# Module Categories over Quasi-Hopf Algebras and Weak Hopf Algebras and the Projectivity of Hopf Modules

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## Abstract

The categories of representations of finite dimensional quasi-Hopf algebras [Dri90] and weak Hopf algebras [BNS99] are finite multi-tensor categories. In this thesis I classify exact module categories, as defined by Etingof and Ostrik [Ost03b, EO04], over those tensor categories. Closely associated with this classification is the question whether relative Hopf modules are projective or free over comodule algebras.

For quasi-Hopf algebras I prove that for an $H$-simple $H$-comodule algebra $A$, every finite dimensional quasi-Hopf bimodule in ${}_H\mathcal{M}_A^H$ is projective as an $A$-module, and if $A$ is a coideal subalgebra of $H$, then it is even free. In particular, finite dimensional quasi-Hopf algebras are free over their coideal subalgebras. This is a generalization of Skryabin's results on the freeness and projectivity over comodule algebras in the Hopf algebra case [Skr07], and it contains Schauenburg's quasi-Hopf algebra freeness theorem [Sch04]. I deduce that a module category $\mathcal{M}$ over a finite dimensional quasi-Hopf algebra is exact and indecomposable if and only if it is of the form ${}_A\mathcal{M}$ for some $H$-simple $H$-comodule algebra $A$. For this purpose I prove among other things a structure theorem for quasi-Hopf bimodules over smash products and a bijective correspondence between $H$-stable ideals of an $H$-module algebra $R$ and the $H$-costable ideals of $R\#H$. The results on the exact module categories over quasi-Hopf algebras have a direct application to $H$-comodule algebras and smash products: They imply that $H$-simple $H$-comodule algebras are quasi-Frobenius and that, in the case when $H$ is semisimple, the smash product $R\#H$ of an $H$-simple $H$-module algebra $R$ by $H$ is also a semisimple algebra.

For weak Hopf algebras it is not possible to generalize the Nichols-Zöller-Freeness Theorem [NZ89] or Skryabin's theorems [Skr07]: I construct an example of a weak Hopf algebra which is not free over one of its weak Hopf subalgebras. However, Etingof and Ostrik [EO04] showed that surjective tensor functors between finite multi-tensor categories map projective objects to projective ones. Considering the restriction functor ${}_H\mathcal{M} \to {}_K\mathcal{M}$, where $K \subset H$ are weak Hopf algebras, this implies that weak Hopf algebras are projective over weak Hopf subalgebras. I conjecture that for a quasi-Frobenius $H$-simple $H$-comodule algebra $A$ every weak Hopf module in $\mathcal{M}_A^H$ is a projective $A$-module, and I show that this holds for weak Hopf algebras which are free over their base algebras. In this case, I classify the exact indecomposable module categories over $H$ by quasi-Frobenius $H$-comodule algebras which are simple from the right and have trivial coinvariants. My classification also gives a new and more direct proof for the classification in the Hopf algebra case [AM07].

## Zusammenfassung

Die Darstellungskategorien endlichdimensionaler Quasihopfalgebren [Dri90] und schwacher Hopfalgebren [BNS99] sind endliche Multitensorkategorien. In dieser Arbeit klassifiziere ich exakte Modulkategorien, definiert von Etingof und Ostrik [Ost03b, EO04], über diesen Tensorkategorien. In engem Zusammenhang mit dieser Klassifikation steht die Frage, ob Hopfmoduln über $H$-Comodulalgebren projektiv oder sogar frei sind.

Für Quasihopfalgebren beweise ich, dass für eine $H$-einfache $H$-Comodulalgebra $A$ jeder endlichdimensionale quasi-Hopfbimodul in ${}_H\mathcal{M}_A^H$ als $A$-Modul projektiv ist, und wenn $A$ eine Coidealunteralgebra von $H$ ist, dann ist er sogar frei. Insbesondere sind endlichdimensionale Qausihopfalgebren frei über ihren Coidealunteralgebren. Dies ist eine Verallgemeinerung von Skryabins Resultaten zur Freiheit und Projektivität im Hopfalgebra-Fall [Skr07] und enthält den Freiheitssatz für Quasihopfalgebren von Schauenburg [Sch04]. Hieraus folgere ich, dass eine Modulkategorie $\mathcal{M}$ über einer endlichdimensionalen Hopfalgebra $H$ genau dann exakt und unzerlegbar ist, wenn sie von der Form ${}_A\mathcal{M}$ ist, wobei $A$ eine $H$-einfache $H$-Comodulalgebra ist. Hierfür beweise ich zunächst einen Struktursatz für Quasihopfbimoduln über Smashprodukten und eine bijektive Korrespondenz zwischen den $H$-Idealen einer $H$-Modulalgebra $R$ und den $H$-costabilen Idealen von $R\#H$. Die Betrachtung von exakten Modulkategorien hat darüber hinaus direkte Anwendungen auf $H$-Comodulalgebren und Smashprodukte. Die Resultate implizieren, dass $H$-einfache $H$-Comodulalgebren quasifrobenius sind, und dass im Fall wenn $H$ halbeinfach ist auch das Smashprodukt $R\#H$, einer $H$-einfache $H$-Modulalgebra $R$ über $H$, eine halbeinfache Algebra ist.

Für schwache Hopfalgebren ist es dagegen nicht möglich die Freiheitssätze von Nichols und Zöller [NZ89] oder Skryabin [Skr07] zu verallgemeinern: Ich konstruiere ein Beispiel einer schwachen Hopfalgebra, die nicht frei ist über einer ihrer schwachen Hopfunteralgebren. Jedoch konnten Etingof und Ostrik [EO04] zeigen, dass surjektive Tensorfunktoren zwischen endlichen Multitensorkategorien projektive Objekte auf projektive abbilden. Betrachtet man nun für schwache Hopfalgebren $K \subset H$ den Einschränkungsfunktor ${}_H\mathcal{M} \to {}_K\mathcal{M}$, dann ist dies ein surjektiver Tensorfunktor und $H$ ist somit ein projektiver $K$-Modul. Ich vermute, dass für eine quasifrobenius $H$-einfache $H$-Comodulalgebra $A$ jeder schwache Hopfmodul in $\mathcal{M}_A^H$ ein projektiver $A$-Modul ist, und beweise dies für schwache Hopfalgebren, die frei sind über ihren Basisalgebren. In diesem Fall klassifiziere ich exakte unzerlegbare Modulkategorien über $H$ durch $H$-Comodulalgebren, die quasifrobenius und $H$-einfach von rechts sind und triviale Coinvarianten haben. Meine Klassifikation liefert auch einen neuen und direkteren Beweis für die Klassifikation im Hopfalgebra-Fall [AM07].

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# Introduction

Tensor categories can be seen as a categorification of algebras. The categorification of the concept of a module over an algebra is then a module category over a tensor category. The theory of tensor categories and module categories over tensor categories has applications in different areas of mathematics and theoretical physics, such as conformal field theory, subfactor theory, and representations of (weak) Hopf algebras (see for example [FS03], [FS10], [Ost03b], and references therein).

A *finite tensor category* in the sense of [EO04] is an abelian category $\mathcal{C}$ that is equivalent to the category of representations of a finite dimensional algebra, and which is monoidal, that is it has a tensor structure $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and a unit object subject to certain stability relations. This definition includes the representations of finite groups, Lie algebras, and more generally the representations of finite dimensional Hopf algebras.

The finite dimensional modules over a finite dimensional Hopf algebra form a finite tensor category, where the tensor product is induced by the comultiplication through the diagonal structure on the tensor product over $k$, and the unit object is the ground field $k$ together with the counit. The reconstruction theory for Hopf algebras [JS91, Sch92] shows that many tensor categories are of this form, though not all of them.

In this work I discuss module categories over tensor categories which are representation categories of generalizations of Hopf algebras, namely quasi-Hopf algebras and weak Hopf algebras.

The representation categories of Hopf algebras are strict, that is associativity and unit morphisms are trivial. When omitting the strictness, this leads to the notion of quasi-Hopf algebras, which are non-coassociative generalizations of Hopf algebras. A *quasi-Hopf algebra* $H$, as introduced by Drinfeld [Dri90], is an algebra with a costructure $\Delta$ that is coassociative only up to conjugation by an invertible element $\phi\in H\otimes H\otimes H$, that is for all $h\in H$:

$$\phi(\Delta\otimes\mathrm{id})(\Delta(h)) = (\mathrm{id}\otimes\Delta)(\Delta(h))\phi,$$

and the coassociator $\phi$ is defined in such a way that the category ${}_H\mathcal{M}$ of left

$H$-modules becomes a monoidal category with this associativity constraint. In [Kas95] it is shown that quasi-Hopf algebras are exactly those algebras $H$ with maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$, whose representation categories have a monoidal structure induced by $\Delta$ and $\varepsilon$ on the underlying category of vector spaces.

Etingof and Ostrik have proven [EO04] that actually every finite tensor category with integer Frobenius-Perron dimensions is equivalent to the representation category of a finite dimensional quasi-Hopf algebra. The Frobenius-Perron dimensions are certain invariants for finite tensor categories which have non-negative values on simple objects [EO04, ENO05]. For quasi-Hopf algebras, the Frobenius-Perron dimensions coincide with the vector space dimension, whereas they can differ in general.

*Weak Hopf algebras* generate examples of tensor categories with non-integer Frobenius-Perron dimensions. They are algebras which have a coalgebra structure, where the comultiplication is multiplicative but not unit preserving and dually the counit is not multiplicative. The notion of weak Hopf algebras was introduced and axiomatized by Böhm, Nill, and Szlachányi [BNS99, Nil98, BS96].

As the comultiplication $\Delta$ of a weak Hopf algebra $H$ is not unit preserving, the tensor product $V \otimes W$ of two $H$-modules $V$ and $W$ does not have an $H$-module structure induced by $\Delta$, but the subspace

$$V \otimes W := \Delta(1)(V \otimes W)$$

is a left $H$-module. Also, the ground field $k$ does not have an $H$-module structure since the counit $\varepsilon$ is not multiplicative. However, weak Hopf algebras contain the so-called base algebras $H_t$ and $H_s$, which are separable subalgebras of $H$ and take on the function of the unit object. In this manner the category of representations of a weak Hopf algebra is provided with a monoidal structure. The unit object of this tensor category is no longer necessarily simple nor one dimensional and the vector space dimension of the tensor product of two objects in the category is not equal to the product of the dimension. The representation categories of weak quasi-Hopf algebras produce all finite tensor categories [EO04]. Moreover, if $\mathcal{C}$ is a fusion category, that is a semisimple finite tensor category, then $\mathcal{C}$ is equivalent as a tensor category to the representation category of a semisimple weak Hopf algebra with commutative base [Ost03b].

A *module category* $\mathcal{M}$ over an abelian monoidal category $\mathcal{C}$ is an abelian category together with a functor $- \otimes^{\mathcal{M}} - : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, with certain associativity and unit constraints. If $R$ is an algebra in the category $\mathcal{C}$, then the category $\mathcal{C}_R$ of right $R$-modules in $\mathcal{C}$ is a module category over $\mathcal{C}$. Ostrik [Ost03b] classified semisimple module categories over semisimple monoidal categories and showed that these are all of the form $\mathcal{C}_R$. Later, Etingof and Ostrik

[EO04] studied module categories in the non-semisimple case. Since there is no chance to classify arbitrary module categories over tensor categories (e.g. in the category of vector spaces, every finite dimensional algebra $A$ gives rise to a module category $\mathcal{M}_A$), they introduced the notion of exact module categories over tensor categories, which can be seen as the analogon of projective modules over finite dimensional algebras. More precisely, a module category $\mathcal{M}$ over a finite multi-tensor category $\mathcal{C}$ is said to be *exact*, if for every projective object $P \in \mathcal{C}$ and every object $M \in \mathcal{M}$, $P \otimes^{\mathcal{M}} M$ is a projective object in $\mathcal{M}$. This definition contains the semisimple case in the sense that module categories over semisimple monoidal categories are exact if and only if they are semisimple. Also, every tensor category, regarded as a module category over itself, is exact. The name exact makes sense, as any additive module functor from an exact module category over a tensor category $\mathcal{C}$ to another module category over $\mathcal{C}$ is exact. The classification of exact module categories over a finite tensor category $\mathcal{C}$ works in the same manner as in the semisimple case: Every exact module category over $\mathcal{C}$ is a finite direct product of indecomposable exact module categories and these are of the form $\mathcal{C}_R$ for some algebra $R$ in $\mathcal{C}$ [EO04, EGNO10].

For some examples of finite tensor categories, there exist complete classifications of module categories over them. Among others, module categories over the representations of the Taft-Hopf algebra and over the representations of $kG$ and $kG^*$, where $G$ is a finite group, were classified by Etingof and Ostrik [Ost03b, Ost03a, EO04]. Mombelli was able to specify module categories over the representation categories of pointed Hopf algebras [Mom08, Mom09]. Mombelli's classifications are based on the general classification of module categories over finite dimensional Hopf algebras by Andruskiewitsch and Mombelli [AM07], which states that these are always representation categories of $H$-comodule algebras.

More precisely, when $\mathcal{C}$ is the representation category of a finite dimensional Hopf algebra $H$, and $A$ is an $H$-comodule algebra, then ${}_A\mathcal{M}$ is a module category over $\mathcal{C}$ where the tensor product is induced by the costructure of $A$. If $A$ is $H$-simple, then ${}_A\mathcal{M}$ is indecomposable, and even exact. This is due to Skryabin's projectivity theorem [Skr07, Theorem 3.5], which states that for an $H$-simple $H$-comodule algebra $A$, *Hopf modules* in ${}^H_A\mathcal{M}$ (i.e. $A$-modules which have a compatible $H$-comodule structure) are projective $A$-modules. If now $M$ is a right $A$ module, then $H \otimes M \in {}^H_A\mathcal{M}$ and therefore $H \otimes M$ is projective in ${}_A\mathcal{M}$. On the other hand, Etingof and Ostrik's description of module categories as modules over an algebra in the category implies that every exact indecomposable module category is equivalent to $({}_H\mathcal{M})_R$, for some $H$-module algebra $R$. The category $({}_H\mathcal{M})_R$, in turn, can be shown to be equivalent as a module category to ${}_A\mathcal{M}$, where $A$ is the $H$-comodule algebra $R^{op}\#H^{cop}$. Altogether, Andruskiewitsch and Mombelli [AM07, Theorem 3.3] proved that, up to module category equivalence, actually any indecomposable exact module category is of the form ${}_A\mathcal{M}$ for an $H$-comodule algebra $A$ which is simple in ${}^H_A\mathcal{M}$ and has trivial coinvariants.

Accordingly, one can see that the classification of module categories over generalizations of Hopf algebras is strongly connected to the question, whether Hopf modules over $H$-simple $H$-comodule algebras are projective as $A$-modules.

Ever since Nichols and Zöller had proven their famous Hopf algebra Freeness Theorem [NZ89] about the freeness of relative Hopf modules and in particular of the Hopf algebra itself over Hopf subalgebras, it was an open question whether this also holds for coideal subalgebras. Masuoka [Mas92] confirmed this for quasi-Frobenius coideal subalgebras. In 2004 Skryabin succeeded in generalizing the Hopf algebra Freeness Theorem to arbitrary coideal subalgebras and thereby gave an entirely new proof for the Nichols-Zöller Theorem. Skryabin proved that for a weakly finite Hopf algebra $H$ and a finite dimensional $H$-simple $H$-comodule algebra $A$, for every object $M \in \mathcal{M}_A^H$ there exists an integer $n$ such that a finite direct product of $n$ copies of $M$ is a free $A$-module. Moreover, if $A$ is a right coideal subalgebra of $H$, then it is $H$-simple and Frobenius and every Hopf module $M \in \mathcal{M}_A^H$ is a free $A$-module. In particular, weakly finite Hopf algebras are free over their finite dimensional right coideal subalgebras.

Thus, the generalization of Skryabin's results to quasi-Hopf algebras and weak Hopf algebras is an important step for the classification of module categories over their representation categories.

The goal of this thesis is the classification of module categories over quasi-Hopf algebras and weak Hopf algebras and the investigation of the freeness and projectivity of quasi-Hopf modules and weak Hopf modules over $H$-comodule algebras and in particular of quasi-Hopf algebras and weak Hopf algebras over subalgebras.

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The outline of the thesis is as follows:

In the first part, the notion of module categories over tensor categories is introduced. I sketch a new proof to Andruskiewitsch's and Mombelli's classification of module categories over ordinary Hopf algebras. The proof will be explicated in more detail for the more general case of weak Hopf algebras in the last part.

In Part II, I generalize Skryabin's freeness and projectivity theorem to quasi-Hopf algebras. These results are published in [Hen10]. It includes the quasi-Hopf algebra freeness theorem by Schauenburg [Sch04].

The main difficulties for the generalization of Skryabin's proof to the quasi-Hopf algebra case arise from the fact that the notion of an ordinary Hopf-module does not make sense, since $H$ is not coassociative and $H$-comodules can not be defined. However, due to the *quasi*-coassociativity of $H$ it is possible to define axioms for costructures of bimodules. Furthermore, the notion of a right coideal subalgebra in the quasi-Hopf algebra case had to be clarified. Naturally, the coideal subalgebra together with the costructure of $H$ has to be an $H$-comodule algebra. The question is, whether the coassociator belonging to the coideal subalgebra should coincide with the coassociator of $H$. However, by looking at simple examples one can see that this definition would be too restrictive.

In Theorem 5.1.5, Theorem 5.2.1, and Proposition 6.3.1 I will prove the following:

> *Let $H$ be a finite dimensional quasi-Hopf algebra and $B$ a right $H$-comodule algebra. If $A$ is a finite dimensional $H$-simple $H$-comodule algebra, then every finite dimensional object $M \in {}_B\mathcal{M}_A^H$ is a projective $A$-module. Moreover, $A$ is quasi-Frobenius.*
> *If $A$ is a right coideal subalgebra of $H$, then it is $H$-simple and every finite dimensional object in ${}_B\mathcal{M}_A^H$ and in ${}_A\mathcal{M}_B^H$ is even a free $A$-module. In particular, $H$ is free as a right and left $A$-module. Moreover, $A$ is a Frobenius algebra.*

Based on these results, I classify module categories over the representation category of a finite dimensional quasi-Hopf algebra by $H$-simple $H$-comodule algebras (Theorem 6.2.4):

> *Let $\mathcal{C}$ be the category of representations of a finite dimensional quasi-Hopf algebra $H$ and $\mathcal{M}$ a module category over $\mathcal{C}$. Then $\mathcal{M}$ is an indecomposable exact module category if and only if there exists a finite dimensional $H$-simple $H$-comodule algebra $A$ such that $\mathcal{M}$ is equivalent to the category of finite dimensional left $A$-modules ${}_A\mathcal{M}^{fd}$ as module categories.*

For this purpose, I first prove a structure theorem for smash products. Module algebras over a quasi-Hopf algebra $H$ are algebras in the category ${}_H\mathcal{M}$, and therefore they are not coassociative. Nevertheless, for an $H$-module algebra $R$ it is possible to define a smash product algebra $R\#H$, which is then an associative $H$-comodule algebra [BPvO00, PvO07]. I prove a category equivalence ${}_R({}_H\mathcal{M})_{\tilde{R}} \approx {}_{R\#H}\mathcal{M}^H_{\tilde{R}\#H}$, which also implies a bijective correspondence of $H$-ideals of $R$ and $H$-stable ideals of $R\#H$. Beyond that, the results about module categories over quasi-Hopf algebras have direct applications to $H$-comodule algebras and smash products. More precisely, the results imply that $H$-comodule algebras are quasi-Frobenius and that, whenever $H$ is semisimple, also the smash product $R\#H$ of an $H$-simple $H$-module algebra $R$ by $H$ is semisimple. This can not be shown directly as in the Hopf algebra case.

Part III of this thesis is devoted to the study of weak Hopf algebras and module categories over them, as well as weak Hopf modules. Etingof and Ostrik have shown [EO04, Theorem 2.5] that surjective tensor functors map projective objects to projective ones. If now $K \subset H$ is a weak Hopf subalgebra, then the restriction functor ${}_H\mathcal{M} \to {}_K\mathcal{M}$ is a surjective tensor functor and therefore $H$ is a projective $K$-module. It is not yet known whether this is also true for coideal subalgebras of weak Hopf algebras.

I give an example of a Frobenius weak Hopf algebra which is not free over a certain weak Hopf subalgebra. This example shows that a generalization of the Nichols-Zöller Theorem to weak Hopf algebras is not possible. It also shows that it is not possible to generalize Skryabin's proof of the projectivity of Hopf modules over $H$-simple $H$-comodule algebras. However, I show that it holds for weak Hopf algebras which are free over their bases (Proposition 9.5.1):

> *Let $H$ be a weak Hopf algebra which is a free right module over its base algebra $H_s$ and let $A$ be a quasi-Frobenius $H$-simple $H$-comodule algebra. Then every finite dimensional object in $\mathcal{M}_A^H$ is a projective $A$-module.*

I conjecture that this proposition holds for arbitrary weak Hopf algebras and I classify module categories over all weak Hopf algebras for which the proposition holds, so in particular for weak Hopf algebras which are free over their bases. In Theorem 10.2.6 and Proposition 10.1.4 I show:

> *Let $H$ be a weak Hopf algebra and $\mathcal{C}$ the finite tensor category of finite dimensional left $H$-modules. Assume that for any finite dimensional quasi-Frobenius $H$-simple left $H$-comodule algebra $A$, every finite dimensional object in ${}^H_A\mathcal{M}$ or in ${}^H\mathcal{M}_A$ is a projective left or right $A$-module, respectively. Let $\mathcal{M}$ be a module category over $\mathcal{C}$. Then $\mathcal{M}$ is exact and indecomposable if and only if there exists a finite dimensional quasi-Frobenius left $H$-comodule algebra $B$, which is simple in ${}^H\mathcal{M}_B$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}^{fd}$ as $\mathcal{C}$-module categories.*
> *Moreover, if $A$ and $B$ are left $H$-comodule algebras, then equivalences of module categories ${}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ over $\mathcal{C}$ are in bijective correspondence with equivariant Morita contexts for $A$ and $B$.*

Here, a Morita context $(A, M, Q, B, \alpha, \beta)$ for $H$-comodule algebras $A$ and $B$ is called *equivariant* if $M$ is a weak Hopf bimodule in ${}_B\mathcal{M}_A^H$. In this work it will be shown that endomorphism rings of weak Hopf modules are $H$-comodule algebras and that ${}_A\mathcal{M} \approx {}_B\mathcal{M}$ as module categories over $H$, if and only if there exists an equivariant Morita context for $A$ and $B$. For ordinary Hopf algebras this was shown in [AM07].

One main step for the classification of module categories over weak Hopf algebras is the fact that weak Hopf modules over $H$-simple $H$-comodule algebras are generators. For this I will show that the trace ideal of a weak Hopf module in $M \in \mathcal{M}_A^H$ is an $H$-stable ideal. If now $A$ is $H$-simple and quasi-Frobenius, then the trace ideal of $M$ is equal to $A$, because it is non-zero. This implies that $M$ is a generator for $\mathcal{M}_A$.

The appendix contains some definitions and facts from ring theory which are needed for this work.

## Notations and Assumptions

Except for some indicated sections, $k$ is assumed to be an algebraically closed field with $\mathrm{char}(k) = 0$. If not stated otherwise, all algebras are associative $k$-algebras with unit. Tensor products, if unadorned, are over $k$.

If $R$ and $R'$ are rings then the categories of left, right, or bimodules will be denoted by ${}_R\mathcal{M}$, $\mathcal{M}_R$, or ${}_R\mathcal{M}_{R'}$, respectively. The superscript ${}^{fd}$ indicates the full subcategory of finite dimensional objects. For example, if $A$ is an algebra, then ${}_A\mathcal{M}^{fd}$ is the category of finite dimensional left $A$-modules. We write $M^{(n)}$ for a direct sum of $n$ copies of $M$.

If $\mathcal{C}$ is a category and $X$ is an object of $\mathcal{C}$, we simply write $X \in \mathcal{C}$. If $Y$ is another object of $\mathcal{C}$ we denote the class of morphisms by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. In the case when the objects of $\mathcal{C}$ are modules over a ring $R$, we write $\mathrm{Hom}_R(M, N)$ for $M, N \in \mathcal{C}$. If two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, we write $\mathcal{C} \approx \mathcal{D}$.

By *ideal* we mean two-sided ideal, otherwise they will be called right or left ideals. The set of maximal ideals of a ring $R$ is denoted by $\mathrm{Max}R$ , the Jacobson radical of $R$ is denoted by $\mathrm{Jac}(R)$. If $M$ is a right $R$-module, then $\mathrm{soc}(M)$ denotes the socle of $M$, i.e. the direct sum of the simple submodules of $M$. If $X$ and $Y$ are ideals of a ring $R$, we write

$$XY := \left\{ \sum_i x_i y_i \,|\, x_i \in X, y_i \in Y \right\}.$$

For costructures we use a simplified Sweedler Notation [Swe69, Section 1.2] omitting the summation symbol, e.g. $\Delta(h) =: h_{(1)} \otimes h_{(2)}$.

For an algebra $A$, $A^{op}$ denotes the algebra with opposite multiplication, and for a coalgebra $C$, $C^{cop}$ denotes the coalgebra with opposite comultiplication.

As usual, $\mathbb{N}$ denotes the set of natural numbers without zero.

# Part I

# Module Categories

# Chapter 1

# Tensor Categories and Module Categories

In this chapter the notion of module categories over finite (multi-) tensor categories are presented. For a detailed introduction to monoidal categories and tensor categories the reader is referred to [EGNO10], [Riv72], [Kas95], or [Müg08]; note that in the latter, monoidal categories are called tensor categories. The general definition of module categories over monoidal categories can be found in [Ost03b].

## 1.1 Tensor Categories

**Definition.** [EO04, EGNO10] A **finite multi-tensor category** $\mathcal{C}$ is a finite abelian $k$-linear rigid monoidal category, where the tensor product is bilinear. Here, a **monoidal category** is a category $\mathcal{C}$ equipped with a bifunctor

$$- \otimes^{\mathcal{C}} - : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

which is called the tensor product, a unit object $\mathbb{1}_{\mathcal{C}}$, together with natural isomorphisms $a_{-,-,-} : (- \otimes^{\mathcal{C}} -) \otimes^{\mathcal{C}} - \to - \otimes^{\mathcal{C}} (- \otimes^{\mathcal{C}} -)$ (the associativity constraint), and $r : - \otimes^{\mathcal{C}} \mathbb{1}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ and $l : \mathbb{1}_{\mathcal{C}} \otimes^{\mathcal{C}} - \to \mathrm{id}_{\mathcal{C}}$ (the right and left unit constraints), all of which are subject to the usual pentagon and triangle relations. A monoidal category is said to be **rigid**, if every object has a right dual object and left dual object. An abelian category $\mathcal{C}$ is ***k*-linear** if the sets of morphisms are vector spaces and the composition of morphisms is bilinear; a $k$-linear abelian category is called **finite** if it has only finitely many simple objects (up to isomorphism), every simple object has a projective cover, every object is of finite length, and the morphism spaces are finite dimensional. This is equivalent to $\mathcal{C}$ being the representation category of a finite dimensional algebra.

A finite multi-tensor category $\mathcal{C}$ is called **finite tensor category** if $\mathrm{End}(\mathbb{1}_{\mathcal{C}}) \cong k$.

*Remark* 1.1.1. The tensor product of a finite multi-tensor category is exact in each factor [EGNO10, §1.13]. Moreover, the unit object is a direct sum of non-isomorphic simple objects [EGNO10, §1.15]

It is a well-known fact that the representation category of a finite dimensional Hopf algebra over a field $k$ is a finite tensor category, where the underlying tensor product is the tensor product over $k$. We will see later on that the representation categories of quasi-Hopf algebras and weak Hopf algebras are finite (multi-)tensor categories. However, for weak Hopf algebras, the tensor product is no longer the tensor product over $k$. In particular, the vector space dimension of the tensor product of two objects in the category can differ from the product of the dimensions. This leads to the introduction of a different concept of dimension.

**Lemma and Definition 1.1.2.** [ENO05, EGNO10] Let $\mathcal{C}$ be a finite tensor category. Let $Gr(\mathcal{C})$ be the **Grothendieck ring** of $\mathcal{C}$ (see for example [EGNO10, 1.16]). That is, as an additive group $Gr(\mathcal{C})$ is the free abelian group generated by the set $V_1, \dots, V_n$ of isomorphism classes of simple objects in $\mathcal{C}$, modulo short exact sequences. For $X \in \mathcal{C}$, define $[X] := \sum_i n_i V_i$ in $Gr(\mathcal{C})$, where $n_i$ is the multiplicity with which $V_i$ occurs in the Jordan-Hölder composition series of $X$. Moreover, $Gr(\mathcal{C})$ has a natural multiplication induced by the tensor product: for $i, j = 1, \dots, n$, define $V_i V_j := [V_i \otimes^{\mathcal{C}} V_j] = \sum_k N^i_{j,k} V_k$. The matrix $(N^i_{j,k})_{1 \leq j,k \leq n}$ is called the multiplication matrix of $V_i$. For a simple object $X \in \mathcal{C}$ define the **Frobenius-Perron dimension** $\mathrm{FPdim}(X)$ as the largest positive eigenvalue of the multiplication matrix of the isomorphism class of $X$, which exists and is well-defined by the Frobenius-Perron Theorem (see for example [EGNO10, 1.44]). Then $\mathrm{FPdim} : Gr(\mathcal{C}) \to \mathbb{C}$ is the uniquely defined character which has positive values on simple objects. Moreover, $\mathrm{FPdim}(X)$ is an algebraic integer.

For Hopf algebras and quasi-Hopf algebras, the Frobenius-Perron dimensions for the representation category coincide with the vector space dimensions. On the other hand, tensor categories with integer Frobenius-Perron dimensions are representation categories of quasi-Hopf algebras [EO04].

**Definition.** [EGNO10] Let $\mathcal{C}$ and $\mathcal{D}$ be finite multi-tensor categories. A **quasi-tensor functor** is an exact faithful monoidal functor $F : \mathcal{C} \to \mathcal{D}$ with natural isomorphisms $\xi : F(X \otimes^{\mathcal{C}} Y) \to F(X) \otimes^{\mathcal{D}} F(Y)$. If there is an additional isomorphism $\xi_0 : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$, then $F$ is called a **tensor functor**.

*Remark* 1.1.3. [EGNO10] Quasi-tensor functors preserve Frobenius-Perron dimensions.

## 1.2 Module Categories over Finite Multi-Tensor Categories

In this section, module categories over tensor categories will be introduced. For more details see [Ost03b], [EGNO10] and [AM07].

Let $\mathcal{C}$ be a finite multi-tensor category.

**Definition.** [Ost03b, EGNO10] A (left) **module category over $\mathcal{C}$** is an abelian $k$-linear category $\mathcal{M}$ together with a bifunctor $-\otimes^{\mathcal{M}}- : \mathcal{C}\times\mathcal{M}\to\mathcal{M}$, which is assumed to be bilinear and exact in both variables. Moreover, there are functorial associativity and unit isomorphisms

$$m : (-\otimes^{\mathcal{C}} -)\otimes^{\mathcal{M}} - \;\to\; -\otimes^{\mathcal{M}} (-\otimes^{\mathcal{M}} -),$$
$$\text{and} \qquad \ell : \mathbb{1}\otimes^{\mathcal{M}} - \;\to\; \mathrm{id},$$

which satisfy the pentagon and triangle relations:

$$(\mathrm{id}_X \otimes m_{Y,Z,M})\circ m_{X,Y\otimes^{\mathcal{C}}Z,M}\circ (a_{X,Y,Z}\otimes \mathrm{id}_M) = m_{X,Y,Z\otimes^{\mathcal{M}}M}\circ m_{X\otimes^{\mathcal{C}}Y,Z,M}, \tag{1.1}$$

$$(\mathrm{id}_X\otimes \ell_M)\circ m_{X,\mathbb{1},M} = (r_X\otimes \mathrm{id}_M), \tag{1.2}$$

for $X,Y,Z\in\mathcal{C}$, $M\in\mathcal{M}$. Here, $\mathbb{1}$ denotes the unit of $\mathcal{C}$, $a$ denotes the associativity, and $r$ the left unit constraint of $\mathcal{C}$.

*Remark* 1.2.1. The exactness in the second variable follows automatically from the fact that $\mathcal{C}$ is a monoidal category with duality (cf. Lemma 8.1.5).

*Examples* 1.2.2. (1) $\mathcal{C}$ is a module category over itself.

(2) [Ost03b] Let $R$ be an algebra in $\mathcal{C}$, then the category $\mathcal{C}_R$ of right $R$-modules in $\mathcal{C}$ (see [Ost03b]) is a module category over $\mathcal{C}$. More precisely, for $X\in\mathcal{C}$ and a module $M$ over $R$ in $\mathcal{C}$, the object $X\otimes^{\mathcal{C}_R} M$ is $X\otimes^{C} M$ in $\mathcal{C}$ where the $R$-module structure is induced by the $R$-module structure $\mu$ of $M$, that is it is given by $\mathrm{id}\otimes\mu$. The associativity and unit constraints are the ones of $\mathcal{C}$. If $X,Y\in\mathcal{C}$ and $M\in\mathcal{C}_R$, then $a_{X,Y,M} : (X\otimes^{\mathcal{C}})\otimes^{\mathcal{C}} M\to X\otimes^{\mathcal{C}}(Y\otimes^{\mathcal{C}} M)$ and $\ell_M : \mathbb{1}\otimes^{\mathcal{C}} M\to M$ clearly are $R$-linear.

If $\mathcal{C}$ is the category of left $H$-modules, where $H$ is a Hopf algebra, quasi-Hopf algebra, or a weak Hopf algebra, then an algebra in $\mathcal{C}$ is exactly an $H$-module algebra and $\mathcal{C}_R$ is the category $({}_H\mathcal{M})_R$ of $H$-modules with a compatible $R$-module structure (see Section 4.2 and Section 8.2).

(3) Later (Remark 2.0.2, Lemma 6.2.1 and Lemma 10.1.1) we will see that if $H$ is a Hopf algebra, quasi-Hopf algebra, or a weak Hopf algebra and $A$ is a left $H$-comodule algebra, then the category ${}_A\mathcal{M}$ of left $A$-modules is a module category over the category of representations of left $H$-modules. The structure

of $X \otimes^{{}_A\mathcal{M}} M$, where $X$ is a left $H$-module and $M$ is a left $A$-module, is the diagonal structure. If $A$ is $H$-simple then ${}_A\mathcal{M}$ is exact (see next section) and on the other hand every exact module category over ${}_H\mathcal{M}^{fd}$ is of this form.

**Lemma and Definition 1.2.3.** [Ost03b, AM07] A **$\mathcal{C}$-module functor** between two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{M}'$ is a functor $F$ together with a natural isomorphism

$$c : F(- \otimes^{\mathcal{M}} -) \to - \otimes^{\mathcal{M}'} F(-),$$

such that for $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$ the pentagon and triangle equation

$$(\mathrm{id}_X \otimes c_{Y,M}) \circ c_{X,Y\otimes^{\mathcal{M}}M} \circ F(m_{X,Y,M}) = m'_{X,Y,F(M)} \circ c_{X\otimes Y,M}, \tag{1.3}$$

$$\ell'_{F(M)} \circ c_{\mathbb{1},M} = F(\ell_M) \tag{1.4}$$

are satisfied. If $(F, c) : \mathcal{M} \to \mathcal{M}'$ and $(G, d) : \mathcal{M}' \to \mathcal{M}''$ are $\mathcal{C}$-module functors then $(GF, d \circ G(c)) : \mathcal{M} \to \mathcal{M}''$ is a $\mathcal{C}$-module functor.

A **natural transformation of module functors** $(F, c)$, $(G, d) : \mathcal{M} \to \mathcal{M}'$, is a natural transformation $\eta : F \to G$, such that for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$

$$d_{X,M} \circ \eta_{X\otimes^{\mathcal{M}}M} = (\mathrm{id}_X \otimes \eta_M) \circ c_{X,M}. \tag{1.5}$$

Consequently, two $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{M}'$ are said to be **equivalent as $\mathcal{C}$-module categories**, if there exist mutually inverse $\mathcal{C}$-module functors $(F, c) : \mathcal{M} \to \mathcal{M}'$ and $(G, d) : \mathcal{M}' \to \mathcal{M}$ and natural isomorphisms $\alpha : GF \to \mathrm{id}_{\mathcal{M}}$ and $\beta : FG \to \mathrm{id}_{\mathcal{M}'}$ which satisfy

$$\alpha_{X\otimes^{\mathcal{M}}M} = (\mathrm{id}_X \otimes \alpha_M) \circ (d_{X,F(M)} \circ G(c_{X,M})), \tag{1.6}$$

$$\beta_{X\otimes^{\mathcal{M}}M'} = (\mathrm{id}_X \otimes \beta_{M'}) \circ (c_{X,G(M')} \circ F(d_{X,M'})), \tag{1.7}$$

for $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $M' \in \mathcal{M}'$. In this case $(F, c)$ is called an equivalence of $\mathcal{C}$-module categories.
Note that clearly $(\mathrm{id}_{\mathcal{M}}, \mathrm{id})$ is a $\mathcal{C}$-module functor.

**Lemma and Definition 1.2.4.** [Ost03b] A direct sum of two $\mathcal{C}$-module categories is again a module category, where the tensor product as well as the associativity and unit isomorphisms are coordinate wise. A $\mathcal{C}$-module category $\mathcal{M}$ is said to be **indecomposable**, if it is not equivalent to a direct sum of non-zero module categories. If $\mathcal{M}$ is a finite module category over $\mathcal{C}$, then $\mathcal{M}$ is a finite direct sum of indecomposable module categories.

## 1.3 Exact Module Categories

Let $\mathcal{C}$ be a finite multi-tensor category.

**Definition.** [EO04] A $\mathcal{C}$-module category $\mathcal{M}$ is called **exact**, if it is finite and for any projective object $P \in \mathcal{C}$ and any $M \in \mathcal{M}$, $P \otimes^{\mathcal{M}} M$ is a projective object in $\mathcal{M}$.

*Remark* 1.3.1. (2) If $\mathcal{C}$ is semisimple, then a module category over $\mathcal{C}$ is exact if and only if it is semisimple, which can be seen by tensoring with the unit object.

**Definition.** An abelian category is a **Frobenius category** if every projective object is injective and vice versa.

**Proposition 1.3.2.** [EO04, Corollary 3.6] *Exact module categories are Frobenius categories.*

In particular, if for an algebra $A$, ${}_A\mathcal{M}$ is an exact module category over a finite multi-tensor category, then $A$ is quasi-Frobenius (see Appendix A.3.1).

**Theorem 1.3.3.** [EO04, Ost03b] *If $\mathcal{M}$ is an exact module category over $\mathcal{C}$ and $M$ a generator for $\mathcal{M}$ in the sense of* [EO04]*, that is for every $N \in \mathcal{M}$ there exists an object $X \in \mathcal{C}$ such that* $\mathrm{Hom}_{\mathcal{M}}(X \otimes^{\mathcal{M}} M, N) \neq 0$. *Then $\underline{End}(M)$ is an algebra in $\mathcal{C}$ and*

$$\underline{Hom}(M, -) : \mathcal{M} \to \mathcal{C}_{\underline{End}(M)}$$

*is an equivalence of module categories.*

Here, $\underline{\mathrm{Hom}}(M, N)$ denotes the **internal Hom** (see for example [EGNO10, 2.10] or [Ost03b]), that is it is the representing object in $\mathcal{C}$ of the representable functor $\mathrm{Hom}_{\mathcal{M}}(- \otimes^{\mathcal{M}} M, N)$; and $\underline{\mathrm{End}}(M) := \underline{\mathrm{Hom}}(M, M)$.

*Proof.* [EGNO10, Theorem 2.11.6 and 2.11.5] (see also [AM07, Theorem 1.14 and Remark 1.15]). ☐

**Remark 1.3.4.** [EO04, AM07]

(1) *Every exact module category over $\mathcal{C}$ is a finite direct sum of exact indecomposable module categories.*

(2) *If $\mathcal{M}$ in the theorem is indecomposable, then every non-zero object in $\mathcal{M}$ is a generator.*

(3) *If $\mathcal{M}$ is as in* (2) *and $M \in \mathcal{M}$ is simple, then the theorem implies that $\underline{End}(M)$ is a simple object in $\mathcal{C}_{\underline{End}(M)}$.*

**Corollary 1.3.5.** *Let $\mathcal{M}$ be an indecomposable exact module category over $\mathcal{C}$. There exists an algebra $R$ in $\mathcal{C}$, which is a simple object in $\mathcal{C}_R$ and $\mathcal{M} \approx \mathcal{C}_R$ as $\mathcal{C}$-module categories.*

# Chapter 2

# Module Categories over Hopf Algebras

In this chapter we discuss indecomposable exact module categories over Hopf algebras. We assume that the reader is familiar with the definitions of Hopf algebras, comodules, Hopf modules, smash products, etc. For the definitions, facts, and usual notations we refer to Montgomery's book about Hopf algebras [Mon93].

Let $H$ be a finite dimensional Hopf algebra and let $\mathcal{C}$ be the finite tensor category of finite dimensional left $H$-modules. Andruskiewitsch and Mombelli [AM07] classified indecomposable exact module categories over $\mathcal{C}$. In this chapter a new and more direct proof for this classification will be sketched, which does not use stabilizers. This new proof has the advantage that it can be generalized to module categories over weak Hopf algebras, which will be presented in Chapter 10.

**Theorem 2.0.1.** [AM07, Proposition 1.24 and Theorem 3.3] *If $\mathcal{M}$ is a module category over $\mathcal{C}$, then $\mathcal{M}$ is exact and indecomposable if and only if there exists a left $H$-comodule algebra $B$ which is a simple object in ${}^H\mathcal{M}_B$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}$ as module categories over $\mathcal{C}$. Moreover, if $A$ and $B$ are $H$-comodule algebras then ${}_A\mathcal{M} \approx {}_B\mathcal{M}$ as module categories over $C$ if and only if there exists an equivariant Morita context $(A, M, Q, B, \alpha, \beta)$, that is $(A, M, Q, B, \alpha, \beta)$ is a Morita context for $A$ and $B$ and $M \in {}_A\mathcal{M}_B^H$.*

*Remark* 2.0.2. Recall that the category ${}_A\mathcal{M}$ is a module category over $\mathcal{C}$ with diagonal structure, that is for $X \in \mathcal{C}$ and $M \in {}_A\mathcal{M}$, $X \otimes M \in {}_A\mathcal{M}$ via the costructure of $A$.

**Definition.** Recall that a subspace $Y$ of an $H$-comodule $(X, \delta)$ is called ***H*-costable** if $\delta(Y) \subset Y \otimes H$. A nonzero $H$-comodule algebra is called ***H*-simple** if it does not contain a proper nonzero $H$-costable ideal. Dually, an $H$-module algebra is called ***H*-simple** if it does not contain a proper nonzero $H$-stable ideal.

**Lemma 2.0.3.** *Let $A$ be a finite dimensional $H$-simple $H$-comodule algebra. Let $M$ be a nonzero finite dimensional object in the category $\mathcal{M}_A^H$ of relative $(H, A)$-Hopf modules. Then $M$ is a progenerator in $\mathcal{M}_A$ (see Appendix A.1).*

*Proof.* Since $A$ is $H$-simple, Theorem 5.2.1 or [Skr07, Theorem 3.5] implies that $M$ is a projective right $A$-module. Moreover, $A$ is quasi-Frobenius by [Skr07, Theorem 4.2]. It remains to show that $M$ is a generator in $\mathcal{M}_A$. For this it suffices to show that the trace ideal $T_M$ of $M$ is $A$ (see Appendix A.2.1). We have $M \in ({}_{H^*}\mathcal{M})_A$, where $H^*$ is the dual Hopf algebra of $H$, and $({}_{H^*}\mathcal{M})_A$ is the category of $A$-modules which have an $H$-linear $A$-module structure. By [SvO06, Lemma 1.3] $T_M$ is an $H^*$-stable ideal of $A$ and therefore $H$-costable. The $H$-simplicity of $A$ implies that $T_M$ is either 0 or $A$. But $T_M \neq 0$ since $A$ is quasi-Frobenius (see Appendix A.3.2). □

**Corollary 2.0.4.** *If $A$ is an $H$-simple left $H$-comodule algebra, then ${}_A\mathcal{M}$ is an exact indecomposable module category over $H$.*

*Proof.* If $M \in {}_A\mathcal{M}$ then $H \otimes M \in {}_A\mathcal{M}$ and therefore it is a projective $A$-module by the opcop-version of Lemma 2.0.3. Hence, $P \otimes M$ is projective for every projective left $H$-module. ${}_A\mathcal{M}$ is indecomposable since $A$ is $H$-simple [AM07, Proposition 1.18]. □

**Proposition 2.0.5.** [AM07, 1.19] *If $\mathcal{M}$ is an indecomposable exact module category over $\mathcal{C}$, then there exists a left $H$-comodule algebra $A$, such that $\mathcal{M} \approx {}_A\mathcal{M}$ as $\mathcal{C}$-module categories.*

*Remark* 2.0.6. The proposition follows from Etingof and Ostrik's classification of exact module categories (Corollary 1.3.5). In fact, let $R$ be the algebra in $\mathcal{C}$ from 1.3.5, then $R$ is an $H$-module algebra and $\mathcal{M} \approx \mathcal{C}_R = ({}_H\mathcal{M})_R$ as module categories. Set $A := R^{op}\#H^{cop}$, then $A$ has the desired property which is shown in [AM07, 1.19].

Moreover, we know from 1.3.5 that $R$ is an $H$-simple $H$-module algebra, hence $R^{op}$ is an $H^{cop}$-simple $H^{cop}$-module algebra, and therefore $R^{op}\#H^{cop}$ is an $H^{cop}$-simple right $H^{cop}$-comodule algebra [MS99, Lemma 1.3], that is an $H$-simple left $H$-comodule algebra.

**Proposition 2.0.7.** *Let $R$ be a finite dimensional $H$-simple left $H$-module algebra and set $A = R\#H$. Then there exists an $H$-comodule algebra $B$ and $M \in {}_B\mathcal{M}_A^H$, such that*

$$M \otimes_A - : {}_A\mathcal{M} \to {}_B\mathcal{M}$$

*is an equivalence of categories, and moreover $B$ is a simple object in $\mathcal{M}_B^H$ and the coinvariants $B^{coH}$ are trivial, that is $B^{coH} \cong k$.*

*Proof.* Let $V$ be a simple right $R$-module, then it is well known that $M := V\#H$ is nonzero and simple in $\mathcal{M}_A^H$ [MS99]. By the lemma above it is a progenerator in $\mathcal{M}_A$ and with Morita Theory we obtain the desired category equivalence for $B := \mathrm{End}_A(M)$ (see Appendix A.1.1). $B$ is an $H$-comodule algebra and $M \in {}_B\mathcal{M}_A^H$, for example by [Lin03].

$B = \mathrm{End}_A(V\#H)$ is a simple object in $\mathcal{M}_B^H$. This follows directly from the fact that $V$ is assumed to be a simple $R$-module and the equivalences of categories:

$$\begin{array}{ccccc} \mathcal{M}_R & \approx & \mathcal{M}_A^H & \approx & \mathcal{M}_B^H \\ W & \mapsto & W\#H & \mapsto & Hom_A(V\#H, W\#H), \end{array}$$

where the first equivalence is the known one for Galois extensions (see for example [MS99]) and the second equivalence is induced by the Morita equivalence (see Proposition 8.5.5 for the proof of the more general case of weak Hopf algebras). The coinvariants of $B$ are trivial, since $M$ is simple in $\mathcal{M}_A^H$ and $B^{coH} = \{f \in \mathrm{End}_A(M) \mid f\ H\text{-colinear}\} \cong k$, which is easy to check in the Hopf algebra case. It is shown in 8.5.2 more generally for weak Hopf algebras. □

*Proof of Theorem 2.0.1.* If $B$ is a left $H$-comodule algebra which is simple in ${}^H\mathcal{M}_B$, then ${}_B\mathcal{M}^{fd}$ is an exact indecomposable module category by Corollary 2.0.4. On the other hand, if $\mathcal{M}$ is an exact indecomposable module category, then we know from Proposition 2.0.5 and Remark 2.0.6 that $\mathcal{M} \approx {}_{R^{op}\#H^{cop}}\mathcal{M}$ as module categories over $\mathcal{C}$, for some $H$-simple left $H$-module algebra $R$. Therefore Proposition 2.0.7 renders an equivalence of categories

$$M \otimes_{(R^{op}\#H^{cop})} - \,:\, {}_{(R^{op}\#H^{cop})}\mathcal{M} \approx {}_B\mathcal{M}$$

for some right $H^{cop}$-comodule algebra $B$, which is simple in $\mathcal{M}_B^{H^{cop}}$ and has trivial coinvariants, and $M$ is an object in ${}_B\mathcal{M}^{H^{cop}}_{(R^{op}\#H^{cop})}$. It follows from [AM07, Proposition 1.23 and 1.24] that this is actually an equivalence of module categories over $\mathcal{C}$.

The second part of the theorem is [AM07, Proposition 1.23, Proposition 1.24, and Theorem 1.25]. □

# Part II

# Quasi-Hopf Algebras

# Chapter 3

# Quasi-Hopf Algebras and Quasi-Hopf Bimodules

A quasi-Hopf algebra $H$ is an associative algebra and a quasi-coassociative coalgebra together with a quasi-antipode. The coalgebra structure $\Delta$ is coassociative up to conjugation by an invertible element $\phi_H \in H \otimes H \otimes H$. The coassociator $\phi_H$ gives rise to an isomorphism $\Phi : (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ for left $H$-Modules $U, V$ and $W$ and it is defined in such a way that the category ${}_H\mathcal{M}$ of left $H$-modules becomes a monoidal category with this associativity constraint.

For quasi-Hopf algebras, the notion of an ordinary Hopf-module does not make sense since $H$ is not coassociative and $H$-comodules can not be defined. However, due to the quasi-coassociativity of $H$ it is possible to define axioms for costructures of bimodules. The category ${}_H\mathcal{M}_H^H$ of quasi-Hopf $H$-bimodules was introduced by Hausser and Nill [HN99b, Section 3]. They have proven a structure Theorem for quasi-Hopf bimodules in ${}_H\mathcal{M}_H^H$ which generalizes Larson's and Sweedler's structure Theorem for Hopf modules. Moreover, Hausser and Nill have shown that every finite dimensional quasi-Hopf algebra is a Frobenius algebra [HN99b, Theorem 4.3]. Bulacu and Caenepeel defined the category ${}_H\mathcal{M}_A^H$ of two-sided $(H, A)$-quasi-Hopf modules [BC03b, Section 3.1], where $A$ is an $H$-comodule algebra as defined in [HN99a, Definition 7.1].

In this chapter it is not necessary to assume that $k$ is algebraically closed.

## 3.1 Quasi-Hopf Algebras

**Definition.** [Dri90] A **quasi-bialgebra** is an algebra $H$ together with algebra morphisms $\Delta : H \to H \otimes H$ and $\epsilon : H \to k$ and an invertible element $\phi_H \in$

$H \otimes H \otimes H$ such that for all $h \in H$:

$$\phi_H(\Delta \otimes \mathrm{id})(\Delta(h)) = (\mathrm{id} \otimes \Delta)(\Delta(h))\phi_H, \tag{3.1}$$

$$(\varepsilon \otimes \mathrm{id})(\Delta(h)) = (\mathrm{id} \otimes \varepsilon)(\Delta(h)) = h, \tag{3.2}$$

$$(1 \otimes \phi_H)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_H)(\phi_H \otimes 1) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_H)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi_H), \tag{3.3}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\phi_H) = 1 \otimes 1. \tag{3.4}$$

*Notation.* We use a simplified Sweedler Notation $\Delta(h) =: h_{(1)} \otimes h_{(2)}$, but since $\Delta$ is not coassociative we write

$$\begin{aligned} (\Delta \otimes \mathrm{id})(\Delta(h)) &=: h_{(1,1)} \otimes h_{(1,2)} \otimes h_{(2)} \\ \text{and} \quad (\mathrm{id} \otimes \Delta)(\Delta(h)) &=: h_{(1)} \otimes h_{(2,1)} \otimes h_{(2,2)}, \end{aligned}$$

for every $h \in H$. Furthermore, we denote

$$\phi_H =: \phi_H^{(1)} \otimes \phi_H^{(2)} \otimes \phi_H^{(3)} \quad \text{and} \quad \phi_H^{-1} =: \phi_H^{(-1)} \otimes \phi_H^{(-2)} \otimes \phi_H^{(-3)},$$

again omitting the summation symbol. When no confusion is possible, we just write $\phi$ for $\phi_H$.

*Remark* 3.1.1. Note that the identities (3.1) – (3.4) also imply

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\phi_H) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\phi_H) = 1 \otimes 1. \tag{3.5}$$

We obtain this by applying $\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon \otimes \varepsilon$ respectively $\varepsilon \otimes \varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to (3.3).

**Definition.** A **quasi-Hopf algebra** is a quasi-bialgebra $H$ together with a **quasi-antipode** $(S, \alpha, \beta)$ where $S$ is a bijective anti-algebra morphism of $H$, $\alpha, \beta \in H$, and for all $h \in H$ the following holds:

$$S(h_{(1)})\alpha h_{(2)} = \varepsilon(h)\alpha \quad \text{and} \quad h_{(1)}\beta S(h_{(2)}) = \varepsilon(h)\beta, \tag{3.6}$$

$$\phi_H^{(1)}\beta S(\phi_H^{(2)})\alpha\phi_H^{(3)} = 1 \quad \text{and} \quad S(\phi_H^{(-1)})\alpha\phi_H^{(-2)}\beta S(\phi_H^{(-3)}) = 1. \tag{3.7}$$

*Remarks* 3.1.2. (1) Drinfeld has shown that either one of the identities in (3.7) is redundant [Dri90, Proposition 1.3]. By applying $\varepsilon$ to those identities we see that $\varepsilon(\alpha)$ and $\varepsilon(\beta)$ are invertible and therefore can be assumed to be 1. We obtain $(\varepsilon \circ S)(h)\varepsilon(\alpha) = \varepsilon(h)\varepsilon(\alpha)$ for all $h \in H$ by applying $\varepsilon$ to the first identity of (3.5). Hence, $\varepsilon \circ S = \varepsilon$.

(2) If $H$ is a quasi-Hopf algebra with coassociator $\phi_H$ and quasi-antipode $(S, \alpha, \beta)$ then so are $H^{op}$, $H^{cop}$, and $H^{opcop}$ with $\phi_{H^{op}} = \phi_H^{-1}$, $\phi_{H^{cop}} = \phi_H^{(-3)} \otimes \phi_H^{(-2)} \otimes \phi_H^{(-1)}$, $\phi_{H^{opcop}} = \phi_H^{(3)} \otimes \phi_H^{(2)} \otimes \phi_H^{(1)}$ and $S^{op} = S^{cop} = S^{opcop\,-1} = S^{-1}$, $\alpha^{op} = \beta^{cop} = S^{-1}(\beta)$, $\beta^{op} = \alpha^{cop} = S^{-1}(\alpha)$, $\alpha^{opcop} = \beta$, $\beta^{opcop} = \alpha$.

(3) By some authors, the quasi-antipode $S$ is not assumed to be bijective, but Bulacu and Caenepeel [BC03a, Theorem 2.2] have shown that the bijectivity of $S$ is automatic whenever $H$ is finite dimensional.

(4) Quasi-Hopf algebras are not self-dual. The dual space $H^* = \mathrm{Hom}_k(H,k)$ of a finite dimensional quasi-Hopf algebra $H$ is a so-called **dual quasi-Hopf algebra** [BC03a], which is a coassociative coalgebra, and quasi-associative algebra with the usual structure using the canonical pairing $\mathrm{Hom}_k(H,k) \times H \to k$.

**Proposition 3.1.3.** *If $H$ is a quasi-Hopf algebra then the category $\mathcal{C}$ of left $H$-modules is a rigid monoidal category* [Dri90]. *If $H$ is finite dimensional, then the category of finite dimensional left $H$-modules is a finite tensor category.*

*On the other hand, every finite tensor category with integer Frobenius-Perron dimensions is equivalent to the representation category of a finite dimensional quasi-Hopf algebra* [EO04, Proposition 2.6].

**Lemma and Definition 3.1.4.** [Dri90] A **twist** or **gauge transformation** of a quasi-Hopf algebra $(H, \Delta, \phi)$ is an invertible element $F \in H \otimes H$ with $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$. It induces a new costructure on $H$ by

$$\begin{aligned} \Delta_F(h) &:= F\Delta(h)F^{-1} \\ \phi_F &:= (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\phi_H(\Delta \otimes \mathrm{id})(F^{-1})(F^{-1} \otimes 1). \end{aligned}$$

Then $(H, \Delta_F, \phi_F)$ is again a quasi-Hopf algebra [Dri90, §1] with $S_F = S$ and

$$\alpha_F = S(F^{(-1)})\alpha F^{(-2)} \text{ and } \beta_F = F^{(1)}\beta S(F^{(2)}),$$

where $F =: F^{(1)} \otimes F^{(2)}$ and $F^{-1} =: F^{(-1)} \otimes F^{(-2)}$ omitting the summation symbol. Recall that in general, ordinary Hopf algebras do not remain Hopf algebras under a twist.

**Proposition 3.1.5.** [Dri90] *If $H'$ is obtained from the quasi-Hopf algebra $H$ by a twist, then the representation categories of $H$ and $H'$ are equivalent as monoidal categories.*

*Remark* 3.1.6. Unlike the antipode of a Hopf algebra, the antipode of a quasi-Hopf algebra is not an anti-coalgebra morphism. However, Drinfeld [Dri90] has found a twist transformation $F_S \in H \otimes H$ such that

$$F_S \Delta(S(h)) F_S^{-1} = (S \otimes S)(\Delta^{op}(h)),$$

for all $h \in H$, where $\Delta^{op}(h) = h_{(2)} \otimes h_{(1)}$. Following Drinfeld, we can give an explicit formula for $F_S$ and $F_S^{-1}$. Let

$$\begin{aligned} \gamma &:= S(\phi_H^{(-1)}\phi_H^{(2)})\alpha\phi_H^{(-2)}\phi^{(3)}_{H\,(1)} \otimes S(\phi^{(1)})\alpha\phi_H^{(-3)}\phi^{(3)}_{H\,(2)}, \\ \delta &:= \phi_H^{(1)}\phi^{(-1)}_{H\,(1)}\beta S(\phi_H^{(-3)}) \otimes \phi_H^{(2)}\phi^{(-1)}_{H\,(2)}\beta S(\phi_H^{(3)}\phi_H^{(-2)}), \end{aligned} \tag{3.8}$$

$$\text{then} \qquad \begin{aligned} F_S &= (S \otimes S)(\Delta^{op}(\phi_H^{(-1)}))\,\gamma\,\Delta(\phi_H^{(-2)}\beta S(\phi^{(-3)})), \\ F_S^{-1} &= \Delta(S(\phi_H^{(-1)})\alpha\phi_H^{(-2)})\,\delta\,(S \otimes S)(\Delta^{op}(\phi^{(-3)})). \end{aligned}$$

*Examples* 3.1.7. In the context of a description of the connection between quantum groups and rational conformal field theory, Dijkgraaf, Pasquier, and Roche [DPR90] defined the important example of a non-trivial quasi-Hopf algebra $D^\omega(G)$. As a vector space the quasi-quantum group $D^\omega(G)$ is the Drinfeld double $D(G)$ of a finite group $G$, with a multiplication and comultiplication which are deformed by a normalized 3-cocycle $\omega$. For a detailed definition see also [Kas95, XV.5].

Etingof and Gelaki gave examples of quasi-Hopf algebras which are not twist-equivalent to an ordinary Hopf algebra [EG04, Gel05, EG05]. They classified finite dimensional quasi-Hopf algebras with radical of codimension 2 and more generally of prime codimension.

## 3.2 *H*-Comodule Algebras and Coideal Subalgebras

**Definition.** Let $(H,\phi_H)$ be a quasi-Hopf algebra. A **right *H*-comodule algebra** is a unital algebra $A$ together with an algebra morphism $\rho : A \to A \otimes H$ and an invertible element $\phi_\rho \in A \otimes H \otimes H$ which satisfy

$$\phi_\rho(\rho \otimes \mathrm{id})(\rho(a)) = (\mathrm{id} \otimes \Delta)(\rho(a))\phi_\rho, \tag{3.9}$$

$$(\mathrm{id} \otimes \varepsilon)(\rho(a)) = a, \tag{3.10}$$

$$(1_A \otimes \phi_H)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi_\rho)(\phi_\rho \otimes 1_H) = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi_\rho)(\rho \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho), \tag{3.11}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\phi_\rho) = (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(\phi_\rho) = 1_A \otimes 1_H, \tag{3.12}$$

for all $a \in A$.

Analogously, one may define left $H$-comodule algebras.

*Notation.* Similar as for quasi-Hopf algebras, we write $\rho(a) =: a_{(0)} \otimes a_{(1)}$ and

$$(\rho \otimes \mathrm{id})(\rho(a)) =: a_{(0,0)} \otimes a_{(0,1)} \otimes a_{(1)},$$
$$(\mathrm{id} \otimes \Delta)(\rho(a)) =: a_{(0)} \otimes a_{(1,1)} \otimes a_{(1,2)},$$

for all $a \in A$. Moreover, $\phi_\rho =: \phi_\rho^{(1)} \otimes \phi_\rho^{(2)} \otimes \phi_\rho^{(3)}$ and $\phi_\rho^{-1} =: \phi_\rho^{(-1)} \otimes \phi_\rho^{(-2)} \otimes \phi_\rho^{(-3)}$.

*Remarks* 3.2.1. (1) Because of the generalized pentagon equation (3.11), $\phi_\rho$ has to be nontrivial whenever $\phi_H$ is nontrivial. On the other hand, $\phi_\rho$ may be nontrivial even if $H$ is an ordinary Hopf algebra.
(2) If $(A,\rho,\phi_\rho)$ is a right $H$-comodule algebra, then $(A^{op},\rho,\phi_\rho^{-1})$ is a right $H^{op}$-comodule algebra and $(A,\rho^{cop},\phi_\rho^{(-3)} \otimes \phi_\rho^{(-2)} \otimes \phi_\rho^{(-1)})$ is a left $H^{cop}$-comodule algebra.

**Definition.** Let $H$ be a quasi-Hopf algebra. A subalgebra $K \subset H$ together with an invertible element $\phi_K \in K \otimes H \otimes H$ is called a **right coideal subalgebra** of $H$, if $\Delta(K) \subset K \otimes H$ and $(K,\Delta,\phi_K)$ is a right $H$-comodule algebra.

*Remark* 3.2.2. Let $K$ be a subalgebra of a quasi-Hopf algebra $H$ satisfying $\Delta(K) \subset K \otimes H$ and $\phi_H, \phi_H^{-1} \in K \otimes H \otimes H$ then $K$ is a right coideal subalgebra of $H$. On the other hand it would be too restrictive to define coideal subalgebras in such a way that the coassociator belonging to $K$ coincides with the coassociator of $H$, which would imply $\phi_H \in K \otimes H \otimes H$. For example, a reasonable right coideal subalgebra of the tensor product $H \otimes H'$ of two quasi-Hopf algebras would be $K \otimes k1_{H'}$, where $K$ is a right coideal subalgebra of $H$ with coassociator $\phi_H$. The coassociator of $K \otimes k1_{H'}$ would be $\phi_H^{(1)} \otimes 1_{H'} \otimes \phi_H^{(2)} \otimes 1_{H'} \otimes \phi_H^{(3)} \otimes 1_{H'}$ and is therefore unequal to $\phi_{H \otimes H'} = \phi_H^{(1)} \otimes \phi_{H'}^{(1)} \otimes \phi_H^{(2)} \otimes \phi_{H'}^{(2)} \otimes \phi_H^{(3)} \otimes \phi_{H'}^{(3)}$ unless $\phi_{H'}$ is trivial (cf. [Sch04] for considerations about quasi-Hopf subalgebras and their coassociators).

## 3.3 Quasi-Hopf Bimodules

**Definition.** Let $(H, \phi_H)$ be a quasi-Hopf algebra and $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ right $H$-comodule algebras. A **right $(H, B, A)$-quasi-Hopf bimodule** is a $(B, A)$-bimodule $M$ with a quasi-coaction $\delta_M : M \to M \otimes H$, which is a $(B, A)$-bimodule morphism ($M \otimes H \in {}_B\mathcal{M}_A$ via $\rho'$ and $\rho$) and which satisfies

$$\phi_{\rho'}(\delta_M \otimes \mathrm{id}_H)(\delta_M(m)) = (\mathrm{id}_M \otimes \Delta)(\delta_M(m))\phi_\rho, \tag{3.13}$$

$$(\mathrm{id} \otimes \varepsilon)(\delta_M(m)) = m, \tag{3.14}$$

for all $m \in M$. The $(H, B, A)$-quasi-Hopf bimodules together with $(B, A)$-bimodule morphisms which are $H$-colinear form a category denoted by ${}_B\mathcal{M}_A^H$.

For left $H$-comodule algebras we can define left quasi-Hopf bimodules in the same way. The resulting category will then be denoted by ${}_B^H\mathcal{M}_A$.

*Notation.* Again we write $\delta_M(m) =: m_{(0)} \otimes m_{(1)}$ and

$$(\delta_M \otimes \mathrm{id})(\delta_M(m)) =: m_{(0,0)} \otimes m_{(0,1)} \otimes m_{(1)},$$
$$(\mathrm{id} \otimes \Delta)(\delta_M(m)) =: m_{(0)} \otimes m_{(1,1)} \otimes m_{(1,2)},$$

for all $m \in M$.

*Remark* 3.3.1. If $H$ is regarded as a right $H$-comodule algebra via $\Delta$, then an $(H, H, H)$-quasi-Hopf bimodule is a usual quasi-Hopf $H$-bimodule defined by Hausser and Nill [HN99b, Section 3]. Another special case is the category ${}_H\mathcal{M}_A^H$ of $(H, A)$-quasi Hopf bimodules introduced by Bulacu and Caenepeel [BC03b, Section 3.1].

*Examples* 3.3.2. (1) A right $H$-comodule algebra $A$ is an object of ${}_A\mathcal{M}_A^H$.
(2) Let $M$ be a right $A$-module, then $M. \otimes . H. \in {}_H\mathcal{M}_A^H$, with

$$\delta_{M \otimes H}(m \otimes h) = m\phi_\rho^{(1)} \otimes h_{(1)}\phi_\rho^{(2)} \otimes h_{(2)}\phi_\rho^{(3)}, \tag{3.15}$$

for all $m \in M$, $h \in H$. The dots indicate the bimodule structure of $M \otimes H$, which means that it is given by $g(m \otimes h)a = ma_{(0)} \otimes gha_{(1)}$ for all $m \in M$, $a \in A$, $h, g \in H$.
(3) If $M$ is a left $A$-module, then $.M \otimes .H. \in {}_A\mathcal{M}_H^H$ in the same way, that is with a costructure given by

$$\delta_{M \otimes H}(m \otimes h) = \phi_\rho^{(-1)} m \otimes \phi_\rho^{(-2)} h_{(1)} \otimes \phi_\rho^{(-3)} h_{(2)}, \tag{3.16}$$

for all $m \in M$, $h \in H$.

## 3.4 Antipode Properties

Unless stated otherwise, in the following $(H, \Delta, \varepsilon, \phi_H, S, \alpha, \beta)$ is a quasi-Hopf algebra and $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ are right $H$-comodule algebras.

**Lemma and Definition 3.4.1.** [HN99a, Section 9] we define

$$p_\rho := \phi_\rho^{(-1)} \otimes \phi_\rho^{(-2)} \beta S(\phi_\rho^{(-3)}) \quad \text{and} \quad q_\rho := \phi_\rho^{(1)} \otimes S^{-1}(\alpha \phi_\rho^{(3)}) \phi_\rho^{(2)}.$$

Since $H$ can be regarded as a right $H$-comodule algebra via $\Delta$, we can define $p_R := p_\Delta$ and $q_R := q_\Delta$ and obtain

$$p_R := \phi_H^{(-1)} \otimes \phi_H^{(-2)} \beta S(\phi_H^{(-3)}) \quad \text{and} \quad q_R := \phi_H^{(1)} \otimes S^{-1}(\alpha \phi_H^{(3)}) \phi_H^{(2)}.$$

Hausser and Nill [HN99a, Lemma 9.1] have shown the following relations:

$$\rho(a_{(0)}) p_\rho (1 \otimes S(a_{(1)})) = p_\rho (a \otimes 1), \tag{3.17}$$
$$(1 \otimes S^{-1}(a_{(1)})) q_\rho \rho(a_{(0)}) = (a \otimes 1) q_\rho, \tag{3.18}$$

$$\rho(q_\rho^{(1)}) p_\rho (1 \otimes S(q_\rho^{(2)})) = 1_A \otimes 1_H, \tag{3.19}$$
$$(1 \otimes S^{-1}(p_\rho^{(2)})) q_\rho \rho(p_\rho^{(1)}) = 1_A \otimes 1_H. \tag{3.20}$$

Those relations are a generalization of the fact that for an ordinary Hopf algebra $H$ and a right $H$-comodule algebra $A$ the following holds: $a_{(0)} \otimes a_{(1)} S(a_{(2)}) = a \otimes 1_H$ for every $a \in A$. Similar we can get the analogon of the antipode property for comodules:

**Lemma 3.4.2.** *Let $(M, \delta_M) \in {}_B\mathcal{M}_A^H$. Then*

$$\delta_M(m_{(0)}) p_\rho (1_A \otimes S(m_{(1)})) = p_{\rho'} (m \otimes 1_H), \tag{3.21}$$
$$(1_B \otimes S^{-1}(m_{(1)})) q_{\rho'} \delta_M(m_{(0)}) = (m \otimes 1_H) q_\rho, \tag{3.22}$$

*for every $m \in M$.*

*Proof.* Analogous to the proof of [HN99a, Lemma 9.1]. $\square$

## 3.5 Quasi-Hopf Bimodule Isomorphisms

If $H$ is an ordinary Hopf algebra, $A$ a right $H$-comodule algebra and $M \in {}_A\mathcal{M}^H$ then we have a well known left $A$-linear and right $H$-linear isomorphism

$$
\begin{aligned}
{}_{.}M \otimes H_{.} &\cong {}_{.}M \otimes {}_{.}H_{.} \\
m \otimes h &\mapsto m_{(0)} \otimes m_{(1)} h \\
m_{(0)} \otimes S(m_{(1)})h &\leftarrow\!\shortmid m \otimes h
\end{aligned}
$$

The $A$- and $H$-structures are indicated by the dots. In the latter case, $M \otimes H$ has a diagonal left $A$-structure induced by the costructure of $A$ whereas in the first case $H$ has a trivial left-$H$-module structure. We have a similar isomorphism for quasi-Hopf algebras.

In the following $(H, \Delta, \varepsilon, \phi_H, S, \alpha, \beta)$ is a quasi-Hopf algebra and $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ are right $H$-comodule algebras.

**Lemma 3.5.1.** *Let $M \in {}_B\mathcal{M}_A^H$ with costructure $\delta$, then*

$$
\begin{aligned}
{}_{.}M \otimes H_{.} &\underset{\varphi'}{\overset{\varphi}{\rightleftarrows}} {}_{.}M \otimes {}_{.}H_{.} \qquad \textit{in } {}_B\mathcal{M}_H \\
m \otimes h &\mapsto m_{(0)} p_\rho^{(1)} \otimes m_{(1)} p_\rho^{(2)} h \\
q_{\rho'}^{(1)} m_{(0)} \otimes S(q_{\rho'}^{(2)} m_{(1)})h &\leftarrow\!\shortmid m \otimes h,
\end{aligned}
$$

*where we write $p_\rho =: p_\rho^{(1)} \otimes p_\rho^{(2)}$ and $q_{\rho'} =: q_{\rho'}^{(1)} \otimes q_{\rho'}^{(2)}$ suppressing the summation symbols.*

*Proof.* Let $m \in M$ and $h \in H$, then

$$
\begin{aligned}
\varphi(\varphi'(m \otimes h)) &= q^{(1)}_{\rho'\,(0)} m_{(0,0)} p_\rho^{(1)} \otimes q^{(1)}_{\rho'\,(1)} m_{(0,1)} p_\rho^{(2)} S(q_{\rho'}^{(2)} m_{(1)})h \\
&= \rho'(q_{\rho'}^{(1)}) \delta(m_{(0)}) p_\rho (1 \otimes S(m_{(1)}) S(q_{\rho'}^{(2)})h) \\
&= \rho'(q_{\rho'}^{(1)}) p_{\rho'} (1 \otimes S(q_{\rho'}^{(2)}))(m \otimes h) \\
&= m \otimes h,
\end{aligned}
$$

where we first use equation (3.21) and then equation (3.19). In the same manner we obtain

$$
\begin{aligned}
\varphi'(\varphi(m \otimes h)) &= q_{\rho'}^{(1)} m_{(0,0)} p^{(1)}_{\rho\,(0)} \otimes S(q_{\rho'}^{(2)} m_{(0,1)} p^{(1)}_{\rho\,(1)}) m_{(1)} p_\rho^{(2)} h \\
&= (\mathrm{id} \otimes S)((1 \otimes S^{-1}(p_\rho^{(2)}) S^{-1}(m_{(1)})) q_{\rho'} \delta(m_{(0)}) \rho(p_\rho^{(1)}))\,(1 \otimes h) \\
&= (m \otimes 1)\,(id \otimes S)(1 \otimes S^{-1}(p_\rho^{(2)}) q_\rho \rho(p_\rho^{(1)})\,(1 \otimes h) \\
&= m \otimes h,
\end{aligned}
$$

using (3.22) and (3.20). Obviously, $\varphi$ is left $B$- and right $H$-linear. $\square$

**Corollary 3.5.2.** The lemma implies further isomorphisms which we will need later. Let again $M \in {}_B\mathcal{M}_A^H$ with costructure $\delta$.

(i) Then $.H \otimes M. \cong M. \otimes .H.$ as $(H, A)$-bimodules, where again the bimodule structures are indicated by the dots. The isomorphism and its inverse are given by

$$.H \otimes M. \underset{\theta^{-1}}{\overset{\theta}{\rightleftarrows}} M. \otimes .H.$$
$$h \otimes m \mapsto q_{\rho'}^{(1)} m_{(0)} \otimes h q_{\rho'}^{(2)} m_{(1)}$$
$$h S^{-1}(m_{(1)} p_\rho^{(2)}) \otimes m_{(0)} p_\rho^{(1)} \leftarrow\!\shortmid m \otimes h,$$

This is in fact an $(H, A)$-bimodule isomorphism, since $\theta$ is an *op*-version of $\varphi$ from Lemma 3.5.1.

(ii) It follows that $.H \otimes M. \in {}_H\mathcal{M}_A^H$ and $.H \otimes M. \cong M. \otimes .H.$ in ${}_H\mathcal{M}_A^H$. In fact, we can define a costructure for $.H \otimes M.$ by means of the $(H, A)$-linear isomorphism $\theta$ and the costructure of $M. \otimes .H.$ given in (3.15). That is, for all $m \in M$ and $h \in H$ we define

$$\delta_{H \otimes M}(h \otimes m) := (\theta^{-1} \otimes \mathrm{id})(\delta_{M \otimes H}(\theta(h \otimes m)).$$

(iii) Moreover, $.H \otimes M. \otimes .H. \cong .H \otimes M. \otimes H.$ in ${}_H\mathcal{M}_{A \otimes H}$, where $.H \otimes M. \otimes .H.$ and $.H \otimes M. \otimes H.$ have different bimodule structures indicated by the dots, that is in the first case we have $g(h \otimes m \otimes h')(a \otimes g') = g_{(1)} h \otimes ma \otimes g_{(2)} h' g'$ and in the latter case we have $g(h \otimes m \otimes h')(a \otimes g') = gh \otimes ma \otimes h'g'$, for all $m \in M$, $a \in A$ and $g, g', h, h' \in H$. This follows directly from Lemma 3.5.1 for $M = H$ and more precisely the isomorphism and its inverse are given by the maps

$$.H \otimes M. \otimes .H. \underset{\psi^{-1}}{\overset{\psi}{\rightleftarrows}} .H \otimes M. \otimes H.$$
$$h \otimes m \otimes g \mapsto q_R^{(1)} h_{(1)} \otimes m \otimes S(q_R^{(2)} h_{(2)}) g$$
$$h_{(1)} p_R^{(1)} \otimes m \otimes h_{(2)} p_R^{(2)} g \leftarrow\!\shortmid h \otimes m \otimes g.$$

# Chapter 4

# Coinvariants, Smash Products, and Structure Theorems for Quasi-Hopf Bimodules

If $H$ is a Hopf algebra and $R$ is a left $H$-module algebra, then the category of right $R$-modules $\mathcal{M}_R$ is equivalent to the category of relative Hopf modules over the smash product $M^H_{R\#H}$. The aim of this chapter is an analogous result for quasi-Hopf algebras. Given that module algebras over quasi-Hopf algebras are not associative, we can only consider modules over an algebra in the monoidal category of $H$-modules. For $H$-module algebras $R$ and $\tilde{R}$ we will prove a category equivalence ${}_R({}_H\mathcal{M})_{\tilde{R}} \approx {}_{R\#H}\mathcal{M}^H_{\tilde{R}\#H}$. In particular, the $H$-ideals of an $H$-module algebra $R$ correspond to the $H$-costable ideals of $R\#H$.

In this chapter it is not necessary to assume that $k$ is algebraically closed.

## 4.1 Coinvariants and the Structure Theorem by Hausser and Nill

If $H$ is a usual Hopf algebra, the coinvariants $M^{coH}$ of an Hopf module $M$ are defined as $M^{coH} = \{m \in M | m_{(0)} \otimes m_{(1)} = m \otimes 1\}$. Then $M^{coH} = \{m_{(0)}S(m_{(1)}) | m \in M\}$ and $M^{coH}$ is a left $H$-module with the adjoint action. The Fundamental Theorem for Hopf modules states, that the category of vector spaces is equivalent to the category of Hopf modules, where the equivalence is given by $\mathrm{vect}_k \approx \mathcal{M}^H_H, V \mapsto V \otimes H^{\cdot}_{\cdot}$ with inverse functor $(\ )^{coH}$. In particular, $(V \otimes H)^{coH} = V \otimes 1$.

In the following sections, $H$ is a quasi-Hopf algebra and $(M, \delta) \in {}_H\mathcal{M}^H_H$.

We would like to define coinvariants of $M$ in such a way that we obtain an analogon to the Fundamental Theorem, so in particular $(.V \otimes .H.)^{coH} = V \otimes 1$ for all $V \in {}_H\mathcal{M}$. As the costructure of an object in ${}_H\mathcal{M}_H^H$ is not coassociative, the definition of the coinvariants in the above sense is not very useful. For example, one can see that for $V \in {}_H\mathcal{M}$ we have $\{x \in V \otimes H \mid \delta(x) = x \otimes 1\} = \{v \otimes 1 | v \in V \text{ and } \phi^{(-1)}v \otimes \phi^{(-2)} \otimes \phi^{(-3)} = v \otimes 1 \otimes 1\}$. However, Hausser and Nill [HN99b] defined a projection

$$\begin{aligned} E : M &\to M \\ m &\mapsto q_R^{(1)} m_{(0)} \beta S(q_R^{(2)} m_{(1)}) \end{aligned} \tag{4.1}$$

and showed that the elements of $E(M)$ have a quasi-coinvariant property and that $E(M)$ is a left $H$-module with $H$-action defined by

$$h \blacktriangleright m := E(hm), \qquad h \in H, m \in M. \tag{4.2}$$

More precisely Hausser and Nill proved the following properties for $E$ and the action $\blacktriangleright$:

**Lemma 4.1.1.** [HN99b, Proposition 3.4 and Corollary 3.9] *Let $M \in {}_H\mathcal{M}_H^H$ and let $E$ and $\blacktriangleright$ be as above. Then for all $g, h \in H$, $m \in M$ the following properties hold.*

(i) $h \blacktriangleright E(m) = E(hm) = h \blacktriangleright m$,

(ii) $gh \blacktriangleright m = g \blacktriangleright (h \blacktriangleright m)$,

(iii) $E(E(m)) = E(m)$,

(iv) $E(mh) = E(m)\varepsilon(h)$,

(v) $hE(m) = (h_{(1)} \blacktriangleright E(m))h_{(2)}$,

(vi) $E(m_{(0)})m_{(1)} = m$,

(vii) $E(E(m)_{(0)}) \otimes E(m)_{(1)} = E(m) \otimes 1$,

(viii) $E(m)_{(0)} \otimes E(m)_{(1)} = (\phi^{(-1)} \blacktriangleright E(m))\phi^{(-2)} \otimes \phi^{(-3)}$.

*It follows that $E(M)$ is a left $H$-module via $\blacktriangleright$ and*

$$\begin{aligned} M^{coH} := E(M) &= \{m \in M | E(m_{(0)}) \otimes m_{(1)} = E(m) \otimes 1\} \\ &= \{m \in M | m_{(0)} \otimes m_{(1)} = (\phi^{(-1)} \blacktriangleright m)\phi^{(-2)} \otimes \phi^{(-3)}\}. \end{aligned}$$

*In particular, for $H$ regarded as a quasi-Hopf $H$-bimodule we have $E(h) = \varepsilon(h)1$ and $E(\phi^{(-1)})\phi^{(-2)} \otimes \phi^{(-3)} = 1 \otimes 1$ .*

Using this definition of coinvariants, Hausser and Nill proved a structure theorem for quasi-Hopf bimodules, which generalizes the Fundamental Theorem for Hopf Modules:

**Theorem 4.1.2.** [HN99b, Theorem 3.8 and Proposition 3.11] *Let $H$ be a quasi-Hopf algebra and $M \in {}_H\mathcal{M}_H^H$. Then there is an equivalence of monoidal categories*

$$\begin{aligned} {}_H\mathcal{M}_H^H &\approx {}_H\mathcal{M} \\ M &\mapsto E(M) \\ (f : M \to N) &\mapsto (f|_{E(M)} : E(M) \to E(N)) \\ .V \otimes .H. &\leftarrow\!\shortmid V \\ (g \otimes \mathrm{id} : V \otimes H \to W \otimes H) &\leftarrow\!\shortmid (g : V \to W). \end{aligned}$$

*Remarks* 4.1.3. (1) $f|_{E(M)}$ above is well-defined, since for an ${}_H\mathcal{M}_H^H$-map $f : M \to N$ and $m \in M$ we have $f(E(m)) = E(f(m))$. For $M \in {}_H\mathcal{M}_H^H$ and $V \in {}_H\mathcal{M}$ the isomorphisms in ${}_H\mathcal{M}_H^H$ and ${}_H\mathcal{M}$ are given by:

$$\begin{aligned} M &\cong .E(M) \otimes .H. & V &\cong E(.V \otimes .H.) \\ m &\mapsto E(m_{(0)}) \otimes m_{(1)} & v &\mapsto v \otimes 1 \\ m \cdot h &\leftarrow\!\shortmid m \otimes h & v\varepsilon(h) &\leftarrow\!\shortmid v \otimes h. \end{aligned}$$

In fact, $E(V \otimes H) = V \otimes 1_H$ for all $V \in {}_H\mathcal{M}$.
(2) The monoidal structure of ${}_H\mathcal{M}$ is the usual one. ${}_H\mathcal{M}_H^H$ is a monoidal category with the tensor product over $H$ and if $(M, \delta_M), (N, \delta_N)$ are objects in ${}_H\mathcal{M}_H^H$ then $M \otimes_H N \in {}_H\mathcal{M}_H^H$ with $\delta_{M \otimes_H N}(m \otimes n) = (m_{(0)} \otimes n_{(0)}) \otimes m_{(1)} n_{(1)}$.

## 4.2 $H$-Module Algebras and Modules in ${}_H\mathcal{M}$

**Definition.** [BPvO00] A left $H$-module $R$ is called a **left $H$-module algebra**, if it is an algebra in the monoidal category ${}_H\mathcal{M}$. That is $R$ has a multiplication with a unit $1_R$ and

$$h \cdot (rr') = (h_{(1)} \cdot r)(h_{(2)} \cdot r'), \tag{4.3}$$

$$h \cdot 1_R = \varepsilon(h) 1_R, \tag{4.4}$$

$$r(r'r'') = ((\phi^{(-1)} \cdot r)(\phi^{(-2)} \cdot r'))(\phi^{(-3)} \cdot r''). \tag{4.5}$$

The quasi-Hopf algebra $H$ itself is not an $H$-module algebra, but we can define a new multiplication $\circ$ on $H$ such that $(H, \circ)$ becomes an $H$-module algebra, which will be denoted by $H_\circ$. More generally, we have the following definition.

**Lemma and Definition 4.2.1.** [BPvO00, Proposition 2.2] Let $R$ be an algebra and $\vartheta : H \to R$ an algebra map. We can define a multiplication on $R$ by

$$\begin{aligned} a \circ b &:= \vartheta(\phi^{(1)})\, a\, \vartheta(S(\phi^{(-1)}\phi^{(2)})\alpha\phi^{(-2)}\phi^{(3)}_{(1)})\, b\, \vartheta(S(\phi^{(-3)}\phi^{(3)}_{(2)})) \\ &= \vartheta(\phi^{(1)}\phi^{(-1)}_{(1)})\, a\, \vartheta(S(\phi^{(2)}\phi^{(-1)}_{(2)})\alpha\phi^{(3)}\phi^{(-2)})\, b\, \vartheta(S(\phi^{(-3)})), \quad \text{by (3.3)} \end{aligned}$$

for all $a, b \in R$. Then $R$ with this new multiplication is an $H$-module algebra, where the $H$-module structure is given by the left adjoint action

$$h \triangleright_\vartheta a := \vartheta(h_{(1)})\, a\, \vartheta(S(h_{(2)})),$$

for $h \in H$, $a \in R$. This $H$-module algebra will be denoted by $R^\vartheta$, and $H_\circ$ is just $H^{id_H}$.

An $H$-module algebra is not associative and therefore it does not makes sense to define modules over it. However, we may define modules in the category ${}_H\mathcal{M}$ of left $H$-modules.

**Definition.** Let $R$, $\tilde{R}$ be $H$-module algebras, then we can define the categories ${}_R({}_H\mathcal{M})$, $({}_H\mathcal{M})_R$, and ${}_R({}_H\mathcal{M})_{\tilde{R}}$ of right, left $R$-modules, and $(R, \tilde{R})$-bimodules in ${}_H\mathcal{M}$. That is, $V \in {}_R({}_H\mathcal{M})_{\tilde{R}}$ is a left $H$-module which has a left $R$-structure and a right $\tilde{R}$-structure which satisfy

$$1_R \bullet v = v = v \bullet 1_{\tilde{R}}, \tag{4.6}$$

$$h \cdot (r \bullet v) = (h_{(1)} \cdot r) \bullet (h_{(2)} \cdot v) \ \text{ and } \ h \cdot (v \bullet \tilde{r}) = (h_{(1)} \cdot v) \bullet (h_{(2)} \cdot \tilde{r}), \tag{4.7}$$

$$r \bullet (r' \bullet v) = ((\phi^{(-1)} \cdot r)(\phi^{(-2)} \cdot r')) \bullet (\phi^{(-3)} \cdot v), \tag{4.8}$$

$$(v \bullet \tilde{r}) \bullet \tilde{r}' = (\phi^{(1)} \cdot v) \bullet ((\phi^{(2)} \cdot \tilde{r})(\phi^{(3)} \cdot \tilde{r}')), \tag{4.9}$$

$$(r \bullet v) \bullet \tilde{r} = (\phi^{(1)} \cdot r) \bullet ((\phi^{(2)} \cdot v) \bullet (\phi^{(3)} \cdot \tilde{r})), \tag{4.10}$$

for all $h \in H$, $v \in V$, $r, r' \in R$, and $\tilde{r}, \tilde{r}' \in \tilde{R}$.

## 4.3 Coinvariants for Relative Quasi-Hopf Bimodules

Let $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ be right $H$-comodule algebras and assume there are $H$-comodule algebra maps

$$\gamma : H \to A \text{ and } \gamma' : H \to B.$$

In particular $(\gamma \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = \phi_\rho$ and $(\gamma' \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = \phi_{\rho'}$. Let $M \in {}_A\mathcal{M}_B^H$, then $M \in {}_H\mathcal{M}_H^H$ via $\gamma$ and $\gamma'$ and we can define the coinvariants of $M$ by means of Hausser and Nill's projection

$$E : M \to M, \quad m \mapsto \gamma(q^{(1)}) m_{(0)} \gamma'(\beta S(q^{(2)} m_{(1)})).$$

Here, $q := q_R$ from Section 3.4. $E(M)$ is then a left $H$-module with the action given by $h \blacktriangleright E(m) = E(h \cdot E(m)) = E(h \cdot m)$.

In the same manner we may define the coinvariants $A^{coH} := E(A)$ of the $H$-comodule algebra $A$. We will see that $A^{coH}$ is an $H$-module algebras with the $H$-action $\blacktriangleright$ but with a new multiplication. This was already shown by

[PvO07] with a different proof. Moreover, we will show that $E(M)$ is an object in the category ${}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$ of $(A^{coH}, B^{coH})$-bimodules in ${}_H\mathcal{M}$. Finally we will prove that in this case, the category ${}_A\mathcal{M}_B^H$ of relative quasi-Hopf bimodules is equivalent to the category ${}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$.

In the following let $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ be $H$-comodule algebras with $H$-comodule algebra maps $\gamma : H \to A$ and $\gamma' : H \to B$ and let $M \in {}_A\mathcal{M}_B^H$.

**Lemma 4.3.1.** *For all $a, a' \in A$, $b, b' \in B$, $m \in M$ and $h \in H$ we have:*

(i) $E(mE(b)) = E(mb)$,

(ii) $E(aE(m)) = E(am)$,

(iii) $E(a_{(0)}m) \cdot \gamma'(a_{(1)}) = a \cdot E(m)$,

(iv) $E(m_{(0)} \cdot b)\gamma'(m_{(1)}) = mE(b)$,

(v) $E(E(\gamma(\phi^{(1)})a)E(\gamma(\phi^{(2)})m)\gamma'(\phi^{(3)})b) = E(E(E(a)m)b)$,

(vi) $E(E(\gamma(\phi^{(1)})a)E(\gamma(\phi^{(2)})a')\gamma(\phi^{(3)})m) = E(E(E(a)a')m)$,

(vii) $E(E(\gamma(\phi^{(1)})m)E(\gamma'(\phi^{(2)})b)\gamma'(\phi^{(3)})b') = E(E(E(m)b)b')$,

(viii) $E(1_A) = 1_A$,

(ix) $E(\gamma(h)) = \varepsilon(h)1_A$,

(x) $E(\gamma(\phi^{(-1)}))\gamma(\phi^{(-2)}) \otimes \phi^{(-3)} = 1_A \otimes 1$.

*Proof.* (i)
$$\begin{aligned} E(m \cdot E(b)) &= E(m\gamma'(q^{(1)})b_{(0)}\gamma'(\beta S(q^{(2)}b_{(1)}))) \\ &= E(m\gamma'(q^{(1)})b_{(0)})\varepsilon(\beta S(q^{(2)}b_{(1)}) && \text{by 4.1.1 (iv)} \\ &= E(m \cdot b). \end{aligned}$$

(ii) analogous to (i).

(iii)
$$\begin{aligned} E(a_{(0)}m)\gamma'(a_{(1)}) &= \gamma(q^{(1)})a_{(0,0)}m_{(0)}\gamma'(\beta S(q^{(2)}a_{(0,1)}m_{(1)})a_{(1)}) \\ &= a\gamma(q^{(1)})m_{(0)}\gamma'(\beta S(q^{(2)}m_{(1)})) && \text{by (3.18)} \\ &= a \cdot E(m). \end{aligned}$$

(iv) analogous to (iii) using (3.21).

(v)
$$\begin{aligned} &E(E(\gamma(\phi^{(1)})a)E(\gamma(\phi^{(2)})m)\gamma(\phi^{(3)})b) \\ &= E(E(E(\gamma(\phi^{(1)})a)_{(0)}E(\gamma(\phi^{(2)})m))\gamma'(E(\gamma(\phi^{(1)})a)_{(1)})\gamma'(\phi^{(3)})b) \\ &&& \text{by (iii)} \\ &= E(E(E(\gamma(\phi^{(-1)}\phi^{(1)})a)\gamma(\phi^{(-2)})E(\gamma(\phi^{(-2)})m))\gamma'(\phi^{(-3)})\gamma'(\phi^{(3)})b) \\ &&& \text{by 4.1.1 (viii)} \end{aligned}$$

$= E(E(E(\gamma(\phi^{(-1)}\phi^{(1)})a)\gamma(\phi^{(-2)}\phi^{(2)})m)\gamma'(\phi^{(-3)}\phi^{(3)})b)$ by (ii)
$= E(E(E(a)m)b)$.

(vi) and (vii) are analogous to (v).

(viii) $E(1_A) = \gamma(q^{(1)}\beta S(q^{(2)})) = \gamma(1) = 1_A$.

(ix) For all $h \in H$ we have $E(\gamma(h)) = E(1_A\gamma(h)) = E(1_A)\varepsilon(h) = 1_A\varepsilon(h)$ by (viii) and 4.1.1 (iv).

(x) follows from (ix) and the axioms for the coassociator (3.5). □

**Proposition 4.3.2.** *$A^{coH} = E(A)$ is an $H$-module algebra and $M^{coH} = E(M)$ is an object in ${}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$ with a structure given by*

$$a * a' := E(aa')$$
$$a \bullet m := E(am)$$
$$m \bullet b := E(mb)$$

*for all $a, a' \in E(A)$, $b \in E(B)$, and $m \in E(M)$, and the usual $H$-module structure $\blacktriangleright$.*

*Proof.* Since $A \in {}_A\mathcal{M}_A^H$, it suffices to show that the following holds for $M \in {}_A\mathcal{M}_B^H$ and for all $a, a' \in E(A)$, $b, b' \in E(B)$, $m \in E(M)$ and $h \in H$.
(i) $h \blacktriangleright (a \bullet m) = (h_{(1)} \blacktriangleright a) \bullet (h_{(2)} \blacktriangleright m)$,
(ii) $h \blacktriangleright (m \bullet b) = (h_{(1)} \blacktriangleright m) \bullet (h_{(2)} \blacktriangleright b)$,
(iii) $(a \bullet m) \bullet b = (\phi^{(1)} \blacktriangleright a) \bullet ((\phi^{(2)} \blacktriangleright m) \bullet (\phi^{(3)} \blacktriangleright b))$,
(iv) $(a * a') \bullet m = (\phi^{(1)} \blacktriangleright a) \bullet ((\phi^{(2)} \blacktriangleright a') \bullet (\phi^{(3)} \blacktriangleright m))$,
(v) $(m \bullet b) \bullet b' = (\phi^{(1)} \blacktriangleright m) \bullet ((\phi^{(2)} \blacktriangleright b) * (\phi^{(3)} \blacktriangleright b'))$,
(vi) $1_A \bullet m = m = m \bullet 1_B$.

Let $a, a' \in E(A)$, $b, b' \in E(B)$, $m \in E(M)$ and $h \in H$, then

$$\begin{aligned}(h_{(1)} \blacktriangleright a) \bullet (h_{(2)} \blacktriangleright m) &= E((h_{(1)} \blacktriangleright a) \cdot E(\gamma(h_{(2)}) \cdot m)) \\ &= E((h_{(1)} \blacktriangleright a) \cdot \gamma(h_{(2)}) \cdot m)) && \text{by 4.3.1 (ii)} \\ &= E(\gamma(h) \cdot am) && \text{by 4.1.1 (v)} \\ &= h \blacktriangleright E(a \cdot m) \\ &= h \blacktriangleright (a \bullet m)\end{aligned}$$

and so (i) holds, (ii) is similar.

(iii), (iv) and (v) hold by Lemma 4.3.1 (v), (vi), (vii), since

$$(a \bullet m) \bullet b = E(E(E(a) \cdot m) \cdot b)$$

and

$$(\phi^{(1)} \blacktriangleright a) \bullet ((\phi^{(2)} \blacktriangleright m) \bullet (\phi^{(3)} \blacktriangleright b)) = E(E(\gamma(\phi^{(1)})a)E(\gamma(\phi^{(2)}) \cdot m) \cdot \gamma'(\phi^{(3)})b)$$

and similar for (iv) and (v). $\square$

## 4.4 A Structure Theorem for $(H, A, B)$-Quasi-Hopf Bimodules

Let again $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ be $H$-comodule algebras, $\gamma : H \to A$ and $\gamma' : H \to B$ $H$-comodule algebra maps, and $M \in {}_A\mathcal{M}_B^H$.

Hausser and Nill's structure theorem for quasi-Hopf bimodules (Theorem 4.1.2) implies

$$\begin{aligned} \Psi : M &\stackrel{\cong}{\to} .M^{coH} \otimes .H. \quad \text{in } {}_H\mathcal{M}_H^H \\ m &\mapsto E(m_{(0)}) \otimes m_{(1)}, \end{aligned}$$

and in particular

$$\begin{aligned} \psi : A &\stackrel{\cong}{\to} .A^{coH} \otimes .H. \quad \text{in } {}_H\mathcal{M}_H^H \\ a &\mapsto E(a_{(0)}) \otimes a_{(1)}. \end{aligned}$$

We would like to define a multiplication on $A^{coH} \otimes H$ and a bimodule structure on $M^{coH} \otimes H$, such that the above isomorphism $\psi$ is an $H$-comodule algebra morphism, and $\Psi$ becomes a morphism in ${}_A\mathcal{M}_B^H$. In particular, we would want

$$E(a_{(0)}a'_{(0)}) \otimes a_{(1)}a'_{(1)} = (E(a_{(0)}) \otimes a_{(1)})(E(a'_{(0)}) \otimes a'_{(1)}),$$

for all $a, a' \in A$. We have

$$\begin{aligned} &E(a_{(0)}a'_{(0)}) \otimes a_{(1)}a'_{(1)} \\ &\quad = E(E(a_{(0,0)}\gamma(\phi^{(-1)}))\gamma(a_{(0,1)}\phi^{(-2)})a'_{(0)}) \otimes a_{(1)}\phi^{(-3)}a'_{(1)} \quad \text{by 4.3.1 (iii), (x)} \\ &\quad = E(E(\gamma(\phi^{(-1)})a_{(0)})\gamma(\phi^{(-2)}a_{(1,1)})a'_{(0)}) \otimes \phi^{(-3)}a_{(1,2)}a'_{(1)} \\ &\quad = (\phi^{(-1)} \blacktriangleright E(a_{(0)})) * (\phi^{(-2)}a_{(1,1)} \blacktriangleright a'_{(0)}) \otimes \phi^{(-3)}a_{(1,2)}a'_{(1)}. \end{aligned} \tag{4.11}$$

In the same way:

$$E(a_{(0)}m_{(0)}) \otimes a_{(1)}m_{(1)}$$
$$= (\phi^{(-1)} \blacktriangleright E(a_{(0)})) * (\phi^{(-2)}a_{(1,1)} \blacktriangleright m_{(0)}) \otimes \phi^{(-3)}a_{(1,2)}m_{(1)} \tag{4.12}$$
and
$$E(m_{(0)}b_{(0)}) \otimes m_{(1)}b_{(1)}$$
$$= (\phi^{(-1)} \blacktriangleright E(m_{(0)})) * (\phi^{(-2)}m_{(1,1)} \blacktriangleright b_{(0)}) \otimes \phi^{(-3)}m_{(1,2)}b_{(1)}, \tag{4.13}$$

for $a \in A$, $b \in B$ and $m \in M$. Hence, the desired isomorphism is obtained as follows:

**Lemma and Definition 4.4.1.** By defining a multiplication on $A^{coH} \otimes H$ via

$$(a \otimes h)(a' \otimes g) := (\phi^{(-1)} \blacktriangleright a) * (\phi^{(-2)}h_{(1)} \blacktriangleright a') \otimes \phi^{(-3)}h_{(2)}g \tag{4.14}$$

for all $a, a' \in A^{coH}$, $g, h \in H$, $A^{coH} \otimes H$ becomes an $H$-comodule algebra and $\psi$ from above is turned into an $H$-comodule algebra isomorphism, that is

$$A \cong A^{coH} \otimes H, \quad a \mapsto E(a_{(0)}) \otimes a_{(1)}$$

as $H$-comodule algebras.

*Proof.* $\psi(1) = E(1_A) \otimes 1 = 1_A \otimes 1$ and for $a \in A$, $h \in H$

$$(1_A \otimes 1)(a \otimes h) = (\phi^{(-1)} \blacktriangleright 1_A) * (\phi^{(-2)} \blacktriangleright a) \otimes \phi^{(-3)}h) = (a \otimes h),$$
$$(a \otimes h)(1_A \otimes 1) = (\phi^{(-1)} \blacktriangleright a) * (\phi^{(-2)}h_{(1)} \blacktriangleright 1_A) \otimes \phi^{(-3)}h_{(2)}) = (a \otimes h),$$

by (4.4) and (3.4). Now, $\psi$ is a multiplicative isomorphism by (4.11) and therefore $A^{coH} \otimes H$ becomes an algebra isomorphic to $A$. Since $\psi$ is a morphism in ${}_H\mathcal{M}_H^H$, it is even an $H$-comodule algebra morphism, where $A^{coH} \otimes H$ is an $H$-comodule algebra with the usual $H$-costructure $\rho_{A^{coH} \otimes H}$ given by

$$\rho_{A^{coH} \otimes H}(a \otimes h) = \phi^{(-1)} \blacktriangleright a \otimes \phi^{(-2)}h_{(1)} \otimes \phi^{(-3)}h_{(2)}.$$

The coassociator of $A^{coH} \otimes H$ is $(\psi \otimes \mathrm{id} \otimes \mathrm{id})(\phi_\rho) = (\psi \circ \gamma \otimes \mathrm{id} \otimes \mathrm{id})(\phi)$. ☐

The algebra structure of $A^{coH} \otimes H$ is the smash product structure, which we will define in detail in the next section. Therefore, 4.4.1 contains [PvO07, Theorem 2.5].

*Remark* 4.4.2. Note that the map $j : H \to A^{coH} \otimes H$, $h \mapsto 1_A \otimes h$ corresponds to the $H$-comodule algebra map $\gamma$. In fact, for $h \in H$:

$$\begin{aligned} \psi(\gamma(h)) &= E(\gamma(h)_{(0)}) \otimes \gamma(h)_{(1)} \\ &= E(\gamma(h_{(1)})) \otimes h_{(2)} \\ &= 1_A\varepsilon(h_{(1)}) \otimes h_{(2)} && \text{by 4.3.1 (ix)} \\ &= 1_A \otimes h. \end{aligned}$$

**Lemma and Definition 4.4.3.** Let $V \in {}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$, then we one can define an $(A^{coH} \otimes H, B^{coH} \otimes H)$-bimodule structure on $V \otimes H$ by

$$(a \otimes h)(v \otimes g) := (\phi^{(-1)} \blacktriangleright a) \bullet (\phi^{(-2)} h_{(1)} \cdot v) \otimes \phi^{(-3)} h_{(2)} g$$

and

$$(v \otimes g)(b \otimes h) := (\phi^{(-1)} \cdot v) \bullet (\phi^{(-2)} g_{(1)} \blacktriangleright b) \otimes \phi^{(-3)} g_{(2)} h$$

for all $a \in A^{coH}$, $b \in B^{coH}$, $g, h \in H$ and $v \in V$. Via $\psi$ from 4.4.1, $V \otimes H$ is an $(A, B)$-bimodule. With the usual costructure

$$\delta_{V \otimes H}(v \otimes g) = \phi^{(-1)} \cdot v \otimes \phi^{(-2)} h_{(1)} \otimes \phi^{(-3)} h_{(2)}$$

$V \otimes H$ becomes an object in ${}_A\mathcal{M}_B^H$, and $\Psi : M \to M^{coH} \otimes H$ becomes an isomorphism in ${}_A\mathcal{M}_B^H$.

*Proof.* Let $a, a' \in A^{coH}$, $h, h', g \in H$ and $v \in V$, then

$$\begin{aligned}
&(a \otimes h)((a' \otimes h')(v \otimes g)) \\
&= (\tilde{\phi}^{(-1)} \blacktriangleright a) \bullet (\tilde{\phi}^{(-2)} h_{(1)} \cdot ((\phi^{(-1)} \blacktriangleright a') \bullet (\phi^{(-2)} h'_{(1)} \cdot v)) \\
&\qquad\qquad\qquad\qquad \otimes \tilde{\phi}^{(-3)} h_{(2)} \phi^{(-3)} h'_{(2)} g \\
&= (\tilde{\phi}^{(-1)} \blacktriangleright a) \bullet ((\tilde{\phi}^{(-2)}_{(1)} h_{(1,1)} \phi^{(-1)} \blacktriangleright a') \bullet (\tilde{\phi}^{(-2)}_{(2)} h_{(1,2)} \phi^{(-2)} h'_{(1)} \cdot v)) \\
&\qquad\qquad\qquad\qquad \otimes \tilde{\phi}^{(-3)} h_{(2)} \phi^{(-3)} h'_{(2)} g \\
&= (\tilde{\phi}^{(-1)} \blacktriangleright a) \bullet ((\tilde{\phi}^{(-2)}_{(1)} \phi^{(-1)} h_{(1)} \blacktriangleright a') \bullet (\tilde{\phi}^{(-2)}_{(2)} \phi^{(-2)} h_{(2,1)} h'_{(1)} \cdot v)) \\
&\qquad\qquad\qquad\qquad \otimes \tilde{\phi}^{(-3)} \phi^{(-3)} h_{(2,2)} h'_{(2)} g \\
&= (\phi^{(1)} \tilde{\phi}^{(-1)}_{(1)} \phi^{(-1)} \blacktriangleright a) \bullet ((\phi^{(2)} \tilde{\phi}^{(-1)}_{(2)} \phi^{(-2)} h_{(1)} \blacktriangleright a') \\
&\qquad\qquad \bullet (\phi^{(3)} \tilde{\phi}^{(-2)} \phi^{(-3)}_{(1)} h_{(2,1)} h'_{(1)} \cdot v)) \otimes \tilde{\phi}^{(-3)} \phi^{(-3)}_{(2)} h_{(2,2)} h'_{(2)} g \\
&= (\tilde{\phi}^{(-1)} \blacktriangleright ((\phi^{(-1)} \blacktriangleright a) \bullet (\phi^{(-2)} h_{(1)} \blacktriangleright a'))) \bullet (\tilde{\phi}^{(-2)} (\phi^{(-3)} h_{(2)} h')_{(1)} \cdot v)) \\
&\qquad\qquad\qquad\qquad \otimes \tilde{\phi}^{(-3)} (\phi^{(-3)} h_{(2)} h')_{(2)} g \\
&= ((a \otimes h)(a' \otimes h'))(v \otimes g)
\end{aligned}$$

by (3.1), (3.3) and (4.8). Moreover,

$$
\begin{aligned}
&\delta_{V\otimes H}((a\otimes h)(v\otimes g))\\
&\quad = \delta_{V\otimes H}((\phi^{(-1)} \blacktriangleright a)\bullet(\phi^{(-2)}h_{(1)}\cdot v)\otimes \phi^{(-3)}h_{(2)}g)\\
&\quad = \tilde\phi^{(-1)}\cdot((\phi^{(-1)} \blacktriangleright a)\bullet(\phi^{(-2)}h_{(1)}\cdot v))\\
&\qquad\qquad \otimes \tilde\phi^{(-2)}\phi^{(-3)}_{(1)}h_{(2,1)}g_{(1)}\otimes \tilde\phi^{(-3)}\phi^{(-3)}_{(2)}h_{(2,2)}g_{(2)}\\
&\quad = (\tilde{\tilde\phi}^{(-1)}\tilde\phi^{(-1)} \blacktriangleright a)\bullet(\tilde{\tilde\phi}^{(-2)}\tilde\phi^{(-2)}_{(1)}\phi^{(-1)}h_{(1)}\cdot v)\\
&\qquad\qquad \otimes \tilde{\tilde\phi}^{(-3)}\tilde\phi^{(-2)}_{(2)}\phi^{(-2)}h_{(2,1)}g_{(1)}\otimes \tilde\phi^{(-3)}\phi^{(-3)}h_{(2,2)}g_{(2)}\\
&\quad = (\tilde{\tilde\phi}^{(-1)}\tilde\phi^{(-1)} \blacktriangleright a)\bullet(\tilde{\tilde\phi}^{(-2)}(\tilde\phi^{(-2)}h_{(1)})_{(1)}\phi^{(-1)}\cdot v)\\
&\qquad\qquad \otimes \tilde{\tilde\phi}^{(-3)}(\tilde\phi^{(-2)}h_{(1)})_{(2)}\phi^{(-2)}g_{(1)}\otimes \tilde\phi^{(-3)}h_{(2)}\phi^{(-3)}g_{(2)}\\
&\quad = \tilde\phi^{(-1)} \blacktriangleright a\otimes \tilde\phi^{(-2)}h_{(1)}\otimes\tilde\phi^{(-3)}h_{(2)})(\phi^{(-1)}\cdot v\otimes\phi^{(-2)}g_{(1)}\otimes\phi^{(-3)}g_{(3)})\\
&\quad = \rho_{A^{coH}\otimes H}(a\otimes h)\delta_{V\otimes H}(v\otimes g),
\end{aligned}
$$

by (3.3) and (3.1). The remaining equations follow analogously. □

**Theorem 4.4.4.** *Let $(A,\rho,\phi_\rho)$ and $(B,\rho',\phi_{\rho'})$ be right $H$-comodule algebras and $\gamma : H \to A$ and $\gamma' : H \to B$ $H$-comodule algebra maps. Let $M \in {}_A\mathcal{M}_B^H$. Then*

$$
\begin{aligned}
{}_A\mathcal{M}_B^H &\approx {}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}\\
M &\mapsto M^{coH}\\
f : M \to N &\mapsto f|_{M^{coH}} : M^{coH} \to N^{coH}\\
V\otimes H &\leftarrow\!\shortmid V\\
g\otimes \mathrm{id} : V\otimes H \to W\otimes H &\leftarrow\!\shortmid g : V \to W.
\end{aligned}
$$

*Proof.* For $M \in {}_A\mathcal{M}_B^H$ we have $M^{coH} \in {}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$ and $M^{coH}\otimes H \cong M$ in ${}_A\mathcal{M}_B^H$ by Lemma 4.4.3. Let $f : M \to N$ be a morphism in ${}_A\mathcal{M}_B^H$, then $f|_{M^{coH}}$ is a well-defined morphism $M^{coH} \to N^{coH}$ in ${}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$. In fact, for all $m \in M$

$$
\begin{aligned}
f(E(m)) &= f(\gamma(q^{(1)})\cdot m_{(0)}\cdot\gamma'(\beta S(q^{(2)}m_{(1)})))\\
&= \gamma(q^{(1)})\cdot f(m)_{(0)}\cdot\gamma'(\beta S(q^{(2)}f(m)_{(1)}))) = E(f(m)),
\end{aligned}
$$

and $f|_{M^{coH}}$ is a morphism in ${}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$, since for all $h \in H$, $a \in E(A)$, $b \in E(B)$, $m \in E(M)$

$$
\begin{aligned}
f(h \blacktriangleright m) = E(f(\gamma(h)\cdot m)) = E(\gamma(h)\cdot f(m)) = h \blacktriangleright f(m),\\
f(a\bullet m) = E(f(a\cdot m)) = E(a\cdot f(m)) = a\bullet f(m),\\
f(m\bullet b) = E(f(m\cdot b)) = E(f(m)\cdot b) = f(m)\bullet b.
\end{aligned}
$$

For $V \in {}_{A^{coH}}({}_H\mathcal{M})_{B^{coH}}$ we have $V \otimes H \in {}_A\mathcal{M}_B^H$ by 4.4.3 and $(V \otimes H)^{coH} = V \otimes 1_H \cong V$, since for all $v \in V$ and $h \in H$,

$$\begin{aligned}
E(v \otimes h) &= \psi(\gamma(q^{(1)}))(\phi^{(-1)} \cdot v \otimes \phi^{(-2)} h_{(1)})\psi(\gamma'(\beta S(q_{(2)}\phi^{(-3)} h_{(2)}))) \\
&= (1_A \otimes q^{(1)})(\phi^{(-1)} \cdot v \otimes \phi^{(-2)} h_{(1)})(1_B \otimes \beta S(q_{(2)}\phi^{(-3)} h_{(2)})) \\
&= (1_A \otimes q^{(1)})(\tilde{\phi}^{(-1)}\phi^{(-1)} \cdot v) \bullet (\tilde{\phi}^{(-2)}\phi^{(-2)}_{(1)} h_{(1,1)} \blacktriangleright 1_B) \\
&\qquad\qquad \otimes \tilde{\phi}^{(-3)}\phi^{(-2)}_{(2)} h_{(1,2)}\beta S(q^{(2)}\phi^{(-3)} h_{(2)}) \\
&= (1_A \otimes q^{(1)})(\phi^{(-1)} \cdot v \otimes \phi^{(-2)} h_{(1)}\beta S(q^{(2)}\phi^{(-3)} h_{(2)}) \\
&= q^{(1)}_{(1)}\phi^{(-1)} \cdot v \otimes q^{(1)}_{(2)}\phi^{(-2)}\beta S(\phi^{(-3)})S(q^{(2)})\varepsilon(h) \\
&= q^{(1)}_{(1)}p^{(1)} \cdot v \otimes q^{(1)}_{(2)}p^{(2)} S(q^{(2)})\varepsilon(h) \\
&= v\varepsilon(h) \otimes 1,
\end{aligned}$$

by Remark 4.4.2, (4.4), (3.4), (3.6) and (3.20). Obviously, $g \otimes \mathrm{id} : V \otimes H \to W \otimes H$ is a well-defined morphism in ${}_A\mathcal{M}_B^H$. □

## 4.5 Smash Products for Quasi-Hopf Algebras

**Lemma and Definition 4.5.1.** [BPvO00, PvO07] Let $R$ be a left $H$-module algebra. The **smash product** of $R$ by $H$ is denoted by $R\#H$. It is $R \otimes H$ as a vector space and the multiplication is defined by

$$(r\#h)(r'\#g) := (\phi^{(-1)} \cdot r)(\phi^{(-2)} h_{(1)} \cdot r')\#\phi^{(-3)} h_{(2)} g,$$

for all $r, r' \in R$ and $g, h \in H$. Then $R\#H$ is a right $H$-comodule algebra with an $H$-costructure given by

$$\rho_{R\#H}(r\#h) = \phi^{(-1)} \cdot r\#\phi^{(-2)} h_{(1)} \otimes \phi^{(-3)} h_{(2)},$$

for $r \in R$, $h \in H$.

The map $j : H \to R\#H,\ h \mapsto 1_R\#h$ is an $H$-comodule algebra map and $(R\#H)^{coH} = E(R\#H) = R\#1_H$ is isomorphic to $R$ as $H$-module algebras. Moreover, $i : R \to (R\#H)^j,\ r \mapsto p^{(1)} \cdot r \otimes p^{(2)}$ is an $H$-module algebra map, where $(R\#H)^j$ is defined as in 4.2.1, and $p := p_R$ from Section 3.4.

Bulacu, Panaite and van Oystaeyen [BPvO00] have shown that the smash product for quasi-Hopf algebras satisfies a universal property as it does for Hopf algebras. The smash product is defined as an algebra $B$, together with an algebra morphism $j : H \to B$ and a $k$-linear map $i : A \to B$ such that $i$ is an $H$-module algebra map $A \to B^j$, where $B^j$ is the $H$-module algebra defined in 4.2.1. Moreover, $B$ satisfies a universal property, that is for any algebra $B'$ with an algebra map $v : H \to B'$ and an $H$-module algebra map $u : A \to B'^j$, there exists an algebra morphism $w : B \to B'$ such that $w \circ j = v$ and $w \circ i = u$.

*Remark* 4.5.2. It can easily be seen that $i$ is an $H$-comodule algebra map, for if $g, h \in H$, then

$$\begin{aligned}(1_R\#g)(1_R\#h) &= (\phi^{(-1)}\cdot 1_R)(\phi^{(-2)}g_{(1)}\cdot 1_R)\otimes \phi^{(-3)}g_{(2)}h \\ &= (\varepsilon(\phi^{(-1)})1_R)(\varepsilon(\phi^{(-2)}g_{(1)})1_R)\otimes \phi^{(-3)}g_{(2)}h \\ &= 1_R\#gh \\ \text{and} \qquad 1_R\#h_{(1)}\otimes h_{(2)} &= \phi^{(-1)}\cdot 1_R\#\phi^{(-1)}h_{(1)}\otimes \phi^{(-2)}h_{(2)}.\end{aligned}$$

For all $r \in R$, $h \in H$, we have

$$\begin{aligned}E(r\#h) &= E(r\#1_H)\varepsilon(h) \\ &= (1_R\#q^{(1)})(\phi^{(-1)}\cdot r\#\phi^{(-2)})(1_R\#\beta S(q^{(2)}\phi^{(-3)}))\varepsilon(h) \\ &= q^{(1)}_{(1)}\phi^{(-1)}\cdot r\#q^{(1)}_{(2)}\phi^{(-2)}\beta S(\phi^{(-3)})S(q^{(2)})\varepsilon(h) \\ &= q^{(1)}_{(1)}p^{(1)}\cdot r\#q^{(1)}_{(2)}p^{(2)}S(q^{(2)})\varepsilon(h) \\ &= (r\#1_H)\varepsilon(h),\end{aligned}$$

where $q := q_R$ and $p := p_R$ from Section 3.4 and the last step holds by (3.19).

The $H$-module algebra structure of $R\#1 = (R\#H)^{coH}$ is exactly the one of $R$. In fact,

$$\begin{aligned}(r\#1_H)*(r'\#1_H) &= E((\phi^{(-1)}\cdot r)(\phi^{(-2)}\cdot r')\#\phi^{(-3)}) \\ &= E(rr'\#1_H) \\ &= rr'\#1_H \\ \text{and} \qquad h \blacktriangleright (r\#1_H) &= E((1_R\#h)(r\#1_H)) \\ &= E((\phi^{(-1)}\cdot 1_R)(\phi^{(-2)}h_{(1)}\cdot r)\#\phi^{(-3)}h_{(2)}) \\ &= E(h_{(1)}\cdot r\#1_H)\varepsilon(h_{(2)}) \\ &= h\cdot r\#1_H,\end{aligned}$$

for all $r, r' \in R$, $h \in H$.

If $R = A^{coH}$ for an $H$-comodule algebra $A$ with $H$-comodule algebra map $\gamma : H \to A$, then the smash product structure is exactly the structure on $A^{coH}\otimes H$ as defined in 4.4.1. Altogether, an $H$-comodule algebra $A$ is isomorphic to a smash product if and only if there exists an $H$-comodule algebra map $\gamma : H \to A$.

For an object $V \in {}_R({}_H\mathcal{M})_{\tilde{R}}$ one can therefore define an $(R\#H, \tilde{R}\#H)$-bimodule structure on $V\otimes H$ as in Definition 4.4.3, such that $V\otimes H$ becomes an object in ${}_{R\#H}\mathcal{M}^H_{\tilde{R}\#H}$.

**Lemma and Definition 4.5.3.** Let $R, \tilde{R}$ be left $H$-module algebras and let $V \in {}_R({}_H\mathcal{M})_{\tilde{R}}$. We denote by $\boldsymbol{V\#H}$ the vector space $V \otimes H$ with an $(R\#H, \tilde{R}\#H)$-bimodule structure defined by

$$(r\#h) \cdot (v\#g) := (\phi^{(-1)} \cdot r) \bullet (\phi^{(-2)} h_{(1)} \cdot v)\#\phi^{(-3)} h_{(2)} g, \tag{4.15}$$

$$(v\#g) \cdot (\tilde{r}\#h) := (\phi^{(-1)} \cdot v) \bullet (\phi^{(-2)} g_{(1)} \cdot v)\#\phi^{(-3)} g_{(2)} h, \tag{4.16}$$

for $v \in V$, $r \in R$, $\tilde{r} \in \tilde{R}$, and $g, h \in H$. Moreover, $V\#H \in {}_{R\#H}\mathcal{M}^H_{\tilde{R}\#H}$ with a $H$-costructure defined by

$$\delta_{V\#H}(v\#h) = \phi^{(-1)} v\#\phi^{(-2)} h_{(1)} \otimes \phi^{(-3)} h_{(2)},$$

for $v \in V$ and $h \in H$.

*Remark* 4.5.4. With this definition the following holds for $V \in {}_R({}_H\mathcal{M})_{\tilde{R}}$:

$$(V\#H)^{coH} = V\#1_H \cong V.$$

In fact, $E(v\#h) = (v\#1_H)\varepsilon(h)$ as above. The $((R\#H)^{coH}, (\tilde{R}\#H)^{coH})$-bimodule structure of $(V\#H)^{coH}$ in ${}_H\mathcal{M}$ is exactly the $(R, R')$-bimodule structure of $V$, which can be proven analogously to the Remark 4.5.2.

Altogether, this leads to the structure theorem for smash products:

**Corollary 4.5.5.** *Let $R, \tilde{R}$ be left $H$-module algebras. Applying Theorem 4.4.4 to the $H$-comodule algebras $R\#H$ and $\tilde{R}\#H$ with the $H$-comodule map $i$, induces*

$$\begin{aligned} {}_R({}_H\mathcal{M})_{\tilde{R}} &\approx {}_{R\#H}\mathcal{M}^H_{\tilde{R}\#H} \\ V &\mapsto V\#H \\ f : V \to W &\mapsto f \otimes id : V\#H \to W\#H \\ E(M) &\leftarrow M \\ g|_{E(M)} : E(M) \to E(N) &\leftarrow g : M \to N. \end{aligned}$$

**Remark 4.5.6.** *If $R$ or $\tilde{R}$ are assumed to be trivial, the corollary implies*

$$\begin{gathered} {}_R({}_H\mathcal{M}) \approx {}_{R\#H}\mathcal{M}^H_H \\ \textit{and} \\ ({}_H\mathcal{M})_R \approx {}_H\mathcal{M}^H_{R\#H}. \end{gathered}$$

## 4.6 *H*-Ideals and *H*-Costable Ideals

**Definition.** Let $(A, \rho)$ be an $H$-comodule algebra. An ideal $I \subset A$ is called ***H*-costable** if $\rho(I) \subset I \otimes H$. $A$ is called ***H*-simple** if $A$ is nonzero and does not contain a nonzero, proper $H$-costable ideal.

Let $R$ be an $H$-module algebra. We will call a subobject of $R$ in ${}_R({}_H\mathcal{M})_R$ an ***H*-ideal** of $R$, and subobjects of $R$ in ${}_R({}_H\mathcal{M})$ respectively $({}_H\mathcal{M})_R$ left respectively right $H$-ideals of $R$. $R$ will be called ***H*-simple**, if $R$ is nonzero and it is a simple object in ${}_R({}_H\mathcal{M})_R$.

**Proposition 4.6.1.** *Let $R$ be an $H$-module algebra. The $H$-ideals of $R$ correspond to the $H$-costable ideals of $R\#H$ as follows.*

$$\begin{array}{ccc} \{H\text{-ideals of } R\} & \overset{\Phi}{\underset{\Psi}{\rightleftarrows}} & \{H\text{-costable ideals of } R\#H\} \\ I & \longmapsto & I\#H \\ J^{coH} & \longleftarrow & J \end{array}$$

*are well-defined mutually inverse bijections. In particular $R$ is $H$-simple if and only if $R\#H$ is $H$-simple.*

*Proof.* We identify $R$ with $(R\#H)^{coH} = R\#1_H$ (see Section 4.4). The claim follows by 4.5.5. □

In Chapter 6 we will apply this result to exact module categories over quasi-Hopf algebras and obtain that these are exactly the modules over $H$-simple $H$-comodule algebras. Moreover, we will be able to deduce that for a semisimple quasi-Hopf algebra $H$ and an $H$-simple $H$-module algebra $R$, also the smash product $R\#H$ is semisimple.

## 4.7 Dual Quasi-Hopf Algebras and Quasi-Smash Products

The dual $H^*$ of a quasi-Hopf algebra together with the convolution product is not an associative algebra, but it is an algebra in the monoidal category ${}_H\mathcal{M}_H$ of $(H, H)$-bimodules, where the bimodule structure of $H^*$ is given as usual by $(h \rightharpoonup \varphi \leftharpoonup h')(g) = \varphi(h'gh)$ for all $g, h, h' \in H$ and $\varphi \in H^*$ [HN99b]. Furthermore, it was mentioned in Remark 3.1.2 that $H^*$ is a dual quasi-Hopf algebra as defined in [BN00, Definition 1.4]. In this section we will introduce the smash product for dual quasi-Hopf algebras as defined by Bulacu and Caenepeel [BC03b]. We will see that relative quasi-Hopf bimodules correspond to modules over the smash product in the category of left $H$-modules. In Chapter 5 this will be used to produce finitely generated subobjects of a relative quasi-Hopf bimodule in ${}_H\mathcal{M}_A^H$ and to generalize a freeness result to non-finitely generated objects in ${}_H\mathcal{M}_A^H$.

*Remark* 4.7.1. Let $H$ be a finite dimensional quasi-Hopf algebra and $B$ an $H$-module algebra. Then the objects in $({}_H\mathcal{M})_B$ are exactly the right $(H^*, B)$-Hopf modules defined in [BN00, Definition 2.4].

**Lemma and Definition 4.7.2.** [BC03b, Proposition 2.2] Let $A$ be a right $H$-comodule algebra. We define the **quasi-smash product** $A\overline{\#}H^*$ as $A \otimes H^*$ as a $k$-space and with a multiplication given by

$$(a\overline{\#}\varphi)(b\overline{\#}\psi) = ab_{(0)}\phi_\rho^{(-1)}\overline{\#}(\varphi \leftharpoonup b_{(1)}\phi_\rho^{(-2)}) * (\psi \leftharpoonup \phi_\rho^{(-3)}),$$

for all $a, b \in A$, $\varphi, \psi \in H^*$, where $a\overline{\#}\varphi$ stands for $a \otimes \varphi$. Then $A\overline{\#}H^*$ is a left $H$-module algebra, with unit $1_A \otimes \varepsilon$ and the $H$-action given by

$$h(a\overline{\#}\varphi) = a\overline{\#}(h \rightharpoonup \varphi),$$

for all $h \in H$, $a \in A$ and $\varphi \in H^*$. Moreover, the map

$$i : A \to A\overline{\#}H^*,\ a \mapsto a\overline{\#}\varepsilon$$

is injective and multiplicative.

**Proposition 4.7.3.** [BC03b, Theorem 3.5] *Let $H$ be a finite dimensional quasi-Hopf algebra and $A$ a right $H$-comodule algebra. Then ${}_H\mathcal{M}_A^H \xrightarrow{\mathcal{G}} ({}_H\mathcal{M})_{A\overline{\#}H^*}$ is an equivalence of categories, where $\mathcal{G}$ is the identity on the objects and morphisms. Furthermore, the $A$-module structures of $M \in {}_H\mathcal{M}_A^H$ and $\mathcal{G}(M) \in ({}_H\mathcal{M})_{A\overline{\#}H^*}$ are the same, where $\mathcal{G}(M)$ is a right $A$-module via the map $i$.*

*Proof.* Bulacu and Caenepeel have shown in [BC03b, Theorem 3.5] that $\mathcal{G}(M) = M \in ({}_H\mathcal{M})_{A\overline{\#}H^*}$ if $M \in {}_H\mathcal{M}_A^H$ and that $\mathcal{G}$ is an equivalence. The left $H$-module structure $\succ$ and the right $A\overline{\#}H^*$-action $\prec$ of $M$ are given by

$$\begin{aligned} h \succ m &:= S^2(h)m, \\ m \prec (a\overline{\#}\varphi) &:= \varphi(S^{-1}(S(U^{(1)})F_S^{(2)}m_{(1)}a_{(1)}p_\rho^{(2)}))\, S(U^{(2)})F_S^{(1)}m_{(0)}a_{(0)}p_\rho^{(1)}, \end{aligned}$$

for all $h \in H$, $\varphi \in H^*$, $a \in A$, and $m \in M$. Here, $F_S$ is defined as in (3.8) and $U^{(1)} \otimes U^{(2)} := F_S^{(-1)}S(q_R^{(2)}) \otimes F_S^{(-1)}S(q_R^{(1)})$ omitting the summation symbol.

The claim that the $A$-module structures are the same follows easily from the unit constraints (3.14) and (3.10) of $M$ and $A$, the definitions of the coassociator, and the fact that $F_S$ is a twist. $\square$

# Chapter 5

# Freeness and Projectivity over Quasi-Hopf Comodule Algebras

In 2004 Skryabin succeeded in generalizing the Hopf algebra Freeness Theorem by Nichols and Zöller [NZ89] to coideal subalgebras and thereby gave an entirely new proof for the Nichols-Zöller Theorem. Skryabin [Skr07] proved that for a weakly finite Hopf algebra $H$ and a semilocal $H$-simple $H$-comodule algebra $A$, all relative Hopf modules in $\mathcal{M}_A^H$ are projective $A$-modules, and $A$ is a Frobenius algebra. Moreover, if $K$ is a right coideal subalgebra of $H$, then $K$ is $H$-simple and all $(H, K)$-Hopf modules even are free $K$-modules. In particular, weakly finite Hopf algebras are free over their finite dimensional right coideal subalgebras. In this chapter we generalize Skryabin's freeness theorem to quasi-Hopf algebras as introduced by Drinfeld [Dri90]. This result includes the quasi-Hopf algebra freeness theorem by Schauenburg [Sch04].

More precisely, we prove that if $H$ is a finite dimensional quasi-Hopf algebra and $K$ is a right coideal subalgebra of $H$, then every object in ${}_H\mathcal{M}_K^H$ and ${}_K\mathcal{M}_H^H$ is a free $K$-module and $K$ is a Frobenius algebra. This also holds for finite dimensional objects in ${}_B\mathcal{M}_K^H$ and ${}_K\mathcal{M}_B^H$, where $B$ is an arbitrary $H$-comodule algebra. We can also conclude that relative quasi-Hopf bimodules over $H$-simple $H$-comodule algebras are projective. The fact that $H$-simple $H$-comodule algebras are quasi-Frobenius can not be shown directly as in Skryabin's case. However we will obtain this by results about module categories over quasi-Hopf algebras in the next chapter.

In this chapter it is not necessary to assume that $k$ is algebraically closed.

## 5.1 Freeness over Right Coideal Subalgebras

In the following let $H$ be a quasi-Hopf algebra.

**Lemma 5.1.1.** *Let $(A, \rho)$ and $(B, \rho')$ be right $H$-comodule algebras and let $M \in {}_B\mathcal{M}_A^H$ be finitely generated as a right $A$-module. Let $I$ be an ideal of $A$ which does not contain a nonzero $H$-costable ideal of $A$. Assume that the ring $A/I \otimes H \otimes H^{op}$ is weakly finite* (see Appendix A.5)*. Moreover, assume that there exists a generating system $m_1, \ldots, m_n$ for $M$ in $\mathcal{M}_A$ such that its image $\overline{m_1}, \ldots, \overline{m_n} \in M/MI$ is an $A/I$-basis for $M/MI$. Then $m_1, \ldots, m_n$ is an $A$-basis for $M$.*

*Proof.* We will show the following:

(a) $J := \rho^{-1}(I \otimes H)$ is an $H$-costable ideal of $A$ contained in $I$, hence $J = 0$.

(b) The images of $\delta_{H\otimes M}(1 \otimes m_1), \ldots, \delta_{H\otimes M}(1 \otimes m_n)$ form a basis of $(.H \otimes M. \otimes .H.)/(.H \otimes M. \otimes .H.)(I \otimes H)$ in ${}_H\mathcal{M}_{(A\otimes H)/(I\otimes H)}$.

Assume that (a) and (b) are satisfied and let $a_1, \ldots, a_n \in A$ with $\sum_i m_i a_i = 0$. Then

$$\begin{aligned}\sum_i \delta_{H\otimes M}(1 \otimes m_i)\rho(a_i) &= \sum_i \delta_{H\otimes M}(1 \otimes m_i a_i) \\ &= \delta_{H\otimes M}(1 \otimes (\sum_i m_i a_i)) \\ &= 0,\end{aligned}$$

and (b) implies that $\rho(a_i) \subset I \otimes H$ for all $i$. By (a) it follows that $a_i \in J = 0$. Hence, $m_1, \ldots, m_n$ form a basis of $M$ in $\mathcal{M}_A$.

*Proof of* (a): $J$ is an ideal of $A$ since $\rho$ is an algebra morphism. It is contained in $I$ since $a = a_{(0)}\varepsilon(a_{(1)}) \in I$ for all $a \in J$. Finally, for all $a \in J$ we have

$$(\rho \otimes \mathrm{id})(\rho(a)) = \phi_\rho^{-1}(\mathrm{id} \otimes \Delta)(\rho(a))\phi_\rho \in I \otimes H \otimes H,$$

since $\phi_\rho \in A \otimes H \otimes H$. Hence $\rho(J) \subset J \otimes H$.

*Proof of* (b): By Lemma 3.4.2 we have

$$p_R((1 \otimes m) \otimes 1) = \delta_{H\otimes M}((1 \otimes m)_{(0)})p_\rho(1 \otimes S((1 \otimes m)_{(1)}))$$

for every $m \in M$, since $.H \otimes M. \in {}_H\mathcal{M}_A^H$ by Remark 3.5.2 (ii). Moreover, $p_R((1 \otimes m) \otimes 1) = p_R^{(1)} \otimes m \otimes p_R^{(2)} = \psi^{-1}(1 \otimes m \otimes 1)$, where $\psi$ is defined as in Remark 3.5.2 (iii).Therefore

$$1 \otimes m \otimes 1 = \psi(\delta_{H\otimes M}((1 \otimes m)_{(0)}))\, p_\rho(1 \otimes S((1 \otimes m)_{(1)})),$$

since $\psi$ is right $A \otimes H$-linear.

Obviously, $1 \otimes m_1, \ldots, 1 \otimes m_n$ is a generating system of $.H \otimes M.$, hence for every $x \in H \otimes M$ there exist $a_{i,j} \in A$, $1 \leq i \leq n$, $j \in \mathcal{S}$ such that $x = \sum_{i,j} x_j(1 \otimes m_i)a_{i,j}$, where $(x_j)_{j\in\mathcal{S}}$ is a $k$-basis of $H$. Hence,

$$\psi(\delta_{H\otimes M}(x)) = \sum_{i,j} x_j\, \psi(\delta_{H\otimes M}(1 \otimes m_i))\, \rho(a_{i,j}).$$

It follows that $\psi(\delta_{H\otimes M}(1 \otimes m_1)), \ldots, \psi(\delta_{H\otimes M}(1 \otimes m_n))$ is a generating system of $.H \otimes M. \otimes H. \in {}_H\mathcal{M}_{A\otimes H}$ as so is $1 \otimes m_1 \otimes 1, \ldots, 1 \otimes m_n \otimes 1$. Since $\psi$ is an $(H, A\otimes H)$-bimodule isomorphism, also $\delta_{H\otimes M}(1\otimes m_1), \ldots, \delta_{H\otimes M}(1\otimes m_n)$ is a generating system of $(.H \otimes M.) \otimes .H.$.

We have natural isomorphisms $(A \otimes H)/(I \otimes H) \cong A/I \otimes H$ and $(H \otimes M. \otimes H.)/(H \otimes M. \otimes H.)(I \otimes H) = (H \otimes M. \otimes H.)/(H \otimes MI. \otimes H.) \cong H \otimes M/MI. \otimes H.$, where the latter is right $A/I \otimes H$-linear. Hence,

$$(.H \otimes M. \otimes .H.)/(H \otimes M. \otimes H.)(I \otimes H) \cong .H \otimes M/MI. \otimes H.$$

in ${}_H\mathcal{M}_{A/I\otimes H}$ by Remark 3.5.2 (iii).

Clearly, $1 \otimes \overline{m_1} \otimes 1, \ldots, 1 \otimes \overline{m_n} \otimes 1$ is a basis for $.H \otimes (M/MI). \otimes H.$ in ${}_H\mathcal{M}_{A/I\otimes H}$ and the claim follows since $A/I \otimes H \otimes H^{op}$ is assumed to be weakly finite. $\square$

In the proof of the analogous result for Hopf algebras [Skr07, Lemma 3.1, 3.2 and 3.3], Skryabin could find a suitable basis for the one-sided $A \otimes H$-module $M \otimes H$. Here, the more involved antipode property makes it necessary to work with the two sided $(H, A \otimes H)$-module $H \otimes M \otimes H$.

**Definition.** [Skr07] Let $R$ be a semilocal ring (see Appendix A.5). For each $Q \in \mathrm{Max}R$ put

$$r_Q(M) := \frac{\mathit{length}(M/MQ)}{\mathit{length}(R/Q)}.$$

Note that $r_Q(M)$ is well-defined since $R/Q$ is a simple Artinian ring.

**Lemma 5.1.2.** [Skr07, Lemma 2.4 and Proof of Lemma 3.4] *Let $R$ be a semilocal ring and let $M$ be a finitely generated right $R$-module.*

*(1) Assume there exists a maximal ideal $P \in \mathrm{Max}R$ with $r_P(M) = n \in \mathbb{Z}$ and $r_P(M) \geq r_Q(M)$ for all $Q \in \mathrm{Max}R$. Then there exists a generating system $m_1, \ldots, m_n$ for $M$ such that its image is a basis for $M/MP$ in $M_{R/P}$.*

*(2) Assume that a finite direct sum of copies of $M$ is a free $R$-module. Then $M$ is a free $R$-module of rank $n$ if and only if $M/MQ$ is a free $R/Q$-module of rank $n$ for a maximal ideal $Q \in \mathrm{Max}R$.*

**Lemma 5.1.3.** *Let $(A, \rho)$ and $(B, \rho')$ be right $H$-comodule algebras and assume that $A$ is semilocal. Let $M \in {}_B\mathcal{M}_A^H$ be finitely generated as a right $A$-module.*

*Assume that there exists a maximal ideal $P \subset A$, which does not contain a nonzero $H$-costable ideal of $A$, and satisfies $r_P(M) \geq r_Q(M)$ for all $Q \in \mathrm{Max}A$. Assume furthermore that $A/P \otimes H \otimes H^{op}$ is weakly finite. Then a suitable direct sum $M^t$ of $t \in \mathbb{N}$ copies of $M$ is a free right $A$-module. Moreover, $M$ is a free right $A$-module whenever $r_P(M) \in \mathbb{Z}$.*

*Proof.* First, assume that $r_P(M) = n \in \mathbb{Z}$. Then, by the foregoing Lemma 5.1.2, there exists a generating system $m_1, \dots, m_n$ of $M$, such that $P$ and $m_1, \dots, m_n$ satisfy the requirements of Lemma 5.1.1 and therefore $m_1, \dots, m_n$ is an $A$-basis for $M$. If $r_P(M) \notin \mathbb{Z}$ then $r_P(M^t) = t\, r_P(M) \in \mathbb{Z}$ for a suitable $t \in \mathbb{N}$ and the claim follows by the first part of the proof. □

**Proposition 5.1.4.** *Let $A$ be a semilocal right $H$-comodule algebra which contains a minimal nonzero $H$-costable ideal that is finitely generated as a right ideal. Moreover assume that $A \otimes H \otimes H^{op}$ is weakly finite. Then the following properties of $A$ are equivalent:*

(i) *$A$ is $H$-simple,*

(ii) *none of the maximal ideals of $A$ contains a nonzero $H$-costable ideal,*

(iii) *there exists an ideal $P \in \mathrm{Max}A$ which contains no nonzero $H$-costable ideals.*

*Proof.* (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial.

(ii) $\Rightarrow$ (i) : Let $I$ be a proper $H$-costable ideal of $A$, then there exists an ideal $Q \in \mathrm{Max}A$ with $I \subset Q$ and therefore $I = 0$.

(iii) $\Rightarrow$ (ii) : Let $0 \neq I$ be a minimal $H$-costable ideal of $A$ which is finitely generated as a right $A$-module. Assume there exists a $Q \in \mathrm{Max}A$ which does contain a nonzero $H$-costable ideal $J$ of $A$. Then $IJ \neq 0$ since $P$ contains neither $I$ nor $J$ as they are nonzero and $H$-costable. Hence, $IJ = I$ by the minimality of $I$. It follows that $IQ = I$ that is $r_Q(I) = 0$

We have $I \in {}_A\mathcal{M}_A^H$ and by assumption $I$ is a finitely generated right $A$-module. Therefore, we can apply Lemma 5.1.3 and get that $I^t$ is a free right $A$-module for a suitable $t \in \mathbb{N}$.

Since $I$ is nonzero, the rank of this free $A$-modules is also nonzero. This is a contradiction to $IQ = I$, since $AQ = Q \subsetneq A$. Hence, none of the maximal ideal of $A$ contains a nonzero $H$-costable ideal of $A$. □

**Theorem 5.1.5.** *Let $H$ be a finite dimensional quasi-Hopf algebra, $B$ a right $H$-comodule algebra, and $K$ a right coideal subalgebra of $H$. Then*

(i) *$K$ is $H$-simple;*

(ii) *every finite dimensional object in* ${}_B\mathcal{M}_K^H$ *and every finite dimensional object in* ${}_K\mathcal{M}_B^H$, *is a free as a $K$-module. In particular, $H$ is free as a right and left $K$-module;*

(iii) *$K$ is a Frobenius algebra.*

*Proof.* (i) Let $K^+ := \ker \varepsilon|_K$, then clearly $K^+ \in \mathrm{Max}K$. Assume that $K^+$ contains a nonzero $H$-costable ideal $I$ of $K$, then $HIH \in {}_H\mathcal{M}_H^H$ where $\Delta$ is the quasi-coaction. Recall that $H$ is a simple object in ${}_H\mathcal{M}_H^H$. This is the case since ${}_H\mathcal{M} \cong {}_H\mathcal{M}_H^H$ $.M \mapsto .M \otimes .H^{\cdot}.$ is a category equivalence (see Theorem 4.1.2). Therefore $HIH = H$, since $I \neq 0$ by assumption. But $I \subset K^+$ and therefore $\varepsilon(HIH) = 0$, which leads to a contradiction. Thus, $K^+$ contains no nonzero $H$-costable ideals of $K$ and Proposition 5.1.4 implies that $K$ is $H$-simple.
(ii) Let $M \in {}_B\mathcal{M}_K^H$ be finitely generated as a right $K$-module. By (i) one can find a maximal ideal $P \in \mathrm{Max}K$ which satisfies the requirements of Lemma 5.1.3. Hence, $M^t \cong K^n$ for some $t, n \in \mathbb{N}$, and therefore $(M/MK^+)^t \cong (K/K^+)^n \cong k^n$. Thus, $n = t \cdot \dim(M/MK^+)$ and the Krull-Schmidt-Theorem implies $M \cong K^{\dim(M/MK^+)}$.
Now, let $M \in {}_K\mathcal{M}_B^H$ be finitely generated as a left $K$-module. Recall that $(H^{op}, \phi_H^{-1})$ is a quasi-Hopf algebra and $(B^{op}, \phi_\rho^{-1})$ and $(K^{op}, \phi_K^{-1})$ are right $H^{op}$-comodule algebras and therefore $K^{op}$ is a right coideal subalgebra of $H^{op}$. Thus, $(M, \delta_M) \in {}_K\mathcal{M}_B^H$ if and only if $(M, \delta_M) \in {}_{B^{op}}\mathcal{M}_{K^{op}}^{H^{op}}$ and therefore $M$ is free as a right $K^{op}$-module.

By choosing $B = H$, we get that $H$ itself is a free right and left $K$-module.

(iii) Finite dimensional quasi-Hopf algebras are Frobenius algebras by [HN99b, Theorem 4.3] which implies that $H \cong H^*$ as right $K$-modules. By (ii), $H$ is a free right and left $K$-module, that is $H \cong K^n$ as right and left $K$-modules for some $n \in \mathbb{N}$. Hence, $K^n \cong (K^n)^* \cong (K^*)^n$ as right $K$-modules. The Krull-Schmidt-Theorem implies that $K \cong K^*$ as right $K$-modules. □

*Remarks* 5.1.6. (1) By [Skr07, Proposition 2.2 (f)] $A \otimes H \otimes H^{op}$ is weakly finite whenever $H$ is finite dimensional and $A$ is weakly finite. In the case when $A$ is finite dimensional it would suffice that $H \otimes H^{op}$ is weakly finite, however it is not known which requirements are necessary for this to hold.
(2) Hence, $A$ and $H$ satisfy the requirements of Proposition 5.1.4 whenever $H$ is finite dimensional and $A$ is semilocal, Noetherian and satisfies descending chain condition on $H$-costable ideals. The requirements are also satisfied if $A$ is finite dimensional and $H \otimes H^{op}$ is weakly finite. Part (i) and (ii) of the theorem would therefore also hold under this condition (for $A = K$). But in the case when $H$ is not finite dimensional, we would no longer have that $H$ itself is a free $K$-module since it is not finitely generated. The generalization of the theorem

to non-finitely generated quasi-Hopf bimodules, which is presented in Section 5.3, only works for finite dimensional quasi-Hopf algebras.
(3) Note also that we need the fact that $H$ is a Frobenius algebra to show that $K$ is Frobenius. In the Hopf algebra case, Skryabin can show directly that coideal subalgebras are Frobenius algebras. For that, he uses the fact that if $M$ is a Hopf module in ${}_A\mathcal{M}^H$ then its dual $M^*$ is an object of $\mathcal{M}_A^H$ [Skr07, Lemma 4.1 and Theorem 4.2] and we do not have an analogous result for quasi-Hopf bimodules.
(4) The theorem induces Schauenburg's quasi-Hopf algebra Freeness Theorem [Sch04, Theorem 3.1 and Theorem 3.5], which is a generalization of the Nichols-Zöller-Theorem for quasi-Hopf algebras. He has shown that whenever $K$ is a quasi-Hopf subalgebra of a finite dimensional quasi-Hopf algebra $H$ then all quasi-Hopf bimodules $M \in {}_K\mathcal{M}_K^H$, which are finitely generated as a $K$-module, are free $K$-modules. Additionally, Schauenburg has shown that if $K \subset H$ is a subalgebra which admits a quasi-Hopf algebra structure, then $H$ is a free right $K$-module. The subalgebras of $H$ that admit quasi-Hopf algebra structures are also right coideal subalgebras.

## 5.2 Projectivity over $H$-Comodule Algebras

**Theorem 5.2.1.** *Let $H$ be a finite dimensional quasi-Hopf algebra and let $(A, \rho, \phi_\rho)$ and $(B, \rho', \phi_{\rho'})$ be right $H$-comodule algebras. If $A$ is semilocal and $H$-simple, then for every object $M \in {}_B\mathcal{M}_A^H$, which is finitely generated as a right $A$-module, there exists an integer $t \in \mathbb{Z}$ such that $M^{(t)}$ is a free $A$-module. In particular, $M$ is a projective $A$-module.*

*Proof.* This follows from 5.1.3, where $P \in \mathrm{Max}A$ is chosen to be with maximal rank, that is $r_P(M) \geq r_Q(M)$ for all $Q \in \mathrm{Max}A$. □

For Hopf algebras, Skryabin also shows that $H$-simple $H$-comodule algebras are Frobenius [Skr07, Theorem 3.5]. We have mentioned in the last section that in order to prove this, he uses the fact that if $M$ is a Hopf module in ${}_A\mathcal{M}^H$ then its dual $M^*$ is an object of $\mathcal{M}_A^H$. This does not work for quasi-Hopf algebras. However, with the help of results about module categories over quasi-Hopf algebras, we will be able to prove in the next chapter that $H$-simple $H$-comodule algebras are quasi-Frobenius.

**Corollary 5.2.2.** *If $H$ is a finite dimensional quasi-Hopf algebra, $A$ is a finite dimensional $H$-simple $H$-comodule algebra, and $M$ is a finite dimensional right $A$-module, then $M_. \otimes H_.$ with the diagonal $A$-module structure, is a projective $A$-module. In fact, $M_. \otimes {}_.H_.$ is an object in ${}_H\mathcal{M}_A^H$ by 3.3.2 and therefore the theorem applies.*

The opcop version of the corollary implies that the category of left modules over an $H$-simple left $H$-comodule algebra is an exact module category over $H$, which will be discussed in the next chapter. We will see that actually every exact module category is of this form.

As for Hopf algebras [Skr07, Proposition 5.4], one can deduce a dimension formula for $H$-simple $H$-comodule algebras:

**Proposition 5.2.3.** *Let $H$ be a finite dimensional quasi-Hopf algebra and let $A$ be a finite dimensional $H$-simple $H$-comodule algebra. Let $V$ be a simple right $A$-module and $W$ a finite dimensional right $A$-module. Set $D = \mathrm{End}_A(V)$. Then*

$$\dim_k D \dim_k A \mid \dim_k V \dim_k W \dim_k H.$$

*Proof.* Let $P \in \mathrm{Max}A$ such that $V$ is isomorphic to the unique simple $A/P$-module. Then every $A/P$-module is a direct sum of copies of $V$ and $A/P \cong V^{(d)}$ for $d = \dim_D V$. Let $M = W \otimes H$, then $M/MP \cong V^{(m)}$ for some integer $m$ and therefore $(M/MP)^{(d)} \cong V^{(dm)} \cong (A/P)^{(m)}$, thus it is a free $A/P$-module of rank $m$. Moreover $M \in {}_H\mathcal{M}_A^H$ and Lemma 5.2.1 together with Lemma 5.1.2 (2) implies that $M^d$ is a free $A$-module of rank $m$. Hence,

$$\begin{aligned} & \dim_k A\, m = d \dim_k M = \dim_D V \dim_k W \dim_k H \\ \Rightarrow \quad & \dim_k D \dim_k A\, m = \dim_k V \dim_k W \dim_k H. \end{aligned}$$

□

## 5.3 Non-Finitely Generated Quasi-Hopf Bimodules

In the Hopf algebra case, the freeness of infinitely generated Hopf modules can be deduced from the freeness of finitely generated Hopf modules [Skr07, Theorem 3.5]. This is done by means of the Finiteness Theorem for comodules [Mon93, 5.1.1], which uses the coassociativity and therefore can not be transfered to quasi-Hopf algebras. However, we can extend the result of Theorem 5.1.5 (ii) in the case $B = H$ to infinitely generated quasi-Hopf bimodules with the help of the category equivalence ${}_H\mathcal{M}_A^H \approx ({}_H\mathcal{M})_{A\overline{\#}H^*}$ from Proposition 4.7.3, proven by Bulacu and Caenepeel [BC03b, Theorem 3.5].

**Corollary 5.3.1.** *Let $H$ be a finite dimensional quasi-Hopf algebra and $K \subset H$ a right coideal subalgebra. Then every object in both ${}_H\mathcal{M}_K^H$ and ${}_K\mathcal{M}_H^H$ is a free $K$-module.*

*Proof.* Let $M \in {}_H\mathcal{M}_K^H$. By Theorem 5.1.5 we only have to consider the case when $M \in {}_H\mathcal{M}_K^H$ is not finitely generated as a $K$-module. Let $\mathcal{F}$ be the set of subobjects of $M$ in $({}_H\mathcal{M})_{K\overline{\#}H^*}$. Then $\mathcal{F}$ is clearly a set of $K$-submodules of $M$. We can show that $\mathcal{F}$ satisfies the conditions of Lemma 5.3.2 below and therefore $M$ is a free $K$-module.

In fact, let $N \in \mathcal{F}$ and $N \subsetneq M$ and let $m \in M \backslash N$, then

$$X := (H \succ m) \prec K\overline{\#}H^*$$

is a finite dimensional $({}_H\mathcal{M})_{K\overline{\#}H^*}$-subobject of $M$. Hence, $N' := N + X$ is an $({}_H\mathcal{M})_{K\overline{\#}H^*}$-subobject of $M$ which properly contains $N$ and $N'/N \cong X \in ({}_H\mathcal{M})_{K\overline{\#}H^*}$. By Proposition 4.7.3, $N/N'$ is an object of ${}_H\mathcal{M}_K^H$ which is finitely generated as a $K$-module. Hence, $N'/N$ is a free $K$-module by Theorem 5.1.5.

If $M \in {}_K\mathcal{M}_H^H$, then $M \in {}_{H^{op}}\mathcal{M}_{K^{op}}^{H^{op}}$ as above, and $M$ is a free left $K$-module by the first part of the proof. $\square$

**Lemma 5.3.2.** [Skr07, Lemma 2.5] *Let $R$ be a semilocal ring and $M$ a right $R$-module which is not finitely generated. Assume there exists a set $\mathcal{F}$ of $R$-submodules of $M$ such that*

- *$0 \in \mathcal{F}$ and the union of every chain of elements of $\mathcal{F}$ is in $\mathcal{F}$,*
- *for each $N \in \mathcal{F}$ with $N \subsetneq M$ there exists an $N' \in \mathcal{F}$ with $N \subsetneq N'$ and $N'/N$ is a free $R$-module.*

*Then $M$ is a free $R$-module.*

# Chapter 6

# Module Categories over Quasi-Hopf Algebras

We have seen that for a finite dimensional quasi-Hopf algebra $H$ the category of finite dimensional left $H$-modules is a finite tensor category and that any finite tensor category with integer Frobenius-Perron dimensions is of this form. In this chapter we will investigate the exact module categories over those tensor categories.

We will show that a module category over a quasi-Hopf algebra $H$ is exact and indecomposable if and only if it is the representation category of a finite dimensional $H$-simple left $H$-comodule algebra. This classification of module categories over quasi-Hopf algebras also provides us with results on smash products and $H$-comodule algebras. It follows that $H$-simple $H$-comodule algebras are quasi-Frobenius, and if $H$ is semisimple, then they are also semisimple. In particular, if $R$ is an $H$-simple $H$-module algebra and $H$ is semisimple, then $R\#H$ is a semisimple algebra.

In this chapter let $H$ be a finite dimensional quasi-Hopf algebra, and $\mathcal{C} := {}_H\mathcal{M}^{fd}$ the finite tensor category of finite dimensional left $H$-modules.

## 6.1 Exact Module Categories over $H$

Etingof and Ostrik's classification of exact module categories (Theorem 1.3.3 and Corollary 1.3.5) implies the following for module categories over $\mathcal{C}$.

**Lemma 6.1.1.** *If $\mathcal{M}$ is an indecomposable exact module category over $\mathcal{C}$, then there exists a finite dimensional right $H$-simple $H$-module algebra $R$, such that $\mathcal{M}$ is equivalent to $({}_H\mathcal{M}^{fd})_R$ as module categories over $\mathcal{C}$.*

*Proof.* An $H$-module algebra is exactly an algebra in the category $\mathcal{C}$. Hence, there exists an $H$-module algebra $R$ which is simple in $\mathcal{C}_R = ({}_H\mathcal{M}^{fd})_R$, such that $\mathcal{M} \approx ({}_H\mathcal{M}^{fd})_R$ as $H$-module categories. Moreover, $R$ is finite dimensional, as it is an object in $\mathcal{C} = {}_H\mathcal{M}^{fd}$. □

*Remark* 6.1.2. For $X \in \mathcal{C}$, $M \in \mathcal{C}_R = ({}_H\mathcal{M}^{fd})_R$, the $R$-module structure of $X \otimes M$ is given by $(x \otimes m) \cdot r = \phi^{(1)}x \otimes (\phi^{(2)}m)(\phi^{(3)}r)$, since the associativity in $\mathcal{C}$ is induced by multiplication with $\phi$ as described in Chapter 3.

## 6.2 Indecomposable Exact Module Categories and *H*-Simple *H*-Comodule Algebras

**Lemma 6.2.1.** *For a finite dimensional left $H$-comodule algebra $(A, \lambda, \phi_\lambda)$, ${}_A\mathcal{M}^{fd}$ is a module category over $\mathcal{C}$ with the diagonal structure via $\lambda$, that is for $X \in \mathcal{C}$ and $M \in {}_A\mathcal{M}^{fd}$, $.X \otimes .M$ is a left $A$-module with $a(x \otimes m) = a_{(-1)}x \otimes a_{(0)}m$. The associativity $m$ is induced by left multiplication with $\phi_\lambda$ and the unit isomorphism $\ell$ is $\varepsilon \otimes \mathrm{id}$.*

*Proof.* The pentagon equation for module categories (1.3) follows from the left-version of the pentagon equation (3.10) for the coassociator $\phi_\lambda$. The triangle equation (1.4) follows from the unit property (3.12) for $\phi_\lambda$. The $H$-linearity of $m$ follows from the quasi-coassociativity of $H$-comodule algebras (3.9). □

**Proposition 6.2.2.** *Let $(A, \lambda, \phi_\lambda)$ be a finite dimensional $H$-simple left $H$-comodule algebra. Then ${}_A\mathcal{M}^{fd}$ is an indecomposable exact module category over $\mathcal{C}$.*

*Proof.* ${}_A\mathcal{M}^{fd}$ is an indecomposable module category by the lemma below, since $A$ is $H$-simple. The functor $- \otimes M$ is exact for any finite dimensional left $A$-module $M$. Hence, in order to prove that ${}_A\mathcal{M}^{fd}$ is exact, it suffices to show that $.H \otimes .M$ is a projective left $A$-module for every $M \in {}_A\mathcal{M}^{fd}$. This is the opcop-version of Corollary 5.2.2. □

**Lemma 6.2.3.** *Let $(A, \lambda, \phi_\lambda)$ be a finite dimensional left $H$-comodule algebra, then ${}_A\mathcal{M}^{fd}$ is an indecomposable module category if and only if $A$ is an indecomposable $H$-comodule algebra, that is if $I$ and $J$ are $H$-costable ideals of $A$ with $A = J \oplus I$, then $I = 0$ or $J = 0$.*

*Proof.* Analogous to [AM07, Proposition 1.18] □

We see that $H$-simple $H$-comodule algebras produce examples of indecomposable exact module categories and these are in fact all indecomposable exact module categories over $\mathcal{C} = {}_H\mathcal{M}^{fd}$.

**Theorem 6.2.4.** *Let $\mathcal{M}$ be a module category over $\mathcal{C}$, then $\mathcal{M}$ is an indecomposable exact module category if and only if there exists a finite dimensional $H$-simple $H$-comodule algebra $A$ such that $\mathcal{M} \approx {}_A\mathcal{M}^{fd}$ as module categories.*

*Proof.* By Lemma 6.1.1 and Proposition 6.2.2, it remains to prove that for an $H$-module algebra $R$, which is simple in $({}_H\mathcal{M}^{fd})_R$, there exists an $H$-simple left $H$-comodule algebra $A$ with ${}_A\mathcal{M}^{fd} \approx ({}_H\mathcal{M}^{fd})_R$ as module categories over $\mathcal{C}$. We can show that the equivalence holds for $A := R^{op}\#H^{cop}$, which works as in the Hopf algebra case [AM07, Proposition 1.19].

In fact, $R^{op}$ is a left $H^{cop}$-module algebra, which is simple in $({}_H\mathcal{M}^{fd})_R \approx {}_{R^{op}}({}_{H^{cop}}\mathcal{M}^{fd})$ and $\mathcal{F} : ({}_H\mathcal{M}^{fd})_R \approx {}_{R^{op}}({}_{H^{cop}}\mathcal{M}^{fd}) \approx {}_{R^{op}\#H^{cop}}\mathcal{M}^{fd}$ as abelian categories by [BPvO00, Proposition 2.16]. The functor $\mathcal{F}$ is the identity on objects and morphisms, and if $M \in ({}_H\mathcal{M}^{fd})_R$ then $M \in {}_{R^{op}\#H^{cop}}\mathcal{M}^{fd}$ with $(r\#h) \blacktriangleright m := (hm)r$. As $R^{op}\#H^{cop}$ is a right $H^{cop}$-comodule algebra, it is a left $H$-comodule algebra with opposite costructure given by $\lambda(r\#h) = \phi^{(1)}h_{(1)} \otimes \phi^{(3)}r\#\phi^{(2)}h_{(2)}$.

Then, together with the natural isomorphism $c$ which is the identity, $\mathcal{F}$ is furthermore an equivalence of module categories over $\mathcal{C}$. We have to show that for all $X \in \mathcal{C}$, $M \in ({}_H\mathcal{M}^{fd})_R$, $c_{X,M} : \mathcal{F}(X \otimes M) \to X \otimes \mathcal{F}(M)$ is a morphism of $A$-modules. For $x \in X, m \in M, h \in H, r \in R$:

$$
\begin{aligned}
&c_{X,M}((r\#h) \blacktriangleright (x \otimes m)) \\
&\qquad = \mathrm{id}_{X,M}((h(x \otimes m)) \cdot r) \\
&\qquad = (h_{(1)}x \otimes h_{(2)}m) \cdot r \\
&\qquad = \phi^{(1)}h_{(1)}x \otimes (\phi^{(2)}h_{(2)}m)(\phi^{(3)}r) && R\text{-module structure of } X \otimes M \text{ in } \mathcal{C} \\
&\qquad = \phi^{(1)}h_{(1)}x \otimes (\phi^{(3)}r\#\phi^{(2)}h_{(2)}) \blacktriangleright m \\
&\qquad = (r\#h)(x \otimes m) && A\text{-module structure of } X \otimes M \\
&\qquad = (r\#h)c_{X,M}(x \otimes m).
\end{aligned}
$$

Since $R^{op}$ is a right $H$-simple left $H^{cop}$-module algebra and therefore $H$-simple, Proposition 4.6.1 implies that $A = R^{op}\#H^{cop}$ is an $H^{cop}$-simple right $H^{cop}$-comodule algebra, hence an $H$-simple left $H$-comodule algebra. □

The classification in [AM07] for ordinary Hopf algebras is stronger: Andruskiewitsch and Mombelli classify indecomposable exact module categories by $H$-comodule algebras which are simple in ${}^H\mathcal{M}_A$ and have trivial coinvariants. A transformation of this stronger classification to quasi-Hopf algebras is not meaningful, as the coinvariants of an $H$-comodule algebra are in general not defined, nor is the category ${}^H\mathcal{M}_A$.

## 6.3 Applications to $H$-Comodule Algebras and Smash Products

In the last chapter we discussed that Skryabin's proof for the fact that $H$-simple $H$-comodule algebras are quasi-Frobenius can not be transfered to the quasi-

Hopf algebra case (for the definition of quasi-Frobenius rings see Appendix A.3). However, we obtain this fact by means of the above result about module categories over quasi-Hopf algebras:

**Proposition 6.3.1.** *Let $H$ be a finite dimensional quasi-Hopf algebra. If $A$ is a finite dimensional $H$-simple $H$-comodule algebra then $A$ is quasi-Frobenius. If $H$ is moreover a semisimple quasi-Hopf algebra then $A$ is also semisimple.*

*Proof.* By the proposition above, ${}_A\mathcal{M}^{fd}$ is an exact module category, and therefore a Frobenius category by 1.3.2, and in particular $A$ is quasi-Frobenius.

If $H$ is semisimple then $\mathcal{C}$ is a semisimple tensor category. Hence, ${}_A\mathcal{M}^{fd}$ is semisimple by 1.3.1 (2), as it is exact. In particular, $A$ is a semisimple object in ${}_A\mathcal{M}^{fd}$ and therefore a semisimple algebra. $\square$

**Corollary 6.3.2.** *If $H$ is a semisimple quasi-Hopf algebra and $R$ a finite dimensional $H$-simple left $H$-module algebra, then $R\#H$ is semisimple.*

*Proof.* $R\#H$ is an $H$-simple right $H$-module algebra and therefore an $H^{cop}$-simple left $H^{cop}$-module algebra. The proposition implies that $R\#H$ is semisimple. $\square$

For a semisimple Hopf algebra $H$, the semisimplicity of the smash product of an $H$-simple $H$-module algebra by $H$ can be deduced directly from Theorem 5.2.1 by means of the existence of a normalized integral. In fact, if $V$ is a left $R\#H$-module, then $M \in \mathcal{M}_R^{H^*}$ and therefore it is a projective $A$-module and thus a projective $R\#H$-module [CF86]. These arguments do not hold for quasi-Hopf algebras.

# Part III

# Weak Hopf Algebras

# Chapter 7

# Weak Hopf Algebras and their Representations

Weak Hopf algebras as introduced by Böhm, Nill, and Szlachányi [BNS99], are generalized Hopf algebras, where the comultiplication is no longer unit preserving and the counit is not multiplicative. However, comultiplication and counit satisfy certain conditions, such that the category of representations of a weak Hopf algebra is still monoidal. Moreover, weak Hopf algebras have an antipode endowing this category with duality.

The category of representations of a (finite dimensional) weak Hopf algebra is a finite multi-tensor category. On the other hand, every finite tensor category is equivalent to the representation category of a weak quasi-Hopf algebra [EO04, Proposition 2.7]; and if the category is semisimple, then it is equivalent to the representation category of a semisimple weak Hopf algebra with commutative base [Ost03b, Theorem 4.1].

## 7.1 Weak Hopf Algebras

**Definition.** A **weak bialgebra** is a *finite dimensional non-zero* unital algebra $H$, which is a coalgebra with comultiplication $\Delta$ and counit $\varepsilon$, satisfying the following properties for $f, g, h \in H$:

$$\Delta(gh) = \Delta(g)\Delta(h), \tag{7.1}$$

$$1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = 1_{(1)} \otimes 1_{(2)}1'_{(1)} \otimes 1'_{(2)} = 1_{(1)} \otimes 1'_{(1)}1_{(2)} \otimes 1'_{(2)}, \tag{7.2}$$

and

$$\varepsilon(fgh) = \varepsilon(fg_{(1)})\varepsilon(g_{(2)}h) = \varepsilon(fg_{(2)})\varepsilon(g_{(1)}h). \tag{7.3}$$

$H$ is called a **weak Hopf algebra** if there is an **antipode** $S$ which is a $k$-linear map and satisfies

$$h_{(1)}S(h_{(2)}) = \varepsilon(1_{(1)}h)1_{(2)}, \qquad S(h_{(1)})h_{(2)} = 1_{(1)}\varepsilon(h1_{(2)}), \tag{7.4}$$
$$S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h). \tag{7.5}$$

Here and in the following we use a simplified Sweedler notation $\Delta(h) =: h_{(1)} \otimes h_{(2)}$ again, omitting the summation symbols. To differentiate the different summation indices we write 1 and $1'$ etc. for different copies of 1.

*Remark* 7.1.1. By some authors (e.g. [Nik02]) weak Hopf algebras are not assumed to be finite dimensional. However, [BNS99] have pointed out, that it makes sense to restrict the definition to the finite dimensional case, because then the definition is self-dual, that is $H^* := \mathrm{Hom}_k(H, k)$ is again a weak Hopf algebra with the usual dual structure, that is

$$(\varphi * \psi)(h) = \varphi(h_{(1)})\psi(h_{(2)}),$$
$$\varphi_{(1)}(h)\varphi_{(2)}(g) = \varphi(gh),$$
$$S(\varphi)(h) = \varphi(S(h)),$$

for $\varphi, \psi \in H^*$, $g, h \in H$. Moreover $(H^*)^* \cong H$. In particular, properties (7.2) and (7.3) are dual to each other.

Here, we will also always work with finite dimensional weak Hopf algebras.

**Definition.** Define $\varepsilon_s$ and $\varepsilon_t$, the **source** and **target map** of a weak Hopf algebra $H$, as follows:

$$\varepsilon_t(h) := h_{(1)}S(h_{(2)}) = \varepsilon(1_{(1)}h)1_{(2)}, \tag{7.6}$$
$$\varepsilon_s(h) := S(h_{(1)})h_{(2)} = 1_{(1)}\varepsilon(h1_{(2)}), \tag{7.7}$$

for all $h \in H$. Their images $H_t := \varepsilon_t(H)$ and $H_s := \varepsilon_s(H)$ are called the bases or **base algebras** of $H$.

**Lemma 7.1.2.** [NV02, 2.2.2] *Let $H$ be a weak Hopf algebra. Then*

$$\varepsilon_t \circ \varepsilon_t = \varepsilon_t, \qquad \varepsilon_s \circ \varepsilon_s = \varepsilon_s, \tag{7.8}$$

*and $H_t$ and $H_s$ are unital subalgebras of $H$ with*

$$\Delta(H_t) \in H \otimes H_t, \qquad \Delta(H_s) \in H_s \otimes H, \tag{7.9}$$
$$\Delta(1) \in H_s \otimes H_t. \tag{7.10}$$

*Moreover, $H_t$ and $H_s$ commute with each other.*

**Lemma 7.1.3.** [BNS99, 2.9 and 2.10] *The antipode of a weak Hopf algebras is, as in the Hopf algebra case, a unit preserving and counit preserving anti-algebra and anti-coalgebra morphism; and it is bijective. Furthermore it satisfies:*

$$\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s \quad \text{and} \quad \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t \tag{7.11}$$

analogously $$\varepsilon_t \circ S^{-1} = S^{-1} \circ \varepsilon_s \quad \text{and} \quad \varepsilon_s \circ S^{-1} = S^{-1} \circ \varepsilon_t.$$

*The restrictions of $S$ to $H_t$ and $H_s$ induce bijections:*

$$S|_{H_t} : H_t \to H_s \quad \text{and} \quad S|_{H_s} : H_s \to H_t.$$

**Lemma 7.1.4.** [BNS99, Section 3.1] *Let $H$ be a weak Hopf algebra.*
*$H^{op}$ is a weak Hopf algebra with antipode $S^{-1}$, target map $S^{-1} \circ \varepsilon_s$ and source map $S^{-1} \circ \varepsilon_t$, $(H^{op})_t = H_t$, $(H^{op})_s = H_s$;*
*$H^{cop}$ is a weak Hopf algebra with antipode $S^{-1}$, target map $S^{-1} \circ \varepsilon_t$ and source map $S^{-1} \circ \varepsilon_s$, $(H^{cop})_t = H_s$, $(H^{cop})_s = H_t$;*
*$H^{opcop}$ is a weak Hopf algebra with antipode $S$, target map $\varepsilon_s$ and source map $\varepsilon_t$, $(H^{opcop})_t = H_s$, $(H^{opcop})_s = H_t$.*

In the following lemma, properties of weak Hopf algebras are gathered, that will be used frequently in the following chapters. Their proofs are straightforward or can be found in [BNS99, (2.2a,b), (2.5a,b), (2.13a,b), (2.23a,b), (2.24a,b), (2.30a-d), and Proposition 2.11] and [NV02, Proposition 2.2.1].

**Lemma 7.1.5.** *Let $(H, \Delta, \varepsilon, S)$ be a weak Hopf algebra, $g, h \in H$, $x \in H_s$, $y \in H_t$.*

$$h \in H_t \Leftrightarrow \Delta(h) = 1_{(1)}h \otimes 1_{(2)}, \qquad h \in H_s \Leftrightarrow \Delta(h) = 1_{(1)} \otimes h1_{(2)}, \tag{7.12}$$

$$\varepsilon_t(h) = \varepsilon(S(h)1_{(1)})1_{(2)}, \qquad \varepsilon_s(h) = 1_{(1)}\varepsilon(1_{(2)}S(h)), \tag{7.13}$$

$$\varepsilon_t(h) = S(1_{(1)})\varepsilon(1_{(2)}h), \qquad \varepsilon_s(h) = \varepsilon(h1_{(1)})S(1_{(2)}), \tag{7.14}$$

$$\varepsilon(g\,\varepsilon_t(h)) = \varepsilon(gh), \qquad \varepsilon(\varepsilon_s(g)h) = \varepsilon(gh), \tag{7.15}$$

$$\varepsilon_t(g\,\varepsilon_t(h)) = \varepsilon_t(gh), \qquad \varepsilon_s(\varepsilon_s(g)h) = \varepsilon_s(gh), \tag{7.16}$$

$$\varepsilon_t(\varepsilon_t(g)h) = \varepsilon_t(g)\varepsilon_t(h), \qquad \varepsilon_s(g\varepsilon_s(h)) = \varepsilon_s(g)\varepsilon_s(h), \tag{7.17}$$

$$g\varepsilon_t(h) = \varepsilon(g_{(1)}h)g_{(2)}, \qquad \varepsilon_s(g)h = h_{(1)}\varepsilon(gh_{(2)}), \tag{7.18}$$

$$\varepsilon_t(g)h = \varepsilon(\varepsilon_t(g)h_{(1)})h_{(2)}, \qquad g\varepsilon_s(h) = g_{(1)}\varepsilon(g_{(2)}\varepsilon_s(h)), \tag{7.19}$$

$$h_{(1)} \otimes \varepsilon_t(h_{(2)}) = 1_{(1)}h \otimes 1_{(2)}, \qquad \varepsilon_s(h_{(1)}) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)}, \tag{7.20}$$

$$\varepsilon_t(h_{(1)}) \otimes h_{(2)} = S(1_{(1)}) \otimes 1_{(2)}h, \qquad h_{(1)} \otimes \varepsilon_s(h_{(2)}) = h1_{(1)} \otimes S(1_{(2)}), \tag{7.21}$$

$$h_{(1)}x \otimes h_{(2)} = h_{(1)} \otimes h_{(2)}S(x), \qquad h_{(1)} \otimes yh_{(2)} = S(y)h_{(1)} \otimes h_{(2)}, \tag{7.22}$$

$$1_{(1)} \otimes \varepsilon_t(h1_{(2)}) = \varepsilon_s(1_{(1)}h) \otimes 1_{(2)}, \tag{7.23}$$

$$1_{(1)} \otimes 1'_{(1)} \otimes 1_{(2)}1'_{(2)} = 1_{(1)} \otimes \varepsilon_s(1_{(2)}) \otimes 1_{(3)}, \tag{7.24}$$

$$1_{(1)}1'_{(1)} \otimes 1_{(2)} \otimes 1'_{(2)} = 1_{(1)} \otimes \varepsilon_t(1_{(2)}) \otimes 1_{(3)}. \tag{7.25}$$

**Corollary 7.1.6.** *$H_s$ and $H_t$ are separable algebras, and in particular semisimple, with separability elements $1_{(1)} \otimes S(1_{(2)})$ and $S(1_{(1)}) \otimes 1_{(2)}$, respectively.*

*Proof.* Equations (7.22) imply in particular that for all $x \in H_s$ and for all $y \in H_t$

$$(x \otimes 1)(1_{(1)} \otimes S(1_{(2)})) = (1_{(1)} \otimes S(1_{(2)}))(1 \otimes x),$$
$$(y \otimes 1)(S(1_{(1)}) \otimes 1_{(2)}) = (S(1_{(1)}) \otimes 1_{(2)})(1 \otimes y),$$

and $S(1_{(1)})1_{(2)} = 1_{(1)}S(1_{(2)}) = 1$. □

*Example* 7.1.7. The easiest example of a weak Hopf algebra is the groupoid algebra: Let $\mathcal{G}$ be a groupoid, that is a small category, in which every morphism is invertible, and let $\mathcal{X}$ be the set of objects in $\mathcal{G}$. The groupoid algebra $k\mathcal{G}$ is generated by morphisms $g \in \mathcal{G}$, where the product of two morphisms is equal to their composition whenever it is defined, and it is zero otherwise. Comultiplication, counit and antipode are defined as for group algebra as follows: $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$. Then

$$\varepsilon_t(g) = gg^{-1} = \mathrm{id}_{\mathrm{target}(g)} \qquad \text{and} \qquad \varepsilon_s(g) = g^{-1}g = \mathrm{id}_{\mathrm{source}(g)},$$

which explains the name target and source map. Hence,

$$(k\mathcal{G})_t = (k\mathcal{G})_s = \mathrm{span}\{\mathrm{id}_x \,|\, x \in \mathcal{X}\}.$$

## 7.2 Representations of Weak Hopf Algebras and their Reconstruction

**Proposition 7.2.1.** [NTV03, EGNO10] *Let $H$ be a weak Hopf algebra. The category $\mathcal{C}$ of finite dimensional left $H$-modules is a finite multi-tensor category. The tensor product is given by*

$$V \otimes W := 1_{(1)}V \otimes 1_{(2)}W$$

*for $V, W \in \mathcal{C}$. The associativity is the identity. The unit object is $H_t$, with an $H$-action given by*

$$h \rightharpoonup x := \varepsilon_t(hx) \overset{(7.12)}{=} h_{(1)}xS(h_{(2)}), \tag{7.26}$$

*together with natural isomorphisms given for $V \in \mathcal{C}$ by*

$$l_V : \quad H_t \otimes V \to V \qquad\qquad l_V^{-1} : V \to H_t \otimes V$$
$$1_{(1)} \cdot x \otimes 1_{(2)} v \mapsto xv \qquad\qquad v \mapsto S(1_{(1)}) \otimes 1_{(2)} v$$

and

$$r_V : \quad V \otimes H_t \to V \qquad\qquad r_V^{-1} : V \to V \otimes H_t$$
$$1_{(1)} v \otimes 1_{(2)} x \mapsto S^{-1}(x) v \qquad\qquad v \mapsto 1_{(1)} v \otimes 1_{(2)}.$$

*The right dual object of an object $V \in \mathcal{C}$ is the dual space $V^* = \mathrm{Hom}_k(V, k)$ with an $H$-action given by $(h \cdot \phi)(g) = \phi(S(h)v)$. Evaluation and coevaluation are given by*

$$ev_V : \quad V^* \otimes V \to H_t \qquad\qquad db_V : H_t \to V \otimes V^*$$
$$\sum_i \phi_i \otimes v_i \mapsto \sum_i \phi_i(1_{(1)} v_i) 1_{(2)} \qquad\qquad x \mapsto \sum_j 1_{(1)} x v_j \otimes 1_{(2)} \cdot \phi_j,$$

*where $\{v_j, \phi^j\}_j$ are dual bases of $V$ and $V^*$.*

*Remarks* 7.2.2. (1) It was shown in [NTV03] that $\mathcal{C}$ is a monoidal category with duality.
(2) One can easily check, that the tensor product defined above is exact in both variables. However, this also follows directly from (1), see for example [EGNO10, 1.13].
(3) It can again be deduced from (1) and [EGNO10, 1.15] that the unit object is semisimple and more precisely $H_t \cong \bigoplus_i U_i$ as $H$-modules, where the $U_i's$ are pairwise non-isomorphic simple left $H$-modules. Vecsernyés [Vec03, Theorem 2.4] has shown directly that $H_t$ and $H_s$ are semisimple left respectively right $H$-modules.
(4) Böhm, Nill, and Szlachányi [BNS99] have proven that weak Hopf algebras are quasi-Frobenius by means of their structure theorem for weak Hopf modules. However, this can also be deduced directly from the fact that $\mathcal{C}$ is a finite tensor category [EO04, Proposition 2.3].

If $H$ is a weak Hopf algebra and $M \in {}_H\mathcal{M}$, then $M$ is a left $H_t$- and $H_s$-module. Since $H_t$ and $H_s$ commute and the restriction of $S$ gives an isomorphism ${H_t}^{op} \cong H_s$, $M$ is an $H_t$-bimodule. This induces a tensor functor

$$\mathrm{base}_H : {}_H\mathcal{M}^{fd} \to {}_{H_t}\mathcal{M}_{H_t}.$$

On the other hand, if $\mathcal{C}$ is a multi-tensor category and $F : \mathcal{C} \to {}_A\mathcal{M}_A^{fd}$ a tensor functor into the category of finite dimensional bimodules over a separable algebra $A$, then one can reconstruct a weak Hopf algebra with base $A$, and $\mathcal{C}$ is equivalent as a tensor category to the category of representations of this weak Hopf algebra.

As $\overline{F} = \operatorname{forget} F : \mathcal{C} \to \mathrm{vect}_k$ is an exact and faithful functor, there is an equivalence of categories $\mathcal{C} \approx {}_H\mathcal{M}$ for $H = \operatorname{End}(\overline{F})$ the algebra of natural transformations $\overline{F} \to \overline{F}$ [JS91]. Szlachányi has shown in [Szl00] that $H$ has a bialgebroid structure over $A$ and $\mathcal{C} \approx {}_H\mathcal{M}$ as monoidal categories and that therefore $H$ is a weak Hopf algebra with base $A$ in the following way:

**Theorem 7.2.3.** [Szl00, JS91] *Let $\mathcal{C}$ be a finite tensor category and $A$ a finite dimensional separable algebra with separability element $\sum_i x_i \otimes y_i \in A \otimes A^{op}$. Let $(F, \xi, \xi_0) : \mathcal{C} \to {}_A\mathcal{M}_A^{fd}$ be a tensor functor. Denote by $\overline{F}$ the functor* $\operatorname{forget} F : \mathcal{C} \to \mathrm{vect}_k$*, then $H := \operatorname{End}_k(F) = \operatorname{Nat}(\overline{F}, \overline{F})$, the algebra of natural transformations $\overline{F} \to \overline{F}$, is a weak Hopf algebra with base $A$. The comultiplication is given by*

$$\begin{aligned} \Delta : H \overset{\overline{\Delta}}{\to} H \otimes_A H \overset{\Xi}{\to} H \otimes_k H \quad \textit{with} \\ \overline{\Delta}(\eta) = \xi \circ \eta \circ \xi^{-1} \\ \Xi(\eta \otimes \eta') = \sum_i \eta x_i \otimes y_i \eta' := \sum_i s(x_i) \circ \eta \otimes t(y_i) \circ \eta', \end{aligned}$$

*where $s : A^{op} \to H$ and $t : A \to H$ are the algebra morphisms induced by the $A$-bimodule structures of each $F(X)$, $X \in \mathcal{C}$. The counit is given by*

$$\varepsilon : H \to k, \eta \mapsto tr_A(\xi_0 \circ \eta_{\mathbb{1}} \circ \xi_0^{-1})$$

*and the antipode is given by*

$$S : H \to H, \eta \mapsto (\eta_{X^*})^*.$$

*Moreover, there is an equivalence of tensor categories*

$$I_F : \mathcal{C} \to {}_H\mathcal{M}^{fd}, X \mapsto (F(X), \mu_X),$$

*with $\mu_X : H \otimes F(X) \to F(X)$, $\eta \otimes x \mapsto \eta_X(x)$ , such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "I_F"] \arrow[dr, "F"'] & & {}_H\mathcal{M}^{fd} \arrow[dl, "\mathrm{base}_H"] \\ & {}_A\mathcal{M}_A^{fd} & \end{tikzcd} .$$

# Chapter 8

# Weak Hopf Modules

In the following, let $H$ be a (finite dimensional) weak Hopf algebra.

In this chapter, the notions of $H$-comodule algebras and weak Hopf modules over them are introduced, which were defined by Böhm [Böh00]. The dual concepts for $H$-comodule algebra and weak Hopf modules are $H$-module algebras [Nik00] and the category $({}_H\mathcal{M})_R$ of modules over them in the category of $H$-modules. A structure theorem for weak Doi-Hopf modules by Zhang and Zhu [ZZ04] will be presented, which is a generalization of the structure theorem for weak Hopf modules in [BNS99]. Properties of the weak smash product $R\#H$ [Nik00] will be reviewed, from which a bijective correspondence of the $H$-stable ideals of the $H$-module algebra $R$ and the $H$-costable ideals of the $H$-comodule algebra $R\#H$ can be deduced.

The structure theorem for weak Hopf modules implies that any object in $\mathcal{M}_H^H$ is a projective $H$-module. I expect that this is also true for $H$-simple $H$-comodule algebras. For the case when $H$ is free over its base algebra, this will be proven in the next chapter .

Moreover, it will be shown that the trace ideals of weak Hopf modules are $H$-costable ideals of the $H$-comodule algebra $A$ and in the case when $A$ is $H$-simple and quasi-Frobenius one can deduce that nonzero weak Hopf modules are generators for $\mathcal{M}_A$.

Finally, a Morita equivalence of weak Hopf module categories will be formulated. It will be shown that for a weak Hopf module $M \in \mathcal{M}_A^H$, the endomorphism ring $B := \mathrm{End}_A(M)$ is an $H$-comodule algebra and the coinvariants of $B$ coincide with the $H$-linear morphisms, and in particular $B^{coH}$ is trivial, whenever $M$ is simple in $\mathcal{M}_A^H$ . Moreover, if $M \in \mathcal{M}_A^H$ is a progenerator for $\mathcal{M}_A$ then a category equivalence $\mathcal{M}_A \approx \mathcal{M}_B$ induces a category equivalence $\mathcal{M}_A^H \approx \mathcal{M}_B^H$.

## 8.1 *H*-Comodule Algebras and Weak Relative Hopf Modules

**Definition.** [Böh00] An algebra $A$, which is also a right $H$-comodule with costructure $\rho$, is called a **right *H*-comodule algebra** if

$$\rho(ab) = \rho(a)\rho(b), \tag{8.1}$$
$$\rho(1^A) \in A \otimes H_t. \tag{8.2}$$

**Lemma 8.1.1.** *Let $A$ be an $H$-comodule algebra and $a \in A$. Then*

$$1^A_{(0)} \otimes 1_{(1)}1^A_{(1)} \otimes 1_{(2)} = 1^A_{(0)} \otimes 1^A_{(1)}1_{(1)} \otimes 1_{(2)} = 1^A_{(0)} \otimes 1^A_{(1)} \otimes 1^A_{(2)}; \tag{8.3}$$
$$a_{(0)} \otimes \varepsilon_t(a_{(1)}) = 1^A_{(0)}a \otimes 1^A_{(1)}; \tag{8.4}$$
$$1^A_{(0)}1'^A_{(0)} \otimes 1^A_{(1)} \otimes 1'^A_{(1)} = 1^A_{(0)} \otimes \varepsilon_t(1^A_{(1)}) \otimes 1^A_{(2)}. \tag{8.5}$$

*Proof.* (8.3): The first equation holds by (8.2) and since $\Delta(1) \in H_s \otimes H_t$ and $H_t$ and $H_s$ commute. Hence

$$1^A_{(0)} \otimes 1^A_{(1)} \otimes 1^A_{(2)} = 1^A_{(0)} \otimes \Delta(1^A_{(1)}) = 1^A_{(0)} \otimes 1_{(1)}1^A_{(1)} \otimes 1_{(2)},$$

by (8.2) and (7.12).

(8.4): $$a_{(0)} \otimes \varepsilon_t(a_{(1)}) = a_{(0)} \otimes \varepsilon(1_{(1)}a_{(1)})1_{(2)} = 1^A_{(0)}a_{(0)} \otimes \varepsilon(1_{(1)}1^A_{(1)}a_{(1)})1_{(2)}$$
$$= 1^A_{(0)}a_{(0)}\varepsilon(1^A_{(1)}a_{(1)}) \otimes 1^A_{(2)} = 1^A_{(0)}a \otimes 1^A_{(1)},$$
by (8.1) and (8.3).

(8.5): follows from (8.4). □

*Remark* 8.1.2. In the same way one may define a **left *H*-comodule algebra** $A$ to be an algebra which is a left $H$-comodule with costructure $\lambda$ and satisfies:

$$\lambda(ab) = \lambda(a)\lambda(b), \tag{8.6}$$
$$\lambda(1^A) \in H_s \otimes A. \tag{8.7}$$

Then the following equations are satisfied:

$$1_{(1)} \otimes 1_{(2)}1^A_{(-1)} \otimes 1^A_{(0)} = 1_{(1)} \otimes 1^A_{(-1)}1_{(2)} \otimes 1^A_{(0)} = 1^A_{(-2)} \otimes 1^A_{(-1)} \otimes 1^A_{(0)}, \tag{8.8}$$
$$\varepsilon_s(a_{(-1)}) \otimes a_{(0)} = 1^A_{(-1)} \otimes a1^A_{(0)}, \tag{8.9}$$
$$1'^A_{(-1)} \otimes 1^A_{(-1)} \otimes 1^A_{(0)}1'^A_{(0)} = 1^A_{(-2)} \otimes \varepsilon_s(1^A_{(-1)}) \otimes 1^A_{(0)}, \tag{8.10}$$

for $a \in A$.

Let now $(A, \rho)$ be a right $H$-comodule algebra.

**Definition.** [Böh00] A **weak $(H, A)$-Hopf module** is a right $H$-comodule $M$ with costructure $\delta$, which has a right $A$-module structure such that

$$\delta(ma) = \delta(m)\rho(a), \tag{8.11}$$

for $m \in M$, $a \in A$. The category of weak relative Hopf modules together with $H$-colinear and $A$-linear morphism will be denoted by $\mathcal{M}_A^H$. In the same way one may define the category ${}_A\mathcal{M}^H$, and if $A$ is a left $H$-comodule algebra, also ${}^H\mathcal{M}_A$ and ${}_A^H\mathcal{M}$.

By considering $H$ as a right $H$-comodule algebra in the usual way, the above definition coincides with the definition of the category of weak right Hopf modules $\mathcal{M}_H^H$ as defined in [BNS99].

Formulas (7.20) and (7.21) can be extended to elements in relative Hopf modules:

**Lemma 8.1.3.** *Let $A$ be a right $H$-comodule algebra. If $M \in \mathcal{M}_A^H$, then for all $m \in M$:*

$$m_{(0)} \otimes \varepsilon_s(m_{(1)}) = m1^A_{(0)} \otimes S(1^A_{(1)}). \tag{8.12}$$

*If $M \in {}_A\mathcal{M}^H$, then for all $m \in M$:*

$$m_{(0)} \otimes \varepsilon_t(m_{(1)}) = 1^A_{(0)}m \otimes 1^A_{(1)}. \tag{8.13}$$

*Let now $A$ be a left $H$-comodule algebra. If $M \in {}_A^H\mathcal{M}$, then for all $m \in M$:*

$$\varepsilon_t(m_{(-1)}) \otimes m_{(0)} = S(1^A_{(-1)}) \otimes 1^A_{(0)}m. \tag{8.14}$$

*If $M \in {}^H\mathcal{M}_A$, then for all $m \in M$:*

$$\varepsilon_s(m_{(-1)}) \otimes m_{(0)} = 1^A_{(-1)} \otimes m1^A_{(0)}. \tag{8.15}$$

*In particular, elements of right or left $H$-comodule algebras satisfy these equations.*

*Proof.* (8.12): Let $M \in \mathcal{M}_A^H$ and $m \in M$. Then

$$\begin{aligned} m_{(0)} \otimes \varepsilon_s(m_{(1)}) &= m_{(0)}1^A_{(0)} \otimes 1_{(1)}\varepsilon(m_{(1)}1^A_{(1)}1_{(2)}) \\ &= m_{(0)}1^A_{(0)}\varepsilon(m_{(1)}1^A_{(1)}) \otimes 1_{(1)}\varepsilon(1^A_{(2)}1_{(2)}) \\ &= m1^A_{(0)} \otimes \varepsilon_s(1^A_{(1)}) = m1^A_{(0)} \otimes S(1^A_{(1)}), \end{aligned}$$

by (7.3) and (7.8) since $1^A_{(1)} \otimes 1^A_{(2)} \in A \otimes H_t$.

(8.13): Let $M \in {}_A\mathcal{M}^H$ and $m \in M$. Then

$$\begin{aligned} m_{(0)} \otimes \varepsilon_t(m_{(1)}) &= 1^A_{(0)}m_{(0)} \otimes \varepsilon(1_{(1)}1^A_{(1)}m_{(1)})1_{(2)} \\ &= 1^A_{(0)}m_{(0)}\varepsilon(1^A_{(1)}m_{(1)}) \otimes 1^A_{(2)} = 1^A_{(0)}m \otimes 1^A_{(1)}, \end{aligned}$$

by (8.3).

The other two equations are the opcop versions. $\square$

**Lemma and Definition 8.1.4.** Let $A$ be a right $H$-comodule algebra, $M$ a right $A$-module and $X$ a right $H$-module. Write

$$M \otimes^A X := M1_{(0)} \otimes X1_{(1)}.$$

Then $M \otimes^A X$ is a right $A$-module with diagonal $A$-module structure. That is, for all $m \in M$, $x \in X$ and $a \in A$:

$$(m1^A_{(0)} \otimes x1^A_{(1)})a = m1^A_{(0)}a_{(0)} \otimes x1^A_{(1)}a_{(1)} = ma_{(0)} \otimes xa_{(1)}.$$

In the same way $M \otimes^A X$ is defined for a left $A$-module $M$ and a left $H$-module $X$. Moreover, $M \otimes^A H \in \mathcal{M}^H_A$ and $M \otimes^A H \in {}_A\mathcal{M}^H$ where in both cases the costructure is given by $\Delta$.

**Lemma 8.1.5.** *Let $M$, $\tilde{M}$ be right $A$-modules and $X$ a finite dimensional right $H$-module.*

(i) $\mathrm{Hom}_A(M \otimes^A X, \tilde{M}) \cong \mathrm{Hom}_A(M, \tilde{M} \otimes^A X^*)$, *where* $X^* = \mathrm{Hom}_k(H, k)$ *is the left dual of $X$ in the category of finite dimensional right $H$-modules, that is it is a right $H$-module via $(\phi \cdot h)(x) = \phi(xS^{-1}(h))$.*

(ii) $- \otimes^A X$ *is an exact functor for $A$-modules.*

(iii) *If $M$ is a projective right $A$-module, then so is $M \otimes^A X$.*

*Proof.* The opcop version of this is part of the result that ${}_A\mathcal{M}^{fd}$ is an ${}_H\mathcal{M}^{fd}$-module category whenever $A$ is a left $H$-comodule algebra. This will be discussed in Chapter 10. It follows from the fact that every object $X \in \mathcal{M}^{fd}_H$ has a right dual defined as in (i) with evaluation $ev_X$ and coevaluation $db_X$; and then one has the usual isomorphism:

$$\begin{array}{ccc} \mathrm{Hom}_A(M \otimes^A X, \tilde{M}) & \xrightarrow{\cong} & \mathrm{Hom}_A(M, \tilde{M} \otimes^A X^*) \\ f & \mapsto & (f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes db_X) \\ (\mathrm{id} \otimes ev_X) \circ (g \otimes \mathrm{id}) & \leftarrow\!\shortmid & g \end{array}$$

omitting the unit isomorphisms. Hence, $- \otimes^A X^*$ is left adjoint to $- \otimes^A X$, and in the same way we obtain a right adjoint $- \otimes^A {}^*X$, where ${}^*X$ is the right dual of $X$. This implies (ii). Hence, for a projective module $M$, $\mathrm{Hom}_A(M \otimes^A X, -)$ is an exact functor, and (iii) follows. $\square$

**Lemma 8.1.6.** *Let $A$ be a right $H$-comodule algebra and $M \in \mathcal{M}^H_A$.*

(i) $M \in {}_{H_s}\mathcal{M}^H_A$ *with left $H_s$-module structure given by* $x \triangleright m = m_{(0)}\varepsilon(xm_{(1)})$.

(ii) $H \otimes_{H_s} M \cong M \otimes^A H$ *as right $A$-modules, where $H \otimes_{H_s} M$ is a right $A$-module with* $(h \otimes m)a = h \otimes ma$.

*Proof.* (i) Let $x, x' \in H_s$, $m \in M$, and $a \in A$. Clearly, $1 \triangleright m = m$ and

$$\begin{aligned}
x \triangleright (x' \triangleright m) &= m_{(0)}\varepsilon(xm_{(1)})\varepsilon(x'm_{(2)}) \\
&= m_{(0)}\varepsilon(x\varepsilon_t(m_{(1)}))\varepsilon(x'm_{(2)})) && \text{by (7.15)} \\
&= m_{(0)}\varepsilon(xS(1_{(1)}))\varepsilon(1_{(2)}x'm_{(1)}) && \text{by (7.21) and Lemma 7.1.2} \\
&= m_{(0)}\varepsilon(x1_{(1)})\varepsilon(1_{(2)}x'm_{(1)}) && \text{by (7.11) and (7.14)} \\
&= m_{(0)}\varepsilon(xx'm_{(1)}) = (xx') \triangleright m.
\end{aligned}$$

$M$ is an $(H_s, A)$-bimodule, since

$$\begin{aligned}
x \triangleright (ma) &= m_{(0)}a_{(0)}\varepsilon(xm_{(1)}a_{(1)}) = m_{(0)}a_{(0)}\varepsilon(xm_{(1)}\varepsilon_t(a_{(1)})) && \text{by (7.11)} \\
&= m_{(0)}a\varepsilon(xm_{(1)}) = (x \cdot m)a. && \text{by (8.13)}
\end{aligned}$$

The $H$-comodule structure is compatible with this $H_s$-module structure. In fact

$$\begin{aligned}
\delta(x \triangleright m) &= m_{(0)} \otimes m_{(1)}\varepsilon(xm_{(2)}) = m_{(0)} \otimes xm_{(1)} && \text{by (7.18)} \\
&= m_{(0)}\varepsilon(1_{(1)}m_{(1)}) \otimes x1_{(2)}m_{(2)} = \Delta(x)\delta(m).
\end{aligned}$$

(ii) The isomorphism and its inverse is given by

$$\begin{aligned}
.H \otimes_{H_s} M. &\cong M. \circledcirc .H. \\
h \otimes_{H_s} m &\overset{\psi}{\mapsto} m_{(0)} \otimes hm_{(1)} \\
\sum_i h_i S^{-1}(m_{i(1)}) \otimes_{H_s} m_{i(0)} &\overset{\varphi}{\hookleftarrow} \sum_i m_i \otimes h_i.
\end{aligned}$$

In particular,

$$\begin{aligned}
\varphi(m1^A_{(0)} \otimes h1^A_{(1)}) &= h1^A_{(2)}S^{-1}(1^A_{(1)})S^{-1}(m_{(1)}) \otimes_{H_s} m_{(0)}1^A_{(0)} \\
&= hS^{-1}(\varepsilon_t(1^A_{(1)}))S^{-1}(m_{(1)}) \otimes_{H_s} m_{(0)}1^A_{(0)} \\
&= hS^{-1}(m_{(1)}) \otimes_{H_s} m_{(0)},
\end{aligned}$$

for $h \in H$, $m \in M$. Let $x \in H_s$ and $h_i \in H$, $m_i \in M$ for $i \in I$, such that $\sum_{i \in I} m_i \otimes h_i \in M \circledcirc^A H$. $\psi$ is well-defined, since

$$\begin{aligned}
\psi(h \otimes x \triangleright m) &= \psi(h \otimes m_{(0)}\varepsilon(xm_{(1)})) \\
&= m_{(0)} \otimes hm_{(1)}\varepsilon(xm_{(2)}) = m_{(0)} \otimes hxm_{(1)}, && \text{by (7.18).}
\end{aligned}$$

Moreover

$$\begin{aligned}
\varphi(\psi(h \otimes m)) &= hm_{(2)}S^{-1}(m_{(1)}) \otimes_{H_s} m_{(0)} = hS^{-1}(\varepsilon_t(m_{(1)})) \otimes_{H_s} m_{(0)} \\
&= hS^{-1}(1_{(2)}) \otimes_{H_s} 1_{(1)} \bullet m = hS^{-1}(1_{(2)})1_{(1)} \otimes_{H_s} m = h \otimes_{H_s} m,
\end{aligned}$$

by (8.14) and (7.10); and

$$\begin{aligned}\psi(\varphi(\sum_i m_i \otimes h_i)) &= \sum_i m_{i(0)} \otimes h_i S^{-1}(m_{i(2)}) m_{i(1)} \\ &= \sum_i m_{i(0)} \otimes h_i S^{-1}(\varepsilon_s(m_{i(1)})) \\ &= \sum_i m_i 1^A_{(0)} \otimes h_i S^{-1}(S(1^A_{(1)})) \\ &= \sum_i m_i \otimes h_i, \qquad \text{by (8.12).}\end{aligned}$$ □

## 8.2 Algebras and Modules in ${}_H\mathcal{M}$

The dual concept of $H$-comodule algebras and weak Hopf modules are $H$-module algebras and modules over them, that have a compatible $H$-module structure.

**Definition.** An algebra $A$, which is a left $H$-module, is a **left *H*-module algebra** [Nik00] if for all $a, b \in A$, $h \in H$

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \tag{8.16}$$

$$h \cdot 1^A = \varepsilon_t(h) \cdot 1^A. \tag{8.17}$$

A left $H$-module has a **compatible** $A$-module structure if it is a left $A$-module satisfying

$$h \cdot (ma) = (h_{(1)} \cdot m)(h_{(2)} \cdot a), \tag{8.18}$$

for all $m \in M$, $a \in A$, $h \in H$. We denote the category of $H$-modules with compatible $A$-module structures together with $H$-linear, $A$-linear morphisms by $({}_H\mathcal{M})_A$. In the same way we may define ${}_A({}_H\mathcal{M})$.

*Remark* 8.2.1. An $H$-module algebra is exactly an algebra in the category ${}_H\mathcal{M}$, and $({}_H\mathcal{M})_A$ are the right $A$-modules in the category of left $H$-modules. These are exactly the dual versions of $H$-comodule algebras and Hopf modules and as for finite dimensional Hopf algebras, it is straightforward to check that $A$ is a right $H$-comodule algebra if and only if it is a left $H^*$-module algebra; and $\mathcal{M}_A^H \approx ({}_{H^*}\mathcal{M})_A$ are equivalent categories. (Note that as always, $H$ is finite dimensional.) For instance, if $A$ is a right $H$-comodule algebra, then it is a a left $H^*$-module algebra with the usual $H^*$-module structure $\varphi \rightharpoonup a = a_{(0)}\varphi(a_{(1)})$. In fact,

$$\varphi \rightharpoonup 1^A \overset{(8.2)}{=} 1^A_{(0)}\varphi(\varepsilon_t(1^A_{(1)})) = 1^A_{(0)}\varepsilon(1_{(1)}1^A_{(1)})\varphi(1_{(2)}) = (\varepsilon_{(1)} * \varphi)(1)\,\varepsilon_{(2)} \rightharpoonup 1^A$$

and

$$\varphi \rightharpoonup (ab) \overset{(8.1)}{=} a_{(0)}b_{(0)}\varphi(a_{(1)}b_{(1)}) = (\varphi_{(1)} \rightharpoonup a)(\varphi_{(2)} \rightharpoonup b),$$

for $a, b \in A$, $\varphi \in H^*$.

**Lemma 8.2.2.** *Let $A$ be an $H$-module algebra, then $A$ is a right $H_t$-module via*

$$a \triangleleft y = S^{-1}(y) \cdot a = a(y \cdot 1^A). \tag{8.19}$$

*Proof.* Since $S^{-1}$ is an anti-algebra morphism, it suffices to show the second equation. For all $y \in H_t$, $a \in A$:

$$S^{-1}(y) \cdot a = a(S^{-1}(y) \cdot 1^A) = a(\varepsilon_t(S^{-1}(y)) \cdot 1^A) = a(S(S^{-1}(y)) \cdot 1^A),$$

by (7.12) and Lemma 7.1.3. □

**Lemma 8.2.3.** *Let $A$ be an $H$-module algebra and $V \in \mathcal{M}_A$.*

(i) *$V \in {}_{H_s}\mathcal{M}$ via $x \triangleright v := v(S(x) \cdot 1^A) = v(x \cdot 1^A)$; and if $V \in ({}_H\mathcal{M})_A$ then this $H_s$-structure coincides with the structure induced by the $H$-module structure.*

(ii) *$H \otimes_{H_s} V \in ({}_H\mathcal{M})_A$ with an $A$-module structure given by $(h \otimes_{H_s} v) \bullet a = h_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot a)$ and the $H$-action is given by $g(h \otimes_{H_s} v) = gh \otimes_{H_s} v$.*

(iii) *If $V \in ({}_H\mathcal{M})_A$, then $H \otimes_{H_s} V \to V, h \otimes_{H_s} v \mapsto h \cdot v$ is right $A$-linear.*

(iv) *If $V \in ({}_H\mathcal{M})_A$, then $H \mathbin{\bar{\otimes}} V := \Delta(1)(H \otimes V) \in ({}_H\mathcal{M})_A$, where the action of $A$ is the $A$-action on $V$, and the $H$-action is diagonal.*

(v) *$\mathrm{Hom}_{H_s}(H, V) \in ({}_H\mathcal{M})_A$, where the $H$-module structure is given by $(h \cdot \zeta)(g) = \zeta(gh)$, for $g, h \in H$ and $\zeta \in \mathrm{Hom}_{H_s}(H, V)$ and the right $A$-module is defined by $(\zeta \bullet a)(h) = \zeta(h_{(1)})(h_{(2)} \cdot a)$.*

*Proof.* Let $x, x' \in H_s$, $y \in H_t$ $g, h \in H$, $v \in V$, $\zeta \in \mathrm{Hom}_{H_s}(H, V)$.

(i) $v(x \cdot 1^A) = v(\varepsilon_t(x) \cdot 1^A) = v(S(x) \cdot 1^A)$ by Lemma 7.1.2; and

$$\begin{aligned} xx' \triangleright v &= v(S(x')S(x) \cdot 1^A) = v(1_{(1)}S(x') \cdot 1^A)(1_{(2)}S(x) \cdot 1^A) \\ &= (v(S(x') \cdot 1^A))(S(x) \cdot 1^A) = x \triangleright (x' \triangleright v). \end{aligned}$$

If $V \in ({}_H\mathcal{M})_A$, then $x \cdot v = (1_{(1)} \cdot v)(1_{(2)}x \cdot 1^A) = v(x \cdot 1^A)$ by (7.12).

(ii) The $A$-module structure is well-defined since

$$\begin{aligned} (hx \otimes_{H_s} v) \bullet a &= h_{(1)} \otimes_{H_s} v(S(x)S(h_{(2)}) \cdot a) \\ &= h_{(1)} \otimes_{H_s} v(S(x) \cdot 1^A)(S(h_{(2)}) \cdot a) && \text{by Lemma 7.1.2} \\ &= h_{(1)}(x \triangleright v)(S(h_{(2)}) \cdot a) = (h \otimes_{H_s} x \triangleright v)a; \end{aligned}$$

$$
\begin{aligned}
(h \otimes_{H_s} v) \bullet 1^A &= h_{(1)} \otimes_{H_s} v(\varepsilon_t(S(h_{(2)})) \cdot 1^A) \\
&= h_{(1)} \otimes_{H_s} v(S(\varepsilon_s(h_{(2)})) \cdot 1^A) && \text{by Lemma 7.1.2} \\
&= h1_{(1)} \otimes_{H_s} v(S^2(1_{(2)}) \cdot 1^A) && \text{by (7.21)} \\
&= h1_{(1)} \otimes_{H_s} S(1_{(2)}) \triangleright v \\
&= h1_{(1)} S(1_{(2)}) \otimes_{H_s} v = h \otimes_{H_s} v;
\end{aligned}
$$

$$
\begin{aligned}
((h \otimes_{H_s} v) \bullet a) \bullet b) &= h_{(1)} \otimes_{H_s} v(S(h_{(3)}) \cdot a)(S(h_{(2)}) \cdot b) \\
&= h_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot (ab)) = (h \otimes_{H_s} v) \bullet (ab).
\end{aligned}
$$

The $A$-module structure is compatible with the $H$-module structure since

$$
\begin{aligned}
g \cdot ((h \otimes_{H_s} v) \bullet a) &= gh_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot a) \\
&= g_{(1)} h_{(1)} \otimes_{H_s} v(S(h_{(2)}) S(g_{(2)}) g_{(3)} \cdot a) && \text{by (7.21)} \\
&= (g_{(1)} h \otimes_{H_s} v) \bullet (g_{(2)} \cdot a).
\end{aligned}
$$

(iii) The map is right $A$-linear, since

$$
\begin{aligned}
(h \otimes_{H_s} v)a = h_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot a) &\mapsto (h_{(1)} \cdot v)(h_{(2)} S(h_{(3)}) \cdot a) \\
&\overset{(7.20)}{=} (1_{(1)} h \cdot v)(1_{(2)} \cdot a) = (h \cdot v)a.
\end{aligned}
$$

(iv) The $A$-module structure is well-defined since

$$
\begin{aligned}
(1_{(1)} h \otimes 1_{(2)} \cdot v)a &= 1_{(1)} h \otimes (1_{(2)} \cdot v)a = 1_{(1)} h \otimes (1'_{(1)} 1_{(2)} \cdot v)(1'_{(2)} a) \\
&= 1_{(1)} h \otimes 1_{(2)} \cdot (va) \in H \otimes V.
\end{aligned}
$$

The $A$-module structure is compatible with the $H$-module structure since

$$
\begin{aligned}
g \cdot ((1_{(1)} h \otimes 1_{(2)} \cdot v)a) &= g_{(1)} h \otimes g_{(2)} \cdot (va) = g_{(1)} h \otimes (g_{(2)} \cdot v)(g_{(3)} \cdot a) \\
&= (g_{(1)} \cdot (1_{(1)} h \otimes 1_{(2)} \cdot v))(g_{(2)} \cdot a).
\end{aligned}
$$

(v) The $A$-module structure is well-defined since

$$
\begin{aligned}
(\zeta \bullet a)(xh) &\overset{(7.12)}{=} \zeta(h_{(1)})(xh_{(2)} \cdot a) \overset{(7.12)}{=} \zeta(h_{(1)})(h_{(2)} \cdot a)(x \cdot 1^A) \\
&= x \triangleright (\zeta(h_{(1)})(h_{(2)} \cdot a)) = x \triangleright ((\zeta \bullet a)(h));
\end{aligned}
$$

$$
\begin{aligned}
(\zeta \bullet 1^A)(h) \quad &= \quad \zeta(h_{(1)})(\varepsilon_t(h_{(2)}) \cdot 1^A) \overset{(7.20)}{=} \zeta(1_{(1)} h)(1_{(2)} \cdot 1^A) \\
&\overset{\zeta H_s\text{-linear}}{=} (1_{(1)} \triangleright \zeta(h))(1_{(2)} \cdot 1^A) = \zeta(h)(1_{(1)} \cdot 1^A)(1_{(2)} \cdot 1^A) = \zeta(h);
\end{aligned}
$$

$$((\zeta \bullet a) \bullet b)(h) = \zeta(h_{(1)})(h_{(2)} \cdot a)(h_{(3)} \cdot b) = (\zeta \bullet ab)(h).$$

The $A$-module structure is compatible with the $H$-module structure since

$$
\begin{aligned}
(h \cdot (\zeta \bullet a))(g) &= (\zeta \bullet a)(gh) = \zeta(g_{(1)} h_{(1)})(g_{(2)} h_{(2)} \cdot a) \\
&= ((h_{(1)} \cdot \zeta)(g_{(1)}))(g_{(2)} \cdot (h_{(2)} a)) = ((h_{(1)} \cdot \zeta) \bullet (h_{(2)} \cdot a))(g). \square
\end{aligned}
$$

**Lemma 8.2.4.** *If $V \in ({}_H\mathcal{M})_A$, then $H \otimes_{H_s} V \cong H \oslash V$. as objects in $({}_H\mathcal{M})_A$.*

*Proof.* The isomorphism is given by

$$\begin{aligned} H \otimes_{H_s} V &\xrightarrow{\cong} H \oslash V. \\ h \otimes v &\overset{\psi}{\mapsto} h_{(1)} \otimes h_{(2)} \cdot v \\ \sum_i h_{i(1)} \otimes S(h_{i(2)}) \cdot v_i &\overset{\varphi}{\leftarrow\!\shortmid} \sum_i h_i \otimes v_i, \end{aligned}$$

In particular,

$$\begin{aligned} \varphi(1_{(1)}h \otimes 1_{(2)} \cdot v) &= 1_{(1)}h_{(1)} \otimes_{H_s} S(1_{(2)}h_{(2)})1_{(3)} \cdot v \\ &= 1_{(1)}h_{(1)} \otimes_{H_s} S(h_{(2)})\varepsilon_s(1_{(2)}) \cdot v \\ &= 1_{(1)}h_{(1)} \otimes_{H_s} S(h_{(2)})S(1_{(2)}) \cdot v = h_{(1)} \otimes S(h_{(2)}) \cdot v, \end{aligned}$$

for $v \in V$, $h \in H$, by Lemma 7.1.2. Let $x \in H_s$ and $h_i \in H$, $v_i \in V$ for all $i \in I$ such that $\sum_i h_i \otimes v_i \in H \oslash V$. Then $\psi$ is well-defined, since $\psi(hx \otimes_{H_s} v) = h_{(1)} \otimes h_{(2)}xv$ by (7.12). It is right $A$-linear since

$$\begin{aligned} \psi((h \otimes_{H_s} v) \bullet a) &= \psi(h_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot a)) \\ &= h_{(1)} \otimes h_{(2)} \cdot (v(S(h_{(3)})) \cdot a)) \\ &= h_{(1)} \otimes (h_{(2)} \cdot v)(h_{(3)}S(h_{(4)}) \cdot a) \\ &= h_{(1)} \otimes (1_{(1)}h_{(2)} \cdot v)(1_{(2)} \cdot a) \qquad \text{by (7.20)} \\ &= h_{(1)} \otimes (h_{(2)} \cdot v)a = \psi(h \otimes_{H_s} v)a. \end{aligned}$$

Obviously, $\psi$ is left $H$-linear. Finally,

$$\begin{aligned} \varphi(\psi(h \otimes_{H_s} v)) &= h_{(1)} \otimes S(h_{(2)})h_{(3)} \cdot v \overset{(7.21)}{=} h_{(1)}1_{(1)} \otimes_{H_s} S(1_{(2)}) \cdot v \\ &= h_{(1)}1_{(1)}S(1_{(2)}) \otimes_{H_s} v = h \otimes v; \end{aligned}$$

$$\begin{aligned} \psi(\varphi(\sum_i h_i \otimes v_i)) &= \sum_i h_{i(1)} \otimes h_{i(2)}S(h_{i(3)}) \cdot v_i \\ &\overset{(7.20)}{=} \sum 1_{(1)}h_i \otimes 1_{(2)} \cdot v_i = \sum_i h_i \otimes v_i. \end{aligned}$$

□

**Lemma 8.2.5.** *Let $V, W \in \mathcal{M}_A$.*

(i) $\mathrm{Hom}_A(V, \mathrm{Hom}_{H_s}(H, W)) \cong \mathrm{Hom}_A(H \otimes_{H_s} V, W)$*.*

(ii) *If $V$ is an injective $A$-module, then so is* $\mathrm{Hom}_{H_s}(H, V)$*.*

*Proof.* (i) The isomorphism is given by

$$\begin{aligned} \mathrm{Hom}_A(V, \mathrm{Hom}_{H_s}(H, W)) &\xrightarrow{\cong} \mathrm{Hom}_A(H \otimes_{H_s} V, W) \\ f \quad &\mapsto \tilde{f} \quad h \otimes_{H_s} v \mapsto f(v)(h) \\ \tilde{g} \quad v \mapsto (h \mapsto g(h \otimes_{H_s} v)) &\leftarrow\!\shortmid \quad g \end{aligned}$$

Let $f \in \mathrm{Hom}_A(V, \mathrm{Hom}_{H_s}(H, W))$ and $g \in \mathrm{Hom}_A(H \otimes_{H_s} V, W)$, then $\tilde{\tilde{f}}(v)(h) = f(v)(h)$ and $\tilde{\tilde{g}}(h \otimes_{H_s} v) = g(h \otimes_{H_s} v)$. Moreover, $\tilde{f}$ is well-defined, since

$$\begin{aligned}
\tilde{f}(h \otimes_{H_s} (x \triangleright v)) &= f(x \triangleright v)(h) = f(v(x \cdot 1^A))(h) \\
&= (f(v) \bullet (x \cdot 1^A))(h) = (f(v)(h_{(1)}))(h_{(2)}x \cdot 1^A) \\
&= (f(v)(h_{(1)}x_{(1)}))(\varepsilon_t(h_{(2)}x_{(2)}) \cdot 1^A) && \text{by (7.12)} \\
&= (f(v)(1_{(1)}hx))(1_{(2)} \cdot 1^A) && \text{by (7.20)} \\
&= (1_{(1)} \triangleright f(v)(hx))(1_{(2)} \cdot 1^A) \\
&= (f(v)(hx))(1_{(1)} \cdot 1^A)(1_{(2)} \cdot 1^A) \\
&= f(v)(hx) = \tilde{f}(hx \otimes_{H_s} v),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{f}((h \otimes_{H_s} v) \bullet a) &= \tilde{f}(h_{(1)} \otimes_{H_s} v(S(h_{(2)}) \cdot a)) = f(v(S(h_{(2)}) \cdot a))(h_{(1)}) \\
&= (f(v) \bullet (S(h_{(2)}) \cdot a))(h_{(1)}) = (f(v)(h_{(1)}))(h_{(2)}S(h_{(3)}) \cdot a) \\
&= (f(v)(1_{(1)}h))(1_{(2)} \cdot a) \qquad \text{by (7.20)} \\
&= (1_{(1)} \triangleright f(v)(h))(1_{(2)} \cdot a) = (f(v)(h))(1_{(1)} \cdot 1^A)(1_{(2)} \cdot a) \\
&= (f(v)(h))a = \tilde{f}(h \otimes_{H_s} v)a;
\end{aligned}$$

and $\tilde{g}$ is well-defined since

$$\begin{aligned}
\tilde{g}(v)(xh) &= g(xh \otimes_{H_s} v) = g(x \cdot (h \otimes_{H_s} v)) \\
&= g((1_{(1)} \cdot (h \otimes_{H_s} v)) \bullet (1_{(2)}x \cdot 1^A)) && \text{by Lemma 8.2.3 and (7.12)} \\
&= g(h \otimes_{H_s} v)) \bullet (x \cdot 1^A) = x \bullet (\tilde{g}(v)(h)),
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{g}(v) \bullet a)(h) &= \tilde{g}(v)(h_{(1)})(h_{(2)} \cdot a) = g(h_{(1)} \otimes_{H_s} v)(h_{(2)} \cdot a) \\
&= g((h_{(1)} \otimes_{H_s} v) \bullet (h_{(2)} \cdot a)) = g(h_{(1)} \otimes_{H_s} v(S(h_{(2)})h_{(3)} \cdot a)) \\
&= g(h1_{(1)} \otimes_{H_s} v(S(1_{(2)}) \cdot a)) \qquad \text{by (7.21)} \\
&= g(h \otimes_{H_s} v(S(1_{(2)}) \cdot a)(S(1_{(1)}) \cdot 1^A)) \\
&= g(h \otimes va) = \tilde{g}(va)(h),
\end{aligned}$$

for $v \in V$, $a \in A$ , $h \in H$, $x \in H_s$.

(ii) $\mathrm{Hom}_A(H \otimes_{H_s} -, V)$ is exact since $V$ is injective and since $H_s$ is semisimple and hence $H \otimes_{H_s} -$ is exact. Hence, $\mathrm{Hom}_A(-, \mathrm{Hom}_{H_s}(H, V))$ is exact by (i). $\square$

## 8.3 Weak Hopf Modules over *H*-Simple *H*-Comodule Algebras Are Generators

The aim of this section is to prove that trace ideals of relative weak Hopf modules are $H$-costable ideals. Therewith, one can deduce that nonzero relative weak Hopf modules over an $H$-simple quasi-Frobenius $H$-comodule algebra $A$ are generators, as their trace ideals have to be equal to $A$. We will prove the dual version of this.

### 8.3.1 *H*-stable and *H*-costable subspaces

Let $C$ be a right $H$-comodule with costructure $\delta$. A subspace $X$ of $C$ is said to be ***H*-costable** if $\delta(X) \subset X \otimes H$. Dually, a subspace $X$ of an $H$-module $V$ is said to be ***H*-stable** if for all $x \in X$ and $h \in H$, $h \cdot x \in X$.

An $H$-(co)module algebra $A$ is called ***H*-simple** if $A \neq 0$ and $A$ does not contain proper nonzero $H$-(co)stable ideals. $A \neq 0$ is called ***H*-simple from the right** or **left** if it does not contain proper nonzero $H$-(co)stable right or left ideals, respectively.

### 8.3.2 Trace Ideals of Weak Hopf Modules

In the following, if $M$ is a module over an algebra $A$, then $T_M$ denotes the trace ideal of $M$ in $A$ (see Appendix A.2 for the definition).

**Proposition 8.3.1.** *Let $A$ be a left $H$-module algebra and $M \in ({}_H\mathcal{M})_A$, then $T_M$ is an $H$-stable ideal of $A$.*

*Proof.* Let $f \in \mathrm{Hom}_A(M, A)$, then $f$ is left $H_s$-linear. In fact,

$$f(x \cdot m) \overset{(7.12),(8.18)}{=} f(m(x \cdot 1^A)) = f(m)(x \cdot 1^A) \overset{(7.12),(8.16)}{=} x \cdot f(m).$$

For every $h \in H$ we may define an $A$-linear morphism

$$\begin{array}{llll} \Phi_h : M \longrightarrow & H \otimes M. \xrightarrow[\cong]{\phi} & H \otimes_{H_s} M \overset{\mathrm{id} \otimes_{H_s} f}{\longrightarrow} & H \otimes_{H_s} A \to A \\ \quad m \mapsto 1_{(1)}h \otimes 1_{(2)} \cdot m & & & g \otimes_{H_s} a \mapsto g \cdot a \end{array}$$

where the isomorphism $\phi$ is the one from Lemma 8.2.4, and the last morphisms is $A$-linear by Lemma 8.2.3 (iii).

Let now $g \in H$ and $m \in M$, then there exist $h_i \in H$ and $m_i \in M$ with $\phi(\sum_i 1_{(1)}h_i \otimes 1_{(2)} \cdot m_i) = g \otimes_{H_s} m$, where $\phi$ is the isomorphism $H \otimes M \overset{\cong}{\to} H \otimes_{H_s} M$. And then

$$g \cdot f(m) = \sum_i \Phi_{h_i}(m_i) \in \sum_{f \in \mathrm{Hom}_A(M,A)} f(M). \qquad \square$$

**Corollary 8.3.2.** *Let $A$ be an $H$-simple left $H$-module algebra which is right Kasch and let $0 \neq M \in ({}_H\mathcal{M})_A$ be a finite dimensional object. Then $M$ is a generator for $\mathcal{M}_A$*

For the definition of right Kasch rings see Appendix A.3.

*Proof.* $T_M \neq 0$ because $A$ is right Kasch (see A.3.2). Hence, $T_M = A$ by the foregoing lemma since $A$ is $H$-simple. This implies the claim (see A.2.1). □

**Corollary 8.3.3.** *If $A$ is a finite dimensional $H$-simple left $H$-module algebra which is right Kasch, then $A$ is quasi-Frobenius.*

*Proof.* Let $V$ be a nonzero finite dimensional injective right $A$-module. By Lemma 8.2.5, $M := \mathrm{Hom}_{H_s}(H, V)$ is an injective $A$-module and $M \in ({}_H\mathcal{M})_A$. $M$ is nonzero, since $H_s$ is an $H_s$-direct summand of $H$. As $A$ is right Kasch, $M$ is a generator in $\mathcal{M}_A$ by 8.3.2. Hence, for some $n$ there is a surjection $M^{(n)} \twoheadrightarrow A$ which splits, that is $A$ is a direct summand of the injective $A$-module $M^{(n)}$, and therefore right self-injective. □

The dual versions of these results are:

**Proposition 8.3.4.** *Let $A$ be a right $H$-comodule algebra and $M \in \mathcal{M}_A^H$ finite dimensional.*

(i) *$T_M$ is an $H$-costable ideal of $A$.*

(ii) *If $A$ is $H$-simple and right Kasch and $M \neq 0$, then $M$ is a generator for $\mathcal{M}_A$.*

(iii) *If $A$ is finite dimensional, $H$-simple and right Kasch, then it is quasi-Frobenius.*

## 8.4 A Structure Theorem for Weak Hopf Modules and Smash Products

### 8.4.1 Invariants and Coinvariants

**Definition.** Let $A$ be a right $H$-comodule algebra and $M \in \mathcal{M}_A^H$ or $M \in {}_A\mathcal{M}^H$ with costructure $\delta$. We define the set of **coinvariants** of $M$ as

$$M^{coH} := \{m \in M \mid \delta(m) \in M \otimes H_t\}. \tag{8.20}$$

If $A$ is a left $H$-comodule algebra and $M \in {}^H\mathcal{M}_A$ or $M \in {}^H_A\mathcal{M}$ with costructure $\delta$, we define the set of coinvariants of $M$ as

$$^{coH}M := \{m \in M \mid \delta(m) \in H_s \otimes M\}. \tag{8.21}$$

**Lemma 8.4.1.** *Let $A$ be a right $H$-comodule algebra. If $M \in \mathcal{M}_A^H$, then*

$$M^{coH} = \{m \in M \mid m_{(0)} \otimes m_{(1)} = m1^A_{(0)} \otimes 1^A_{(1)}\}.$$

*If $M \in {}_A\mathcal{M}^H$, then*

$$M^{coH} = \{m \in M \mid m_{(0)} \otimes m_{(1)} = 1^A_{(0)}m \otimes 1^A_{(1)}\}.$$

*Let now $A$ be a left $H$-comodule algebra. If $M \in {}^H\mathcal{M}_A$, then*

$${}^{coH}M = \{m \in M \mid m_{(-1)} \otimes m_{(0)} = 1^A_{(-1)} \otimes m1^A_{(0)}\}.$$

*If $M \in {}^H_A\mathcal{M}$, then*

$${}^{coH}M = \{m \in M \mid m_{(-1)} \otimes m_{(0)} = 1^A_{(-1)} \otimes 1^A_{(0)}m\}.$$

*Proof.* The first equation is [ZZ04, Lemma 2.1], the second and third follow from Lemma 8.1.3, and the last equation is the opcop version of [ZZ04, Lemma 2.1]. □

**Corollary 8.4.2.** *If $A$ is an $H$-comodule algebra, the following holds for a coinvariant element $a \in A^{coH}$:*

$$\rho(a) = 1^A_{(0)}a \otimes 1^A_{(1)} = a1^A_{(0)} \otimes 1^A_{(1)}.$$

*Example* 8.4.3. Let $M$ be a right $A$-module. Then

$$(M \otimes^A H)^{coH} = \{m1^A_{(0)} \otimes 1^A_{(1)} \mid m \in M\}.$$

In fact, if $\sum_i m_i \otimes h_i \in (M \otimes^A H)^{coH}$, then

$$\sum_i m_i \otimes h_{i(1)} \otimes h_{i(2)} = \sum_i m_i 1^A_{(0)} \otimes h_i 1^A_{(1)} \otimes 1^A_{(2)}.$$

By applying $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}$ we obtain

$$\begin{aligned}\sum_i m_i \otimes h_i &= \sum_i m_i 1^A_{(0)} \otimes \varepsilon(h_i 1^A_{(1)})1^A_{(2)} \overset{(7.15)}{=} \sum_i m_i 1^A_{(0)} \otimes \varepsilon(h_i \varepsilon_t(1^A_{(1)}))1^A_{(2)} \\ &\overset{(8.5)}{=} \sum_i m_i 1^A_{(0)} 1^{A\,\prime}_{(0)} \otimes \varepsilon(h_i 1^A_{(1)})1^{A\,\prime}_{(1)} = \sum_i m_i \varepsilon(h_i) 1^A_{(0)} \otimes 1^A_{(1)}.\end{aligned}$$

In the special case when $A = H$ is the weak Hopf module algebra itself then, as for Hopf algebras, there exists a projection $E : M \to M^{coH}$:

**Lemma and Definition 8.4.4.** Let $M \in \mathcal{M}_H^H$ and define

$$\begin{aligned} E : M &\to M \\ m &\mapsto m_{(0)}S(m_{(1)}). \end{aligned}$$

Then $E(M) = M^{coH}$ and $E(m) = m$ for all $m \in M^{coH}$.

*Proof.* $E(M) \subset M^{coH}$ is shown in the proof of [ZZ04, Theorem 2.2]. If $m \in M^{coH}$, then $E(m) = m1_{(1)}S(1_{(2)}) = m$ by Lemma 8.4.1. □

The dual version of the coinvariants of a weak Hopf module are the invariants of objects in $({}_H\mathcal{M})_A$:

**Lemma and Definition 8.4.5.** Let $A$ be a left $H$-module algebra and $M \in ({}_H\mathcal{M})_A$. The **invariants** of $M$ are defined as

$$M^H := \{m \in M \mid h \cdot m = \varepsilon_t(h) \cdot m \ \forall h \in H\}. \tag{8.22}$$

Then $M^{coH} = M^{H^*}$. In fact,

$$\begin{aligned} &\{m \in M \mid \varphi \rightharpoonup m = \varepsilon_t(\varphi) \rightharpoonup m \, \forall \varphi \in H^* \} \\ =&\{m \in M \mid m_{(0)}\varphi(m_{(1)}) = m_{(0)}\varepsilon(1_{(1)}m_{(1)})\varphi(1_{(2)}) \, \forall \varphi \in H^* \} \\ =&\{m \in M \mid m_{(0)} \otimes m_{(1)} = m_{(0)} \otimes \varepsilon_t(m_{(1)}) \}. \end{aligned}$$

### 8.4.2 A Structure Theorem

The structure theorem for Hopf modules over ordinary Hopf algebras states a category equivalence between vector spaces and Hopf modules. It implies in particular that every Hopf module over a Hopf algebra $H$ is a free $H$-module. For weak Hopf modules, the freeness has to be replaced by projectivity. Böhm, Nill, and Szlachányi [BNS99, Theorem 3.9] proved a structure Theorem for weak Hopf modules in $\mathcal{M}_H^H$ which was generalized by Zhang and Zhu [ZZ04] to $H$-comodule algebras $A$ which allow an $H$-comodule algebra map $\gamma : H \to A$. In this case, $M \in \mathcal{M}_A^H$ implies $M \in \mathcal{M}_H^H$ via $\gamma$.

**Theorem 8.4.6.** [ZZ04, Theorem 2.2 ] *Let $A$ be a right $H$-comodule algebra and $\gamma : H \to A$ an $H$-comodule algebra map. Let $M \in \mathcal{M}_A^H$. Then*

$$\begin{aligned} M^{coH} \otimes_{A^{coH}} A \quad &\cong \quad M \qquad \text{in } \mathcal{M}_A^H \\ m \otimes a &\mapsto ma \\ m_{(0)}\gamma(S(m_{(1)})) \otimes m_{(2)} &\leftarrow\!\shortmid m. \end{aligned}$$

*Here, the $H$-comodule structure and the the $A$-module structure of $M^{coH} \otimes_{A^{coH}} A$ are those of $A$.*

**Corollary 8.4.7.** *If $M \in \mathcal{M}_H^H$, then $M$ is a projective right $H$-module and it is a free right $H$-module if $M^{coH}$ is a free right $H_t$-module.*

*Proof.* By the theorem, $M \cong M^{coH} \otimes_{H^{coH}} H$ where $H^{coH} = H_t$ is a semisimple algebra and therefore $M^{coH}$ is a direct summand of a free $H_t$-module. Hence, $M \cong M^{coH} \otimes_{H_t} H$ is a direct summand of a free $H$-module. □

**Lemma 8.4.8.** *Let $A$ and $\gamma$ be as in the theorem. Let $V$ be a right $A^{coH}$-module, then $V \otimes_{A^{coH}} A \in \mathcal{M}_A^H$ and*

$$V \cong (V \otimes_{A^{coH}} A)^{coH} \quad v \mapsto v \otimes 1^A.$$

*Proof.* Let $E_A$ and $E$ be the map from 8.4.4 for $A$ and $V \otimes_{A^{coH}} A$ regarded as objects in $\mathcal{M}_H^H$ via $\gamma$, respectively. We know $E(V \otimes_{A^{coH}} A) = (V \otimes_{A^{coH}} A)^{coH}$ and $E_A(A) = A^{coH}$ and we have $E(V \otimes_{A^{coH}} A) = V \otimes_{A^{coH}} 1^A$. In fact, if $v \in V$ and $a \in A$, then

$$\begin{aligned} E(v \otimes_{A^{coH}} a) &= v \otimes_{A^{coH}} a_{(0)}\gamma(S(a_{(1)})) \\ &= v \otimes_{A^{coH}} E_A(a) = vE_A(a) \otimes_{A^{coH}} 1^A. \end{aligned}$$ □

Hence, together with this lemma the theorem above implies a category equivalence between the category of right $A^{coH}$-modules and the category $\mathcal{M}_A^H$:

**Theorem 8.4.9.** *Let $A$ be a right $H$-comodule algebra and $\gamma : H \to A$ an $H$-comodule algebra map. Then*

$$\begin{array}{ccc} \mathcal{M}_{A^{coH}} & \approx & \mathcal{M}_A^H \\ V & \mapsto & V \otimes_{A^{coH}} A \\ M^{coH} & \leftarrow\!\shortmid & M\,. \end{array}$$

### 8.4.3 Weak Smash Products

As for ordinary Hopf algebras, a special case of $H$-comodule algebras which allow an $H$-comodule algebra map $\gamma : H \to A$, are smash products (in fact, such algebras are always smash products, see [Zha10]), and thus the structure theorem can be applied to smash products.

**Lemma and Definition 8.4.10.** [Nik00] Let $R$ be a right $H$-module algebra, then the **smash product** $R\#H$ is $R \otimes_{H_t} H$ as a vector space where $R$ is a right $H_t$-module as in 8.2.2. $R\#H$ is an $H$-comodule algebra with unit $1_R\#1$ and with multiplication and costructure given by

$$(r\#h)(s\#g) = r(h_{(1)} \cdot s)\#h_{(2)}g, \tag{8.23}$$

$$\rho(r\#h) = r\#h_{(1)} \otimes h_{(2)}, \tag{8.24}$$

for $r, r' \in R$, $h, h' \in H$. The map $H \to R\#H$, $h \mapsto 1_R\#h$ is an $H$-comodule algebra map. $R \to R\#H$, $r \mapsto r\#1$ is an algebra inclusion and $(R\#H)^{coH} = R\#1 \cong R$.

In the following we will associate elements of $R\#1$ with elements of $R$ and elements of $1\#H$ will be associated to elements of $H$, and we will write $rh$ for

$r\#h$ and then $hr = (h_{(1)} \cdot r)h_{(2)}$.

Let $A := R\#H$, and $V \in \mathcal{M}_R$, then $V\#H := V \otimes_R A$ is an object in $\mathcal{M}_A^H$, where the module and comodule structure are the one of $A$. Again we write $(vr)h$ for $v \otimes_R r\#h = vr \otimes_R 1_R\#h$. Then

$$E(vh) = vh_{(1)}S(h_{(2)}) = v\varepsilon_t(h) = v(1_R \triangleleft \varepsilon_t(h)) \in V,$$

for all $v \in V$ and $h \in H$, where $\triangleleft$ is the $H_t$-action on $R$ as defined in (8.19); and therefore $(V\#H)^{coH} = E(V\#H) \cong V$. With the structure theorem for weak Hopf modules 8.4.6 we obtain the following:

**Corollary 8.4.11.** *If $R$ is a left $H$-module algebra and $A := R\#H$, then*

$$\begin{array}{rcl} \mathcal{M}_R & \approx & \mathcal{M}_{R\#H}^H \\ V & \mapsto & V\#H \\ M^{coH} & \leftarrow\!\shortmid & M \end{array}$$

*are quasi-inverse equivalences, where*

$$\begin{aligned} M &\cong M^{coH}\#H \qquad \text{in } \mathcal{M}_{R\#H}^H \\ m &\mapsto E(m_{(0)})m_{(1)} \\ ma &\leftarrow\!\shortmid m \otimes_R a. \end{aligned}$$

As for Hopf algebras [MS99] one can use this equivalence to show that the $H$-stable ideals of $R$ correspond to the $H$-costable ideals of $A := R\#H$ and in particular, $R$ is $H$-simple if and only if $A$ is $H$-simple.

**Proposition 8.4.12.** *Let $R$ be an $H$-module algebra and $A := R\#H$, then*

$$\{H\text{-stable ideals of } R\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{H\text{-costable ideals of } A\}$$
$$\begin{array}{rcl} I & \longmapsto & I\#H \\ J^{coH} & \longleftarrow\!\shortmid & J \end{array}$$

*are well-defined mutually inverse bijections.*

*Proof.* If $I$ is an $H$-stable ideal of $R$, then it is straightforward to check that $I\#H$ is an $H$-costable ideal of $R\#H$, since if $r$ or $r' \in I$, then $(rh)(r'h') = (r(h_{(1)} \cdot r'))h_{(2)}h' \in I\#H$. On the other hand, let $J$ be an $H$-costable ideal of $A$. It is clear that $J^{coH} = J \cap (R\#H)^{coH} = J \cap R$ is an ideal of $R$. It is $H$-stable since for $r \in J \cap R = J^{coH}$ and $h \in H$ one has $h \cdot r \in R$ and:

$$\begin{aligned} h \cdot r &= (h_{(1)} \cdot r)(\varepsilon_t(h_{(2)}) \cdot 1_R) \\ &= (h_{(1)} \cdot r) \triangleleft \varepsilon_t(h_{(2)}) \\ &= (h_{(1)} \cdot r)\varepsilon_t(h_{(2)}) \\ &= (h_{(1)} \cdot r)h_{(2)}S(h_{(3)}) = h_{(1)}(rS(h_{(2)})) \in J, \end{aligned}$$

where $\triangleleft$ is the right $H_t$-action on $R$ as defined in (8.19). Hence, $h \cdot r \in J \cap R = J^{coH}$. Finally, $(I\#H)^{coH} = I$ and $J^{coH}\#H = (J^{coH}\#1)(1_R\#H) \subset J$, and the Proposition 8.4.11 implies

$$J^{coH}\#H \cong J \in \mathcal{M}_A^H. \qquad \square$$

**Lemma 8.4.13.** *Let $R$ be a right $H$-module algebra. Then $R^{op}$ is a right $H^{cop}$-module algebra and $({}_H\mathcal{M})_R \approx {}_{R^{op}\#H^{cop}}\mathcal{M}$. The functor is the identity on objects and morphisms, where the action of $R^{op}\#H^{cop}$ on $M \in ({}_H\mathcal{M})_R$ is given by $(r\#h) \blacktriangleright m = (h \cdot m)r$, for all $r \in R$, $h \in H$ and $m \in M$.*

*Proof.* Recall that the target map of $H^{cop}$ is $S^{-1} \circ \varepsilon_t$. We have that $h \cdot 1_R = \varepsilon_t(h) \cdot 1_R = \varepsilon_t(S^{-1}(\varepsilon_t(h))) \cdot 1_R = S^{-1}(\varepsilon_t(h)) \cdot 1_R$, hence $R^{op}$ is in fact an $H^{cop}$-module algebra.

Let $M \in ({}_H\mathcal{M})_R$. The $R^{op}\#H^{cop}$-module structure is well-defined, for if $x \in (H^{cop})_t = H_s$, $r \in R$, $h \in H$, $m \in M$, then

$$\begin{aligned}(r\#xh) \blacktriangleright m &\overset{(7.12)}{=} (1_{(1)}xh \cdot m)(1_{(2)} \cdot r)\\ &\overset{(7.22)}{=} (1_{(1)}h \cdot m)(1_{(2)}S(x) \cdot r) = ((r \triangleleft x)\#h) \blacktriangleright m,\end{aligned}$$

because the right $H_s$-module structure of $R$ is given by $r \triangleleft x = S(x) \cdot r$. Moreover, for $h' \in H$ and $r' \in R$:

$$\begin{aligned}(r\#h) \blacktriangleright ((r'\#h') \blacktriangleright m) &= (h \cdot ((h' \cdot m)r))r = (h_{(1)}h' \cdot m)(h_{(2)} \cdot r')r\\ &= ((h_{(2)}r')r\#h_{(1)}h') \blacktriangleright m = ((r\#h)(r'\#h)) \blacktriangleright m. \square\end{aligned}$$

## 8.5 Morita Theory for $H$-Comodule Algebras and Weak Hopf Modules

In this section it will be proven that if $M$ is a weak Hopf module in $\mathcal{M}_A^H$, then $B := \mathrm{End}_A(M)$ is an $H$-comodule algebra and $M \in {}_B\mathcal{M}_A^H$. Moreover, if $M$ is a progenerator for $\mathcal{M}_A$ then the Morita equivalence $\mathcal{M}_A \approx \mathcal{M}_B$ also induces an equivalence of the categories of relative weak Hopf modules $\mathcal{M}_A^H \approx \mathcal{M}_B^H$.

### 8.5.1 Endomorphism Rings of Weak Hopf Modules

The endomorphism ring of a relative weak Hopf module is an $H$-comodule algebra. Moreover, its coinvariants are exactly the $A$-module maps which are also $H$-colinear. Here, we prove the dual version.

**Lemma 8.5.1.** *Let $A$ be a left $H$-module algebra and $M \in ({}_H\mathcal{M})_A$.*

(i) $B := \operatorname{End}_A(M)$ *is a left $H$-module algebra with an $H$-module structure given by*

$$(h \bullet f)(m) := h_{(1)} f(S(h_{(2)}) \cdot m) \tag{8.25}$$

*for $f \in B$, $h \in H$, $m \in M$.*

(ii) *The invariants $B^H$ of $B$ are exactly the $H$-linear morphisms in $\operatorname{End}_A(M)$, denoted by ${}_H\operatorname{End}_A(M)$. In particular, $B^H \cong k$ if $M$ is simple in $({}_H\mathcal{M})_A$.*

(iii) $\operatorname{End}_A(A_A) \cong A$ *as $H$-module algebras, and in particular, $A^H \cong k$ if $A$ is simple in $({}_H\mathcal{M})_A$.*

(iv) $Q := \operatorname{Hom}_A(M, A) \in ({}_H\mathcal{M})_B$.

(v) $M \in {}_B({}_H\mathcal{M})_A$ *and* $Q \in {}_A({}_H\mathcal{M})_B$.

*Proof.* (i) Let $f \in \operatorname{End}_A(M)$ and $h \in H$, then $h \bullet f \in \operatorname{End}_A(M)$ since for all $m \in M$ and $a \in A$:

$$\begin{aligned}
(h \bullet f)(ma) &= h_{(1)} \cdot f(S(h_{(2)}) \cdot (ma)) \overset{7.1.3}{=} h_{(1)} \cdot (f(S(h_{(3)}) \cdot m))(S(h_{(2)}) \cdot a) \\
&= (h_{(1)} \cdot f(S(h_{(4)}) \cdot m))(h_{(2)} S(h_{(3)}) \cdot a) \\
&\overset{(7.20)}{=} (1_{(1)} h_{(1)} \cdot f(S(h_{(2)}) \cdot m))(1_{(2)} \cdot a) = ((h \bullet f)(m))a.
\end{aligned}$$

This is a well-defined $H$-module structure, since

$$\begin{aligned}
(1 \bullet f)(m) = 1_{(1)} \cdot f(S(1_{(2)}) \cdot m) &\overset{7.1.2 \text{ and } (7.12)}{=} 1_{(1)} \cdot f(m(S(1_{(2)}) \cdot 1^A)) \\
&= 1_{(1)} \cdot (f(m))(S(1_{(2)}) \cdot 1^A)) = (1_{(1)} \cdot f(m))(\varepsilon_t(1_{(2)}) \cdot 1^A)) \\
&= (1_{(1)} \cdot f(m))(1_{(2)} \cdot 1^A) = f(m),
\end{aligned}$$

$$\begin{aligned}
(h \bullet (h' \bullet f))(m) &= h_{(1)} (h' \bullet f)(S(h_{(2)}) \cdot m) \\
&= h_{(1)} h'_{(1)} \cdot f(S(h'_{(2)}) S(h_{(2)}) \cdot m) = (hh' \bullet f)(m).
\end{aligned}$$

for $f \in \operatorname{End}_A(M)$, $h, h' \in H$, $m \in M$. This $H$-module structure is compatible with the algebra structure since for another $f' \in \operatorname{End}_A(M)$:

$$\begin{aligned}
(h_{(1)} \bullet f) \circ (h_{(2)} \bullet f')(m) &= h_{(1)} f(S(h_{(2)}) h_{(3)} \cdot f'(S(h_{(4)}) \cdot m)) \\
&\overset{(7.20)}{=} h_{(1)} f(1_{(1)} \cdot f'(S(h_{(2)} 1_{(2)}) \cdot m)) \\
&= h_{(1)} f((1 \bullet f')(S(h_{(2)}) \cdot m)) = (h \bullet (f \circ f'))(m)
\end{aligned}$$

$$\text{and } (h \bullet \mathrm{id})(m) = h_{(1)} S(h_{(2)}) m = (\varepsilon_t(h) \bullet \mathrm{id})(m).$$

(ii) Let $f \in {}_H\operatorname{End}_A(M)$, then

$$\begin{aligned}
(h \bullet f)(m) &= h_{(1)} S(h_{(2)}) \cdot f(m) = \varepsilon_t(h) \cdot (1 \bullet f)(m) \\
&= 1_{(1)} \varepsilon_t(h) \cdot f(S(1_{(2)}) \cdot m) \overset{(7.12)}{=} (\varepsilon_t(h) \bullet f)(m)
\end{aligned}$$

for all $h \in H$ and $m \in M$, that is $f \in (\mathrm{End}_A(M))^H$. If on the other hand $f \in (\mathrm{End}_A(M))^H$, then

$$\begin{aligned}
h \cdot f(m) &= h \cdot (1 \bullet f)(m) = h1_{(1)} \cdot f(S(1_{(2)}) \cdot m) \\
&= h_{(1)} \cdot f(S(h_{(2)})h_{(3)} \cdot m) = (h_{(1)} \bullet f)(h_{(2)} \cdot m) && \text{by (7.12)} \\
&= (\varepsilon_t(h_{(1)}) \bullet f)(h_{(2)} \cdot m) = (S(1_{(1)}) \bullet f)(1_{(2)}h \cdot m) && \text{by (7.21)} \\
&= (\varepsilon_t(1_{(1)}) \bullet f)(1_{(2)}h \cdot m) = (1_{(1)} \bullet f)(1_{(2)}h \cdot m) && \text{by Lemma 7.1.2} \\
&= 1_{(1)} \cdot f(S(1_{(2)})1_{(3)}h \cdot m) = 1_{(1)} \cdot f(S(1_{(2)})h \cdot m) && \text{by Lemma 7.1.2} \\
&= (1 \bullet f)(h \cdot m) = f(h \cdot m),
\end{aligned}$$

for all $h \in H$, $m \in M$, hence $f$ is $H$-linear.

(iii) The algebra isomorphism

$$\begin{aligned}
\varphi : A &\to \mathrm{End}_A(A_A) \\
a &\mapsto (b \mapsto ab)
\end{aligned}$$

is an $H$-module algebra morphism, since for $h \in H$ and $a, b \in A$:

$$\begin{aligned}
(h \bullet \varphi(a))(b) &= h_{(1)} \cdot (a(S(h_{(2)}) \cdot b)) = (h_{(1)} \cdot a)(h_{(2)}S(h_{(3)}) \cdot b) \\
&\overset{(7.20)}{=} (1_{(1)}h \cdot a)(1_{(2)} \cdot b) = (h \cdot a)b = (\varphi(h \cdot a))(b).
\end{aligned}$$

(iv) Clearly, $Q := \mathrm{Hom}_A(M, A)$ is a right $\mathrm{End}_A(M)$-module via composition. It is an $H$-module via

$$(h \bullet g)(m) = h_{(1)} \cdot g(S(h_{(2)}) \cdot m) \tag{8.26}$$

for $g \in \mathrm{Hom}_A(M, A)$. The proof works analogous to (iii). And also similar to (iii) we get that the structures are compatible.

(v) $M$ is an $(\mathrm{End}_A(M), A)$-bimodule via $fm = f(m)$. The structure is compatible with the $H$-module structure since if $f \in \mathrm{End}_A(M)$, $h \in H$, $m \in M$, then

$$\begin{aligned}
(h_{(1)} \bullet f)(h_{(2)} \cdot m) &= h_{(1)} \cdot f(S(h_{(2)})h_{(3)} \cdot m) \overset{(7.21)}{=} h1_{(1)} \cdot f(S(1_{(2)}) \cdot m) \\
&= h \cdot (1 \bullet f)(m) = h \cdot f(m) = h \cdot \, fm.
\end{aligned}$$

The $A$-module structure on $\mathrm{Hom}_A(M, A)$ is given by $(ag)(m) = ag(m)$. This structure clearly interchanges with the $\mathrm{End}_A(M)$-module structure. It is compatible with the $H$-module structure since if $g \in \mathrm{Hom}_A(M, A)$, $h \in H$, $m \in M$, and $a \in A$, then

$$\begin{aligned}
((h_{(1)} \cdot a)(h_{(2)} \bullet g))(m) &= (h_{(1)} \cdot a)(h_{(2)} \cdot g(S(h_{(3)}) \cdot m)) \\
&= h_{(1)} \cdot (ag(S(h_{(2)}) \cdot m)) = (h \bullet (ag))(m). \qquad \square
\end{aligned}$$

**Corollary 8.5.2.** *The dual version of the proposition implies for a right $H$-comodule algebra $A$ and $M \in \mathcal{M}_A^H$:*

(i) $B := \mathrm{End}_A(M)$ *is a right $H$-comodule algebra with an $H$-comodule structure defined by* $f_{(0)}(m) \otimes f_{(1)} = f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)} S(m_{(1)})$.

(ii) *The coinvariants $B^{coH}$ of $B$ are exactly the $H$-colinear morphisms in* $\mathrm{End}_A(M)$, *denoted by* $\mathrm{End}_A^H(M)$. *In particular,* $B^{coH} \cong k$ *if $M$ is a simple object in* $\mathcal{M}_A^H$.

(iii) $\mathrm{End}_A(A_A) \cong A$ *as $H$-comodule algebras, and in particular,* $A^{coH} \cong k$ *if $A$ is simple in* $\mathcal{M}_A^H$.

(iv) $Q := \mathrm{Hom}_A(M, A) \in \mathcal{M}_B^H$.

(v) $M \in {}_B\mathcal{M}_A^H$ *and* $Q \in {}_A\mathcal{M}_B^H$.

*Remark* 8.5.3. The fact that the coinvariants of $A$ coincide with $\mathrm{End}_A^H(A)$ was already shown in [Zha10, Proposition 2.5]. More precisely, it is shown that for $M \in \mathcal{M}_A^H$

$$\mathrm{Hom}_A^H(A, M) \to M^{coH}, \quad f \mapsto f(1_A)$$

is an isomorphism of vector spaces with inverse $m \mapsto (a \mapsto ma)$, and it is multiplicative in the case when $M = A$.

### 8.5.2 Morita Equivalence for Weak Hopf Modules

Morita theory (see Appendix A.1) states that a progenerator in $M \in \mathcal{M}_A$ induces an equivalence of categories $\mathrm{Hom}_A(M, -) : {}_A\mathcal{M} \to {}_B\mathcal{M}$, where $B := \mathrm{End}_A(M)$. If now $A$ is an $H$-comodule algebra and $M \in \mathcal{M}_A^H$, then this equivalence can be restricted to an equivalence of the categories of weak Hopf modules $\mathcal{M}_A^H \approx \mathcal{M}_B^H$. Again, we prove the dual version.

**Proposition 8.5.4.** *Let $A$ be a right $H$-module algebra and $M \in ({}_H\mathcal{M})_A$. Assume furthermore that $M$ is a progenerator in $\mathcal{M}_A$. Let $B$ and $Q$ be defined as in Lemma 8.5.1. Then*

$$\begin{aligned} \mathrm{Hom}_A(M, -) : ({}_H\mathcal{M})_A &\to ({}_H\mathcal{M})_B, \\ \mathrm{Hom}_B(Q, -) : ({}_H\mathcal{M})_B &\to ({}_H\mathcal{M})_A \end{aligned}$$

*are well-defined mutually inverse functors.*

*Proof.* We know from Morita theory that the functors $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_B(Q, -)$ are well-defined mutually inverse functors ${}_A\mathcal{M} \leftrightarrow {}_B\mathcal{M}$. We have to show that the restrictions to $({}_H\mathcal{M})_A$ and $({}_H\mathcal{M})_B$ are well-defined.

Let $N \in ({}_H\mathcal{M})_A$, then $\mathrm{Hom}_A(M, N) \in ({}_H\mathcal{M})_B$ with an $H$-module structure given by

$$(h \bullet f)(m) = h_{(1)} \cdot f(S(h_{(2)}) \cdot m).$$

The $B$-module structure is as usual the composition of homomorphisms. The proof is analogous to Lemma 8.5.1 (iv). If $\varphi : N \to N'$ is a morphism in $({}_H\mathcal{M})_A$, then $\mathrm{Hom}_A(M, \varphi)$ is also $H$-linear.

On the other hand, if $N \in ({}_H\mathcal{M})_B$, then $\mathrm{Hom}_B(Q, N)$ is an $A$-module via $((fa)(q)) = f(aq)$ as usual, where $(aq)(m) = aq(m)$ as in Lemma 8.5.1. It is an $H$-module via

$$(h \bullet f)(q) = h_{(1)} \cdot f(S(h_{(2)}) \bullet q),$$

where $Q \in ({}_H\mathcal{M})_B$ with $(h \bullet q)(m) = h_{(1)} \cdot q(S(h_{(2)}) \cdot m)$ as in Lemma 8.5.1. The structures are compatible:

$$\begin{aligned}
(h_{(1)} \bullet f)(h_{(2)} \cdot a)(q) &= (h_{(1)} \bullet f)((h_{(2)} \cdot a)q) \\
&= h_{(1)} \cdot f((S(h_{(3)})h_{(4)} \cdot a)(S(h_{(2)}) \bullet q)) \\
&= h_{(1)} \cdot f((S(1_{(2)}) \cdot a)(S(h_{(2)}1_{(1)}) \bullet q) \qquad \text{by (7.21)} \\
&= h_{(1)} \cdot f(a(S(h_{(2)}) \bullet q)) = (h \bullet (fa))(q),
\end{aligned}$$

for $h \in H$, $n \in N$, $m \in M$, and $q \in Q = \mathrm{Hom}_A(M, A)$. And again, if $\varphi : N \to N'$ is an $({}_H\mathcal{M})_B$-morphism, then $\mathrm{Hom}_B(M, \varphi)$ is also $H$-linear.

If $N \in ({}_H\mathcal{M})_A$, then

$$\begin{aligned}
N &\cong \mathrm{Hom}_B(Q, \mathrm{Hom}_A(M, N)) \qquad \text{in } ({}_H\mathcal{M})_A \\
n &\mapsto F_n, \qquad \text{where } F_n(q)(m) = nq(m).
\end{aligned}$$

This is an isomorphism of $A$-modules by Morita theory. It is $H$-linear since

$$\begin{aligned}
((h \bullet F_n)(q))(m) &= (h_{(1)} \bullet (F_n(S(h_{(2)}) \bullet q))(m) \\
&= h_{(1)} \cdot ((F_n(S(h_{(3)}) \bullet q))(S(h_{(2)}) \cdot m)) \\
&= h_{(1)} \cdot (n(S(h_{(4)}) \cdot q(S^2(h_{(3)})S(h_{(2)}) \cdot m))) \\
&= (h_{(1)} \cdot n)(h_{(2)}S(1_{(2)}h_{(3)}) \cdot q(S^2(1_{(1)}) \cdot m)) \qquad \text{by (7.21)} \\
&= (1_{(1)}h \cdot n)(1_{(2)}(S(1) \bullet q)(m)) \qquad \text{by (7.20)} \\
&= (h \cdot n)q(m) = (F_{h \cdot n}(q))(m),
\end{aligned}$$

for $h \in H$, $n \in N$, $m \in M$, and $q \in Q = \mathrm{Hom}_A(M, A)$. Finally if $N \in ({}_H\mathcal{M})_B$ then

$$\begin{aligned}
N &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(Q, N)) \qquad \text{in } ({}_H\mathcal{M})_B \\
n &\mapsto F_n, \qquad \text{where } F_n(m)(q) = n(mq).
\end{aligned}$$

As usual, $mq \in B$ with $(mq)(\tilde{m}) = m(q(\tilde{m}))$.

By Morita, this is an isomorphism of $B$-modules; and it is $H$-linear for if $h \in H$, $n \in N$, $m, \tilde{m} \in M$, and $q \in Q = \mathrm{Hom}_A(M, A)$ then

$$\begin{aligned}
((h \bullet F_n)(m))(q) &= (h_{(1)} \bullet F_n(S(h_{(2)}) \cdot m))(q) \\
&= h_{(1)} \cdot ((F_n(S(h_{(3)}) \cdot m))(S(h_{(2)}) \bullet q)) \\
&= h_{(1)}(n((S(h_{(3)}) \cdot m)(S(h_{(2)}) \bullet q))) \\
&= h_{(1)} \cdot (n(S(h_{(2)}) \bullet (mq))) && \text{by } (*) \text{ below} \\
&= (h_{(1)} \cdot n)(h_{(2)}S(h_{(3)}) \bullet (mq)) \\
&= (1_{(1)}h \cdot n)(1_{(2)}(mq)) && \text{by (7.20)} \\
&= (F_{h \cdot n}(m))(q),
\end{aligned}$$

where $(*)$ holds since

$$\begin{aligned}
(h \bullet (mq))(\tilde{m}) &= h_{(1)}(mq(S(h_{(2)}) \cdot \tilde{m})) \\
&= (h_{(1)} \cdot m)(h_{(2)} \cdot q(S(h_{(3)}) \cdot \tilde{m})) \\
&= (h_{(1)} \cdot m)((h_{(2)} \bullet q)(\tilde{m})) \\
&= ((h_{(1)} \cdot m)(h_{(2)} \bullet q))(\tilde{m}).
\end{aligned}$$

□

**Corollary 8.5.5.** *Let $A$ be a right $H$-comodule algebra and $M \in \mathcal{M}_A^H$. Assume furthermore that $M$ is a progenerator for $\mathcal{M}_A$. Let $B$ and $Q$ be defined as in Lemma 8.5.2. Then*

$$\begin{aligned}
\mathrm{Hom}_A(M, -) &: \mathcal{M}_A^H \to \mathcal{M}_B^H, \\
\mathrm{Hom}_B(Q, -) &: \mathcal{M}_B^H \to \mathcal{M}_A^H
\end{aligned}$$

*are well-defined mutually inverse functors.*

# Chapter 9

# Projectivity and Freeness over $\boldsymbol{H}$-Comodule Algebras

Etingof and Ostrik have proven that surjective tensor functors map projective objects to projective ones [EO04, Theorem 2.5] and this implies in particular that weak Hopf algebras are projective over their weak Hopf subalgebras. However, one can not expect a weak Hopf algebra version of the Hopf algebra Freeness Theorem by Nichols and Zöller [NZ89] or even of Skryabin's freeness theorem for coideal subalgebras [Skr07]. In this chapter an example of a Frobenius weak Hopf algebra will be constructed, which has a non-Frobenius weak Hopf subalgebra over which it is not free.

We will see that weak Hopf algebras, which are free over their bases, are Frobenius. And in this case weak Hopf modules over quasi-Frobenius $H$-simple $H$-comodule algebras are projective. This is a weak Hopf algebra version of [Skr07, Theorem 3.5]. For an ordinary Hopf algebra $H$ Skryabin has proven that if $M$ is a Hopf module over an $H$-simple $H$-comodule algebra $A$, then there exists a natural number $n$ such that a direct sum of $n$ copies of $M$ is a free $A$-module. It will not be possible to generalize Skryabin's proof to arbitrary weak Hopf algebras, since my example of a weak Hopf algebra which is not free over a certain weak Hopf subalgebra, is also a counter example for Skryabin's stronger result. Nevertheless, I conjecture that for any weak Hopf algebra $H$, weak Hopf modules over quasi-Frobenius $H$-simple $H$-comodule algebras are projective. In the next chapter module categories over weak Hopf algebras that satisfy this conjecture will be classified.

As mentioned before, Skryabin has also shown that, in the Hopf algebra case, coideal subalgebras are a special case of quasi-Frobenius $H$-simple $H$-comodule algebras. For weak Hopf algebras this is not true in general. The easiest example would be to consider the weak Hopf algebra itself, which is not $H$-simple in many cases. Moreover, Skryabin could prove in [Skr07, Theorem 4.2] that in fact all finite dimensional $H$-simple $H$-comodule algebras are quasi-

Frobenius (actually even Frobenius). It is not known whether this is also true for weak Hopf algebras.

## 9.1 $H$-Costable Ideals in Weak Hopf Algebras

**Proposition 9.1.1.** *Let $H$ be a weak Hopf algebra. Then the following properties are equivalent:*

(1) *$H$ is $H$-simple in the sense that is it does not contain a nonzero proper $H$-costable ideal;*

(2) *$H_t$ is a simple object in ${}_H\mathcal{M}$, where the $H$-module structure of $H_t$ is given by $h \rightharpoonup x = \varepsilon_t(hx)$ as in Propostion 7.2.1.*

*Proof.* Vecsernyés [Vec03, Theorem 2.4] has shown that $H_t \cong \bigoplus_{\alpha\in\mathcal{A}} H_t\alpha$ as left $H$-modules, where $\mathcal{A}$ is the set of primitive orthogonal idempotents of $H_t \cap \mathrm{Center}(H)$. Moreover, $H_t\alpha$ is a simple $H$-module for each $\alpha \in \mathcal{A}$.

If $H_t$ is not a simple $H$-module, than there exist $\alpha \neq \beta \in \mathcal{A}$. Then $J := H\alpha$ is a nonzero $H$-costable ideal in $H$ by (7.12), since $\alpha \in H_t \cap \mathrm{Center}(H)$. It is a proper ideal, as $\beta \notin J$.

On the other hand, assume now that $H_t$ is a simple $H$-module and let $J$ be a nonzero $H$-costable ideal in $H$. Then $J$ is an $\mathcal{M}_H^H$-subobject of $H$, and in particular $J \cong J^{coH} \otimes_{H_t} H$ by the structure theorem for weak Hopf modules (see Theorem 8.4.6 or [BNS99, Theorem 3.9]). However, $J^{coH} = J \cap H^{coH} = J \cap H_t$ and it is an $H$-submodule of $H_t$. In fact, if $x \in J \cap H_t$ and $h \in H$, then $h \rightharpoonup x = \varepsilon_t(hx) = h_{(1)}xS(h_{(2)}) \in J$ (see Proposition 7.2.1). Thus, the simplicity of $H_t$ yields $J^{coH} = H_t$, and therefore $J = H$. □

## 9.2 Frobenius Weak Hopf Algebras and Freeness over the Bases

Unlike Hopf algebras, weak Hopf algebras are not Frobenius in general [IK09]. However, they are quasi-Frobenius (see Appendix A.3 for the definition), which was shown by Böhm, Nill, and Szlachányi [BNS99, Theorem 3.11]. They also gave necessary and sufficient conditions for weak Hopf algebras to be Frobenius:

**Theorem 9.2.1.** [BNS99, Theorem 3.16] *Define the space of right integrals in $H$ by*

$$\mathcal{I}^R(H) := \{r \in H \mid rh = r\varepsilon_s(h)\,\forall h \in H\}.$$

*Then the following conditions are equivalent:*

(i) *$H$ is a Frobenius algebra;*

(ii) $\dim_k \mathcal{I}^R(H) = \dim_k H_t$*;*

(iii) *There exist non-degenerate integrals in $H$;*

(iv) *$H^*$ is a Frobenius algebra.*

**Proposition 9.2.2.** *Let $H$ be a weak Hopf algebra. If $H$ is a free right $H_s$- or $H_t$-module, then $H$ is a Frobenius algebra.*

*Proof.* Assume that $H$ is a free right $H_s$-module. Then $\dim_k H^* = \dim_k H = \mathrm{rank}_{H_s} H \ \dim_k H_s$ and $H^* \cong H \otimes_{H_s} \mathcal{I}^R(H^*) \cong \mathcal{I}^R(H^*)^{(\mathrm{rank}_{H_s} H)}$ as vector spaces by the opcop version of [BNS99, Theorem 3.9]. Hence, $\dim_k H^* = \dim_k \mathcal{I}^R(H^*)\ \mathrm{rank}_{H_s} H$. This implies $\dim_k \mathcal{I}^R(H^*) = \dim_k H_s = \dim_k H_t^*$ [BNS99, Lemma 2.6] and from the theorem follows that $H$ is Frobenius.
The case when $H$ is a free right $H_t$-module is the cop-version. □

*Remark* 9.2.3. The converse of the lemma does not hold. Nikshych and Vainerman gave an example of a semisimple (so in particular Frobenius) weak Hopf algebra which is not a free $H_t$- or $H_s$-module [NV00, Example 7.3]. This is also an example of a weak Hopf algebra, which has an $H$-comodule algebra over which it is not free.

Iovanov and Kadison [IK09] gave a different condition for a weak Hopf algebra to be Frobenius:

**Proposition 9.2.4.** [IK09, Theorem 2.2] *Let $H$ be a weak Hopf algebra and $H_t$ its base algebra. Assume that all simple $H_t$-modules have the same dimension, then $H$ is a Frobenius algebra.*

Furthermore, Iovanov and Kadison have constructed a weak Hopf algebra which is not a Frobenius algebra:

*Example* 9.2.5. [IK09] Let $B$ be the Taft Hopf algebra $k < g, x \,|\, g^p = 1,\ x^p = 0,\ xg = \lambda gx >$, where $p$ is prime and $\lambda$ is a primitive $p$-th root of unity, $g$ is a group-like element, and $x$ is skew-primitive with $\Delta(x) = g \otimes x + x \otimes 1$ and $S(x) = -g^{-1}x$. The Jacobson radical $J_B$ is $Bx$, as it is obviously a nilpotent ideal, and so the simple right $B$-modules are the one-dimensional modules $V_0 := k_\varepsilon, V_1, \ldots, V_{p-1}$, where for all $i$ and $v_i \in V_i$:

$$g \cdot v_i = \lambda^i v_i, \qquad\qquad x \cdot v_i = 0.$$

Denote by $D$ the Hopf algebra $k < g \,|\, g^p = 1 >$. Let $\mathcal{B}$ denote the finite tensor category of finite dimensional left $B$-modules. Let $A = M_{d_0}(k) \times \ldots \times M_{d_{p-1}}(k)$ be a separable algebra, denote by $S_0, \ldots S_{p-1}$ the corresponding non-isomorphic simple right $A$-modules with $\dim_k S_i = d_i$. Let $\hat{S}_i := \mathrm{Hom}_k(S_i, k)$ with the usual left $A$-module structure. Then the $\hat{S}_i \otimes S_j$, $0 \le i, j \le p-1$ are the simple objects in ${}_A\mathcal{M}_A$. Define a functor $F : \mathcal{B} \to {}_A\mathcal{M}_A$ into the monoidal category of $A$-bimodules

$$\begin{array}{rcl} F: \mathcal{B} & \xrightarrow{F_1} \mathcal{M}_D^{fd} \xrightarrow{F_2} {}_D\mathcal{M}_D^{fd} \xrightarrow{F_3} & {}_A\mathcal{M}_A^{fd} \\ F_1({}_BV) & = & {}_DV \\ F_2(V_k) & = & \bigoplus_{j=0}^{p-1} V_j^* \otimes V_{k+j} \\ F_3(V_i^* \otimes V_j) & = & \hat{S}_i \otimes S_j. \end{array}$$

Here, the tensor structure of the categories of bimodules is given by the tensor product over the algebra, whereas the tensor structure of the categories of modules over a Hopf algebra is the tensor product over $k$ with diagonal structure and $V^*$ indicates the right dual in that category.

**Proposition 9.2.6.** [IK09, Proposition 2.5] *$F$ above is a tensor functor and the reconstructed weak Hopf algebra $K := \mathrm{End}_k(F)$ is not Frobenius unless $d_i = d_j$ for all $i$ and $j$.*

Note that the functor $F_2$ is defined slightly different then in [IK09], because otherwise $F$ would not map $V_0$ to $A$ (the unit object of ${}_A\mathcal{M}_A$). In fact, if $F_2$ is defined as in [IK09], then $\dim(F(V_0)) = \sum_i d_i d_{p-i} \neq \dim(A) = \sum_i d_i^2$.

However the proof works analogously and the arguments to show that $F_2$ is a tensor functor are the same. We only have to check that if for all $k$: $\sum_j d_j d_{k+j} = \dim_k(F(V_k)) = \dim_k(F(\mathrm{soc}(P_B(V_k)))) = \dim(F(V_{k+1})) = \sum_j d_j d_{k+1+j}$, then all the $d_i$'s are equal (see proof of Proposition 9.4.2 below). This is in fact true, since in particular $\sum_j d_j^2 = \sum_j d_j d_{k+j}$ for all $k$ and the Cauchy-Schwarz inequality implies that $d_i = d_j$ for all $i$ and $j$.

## 9.3 Projectivity over Weak Hopf Subalgebras

**Definition.** [NV02, Section 2.1] A **morphism of weak Hopf algebras** $H$ and $H'$ is an algebra and coalgebra map $\alpha : H \to H'$ with $S' \circ \alpha = \alpha \circ S$. A subalgebra $K \subset H$ is a **weak Hopf subalgebra** if the inclusion map is a weak Hopf algebra morphism. In particular, $K_t = H_t$ and $K_s = H_s$.

**Definition.** [EO04] Let $\mathcal{C}$ and $\mathcal{D}$ be multi-tensor categories and $F : \mathcal{C} \to \mathcal{D}$ a tensor functor. $F$ is called **surjective** if all objects of $\mathcal{B}$ are sub-quotients of images of objects of $\mathcal{C}$.

**Theorem 9.3.1.** [EO04, Theorem 2.5 and Section 3] *Let $\mathcal{C}$ and $\mathcal{D}$ be multi-tensor categories and $F : \mathcal{C} \to \mathcal{D}$ a surjective tensor functor. Then $F$ maps projective objects to projective ones.*

**Lemma 9.3.2.** *Let $K \subset H$ be a weak Hopf subalgebra. Then the functor*

$$\text{restr.} : {}_H\mathcal{M}^{fd} \to {}_K\mathcal{M}^{fd}, \qquad {}_HV \mapsto {}_KV$$

*is a surjective tensor functor.*

*Proof.* If $V$ is a left $K$-module, then ${}_KV \cong K \otimes_K V$ as left $K$-modules, and this is a quotient of $K \otimes V$, which in turn is a $K$-submodule of $H \otimes V$. □

**Theorem 9.3.3.** *If $H$ is a weak Hopf algebra and $K$ a weak Hopf subalgebra of $H$, then $H$ is a projective right and left $K$-module.*

*Proof.* Theorem 9.3.1 implies that the functor restr. from the lemma maps projective objects to projective ones. □

## 9.4 Freeness over Weak Hopf Subalgebras — a Counterexample

Suppose the Nichols-Zöller Theorem would hold for weak Hopf algebras. Then for a weak Hopf subalgebra $K \subset H$ there would exist an $n \in \mathbb{N}$ with $H \cong K^{(n)}$ as left and right $K$-modules. If now $H$ is Frobenius, then so would be $K$, because then $(K^*)^{(n)} \cong H^* \cong H \cong K^{(n)}$ as right $K$-modules. In this section we will construct a Frobenius weak Hopf algebra $H$, which has a non-Frobenius weak Hopf subalgebra $K$, and therefore $H$ can not be a free right and left $K$-module. In our example, $K$ is the reconstructed weak Hopf algebra from Proposition 9.2.6.

**Lemma 9.4.1.** *Let $\mathcal{C}$ and $\mathcal{B}$ be finite tensor categories and $G : \mathcal{C} \to \mathcal{B}$ a surjective tensor functor which satisfies*

$$(*) \qquad G(V \otimes W) = G(V) \otimes G(W), \qquad G(\mathbb{1}_{\mathcal{C}}) = \mathbb{1}_{\mathcal{B}}, \qquad G(V^*) = G(V)^*,$$

*for all $V, W \in \mathcal{C}$. Let $A$ be a separable algebra and $F : \mathcal{B} \to {}_A\mathcal{M}_A^{fd}$ a tensor functor. Let $K := \mathrm{End}_k(F)$ and $H := \mathrm{End}_k(FG)$ be the reconstructed weak Hopf algebras as in Theorem 7.2.3. Then $K$ is a weak Hopf subalgebra of $H$ with $K \hookrightarrow H$, $\eta \mapsto \tilde{\eta}$, where $\tilde{\eta}_X = \eta_{G(X)}$. Moreover, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C} \arrow[rrr, "I_{FG}"] \arrow[dr, "G"'] & & & {}_H\mathcal{M}^{fd} \arrow[dl, "\text{restr.}"'] \arrow[ddll, bend left, "\mathrm{base}_H"] \\
& \mathcal{B} \arrow[r, "I_F"] \arrow[d, "F"'] & {}_K\mathcal{M}^{fd} \arrow[dl, "\mathrm{base}_K"] & \\
& {}_A\mathcal{M}_A. & &
\end{tikzcd}$$

*Proof.* The map $K \hookrightarrow H$ is an inclusion since $G$ is surjective. Clearly, it is an algebra morphism. Moreover, $\Delta_H(\tilde{\eta}) = \Delta_K(\eta)$, $\varepsilon_H(\tilde{\eta}) = \varepsilon_K(\eta)$, and $S_H(\tilde{\eta}) = S_K(\eta)$ because of (*); hence it is a weak Hopf algebra morphism.

For the commutativity of the diagram, we only have to check that the upper quadrangle commutes. In fact, the lower and the outer triangle commute by reconstruction theory (Theorem 7.2.3), and obviously also the right triangle commutes. The quadrangle commutes since restr.$(FG(V), \mu_V) = (F(G(V)), \mu_{G(V)})$ for all $V \in \mathcal{C}$, and therefore restr. $I_{FG} = I_F\, G$. □

**Proposition 9.4.2.** *Let $B$ be as in Proposition 9.2.6 and let $C$ be the Hopf algebra $k < g, x, y \,|\, g^p = 1,\, x^p = y^p = 0,\, xg = \lambda gx, yg = \lambda^{-1} gy, xy = \lambda yx >$, where the Hopf algebra structure of $g$ and $x$ is as for $B$ and also $y$ is $(g,1)$-skew-primitive. Hence $B \subset C$ is a Hopf subalgebra. Denote by $\mathcal{C}$ and $\mathcal{B}$ the finite tensor categories of finite dimensional left $C$-modules and $B$-modules, respectively. Let the algebra $A$, the $d_i$'s, and the tensor functor $F : \mathcal{B} \to {}_A\mathcal{M}_A^{fd}$ be defined as in Proposition 9.2.6, and denote by $G$ the restriction functor* restr. $: \mathcal{C} \to \mathcal{B}$. *Let $H := \mathrm{End}_k(FG)$ and $K := \mathrm{End}_k(F)$ be the reconstructed weak Hopf algebras with base $A$. Then*

(i) *$H$ is a Frobenius algebra;*

(ii) *If there exist $i$ and $j \in \{0, \dots, p-1\}$ with $d_i \neq d_j$, then for every $m \in \mathbb{N}$, $H^{(m)}$ is not a free left $K$-module.*

*Proof.* $G$ is a surjective tensor functor by 9.3.2 and it satisfies (*). Therefore $K$ is a weak Hopf subalgebra of $H$ and the diagram from above commutes.

The Jacobson radical $J_C$ of $C$ is $Cx + Cy$ (as it is nilpotent). Denote by $V_0, \dots, V_{p-1}$ the simple $C$ or $B$ modules (as in 9.2.6) with $x \cdot v_i = y \cdot v_i = 0$ and $g \cdot v_i = \lambda^i v_i$ for all $i$, and $v_i \in V_i$. Note that $V_i \otimes V_j \cong V_{i+j}$. Denote by $P_C(V_i)$ and $P_B(V_i)$ the corresponding projective covers in $\mathcal{C}$ and $\mathcal{B}$, respectively. By [IK09, Example 2.4] we know that for all $i$

$$\mathrm{soc}(P_B(V_i)) \cong V_{i+1},$$

where $V_p = V_0$. In contrast, $C$ is symmetric (see Appendix A.4) by [Lor97, 2.5 and 4.2], since $C$ is unimodular with left and right integral $\sum_{i=0}^{p-1} g^i x^{p-1} y^{p-1}$ and $S^2$ is inner. Hence, for all $i$

$$\mathrm{soc}(P_C(V_i)) \cong V_i.$$

To see that $H$ is Frobenius, we have to check that for all principle indecomposable modules $P$ of $H$, $\dim(P/\mathrm{Jac}(H)P) = \dim(\mathrm{soc}(P))$ (see Appendix A.4.1). The diagram in Lemma 9.4.1 implies that this is true since for all $i = 0, \dots, p-1$ $\dim(FG(\mathrm{soc}(P_C(V_i))) = \dim(F(V_i)) = \dim(F(P_C(V_i)/J_C P_C(V_i)).$

In [Lor97, 4.2] and [IK09, Example 2.4] it was shown that

$$P_C(V_0) = k < c^{(r,s)} | 0 \leq r, s \leq p-1 >, \text{ where } \quad c^{(r,s)} = \sum_{i=0}^{p-1} x^r y^s g^i;$$

$$\text{and } \forall r: \quad P_B(V_r) = k < b_r^{(0)}, b_r^{(1)}, \dots, b_r^{(p-1)} >, \text{ where } \quad b_r^{(s)} := \sum_{i=0}^{p-1} \lambda^{i(s-r)} g^i x^s.$$

Moreover, the Jacobson radicals $J_C$ and $J_B$ are Hopf ideals and therefore

$$P_C(V_i) \cong P_C(V_0 \otimes V_i) \cong P_C(V_0) \otimes V_i \quad \text{and} \quad P_B(V_i) \cong P_B(V_0) \otimes V_i$$

for all $i$ (see for example [Lor97, 3.3]). Hence $G(P_C(V_0)) \cong \bigoplus_{j=0}^{p-1} P_B(V_j)$, since $x \cdot c^{(s,r)} = c^{(s+1,r)}$, $g \cdot c^{(s,r)} = \lambda^{s-r} c^{(s,r)}$ and on the other hand $x \cdot b_r^{(s)} = b_r^{(s+1)}$, $g \cdot b_r^{(s)} = \lambda^{s-r} b_r^{(s)}$. This yields

$$G(P_C(V_i)) \cong G(P_C(V_0) \otimes V_i) \cong G(P_C(V_0)) \otimes V_i \cong \bigoplus_{j=0}^{p-1} P_B(V_j) \otimes V_i \cong \bigoplus_{j=0}^{p-1} P_B(V_j)$$

for all $i$.

Suppose that $H^{(m)}$ is a free left $K$-module for some $m \in \mathbb{N}$, that is $H^{(m)} \cong K^{(n)}$ as $K$-modules for some $n$. Then

$$\bigoplus_{i=0}^{p-1} \text{restr.}(I_{FG}(P_C(V_i)))^{(m\,n_i)} \cong \text{restr.}({}_H H^{(m)}) \cong {}_K K^{(n)} \cong \bigoplus_{i=0}^{p-1} I_F(P_B(V_i))^{(n\,n_i)},$$

where $n_i = \dim(I_F(V_i)) = \dim(F(V_i)) = \sum_{j=0}^{p-1} d_j d_{i+j}$ and indices are modulo $p$. The foregoing observation and the commutativity of the diagram in Lemma 9.4.1 imply

$$\begin{aligned} \bigoplus_{i=0}^{p-1} \text{restr.}(I_{FG}(P_C(V_i)))^{(m\,n_i)} &\cong \bigoplus_{i=0}^{p-1} (I_F G(P_C(V_i)))^{(m\,n_i)} \\ &\cong \bigoplus_{i=0}^{p-1} (I_F(P_B(V_i)))^{(\sum_j m\,n_j)}. \end{aligned}$$

Hence if $H^{(m)}$ is a free $K$ module of rank $n$ then $\sum_j m\,n_j = n\,n_i$ for all $i$ and therefore $n_0, \dots n_{p-1}$ are pairwise equal. In particular, $\sum_i d_i^2 = \sum_i d_i d_{i+j}$ for all $j$; and by Cauchy-Schwarz $d_i = d_j$ for all $i$ and $j$. $\square$

*Remark* 9.4.3. In this example, $H$ is a weak Hopf module in $\mathcal{M}_K^H$ and $K$ is a quasi-Frobenius $H$-simple $H$-comodule algebra. In fact, $H_t = K_t$ is a simple object in ${}_K\mathcal{M}^{fd}$, since it is the unit object, and the unit object in $\mathcal{B}$ is simple. By Lemma 9.1.1, $K$ is $K$-simple as a $K$-comodule algebra, and therefore also

$H$-simple as an $H$-comodule algebra. $K$ is quasi-Frobenius, because weak Hopf algebras are quasi-Frobenius by [BNS99, Theorem 3.11]. Hence, as I have already mentioned in the introduction of this chapter, for arbitrary weak Hopf algebras [Skr07, Theorem 3.5] is not true in general, and therefore in particular its proof can not be transferred to weak Hopf algebras. However, for the case when $H$ is free as a right $H_s$-module, we can prove that if $A$ is a finite dimensional $H$-simple $H$-comodule algebra and $M \in \mathcal{M}_A^H$, then a finite direct sum of copies of $M$ is a free $A$-module. This will be done in the next section with a different proof then that of [BNS99, Theorem 3.11].

## 9.5 Projectivity of Weak Hopf Modules over Quasi-Frobenius $H$-Simple $H$-Comodule Algebras

**Proposition 9.5.1.** *Let $H$ be a weak Hopf algebra which is free as a right $H_s$-module. Let $A$ be a finite dimensional $H$-simple right $H$-comodule algebra which is quasi-Frobenius. Then every finite dimensional object $M \in \mathcal{M}_A^H$ is a projective $A$-module.*

The proof of the proposition uses the same idea as the proof of the Nichols-Zöller-Theorem for Hopf algebras.

*Proof.* Let $t$ be the rank of $H$ as an $H_s$-module. Let $0 \neq M \in \mathcal{M}_A^H$. It is known from 8.3.2 that $M$ is a generator for $\mathcal{M}_A$ and so in particular $M$ is faithful. It is a well known fact that since $A$ is quasi-Frobenius there exists an integer $r$ such that $M^{(r)} \cong F \oplus E$, where $F$ is a free $A$-module of rank $n \neq 0$ and and $E$ is not faithful, [CR62, §59] (analogous to [NZ89, Proposition 4]). The isomorphisms in Lemma 8.1.6 and the exactness of $V \underline{\otimes}^A -$ for a right $A$-module $V$, and of $- \underline{\otimes}^A H$ (see 8.1.5 and proof of 10.1.1) imply

$$
\begin{array}{l}
\qquad\quad \underbrace{M^{(r)} \underline{\otimes}^A H} \cong (H \otimes_{H_s} M.)^{(r)} \cong M^{(tr)} \cong F^{(t)} \oplus E^{(t)} \\
\qquad\qquad\quad \| \wr \\
\overbrace{\underbrace{F \underline{\otimes}^A H} \quad \oplus \quad E \underline{\otimes}^A H} \\
\qquad \| \wr \\
(A \underline{\otimes}^A H)^{(n)} \cong (H \otimes_{H_s} A.)^{(n)} \cong F^{(t)}
\end{array}
$$

which implies that $E \underline{\otimes}^A H \cong E^{(t)}$ by the Krull-Schmidt theorem. However, if $E \neq 0$, then $0 \neq E \underline{\otimes}^A H \in \mathcal{M}_A^H$ is again a generator for $\mathcal{M}_A$ by Corollary 8.3.2, which is a contradiction to the fact that $E$ is not faithful. Hence, $E = 0$ and $M^{(r)}$ is a free $A$-module. $\square$

*Remark* 9.5.2. In the proposition it suffices to assume that $A$ is right Kasch, because finite dimensional $H$-simple $H$-comodule algebras which are right Kasch (see Appendix A.3) are quasi-Frobenius by Corollary 8.3.3.

**Corollary 9.5.3.** *Let $H$ be a weak Hopf algebra which is a free left $H_s$-module and let $A$ be a finite dimensional quasi-Frobenius $H$-simple right $H$-comodule algebra. Let $M$ be a nonzero finite dimensional object in $\mathcal{M}_A^H$, then $M$ is a progenerator for $\mathcal{M}_A$.*

*Proof.* $M$ is a projective right $A$-module by the theorem. It is a generator by Corollary 8.3.2. □

**Conjecture 9.5.4.** *Let $H$ be a weak Hopf algebra and let $A$ be a finite dimensional $H$-simple right $H$-comodule algebra which is quasi-Frobenius. Then every finite dimensional object $M \in \mathcal{M}_A^H$ is a projective $A$-module.*

In the following chapter, exact module categories over weak Hopf algebras that satisfy this conjecture, will be classified. Hence, if the conjecture could be confirmed, then a full classification of exact module categories over any weak Hopf algebra could be achieved by means of the results in Theorem 10.2.6, in the sense that Andruskiewitsch and Mombelli's classification for the Hopf algebra case [AM07] could be generalized to arbitrary weak Hopf algebras.

# Chapter 10

# Module Categories over Weak Hopf Algebras

As we have seen in Chapter 7, the category $\mathcal{C}$ of finite dimensional left modules over a weak Hopf algebra $H$ is a finite multi-tensor category. Thus, it makes sense to define module categories over $\mathcal{C}$ and in this case we will refer to those as module categories over the weak Hopf algebra $H$.

In the following let $H$ be a (finite dimensional) weak Hopf algebra, and $\mathcal{C} := {}_H\mathcal{M}^{fd}$ the finite multi-tensor category of finite dimensional left $H$-modules.

## 10.1 Module Categories Induced by $H$-Comodule Algebras

In this section we will see that $H$-comodule algebras induce module categories over $H$, and we will investigate under which condition two such module categories are equivalent.

**Lemma 10.1.1.** *For a finite dimensional left $H$-comodule algebra $A$, ${}_A\mathcal{M}^{fd}$ is a module category over $\mathcal{C}$ with the diagonal structure. That is, as in Lemma 8.1.4, if $X \in \mathcal{C}$ and $M \in {}_A\mathcal{M}^{fd}$, then $X \otimes^A M$ is a left $A$-module via*

$$a(\sum_i x_i \otimes m_i) = \sum_i a_{(-1)}x_i \otimes a_{(0)}m_i,$$

*for $a \in A$, $\sum_i x_i \otimes m_i \in X \otimes^A M$. The associativity $m$ is the identity and the unit isomorphism is given by*

$$\ell_M : H_t \otimes^A M \to M, \quad \sum_i y_i \otimes m_i \mapsto \sum_i m_i \varepsilon(y_i).$$

*Proof.* For a finite dimensional left $A$-module $M$ the functor $-\otimes^A M$ is exact. In fact, if $X\in\mathcal{C}$, then $X\otimes^A M$ is a subspace of $X\otimes M$. Let now

$$0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0$$

be a short exact sequence in $\mathcal{C}$. Consider the sequence

$$0\to X\otimes^A M\xrightarrow{f\otimes\mathrm{id}|_{X\otimes^A M}} Y\otimes^A M\xrightarrow{g\otimes\mathrm{id}|_{Y\otimes^A M}} Z\otimes^A M\to 0$$

in ${}_A\mathcal{M}^{fd}$. Then $\ker(f\otimes\mathrm{id}|_{X\otimes^A M})\subset\ker f\otimes\mathrm{id}=0$, and $(g\otimes\mathrm{id}|_{Y\otimes^A M})\circ(f\otimes\mathrm{id}|_{X\otimes^A M})=gf\otimes\mathrm{id}|_{X\otimes^A M}=0$. Moreover, if $\sum_i y_i\otimes m_i=:y\in\ker(g\otimes\mathrm{id}|_{Y\otimes^A M})\subset\ker(g\otimes\mathrm{id})$, then there exist $\sum_j x_j\otimes m_j=:x\in X\otimes M$ with $(f\otimes\mathrm{id})(x)=y$, and therefore $(f\otimes\mathrm{id})(\sum_j 1^A_{(-1)}x_j\otimes 1^A_{(0)}m_j)=\sum_j 1^A_{(-1)}f(x_j)\otimes 1^A_{(0)}m_j=\sum_i 1^A_{(-1)}y_i 1^A_{(0)}\otimes m_i=y$. With a similar argument, one can show that $g\otimes\mathrm{id}|_{Y\otimes^A M}$ is surjective.

Obviously, the pentagon property is satisfied. It remains to check the properties of the unit morphism $\ell$. For $M\in{}_A\mathcal{M}^{fd}$, $\ell_M$ is $A$-linear since

$$\begin{aligned}\sum_i a_{(0)}m_i\varepsilon(a_{(-1)}\rightharpoonup y_i)&=\sum_i a_{(0)}m_i\varepsilon(a_{(-1)}y_i)\\&=\sum_i a_{(0)}1^A_{(0)}m_i\varepsilon(a_{(-1)}1^A_{(-1)})(1^A_{(-2)}y_i)\\&=\sum_i a1^A_{(0)}m_i\varepsilon(1^A_{(-1)}y_i) &&\text{by (7.3)}\\&=a(\sum_i 1^A_{(0)}m_i\varepsilon(1^A_{(-1)}\rightharpoonup y_i))\\&=a(\sum_i m_i\varepsilon(y_i))\end{aligned}$$

for $\sum_i y_i\otimes m_i\in H_t\otimes^A M$, where $H_t$ is the unit object of $\mathcal{C}$ with $H$-action $\rightharpoonup$ (see Proposition 7.2.1). We also have to show that $\ell$ satisfies (1.2), that is for $X\in\mathcal{C}$ and $M\in{}_A\mathcal{M}^{fd}$: $\mathrm{id}_X\otimes\ell_M=r_X\otimes\mathrm{id}_M$, where $r_X$ is the right unit constraint of $\mathcal{C}$ as in Proposition 7.2.1. In fact for $x\in X$, $y\in H_t$ and $m\in M$:

$$\begin{aligned}&(r_X\otimes\mathrm{id}_M)(1^A_{(-2)}x\otimes 1^A_{(-1)}\rightharpoonup y\otimes 1^A_{(0)}m)\\&=S^{-1}(\varepsilon_t(1^A_{(-1)}y))1^A_{(-2)}x\otimes 1^A_{(0)}m\\&=S^{-1}(\varepsilon_t(1^A_{(-1)}))1^A_{(-2)}S^{-1}(y)x\otimes 1^A_{(0)}m &&\text{by 7.1.2 and (7.22)}\\&=\varepsilon_s(S^{-1}(1^A_{(-1)})1^A_{(-2)})S^{-1}(y)x\otimes 1^A_{(0)}m &&\text{by 7.1.2, (8.7), (7.16), (7.17)}\\&=1^A_{(-1)}S^{-1}(y)x\otimes 1^A_{(0)}m &&\text{by 7.1.2 and (8.7)}\\&=1^A_{(-2)}S^{-1}(y)x\otimes 1^A_{(0)}m\varepsilon(1^A_{(-1)})\\&=1^A_{(-2)}x\otimes 1^A_{(0)}m\varepsilon(1^A_{(-1)}y) &&\text{by (7.22)}\\&=(\mathrm{id}_X\otimes\ell_M)(1^A_{(-2)}x\otimes 1^A_{(-1)}\rightharpoonup y\otimes 1^A_{(0)}m).\end{aligned}$$

□

### 10.1.1 Morita Contexts for Module Categories over $H$ Arising from $H$-Comodule Algebras

In the following sections we want to formulate a Morita equivalence for $H$-comodule algebras in the sense that we can decide under which conditions a Morita context $(A, M, Q, B, \alpha, \beta)$ for ${}_A\mathcal{M}^{fd} \approx {}_B\mathcal{M}^{fd}$, where $A$ and $B$ are $H$-comodule algebras, induces a $\mathcal{C}$-module category equivalence. As it turns out, this is the case whenever $M$ is a weak Hopf module. We have seen in Chapter 8 that in this case the Morita equivalence also induces an equivalence of the categories of relative weak Hopf modules over $A$ and $B$. The Hopf algebra version of the following results were proven in [AM07, Proposition 1.23 and 1.24].

**Lemma 10.1.2.** *Let $A$ and $B$ be finite dimensional left $H$-comodule algebras and $M$ a finite dimensional object ${}^H_B\mathcal{M}_A$, then $M \otimes_A - : {}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ is a $\mathcal{C}$-module functor, together with the natural isomorphism*

$$c^M : M \otimes_A (- \oslash^A -) \to - \oslash^B (M \otimes_A -),$$

*which is defined by*

$$\begin{aligned} c^M_{X,V} : M \otimes_A (X \oslash^A V) &\to X \oslash^B (M \otimes_A V) \qquad (10.1)\\ m \otimes_A (1^A_{(-1)}x \otimes 1^A_{(0)}v) &\mapsto m_{(-1)}1^A_{(-1)}x \otimes m_{(0)} \otimes_A 1^A_{(0)}v \\ &= m_{(-1)}x \otimes m_{(0)} \otimes_A v, \end{aligned}$$

*for $X \in \mathcal{C}$, $V \in {}_A\mathcal{M}^{fd}$.*

*Proof.* The inverse of $c^M_{X,V}$ is given by

$$\begin{aligned} 1^B_{(-1)}x \otimes 1^B_{(0)}m \otimes_A v \mapsto\ & 1^B_{(0)}m_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(1^B_{(-1)}m_{(-1)})1^B_{(-2)}x \otimes 1^A_{(0)}v) \\ =\ & 1^B_{(0)}m_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(m_{(-1)})S^{-1}(\varepsilon_s(1^B_{(-1)}))x \otimes 1^A_{(0)}v) \\ \overset{(8.7)}{=}\ & m_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(m_{(-1)})x \otimes 1^A_{(0)}v). \qquad (10.2) \end{aligned}$$

These maps are actually mutually inverse, since for all $m \in M$, $x \in X$ and $v \in V$:

$$\begin{aligned} & m_{(0)}\otimes_A(1^A_{(-1)}S^{-1}(m_{(-1)})m_{(-2)}x \otimes 1^A_{(0)}v) \\ &= m_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(\varepsilon_s(m_{(-1)}))x \otimes 1^A_{(0)}v) \\ &= m1'^A_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(1'^A_{(-1)})x \otimes 1^A_{(0)}v) && \text{by } (8.15) \\ &= m \otimes_A (1'^A_{(-1)}S^{-1}(1'^A_{(-2)})x \otimes 1'^A_{(0)}v && \text{by } (8.1) \\ &= m \otimes_A (1^A_{(-1)}x \otimes 1^A_{(0)}v), \end{aligned}$$

and

$$\begin{aligned} & m_{(-1)}S^{-1}(m_{(-2)})x \otimes m_{(0)} \otimes_A v \\ &= S^{-1}(\varepsilon_t(m_{(-1)}))x \otimes m_{(0)} \otimes_A v \\ &= 1^B_{(-1)}x \otimes 1^B_{(0)}m \otimes_A v. && \text{by } (8.14) \end{aligned}$$

They are well-defined, since for $a \in A$

$$\begin{aligned}
&c^M_{X,V}(ma \otimes_A (1^A_{(-1)}x \otimes 1^A_{(0)}v))\\
&\qquad = m_{(-1)}a_{(-1)}x \otimes m_{(0)}a_{(0)} \otimes_A v\\
&\qquad = m_{(-1)}a_{(-1)}x \otimes m_{(0)} \otimes_A a_{(0)}v\\
&\qquad = c^M_{X,V}(m \otimes_A (a_{(-1)}x \otimes a_{(0)}v))
\end{aligned}$$

and

$$\begin{aligned}
&(c^M_{X,V})^{-1}(1^B_{(-1)}x \otimes 1^B_{(0)}ma \otimes_A v)\\
&\qquad = m_{(0)}a_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(m_{(-1)}a_{(-1)})x \otimes 1^A_{(0)}v)\\
&\qquad = m_{(0)} \otimes_A (a_{(-1)}S^{-1}(a_{(-2)})S^{-1}(m_{(-1)})x \otimes a_{(0)}v)\\
&\qquad = m_{(0)} \otimes_A (1^A_{(-1)}S^{-1}(m_{(-1)})x \otimes 1^A_{(0)}av) \qquad \text{by (8.14)}\\
&\qquad = (c^M_{X,V})^{-1}(1^B_{(-1)}x \otimes 1^B_{(0)}m \otimes_A av).
\end{aligned}$$

Obviously, $c^M_{X,V}$ is a $B$-module morphism. It remains to shown that $c^M$ satisfies (1.3) and (1.4), that is for $V \in {}_A\mathcal{M}^{fd}$, $X, X' \in \mathcal{C}$ we have to check that $(\mathrm{id}_X \otimes c^M_{X',V}) \circ (c^M_{X,X'\otimes^A V}) = c^M_{X\otimes X',V}$ and $\ell_{M\otimes V} \circ c^M_{H_t,V} = \mathrm{id}_M \otimes \ell_V$. This holds, for if $x \in X$, $x' \in X'$, $y \in H_t$, $m \in M$ and $v \in V$ then

$$\begin{aligned}
(\mathrm{id}_X \otimes c^M_{X',V}) \circ (c^M_{X,X'\otimes^A V})&(m \otimes_A (1^A_{(-2)}x \otimes 1^A_{(-1)}x' \otimes 1^A_{(0)}v))\\
&= m_{(-2)}x \otimes m_{(-1)}x' \otimes m_{(0)} \otimes_A v\\
&= m_{(-1)}(1_{(1)}x \otimes 1_{(2)}x') \otimes m_{(0)} \otimes_A v\\
&= c^M_{X\otimes X',V}(m \otimes_A (1^A_{(-2)}x \otimes 1^A_{(-1)}x' \otimes 1^A_{(0)}v))
\end{aligned}$$

and

$$\begin{aligned}
\ell_{M\otimes V}(c^M_{H_t,V}&(m \otimes_A (1^A_{(-1)} \rightharpoonup y \otimes 1^A_{(0)}v)))\\
&= m_{(0)} \otimes_A v\varepsilon(\varepsilon_t(m_{(-1)}y)) && \text{by (7.26)}\\
&= m_{(0)}1^A_{(0)} \otimes_A v\varepsilon(m_{(-1)}1^A_{(-1)})\varepsilon(1^A_{(-2)}y) && \text{by (7.3)}\\
&= m \otimes_A 1^A_{(0)}v\varepsilon(1^A_{(-1)}y)\\
&= (\mathrm{id}_M \otimes \ell_V)(m \otimes 1^A_{(-1)} \cdot y \otimes 1^A_{(0)}v).
\end{aligned}$$

□

**Proposition 10.1.3.** *Let $A$ be a left $H$-comodule algebra, and $M \in {}^H\mathcal{M}_A$. Assume furthermore that $M$ is a progenerator in $\mathcal{M}_A$ and let $B := \mathrm{End}_A(M)$ and $Q := \mathrm{Hom}_A(M, A)$. Then $(M \otimes_A -, c^M) : {}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ and $(Q \otimes_B -, c^Q) : {}_B\mathcal{M}^{fd} \to {}_A\mathcal{M}^{fd}$ are quasi inverse $\mathcal{C}$-module category equivalences, where $c^M$ and $c^Q$ are defined as in the lemma above. The natural isomorphisms $\alpha : Q \otimes_B M \otimes_A - \to \mathrm{id}_{{}_A\mathcal{M}}$ and $\beta : M \otimes_A Q \otimes_B - \to \mathrm{id}_{{}_B\mathcal{M}}$ are* *given by*

$$\alpha_V : Q \otimes_B M \otimes_A V \xrightarrow{\cong} V$$
$$q \otimes_B m \otimes_A v \mapsto q(m)v$$

and

$$\beta_W : M \otimes_A Q \otimes_B W \xrightarrow{\cong} W$$
$$m \otimes_A q \otimes_B w \mapsto (mq)w,$$

*where* $mq \in B$ *with* $(mq)(m') := mq(m')$.

*Proof.* $B$ is a left $H$-comodule algebra and $M \in {}^H_B\mathcal{M}_A$ and $Q \in {}^H_A\mathcal{M}_B$ by the cop-version of Corollary 8.5.2. By the lemma above, $(M \otimes_A -, c^M)$ and $(Q \otimes_B -, c^Q)$ are $\mathcal{C}$-module functors. They are mutually inverse category equivalences and $\alpha$ and $\beta$ are well-defined natural isomorphisms, by Morita theory. It remains to show that $\alpha$ and $\beta$ satisfy (1.6) and (1.7), that is for $X \in \mathcal{C}$, $V \in {}_A\mathcal{M}^{fd}$ and $W \in {}_B\mathcal{M}^{fd}$ the following equations should hold:

$$\alpha_{X \otimes^A V} = (\mathrm{id}_X \otimes \alpha_V)(c^Q_{X, M \otimes_A V} \circ (\mathrm{id}_Q \otimes c^M_{X,V}))$$

and $$\beta_{X \otimes^B W} = (\mathrm{id}_X \otimes \beta_W)(c^M_{X, Q \otimes_B W} \circ (\mathrm{id}_M \otimes c^Q_{X,W})).$$

If $m \in M$, $q \in Q$, $x \in X$, $v \in V$, and $w \in W$, then

$$\begin{aligned}
&(\mathrm{id}_X \otimes \alpha_V)(c^Q_{X, M \otimes_A V} \circ (\mathrm{id}_Q \otimes c^M_{X,V}))(q \otimes_B m \otimes_A (1^A_{(-1)} x \otimes 1^A_{(0)} v)) \\
&\quad = q_{(-1)} m_{(-1)} x \otimes q_{(0)}(m_{(0)}) v \\
&\quad = q(m)_{(-1)} x \otimes q(m)_{(0)} v \qquad (*) \\
&\quad = \alpha_{X \otimes^A V}(q \otimes_B m \otimes_A (1^A_{(-1)} x \otimes 1^A_{(0)} v)),
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{id}_X \otimes \beta_W)(c^M_{X, Q \otimes_A W} \circ (\mathrm{id}_M \otimes c^Q_{X,W}))(m \otimes_A q \otimes_Q (1^B_{(-1)} x \otimes 1^B_{(0)} w)) \\
&\quad = m_{(-1)} q_{(-1)} x \otimes (m_{(0)} q_{(0)}) w \\
&\quad = (mq)_{(-1)} x \otimes (mq)_{(0)} w \qquad (**) \\
&\quad = \beta_{X \otimes^B W}(m \otimes_A q \otimes_Q (1^B_{(-1)} x \otimes 1^B_{(0)} w)),
\end{aligned}$$

Here (*) is the dual version of

$$\begin{aligned}
(h_{(1)} \bullet q)(h_{(2)} \cdot m) &\overset{(8.26)}{=} h_{(1)} \cdot q(S(h_{(2)}) h_{(3)} \cdot m) \\
&\overset{(7.21)}{=} h 1_{(1)} \cdot q(S(1_{(2)}) \cdot m) \overset{(8.26)}{=} h \cdot q(m),
\end{aligned}$$

and (**) is the dual version of

$$\begin{aligned}
(h_{(1)} \cdot m)(h_{(2)} \bullet q)(\tilde{m}) &\overset{(8.26)}{=} (h_{(1)} \cdot m)(h_{(2)} \cdot q(S(h_{(3)}) \cdot \tilde{m})) \\
&\overset{(8.18)}{=} h_{(1)} \cdot ((mq)(S(h_{(2)}) \cdot \tilde{m}) \overset{(8.25)}{=} (h \bullet (mq))(\tilde{m}).
\end{aligned}$$

where $A$ and therefore also $B$ are now $H$-module algebras and $M \in {}_B({}_H\mathcal{M})_A$ and $Q \in {}_A({}_H\mathcal{M})_B$ (see 8.5.1); and $h \in H$, $q \in Q$ and $m, \tilde{m} \in M$. $\square$

**Definition.** As in [AM07] a Morita context $(A, M, Q, B, \alpha, \beta)$ (see Appendix A.1) for $H$-comodule algebras $A$ and $B$ will be called an **equivariant Morita context** if $M \in {}^H_B\mathcal{M}_A$.

**Proposition 10.1.4.** *Let $A$ and $B$ be left $H$-comodule algebras. Equivalences of module categories ${}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ over $\mathcal{C}$ are in bijective correspondence with equivariant Morita contexts for $A$ and $B$.*

*Proof.* If $(A, M, B, Q, \alpha, \beta)$ is an equivariant Morita context for $A$ and $B$, then $(M \otimes_A -, c^M) : {}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ is an equivalence of module categories over $\mathcal{C}$ by the foregoing proposition.

Conversely, let $(F, c) : {}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$ be an equivalence of $\mathcal{C}$-module categories. With Morita theory we obtain a Morita context $(A, M, Q, B, \alpha, \beta)$ for $A$ and $B$. We will show that $M \in {}^H_B\mathcal{M}_A$. Define a costructure on $M$ by

$$\delta_M : M \to H \otimes^A (M \otimes_A A) \cong H \otimes^A M, \qquad m \mapsto c_{H,A}(m \otimes_A 1^A_{(-1)} \otimes 1^A_{(0)}).$$

Let $X \in \mathcal{C}$ and $V \in \mathcal{M}$. For $x \in X$ and $v \in V$ define morphisms $f_{(x)} : H \to X$, $h \mapsto hx$ in $\mathcal{C}$ and $g_v : A \to V$, $a \mapsto av$ in $\mathcal{M}$. As $c$ is a natural isomorphism the following diagram is commutative:

$$\begin{tikzcd}
M \otimes_A (H \otimes^A A) \arrow[r, "c_{H,A}"] \arrow[d, "(F(-\otimes^A -))(f_{(x)}, g_{(v)})"'] & H \otimes^B (M \otimes_A A) \arrow[d, "(-\otimes^B F(-))(f_{(x)}, g_{(v)})"] \\
M \otimes_A (X \otimes^A V) \arrow[r, "c_{X,V}"] & X \otimes^B (M \otimes_A V)
\end{tikzcd}.$$

With the definition of $\delta_M$ this implies

$$c_{X,V}(m \otimes 1^A_{(-1)}x \otimes 1^A_{(0)}v) = m_{(-1)}x \otimes m_{(0)} \otimes_A v, \tag{10.3}$$

for all $m \in M$. Hence, if we can shown that $\delta_M$ is a weak Hopf bimodule structure for $M$, then $c$ equals $c^M$ defined in (10.1). On the other hand, if $M$ is a weak Hopf bimodule in ${}^H_B\mathcal{M}_A$, then for $m \in M$:

$$c^M_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)}) = m_{(-1)} \otimes m_{(0)}$$

and we would have the desired bijective correspondence. Thus it remains to show that $\delta_M$ is actually a comodule structure for $M$ and that it is $(B, A)$-bilinear. Let $m \in M$, then $(\Delta \otimes \mathrm{id})(\delta_M(m))$ because

$$\begin{aligned}
&(\Delta \otimes \mathrm{id})(c_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) \\
&= c_{H \otimes H, A}(m \otimes 1^A_{(-2)} \otimes 1^A_{(-1)} \otimes 1^A_{(0)}) && \text{by the diagram below} \\
&= (\mathrm{id} \otimes c_{H,A})(c_{H, H \otimes A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)}(1'^A_{(-1)} \otimes 1'^A_{(0)})) && \text{by (1.3) and (8.6)} \\
&= (\mathrm{id} \otimes c_{H,A})(\delta_M(m) \otimes 1^A_{(-1)} \otimes 1^A_{(0)}) && \text{by (10.3).}
\end{aligned}$$

The following diagram commutes by the naturality of $c$:

$$\begin{tikzcd} M \otimes_A (H \oslash^A A) \arrow[r, "c_{H,A}"] \arrow[d, "{(F(-\oslash^A A))(\Delta) \atop =\mathrm{id}\otimes\Delta\otimes\mathrm{id}}"'] & H \oslash^B (M \otimes_A A) \arrow[d, "{(-\oslash^B F(A))(\Delta) \atop =\Delta\otimes\mathrm{id}\otimes\mathrm{id}}"] \\ M \otimes_A (H \oslash H \oslash^A A) \arrow[r, "c_{H\oslash H,A}"] & H \oslash H \oslash^B (M \otimes_A A) \end{tikzcd} .$$

In the same way the diagram

$$\begin{tikzcd} M \otimes_A (H \oslash^A A) \arrow[r, "c_{H,A}"] \arrow[d, "{(F(-\oslash^A A))(\varepsilon_t) \atop =\mathrm{id}\otimes\varepsilon_t\otimes\mathrm{id}}"'] & H \oslash^B (M \otimes_A A) \arrow[d, "{(-\oslash^B F(A))(\varepsilon_t) \atop =\varepsilon_t\otimes\mathrm{id}\otimes\mathrm{id}}"] \\ M \otimes_A (H_t \oslash^A A) \arrow[r, "c_{H_t,A}"] & H_t \oslash^B (M \otimes_A A) \end{tikzcd} .$$

commutes and therefore

$$\begin{aligned} &(\varepsilon \otimes \mathrm{id})(c_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) \\ &\qquad = \ell_M(\varepsilon_t \otimes \mathrm{id})(c_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) \\ &\qquad = \ell_M(c_{H_t,A}(\mathrm{id} \otimes \varepsilon_t \otimes \mathrm{id})(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) \\ &\qquad = (\mathrm{id} \otimes \ell_M)((\mathrm{id} \otimes \varepsilon_t \otimes \mathrm{id})(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) \qquad \text{by (1.4)} \\ &\qquad = m \otimes 1^A_{(0)}\varepsilon(1^A_{(-1)}) = m \otimes 1^A, \end{aligned}$$

for every $m \in M$, which implies $(\varepsilon \otimes \mathrm{id}) \circ \delta_M = \mathrm{id}_M$. Finally, $\delta_M$ is $A$-linear and $B$-linear because

$$c_{H,A}(bm \otimes 1^A_{(-1)} \otimes 1^A_{(0)}) = c_{H,A}(b(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)})) = b\, c_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)}),$$

since $c$ is $B$-linear; and

$$\begin{aligned} c_{H,A}(ma \otimes 1^A_{(-1)} \otimes 1^A_{(0)}) &= c_{H,A}(m \otimes a_{(-1)} \otimes a_{(0)})) \\ &= m_{(-1)}a_{(-1)} \otimes m_{(0)} \otimes a_{(0)} \qquad \text{by (10.3)} \\ &= c_{H,A}(m \otimes 1^A_{(-1)} \otimes 1^A_{(0)}))a \end{aligned}$$

for $m \in M$, $a \in A$ and $b \in B$. $\square$

### 10.1.2 Module Categories Induced by Smash-Products

**Proposition 10.1.5.** *Let $R$ be a finite dimensional left $H$-module algebra and set $A = R\#H$. Assume that there exists a simple right $R$-module $V$ such that $M := V\#H$ is a progenerator in $\mathcal{M}_A$. Then for $B := \mathrm{End}_A(M)$ we obtain an equivalence of $H^{cop}$-module categories*

$$M \otimes_A - : \ {}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd},$$

*and moreover $B$ is a simple object in $\mathcal{M}^H_B$ and $B^{coH} \cong k$.*

*Proof.* With Morita theory we obtain an equivalence of categories

$$M \otimes_A - : {}_A\mathcal{M} \to {}_B\mathcal{M}.$$

By Proposition 10.1.3, it follows that this is actually an equivalence of $H^{cop}$-module categories, since $M \in {}_B\mathcal{M}_A^H$; and $B$ is an $H$-comodule algebra with trivial coinvariants by Proposition 8.5.2.

It remains to show that $B = \mathrm{End}_A(M)$ is a simple object in $\mathcal{M}_B^H$. This follows directly by the fact that $V$ is assumed to be a simple $R$-module and the equivalences of categories (Lemma 8.5.5 and Proposition 8.4.12):

$$\begin{array}{ccccc} \mathcal{M}_R & \approx & \mathcal{M}_A^H & \approx & \mathcal{M}_B^H \\ W & \mapsto & W\#H & \mapsto & \mathrm{Hom}_A(V\#H, W\#H). \end{array} \qquad \square$$

*Remark* 10.1.6. Assume that in the proposition $H$ is a free right $H_s$-module and $R$ is a simple $H$-module algebra and moreover $A := R\#H$ is quasi-Frobenius. Then for every simple right $R$-module $V$, $M := V\#H \in \mathcal{M}_A^H$ is a progenerator by Corollary 9.5.3.

## 10.2 Indecomposable Exact Module Categories over Weak Hopf Algebras

In this section, indecomposable exact module categories over weak Hopf algebras, that satisfy Conjecture 9.5.4, will be classified. This is a generalization of [AM07, Theorem 3.3]. However, a new proof for this result is given, which does not use the notion of stabilizers. The Hopf algebra version of this proof is presented in Chapter 2.

Let again $H$ be a (finite dimensional) weak Hopf algebra and $\mathcal{C}$ the finite multi-tensor category of finite dimensional left $H$-modules.

### 10.2.1 Etingof and Ostrik's Classification of Exact Module Categories in the Weak Hopf algebra Case

Algebras in the category $\mathcal{C}$ are exactly the finite dimensional $H$-module algebras. Hence for a finite dimensional $H$-module algebra $R$, the category $({}_H\mathcal{M})_R$ is a $\mathcal{C}$-module category, where the $R$-module structure for the tensor product $X \otimes M$ of a left $H$-module $X$ and $M \in ({}_H\mathcal{M})_R$ is given by

$$\begin{aligned}(1_{(1)}x \otimes 1_{(2)} \cdot m) \bullet r &= 1_{(1)}x \otimes (1_{(2)} \cdot m)r \\ &= 1_{(1)}x \otimes (1'_{(1)}1_{(2)} \cdot m)(1'_{(2)} \cdot r) = 1_{(1)}x \otimes 1_{(2)} \cdot (mr)\end{aligned}$$

for $a \in A$, $m \in M$ and $x \in X$.

Proposition 1.3.5 implies the following:

**Lemma 10.2.1.** *Every indecomposable exact module category over $H$ is equivalent to $({}_H\mathcal{M}^{fd})_R$ for some $H$-module algebra $R$, which is a finite dimensional and $H$-simple.*

*Proof.* By 1.3.5 $\mathcal{M} \approx \mathcal{C}_R = ({}_H\mathcal{M}^{fd})_R$ as $\mathcal{C}$-module categories, where $R$ is an $H$-module algebra which is simple in $\mathcal{C}_R$, that is $R$ is right $H$-simple and in particular it is $H$-simple. $R$ is finite dimensional, as it is an object in $\mathcal{C} = {}_H\mathcal{M}^{fd}$. □

**Proposition 10.2.2.** *If $M$ is an indecomposable exact module category over $H$, then there exists a finite dimensional quasi-Frobenius $H$-simple $H$-comodule algebra $A$ such that $\mathcal{M} \approx {}_A\mathcal{M}^{fd}$ as module categories.*

*Proof.* In view of the lemma, it suffices to prove that for a finite dimensional $H$-simple left $H$-module algebra $R$, there exists a finite dimensional $H$-simple left $H$-comodule algebra $A$ with ${}_A\mathcal{M}^{fd} \approx ({}_H\mathcal{M}^{fd})_R$ as $\mathcal{C}$-module categories. We can show that the desired equivalence holds for $A := R^{op}\#H^{cop}$, which works as in the Hopf algebra case [AM07, Proposition 1.19].

In fact, $R^{op}$ is a left $H^{cop}$-module algebra and

$$F : ({}_H\mathcal{M}^{fd})_R \approx {}_{R^{op}\#H^{cop}}\mathcal{M}^{fd}$$

as abelian categories by Lemma 8.4.13. Since $A := R^{op}\#H^{cop}$ is a right $H^{cop}$-comodule algebra, it is a left $H$-comodule algebra with opposite costructure given by $\lambda(r\#h) = h_{(1)} \otimes r\#h_{(2)}$.

Then, $F$ is an equivalence of $\mathcal{C}$-module categories together with the natural isomorphism $c$ which is the identity. It suffices to show that for all $X \in \mathcal{C}$, $M \in ({}_H\mathcal{M}^{fd})_R$, $c_{X,M} : F(X \otimes M) \to X \otimes^A F(M)$ is a morphism of $A$-modules. For $x \in X, m \in M, h \in H, r \in R$:

$$\begin{aligned}
&c_{X,M}((r\#h) \blacktriangleright (1_{(1)}x \otimes 1_{(2)} \cdot m)) \\
&\quad = id_{X,M}((h(1_{(1)}x \otimes 1_{(2)}m)) \bullet r) \\
&\quad = (h_{(1)}x \otimes h_{(2)} \cdot m) \bullet r \\
&\quad = h_{(1)}x \otimes (h_{(2)} \cdot m)r && R\text{-module structure of } X \otimes^A M \text{ in } \mathcal{C} \\
&\quad = h_{(1)}x \otimes (r\#h_{(2)}) \blacktriangleright m \\
&\quad = (r\#h)(1_{(1)}x \otimes 1_{(2)} \cdot m) && A\text{-module structure of } X \otimes^A M \\
&\quad = (r\#h)c_{X,M}(1_{(1)}x \otimes 1_{(2)} \cdot m).
\end{aligned}$$

$R$ is a left $H$-module algebra which is simple in $({}_H\mathcal{M}^{fd})_R$, therefore $R^{op}$ is an $H^{cop}$-simple left $H^{cop}$-module algebra, and Proposition 8.4.12 implies that

$A = R^{op}\#H^{cop}$ is an $H^{cop}$-simple right $H^{cop}$-comodule algebra, hence an $H$-simple left $H$-comodule algebra. Finally, $A$ is quasi-Frobenius by Remark 1.3.1 since ${}_A\mathcal{M}^{fd}$ is exact. □

### 10.2.2 Classification of Exact Module Categories by *H*-Comodule Algebras

Let $H$ be a weak Hopf algebra and $A$ an $H$-simple left $H$-comodule algebra. In the last chapter a conjecture was formulated (Conjecture 9.5.4) which states that every finite dimensional object $M \in \mathcal{M}_A^H$ is a projective right $A$-module, in particular the opcop version of this would imply that every object of the form $H \otimes^A V$, where $V$ is a finite dimensional left $A$-module, is a projective left $A$-module. This is satisfied under certain conditions, and in these cases a classification of module categories over $H$ is possible.

**Lemma 10.2.3.** *Let $H$ be a (finite dimensional) weak Hopf algebra, and $A$ a finite dimensional $H$-simple left $H$-comodule algebra. Assume that one of the following properties is satisfied:*

(i) *$H$ is a free left $H_t$-module;*

(ii) *$H$ is semisimple and* **pseudo-unitary** [Nik04]*, that is the categorical dimension of $\mathcal{C} = {}_H\mathcal{M}^{fd}$ equals its Frobenius-Perron dimension.*

*Then every finite dimensional object $M \in {}^H_A\mathcal{M}$ is a projective left $A$-module.*

*Proof.* (i) is the opcop-version of 9.5.1.
(ii) If $H$ is semisimple and pseudo-unitary, then so is its dual $H^*$ [Nik04, Proposition 3.1.5 and Corollary 5.2.6]. $A$ is an $H^*$-module algebra. By [Nik04, Theorem 6.1.3], $\mathrm{Jac}(A)$ is an $H^*$-stable ideal and therefore an $H$-costable ideal. Since $A$ is $H$-simple, this implies $\mathrm{Jac}(A) = 0$ and therefore $A$ is a semisimple algebra. □

**Proposition 10.2.4.** *Assume that $H$ is a (finite dimensional) weak Hopf algebra and $A$ is a finite dimensional $H$-simple $H$-comodule algebra, such that for every finite dimensional left $A$-module $V$, $H \otimes^A V$ is a projective $A$-module. Then ${}_A\mathcal{M}^{fd}$ is an indecomposable exact module category over $H$.*

*Proof.* ${}_A\mathcal{M}^{fd}$ is an indecomposable module category by the lemma below, since $A$ is $H$-simple. Let $V$ be a right $A$-module, then $- \otimes^A V$ is an exact functor by 10.1.1. Therefore, if $P$ is a projective object in $\mathcal{C}$, then $P \otimes^A V$ is a direct sum of summands of $H \otimes^A V$, and the claim follows by the assumption about objects of this form. □

**Lemma 10.2.5.** *Let $H$ be a (finite dimensional) weak Hopf algebra and let $A$ be a finite dimensional left $H$-comodule algebra, then ${}_A\mathcal{M}^{fd}$ is an indecomposable module category if and only if $A$ is an indecomposable $H$-comodule algebra, that is if $I$ and $J$ are $H$-costable ideals of $A$ with $A = J \oplus I$, then $I = 0$ or $J = 0$.*

*Proof.* Analogous to [AM07, Proposition 1.18] □

Thus, in the above cases from Lemma 10.2.3, quasi-Frobenius $H$-simple $H$-comodule algebras produce examples of indecomposable exact module categories over $H$. These are in fact all the indecomposable exact module categories over $H$, as we will see in the next theorem.

**Theorem 10.2.6.** *Let $H$ be a (finite dimensional) weak Hopf algebra and assume that for any finite dimensional quasi-Frobenius $H$-simple left $H$-comodule algebra $A$, every finite dimensional object in ${}^H_A\mathcal{M}$ or in ${}^H\mathcal{M}_A$ is a projective left or right $A$-module, respectively. Let $\mathcal{M}$ be a module category over $H$, then $\mathcal{M}$ is exact and indecomposable, if and only if there exists a finite dimensional quasi-Frobenius left $H$-comodule algebra $B$, which is simple in ${}^H\mathcal{M}_B$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}^{fd}$ as $\mathcal{C}$-module categories.*

*Proof.* Let $R$ and $A := R^{op}\#H^{cop}$ be as in the proof of 10.2.2. Every finite dimensional nonzero object in $\mathcal{M}_A^{H^{cop}}$ is a projective right $A$-module and therefore a progenerator by Proposition 8.3.4. Hence Proposition 10.1.5 gives us an equivalence of $H$-module categories ${}_A\mathcal{M}^{fd} \to {}_B\mathcal{M}^{fd}$, where $B$ is a finite dimensional right $H^{cop}$-simple right $H^{cop}$-comodule algebra with trivial coinvariants, hence it is a right $H$-simple left $H$-comodule algebra. Again, $B$ is quasi-Frobenius by Remark 1.3.1.

On the other hand, if $B$ is a left $H$-comodule algebra which is simple in ${}_B\mathcal{M}^H$, then ${}_B\mathcal{M}^{fd}$ is an exact indecomposable module category over $H$ by Proposition 10.2.4, since all finite dimensional objects in ${}^H_B\mathcal{M}$ are projective $B$-modules by the assumption about $H$. □

**Corollary 10.2.7.** *Let $H$ be a (finite dimensional) weak Hopf algebra which is free over its bases, that is it is free as a left and right $H_t$-module. Let $\mathcal{M}$ be a module category over $H$, then $\mathcal{M}$ is exact and indecomposable, if and only if there exists a finite dimensional quasi-Frobenius left $H$-comodule algebra $B$, which is simple in $\mathcal{M}_B^H$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}^{fd}$ as $\mathcal{C}$-module categories.*

*Proof.* By Lemma 10.2.3 (i) and its op version the requirements of the theorem are satisfied. □

**Corollary 10.2.8.** *Let $H$ be a (finite dimensional) weak Hopf algebra which is semisimple and pseudo unitary. Let $\mathcal{M}$ be a module category over $H$, then $\mathcal{M}$ is semisimple and indecomposable, if and only if there exists a finite dimensional* *semisimple left $H$-comodule algebra $B$, which is simple in $\mathcal{M}_B^H$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}^{fd}$ as $\mathcal{C}$-module categories.*

*Proof.* Recall that a module category over a semisimple tensor category is exact if and only if it is semisimple (Remark 1.3.1). We can apply the theorem since an $H$-simple $H$-comodule algebra is semisimple by [Nik04] (see proof of Lemma 10.2.3 (ii)). $\square$

Based on Conjecture 9.5.4 the theorem leads to the following:

**Conjecture 10.2.9.** *Let $H$ be a (finite dimensional) weak Hopf algebra and let $\mathcal{M}$ be a module category over $H$. $\mathcal{M}$ is exact and indecomposable, if and only if there exists a finite dimensional quasi-Frobenius left $H$-comodule algebra $B$, which is simple in ${}^H\mathcal{M}_B$ and has trivial coinvariants, such that $\mathcal{M} \approx {}_B\mathcal{M}^{fd}$ as $\mathcal{C}$-module categories.*

# Appendix

# Appendix A

# Some Ring Theoretic Facts

In this chapter we recall some ring theoretic definitions and facts. For more details and proofs the reader is referred to [Lam98, §1 , §15, §16, and §18].

## A.1 Morita Equivalence

Let $R$ be a ring. A **generator** (for $\mathcal{M}_R$) is a right $R$-module $M$ such that for every $V \in \mathcal{M}_R$, $V$ is an epimorphic image of some direct sum $M^{(n)}$. If $M$ is moreover finitely generated projective, it is called a **progenerator** (for $\mathcal{M}_R$).

Let $M \in \mathcal{M}_R$. Set $S := \mathrm{End}_R(M)$ and $Q := \mathrm{Hom}_R(M, R)$, then $M \in {}_S\mathcal{M}_R$ and $Q \in {}_R\mathcal{M}_S$, where the left $R$-action of $Q$ is given by $(rq)(m) = rq(m)$ and the right $S$-action is defined by $(qs)(m) = q(s(m))$. We may define an $(R, R)$-bimodule morphism

$$\alpha : Q \otimes_S M \to R, \qquad q \otimes m \mapsto q(m),$$

and an $(S, S)$-bimodule morphism

$$\beta : M \otimes_R Q \to S, \qquad m \otimes q \mapsto mq,$$

where $(mq)(\tilde{m}) := mq(\tilde{m})$. We call $(R, M, Q, S, \alpha, \beta)$ the **Morita context** for $M \in \mathcal{M}_R$.

**Theorem A.1.1.** *Let $M$ be a progenerator for $\mathcal{M}_R$, then $\alpha$ and $\beta$ are isomorphisms and*

$$\begin{aligned} - \otimes_R Q : \mathcal{M}_R \to \mathcal{M}_S \quad &\text{and} \quad - \otimes_S M : \mathcal{M}_S \to \mathcal{M}_R \\ M \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M} \quad &\text{and} \quad Q \otimes_S - : {}_S\mathcal{M}_S \to {}_R\mathcal{M} \end{aligned}$$

*are mutually inverse category equivalences, respectively. Moreover*

$$\begin{aligned} - \otimes_R Q \cong \mathrm{Hom}_R(M_R, -) \quad &\text{and} \quad - \otimes_S M \cong \mathrm{Hom}_S(Q_S, -) \\ M \otimes_R - \cong \mathrm{Hom}_R({}_RM, -) \quad &\text{and} \quad Q \otimes_S - \cong \mathrm{Hom}_S({}_SQ, -). \end{aligned}$$

## A.2 Trace Ideals

Let $M$ be a right module over a ring $R$.

$$T_M := \sum_{f \in \mathrm{Hom}_R(M,R)} f(M)$$

is called the **trace ideal of $M$ in $R$**.

**Proposition A.2.1.** *The following properties for a right $R$-module $M$ are equivalent:*

(i) *$M$ is a generator;*

(ii) *$R$ is a direct summand of $M^{(n)}$ for some $n \in \mathbb{N}$;*

(iii) *$T_M = R$.*

## A.3 Quasi-Frobenius and Kasch Rings

A ring $R$ is said to be **quasi-Frobenius** if it is right self-injective and right Noetherian or, equivalently, if it is left self injective and left Noetherian. $R$ is called **right Kasch**, if every simple right $R$-module can be embedded into $R$.

**Proposition A.3.1.** *The following properties for a ring $R$ are equivalent:*

(i) *$R$ is quasi-Frobenius;*

(ii) *$R$ is right Kasch and every principle indecomposable has a simple socle;*

(iii) *A right $R$-module is projective if and only if it is injective.*

Here, the socle of a module $M$ is the direct sum of simple submodules of $M$ and will be denoted by $\mathrm{soc}(M)$.

**Remark A.3.2.** *Let $R$ be right Kasch and $0 \neq M \in \mathcal{M}_R^{fd}$, then $\mathrm{Hom}_R(M, R) \neq 0$, and in particular $T_M \neq 0$.*

## A.4 Frobenius and Symmetric Algebras

Let $R$ be a finite dimensional algebra and let $R^* := \mathrm{Hom}_k(R, k)$, which has a natural structure of an $(R, R)$-bimodule. Then $R$ is a **Frobenius algebra** if $R^* \cong R$ as right (or equivalently left) $R$-modules. $R$ is said to be **symmetric** if $R^* \cong R$ as $(R, R)$-bimodules.

Clearly, if $R$ is Frobenius then it is self-injective and therefore also quasi-Frobenius. We know, that for quasi-Frobenius algebras, the socles of principle indecomposable modules are simple. For Frobenius and symmetric algebras we have additional conditions for these socles. We have the following equivalences for a finite dimensional algebra $R$:

**Proposition A.4.1.** (1) *R is Frobenius if and only if it is quasi-Frobenius and*

$$\dim(P/Jac(R)P) = \dim(\mathrm{soc}(P))$$

*for every principle indecomposable module $P$.*

(2) *R is symmetric if and only if it is quasi-Frobenius and*

$$P/\mathrm{Jac}(R)P \cong \mathrm{soc}(P)$$

*for every principle indecomposable module $P$.*

## A.5 Semilocal and Weakly Finite Rings

A ring $R$ is called **semilocal** if $R/\mathrm{Jac}(R)$ is a semisimple ring, where $\mathrm{Jac}(R)$ denotes the Jacobson radical of $R$. The set $\mathrm{Max}R$ of maximal ideals of a semilocal ring $R$ is finite and $\mathrm{Jac}(R) = \bigcap_{P \in \mathrm{Max}R} P$.

A ring $R$ is called **weakly finite** if for every $n \in \mathbb{N}$ and every pair of $n \times n$-matrices $X, Y \in \mathbb{M}_n(R)$, $XY = 1$ implies $YX = 1$.

**Proposition A.5.1.** *Let $R$ be a ring. If $R$ is either*

- *commutative,*
- *semilocal, or*
- *Noetherian*

*then $R$ is weakly finite. In particular, finite dimensional algebras are weakly finite.*

**Proposition A.5.2.** *The following properties of a ring $R$ are equivalent:*

(i) *$R$ is weakly finite;*

(ii) *if $M$ is a finitely generated free $R$-module of rank $n$, then every generating system for $M$ consisting of $n$ elements is a basis.*

# Bibliography


[AM07] N. Andruskiewitsch and J. M. Mombelli. On module categories over finite-dimensional Hopf algebras. *J. Algebra*, 314:383–418, 2007.

[BC03a] D. Bulacu and S. Caenepeel. Integrals for (dual) quasi-Hopf algebras. Applications. *J. Algebra*, 266:552–583, 2003.

[BC03b] D. Bulacu and S. Caenepeel. Two-sided two-cosided Hopf modules and Doi-Hopf modules for quasi-Hopf algebras. *J. Algebra*, 270:55–95, 2003.

[BN00] D. Bulacu and E. Nauwelaerts. Relative Hopf modules for (dual) quasi-Hopf algebras. *J. Algebra*, 229:632–659, 2000.

[BNS99] G. Böhm, F. Nill, and K. Szlachányi. Weak Hopf algebras I. Integral theory and $C^*$-structure. *J. Algebra*, 221:385–438, 1999.

[Böh00] G. Böhm. Doi-Hopf modules over weak Hopf algebras. *Comm. Alg.*, 28:4687–4698, 2000.

[BPvO00] D. Bulacu, F. Panaite, and F. van Oystaeyen. Quasi-Hopf algebra actions and smash products. *Comm. Alg.*, 28:631–651, 2000.

[BS96] G. Böhm and K. Szlachányi. A coassociative $C^*$-quantum group with nonintegral dimensions. *Lett. in Math. Phys.*, 35:437–456, 1996.

[CF86] M. Cohen and D. Fishman. Hopf algebra actions. *J. Algebra*, 100:363–379, 1986.

[CR62] C.W. Curtis and I. Reiner. *Representation Theory of Finite Groups and Associative Algebras.* Interscience, New York, 1962.

[DPR90] R. Dijkgraaf, V. Pasquier, and P. Roche. Quasi-Hopf algebras, group cohomology and orbifold models. *Nuclear Phys. B Proc. Suppl.*, 18B:60 – 72, 1990.

[Dri90] V.G. Drinfeld. Quasi-Hopf algebras. *Leningrad Math. J.* , 1:1419 – 1457, 1990.

[EG04] P. Etingof and S. Gelaki. Finite dimensional quasi-Hopf algebras with radical of codimension 2. *Mathematical Research Letters*, 11:685–696, 2004.

[EG05] P. Etingof and S. Gelaki. On radically graded finite dimensional quasi-Hopf algebras. *Mosc. Math. J.*, 5:371–378, 2005.

[EGNO10] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. Tensor categories. 2010. Lecture notes, http://www-math.mit.edu/ etingof/tenscat.pdf.

[ENO05] P. Etingof, D. Nikshych, and V. Ostrik. On fusion categories. *Annals of Mathematics*, 162:581–642, 2005.

[EO04] P. Etingof and V. Ostrik. Finite tensor categories. *Mosc. Math. J.*, 4:782–783, 2004.

[FS03] J. Fuchs and C. Schweigert. Category theory for conformal boundary conditions. *Fields Inst. Commun.*, 39:25–71, 2003.

[FS10] J. Fuchs and C. Schweigert. Hopf algebras and finite tensor categories in conformal field theory. 2010. arXiv: hep-th/1004.3405.

[Gel05] S. Gelaki. Basic quasi-Hopf algebras of dimension $n^3$. *J. Pure Appl. Algebra*, 198:165–174, 2005.

[Hen10] H. Henker. Freeness of quasi-Hopf algebras over right coideal subalgebras. *Comm. Alg.*, 38:876–889, 2010.

[HN99a] F. Hausser and F. Nill. Diagonal crossed products by duals of quasi-quantum groups. *Rev. Math Phys.*, 11:553–629, 1999.

[HN99b] F. Hausser and F. Nill. Integral theory for quasi-Hopf algebras. 1999. arXiv: math/9904164.

[IK09] M.C. Iovanov and L. Kadison. When weak quasi-Hopf algebras are Frobenius. *Proc. Amer. Math. Soc.*, 138:837–845, 2009.

[JS91] A. Joyal and R. Street. An introduction to Tannaka duality and quantum groups. *Category theory (Como, 1990), Lecture Notes in Math.*, 1488:413–492, 1991.

[Kas95] C. Kassel. *Quantum Groups,* vol. 155 of GTM. Springer, New York, 1995.

[Lam98] T.Y. Lam. *Lectures on Modules and Rings.* Springer, 1998.

[Lin03] V. Linchenko. Nilpotent subsets of Hopf module algebras. *Groups, Rings, Lie and Hopf algebras*, pages 121–127, 2003.

[Lor97] M. Lorenz. Representations of finite dimensional Hopf algebras. *J. Algebra*, 188:476–505, 1997.

[Mas92] A. Masuoka. Freeness of Hopf algebras over coideal subalgebras. *Comm. Alg.*, 20:1353–1373, 1992.

[Mom08] M. Mombelli. Module categories over pointed Hopf algebras. 2008. arXiv 0811.4090.

[Mom09] M. Mombelli. Representations of tensor categories coming from quantum linear spaces. 2009. arXiv 0907.4517.

[Mon93] S. Montgomery. *Hopf algebras and Their Action on Rings.* CBSM Regional Conference Series in Mathematics, Vol. 82, American Mathematical Society, 1993.

[MS99] S. Montgomery and H.-J. Schneider. Prime ideals in Hopf Galois extensions. *Isr. J. Math*, 112:187–235, 1999.

[Müg08] M. Müger. Tensor categories: A selective guided tour. 2008. arXiv: 0804.3587.

[Nik00] D. Nikshych. A duality theorem for quantum groupoids. *Contemp. Math. New trends in Hopf algebra theory*, 267:237–243, 2000.

[Nik02] D. Nikshych. On the structure of weak Hopf algebras. *Adv. Math.*, 170:257–286, 2002.

[Nik04] D. Nikshych. Semisimple weak Hopf algebras. *J. Algebra*, 275:639–667, 2004.

[Nil98] F. Nill. Axioms for weak bialgebras. 1998. arXiv: math/9805104.

[NTV03] D. Nikshych, V. Tuarev, and L. Vainerman. Invariants of knots and 3-manifolds from quantum groupoids. *Topology Appl.*, 127:91–123, 2003.

[NV00] D. Nikshych and L. Vainerman. A characterization of depth 2 subfactors of $II_1$ factors. *J. Funct.-Analysis*, 171:278–307, 2000.

[NV02] D. Nikshych and L. Vainerman. Finite quantum groups and their applications. New directions in Hopf algebras *MSRI Publications*, 43:211–262, 2002.

[NZ89] W.D. Nichols and M.B. Zöller. A Hopf algebra freeness theorem. *Amer. J. Math.* , 111:381–385, 1989.

[Ost03a] V. Ostrik. Module categories over the Drinfeld double of a finite group. *Int. Math. Res. Not.*, 27:1507–1520, 2003.

[Ost03b] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8:177–206, 2003.

[PvO07] F. Panaite and F. van Oystaeyen. A structure theorem for quasi-Hopf comodule algebras. *Proc. Amer. Math. Soc.*, 135:1669–1677, 2007.

[Riv72] N. Saavedra Rivano. *Catégories Tannakiennes. Lecture Notes in Mathematics 265.* Springer, 1972.

[Sch92] P. Schauenburg. Tannaka duality for arbitrary Hopf algebras. *Algebra Berichte*, 66, Verlag Reinhard Fischer, München, 1992.

[Sch04] P. Schauenburg. A quasi-Hopf algebra freeness theorem. *Proc. Amer. Math. Soc.*, 132:965–972, 2004.

[Skr07] S. Skryabin. Projectivity and freeness over comodule algebras. *Trans. Amer. Math. Soc.*, 359:2597–2623, 2007.

[SvO06] S. Skryabin and F. van Oystaeyen. The Goldie Theorem for $H$-semiprime algebras. *J. Algebra*, 305:292–320, 2006.

[Swe69] M.E. Sweedler. *Hopf Algebras.* Benjamin, 1969.

[Szl00] K. Szlachányi. Finite quantum groupoids and inclusions of finite type. 2000. arXiv math/0011036.

[Vec03] P. Vecsernyés. Larson-Sweedler theorem and the role of grouplike elements in weak Hopf algebras. *J. Algebra*, 270:471–520, 2003.

[Zha10] L. Zhang. The structure theorems of weak Hopf algebras. *Comm. Alg.*, 38:1269–1281, 2010.

[ZZ04] L. Zhang and S. Zhu. Fundamental theorem of weak Doi-Hopf modules and semisimple weak smash product Hopf algebras. *Comm. Alg.*, 32:3403–3415, 2004.

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