin\mathrm{Class}(n+1)$.

b) *If $T(n,\Omega,\rho)\cap\Omega\subset\alpha$, then*
$\mathrm{tp}(\alpha,\Omega)\leqslant\begin{cases}\Omega(+^{n-1})\cdots(+^1)2 & \text{iff } \alpha\notin\mathrm{Lim}(\mathrm{Class}(n))\\ \eta(n,\Omega,\rho) & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\in\mathrm{Lim}\\ \rho+1 & \text{iff } \alpha\in\mathrm{Lim}(\mathrm{Class}(n))\wedge l(n,\alpha,m(\alpha))\notin\mathrm{Lim}\end{cases}$

Proof. Let n, Ω, α and ν be as stated.

Suppose $\rho\in[\Omega,\Omega(+^n))$ is such that $\forall\delta\in(\nu,\alpha).\mathrm{tp}(\delta,\Omega)<\rho$. **(a0*)**

a)
To show $\alpha\notin\mathrm{Class}(n+1)$. **(a1*)**

Assume the opposite, assume that $\alpha\in\mathrm{Class}(n+1)$. **(a2*)**

Consider the canonical sequence $(\gamma_k(n,\Omega))_{k\in[1,\omega)}$ of $\Omega(+^n)$ as given in definition 4.16. Since by proposition 4.17 $(\gamma_k(n,\Omega))_{k\in[1,\omega)}\subset(\Omega,\Omega(+^n))$ and $\gamma_k(n,e)\underset{\mathrm{cof}}{\hookrightarrow}\Omega(+^n)$, then let $i\in\omega$ be such that $\gamma_i(n,\Omega)>\rho$. **(a3*)**
On the other hand, note (a2*) implies $\alpha<_1\gamma_i(n,\alpha)+1$; subsequently, by proposition 4.17 and lemma 4.15, there exists of a sequence $(\delta_j)_{j\in J}\subset\mathrm{Class}(n)\cap\alpha$ **(a4*)** such that
$\forall j\in J.T(n,\alpha,\gamma_i(n,\alpha))\cap\alpha\subset\delta_j$,
$\delta_j\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ and $\forall j\in J.m(\delta_j)=\gamma_i(n,\alpha)[g(n,\alpha,\delta_j)]\underset{\text{prop. 4.17}}{=}\gamma_i(n,\delta_j)\in(\delta_j,\delta_j(+^n))$. **(a5*)**

By (a4*) and (a5*), let $j_0\in J$ be such that $\delta_{j_0}\in(v,\alpha)$. **(a6*)**.
Note $\delta_{j_0}\underset{\text{by (a5*)}}{\in}\mathrm{Class}(n)\backslash\mathrm{Class}(n+1)$. This way,
$\mathrm{tp}(\delta_{j_0},\Omega)=m(\delta_{j_0})[g(n,\delta_{j_0},\Omega)]\underset{\text{by (a5*)}}{=}\gamma_i(n,\delta_{j_0})[g(n,\delta_{j_0},\Omega)]\underset{\text{prop. 4.17}}{=}\gamma_i(n,\Omega)\underset{\text{by (a3*)}}{>}\rho$. Contradiction with (a0*).

Thus (a1*) holds.

b)
Suppose $T(n,\Omega,\rho)\cap\Omega\subset\alpha$. **(*b1)**

Let's see that $\mathrm{tp}(\alpha,\Omega)$ is bounded as stated in the claim of the theorem.

Suppose $\alpha\notin\mathrm{Lim}(\mathrm{Class}(n))$. Then $\forall u\in[1,\omega).\alpha\in\mathrm{Class}(u)\Longrightarrow u\leqslant n$. This clearly implies that $\mathrm{tp}(\alpha,\Omega)\leqslant\Omega(+^{n-1})\cdots(+^1)2$.

So let's suppose from now on that $\alpha\in\mathrm{Lim}(\mathrm{Class}(n))$. **(0)**

Consider the ordinals $\beta(\alpha):=\begin{cases}\sigma_2 & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1>\ldots>\sigma_q\}\wedge q\geqslant 2\\ \lambda(n+1,\alpha) & \text{iff } f(n+1,\lambda(n+1,\alpha))=\{\alpha=\sigma_1\}\end{cases}$ and $\gamma:=\sup\{m(\delta)[g(n,\delta,\alpha)]\mid\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n)\}$.

Note $(\beta(\alpha),\alpha)\subset(v,\alpha)$ and so by (a0*) we obtain
$\forall\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n).m(\delta)[g(n,\delta,\alpha)][g(n,\alpha,\Omega)]=m(\delta)[g(n,\delta,\Omega)]=\mathrm{tp}(\delta,\Omega)<\rho$; from this and (*b1) we get $\forall\delta\in(\beta(\alpha),\alpha)\cap\mathrm{Class}(n).m(\delta)[g(n,\delta,\alpha)]<\rho[g(n,\Omega,\alpha)]$. Thus
$\gamma\leqslant\rho[g(n,\Omega,\alpha)]$ and then
$\gamma[g(n,\alpha,\Omega)]\leqslant\rho[g(n,\Omega,\alpha)][g(n,\alpha,\Omega)]=\rho$. Therefore:
$\eta(n,\Omega,\gamma[g(n,\alpha,\Omega)])\underset{\text{by prop. 6.8}}{\leqslant}\eta(n,\Omega,\rho)$; **(1)** and
$\gamma[g(n,\alpha,\Omega)]+1\leqslant\rho+1$. **(2)**

Finally, by (0), (1), (2) and previous proposition 7.31, we conclude
$\mathrm{tp}(\alpha,\Omega)=\begin{cases}\eta(n,\Omega,\gamma[g(n,\alpha,\Omega)])\leqslant\eta(n,\Omega,\rho) & \text{if } l(n,\alpha,m(\alpha))\in\mathrm{Lim}\\ \gamma[g(n,\alpha,\Omega)]+1\leqslant\rho+1 & \text{if } l(n,\alpha,m(\alpha))\notin\mathrm{Lim}\end{cases}$. □

7.3.3 $O_2\geqslant|\mathrm{ID}_1|$

Our goal in this subsection is proposition 7.35, whose corollary is $O_2\geqslant|\mathrm{ID}_1|$ (corollary 7.36).

Proposition 7.33. $\forall n\in\omega.\forall\alpha\in\Omega_{n+1}(+^1).\forall\xi\in C_n(\alpha).\mathrm{Ep}(\xi)\backslash\{\Omega_i\mid i\geqslant 1\}\subset\psi_n\alpha$.

Proof. Let $n\in\omega$ and $\alpha\in\Omega_{n+1}$ be arbitrary. Let $X:=\{\beta\in\mathrm{OR}\mid\mathrm{Ep}(\beta)\backslash\{\Omega_i\mid i\geqslant 1\}\subset\psi_n\alpha\}$. We proceed by induction on the inductive definition of $C_n(\alpha)$; in fact, we will use the version of the induction principle given by theorem 7.13.

1. $\Omega_n\subset X$ holds because $\Omega_n=\psi_n 0\leqslant\psi_n\alpha$.

2. Suppose $\xi,\delta\in X\cap C_n(\alpha)$. Then clearly
$\mathrm{Ep}(\xi+\delta)\backslash\{\Omega_i\mid i\geqslant 1\}\subset(\mathrm{Ep}(\xi)\backslash\{\Omega_i\mid i\geqslant 1\})\cup(\mathrm{Ep}(\delta)\backslash\{\Omega_i\mid i\geqslant 1\})\subset\psi_n\alpha$. So $\xi+\delta\in X$.

3. Suppose $\xi=\psi_u\delta$ for some $\delta\in\alpha\cap X\cap C_n(\alpha)\cap C_u(\delta)$ and some $u\in[0,\omega]$.
+ Case $u\geqslant n+1$. Since $\delta\in\alpha\in\Omega_{n+1}(+^1)\subset\Omega_{u+1}(+^1)$, then $\xi=\psi_u\delta\underset{\text{prop. 7.21}}{=}\omega^{\Omega_u+\delta}$ and therefore $\mathrm{Ep}(\xi)\backslash\{\Omega_i\mid i\geqslant 1\}=\mathrm{Ep}(\delta)\backslash\{\Omega_i\mid i\geqslant 1\}\underset{\text{because }\delta\in X}{\subset}\psi_n\alpha$.
+ Case $u\leqslant n$. Then $\xi=\psi_u\delta\leqslant\psi_n\delta\underset{\text{because }\delta\in\alpha\cap C_n(\alpha)\text{ and prop. 7.22}}{<}\psi_n\alpha$. So $\mathrm{Ep}(\xi)\subset\psi_n\alpha$ and particularly $\xi\in X$.

From 1, 2 and 3 and theorem 7.13 we conclude $C_n(\alpha)\subset X$, that is, the theorem holds. □

Lemma 7.34.
$\forall n\in\omega.\forall\alpha\in[\Omega_{n+1},\Omega_{n+1}(+^1))$.
$\psi_{n+1}\alpha<\Omega_{n+1}(+^1)\wedge$
$\forall\zeta\in C_n(\alpha)\cap(\Omega_n,\Omega_{n+1}).\mathrm{tp}(\zeta,\Omega_{n+1})<\eta(\psi_{n+1}\alpha)=\eta(1,\Omega_{n+1},\psi_{n+1}\alpha)$.

Proof. Let $n\in\omega$. We now proceed by induction on $([\Omega_{n+1},\Omega_{n+1}(+^1)),<)$.

Let $\alpha\in[\Omega_{n+1},\Omega_{n+1}(+^1))$.

Assume
$\forall\beta\in[\Omega_{n+1},\Omega_{n+1}(+^1))\cap\alpha$.
$\psi_{n+1}\beta<\Omega_{n+1}(+^1)\wedge$
$\forall\zeta\in C_n(\beta)\cap(\Omega_n,\Omega_{n+1}).\mathrm{tp}(\zeta,\Omega_{n+1})<\eta(\psi_{n+1}\beta)$. **(IH)**

First note $\psi_{n+1}\alpha\underset{\text{prop. 7.21}}{=}\omega^{\Omega_{n+1}+\alpha}<\Omega_{n+1}(+^1)$. **(a0*)**

On the other hand, let $X:=\{\xi\in\mathrm{OR}\mid\xi\in(\Omega_n,\Omega_{n+1})\Longrightarrow\mathrm{tp}(\xi,\Omega_{n+1})<\eta(\psi_{n+1}\alpha)\}$. Now we show that $C_0(\alpha)\subset X$; for this purpose, we will use the induction principle given by theorem 7.13.

1. $\Omega_n\subset X$ clearly.

2. Suppose $\xi,\delta\in X\cap C_n(\alpha)$.
Suppose $\xi+\delta\in(\Omega_n,\Omega_{n+1})$. Then it is easy to see that

$$\mathrm{tp}(\xi+\delta,\Omega_{n+1})=\begin{cases}\mathrm{tp}(\delta,\Omega_{n+1})<\eta(\psi_{n+1}\alpha) & \text{iff } \xi+\delta=\delta\\ \Omega_{n+1}<\Omega_{n+1}2\leqslant\eta(\psi_{n+1}\alpha) & \text{iff } \xi<\xi+\delta>\delta\\ \mathrm{tp}(\xi,\Omega_{n+1})<\eta(\psi_{n+1}\alpha) & \text{iff } \xi+\delta=\xi\end{cases}.$$ Thus $\xi+\delta\in X$.

3. Suppose $\xi=\psi_u\delta$ for some $\delta\in X\cap C_n(\alpha)\cap C_u(\delta)\cap\alpha$ and some $u\in[0,\omega]$.

Suppose $\xi\in(\Omega_n,\Omega_{n+1})$. **(0*)**.

Then $u=n$ and we have $\xi=\psi_n\delta$ for some $\delta\in X\cap C_n(\alpha)\cap C_n(\delta)\cap\alpha$. **(1*)**.

\+ Case $\delta<\Omega_{n+1}$. Then, by (1*) and prop. 7.24, $\delta\in\Omega_n(+^1)$. Therefore $\xi=\psi_n\delta \underset{\text{prop. 7.21}}{=} \omega^{\Omega_n+\delta} \underset{\text{by (0*)}}{\subset} (\Omega_n,\ \Omega_n(+^1))$ and then $\mathrm{tp}(\xi,\ \Omega_{n+1})<\Omega_{n+1}2\leqslant\eta(\psi_{n+1}\alpha)$. Thus $\xi\in X$ in this case.

\+ Case $\delta=\Omega_{n+1}$. Then $\xi=\psi_n\delta=\Omega_n(+^1)$ and then $\mathrm{tp}(\xi,\ \Omega_{n+1})=\mathrm{tp}(\Omega_n(+^1),\ \Omega_{n+1})=\Omega_{n+1}2<\omega^{\Omega_{n+1}+\Omega_{n+1}}\leqslant\omega^{\Omega_{n+1}+\alpha} \underset{\text{by prop. 7.21}}{=} \psi_{n+1}\alpha\leqslant\eta(\psi_{n+1}\alpha)$. Thus $\xi\in X$ in this case.

\+ Case $\delta>\Omega_{n+1}$. Then $\delta \underset{\text{by (1*)}}{\in} (\Omega_{n+1},\alpha)\subset[\Omega_{n+1},\Omega_{n+1}(+^1))$ and then, by our (IH) applied to δ, we get
$\psi_{n+1}\delta<\Omega_{n+1}(+^1)\wedge$
$\forall\zeta\in C_n(\delta)\cap(\Omega_n,\Omega_{n+1}).\mathrm{tp}(\zeta,\Omega_{n+1})<\eta(\psi_{n+1}\delta)<\Omega_{n+1}(+^1)$. **(2*)**

To show $\mathrm{tp}(\xi,\Omega_{n+1})=\mathrm{tp}(\psi_n\delta,\Omega_{n+1})<\eta(\psi_{n+1}\alpha)$. **(b1*)**

First note that, since $\psi_n\delta=C_n(\delta)\cap\Omega_{n+1}$, then $C_n(\delta)\cap(\Omega_n,\Omega_{n+1})=(\Omega_n,\ \psi_n\delta)$. **(3*)**.
Moreover, $\mathrm{Ep}(\eta(\psi_{n+1}\delta))\cap\Omega_{n+1}\subset\mathrm{Ep}(\psi_{n+1}\delta)\cap\Omega_{n+1}=\mathrm{Ep}(\omega^{\Omega_{n+1}+\delta})\cap\Omega_{n+1}\subset$
$\mathrm{Ep}(\delta)\cap\Omega_{n+1} \underset{\text{prop. 7.33}}{\subset} \psi_n\delta$. This, (0*), (2*) and (3*) imply, by proposition 7.32,
$\xi=\psi_n\delta<\Omega_n(+^2)$ and

$$\mathrm{tp}(\psi_n\delta,\Omega_{n+1})\leqslant\left\{\begin{array}{ll}\Omega_{n+1}2 & \text{iff } \psi_n\delta\notin\mathrm{Lim}(\mathrm{Class}(1))\\ \eta(\eta(\psi_{n+1}\delta)) & \text{iff } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(1))\wedge l(1,\psi_n\delta,m(\psi_n\delta))\in\mathrm{Lim}\\ \eta(\psi_{n+1}\delta)+1 & \text{iff } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(1))\wedge l(1,\psi_n\delta,m(\psi_n\delta))\in\mathrm{Lim}\end{array}\right\}=$$

$$\underset{\text{by prop.2.26}}{=}\left\{\begin{array}{ll}\Omega_{n+1}2 & \text{iff } \psi_n\delta\notin\mathrm{Lim}(\mathrm{Class}(1))\\ \eta(\psi_{n+1}\delta) & \text{iff } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(1))\wedge l(1,\psi_n\delta,m(\psi_n\delta))\in\mathrm{Lim}\\ \eta(\psi_{n+1}\delta)+1 & \text{iff } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(1))\wedge l(1,\psi_n\delta,m(\psi_n\delta))\in\mathrm{Lim}\end{array}\right\}\leqslant$$

$\leqslant\eta(\psi_{n+1}\delta)+1$. **(b2*)**
But for any $x\in[\Omega_{n+1},\ \Omega_{n+1}(+^1))\cap\mathbb{P}$, $\eta(x)<\min\{y\in\mathbb{P}\mid y>x\}$. From this observation we have that
$\eta(\psi_{n+1}\delta)+1<\min\{y\in\mathbb{P}\mid\psi_{n+1}\delta>x\} \underset{\substack{\text{because } \delta\in C_n(\delta)\Longrightarrow\delta\in C_{n+1}(\delta)\\ \text{and by prop.7.20}}}{=}$
$(\psi_{n+1}(\delta+1))\leqslant\psi_{n+1}\alpha\leqslant\eta(\psi_{n+1}\alpha)$. **(b3*)**

(b2*) and (b3*) show that (b1*) holds. Thus, $\xi\in X$ in this case too.

Finally, from 1, 2 and 3 and the induction principle given by theorem 7.13, we conclude that $C_n(\alpha)\subset X$. This and (a0*) prove that the whole theorem holds. □

Proposition 7.35. *Let $n\in\omega$. Then for any $\alpha\in[\Omega_{n+1},\Omega_{n+1}(+^1))$, $\psi_n\alpha<\Omega_n(+^2)$.*

Proof. Let $\alpha\in[\Omega_{n+1},\Omega_{n+1}(+^1))$.
From the equality $\psi_n\alpha \underset{\text{prop. 7.20}}{=} C_n(\alpha)\cap\Omega_{n+1}$ and previous lemma 7.34 we get
$\forall\xi\in(\Omega_n,\ \psi_n\alpha).\mathrm{tp}(\xi,\ \Omega_{n+1})<\eta(\psi_{n+1}\alpha)$. This implies that $\psi_n(\alpha)\leqslant\Omega_n(+^2)$, but since by lemma 7.32 $\psi_n(\alpha)\notin\mathrm{Class}(2)$, then $\psi_n(\alpha)<\Omega_n(+^2)$. □

Corollary 7.36. $\forall n\in\omega.\psi_n(\Omega_{n+2})\leqslant\Omega_n(+^2)$. *In particular* $\psi_0(\Omega_2)=|\mathrm{ID}_1|\leqslant O_2$.

Proof. Let $n\in\omega$. Consider the sequence $(\xi_i)_{i\in\omega}$ defined recursively as
$\xi_0:=\Omega_{n+1}+1$, $\xi_{i+1}:=\omega^{\xi_i}$. Then $\forall i\in\omega.\xi_i \underset{\text{Easy}}{\in} C_n(\xi_i)\cap[\Omega_{n+1},\ \Omega_{n+1}(+^1))$ and therefore, by previous proposition 7.35, $\forall i\in\omega.\psi_n(\xi_i)<\Omega_n(+^2)$. From this and the fact that
$\psi_n(\xi_i)\underset{\text{cof}}{\longrightarrow}\psi_n(\Omega_{n+1}(+^1))=\psi_n(\Omega_{n+2})$ we conclude $\psi_n(\Omega_{n+2})\leqslant\Omega_n(+^2)$. □

7.3.4 $O_2 \leqslant |\mathrm{ID}_1|$

Because of certain technical problems working with the previously introduced version of the psi functions, in this section we will work with another variant of them that we will denote as $(\psi_n^1)_{n\in[0,\omega]}$.

Definition 7.37. *By recursion on OR, the functions ψ_u^1 and C_u^1 (for any $u\in[0,\omega]$) are simultaneously defined in the following way: Suppose that $C_u^1(\xi)$ and $\psi_u^1(\xi)$ are defined for all $\xi<\alpha$ and for all $u\in[0,\omega]$.*

Then, for any $v\in[0,\omega]$, $\psi_v^1\alpha := \min\{\gamma\in\mathrm{OR} \mid \gamma\notin C_v^1(\alpha)\}$, where the set $C_v^1(\alpha)$ is inductively defined by the following clauses:

- ***(C1).*** $\{\Omega_v \mid v\in[0,\omega]\}\subset C_v^1(\alpha)$
- ***(C2).*** $\Omega_v\subset C_v^1(\alpha)$
- ***(C3).*** $\sigma,\delta\in C_v^1(\alpha)\wedge\xi=\omega^\sigma\dot{+}\delta\Longrightarrow\xi\in C_v^1(\alpha)$
- ***(C4).*** $n\in[0,\omega]\wedge\sigma\in\alpha\cap C_v^1(\alpha)\cap C_n^1(\sigma)\Longrightarrow\psi_n^1\sigma\in C_v^1(\alpha)$

Definition 7.38. *For $n\in[0,\omega]$, let $\mathcal{G}_v^1 := \{\alpha\in\mathrm{OR}\mid\alpha\in C_v^1(\alpha)\}$.*

Proposition 7.39. *Let $v\in[0,\omega]$ and $\alpha,\beta,\gamma,\xi,\delta\in\mathrm{OR}$. Then*

- *a)* $\alpha\leqslant\beta\Longrightarrow C_v^1(\alpha)\subset C_v^1(\beta)$ *and* $\psi_v^1\alpha\leqslant\psi_v^1\beta$
- *b)* $\psi_v^1\alpha\subset C_v^1(\alpha)$.
- *c)* $\Omega_v<\psi_v^1\alpha<\Omega_{v+1}$.
- *d)* $\psi_v^1[\alpha\cap\mathcal{G}_v^1]\subset\psi_v^1(\alpha)$
- *e)* $\psi_v^1\alpha\in\mathbb{E}$.
- *f)* $\beta\in\mathbb{E}\wedge\psi_v^1[\alpha\cap\mathcal{G}_v^1]\subset\beta\Longrightarrow C_v^1(\alpha)\cap\Omega_{v+1}\subset\beta$
- *g)* $\psi_v^1(\alpha)=C_v^1(\alpha)\cap\Omega_{v+1}$
- *h)* $\gamma\in C_v^1(\alpha)\wedge\gamma=\omega^\sigma\dot{+}\delta\Longrightarrow\sigma,\delta\in C_v^1(\alpha)$
- *i)* $\gamma<\Omega_{v+1}(+^1)\Longrightarrow\gamma\in C_v^1(\alpha)\Longleftrightarrow\mathrm{Ep}(\gamma)\cap\Omega_{v+1}\subset\psi_v^1(\alpha)$
- *j)* $C_v^1(\alpha)$ *is closed under* $+$ *and* $\lambda x.\omega^x$.
- *k)* *If* $\alpha_0<\alpha$ *and* $[\alpha_0,\alpha)\cap C_v^1(\alpha_0)=\emptyset$, *then* $C_v^1(\alpha_0)=C_v^1(\alpha)$

Proof. Proof as in Buchholz notes [5]. □

Proposition 7.40. *Let $v\in[0,\omega]$ and $\alpha,\beta\in\mathrm{OR}$ with $\alpha<\beta$. Then*

- *a)* $\psi_v^1(\alpha)=\min\{\xi\in\mathbb{E}\mid\psi_v[\alpha\cap\mathcal{G}_v]\subset\xi\}$
- *b)* $\alpha\in\mathrm{Lim}\Longrightarrow\psi_v^1(\alpha)=\sup\psi_v^1[\alpha]=\sup\psi_v^1[\alpha\cap\mathcal{G}_v]$
- *c)* $\mathcal{G}_v^1\cap[\alpha,\beta)=\emptyset\Longrightarrow C_v^1(\beta)=C_v^1(\alpha)$.
- *d)* $\mathcal{G}_v^1\cap[\alpha,\beta)\neq\emptyset\Longleftrightarrow\psi_v\alpha<\psi_v\beta$.
- *e)* $\alpha\in\beta\cap C_v^1(\beta)\Longrightarrow\psi_v^1\alpha<\psi_v^1\beta$.

Proof. Proof as in Buchholz notes [5]. □

Proposition 7.41. *Let* $v\in[0,\omega]$ *and* $\alpha,\beta,\gamma,\xi<\Omega_{v+1}(+^1)$. *Then*

a) $\gamma\dot{+}\alpha\in\mathcal{G}^1_v\Longrightarrow\gamma\in\mathcal{G}^1_v$

b) $\gamma\in\mathbb{E}\cap(\Omega_v,\psi^1_v(\alpha))\Longrightarrow\gamma=\psi^1_v(\xi)$ *with* $\xi\in\alpha\cap\mathcal{G}^1_v$

c) $\gamma+\omega^\beta\leqslant\alpha<\gamma+\omega^{\beta+1}\wedge\alpha\in\mathcal{G}^1_v\Longrightarrow\mathrm{Ep}(\beta)\cap\Omega_{v+1}\subset\psi^1_v(\gamma+\omega^\beta)$

d) $\gamma\dot{+}\omega^{\beta+1}\in\mathcal{G}^1_v\Longrightarrow\gamma\dot{+}\omega^\beta n\in\mathcal{G}^1_v$ *for all* $n\in\omega$.

e) $\gamma\dot{+}\omega^\beta\in\mathcal{G}^1_v\wedge\beta\in\mathrm{Lim}\wedge\xi\in(\gamma+\omega^\beta)\cap\mathcal{G}^1_v\Longrightarrow\exists\sigma<\beta.\xi<\gamma\dot{+}\omega^\sigma\in\mathcal{G}^1_v$

Proof. Proof as in [5]. □

The following is the main lemma of this subsection. The proof carried out here uses several ideas appearing in Buchholz notes [5].

Lemma 7.42.
$\forall n\in\omega.\forall\alpha\forall\gamma\forall\beta.\alpha=\gamma\dot{+}\omega^\beta\wedge\alpha\in\mathcal{G}^1_n\cap\Omega_{n+1}(+^1)\wedge\beta\in[\Omega_{n+1},\Omega_{n+1}(+^1))\Longrightarrow$
$\mathrm{Ep}(\beta)\cap\Omega_{v+1}\subset\psi^1_n\alpha\wedge\psi^1_n\alpha\leqslant_1\psi^1_n\alpha+\beta[g(1,\Omega_{n+1},\psi^1_n\alpha)]$

Proof. Let $n\in\omega$. By induction on $(\mathrm{OR},<)$ we prove:
$\forall\alpha\forall\gamma\forall\beta.\alpha=\gamma\dot{+}\omega^\beta\wedge\alpha\in\mathcal{G}^1_n\wedge\beta\in[\Omega_{n+1},\Omega_{n+1}(+^1))\Longrightarrow$
$\mathrm{Ep}(\beta)\cap\Omega_{v+1}\subset\psi^1_n\alpha\wedge\psi^1_n\alpha\leqslant_1\psi^1_n\alpha+\beta[g(1,\Omega_{n+1},\psi^1_n\alpha)]$ **(*)**

Let $\alpha\in\mathrm{OR}$.
Suppose (*) holds for any $\alpha'<\alpha$. **(IH)**

Suppose $\alpha=\gamma\dot{+}\omega^\beta$ for some $\gamma,\beta\in\mathrm{OR}$ and that $\alpha\in\mathcal{G}^1_n$ and $\beta\in[\Omega_{n+1},\Omega_{n+1}(+^1))$. **(s1)**
Note $\mathrm{Ep}(\beta)\cap\Omega_{v+1}\subset\psi^1_n\alpha$ follows from clause *c)* of proposition 7.41. **(s2)**. So we only have to see that $\psi^1_n\alpha\leqslant_1\psi^1_n\alpha+\beta[g(1,\Omega_{n+1},\psi^1_n\alpha)]$.

We have certain subcases:

Subcase $\beta=\Omega_{n+1}$.
Easy: Since $\psi^1_n\alpha\in\mathbb{E}$, then $\psi^1_n\alpha\leqslant_1\psi^1_n\alpha+\psi^1_n\alpha=\psi^1_n\alpha+\Omega_{n+1}[g(1,\Omega_{n+1},\psi^1_n\alpha)]$.

Subcase $\beta=l+1$.

Consider the sequence $(\alpha_m)_{m\in\omega}$ defined as $\alpha_m:=\gamma\dot{+}\omega^l(m+1)$. Then
$\alpha_m\underset{\mathrm{cof}}{\hookrightarrow}\alpha$ **(u1)**; moreover, $\alpha=\gamma\dot{+}\omega^{l+1}\in\mathcal{G}^1_n$ implies, by clause *d)* of proposition 7.41,
$\forall m\in\omega.\alpha_m\in\mathcal{G}^1_n\cap\alpha$. **(u2)**. Therefore
$\psi^1_n\alpha\underset{\text{clause } b) \text{ of prop. 7.40}}{=}\sup\{\psi^1_n\xi\mid\xi\in\mathcal{G}^1_n\cap\alpha\}=\sup\{\psi^1_n(\alpha_m)\mid m\in\omega\}$. **(a0)**

On the other hand, note $l\geqslant\Omega_{n+1}$ and then, by (u2) and our (IH)
$\forall m\in\omega.\psi^1_n(\alpha_m)\in\psi^1_n(\alpha_m)\leqslant_1\psi^1_n(\alpha_m)+l[g(1,\Omega_{n+1},\psi^1_n(\alpha_m))]$. **(a4)**

Now, from $\psi^1_n\alpha\in\mathbb{E}$ and (a4) we get
$\forall m\in\omega$.
$\psi^1_n(\alpha_m)\leqslant_1\psi^1_n(\alpha_m)+l[g(1,\Omega_{n+1},\psi^1_n(\alpha_m))]=(\psi^1_n\alpha+l[g(1,\Omega_{n+1},\psi^1_n\alpha)])[g(1,\psi^1_n\alpha,\psi^1_n(\alpha_m))]$,
which implies, by (a0) and our cofinality properties for Class(1) (see proposition 2.40),
$\psi^1_n(\alpha)\leqslant_1\psi^1_n\alpha+l[g(1,\Omega_{n+1},\psi^1_n\alpha)]+1=$
$\psi^1_n\alpha+(l+1)[g(1,\Omega_{n+1},\psi^1_n\alpha)]=\psi^1_n\alpha+\beta[g(1,\Omega_{n+1},\psi^1_n\alpha)]$. Thus the theorem holds in this subcase.

Subcase $\beta \in (\Omega_{n+1}, \Omega_{n+1}(+^1)) \cap$ Lim.

Since $\gamma\dot{+}\omega^\beta \in \mathcal{G}_n^1$ and $0 \in \gamma\dot{+}\omega^\beta \cap \mathcal{G}_n^1$, then clause $e)$ of proposition 7.41 implies that the set $\{\sigma < \beta \mid \gamma\dot{+}\omega^\sigma \in \mathcal{G}_n\} \neq \emptyset$. Moreover, for $\delta := \sup\{\sigma < \beta \mid \gamma\dot{+}\omega^\sigma \in \mathcal{G}_n^1\}$, **(c1)**
clause $e)$ of proposition 7.41 implies $\delta \notin \{\sigma < \beta \mid \gamma\dot{+}\omega^\sigma \in \mathcal{G}_n^1\}$, from which follows the existence of an increasing sequence $(\sigma_i)_{i\in I} \subset \delta$ such that $\sigma_i \underset{\text{cof}}{\longrightarrow} \delta$ and $\forall i \in I.\gamma\dot{+}\omega^{\sigma_i} \in \mathcal{G}_n^1$. **(c2)**

To show $\psi_n^1\alpha = \psi_n^1(\gamma\dot{+}\omega^\delta) = \sup\{\psi_n^1(\gamma\dot{+}\omega^{\sigma_i}) \mid i \in I\}$. **(c3)**

The right hand side equality in (c3) clearly follows from (c2). So let's prove the left hand side equality. If $\delta = \beta$ then (c3) clearly holds. So suppose $\delta < \beta$. Let $\xi \in [\gamma\dot{+}\omega^\delta, \gamma\dot{+}\omega^\beta)$. Then it is not possible that $\xi \in \mathcal{G}_n^1$, otherwise, by clause $e)$ of proposition 7.41, there exists $\rho < \beta$ such that
$\gamma\dot{+}\omega^\delta \leqslant \xi < \gamma\dot{+}\omega^\rho \in \mathcal{G}_n^1$. This implies $\delta < \rho \in \{\sigma < \beta \mid \gamma\dot{+}\omega^\sigma \in \mathcal{G}_n^1\}$. Contradiction with (c1). The previous shows that $[\gamma\dot{+}\omega^\delta, \gamma\dot{+}\omega^\beta) \cap \mathcal{G}_n^1 = \emptyset$ which implies, by clause $c)$ of proposition 7.40, $C_n^1(\gamma\dot{+}\omega^\delta) = C_n^1(\gamma\dot{+}\omega^\beta)$ and therefore
$\psi_n^1(\gamma\dot{+}\omega^\delta) = C_n^1(\gamma\dot{+}\omega^\delta) \cap \Omega_{n+1} = C_n^1(\gamma\dot{+}\omega^\beta) \cap \Omega_{n+1} = \psi_n^1(\gamma\dot{+}\omega^\beta) = \psi_n^1\alpha$. Thus (c3) holds.

Now, let's see that $\exists \rho \in [\Omega_{n+1}, \beta).\gamma\dot{+}\omega^\rho \in C_n^1(\gamma\dot{+}\omega^\rho)$. **(f0)**

Since $0 \in \mathcal{G}_n^1 \cap (\gamma\dot{+}\omega^\beta)$ and $\gamma\dot{+}\omega^\beta \in \mathcal{G}_n^1$, then, by clause $e)$ of proposition 7.41, there exists some
$\xi < \beta$ such that $\gamma\dot{+}\omega^\xi \in C_n^1(\gamma\dot{+}\omega^\xi)$. If $\xi \geqslant \Omega_{n+1}$ then (f0) holds. So suppose $\xi < \Omega_{n+1}$. Then, using prop. 7.39, we get $\gamma \in C_n^1(\gamma\dot{+}\omega^\xi) \subset C_n^1(\gamma\dot{+}\omega^{\Omega_{n+1}}) \ni \Omega_{n+1}$; therefore, since $C_v^1(\gamma\dot{+}\omega^{\Omega_{n+1}})$ is closed under $+$ by prop. 7.39, $\gamma\dot{+}\omega^{\Omega_{n+1}} \in C_n^1(\gamma\dot{+}\omega^{\Omega_{n+1}})$.

Thus (f0) holds.

Because of (f0), we can assume without loss of generality that $\forall i \in I.\sigma_i \geqslant \Omega_{n+1}$. **(f1)**.
Now, from (f1) and our (IH) applied to $\gamma\dot{+}\omega^{\sigma_i} \in \mathcal{G}_n^1 \cap \alpha$ for any $i \in I$, we conclude that
$\forall i \in I.\psi_n^1(\gamma\dot{+}\omega^{\sigma_i}) \leqslant_1 \psi_n^1(\gamma\dot{+}\omega^{\sigma_i}) + \sigma_i[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_i}))]$. **(f2)**

On the other hand, since $\forall i \in I.\sigma_i \underset{\text{using prop. 7.39}}{\in} C_n^1(\gamma\dot{+}\omega^{\sigma_i}) \subset C_n^1(\alpha) \ni \beta \wedge \sigma_i < \delta \leqslant \beta < \Omega_{n+1}(+^1)$, then for any $i \in I$, $(\mathrm{Ep}(\sigma_i) \cap \Omega_{n+1}) \cup (\mathrm{Ep}(\beta) \cap \Omega_{n+1}) \underset{\text{clause } i) \text{ of prop. 7.39}}{\subset} \psi_n^1\alpha$ and so the substitutions $\sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$ and $\beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$ are well defined and are such that
$\sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)] \in [\psi_n^1\alpha, (\psi_n^1\alpha)(+^1)) \ni \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$.

To show $\sup\{\sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)] \mid i \in I\} = \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. **(f4)**

Since $\forall i \in I.\sigma_i < \beta$, then $\forall i \in I.\sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)] < \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$; therefore
$\xi := \sup\{\sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)] \mid i \in I\} \leqslant \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. **(f5)**

Now we show that $\xi \not< \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. **(f6)**

Suppose that $\xi < \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. Then we have
$\forall i \in I.\sigma_i \leqslant \xi[g(1, \psi_n^1\alpha, \Omega_{n+1})] < \beta$. **(f7)**.

On the other hand, by proposition 2.10, $\mathrm{Ep}(\xi[g(1, \psi_n^1\alpha, \Omega_{n+1})]) \cap \Omega_{n+1} = \mathrm{Ep}(\xi) \cap \psi_n^1\alpha \subset \psi_n^1\alpha$; therefore, by clause $i)$ of proposition 7.39, $\xi[g(1, \psi_n^1\alpha, \Omega_{n+1})] \in C_n^1(\alpha)$. This, the fact that $\gamma \underset{\text{from } h) \text{ of prop.7.39}}{\in} C_n^1(\gamma\dot{+}\omega^\beta) = C_n^1(\alpha)$ and the right hand side inequality of (f7) imply by the definition of $C_n^1(\alpha)$ that $\gamma\dot{+}\omega^{\xi[g(1, \psi_n^1\alpha, \Omega_{n+1})]} \in C_n^1(\alpha) \cap (\gamma\dot{+}\omega^\beta) = C_n^1(\alpha) \cap \alpha$. From the latter and proposition 7.40 we get $\psi_n^1(\gamma\dot{+}\omega^{\xi[g(1, \psi_n^1\alpha, \Omega_{n+1})]}) < \psi_n^1\alpha$. **(f8)**. But from (f8) and (c3) it follows the existence of some $k \in I$ such that
$\psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) > \psi_n^1(\gamma\dot{+}\omega^{\xi[g(1, \psi_n^1\alpha, \Omega_{n+1})]}) \underset{\text{by (f7) and the monotonicity of } \psi_n^1}{\geqslant} \psi_n^1(\gamma\dot{+}\omega^{\sigma_k})$. Contradiction.
Thus $\xi \not< \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. This concludes the proof of (f6).

(f6) and (f5) show that (f4) holds.

Finally, we can show that $\psi_n^1\alpha \leqslant_1 \psi_n^1\alpha + \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. **(f9)**

Since $(\sigma_i)_{i\in I}$ is increasing, then for any $i, k \in I$ with $i < k$, $\sigma_i \in C_n^1(\gamma\dot{+}\omega^{\sigma_i}) \subset C_n^1(\gamma\dot{+}\omega^{\sigma_k}) \ni \sigma_k$ and therefore by clause $i)$ of proposition 7.39, $\mathrm{Ep}(\sigma_i) \cap \Omega_{n+1} \subset \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) \supset \mathrm{Ep}(\sigma_k) \cap \Omega_{n+1}$; because of this, the substitutions $\sigma_i[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))]$ and $\sigma_k[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))]$ are well defined and are such that $\psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) \leqslant \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) + \sigma_i[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))] \leqslant$ $\psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) + \sigma_k[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))]$. From the previous, (f2) and $\leqslant_1$ -connectedness we conclude that
$\forall k \in I. \forall i \in I \cap k.$
$\psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) \leqslant_1 \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}) + \sigma_i[g(1, \Omega_{n+1}, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))] =$
$(\psi_n^1\alpha + \sigma_i[g(1, \Omega_{n+1}, \psi_n^1\alpha)])[g(1, \psi_n^1\alpha, \psi_n^1(\gamma\dot{+}\omega^{\sigma_k}))]$. **(g1)**

Finally, from (g1) and our cofinality properties for Class(1) (see proposition 2.40) we get $\forall i \in I. \psi_n^1\alpha \leqslant_1 \psi_n^1\alpha + \sigma_i[g(1, \Omega_{n+1}, \psi_n\alpha)] + 1$. Note this and (f4) imply, by $\leqslant_1$ -continuity, that $\psi_n^1\alpha \leqslant_1 \psi_n^1\alpha + \beta[g(1, \Omega_{n+1}, \psi_n^1\alpha)]$. This shows (f9).

(f9) and (f3) show that the theorem holds also for this subcase. □

Reminder: For an arbitrary $e \in \mathbb{E}$, the sequence $(\omega_k(e))_{k\in\omega}$ was defined in previous chapters as: $\omega_0(e) := e + 1$, $\omega_{i+1}(e) := \omega^{\omega_i(e)}$.

Corollary 7.43.

1. $\forall n \in \omega. \psi_n^1(\Omega_{n+1}(+^1)) \in \mathrm{Class}(2)$.
2. $\forall n \in \omega. \psi_n^1(\Omega_{n+1}(+^1)) \geqslant \Omega_n(+^2)$.

Proof.
1
Let $n \in \omega$. Let's abbreviate $\rho := \psi_n^1(\Omega_{n+1}(+^1))$.

Since $\rho \in \mathbb{E}$, consider the sequence $(\omega_k(\rho))_{k\in\omega}$.

To show $\forall k \in \omega. \rho \leqslant_1 \omega_k(\rho) + 1$. **(1*)**

Let $k \in \omega$ be arbitrary.
Let $s \in \omega$ be also arbitrary. Then
$\omega_{k+s+1}(\Omega_{n+1}) = \omega^{\omega_{k+s}(\Omega_{n+1})} \underset{\text{easy}}{\in} \mathcal{G}_n \cap \Omega_{n+1}(+^1) \wedge \omega_{k+s}(\Omega_{n+1}) \in [\Omega_{n+1}, \Omega_{n+1}(+^1))$ and therefore, by previous lemma 7.42,
$\psi_n^1(\omega_{k+s+1}(\Omega_{n+1})) \leqslant_1 \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})) + \omega_{k+s}(\Omega_{n+1})[g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))]$; this and the fact that
$\omega_k(\Omega_{n+1})[g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))] \leqslant$
$\psi_n^1(\omega_{k+s+1}(\Omega_{n+1})) + \omega_{k+s}(\Omega_{n+1})[g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))]$ imply by $\leqslant_1$ -connectedness that $\psi_n^1(\omega_{k+s+1}(\Omega_{n+1})) \leqslant_1 \omega_k(\Omega_{n+1})[g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))]$. Since the previous holds for any $s \in \omega$, we have shown:
$\forall s \in \omega. \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})) \leqslant_1 \omega_k(\Omega_{n+1})[g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))] \underset{\text{easy}}{=}$
$\omega_k(\rho)[g(1, \rho, \Omega_{n+1})][g(1, \Omega_{n+1}, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))] =$
$\omega_k(\rho)[g(1, \rho, \psi_n^1(\omega_{k+s+1}(\Omega_{n+1})))]$. **(2*)**

Concluding, note that the sequence $(\delta_s)_{s\in\omega}$ defined as $\delta_s := \psi_n^1(\omega_{k+s+1}(\Omega_{n+1}))$ is such that $(\delta_s) \subset \rho \cap \mathbb{E}$, $\delta_s \underset{\text{cof}}{\hookrightarrow} \psi_n^1(\Omega_{n+1}(+^1)) = \rho$ and, by (2*), $\forall s \in \omega.\delta_s \leqslant \omega_k(\rho)[g(1, \rho, \delta_s)]$. Therefore, by our cofinality properties in Class(1) (see proposition 2.40), $\rho \leqslant \omega_k(\rho) + 1$. Since this was done for an arbitrary $k \in \omega$, then we have shown (1*).

Finally, from (1*) and the fact that the sequence $(\omega_k(\rho) + 1)_{k\in\omega}$ is cofinal in $\rho(+^1)$, we get by $\leqslant_1$-continuity that $\rho \leqslant_1 \rho(+^1)$. Thus $\rho \in \text{Class}(2)$.

2
Let $n \in \omega$. Since $\Omega_n \underset{\text{by prop. 7.39}}{<} \psi_n^1(\Omega_{n+1}(+^1)) \underset{\text{by 1.}}{\in} \text{Class}(2)$, then
$\Omega_n(+^2) = \min\{\xi \in \text{Class}(2) | \xi > \Omega_n\} \leqslant \psi_n^1(\Omega_{n+1}(+^1))$. □

Corollary 7.44. $\forall n \in \omega.\psi_n(\Omega_{n+2}) = \Omega_n(+^2)$. *In particular* $|\text{ID}_1| = \psi_0(\Omega_2) = O_2$.

Proof. Let $n \in \omega$. Then
$\Omega_n(+^2) \underset{\text{corollary 7.36}}{\geqslant} \psi_n(\Omega_{n+2}) = \psi_n(\Omega_{n+1}(+^1)) \geqslant \psi_n^1(\Omega_{n+1}(+^1)) \underset{\text{corollary 7.43}}{\geqslant} \Omega_n(+^2)$.
Thus $\psi_n(\Omega_{n+2}) = \Omega_n(+^2)$. □

7.4 Conjecture involving $|\text{ID}_n|$ and the ordinals O_{n+1}

Motivated by corollaries 7.10, 7.11 and 7.44, I conjecture that $\forall n \in \omega \forall m \in [1, \omega).\psi_n(\Omega_{n+m}) = \Omega_n(+^m)$. In particular this would mean $\forall n \in [1, \omega).O_{n+1} = |\text{ID}_n|$ and as an easy consequence $O_\omega = |\Pi_1^1\text{-CA}_0|$. Once more, the reader is warned that the assertion $\forall n \in \omega \forall m \in [1, \omega).\psi_n(\Omega_{n+m}) = \Omega_n(+^m)$ is a conjecture: There are many technical difficulties that one needs to overcome in order to achieve such a goal (for example one might need to consider different versions of the ψ_v functions). The best result in such direction that I have is the next "lemma" whose "proof" assumes the validity of some results which have NOT been justified. I dare to include this "result" because my impression is that, if not exactly as it is stated, a variation of it should work.

The following are the NOT proven assumptions used in next lemma 7.46.

Let $n \in \omega$ and $k \in [1, \omega)$ and $r \in [\Omega_{n+k}, \Omega_{n+k}(+^1))$.
W^r is an ordinal in $[\Omega_{n+k}, \Omega_{n+k}(+^1))$ depending on r (and possibly on k) such that:
1. $\forall r \in [\Omega_{n+k}, \Omega_{n+k}(+^1)).\psi_{n+1}(W^r) \in \text{Class}(k-1)$.
2. $\forall r, s \in [\Omega_{n+k}, \Omega_{n+k}(+^1)).r \leqslant s \Longrightarrow W^r \leqslant W^s$.
3. $\forall \delta \in C_n(\delta) \cap [\Omega_{n+k}, \Omega_{n+k}(+^1)).\psi_{n+1}W^\delta < \psi_{n+1}W^{\delta+1}$.
4. $\forall \delta \in C_n(\delta) \cap [\Omega_{n+k}, \Omega_{n+k}(+^1)).T(k, \Omega_{n+1}, \eta(k, \Omega_{n+1}, \psi_{n+1}(W^\delta))) \cap \Omega_{n+1} \subset \psi_n\delta$.

Remark 7.45. Note that if we define $W^r := \Omega_{n+1}^r$, then 2 clearly holds. Moreover, for the ψ_v^1 functions holds: $\forall \delta \in C_n^1(\delta).\psi_{n+1}^1 W^\delta < \psi_{n+1}^1 W^{\delta+1}$. This is because
$\delta \in C_n^1(\delta) \cap [\Omega_{n+k}, \Omega_{n+k}(+^1)) \Longrightarrow \delta \in C_{n+1}^1(W^{\delta+1}) \cap W^{\delta+1} \Longrightarrow$
$W^\delta \in C_{n+1}(W^{\delta+1}) \Longrightarrow \psi_{n+1}^1 W^\delta < \psi_{n+1}^1 W^{\delta+1}$.
So assumptions 2 and 3 do not look so hard.

Assumption 1 looks more problematic, however, one should note that assumption 1 essentially tell us that we "need to understand first the ordinals $\psi_{n+1}(r)$ for $r \in [\Omega_{n+k}, \Omega_{n+k}(+^1))$"; in particular "from our induction hypothesis" we should know that
$\{\psi_{n+1}(W^r) \mid r \in [\Omega_{n+k}, \Omega_{n+k}(+^1))\} \subset \text{Class}(k-1)$.

Assumption 4 is quite technical (and annoying). It is this condition the reason why one might need to consider a different version of the ψ_v functions such that this assumption holds. But this is actually something not new: In order to prove corollary 7.43, we used the functions ψ_v^1 (instead of the functions ψ_v) and we just mentioned that the reason for that were certain technical problems we had. Well, to be more precise, the problem is that clause *i*) of proposition 7.39 does not hold with the ψ_v functions; this causes serious problems because clause *i*) of proposition 7.39 is used in the proofs of clauses *c*), *d*) and *e*) of proposition 7.41 and in the proof of lemma 7.42.

Because of the previous observations, assumption 4 seems to be the general form of a difficulty we had earlier.

Lemma 7.46.
$\forall k \in [1,\omega).\forall n \in \omega.\forall \alpha \in [\Omega_{n+k}, \Omega_{n+k}(+^1)).$
$\psi_{n+1}(W^\alpha) < \Omega_{n+1}(+^k) \wedge$
$\forall \xi \in C_n(\alpha) \cap (\Omega_n, \Omega_{n+1}).\mathrm{tp}(\xi, \Omega_{n+1}) < \eta(k, \Omega_{n+1}, \psi_{n+1}(W^\alpha)).$

Proof. (The steps that are NOT justified appear ***emphasized***).

Let $A := \{(k,n,\alpha) \mid k \in [1,\omega) \wedge n \in \omega \wedge \alpha \in [\Omega_{n+k}, \Omega_{n+k}(+^1))\}$. Let $<_A$ be the lexicographical order of A induced by the usual order in the ordinals. Then $(A, <_A)$ is a well order. We prove the theorem by induction on $(A, <_A)$.

Let $(k,n,\alpha) \in A$ and suppose
$\forall (k',n',\alpha') \in A.(k',n',\alpha') <_A (k,n,\alpha) \Longrightarrow$
$\psi_{n'+1}(W^{\alpha'}) < \Omega_{n'+1}(+^{k'}) \wedge$
$\forall \xi' \in C_{n'}(\alpha') \cap (\Omega_{n'}, \Omega_{n'+1}).\mathrm{tp}(\xi', \Omega_{n'+1}) < \eta(k', \Omega_{n'+1}, \psi_{n'+1}(W^{\alpha'})).$ **(IH)**

To show $\psi_{n+1}(W^\alpha) < \Omega_{n+1}(+^k)$. **(a0)**

Case $k=1$. Then $\alpha \in [\Omega_{n+1}, \Omega_{n+1}(+^1))$; so
$\psi_{n+1}(W^\alpha) \underset{\textbf{\textit{By assumption } } W^\alpha < \Omega_{n+1}(+^1)}{<} \Omega_{n+1}(+^1) = \Omega_{n+1}(+^k).$ **(a1)**

Case $k \geqslant 2$. Let $\gamma_0 \in (W^\alpha, \Omega_{n+1}(+^1))$ be such that $\psi_{n+1}(W^\alpha) < \psi_{n+1}(\gamma_0)$ **(a2)** (γ_0 exists because ***by assumption*** $\boldsymbol{W^\alpha < \Omega_{n+1}(+^1)}$). Since $(k-1, n+1, \gamma_0) \in A$ and $(k-1, n+1, \gamma_0) <_A (k,n,\alpha)$, then, from our (IH) applied to $(k-1, n+1, \gamma_0)$, we have that
$\psi_{n+2}(W^{\gamma_0}) < \Omega_{n+2}(+^{k-1}) \wedge$
$\forall \xi \in C_{n+1}(\gamma_0) \cap (\Omega_{n+1}, \Omega_{n+2}).\mathrm{tp}(\xi, \Omega_{n+2}) < \eta(k-1, \Omega_{n+2}, \psi_{n+2}(W^{\gamma_0})) < \Omega_{n+2}(+^{k-1})$; now, since $\Omega_{n+1}(+^k) = \min\{\zeta \in \mathrm{OR} \mid \Omega_{n+1} < \zeta \wedge \mathrm{tp}(\zeta, \Omega_{n+2}) \geqslant \Omega_{n+2}(+^{k-1})\}$, then the previous means $C_{n+1}(\gamma_0) \cap (\Omega_{n+1}, \Omega_{n+2}) \subset \Omega_{n+1}(+^k)$ and therefore
$\psi_{n+1}(W^\alpha) \underset{\text{by (a2)}}{<} \psi_{n+1}(\gamma_0) = \sup(C_{n+1}(\gamma_0) \cap (\Omega_{n+1}, \Omega_{n+2})) \leqslant \Omega_{n+1}(+^k).$ **(a3)**.

(a1) and (a3) show that (a0) holds.

To show $\forall \xi \in C_n(\alpha) \cap (\Omega_n, \Omega_{n+1}).\mathrm{tp}(\xi, \Omega_{n+1}) < \eta(k, \Omega_{n+1}, \psi_{n+1}(W^\alpha)).$ **(b1)**

Let $X := \{\beta \in \mathrm{OR} \mid \beta \in (\Omega_n, \Omega_{n+1}) \Longrightarrow \mathrm{tp}(\beta, \Omega_{n+1}) < \eta(k, \Omega_{n+1}, \psi_{n+1}(W^\alpha))\}$. For the proof of (b1) we proceed by induction on the inductive definition of $C_n(\alpha)$; in fact, we will use the version of the induction principle given by theorem 7.13.

1. Clearly $\Omega_n \subset X$.

2. Suppose $\xi, \delta \in X \cap C_n(\alpha)$.

Suppose $\xi+\delta\in(\Omega_n,\Omega_{n+1})$. Then it is easy to see that

$$\mathrm{tp}(\xi+\delta,\Omega_{n+1})=\begin{cases}\mathrm{tp}(\delta,\Omega_{n+1})<\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha)) & \text{iff } \xi+\delta=\delta\\ \Omega_{n+1}<\Omega_{n+1}2\leqslant\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha)) & \text{iff } \xi<\xi+\delta>\delta\\ \mathrm{tp}(\xi,\Omega_{n+1})<\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha)) & \text{iff } \xi+\delta=\xi\end{cases}.$$

So $\xi+\delta\in X$.

3. Suppose $\xi=\psi_u\delta$ for some $\delta\in\alpha\cap X\cap C_n(\alpha)\cap C_u(\delta)$ and some $u\in[0,\omega]$.
Suppose $\xi\in(\Omega_n,\Omega_{n+1})$. **(b2)**
Then $u=n$ and we have $\xi=\psi_n\delta$ with $\delta\in\alpha\cap X\cap C_n(\alpha)\cap C_n(\delta)$. **(b3)**

Case $\delta<\Omega_n(+^1)$. Then $\delta\underset{\text{by (b2)}}{>}0$ and we have $\xi=\psi_n\delta=\omega^{\Omega_n+\delta}\in(\Omega_n,\Omega_n(+^1))$. This implies $\mathrm{tp}(\xi,\Omega_{n+1})<\Omega_{n+1}(+^1)2\leqslant\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha))$. **(b4)**

Case $\delta\geqslant\Omega_n(+^1)$. Then, $\delta\underset{\text{by (b3)}}{\in}C_n(\delta)\cap[\Omega_n(+^1),\alpha)\subset\Omega_{n+k}(+^1)$; this implies, the existence of some $m\in[1,k]$ such that $\delta\in[\Omega_{n+m},\Omega_{n+m}(+^1))$. **(b5)**

Now, note (b5) means $(m,n,\delta)\in A$; moreover, since $m\in[1,k]$ and $\delta\underset{\text{by (b3)}}{<}\alpha$, then we have that $(m,n,\delta)<_A(k,n,\alpha)$. Therefore, from our (IH) applied to (m,n,δ) we get
$\psi_{n+1}(W^\delta)<\Omega_{n+1}(+^m)\wedge$
$\forall\gamma\in C_n(\delta)\cap(\Omega_n,\Omega_{n+1}).\mathrm{tp}(\gamma,\Omega_{n+1})<\eta(m,\Omega_{n+1},\psi_{n+1}(W^\delta))<\Omega_{n+1}(+^m)$. **(b6)**

Subcase $m<k$.
By (6) and proposition 7.32, $\psi_n(\delta)\notin\mathrm{Class}(m+1)$. Thus
$\mathrm{tp}(\psi_n(\delta),\Omega_{n+1})<\Omega_{n+1}(+^m)\leqslant\Omega_{n+1}(+^{k-1})\leqslant\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha))$.

Subcase $m=k$.
Then (b3) and (b5) assert $\delta\in\alpha\cap X\cap C_n(\alpha)\cap C_n(\delta)\cap[\Omega_{n+k},\Omega_{n+k}(+^1))$ **(c1)**
and (b6) is actually
$\psi_{n+1}(W^\delta)<\Omega_{n+1}(+^k)\wedge$
$\forall\gamma\in C_n(\delta)\cap(\Omega_n,\Omega_{n+1}).\mathrm{tp}(\gamma,\Omega_{n+1})<\eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta))$. **(c2)**

Since by assumption $\boldsymbol{T(k,\Omega_{n+1},\eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta)))\cap\Omega_{n+1}\subset\psi_n\delta}$**,** then $\mathrm{tp}(\psi_n\delta,\Omega_{n+1})\leqslant$

$$\leqslant\begin{cases}\Omega_{n+1}(+^{k-1})...(+^1)2 & \text{if } \psi_n\delta\notin\mathrm{LimClass}(k)\\ \eta(k,\Omega_{n+1},\eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta))) & \text{if } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(k))\wedge l(k,\alpha,m(\psi_n\delta))\in\mathrm{Lim}\\ \eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta))+1 & \text{if } \psi_n\delta\in\mathrm{Lim}(\mathrm{Class}(k))\wedge l(k,\alpha,m(\psi_n\delta))\notin\mathrm{Lim}\end{cases}$$

$$\leqslant\eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta))+1$$
$$\underset{(**)}{<}\psi_{n+1}(W^{\delta+1})\underset{(***)}{\leqslant}\psi_{n+1}(W^\alpha)\leqslant\eta(k,\Omega_{n+1},\psi_{n+1}(W^\alpha))$$

(**) By our assumption $\boldsymbol{\forall r\in[\Omega_{n+k},\Omega_{n+k}(+^1)).\psi_{n+1}(W^r)\in\mathrm{Class}(k-1)}$ we have that $\psi_{n+1}(W^\delta)\in\mathrm{Class}(k-1)$. Then $\eta(k,\Omega_{n+1},\psi_{n+1}(W^\delta))+1$ is smaller than the next ordinal in $\mathrm{Class}(k-1)$ which is less or equal than $\psi_{n+1}(W^{\delta+1})$ by our assumptions
$\boldsymbol{\forall\delta\in C_n(\delta).\psi_{n+1}W^\delta<\psi_{n+1}W^{\delta+1}}$ ***and*** $\boldsymbol{\forall r\in[\Omega_{n+k},\Omega_{n+k}(+^1)).\psi_{n+1}(W^r)\in\mathrm{Class}(k-1)}$**.**

(***) By our assumption $\boldsymbol{\forall r,s\in[\Omega_{n+k},\Omega_{n+k}(+^1)).r\leqslant s\Longrightarrow W^r\leqslant W^s}$**.** □

Proposition 7.47. $\forall n\in\omega.\forall m\in[1,\omega).\forall\alpha\in[\Omega_{n+m},\Omega_{n+m}(+^1)).\psi_n\alpha<\Omega_n(+^{m+1})$.

Proof. It follows from previous lemma 7.46 in a similar way as proposition 7.35. □

Corollary 7.48. $\forall n \in \omega \forall m \in [1, \omega). \psi_n(\Omega_{n+m}) \leqslant \Omega_n(+^m)$. *In particular* $|\mathrm{ID}_m| = \psi_0(\Omega_{m+1}) \leqslant O_m$.

Proof. Let $n \in \omega$.
Case $m = 1$, then it is a known fact that $\psi_n(\Omega_{n+1}) = \Omega_n(+^1)$.
Case $m \geqslant 2$. The claim follows from previous proposition 7.47 in a similar way as corollary 7.36. $\square$

Corollary 7.49. $O_\omega \geqslant |\Pi^1_1\text{-CA}_0|$.

Proof. $O_\omega = \sup \{O_i \mid i \in \omega\} \underset{\text{by previous corollary 7.48}}{\geqslant} \sup \{\psi_0(\Omega_{m+1}) \mid i \in \omega\} = \psi_0(\Omega_\omega) = |\Pi^1_1\text{-CA}_0|$. $\square$

Appendix A
Restriction of isomorphisms

Proposition A.1. *Let* $(C, \bar{R}^C, \bar{f}^C, \bar{c})$, $(Q, \bar{R}^Q, \bar{f}^Q, \bar{q})$ *be structures of a language* L. *Suppose* $(B, \bar{R}^B, \bar{f}^B, \bar{b}) \subset (C, \bar{R}^C, \bar{f}^C, \bar{c})$, *that is,* $B \subset C$, $R^B = R^C \cap B^n$ *for any* n*-ary relation* R^C, $f^B = f^C|_B$ *for any function* f^C *and any distinguished element* b *of* B *is a distinguished element of* C.

Suppose $h: (C, \bar{R}^C, \bar{f}^C, \bar{c}) \longrightarrow (h[C], \bar{R}^{h[C]}, \bar{f}^{h[C]}, \overline{h(c)}) \subset (Q, \bar{R}^Q, \bar{f}^Q, \bar{q})$ *is an isomorphism.*

Then $h|_B: (B, \bar{R}^B, \bar{f}^B, \bar{b}) \longrightarrow (h[B], \bar{R}^{h[B]}, \bar{f}^{h[B]}, \overline{h(b)})$ *is an isomorphism.*

Proof. For any $a_1, ..., a_n \in B$ and any relation R^B we have
$R^B(a_1, ..., a_n) \Longleftrightarrow R^C(a_1, ..., a_n) \Longleftrightarrow R^{h[C]}(h(a_1), ..., h(a_n)) \Longleftrightarrow R^{h[B]}(h|_B(a_1), ..., h|_B(a_n))$.

Clearly $b \in B$ is a distinguished element iff $h(b) = h|_B(b) \in h[B]$ is a distinguished element.

Let's see that the operations behave also correctly (of course the problem is with the closure of such operations):

Let $a_1, ..., a_n \in B$. Suppose $f^C(a_1, ..., a_n) = f^B(a_1, ..., a_n) \in B$. Then
$f^{h[C]}(h(a_1), ..., h(a_n)) \in h[C]$ and $f^{h[C]}(h(a_1), ..., h(a_n)) = h(f^C(a_1, ..., a_n)) = h(f^B(a_1, ..., a_n))$.
Clearly $h(a_1), ..., h(a_n) \in h[B] \subset h[C]$ and so from the previous equalities we have
$f^{h[B]}(h|_B(a_1), ..., h|_B(a_n)) = f^{h[C]}(h(a_1), ..., h(a_n)) = h(f^B(a_1, ..., a_n)) \in h[B]$.

Now suppose $f^{h[B]}(h|_B(a_1), ..., h|_B(a_n)) \in h[B]$. Then there exists $a \in B \subset C$ such that
$h(a) = f^{h[B]}(h|_B(a_1), ..., h|_B(a_n))$. (A)

On the other hand, $f^{h[C]}(h(a_1), ..., h(a_n)) = f^{h[B]}(h|_B(a_1), ..., h|_B(a_n)) \in h[B] \subset h[C]$; then $f^C(a_1, ..., a_n)) \in C$ and $h(f^C(a_1, ..., a_n)) = f^{h[C]}(h(a_1), ..., h(a_n)) = f^{h[B]}(h|_B(a_1), ..., h|_B(a_n))$.
From this and (A) we have found that $h(f^C(a_1, ..., a_n)) = h(a)$ and therefore, since h is bijective,

$f^C(a_1, ..., a_n) = a \in B$. $\square$

Bibliography

[1] H. Bachmann. *Die Normalfunktionen und das Problem der ausgezeichneten Folgen von Ordinalzahlen*. Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, 1950.

[2] J. Bridge. *A simplification of the Bachmann method for generating large countable ordinals*. The Journal of Symbolic Logic, 1975.

[3] W. Buchholz. *Notes on the comparison of* $\hat{\Theta}_\alpha$ *and* F_α *for* $\alpha < \epsilon_{\Omega+1}$. 1981.

[4] W. Buchholz. A new system of proof-theoretic ordinal functions. *Annals of Pure and Applied Logic*, 1986.

[5] W. Buchholz. Notes on ordinals. 28.12.2007.

[6] W. Buchholz, S. Feferman, W. Pohlers, and Sieg W. *Iterated Inductive Definitions and Subsystems of Analysis: Recent proof-theoretical studies*. Lecture Notes in Mathematics. Springer, Berlin Heidelberg New York, 1981.

[7] W. Buchholz and K. Schütte. *Proof theory of impredicative subsystems of analysis*. Bibliopolis, 1988.

[8] T. Carlson. *Knowledge, machines, and the consistency of Reinhardt's strong mechanistic thesis*. Arch. Math. Logic, 1999.

[9] T. Carlson. Ordinal arithmetic and $\sum_1$ -elementarity. *Archive for Mathematical Logic*, 1999.

[10] T. Carlson. Elementary patterns of resemblance. *Annals of Pure and Applied Logic*, 2001.

[11] W. Pohlers. *Proof theory. An introduction*. Lect. Notes Math. Springer, Berlin Heidelberg New York, 1989.

[12] W. Pohlers. *Proof Theory: The first Step into Impredicativity*. Springer, 2008.

[13] M. Rathjen and A. Weiermann. *Proof-theoretic investigations on Kruskal's theorem*. Ann. Pure and Appl. Logic., 1993.

[14] H. Schwichtenberg. *Logic Lecture Notes*. 2004.

[15] K. Schütte. *Proof theory*. Springer-Verlag, 1977.

[16] K. Schütte and S.G. Simpson. *Ein in der reinen Zahlentheorie unbeweisbarer Satz über endliche Folgen von natürlichen Zahlen*. Archiv mathematische Logik Grundlagenforschung, 1985.

[17] A. Setzer. *Ordinalzahlanalyse prädikativer Theorien unter Verwendung von Fundamentalfolgen*. Diplomarbeit, Universität München., 1990.

[18] G. Wilken. *The Bachmann-Howard structure in terms of* $\sum_1$ *-elementarity*. Archive for Mathematical Logic, 2006.

[19] G. Wilken. *Ordinal Arithmetic based on Skolem hulling*. Annals of Pure and Applied Logic, 2007.

[20] G. Wilken. $\sum_1$ *-elementarity and Skolem hull operators*. Annals of Pure and Applied Logic, 2007.