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Two-dimensional foliations on four-manifolds

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Abstract

We study two-dimensional foliations on four-manifolds and examine properties of their closed leaves. After considering the general case of smooth foliations, we focus on foliations with symplectic leaves and then on symplectic pairs. In both cases certain restrictions on the underlying distributions and on the closed leaves of such foliations are derived.

We further study the geometry of characteristic classes of surface bundles with and without flat structures. For general surface bundles we show that the MMM-class are hyperbolic in the sense of Gromov and deduce certain restrictions on the topology of bundles under the assumption that the base is a product or that the bundle is holomorphic. We further consider characteristic classes of flat bundles, whose horizontal foliations have closed leaves and compute the abelianisation of the diffeomorphism group of a compact surface with marked points. When the foliations have a transverse symplectic structure, we show the non-triviality of certain derived characteristic classes in leaf-wise cohomology. For bundles with boundary we show that there is a relationship between the geometry of a flat structure and the topology of the boundary.

We also introduce the relation of symplectic cobordism amongst transverse knots. Specialising to the case of symplectic concordance we produce an infinite family of knots that show that this relation is not symmetric, in stark contrast to its smooth counterpart.

Zusammenfassung

Wir beschäftigen uns mit zwei-dimensionalen Blätterungen auf Viermanigfaltigkeiten und untersuchen die Eigenschaften ihrer abgeschlossenen Blätter. Nachdem wir den Fall von glatten Blätterungen betrachtet haben, konzentrieren wir uns auf Blätterungen mit symplektischen Blättern und anschließend auf symplektische Paare. Für diese beiden Fälle zeigen wir, dass die zugrundeliegenden Distributionen und abgeschlossenen Blätter solcher Blätterungen gewissen Beschränkungen unterliegen.

Weiterhin untersuchen wir die Geometrie der charakteristischen Klassen von Flächenbündeln mit und ohne flache Strukturen. Für allgemeine Flächenbündel zeigen wir, dass die MMM-Klassen hyperbolisch im Sinne von Gromov sind. Unter der Annahme, dass die Basis ein Produkt oder das Bündel holomorph ist, leiten wir außerdem gewisse Einschränkungen an die Topologie solcher Bündel her. Des weiteren behandeln wir charakteristische Klassen flacher Bündel, deren horizontale Blätterungen abgeschlossene Blätter besitzen und berechnen die Abelianisierung der Diffeomorphismengruppe einer kompakten Fläche mit markierten Punkten. Wenn die Blätterungen eine transversale symplektische Struktur aufweisen, zeigen wir, dass gewisse sekundäre charakteristische Klassen in der blattweisen Kohomologie nicht trivial sind. Für Bündel mit Rand leiten wir eine Beziehung zwischen der Geometrie einer flachen Struktur und der Topologie des Randes her.

Schließlich führen wir die Relation des symplektischen Kobordismus für transversale Knoten ein. Im Spezialfall der symplektischen Konkordanz zeigen wir mittels einer unendlichen Familie von Knoten, dass diese Relation im Gegensatz zur glatten Konkordanz nicht symmetrisch ist.

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Chapter 1

Introduction

The motivating theme of this thesis is the study of 2-dimensional foliations on 4-manifolds. In studying examples of such foliations the geometry of surface bundles consequently took on a prominent role and hence the majority of results we present are in some way related to surface bundles. Another recurring theme is that of 4-dimensional symplectic topology and as such we also consider the topology of certain symplectic surfaces in the form of cobordisms of transverse knots in contact manifolds.

1.1 Foliations and distributions on 4-manifolds

The question of whether a given manifold admits a foliation of a given dimension is a very difficult one in general. An obvious necessary condition for the existence of a q-dimensional foliation is the existence of a q-dimensional distribution. In dimensions 1 and 2 this is in fact also sufficient. For dimension 1 this is obvious and in dimension 2 it follows from Thurston's h-principle, which says that on a manifold of dimension at least 4, any oriented 2-dimensional distribution is homotopic to an integrable one (cf. [Th2]). Thus the existence of 2-dimensional foliations reduces to the problem of the existence of oriented 2-plane distributions. An oriented 2-plane field is then a section of the Grassmanian bundle of 2-planes and the existence of such sections is a homotopy problem that can be expressed in terms of obstruction theory.

In the case of oriented 4-manifolds the existence of an oriented 2-dimensional distribution is equivalent to a splitting of the tangent bundle as the Whitney sum of two oriented rank-2 subbundles ξ_1, ξ_2. As these bundles are oriented there is an almost complex structure on TM so that both bundles are complex subbundles. Thus the Whitney sum formula yields certain equations that the Chern classes of ξ_1, ξ_2 must satisfy. Since the Euler class of an oriented rank-2 bundle is the same as its first Chern class, these equations may be written as follows:

$$e(\xi_1) \smile e(\xi_2) = c_2(M) \text{ and } e^2(\xi_1) + e^2(\xi_2) = c_1^2(M).$$

Another necessary condition that these classes satisfy is that $e(\xi_1) + e(\xi_2)$ reduce to the second Stiefel-Whitney class $w_2(M)$ in mod 2 cohomology. The latter condition combined with the above equations give what we shall call the distribution equations. It is an old result of Hirzebruch and Hopf that the existence of a pair of cohomology classes e_1, e_2 that satisfy the distribution equations implies the existence of a pair of complementary 2-plane fields whose Euler classes are e_1, e_2 respectively (cf. [HH]). The path to this result suggests an alternate form of the cohomological equations above that is particularly useful for performing calculations (Proposition 2.2.4). Using these equations one can give precise conditions under which a 4-manifold admits 2-plane distributions in terms of the Euler characteristic and signature of the manifold (Theorem 2.2.8). So in particular the existence of foliations on 4-manifolds depends only on the homotopy type of the given manifold.

Given that 2-dimensional foliations are abundant on 4-manifolds as soon as certain weak topological conditions are satisfied, it is natural to try to understand the geometry of these foliations more closely. In particular, we investigate the closed leaves of such foliations. This problem has been studied by Mitsumatsu and Vogt, whose approach is based on a stronger form of the result of Thurston mentioned above. For what Thurston actually proved is what is known as a relative h-principle. That is, if ξ is a distribution of 2-planes that is integrable on a neighbourhood of a compact set K, then ξ is homotopic to an integrable distribution that agrees with ξ on K.

Now if an embedded surface Σ can be realised as a leaf of a foliation on a 4-manifold M, then the foliation defines a flat connection on the normal bundle ν_Σ of Σ via the Bott construction. The classical Milnor-Wood inequality then implies that the Euler class of ν_Σ satisfies $|e(\nu_\Sigma)| \leq g(\Sigma) - 1$. This inequality is then an obvious necessary condition for a given surface to be realisable as a leaf of a foliation. Using the relative h-principle Mitsumatsu and Vogt showed that Σ can be made a leaf of a foliation if and only if its normal bundle satisfies the Milnor-Wood inequality and there exist cohomology classes e_1, e_2 satisfying the necessary cohomological conditions described above as well as the following equations:

$$e_1([\Sigma]) = e(\nu_\Sigma) = [\Sigma]^2 \text{ and } e_2([\Sigma]) = 2 - 2g(\Sigma).$$

By using the alternate form of the distribution equations, we will generate many examples of distributions where one has great flexibility in solving the additional equations needed to realise a given surface Σ as a leaf. These examples can then be used to answer certain questions posed in [MV] (cf. Section 2.3).

In order to obtain stronger results on the geometry of leaves of foliations in view of the h-principles at hand, one needs to consider more restricted classes of foliations. If one considers foliations that are complex analytic, then a closed leaf must have trivial normal bundle (Proposition 2.4.1). So complex foliations are too rigid to have interesting closed leaves. Another interesting class of foliations are those that are symplectic, in the sense that each leaf is symplectic with respect to some symplectic form. If a foliation is symplectic, then there are no longer any local obstructions as in the case of complex analytic foliations. There are however subtle restrictions on the Euler classes of the underlying distributions of symplectic foliations that come from Seiberg-Witten Theory. In particular, if e_1, e_2 are the Euler classes of the underlying distribution of the foliation and its orthogonal complement, then either $e_1 + e_2$ or $e_1 - e_2$ is $\pm c_1(K)$, where $c_1(K)$ is the canonical class associated to the symplectic form. Moreover, if $b_2^+(M)$ is at least two, then there are only finitely many possibilities for $c_1(K)$ (Proposition 2.4.6).

A further special case of symplectic foliations are so-called symplectic pairs. We refer to the main text for a general definition, but in the case of 4-manifolds a symplectic pair is equivalent to a pair of closed 2-forms ω_1, ω_2 such that

$$\omega_1^2 = \omega_2^2 = 0 \text{ and } \omega_1 \wedge \omega_2 \neq 0.$$

The kernel foliations of the forms ω_1, ω_2 are then symplectic foliations with respect to the symplectic forms $\omega_1 \pm \omega_2$. In the case of symplectic pairs the restrictions on the Euler classes are even stronger. In particular, under the assumption that $b_2^{\pm}(M) > 1$, the possibilities for the Euler classes of both foliations are finite (Proposition 2.5.2). In the case of a product of Riemann surfaces of genus greater than two, the possible canonical classes are known and one can calculate that the Euler classes of the kernel distributions of a symplectic pair are essentially unique (see Example 2.5.3). These results then give cohomological obstructions for surfaces to be realisable as leaves of symplectic pairs.

There are also more subtle geometric restrictions on the way a leaf of a symplectic pair can be embedded in a given 4-manifold. As noted above each leaf L of the kernel foliations of a symplectic pair is symplectic with respect to both of the symplectic forms $\omega_1 \pm \omega_2$. This fact can be exploited to show that a piece of the leaf L cannot be locally isotopic to a piece of a Milnor fibre, under certain assumptions either on the topology of the manifold or on the homology class represented by L (Proposition 2.5.13).

1.2 Surface bundles and their characteristic classes

The topology of surface bundles is a rich and well-studied area of mathematics. By a classical result of Earle and Eells, the classifying space of oriented surface bundles is an Eilenberg-MacLane space $B\Gamma_h = K(\Gamma_h, 1)$, whose fundamental group is the mapping class group of Σ_h. The group Γ_h is defined as the group of orientation preserving diffeomorphisms of Σ_h modulo isotopy. In this way a given surface bundle $\Sigma_h \to E \to B$ is classified by its holonomy representation $\pi_1(B) \to \Gamma_h$, which induces the classifying map of E.

The cohomology of $B\Gamma_h$ is important not only in the theory of surface bundles but also in the theory of moduli spaces of Riemann surfaces, since the rational cohomologies of $B\Gamma_h$ and the moduli space of genus h Riemann surfaces $\mathcal{M}_h$ are isomorphic. There are a plethora of non-trivial cohomology classes in $H^*(B\Gamma_h)$ called Mumford-Miller-Morita (MMM) classes in honour of their discoverers. To define these we let $e(E)$ denote the vertical Euler class of the oriented rank-2 vector bundle of vectors that are tangent to the fibres of E. The k-th MMM-class is then defined as $e_k(E) = \pi_! e^{k+1}(E)$, where $\pi_!$ denotes the transfer homomorphism given by integration along the fibre. The geometry of these characteristic classes is very interesting and we examine several aspects of this in this thesis.

Our main motivation for studying the MMM-classes is a conjecture due to Morita that the classes e_k are bounded in the sense of Gromov. We recall that a cohomology class is bounded in the sense of Gromov if it has a representative singular cocycle that is bounded on singular simplices. The vertical Euler class e is bounded (Proposition 3.1.7) and we give a new proof of this result, which is originally due to Morita. Our proof uses the adjunction inequality coming from Seiberg-Witten theory and is independent of Morita's original argument. The boundedness of e then implies that the self-intersection number of a section of a bundle over a 2-dimensional base is bounded in terms of the genus of the fibre and base. Mitsumatsu and Vogt conjectured that a similar bound should exist for more general multisections. We verify this conjecture in the case of pure sections (Proposition 3.1.10). However, we also show that a stable version of the conjecture of Mitsumatsu and Vogt, which is technically stronger than their original question, is in fact false (Corollary 3.1.15).

Returning to the question of Morita concerning the boundedness of the MMM-classes one can use the boundedness of the vertical Euler class to see that if e_k is non-trivial for a bundle, whose base is a $2k$-dimensional manifold, then the total space must have non-zero simplicial volume. The condition that a manifold has non-vanishing simplicial volume is equivalent to the fact that the Poincaré dual of the fundamental class is bounded as a cohomology class in the above sense. If the non-vanishing of the simplicial volume of the total space of a surface bundle implied that the base also had non-vanishing simplicial volume, then Morita's conjecture would follow. However, in full generality this does not hold with counterexamples given by hyperbolic surfaces bundles over the circle. The fundamental group of the circle is amenable and a general fact about the simplicial volume is that it vanishes for manifolds with amenable fundamental group. The examples above are in fact the exception and this leads us to show that the simplicial volume of a surface bundle over a base with amenable fundamental group is trivial as soon as the dimension of B is greater than 1 (Theorem 3.2.2). As a corollary of this theorem, we reprove a result of Morita that the MMM-classes vanish on amenable groups (Theorem 3.2.3).

There is a weaker notion of boundedness for cohomology classes that is also due to Gromov, namely hyperbolicity. For a compact manifold a cohomology class is hyperbolic if the pullback of a de Rham representative to the universal cover has a C^0-bounded primitive with respect to the pullback metric. The hyperbolicity condition can be reexpressed in a manner that does not involve metrics and can be generalised to arbitrary spaces (cf. Section 3.3). Although hyperbolicity is strictly weaker than boundedness, it does imply vanishing on amenable subgroups and as further evidence for Morita's conjecture, we prove that the MMM-classes are hyperbolic (Theorem 3.3.8).

Another natural question concerning the MMM-classes is the following: given a base manifold B, for which MMM-classes does there exist a surface bundle over B so that $e_k(E)$ is non-trivial? This question can be rephrased as a question about the representability of certain homology classes in $H_*(B\Gamma_h)$. We restrict our attention to the case where $B = M_1 \times M_2$ is a non-trivial product of two manifolds and show that e_k vanishes for such manifolds if the dimension of both of the factors is less than $2k$ (Theorem 3.2.5). As an application of the results discussed above, we then deduce strong restrictions on the topology of bundles that are holomorphic, in the sense that both base and total space are complex and the bundle projection is holomorphic. In particular, we show that if the fundamental group of the base of a holomorphic bundle is amenable, then the total space is virtually a product, and if the base is a product of Riemann surfaces, then e_1^2 must be trivial (Section 3.4).

1.3 Flat surface bundles

An important special case of fibre bundles from the perspective of foliation theory are flat bundles. Recall that a fibre bundle E over a manifold B is called flat if it admits a foliation that is everywhere complementary to the fibres. An equivalent formulation of this condition in the case of surface bundles is that the holonomy homomorphism $\pi_1(B) \to \Gamma_h$ lifts to the group of orientation preserving diffeomorphisms $Diff^+(\Sigma_h)$. In general, there are cohomological restrictions to the existence of flat structures, since the Bott vanishing theorem implies that the MMM-classes $e_k(E)$ vanish for flat bundles if $k \geq 3$. However, in low dimensions flatness is not as restrictive. If the dimension of the base is one, then rather trivially the total space of any bundle admits a horizontal foliation, which may be thought of as a horizontal flow. If B is a surface, then any bundle admits a horizontal foliation after stabilisation by a result of Kotschick and Morita.

A closed leaf of the horizontal foliation of a flat bundle E intersects each fibre in a finite number points. Thus the existence of such a closed leaf is equivalent to a reduction of the holonomy group of E to group $Diff^+(\Sigma_{h,k})$ that consists of orientation preserving diffeomorphisms fixing k marked points. Characteristic classes of flat bundles with closed leaves can then be considered as elements in the group cohomology of $Diff^+(\Sigma_{h,k})$ considered as a discrete group. This cohomology is of course difficult to understand in general and we content ourselves with the low dimensional cases. As such, we compute that the first homology of $Diff^+_\delta(\Sigma_{h,k})$ is $\mathbb{R}^+ \times \mathbb{Z}_2$ under the assumption that h is at least three and k is at least two (Proposition 4.1.13). We also compute the abelianisation of the group of compactly supported diffeomorphism on $\mathbb{R}^2$ fixing the origin. This result is originally due to Fukui, although his proof seems to be incomplete (see Section 4.1.2).

For flat surface bundles one obtains a natural characteristic class by restricting the vertical Euler class to a closed leaf. If the base is a surface, this corresponds to the self-intersection number of the leaf and we show that there exist foliations for which these self-intersection numbers are non-trivial (Proposition 4.1.5). Moreover, if we instead assume that a given bundle admits a section that has self-intersection divisible by $2h-2$, then this section can be made a leaf of a horizontal foliation after stabilisation (Theorem 4.1.7). This result means that obstructions to the existence of certain horizontal foliations given in [BCS] are not stable in the sense that they disappear after one performs a certain number of stabilisations.

We also study the closed leaves of flat bundles whose horizontal foliations admit transverse symplectic structures. For codimension 2 foliations a transverse symplectic structure is equivalent to the fact that the foliation can be defined as the kernel of a closed 2-form. A transverse symplectic form on a flat surface bundle is then equivalent to a holonomy invariant symplectic form on the fibres, which means that the holonomy map of the bundle lies in the group $Symp(\Sigma_h)$ of symplectomorphisms of Σ_h. Similarly, a flat symplectic bundle with a closed leaf is equivalent to a holonomy representation in the group $Symp(\Sigma_{h,k})$ of symplectomorphisms that fix k marked points. We again concentrate on the case where the base of the bundle is a surface showing that there are indeed flat symplectic surface bundles having closed leaves of non-zero self-intersection (Proposition 4.2.9). The most interesting examples of such foliations occur as the horizontal foliations of symplectically flat sphere bundles, for which any closed leaf with non-zero self-intersection number must be unique

(Section 4.2.1). As a further consequence of Proposition 4.2.9 we deduce that there exist symplectic pairs on 4-manifolds both of whose kernel foliations have leaves with non-zero intersection numbers (Corollary 4.2.10).

1.4 Surface bundles and extended Hamiltonian groups

Given an arbitrary circle bundle it is a basic question to ask whether it bounds a surface bundle. This question may be viewed as the fibred analogue of the bordism problem for closed manifolds. If the circle bundle is assumed to be flat, then one can consider the problem of extending the flat structure to the interior of such a fibred null-cobordism. If the fibre is assumed to be a disc, then there is a dichotomy depending on whether one requires that the foliation be symplectic or not. For in the smooth case, it is a simple matter to show that any flat circle bundle over a surface admits a flat disc bundle filling after stabilisation. However, in the symplectic case the Euler class provides an obstruction by a result of Tsuboi. In fact, Tsuboi gives a formula for computing the Euler class of a flat circle bundle in terms of the Calabi invariant of certain extensions of the boundary holonomy to the interior of a disc (Theorem 5.2.1).

Tsuboi's result can be reformulated in terms of the five-term exact sequence in group cohomology. The advantage of this reformulation is that it can easily be generalised to the case where the fibre of the filling is an arbitrary Riemann surface with one boundary component. After suitably generalising the Calabi map we shall extend Tsuboi's formula to the extended Hamiltonian group of a Riemann surface with one boundary component (Theorem 5.2.10). Here the extended Hamiltonian group is a subgroup of the symplectomorphism group which is defined as the kernel of a certain crossed homomorphism that is an extension of the ordinary Flux homomorphism in symplectic geometry.

As a consequence we see that the Euler class gives a obstruction to filling a circle bundle by a flat surface bundle with holonomy in the extended Hamiltonian group. We contrast this result with the fact that any flat circle bundle can be filled by a flat symplectic bundle after stabilisation (Theorem 5.1.4). As a final application of these methods we derive a Tsuboi-type formula for the first MMM-class of a surface bundle with boundary (Corollary 5.3.3).

As previously mentioned, the Bott vanishing theorem implies that the MMM-classes e_k vanish on flat bundles if k is a least three. On the other hand, there exist flat bundles for which the first MMM-class is non-trivial by a result of Kotschick and Morita. These bundles can in fact be chosen to have symplectic holonomy. This led Kotschick and Morita to ask whether the second MMM-class can be non-trivial for flat surfaces bundles or flat bundles with symplectic holonomy. We shall answer this question under the assumption that the holonomy group lies in the extended Hamiltonian group. For as in the case of a surface with boundary one can also define an extended Flux homomorphism on $Symp(\Sigma_h)$, whose kernel is the extended Hamiltonian group $\widetilde{Ham}(\Sigma_h)$. We show that the class e_2 is trivial when considered as an element in the group cohomology of $\widetilde{Ham}(\Sigma_h)$ and that the powers e_1^k are also trivial if $k \geq 2$ (Theorem 5.4.4).

1.5 Characteristic classes of symplectic foliations

The Pontryagin classes of the normal bundle to a foliation $\mathcal{F}$ define characteristic classes, which depend only on the homotopy class of the underlying distribution. If the foliation has a transverse volume form ω, then for certain polynomials P in the Pontryagin classes we show that there exist well-defined foliated cohomology classes γ_P such that $P(\Omega) = \omega \wedge \gamma_P$ (Proposition 6.1.3). Similar factorisations for transversally symplectic foliations were obtained in [KM3] via computations in the Gelfand-Fuks cohomology of formal Hamiltonian vector fields. By working with Bott connections we obtain a more concrete construction that is immediately applicable to foliations that are only transversally volume preserving.

Although one expects that the foliated cohomology classes γ_P discussed above contain more information than the Pontryagian classes themselves, this is by no means immediate from their definition alone. We consider this problem by starting with the first non-trivial case. That is, we assume that we have a transversally symplectic codimension 2 foliation on a 4-manifold. In this case the first Pontryagin class of the normal bundle to the foliation has a factorisation $p_1(\nu_F) = \omega \wedge \gamma$.

In general, any foliated cohomology class yields a well-defined class in ordinary cohomology when restricted to a leaf. We show that there exist symplectically foliated $\mathbb{R}^2$-bundles that have closed leaves L, such that the restriction of γ to L is non-trivial. Moreover, we may assume that such a bundle is topologically trivial (Theorem 6.3.7). We conclude that the class γ contains information that is sensitive to the geometry of the foliation and not just its homotopy class as a distribution. The examples that we obtain are of differentiability class C^k for finite k and we unfortunately cannot obtain smooth examples using our methods.

1.6 Symplectic cobordism and transverse knots

In the general theory of knots (or links) in S^3, or more generally in an arbitrary 3-manifold, one usually considers knots up to isotopy. A less restrictive equivalence relation on the set of knots is that of cobordism or concordance. Here two oriented links K_0, K_1 in S^3 are cobordant if there is a properly embedded, oriented surface Σ in $S^3 \times [0,1]$ such that the intersection of Σ with $S^3 \times \{i\}$ is K_i. Two knots are concordant if they are cobordant via an annulus. The cobordism relation is trivial for the 3-sphere, since every knot bounds an embedded surface and, hence, every knot is null-cobordant. The concordance relation is however far from trivial and has been a subject of intense study amongst knot theorists for many years.

We shall consider similar relations in the setting of contact topology. There are two natural classes of knots that one studies in the presence of a contact structure. The first are transverse knots that are everywhere transverse to the contact distribution and the second are Legendrian which are everywhere tangent. The analogous notion of cobordism for transverse knots is symplectic cobordism. Two transverse knots K_0, K_1 in a contact manifold M will be symplectically cobordant if there is a properly embedded symplectic surface Σ in the symplectisation $(M \times \mathbb{R}, d(e^t\lambda))$ whose negative/positive ends are $K_0 \times (-\infty, -R)$ and $K_1 \times (R, \infty)$ respectively. As in the smooth case, two transverse knots are symplectically concordant if they are cobordant via an annulus. This notion is the transverse analogue of

Lagrangian cobordism/concordance as introduced by Chantraine.

For ordinary smooth cobordism every knot is null-bordant. However this is no longer true for transverse knots in S^3 with its standard contact structure. In fact, a link is symplectically null-bordant if and only if it is quasipositive by a result of Boileau and Orevkov. Another important property of null-bordisms for knots in S^3 is that they minimise the genus for all smooth slicing surfaces in the 4-ball (Theorem 7.2.3). In fact, if Σ is a symplectic cobordism from K_0 to K_1, then one has the following equation for the slice genera of the knots

$$2 - 2g(K_1) = 2 - 2g(K_0) + \chi(\Sigma).$$

This means that if K_0 is symplectically cobordant to K_1 by a cobordism with negative Euler characteristic, then the opposite relation cannot hold. This implies an asymmetry in the symplectic cobordism relation that is not present in the smooth setting.

We have seen that the slice genus obstructs symmetry for the symplectic cobordism relation. Hence we consider the question of symmetry for symplectic concordance instead, since two concordant knots necessarily have the same slice genus. However, this relation also fails to be symmetric and we produce an infinite family of examples K_n that are not symplectically concordant to the unknot K_0, but which are symplectically null-concordant meaning that there is a symplectic concordance from the unknot to K_n.

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Chapter 2

Distributions and leaves of foliations

We first review basic facts concerning 2-plane distributions on 4-manifolds. We then discuss the consequences of the Milnor-Wood inequality on closed leaves of 2-dimensional foliations and recall the cohomological criteria of Mitsumatsu and Vogt that give necessary and sufficient conditions for the realisability of a surface as a leaf of a foliation. After showing that there is a great deal of flexibility in satisfying these cohomological criteria, we give a series of examples that answer two questions posed in [MV] concerning the topological properties of closed leaves of foliations (cf. Examples 2.3.4 and 2.3.5). We next consider special classes of foliations and their closed leaves, focusing mainly on symplectic foliations and symplectic pairs. In particular, we derive restrictions on the local geometry of leaves of symplectic pairs.

2.1 Conventions

All manifolds are connected and smooth. We shall only consider oriented distributions, which will always be smooth. Unless otherwise stated all (co)homology groups will be taken to have integral coefficients.

2.2 The distribution equations on a 4-manifold

It is a basic question, whether a given manifold M admits a foliation. For 1-dimensional foliations on compact manifolds, this is equivalent to the vanishing of the Euler characteristic. In general, a necessary condition for M to admit a k-dimensional foliation is that it first admits a k-dimensional distribution. For 2-dimensional foliations of codimension greater than 1, this is also sufficient. In fact more is true and one has the following (relative) h-principle which is due to Thurston.

Theorem 2.2.1 ([Th2], Cor. 3)**.** *Let M be an oriented manifold of dimension greater than 3 and let ξ be an oriented distribution of 2-planes. Then ξ is homotopic to a foliation. Furthermore, if ξ is integrable in a neighbourhood of a compact set $K \subset M$ then this homotopy may be taken relative to K.*

We shall now explain how oriented distributions correspond to sections of certain 2-sphere bundles over M, for a more detailed account we refer to [HH].

We first consider the double covering

$$\begin{tikzcd} SO(4) \arrow[r, "p"] & SO(3)_+ \times SO(3)_- \\ SO(2) \times SO(2) \arrow[u] \arrow[r, "p"] & SO(2) \times SO(2) \arrow[u] \end{tikzcd}$$

that one obtains by considering the natural representation of $SO(4)$ on Λ^2, which then gives induced representations on $\Lambda^2_\pm$, the space or self-, resp. anti-self-dual 2-forms. The subgroup $SO(2) \times SO(2)$ in $SO(4)$ denotes the group of transformations that preserve the decomposition $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ and are orientation preserving on each factor. The image under p of this subgroup fixes a 1-dimensional subspace in Λ^2_+, Λ^2_- respectively that we may describe explicitly. To this end we pick oriented bases $\{e_1, e_2\}$ and $\{e_3, e_4\}$ for the two $\mathbb{R}^2$-factors and let $\{e^1, e^2\}$ and $\{e^3, e^4\}$ denote the corresponding dual bases. The invariant subspaces are then given by the span of the following 2-forms:

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 \in \Lambda^2_+$$

$$\bar{\omega} = e^1 \wedge e^2 - e^3 \wedge e^4 \in \Lambda^2_-.$$

We then identify the image of $SO(2)\times SO(2)$ with the set of factor-wise orientation preserving transformations of

$$\omega^\perp \oplus \bar{\omega}^\perp \subset \Lambda^2_+ \oplus \Lambda^2_-.$$

After choosing an explicit basis for $\omega^\perp$ and $\bar{\omega}^\perp$ and identifying $SO(2) = U(1) \subset \mathbb{C}$ in the usual fashion, one computes that the map

$$SO(2) \times SO(2) \xrightarrow{p} SO(2) \times SO(2),$$

is given by

$$(e^{i\phi_1}, e^{i\phi_2}) \mapsto (e^{i(\phi_1 - \phi_2)}, e^{i(\phi_1 + \phi_2)}).$$

An oriented 2-plane distribution ξ is equivalent to a reduction of the structure group of TM to $SO(2) \times SO(2)$ and under the correspondence above this in turn reduces the structure group of $\Lambda^2_+(M) \oplus \Lambda^2_-(M)$ to $SO(2) \times SO(2)$. However, such a reduction is equivalent to having a pair of sections in the product of unit sphere bundles $S(\Lambda^2_+(M)) \times S(\Lambda^2_-(M))$. We shall denote such a pair of sections by (α_+, α_-).

Conversely, given such a pair of non-vanishing sections of unit length we obtain an oriented distribution as the kernel of the following 2-form $\alpha = \alpha_+ - \alpha_-$. For in terms of the decomposition into self- and anti-self-dual parts we have

$$0 \neq ||\alpha_+||^2 = \alpha_+ \wedge \alpha_+ = -\alpha_- \wedge \alpha_- = ||\alpha_-||^2$$

and this is equivalent to the fact $\alpha^2 = 0$, that is α is of constant rank and the kernel distribution is well-defined. We also have an analogously defined form $\alpha^\perp$ for $\xi^\perp$ and it is not hard to see that $\alpha^\perp = \alpha_+ + \alpha_-$. We then orient $\xi = \mathrm{Ker}(\alpha)$ so that $\alpha^\perp|_\xi > 0$. By construction if we started with an oriented distribution ξ, then $(\alpha_+, \alpha_-) = (\omega, \bar{\omega})$ and under the above correspondence we obtain our original distribution again.

Now a pair of non-vanishing sections (α_+, α_-) gives splittings

$$\Lambda^2_\pm(M) = \mathbb{R}\alpha_\pm \oplus L_\pm,$$

and there is an explicit relationship between the Euler class of ξ and its complement $\xi^\perp$ and the Euler classes of $L_\pm$. We shall denote the Euler classes of $\xi, \xi^\perp$ by e_1, e_2 and those of $L_\pm$ by $K_\pm$. Our description of the map p on $SO(2) \times SO(2)$ yields that as $SO(2)$-bundles

$$\xi \otimes \xi \cong L_+ \otimes L_-$$

$$\xi^\perp \otimes \xi^\perp \cong L_+^{-1} \otimes L_-,$$

and, hence,

$$2e_1 = K_+ + K_-$$

$$2e_2 = -K_+ + K_-.$$

Thus we see that the existence of a distribution is equivalent to the existence of sections of certain associated bundles. The existence of sections of S^2-bundles can be formulated purely in cohomological terms and this is the content of the following theorem, which is attributed to Massey in [DW].

Theorem 2.2.2 (Massey)**.** *A 2-sphere bundle over a compact, oriented 4-manifold $S^2 \to E \to M$ admits a section if and only if there is a class $\gamma \in H^2(M)$ such that*

$$\gamma \smile \gamma = p_1(E)$$

$$\gamma \equiv w_2(M) \ \mathit{mod}\ 2.$$

Moreover, in this case the associated $\mathbb{R}^3$-bundle splits as $E = \mathbb{R} \oplus L$ where L is an oriented rank-2 bundle with Euler class $e(L) = \gamma$.

In our case we can compute the first Pontryagin classes of $S(\Lambda^2_\pm(M))$ in terms of the characteristic classes of M. To this end we prove the following lemma, which is valid for any oriented, rank four vector bundle.

Lemma 2.2.3. *Let $E \to M$ be an oriented real vector bundle of rank four. And let $\Lambda^2_\pm(E)$ be the associated bundles of self-, resp. anti-self-dual 2-forms. Then the following holds modulo torsion:*

$$p_1(\Lambda^2_\pm(E)) = \pm 2e(E) + p_1(E).$$

Proof. We consider the map

$$SO(4) \xrightarrow{p} SO(3)_+ \times SO(3)_-$$

and let $p_\pm$ be the composition with the projections to each factor. We then have induced maps on classifying spaces

$$BSO(4) \xrightarrow{p_\pm} BSO(3)_\pm.$$

We further let $\Lambda^2_\pm$ denote the bundles of self-, resp. anti-self-dual 2-forms associated to the universal bundle $ESO(4)$, which are then classified by the maps $p_\pm$. Since $H^4(BSO(4))$ is generated by p_1 and the Euler class e modulo torsion, there are constants $\lambda_\pm, \mu_\pm \in \mathbb{Z}$ so that the following holds in $H^4(BSO(4))/Tor$:

$$p_1(\Lambda^2_\pm) = p_\pm^* p_1(ESO(3)) = \lambda_\pm e + \mu_\pm p_1.$$

Now let $E = TM$ be the tangent bundle of a symplectic manifold (M, ω). Then for an ω-compatible almost complex structure J, one has the following isomorphism of complex bundles

$$\Lambda^2_+(M) \otimes \mathbb{C} \cong K \oplus \bar{K} \oplus \mathbb{C}\omega,$$

where K denotes the canonical bundle given by the chosen almost complex structure (cf. [DKr], Lemma 2.1.57). Thus applying the Whitney sum formula, we obtain the following modulo 2-torsion

$$p_1(\Lambda^2_+(M)) = -c_2(\Lambda^2_+(M) \otimes \mathbb{C}) = c_1(K)^2 = 2e(M) + p_1(M).$$

Applying this calculation to the symplectic manifolds $M_1 = \mathbb{C}P^2$ and $M_2 = S^2 \times S^2$, verifies the formula for an arbitrary bundle E since the vectors $(e(M_1), p_1(M_1))$ and $(e(M_2), p_1(M_2))$ are linearly independent.

Finally, for an oriented manifold we note that $\Lambda^2_-(E) = \Lambda^2_+(\bar{E})$, where $\bar{E}$ denotes E taken with the opposite orientation. Hence, modulo 2-torsion

$$p_1(\Lambda^2_-(E)) = p_1(\Lambda^2_+(\bar{E})) = 2e(\bar{E}) + p_1(\bar{E}) = -2e(E) + p_1(E),$$

which completes the proof. □

As a consequence of of Hirzebruch Signature Theorem, we obtain the following proposition in the case of 4-manifolds.

Proposition 2.2.4 (Characteristic equations 1)**.** *A closed, oriented 4-manifold M admits an oriented 2-plane distribution if and only if there exists a pair $K_+, K_- \in H^2(M)$ such that:*

$$\langle K^2_\pm, [M]\rangle = \pm 2\chi(M) + 3\sigma(M) \tag{2.1}$$

$$K_\pm \equiv w_2(M) \ \textit{mod 2}.$$

Proof. This follows immediately from Lemma 2.2.3 and Theorem 2.2.2 and the fact that $\langle p_1(M), [M]\rangle = 3\sigma(M)$ by the Hirzebruch Signature Theorem. □

It is clear that the first of these equations only depends on K_+, K_- considered as classes in $H^2(M)/Tor$. This holds for the second equation as well. For if $K_\pm$ are elements that reduce modulo 2 to $w_2(M)$, then they are characteristic elements for the cup product pairing on $H^2(M)$, that is

$$\langle \alpha^2, [M]\rangle \equiv \langle \alpha \smile K_\pm, [M]\rangle \text{ mod } 2$$

for all $\alpha \in H^2(M)$. Conversely, any characteristic element $[K] \in H^2(M)/Tor$ has a representative $\bar{K} \in H^2(M)$, whose reduction modulo 2 is $w_2(M)$.

Lemma 2.2.5. *Let $[K] \in H^2(M)/Tor$ be a characteristic element for the intersection form of an oriented 4-manifold M. Then the class $[K]$ has a representative $\bar{K}$ whose reduction modulo 2 is $w_2(M)$.*

Proof. We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Ext(H_1(M),\mathbb{Z}) \arrow[r] \arrow[d] & H^2(M,\mathbb{Z}) \arrow[r, "p"] \arrow[d] & Hom(H_2(M),\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Ext(H_1(M),\mathbb{Z}_2) \arrow[r] & H^2(M,\mathbb{Z}_2) \arrow[r, "p"] & Hom(H_2(M),\mathbb{Z}_2) \arrow[r] & 0. \end{tikzcd}$$

If K is characteristic, then for all integral classes $[S]$

$$\langle K, [S] \rangle \equiv [S]^2 \bmod 2$$

and hence $p(K)$ reduced modulo 2 is the same as $p(w_2(M))$. Moreover the left vertical arrow is surjective and the image consists of the mod 2 reductions of torsion elements in $H^2(M)$. So after adding some torsion we obtain a $\bar{K}$ that is still characteristic and reduces to $w_2(M)$ modulo 2. $\square$

With the help of Lemma 2.2.5 we may give an alternate form of the characteristic equations that will be very useful in conducting calculations below.

Proposition 2.2.6 (Characteristic equations 2)**.** *A closed, oriented 4-manifold M admits an oriented 2-plane distribution if and only if there exists a pair of characteristic elements for the intersection form $K_+, K_- \in H^2(M)$ such that:*

$$\langle K_\pm^2, [M] \rangle = \pm 2\chi(M) + 3\sigma(M). \tag{2.2}$$

We may also express Proposition 2.2.4 above in terms of the Euler classes of the distributions ξ and $\xi^\perp$. The correspondence between the classes e_1, e_2 and K_+, K_- is only valid up to 2-torsion, however this is not a major problem and we obtain the following proposition.

Proposition 2.2.7 (Distribution equations)**.** *Let M be a 4-manifold and let $Tor_2 \subset H^2(M)$ be the subgroup of 2-torsion elements. There exist complementary 2-plane distributions ξ and $\xi^\perp$ on M with Euler classes $e_1, e_2 \in H^2(M)/Tor_2$ if and only if the following equations hold:*

$$\begin{aligned} \langle e_1^2 + e_2^2, [M] \rangle &= 3\sigma(M) \\ \langle e_1 \smile e_2, [M] \rangle &= \chi(M) \\ e_1 + e_2 &\equiv w_2(M) \ \mathit{mod}\ 2. \end{aligned} \tag{2.3}$$

Proof. For the necessity we choose an almost complex structure compatible with the splitting $TM \cong \xi \oplus \xi^\perp$. We set $e_1 = c_1(\xi)$ and $e_2 = c_1(\xi^\perp)$. Then the Whitney sum formula yields

$$\begin{aligned} e_1^2 + e_2^2 &= c_1^2(M) - 2c_2(M) = p_1(M) \\ e_1 \smile e_2 &= c_2(M) = e(M) \\ e_1 + e_2 &= c_1(M). \end{aligned}$$

Finally, we apply the Hirzebruch Signature Theorem and use the fact that $c_1(M)$ reduces to $w_2(M)$ in mod 2 cohomology.

Conversely, if we have solutions e_1, e_2, then we set

$$K_+ = e_1 + e_2, \quad K_- = e_1 - e_2.$$

These classes satisfy the hypotheses of Proposition 2.2.4 and hence we have distributions $\xi, \xi^\perp$, whose Euler classes e_1', e_2' satisfy:

$$2e_1' = K_+ + K_- = 2e_1$$

$$2e_2' = -K_+ + K_- = 2e_2,$$

and the classes e_i and e_i' agree modulo 2-torsion. □

Using the distribution equations above one may give necessary and sufficient conditions for the existence of distributions in terms the Euler characteristic and signature of M. In particular, the existence of a smooth foliation only depends on the homotopy type of M. The following result goes back to Atiyah and Saeki (cf. [Mats]).

Proposition 2.2.8 (Existence of distributions)**.** *Let M be an oriented 4-manifold with indefinite intersection form, then M admits a distribution if and only if*

$$\sigma(M) \equiv 0 \textit{ mod 2} \text{ and } \chi(M) \equiv \sigma(M) \textit{ mod 4}.$$

Proof. By Proposition 2.2.6 it suffices to find characteristic elements $K_+, K_- \in H^2(M)$ for the intersection form such that:

$$\langle K_\pm^2, [M] \rangle = \pm 2\chi(M) + 3\sigma(M).$$

By the Theorem of van der Blij (see [MH], p. 24), if K is characteristic for the intersection form, then the following holds:

$$\langle K^2, [M] \rangle \equiv \sigma(M) \text{ mod } 8.$$

So a necessary condition for a solution of (2.2) is that

$$\begin{aligned} & \sigma(M) \equiv \pm 2\chi(M) + 3\sigma(M) \text{ mod } 8 \\ \iff & 2\chi(M) \equiv \pm 2\sigma(M) \text{ mod } 8 \\ \iff & \chi(M) \equiv \pm\sigma(M) \text{ mod } 4 \\ \iff & \chi(M) \equiv \sigma(M) \text{ mod } 4 \text{ and } \sigma(M) \equiv 0 \text{ mod } 2. \end{aligned}$$

Next we claim that

$$\Sigma = \{K^2 \mid K \text{ is characteristic}\} = \sigma(M) + 8\mathbb{Z},$$

which means that if $\sigma(M) \equiv \pm 2\chi(M) + 3\sigma(M)$ mod 8, then this is sufficient for the existence of a solution of (2.2). We assumed that M has indefinite intersection form, thus by the Hasse-Minkowski classification of indefinite, integral quadratic forms we have to consider two cases according to whether the intersection form that we denote by $Q(\ ,\ )$ is even or odd.

Case 1: $Q(\ ,\ )$ is odd.

In this case $H^2(M,\mathbb{Z})/Tor$ has a basis $e_1, ...e_n$ so that for an element $\alpha = \sum \lambda_i e_i$ we have

$$Q(\alpha,\alpha) = \sum_{i=1}^{b_2^+} \lambda_i^2 - \sum_{j=b_2^+ +1}^{b_2^+ + b_2^-} \lambda_j^2$$

and an element $K = \sum \lambda_i e_i$ is characteristic if and only if $\lambda_i \equiv 1 \bmod 2$, for all $1 \le i \le n$. Then by noting that

$$(2m+1)^2 - (2m-1)^2 = 8m,$$

and defining K by taking

$$\lambda_i = \begin{cases} 2m+1, i = b_2^+ \\ 2m-1, i = b_2^+ + 1 \\ 1 \qquad\quad, \text{otherwise} \end{cases}$$

we see that

$$Q(K,K) = \sigma(M) + 8m,$$

thus proving the claim in the odd case, since m can be chosen arbitrarily.

Case 2: $Q(\ ,\ )$ is even.

In this case the Hasse-Minkowski classification implies that $Q \cong kH \oplus lE_8$ and $K = \sum \lambda_i e_i$ is characteristic if and only if $\lambda_i \equiv 0 \bmod 2$ for all $1 \le i \le n$. Hence $K = (2x, 2y, 0, ..., 0)$ is characteristic for Q and

$$K^2 = 8xy.$$

Thus the set Σ defined above is in fact $8\mathbb{Z} = \sigma(M) + 8\mathbb{Z}$. □

Remark 2.2.9. If the intersection form Q of M is definite, say positive definite, then by Donaldson's Theorem it is diagonalisable. It then follows that there is an integral basis $\{e_i\}$ such that

$$Q(\sum_{i=1}^{b_2} \lambda_i e_i, \sum_{i=1}^{b_2} \lambda_i e_i) = \sum_{i=1}^{b_2} \lambda_i^2$$

and we see that the set of squares of characteristic elements is

$$\Sigma = \{n \mid n \ge b_2(M) \text{ and } n \equiv \sigma \bmod 8\}.$$

So in other words a manifold with a definite intersection form admits a distribution if and only if it satisfies the hypotheses of Proposition 2.2.8 and

$$\begin{aligned} & b_2(M) \le \pm 2\chi(M) + 3\sigma(M) \\ \iff & -2b_2(M) \le \pm 2\chi(M) = \pm 2(2 - 2b_1(M) + b_2(M)) \\ \iff & b_1(M) \le 1 + b_2(M) \text{ and } 1 \le b_1(M). \end{aligned}$$

Remark 2.2.10. We also note that if $b_2^+(M), b_2^-(M) > 1$, then by the proof of Proposition 2.2.8 above there are infinitely many characteristic elements K_i that solve

$$\langle K_i^2, [M]\rangle = 2\chi(M) + 3\sigma(M).$$

Hence, by taking any fixed class L that solves

$$\langle L^2, [M]\rangle = -2\chi(M) + 3\sigma(M),$$

we obtain distributions with infinitely many distinct characteristic pairs (K_+, L). Since these pairs are invariant under homotopy we conclude that M has infinitely many non-homotopic distributions.

2.3 The Milnor-Wood inequality and compact leaves

In this section we will recall the Milnor-Wood inequality for flat $GL^+(2, \mathbb{R})$-bundles over closed surfaces and discuss its consequences for the existence of foliations with closed leaves. We shall then recall the homological criteria of Mitsumatsu and Vogt in [MV] for the existence of a foliation having a given compact surface as a leaf. Then by using the characteristic equations (cf. Proposition 2.2.6), we shall give examples that answer several questions posed in [MV].

We shall first recall some generalities concerning obstruction classes for flat bundles. We let G be any connected Lie group and consider a flat principal G-bundle E over a surface Σ. By the classification of flat bundles, this corresponds to a homomorphism $\rho : \pi_1(\Sigma) \to G$, which is unique up to conjugation by elements in G. We let $\widetilde{G} \xrightarrow{p} G$ denote the universal cover of G and note that the fundamental group of G can be identified with a subgroup of $\widetilde{G}$ in a natural way. If a_i, b_i denote the standard generators of $\pi_1(\Sigma)$, we choose lifts α_i, β_i of $\rho(a_i), \rho(b_i)$ in $\widetilde{G}$. It was shown by Milnor in [Mil] that the obstruction to the existence of a global section, that we denote by $c(E) \in H^2(\Sigma, \pi_1(G))$, is given by

$$c(E) = -\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}....\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}. \tag{2.4}$$

In the case $G = GL^+(2, \mathbb{R})$, we have that $\pi_1(G) = \mathbb{Z}$ and up to multiplication by a constant $c(E)$ is just the Euler class $e(E)$. Another observation is that if G is abelian, then every flat principal G-bundle over a surface admits a section and is hence trivial. For in this case the universal cover is also an abelian Lie group and hence the right-hand side of equation (2.4) is always trivial.

Proposition 2.3.1. *If $E \to \Sigma$ is a flat principal G-bundle over a surface and G is abelian, then E is a trivial bundle.*

The Milnor-Wood inequality expresses a relationship between the Euler class of certain flat bundles over closed surfaces and the genus of the base. This inequality is due to Milnor in the case of $GL^+(2, \mathbb{R})$-bundles and its generalisation to $Homeo^+(S^1)$ is due to Wood. We shall in fact only need Milnor's original inequality in what follows, but in accordance with standard usage we will refer to the following result as the Milnor-Wood inequality.

Theorem 2.3.2 (Milnor-Wood inequality, [Mil])**.** *A $GL^+(2,\mathbb{R})$-bundle $E \to \Sigma$ over a surface of genus $g > 0$ is flat if and only if*

$$|e(\Sigma)| \leq g(\Sigma) - 1.$$

The same holds for flat $SL(2,\mathbb{R})$-bundles.

In [MV] certain criteria were proven for the existence of 2-dimensional foliations with closed leaves on a 4-manifold M. The basic observation is that the Milnor-Wood inequality puts homological restrictions on which classes $[\Sigma] \in H_2(M)$ can occur as leaves of a foliation. To begin with we note that if Σ is a leaf of some foliation $\mathcal{F}$ on M, then there is a connection on the normal bundle $\nu(\mathcal{F})$ that is flat when restricted to leaves of the foliation. This is a so-called Bott connection (cf. Definition 6.1.2 below). Thus, in particular, if Σ is a leaf of some foliation then the normal bundle of Σ is a flat bundle and the Milnor-Wood inequality implies that

$$|[\Sigma]^2| \leq g(\Sigma) - 1.$$

Conversely, there are sufficient conditions for a given embedded surface to be a leaf of some foliation on a manifold M. These are given in the following theorem of Mitsumatsu and Vogt.

Theorem 2.3.3 ([MV], Th. 4.4)**.** *If a compact surface Σ satisfies the Milnor-Wood inequality, then Σ can be realised as a leaf of a foliation $\mathcal{F}$ on M if and only if there are classes e_1, e_2 that solve equations (2.3) and satisfy the following two additional equations:*

$$\langle e_1, [\Sigma] \rangle = \chi(\Sigma), \quad \langle e_2, [\Sigma] \rangle = [\Sigma]^2. \tag{2.5}$$

There are certain questions about the properties of surfaces that are leaves of foliations that would suggest that the condition of being a leaf is in fact restrictive. The first is whether a class $\sigma \in H_2(M)$ knows its foliated genus ([MV], Question 8.8), that is if Σ_1, Σ_2 are leaves of foliations $\mathcal{F}_1, \mathcal{F}_2$ and $[\Sigma_i] = \sigma$ in $H_2(M)$, then we must have $\chi(\Sigma_1) = \chi(\Sigma_2)$. The following examples provide a negative answer to this question.

Example 2.3.4 (The genus of leaves representing a fixed homology class)*.* Suppose we have a manifold M whose signature and Euler characteristic satisfy the congruences of Proposition 2.2.8 so that M admits distributions. Assume further that the intersection form on M is odd and is of the form

$$Q \cong 2\langle 1 \rangle \oplus 2\langle -1 \rangle.$$

For example one can take

$$M = 2\mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2 \# S^1 \times S^3.$$

We choose a basis $v_1, ...v_4$ of $H^2(M)$ so that the intersection form Q has the stipulated from. One then has an explicit family of solutions to equation (2.2) given by setting

$$K_+ = (2t - s, 2y + 1, 2y - 1, 1)$$

$$K_- = (2t + s, 2z + 1, 2z - 1, 1).$$

Here the $s, t \in \mathbb{Z}$ are free parameters, s is odd and y, z are chosen so that equation (2.2) is satisfied. These solutions then yield the following solutions to the distribution equations:

$$e_1 = (2t, y + z + 1, y + z - 1, 1)$$

$$e_2 = (s, z - y, z - y, 0).$$

We let $[\Sigma]$ be a surface of minimal genus representing the class v_1 so that $[\Sigma]^2 = 1$. We set $s = 1$ so that the second equation in (2.5) above is satisfied. Then as t was a free parameter in the set of solutions, we may choose t so that the first equation in (2.5) is also satisfied. Furthermore, we may glue in as many trivial handles as we like to obtain a $\tilde{\Sigma}$ that is homologous to Σ and has arbitrarily large genus. In particular, we can assume that the Milnor-Wood inequality is satisfied and we see that v_1 has representatives of infinitely many genera that can be made a leaf of a foliation. It is easy to see that this example generalises to manifolds M with odd intersection form and $b_2^{\pm}(M) \geq 2$, one simply splits the intersection form

$$Q \cong 2\langle 1 \rangle \oplus 2\langle -1 \rangle \oplus \overline{Q}$$

and augments the solutions for (K_+, K_-) given above by putting odd integers in the remaining entries. In this more general case we may even assume that M is simply connected by setting

$$M = k\mathbb{C}P^2 \# l\overline{\mathbb{C}P}^2,$$

where k and l are both odd and larger than 3.

Another question posed by Mitsumatsu and Vogt is whether the Euler class of a distribution determines whether or not every leaf of a foliation with this Euler class will be genus minimising or not. This is false as the following example shows, providing a negative answer to Question 8.7 of [MV].

Example 2.3.5 (The Euler class does not determine whether the genera of leaves are minimal). We let $M = T^2 \times \Sigma_g$ and endow it with a product symplectic structure so that the two factors are symplectic submanifolds. Then the homology class $\sigma = [T^2 \times pt] + [pt \times \Sigma_g] = T + S$ can be represented by a symplectic surface Σ of genus $g + 1$ that one obtains by resolving the intersection point of $(T^2 \times pt) \bigcup (pt \times \Sigma_g)$. Moreover, by the symplectic Thom conjecture this surface is genus minimising (cf. Theorem 7.2.1) and the Milnor-Wood equality is satisfied if $g \geq 3$.

The intersection form on M is of the form:

$$Q \cong (2g + 1)H$$

and we may take a hyperbolic basis $\{e_i\}$ for $H^2(M)$ with $e_1 = T$, $e_2 = S$. Since $\chi(M) = \sigma(M) = 0$ we have the following solutions of the characteristic equations above

$$K_+ = (-2 + 2g, 0, -2 + 2g', 0, ..., 0)$$

$$K_- = (2+2g, 0, 2+2g', 0, ..., 0),$$

and the corresponding Euler classes are

$$e_1 = (2g, 0, 2g', 0, .., 0)$$

$$e_2 = (2, 0, 2, 0, ..., 0).$$

Thus if we let Σ' be a surface representing the class $\sigma' = (0,0,1,1,0,..,0)$ of genus $g'+1$ and we take g' large, then the Milnor-Wood inequality and equations (2.5) hold for both Σ and Σ'. However, Σ is genus minimising and Σ' is not. Hence the Euler class of a foliation does not determine whether closed leaves of a foliation with the given Euler class is genus minimising or not.

It is natural to ask whether *any* embedded surface can be made a leaf of a foliation. This is not true in general, as was shown by Mitsumatsu and Vogt (cf. [MV], Cor. 7.2). For if L is a leaf of a foliation on $M = S^2 \times \Sigma_g$, then there is a constant B_g such that

$$|[L]^2| < B_g.$$

However, in certain cases there are no such restrictions. In particular, the argument given in Example 2.3.4 is in fact quite general and we record this in the following proposition.

Proposition 2.3.6. *Let M be a manifold with $b_\pm^2(M) > 1$ and let Σ be an embedded surface of genus greater than 1 such that $[\Sigma]^2 = 1$. Then Σ can be made a leaf of a foliation.*

If in addition M is assumed to be symplectic, then any embedded surface with $[\Sigma]^2 = 1$ can be made a leaf of a foliation.

Proof. Set $v_1 = [\Sigma]$. Then, since $[\Sigma]^2 = 1$, we have an orthogonal splitting

$$H^2(M) \cong \langle v_1 \rangle \oplus \langle v_1 \rangle^\perp.$$

One then argues as in Example 2.3.4 to find appropriate solutions of equations (2.5), since the Milnor-Wood inequality is satisfied by assumption.

If M is symplectic with canonical class $c_1(K)$ and $b_\pm^2(M) > 1$, then the adjunction inequality (cf. [GS]) implies that

$$[\Sigma]^2 + |c_1(K).[\Sigma]| \leq 2g(\Sigma) - 2.$$

Hence $g(\Sigma)$ is at least 2 and the Milnor-Wood inequality automatically holds. Thus the previous argument applies and Σ can be made a leaf of a foliation. $\square$

2.4 Special classes of foliations and their closed leaves

We have seen that the condition of being a leaf of a 2-dimensional foliation on a 4-manifold is in general not very restrictive. In view of this we will focus on more special varieties of foliations. We shall start with the case of 1-dimensional complex foliations on complex surfaces, in which case a closed leaf must in fact have self-intersection zero.

Proposition 2.4.1. *If Σ is a leaf of a 1-dimensional complex foliation $\mathcal{F}$ on a complex surface, then $[\Sigma]^2 = 0$.*

Proof. We let $\mathcal{T}M$ denote the holomorphic tangent bundle of M and $\mathcal{T}\mathcal{F} \subset \mathcal{T}M$ the tangent bundle to $\mathcal{F}$. If Σ is a leaf of a foliation then a Bott connection (cf. Definition 6.1.2) will define a flat $\mathbb{C}^*$-linear connection on the normal bundle $\nu(\Sigma)$ of the leaf. Hence by Proposition 2.3.1 the bundle $\nu(\Sigma)$ must be trivial and $[\Sigma]^2 = 0$. □

In this way we see that there are strong local restrictions on closed leaves of complex foliations. We shall next consider symplectic foliations and their closed leaves. Here a symplectic foliation on a symplectic manifold (M, ω) is a foliation such that ω is positive on all leaves. In contrast to the case of complex foliations, there are no longer any local restrictions on the closed leaves of symplectic foliations as soon as the Milnor-Wood inequality is satisfied.

Proposition 2.4.2. *If Σ is a symplectic surface in a symplectic manifold (M, ω), then a neighbourhood of Σ admits a symplectic foliation that has Σ as a leaf if and only if the Milnor-Wood inequality holds.*

Proof. By Theorem 2.3.2 the Milnor-Wood inequality holds precisely when a bundle over Σ is flat as an $SL(2, \mathbb{R})$-bundle. Such a flat structure makes the surface a leaf of a foliation of a regular neighbourhood $\nu(\Sigma)$ of Σ. Since the symplectic form is positive on the closed leaf Σ, it is also positive on all leaves in a small neighbourhood and, hence, the foliation is automatically symplectic on a sufficiently small neighbourhood of Σ. □

Remark 2.4.3. We remark that the leaves of the foliation given in Proposition 2.4.2 can never be complex if the bundle under consideration has non-trivial Euler class, as this would then contradict Proposition 2.4.1. In particular, if J is an almost complex structure with respect to which the leaves are almost complex, then we know that J cannot be integrable.

The condition of being symplectic however does put restrictions on the possible Euler classes of the underlying distribution associated to a symplectic foliation. These restrictions are then no longer of a local but of a global nature.

Proposition 2.4.4. *Let (M, ω) be a symplectic 4-manifold, ξ a distribution of 2-planes that is homotopic to a symplectic distribution and (K_+, K_-) the pair of characteristic elements associated to ξ. Then $K_+ = \pm c_1(K)$, where K denotes the canonical bundle of an ω-compatible almost complex structure.*

Proof. We first take an ω-compatible almost complex structure and let $g_J = \omega(\,\cdot\,, J\,\cdot)$ be the associated metric on M. We have seen that an oriented 2-dimensional distribution ξ on a 4-manifold is equivalent to a pair of self-dual and anti-self-dual forms (α_+, α_-) of norm 1 so that ξ is the kernel of $\alpha = \alpha_+ - \alpha_-$. The distribution ξ is symplectic with respect to ω if and only if $\alpha \wedge \omega \neq 0$ at each point. By taking $-\omega$ if necessary we may assume that $\alpha \wedge \omega > 0$. However, with respect to the metric g_J we have $\omega \in \Omega^2_+(M)$ and self-dual forms pair trivially with anti-self-dual forms, so in fact $\alpha \wedge \omega = \alpha_+ \wedge \omega$ and the condition of being symplectic only depends on the self-dual part of α.

Moreover, if $\alpha_+ \wedge \omega > 0$, then the line of self-dual 2-forms $\alpha_t = (1-t)\alpha_+ + t\omega$ satisfies $\alpha_t \wedge \omega > 0$ at all times $0 \leq t \leq 1$. Hence after a homotopy we may assume that the self-dual part of the form defining a symplectic distribution is in fact ω. This means that the splitting of the self-dual bundle that we obtain from such a distribution is given by

$$\Lambda^2_+(M) = \mathbb{R}\omega \oplus L.$$

We consider the image of L in the complexification

$$\Lambda^2_+(M) \otimes \mathbb{C} \cong \mathbb{C}\omega \oplus K \oplus \bar{K},$$

where K denotes the canonical bundle of the chosen ω-compatible almost complex structure (cf. Lemma 2.2.3). Then, since the image of L is contained in $K \oplus \bar{K}$ and consists of real forms, the projection to the canonical bundle K defines an isomorphism of real oriented vector bundles after possibly changing the orientation on L. Thus we conclude that

$$K_+ = e(L) = \pm e(K) = \pm c_1(K)$$

in the pair of characteristic elements associated to ξ. □

As an immediate corollary we see that there is an abundance of distributions that cannot be homotoped to symplectic distributions on manifolds with large enough $b_2^{\pm}$.

Corollary 2.4.5. *Let M be a symplectic manifold that admits a distribution and assume that $b_2^{\pm}(M) \geq 2$. Then there are infinitely many homotopy classes of 2-plane fields that are not homotopic to symplectic distributions with respect to any symplectic form.*

Proof. We let (K_+, K_-) denote the pair of characteristic elements associated to a given distribution. Proposition 2.4.6 implies that a necessary condition for a distribution to be homotopic to a symplectic distribution is that K_+ is (up to sign) the canonical class of some symplectic form on M. If $b_2^+(M) \geq 2$, then by the results of Taubes on the Seiberg-Witten invariants of symplectic manifolds (cf. [Tau]), the canonical class associated to a symplectic form on M is a Seiberg-Witten basic class. Moreover, if $b_2^+(M) \geq 2$, then the set of Seiberg-Witten basic classes is finite (cf. [GS]). By Remark 2.2.10 above, the assumption that $b_2^{\pm}(M) \geq 2$ means that there are infinitely many pairs (K_i, L) that can occur as solutions of the equations in (2.2) above and we may assume that no K_i occurs as the canonical class of any symplectic form on M. Thus the distributions associated to the pairs (K_i, L) cannot be homotoped to become symplectic with respect to *any* symplectic form. □

We shall now use Proposition 2.4.4 together with some symplectic geometry to derive homological restrictions on the possible closed leaves of symplectic foliations that are much stronger than those for arbitrary smooth foliations.

Proposition 2.4.6. *Let Σ be a leaf of a symplectic foliation $\mathcal{F}$ and (K_+, K_-) the pair of characteristic elements associated to $\mathcal{F}$. Then $K_+ = \pm c_1(K)$ is the canonical class of an ω-compatible almost complex structure and*

$$|[\Sigma]^2| \leq c_1(K).[\Sigma]. \tag{2.6}$$

Proof. That $K_+ = \pm c_1(K)$ follows from Proposition 2.4.4. By the adjunction formula for symplectic surfaces

$$2g(\Sigma) - 2 = [\Sigma]^2 + c_1(K).[\Sigma].$$

Combining this with the Milnor-Wood inequality implies that

$$2|[\Sigma]^2| \leq [\Sigma]^2 + c_1(K).[\Sigma].$$

One then obtains the above inequality by considering the two cases $[\Sigma]^2 \geq 0$ and $[\Sigma]^2 < 0$. □

As a corollary we see that if the canonical class of a symplectic manifold is assumed to be trivial, then any compact leaf of a symplectic foliation must be a torus. Examples of manifolds with vanishing canonical class are given by S^1-bundles over 3-manifolds Y, where Y has the structure of a T^2-bundle over S^1. For example $M = T^4$ has this property.

Corollary 2.4.7. *Let (M, ω) be a symplectic manifold with trivial canonical class and let L be a leaf of a symplectic foliation on M. Then $\chi(L) = [L]^2 = 0$.*

Proof. By assumption $c_1(K) = 0$. Thus Proposition 2.4.6 implies that $|[L]^2| \leq c_1(K).[L] = 0$ and $[L]^2 = 0$. Finally by the adjunction formula for symplectic surfaces we compute

$$-\chi(L) = 2g(L) - 2 = [L]^2 + c_1(K).L = 0.$$ □

Remark 2.4.8. Inequality (2.6) also has the interesting consequence that if a class of non-zero self-intersection can be represented by a symplectic surface that also satisfies the Milnor-Wood inequality, then it cannot be very divisible. This is because the left hand side of

$$|[d\Sigma]^2| \leq c_1(K).[d\Sigma],$$

grows quadratically whereas the right-hand side only grows linearly in d. Now Donaldson proved that a sufficiently large multiple of an integral symplectic form can be represented by a symplectic surface (cf. [Don]). This means that one has a natural source of symplectic surfaces, however the classes in $H_2(M)$ that are represented by such surfaces will be highly divisible and will in general not satisfy the Milnor-Wood inequality.

2.5 Symplectic pairs

A further interesting special case of symplectic foliations on 4-manifolds are symplectic pairs (cf. [BK], [KM1]). We shall first give a general definition and then specialise to the case of 4-manifolds.

Definition 2.5.1 (Symplectic pairs)**.** A *symplectic pair* consists of a pair of closed 2-forms ω_1, ω_2 of constant and complementary ranks $2m, 2k$ respectively, such the form ω_2 is symplectic on the kernel foliation $\mathcal{F}_1 = \mathrm{Ker}(\omega_1^m)$ and ω_1 is symplectic on the kernel foliation $\mathcal{F}_2 = \mathrm{Ker}(\omega_2^k)$.

In the case of a 4-manifold a symplectic pair may be thought of as a pair of closed 2-forms ω_1, ω_2 satisfying the following conditions:

$$\omega_1^2 = 0 = \omega_2^2 \text{ and } \omega_1 \wedge \omega_2 \neq 0. \tag{2.7}$$

The most natural examples of such pairs occur as the horizontal and vertical foliations of flat surface bundles with symplectic holonomy. Other examples occur as S^1-bundles over fibred 3-manifolds, as the natural symplectic form one constructs on such bundles comes from a symplectic pair (cf. [FGM]). Manifolds that admit symplectic pairs are very special as in particular the forms

$$\omega = \omega_1 + \omega_2 \ , \ \bar{\omega} = \omega_1 - \omega_2$$

are both symplectic, but define opposite orientations of M. So if $b_2^{\pm}(M) \geq 2$, then by the results of Taubes (cf. [Tau]) M has non-vanishing Seiberg-Witten invariants in both orientations. In particular, M contains no embedded spheres that represent non-torsion elements in $H_2(M)$. We saw in Proposition 2.4.4 that the possible homotopy classes of symplectic distributions are quite restricted. For a symplectic pair the restrictions on the possible Euler classes are even stronger.

Proposition 2.5.2. *Let M admit a symplectic pair and assume $b_2^{\pm}(M) \geq 2$. Then the Euler classes of the kernel foliations $\mathcal{F}_1, \mathcal{F}_2$ satisfy:*

$$2e(\mathcal{F}_i) = K_1^i + K_2^i,$$

where K_j^i are Seiberg-Witten basic classes of M in one orientation or the other. In particular, there are only finitely many possible Euler classes.

Proof. We choose an ω-compatible almost complex structure that preserves the splitting given by $TM = \mathcal{F}_1 \oplus \mathcal{F}_2$ and take the associated metric. Then with respect to this metric we see that $\omega \in \Omega^2_+(M)$ and $\bar{\omega} \in \Omega^2_-(M)$. The distributions $\mathcal{F}_i$ are symplectic for both ω and $\bar{\omega}$ and by Proposition 2.4.4 we conclude that $K_+ = \pm c_1(K)$ and $\bar{K}_+ = \pm c_1(\bar{K})$, where the bars denote the canonical bundle of an almost complex structure compatible with $\bar{\omega}$. Now since $\Lambda^2_+(\bar{M}) = \Lambda^2_-(M)$, it follows that $\bar{K}_+ = K_-$. If $b_2^+(M) \geq 2$, then the first Chern class of the canonical bundle of any symplectic form is a Seiberg-Witten basic class and the set of Seiberg-Witten basic classes is finite (cf. [GS], [Tau]). By assumption $b_2^+(M) \geq 2$ in both orientations and, hence, the possibilities for K_+ and K_- are finite, which proves the second part of the proposition. □

In certain special cases where the Seiberg-Witten invariants are known, one can compute solutions to equations (2.5) above, in order to understand what classes can be realised as leaves of a foliation coming from a symplectic pair. We illustrate this by considering the case of a product of Riemann surfaces in the following example.

Example 2.5.3. Let $M = \Sigma_h \times \Sigma_g$ be a product of Riemann surfaces of genus $g, h \geq 2$. Then since M and $\bar{M}$ are minimal surfaces of general type and $b_2^{\pm}(M) \geq 2$ the only SW-basic classes on $M, \bar{M}$ are as follows (cf. [GS], p. 91):

$$K_+ = \pm c_1(K) = \pm[(2g-2)[\Sigma_h] + (2h-2)[\Sigma_g]]$$

$$\bar{K}_+ = \pm c_1(\bar{K}) = \pm[(2g-2)[\Sigma_h] - (2h-2)[\Sigma_g]].$$

Thus the only possibilities for the Euler classes of a symplectic pair are those corresponding to the natural symplectic pair given by the product structure:

$$e_1 = \pm(2h-2)[\Sigma_g], e_2 = \pm(2g-2)[\Sigma_h].$$

This puts restrictions on the possible closed leaves of such a symplectic pair. For example one sees that a class $L_{a,b} = a[\Sigma_h] + b[\Sigma_g]$ with $a, b \neq 0$ can satisfy the second equation (2.5) with $\pm e_1, \pm e_2$ as above if and only if $\pm b(2g-2) = 2ab$ and this holds if and only if $a = \pm(g-1)$ by our assumption that $b \neq 0$. One can further construct a symplectic representative $\Sigma_{a,b}$ for $L_{a,b}$ by resolving the double points of a union of a disjoint copies of Σ_h and b disjoint copies of Σ_g. We calculate

$$\chi(\Sigma_{a,b}) = |a|(2-2h) + |b|(2-2g) - 2|ab|.$$

Plugging this into the first equation (2.5) we obtain

$$\pm|a|(2h-2) = \chi(\Sigma_{a,b}) = |a|(2-2h) + |b|(2-2g) - 2|ab|. \tag{2.8}$$

Since $|b|(2g-2) = 2|ab|$, we conclude that $|b|(2-2g) = 0$, which contradicts the assumption that both b and $2-2g$ are non-zero. Since everything is symmetric in h, g and a, b, the same conclusion holds if we swap the roles of e_1 and e_2 above. It follows that the surfaces $\Sigma_{a,b}$ can never be made leaves of foliations that are homotopic to the kernel foliation of a symplectic pair. Moreover, the surfaces are symplectic and, hence, genus minimising in their homology class by the symplectic Thom conjecture (cf. Theorem 7.2.1). This means that equation (2.8) cannot be solved for any representatives of the class $L_{a,b}$ so that no representatives of these classes can be made a leaf of the kernel foliation of a symplectic pair.

2.5.1 Topological constructions of symplectic pairs

There a several topological constructions that one can perform in the symplectic category. The most useful of these is the Gompf sum, which also works for symplectic pairs in certain cases. We first recall the definition of the Gompf sum of two 4-manifolds along a symplectic surface.

Definition 2.5.4 (Gompf sum)**.** Let $(M_1, \omega_1), (M_2, \omega_2)$ be symplectic 4-manifolds and let Σ_1, Σ_2 be embedded symplectic surfaces of the same genus in M_1, M_2 respectively. Assume that $[\Sigma_1]^2 = -[\Sigma_2]^2$. Then after choosing an identification of tubular neighbourhoods $\nu(\Sigma_1), \nu(\Sigma_2)$ of Σ_1, Σ_2 respectively, we can form the normal connected sum:

$$M_1 \#_{\Sigma_1=\Sigma_2} M_2 = \Big(M_1 \setminus \nu(\Sigma_1)\Big) \bigcup_{\partial\nu(\Sigma_1)=\partial\nu(\Sigma_2)} \Big(M_2 \setminus \nu(\Sigma_2)\Big).$$

This manifold is symplectic and is called the Gompf sum of M_1 and M_2.

In order to obtain Gompf sums for symplectic pairs we will only allow sums along compact leaves of the kernel foliations that have neighbourhoods of a very simple form.

Definition 2.5.5 (Projectable leaves)**.** Let $(M_1, \omega_1, \omega_2)$ be a 4-manifold that carries a symplectic pair. Let Σ be a closed leaf of one of the kernel foliations. We say that Σ is projectable if there is an open neighbourhood U of Σ that is diffeomorphic to $\Sigma \times D^2$ such that under this identification

$$\omega_1 = \pi_1^* \omega_\Sigma \text{ and } \omega_2 = \pi_2^* \omega_{D^2},$$

where π_i are the projections onto the factors and $\omega_\Sigma, \omega_{D^2}$ are area forms on Σ and D^2 respectively.

It was noted by Bande and Kotschick that the Gompf sum is compatible with symplectic pairs when one glues along projectable leaves.

Proposition 2.5.6 (Gompf sums for symplectic pairs, [BK])**.** *Let $(M_1, \omega_1, \omega_2), (M_2, \eta_1, \eta_2)$ be 4-manifolds that admit symplectic pairs and assume that the kernel foliations $\mathcal{F}_1$ of ω_1, $\mathcal{F}_2$ of η_1 have projectable, closed leaves Σ_1, Σ_2 of the same genus. Then the Gompf sum $M_1 \#_{\Sigma_1 = \Sigma_2} M_2$ admits a symplectic pair.*

Proof. We choose projectable neighbourhoods of U_1 and U_2 of Σ_1, Σ_2 respectively and identifications of U_i with $\Sigma \times D^2$. After rescaling ω_i and η_i we may assume by Moser stability that

$$\omega_1 = \eta_1 = \pi_1^* \omega_\Sigma \text{ and } \eta_2 = \omega_2 = \pi_2^* dx \wedge dy.$$

Taking the normal connected sum coming from these identifications, we see that the forms ω_i and η_i glue together to give a symplectic pair on the Gompf sum $M_1 \#_{\Sigma_1 = \Sigma_2} M_2$. $\square$

As an application, we will use Proposition 2.5.6 in order to show that the examples of Akhmedov given in [Akh] admit symplectic pairs. These examples are interesting as they have the cohomology of $S^2 \times S^2$ and are not diffeomorphic to surface bundles. Such examples were first found by Bande and Kotschick in [BK], where for example the Kuga surface is shown to admit a symplectic pair, but as it is aspherical it cannot be diffeomorphic to a surface bundle.

Example 2.5.7 (Akhmedov's examples admit symplectic pairs). Let $T^2 \to M \to S^1$ be a torus bundle with orientation preserving monodromy ϕ. After choosing a symplectic form ω on T^2 we may assume that ϕ preserves ω. Furthermore, we may assume that ϕ fixes a neighbourhood of $0 \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

The product $E = M \times S^1$ has the structure of a torus bundle:

$$T^2 \to E = M \times S^1 \to S^1 \times S^1.$$

We then define a symplectic pair on E by letting ω_1 be the pullback of the vertical symplectic form on M and by setting $\omega_2 = p_1^* d\theta_1 \wedge p_2^* d\theta_2$, where p_i are the projections to the two factors of the base torus and $d\theta_i$ are the angular forms on the circle factors. Moreover, the leaf $S = (0 \times S^1) \times S^1$, which is also a section, is projectable as is a T^2-fibre F of the bundle E.

We then perform the Gompf sum with two copies of E along S and F to obtain a manifold Y that admits a symplectic pair. We identify the tori via the description as a product of circles. We let $\Sigma = F\#S$ be the surface obtained as the connect sum of the other section and a fibre and note that Σ is a projectable leaf of the symplectic pair on Y. We again perform the Gompf sum by gluing two copies of Y along Σ. Let a_i, b_i be standard generators of the fundamental group of Σ. We identify the two copies of Σ via a diffeomorphism ψ that sends a_i, b_i to a_{i+1}, b_{i+1} respectively, where the indices are to be interpreted mod 2.

According to [Akh] the resulting manifold Z is spin and has the cohomology of $S^2 \times S^2$. The second cohomology $H^2(Z)$ has a basis P, Q represented by embedded, symplectic surfaces of genus 2. Thus $Z \neq S^2 \times S^2$, since $H^2(S^2 \times S^2)$ has a basis of symplectic 2-spheres P', Q' and any diffeomorphism must take $\{P, Q\}$ to $\{\pm P', \pm Q'\}$, which would contradict the genus minimality of symplectic surfaces (cf. [OS]).

More generally, one may start with a 3-manifold that is fibred with fibre of genus g and one whose fibre is the 2-torus. By performing a similar construction to that described above one obtains examples that also admit symplectic pairs. These examples have the cohomology of a connect sum $\#_{2g-1} S^2 \times S^2$ and, hence, have vanishing first homology (cf. [Akh]). So again these examples cannot be diffeomorphic to surface bundles if one assumes $g \geq 2$.

Another basic construction for symplectic manifolds is that of branched coverings.

Definition 2.5.8 (Branched coverings)**.** A d-fold branched covering is a smooth map $\tilde{X} \xrightarrow{\pi} X$ with critical set $\Sigma \subset X$ called the *branch locus*, such that the restriction $\tilde{X} \setminus \pi^{-1}(\Sigma) \to X \setminus \Sigma$ is a d-fold covering and for each $x \in \pi^{-1}(\Sigma)$ there are local charts $U, V \to \mathbb{R}^2_+ \times \mathbb{C}$ about $x, \pi(x)$ on which π is given by $(p, z) \mapsto (p, z^{d_p})$, for some positive integer d_p called the *branching index* of π at p. A branched covering is called *cyclic* if it is a cyclic covering away from the branching locus.

Remark 2.5.9. Away from the branching locus a branched covering is determined by a finite index subgroup of $\pi_1(X \setminus \Sigma)$. If ν is a regular neighbourhood of Σ and $\tilde{\nu}$ its preimage in $\tilde{X}$, then an S^1-bundle structure on $\partial\nu$ induces one on $\partial\tilde{\nu}$ and there is a unique way to fill this in by a disc bundle.

We further note that our definition allows for the case of manifolds with boundary, in which case we consider properly embedded branching loci.

In the case of cyclic branched coverings we have the following existence result (see [GS], p. 239 ff).

Theorem 2.5.10. *Let $\Sigma \subset X$ be an embedded surface such that $[\Sigma] = d[\Sigma']$ in $H_2(X, \mathbb{Z})$. Then there is a cyclic d-fold branched cover $\tilde{X} \xrightarrow{\pi} X$ with branching locus Σ.*

The branched covering construction can be performed in a symplectic manner if the branching locus is assumed to be symplectic. For symplectic pairs the same holds if one takes branching loci that consist of projectable leaves. The following proposition is a slight variation of Proposition 10 in [Aur].

Proposition 2.5.11 (Branched coverings for symplectic pairs)**.** *Let (X, ω) be a symplectic manifold and let $\Sigma \subset X$ be an embedded symplectic surface. If $\tilde{X} \xrightarrow{\pi} X$ is a covering branched over Σ, then $\tilde{X}$ carries a symplectic form $\tilde{\omega}$ which agrees with $\pi^*\omega$ outside a neighbourhood of Σ.*

Furthermore, if $(M_1, \omega_1, \omega_2)$ is a 4-manifold with a symplectic pair and Σ is a union of projectable leaves, then a branched cover $\tilde{M} \to M$ with branching locus Σ also admits a symplectic pair.

Proof. We will first define an exact 2-form τ on $\tilde{X}$ so that $\pi^*\omega \wedge \tau|_{\tilde{\Sigma}} > 0$. To do this we take a point $x \in \tilde{\Sigma}$ and choose coordinates so that π has the form

$$(z_1, z_2) \mapsto (z_1, z_2^d)$$

on some neighbourhood of x. After choosing some metric on $\tilde{X}$ we let $\rho_x > 0$ be such that the polydisc $B(2\rho_x) \times B(2\rho_x)$ is contained in the coordinate patch chosen above. We let χ be a bump function with support in $B(2\rho_x)$ that is constant 1 on $B(\rho_x)$. We next define

$$\tau_x = d(\chi(z_1)\chi(z_2)x_2 dy_2)$$

and extend by 0 to the rest of $\tilde{X}$. We let $K = \operatorname{Ker} \pi \subset T\tilde{X}|_{\tilde{\Sigma}}$ be oriented so that it intersects $T\tilde{\Sigma}$ positively, then by definition τ_x is non-negative on K and is strictly positive on $B(\rho_x) \times B(\rho_x)$. By compactness there are finitely many x_i so that the form

$$\tau = \sum \tau_{x_i}$$

is strictly positive on K along $\tilde{\Sigma}$, that is $\pi^*\omega \wedge \tau > 0$ on $\tilde{\Sigma}$. We set $\tilde{\omega} = \pi^*\omega + \epsilon\tau$ and compute

$$\tilde{\omega}^2 = \pi^*\omega \wedge \pi^*\omega + \epsilon(\pi^*\omega \wedge \tau + \epsilon\tau \wedge \tau).$$

For small ρ we have that $\pi^*\omega \wedge \tau > 0$ on a tubular neighbourhood $\nu_{2\rho}$ of $\tilde{\Sigma}$. Thus for all sufficiently small ϵ the second term above is positive on $\nu_{2\rho}$. Moreover $\pi^*\omega \wedge \pi^*\omega$ is non-negative and is strictly positive away from the branching locus, thus by choosing ϵ small enough we can ensure that $\tilde{\omega}$ is non-degenerate on the rest of $\tilde{X}$.

If M has a symplectic pair and Σ is projectable, then Σ has trivial normal bundle. Thus we may identify neighbourhoods of Σ and $\tilde{\Sigma}$ with $\Sigma \times D^2$ in such a way that is compatible with the projections defining the symplectic pair on M and so that π has the form $(p, z) \to (p, z^d)$, where z is a complex coordinate on the disc. We let π_2 denote the second projection of this product neighbourhood and set

$$\tilde{\omega}_1 = \pi^*\omega_1$$
$$\tilde{\omega}_2 = \pi^*\omega_2 + \pi_2^*\tau,$$

where τ is any non-negative form that is non-zero at the origin and has compact support in D^2. It is easy to check that this gives the desired symplectic pair on $\tilde{M}$. □

2.5.2 Geometry of leaves of symplectic pairs

In general the kernel foliations of a symplectic pair will be too complicated in a neighbourhood of a leaf to be able to define Gompf sums or branched coverings for arbitrary leaves of symplectic pairs. However, if we perform Gompf sums along leaves of the kernel foliations of a symplectic pair, or take branched covers then the manifolds we obtain will still admit symplectic forms in *both* orientations, which is in itself restrictive as we have seen.

Proposition 2.5.12. *Let $(M_1, \omega_1, \omega_2), (M_2, \eta_1, \eta_2)$ be 4-manifolds that admit symplectic pairs and let $\Sigma_1 \cong \Sigma_2$ be leaves of the kernel foliations $\mathcal{F}_1$ of ω_1, $\mathcal{F}_2$ of η_2 respectively such that $[\Sigma_1]^2 = -[\Sigma_2]^2$. Then the Gompf sum*

$$X = M_1 \#_{\Sigma_1 = \Sigma_2} M_2$$

admits symplectic structures in both orientations. Similarly, any branched covering $\tilde{M} \xrightarrow{\pi} M_1$ branched over Σ_1 also admits symplectic structures compatible with both orientations.

Proof. We form the Gompf sum first with respect to the symplectic forms

$$\omega = \omega_1 + \omega_2$$
$$\eta = \eta_1 + \eta_2$$

to get a symplectic form on

$$X = M_1 \#_{\Sigma_1 = \Sigma_2} M_2.$$

We next take the sum with respect to the symplectic forms

$$\bar{\omega} = -\omega_1 + \omega_2$$
$$\bar{\eta} = \eta_1 - \eta_2$$

to get a symplectic form on

$$\bar{X} = \bar{M}_1 \#_{\Sigma_1 = \Sigma_2} \bar{M}_2,$$

where the bar denotes the manifold X taken with the opposite orientation. Similarly, if $\tilde{M} \xrightarrow{\pi} M_1$, is a branched covering branched along a leaf Σ_1, then Proposition 2.5.11 applied to the symplectic forms ω and $\bar{\omega}$ gives symplectic forms on $\tilde{M}$ that define opposite orientations. □

As an application of Proposition 2.5.12 we will show that there are geometric restrictions on the local structure of leaves of symplectic pairs. The examples we consider were first utilised by Gompf in the context of constructing symplectically aspherical manifolds (cf. [Gom]).

We shall need to recall the definition of Milnor fibres and review their basic properties. Let $p, q, r \in \mathbb{N}$ be positive integers and $\epsilon \in \mathbb{C} \setminus 0$. Then the Milnor fibre $M(p, q, r)$ is defined as

$$M(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = \epsilon\}.$$

We let $M_c(p, q, r)$ denote the intersection of $M(p, q, r)$ with the unit ball in $\mathbb{C}^3$ and note that the interior of this manifold is diffeomorphic to $M(p, q, r)$.
We now list some basic properties of these Milnor fibres (see [Gom] and [GS], pp. 231-33):

1. If $|\epsilon|$ is sufficiently small, then $M_c(p, q, r)$ is the unique cyclic p-fold branched covering of B^4 branched over

$$\Sigma_{q,r} = \{(x, y) \in \mathbb{C}^2 \mid x^q + y^r = \epsilon\} \cap B^4.$$

2. $M_c(2, 2, d)$ for $d \geq 2$ is diffeomorphic to a plumbing of $d - 1$ spheres of square -2.

3. $M_c(p,q,r) \subset M_c(p',q',r')$ if $p \le p', q \le q', r \le r'$.

4. $M_c(p,q,r) = M_c(\sigma(p),\sigma(q),\sigma(r))$ for any permutation $\sigma \in S_3$.

With these preliminaries we will see that a closed leaf of a symplectic pair cannot be locally isotopic to a piece of the curves $\Sigma_{q,r}$ defined above.

Proposition 2.5.13. *Let (M,ω_1,ω_2) be a manifold with a symplectic pair and let Σ be a leaf of one of the kernel foliations. If $[\Sigma] = d[\Sigma']$ is a divisible class in $H_2(M)$ for $d > 2$, then the intersection $\Sigma_c = \Sigma \cap B^4$ with any embedded ball is not isotopic relative ∂B^4 to the curve $\Sigma_{q,r}$. In the case $d = 2$ the same conclusion holds except possibly in the case $q = r = 2$.*

Proof. By Theorem 2.5.10 there exists a d-fold branched covering $\tilde{M}$ branched over Σ. Then by Proposition 2.5.12 we know that $\tilde{M}$ is symplectic in both orientations, and hence cannot contain any (-2)-spheres if $b_2^-(\tilde{M}) \ge 2$. However, if the piece Σ_c is isotopic to $\Sigma_{q,r}$ then by fact (1) above we know that

$$M_c(d,q,r) \subset \tilde{M}.$$

Furthermore, combining facts (2), (3) and (4) we deduce that

$$M_c(d,2,2) \subset M_c(d,q,r) \subset \tilde{M}.$$

Since $M_c(d,2,2)$ is a plumbing of $d-1$ spheres of self-intersection -2, we have a contradiction if $d > 2$. If $d = 2$ then we have

$$M_c(2,q,r) \subset \tilde{M}$$

and the same argument holds unless $q = r = 2$. □

Remark 2.5.14. We note that the condition that the leaf Σ represent a divisible class in Proposition 2.5.12 is not always necessary. In particular, if M is a surface bundle, whose base and fibre are of genus at least two, then after taking a suitable covering we may assume that the pull-back of Σ is divisible by any given d (cf. [Mor1], Prop. 4.3). Since symplectic pairs lift under covering maps and the relevant property of Σ is local, Proposition 2.5.12 holds for arbitrary leaves of symplectic pairs on surface bundles.

As a consequence of Proposition 2.5.13 and the previous remark, we note that any symplectic surface that is obtained by resolving at least one double point in a surface bundle, whose base and fibre are of genus at least two, can never be the leaf of a symplectic pair. A similar conclusion was obtained in Example 2.5.3 for the product of two Riemann surfaces, where the arguments were homological rather than geometric.

Chapter 3

Surface Bundles and their characteristic classes

Morita has conjectured that the MMM-classes are bounded in the sense of Gromov. As noted in [Mor4], a result of Gromov implies that the odd MMM-classes are bounded and so it remains to show that the same holds for the even classes. The starting point for the discussion of boundedness of the MMM-classes is the fact that the vertical Euler class is bounded. We shall give a new proof of this fact using the adjunction inequality coming from Seiberg-Witten theory. The vertical Euler class gives a characteristic class e_k^v of surface bundles with k-multisections, which is then also bounded. Motivated by a question posed by Mitsumatsu and Vogt in [MV], we show that the norms of e_k^v are not bounded independently of k.

As evidence for his conjecture Morita showed that the MMM-classes vanish on amenable groups. By a result of Ivanov amenable subgroups of the mapping class group are virtually abelian and using this observation we give an elementary proof that the MMM-classes vanish on amenable groups, which is independent of the proof in [Mor4]. As further evidence for the boundedness conjecture we then use Morita's original argument to show that the MMM-classes are hyperbolic. The hyperbolicity condition is strictly weaker than boundedness, but does imply vanishing on amenable subgroups.

3.1 Surface bundles and holonomy representations

We begin by recalling some generalities about surface bundles and their sections, which for the most part can be found in [Mor1]. Let $\Gamma_h = Diff^+(\Sigma_h)/Diff_0(\Sigma_h)$ denote the mapping class group of an oriented Riemann surface Σ_h of genus h. By the classical result of Earle and Eells the identity component $Diff_0(\Sigma_h)$ is contractible in the C^∞-topology if $h \geq 2$ (cf. [EE]). Thus the classifying space $BDiff^+(\Sigma_h)$ is homotopy equivalent to $B\Gamma_h$, which is in turn the Eilenberg-MacLane space $K(\Gamma_h, 1)$.

In general, any bundle is determined up to bundle isomorphism by the homotopy class of its classifying map and since $BDiff^+(\Sigma_h)$ is aspherical, a surface bundle $\Sigma_h \to E \to B$ is determined up to bundle isomorphism by the conjugacy class of its *holonomy representation*:

$$\rho : \pi_1(B) \to \Gamma_h.$$

The conjugation ambiguity is a result of the choice of base points. Conversely, any homomorphism $\rho : \pi_1(B) \to \Gamma_h$ induces a map $B \to K(\Gamma_h, 1) = B\Gamma_h$ and thus defines a bundle that has holonomy ρ.

A section of a bundle E is equivalent to a lift of the holonomy map of the bundle to $\Gamma_{h,1}$, which denotes the mapping class group of Σ_h with one marked point. That is E admits a section if and only if there is a lift $\bar{\rho}$ so that the following diagram commutes

$$\begin{tikzcd} & \Gamma_{h,1} \arrow[d] \\ \pi_1(B) \arrow[r, "\rho"] \arrow[ur, dashed, "\bar{\rho}"] & \Gamma_h. \end{tikzcd}$$

Similarly, a k-multisection is the same as a lift of the holonomy map to the mapping class group with k marked points that we denote by $\Gamma_{h,k}$. Here we require only that the set of k marked points be fixed *as a set*, rather than that each marked point itself be fixed by elements of $\Gamma_{h,k}$.

From any bundle with a k-multisection one obtains in a natural way one with a section. For if $S \hookrightarrow E$ is a multisection and if we denote by p the composition of this inclusion with the projection to the base B, then the pullback bundle p^*E has a natural section $\widetilde{S}$ induced by S.

As in the case of surface bundles the classifying space of surface bundles with a section is the Eilenberg-MacLane space $B\Gamma_{h,1} = K(\Gamma_{h,1}, 1)$, if $h \geq 2$. Furthermore, there is a natural exact sequence given by forgetting the marked point

$$1 \to \pi_1(\Sigma_h) \to \Gamma_{h,1} \to \Gamma_h \to 1$$

and one may identify $B\Gamma_{h,1}$ with the the total space $E\Gamma_h$ of the universal bundle over $B\Gamma_h$. Similarly, the classifying space for bundles with a k-multisection is the Eilenberg-MacLane space $B\Gamma_{h,k} = K(\Gamma_{h,k}, 1)$ (cf. [Mor1]).

The bundle of vectors that are tangent to the fibres of the projection $E\Gamma_h \to B\Gamma_h$ is an oriented rank-2 vector bundle. The Euler class of this bundle defines a cohomology class $e \in H^2(E\Gamma_h) = H^2(B\Gamma_{h,1})$, which we will call the vertical Euler class. Alternately, one can define e as the Euler class associated to the central extension

$$1 \to \mathbb{Z} \to \Gamma_h^1 \to \Gamma_{h,1} \to 1,$$

where $\Gamma_h^1 = Diff^c(\Sigma_h^1)/Diff_0^c(\Sigma_h^1)$ denotes the mapping class group of a once punctured, genus h surface. Here the right most map is given by collapsing the boundary to a point and the kernel is generated by a Dehn twist along a curve parallel to the boundary. Under the identification of $E\Gamma_h$ with the classifying space $B\Gamma_{h,1}$ of bundles with a section S, the class e corresponds to the characteristic class given by restricting the vertical Euler class to S.

Furthermore, if the base of the bundle is a surface, this is just the self-intersection number of the section.

We shall now give examples of surface bundles over surfaces with sections of non-zero self-intersection, showing in particular that the class $e \in H^2(E\Gamma_h) = H^2(\Gamma_{h,1})$ is non-trivial. This can be done in several ways and we choose to show that there are sections of non-zero self-intersection in bundles that are *a priori* trivial. The following is a generalisation of Example 8.13 in [MV].

Example 3.1.1 (Multisections in $E = \Sigma_h \times \Sigma_g$). Let $g, h \geq 2$ and let $\Sigma = \Sigma_2$. We then choose a surjective homomorphism $\pi_1(\Sigma) \to \mathbb{Z}_{g-1} \times \mathbb{Z}_{h-1}$. Denote by $\widetilde{\Sigma}$ the normal covering space associated to this homomorphism. Let $G_1 = \mathbb{Z}_{g-1} \times 0$ and $G_2 = 0 \times \mathbb{Z}_{h-1}$ be the two factors of $G = \mathbb{Z}_{g-1} \times \mathbb{Z}_{h-1}$. By construction G_1 and G_2 act freely on $\widetilde{\Sigma}$ and we thus obtain coverings $\widetilde{\Sigma} \stackrel{\pi_i}{\to} \widetilde{\Sigma}/G_i$. We define a map

$$\widetilde{\Sigma} \stackrel{\pi_1 \times \pi_2}{\to} (\widetilde{\Sigma}/G_1) \times (\widetilde{\Sigma}/G_2) = \Sigma_h \times \Sigma_g.$$

First of all we claim that the map $p = \pi_1 \times \pi_2$ is an embedding. For the preimage of a point $p(x) = (\pi_1(x), \pi_2(x))$ is the intersection of the orbits

$$G_1.x \cap G_2.x = (G_1 \cap G_2).x = e.x = x.$$

Moreover, the map p is transverse to the fibres Σ_h, so that the image of p defines a multisection S of E. Next if TF denotes the vertical bundle of E, then the self-intersection of S is given by

$$|[S]^2| = |e(TF)([S])| = |(2-2h)[\Sigma_g].[S]| = |(2-2h).(g-1)| \neq 0.$$

By taking pullbacks we see that for every h there exist bundles with sections $[S]$ so that $[S]^2 \neq 0$, so in particular we deduce that the characteristic class e is non-trivial for all $h \geq 2$.

The sections in Example 3.1.1 have the property that $|[S]^2| = |\chi(S)|$. In fact for fixed genus this self-intersection number is maximal as the following proposition shows.

Proposition 3.1.2. *Let S be a section of a surface bundle $\Sigma_h \to E \to \Sigma_g$ with $h \geq 2$, then $|[S]^2| \leq max\{0, 2g-2\}$.*

Proof. We first consider the case $g \geq 2$. Since the genus of the fibre is greater than 1, we may apply the Thurston construction to obtain a symplectic form on E with respect to which S is symplectic *and* $[S]^2 \geq 0$. Because the genus of fibre and base are positive, the bundle E is aspherical and it follows from Proposition 1 of [Kot1] that

$$|\sigma(E)| \leq \chi(E).$$

This gives a lower bound for $b_2^+(E)$ and $b_2^-(E)$, since

$$2\min\{b_2^+(E), b_2^-(E)\} = b_2(E) - |\sigma(E)|$$

and

$$b_2(E) - |\sigma(E)| = \chi(E) - 2 + 2b_1(E) - |\sigma(E)| \geq 2b_1(E) - 2 \geq 6.$$

Thus in either orientation $b_2^+(E) \geq 3$ and by applying the adjunction inequality (cf. [GS]) we conclude that

$$|[S]^2| \leq |[S]^2| + |K.[S]| \leq 2g(S) - 2.$$

If $g < 2$ then as both S^2 and T^2 have self maps of arbitrary degree we may assume after taking coverings that $|[S]^2|$ is large, say $|[S]^2| > 3$ or that $|[S]^2| = 0$ in which case the inequality is trivially true. In the former case we take the pullback bundle under a degree 1 collapsing map from a genus 2 surface Σ_2 to Σ_g. The resulting bundle has a section S' of self-intersection

$$|[S']^2| > 3 > 2g(\Sigma_2) - 2,$$

which yields a contradiction. □

We contrast the bound given by Proposition 3.1.2 with that given by the Milnor-Wood inequality which provides a better bound for the self-intersection number. In particular, we see that the sections we obtained in Example 3.1.1 above, cannot be realised as a leaf of a foliation as they do not satisfy the Milnor-Wood inequality.

3.1.1 Boundedness of the vertical Euler class

As a consequence of Proposition 3.1.2 we will prove that the class $e \in H^2(\Gamma_{h,1})$ is bounded in the sense of Gromov (cf. [Gro]). This fact is originally due to Morita, whose proof uses the boundedness of the Euler class in $Homeo^+(S^1)$ as proved by Wood (cf. [Mor4]). On the other hand our proof relies on the adjunction inequality in the spirit of [Kot1]. Before giving the proof we recall the definition of the Gromov-Thurston (semi-)norm.

Definition 3.1.3 (Gromov-Thurston norm)**.** Let X be any topological space and let $\alpha \in H_2(X, \mathbb{Z})$. Define the minimal genus $g_{min}(\alpha)$ of α to be the minimal g so that α is representable as the image of the fundamental class under a map $\Sigma_g \to X$ modulo torsion. We define the Gromov-Thurston norm to be

$$||\alpha||_{GT} = \lim_{n\to\infty} \frac{2g_{min}(n\alpha) - 2}{n}.$$

Gromov has also defined another norm on real homology. This is the so-called l^1-norm and is defined as follows.

Definition 3.1.4 (l^1-norm)**.** Let $c = \sum_i \lambda_i \sigma_i$ be a chain in $C_k(X, \mathbb{R})$. We define the l^1-norm of c to be

$$||c||_1 = \sum_i |\lambda_i|.$$

For a class $\alpha \in H_k(X, \mathbb{R})$ we define

$$||\alpha||_1 = \inf \{ \ ||z||_1 \mid \alpha = [z]\}.$$

In degree 2 the Gromov-Thurston norm agrees with the l^1-norm $|| \cdot ||_1$ up to a constant. We record this in the following lemma, a proof of which is given in [BG].

Lemma 3.1.5. *Let $\alpha \in H_2(X, \mathbb{Z})$. Then the following equality holds*

$$||\alpha||_1 = 2||\alpha||_{GT}.$$

If in addition the group $H_2(X, \mathbb{Z})$ is finitely generated, we may extend $||\cdot||_{GT}$ to a semi-norm on $H_2(X, \mathbb{R})$ and the equality of Lemma 3.1.5 holds for this extension. We shall denote this extension again by $||\cdot||_{GT}$.

Lemma 3.1.6. *Let $H_2(X, \mathbb{Z})$ be finitely generated. Then there is a unique extension of $||\cdot||_{GT}$ to $H_2(X, \mathbb{R})$, for which $||\cdot||_1 = 2||\cdot||_{GT}$.*

Proof. One first extends $||\cdot||_{GT}$ to $H_2(X, \mathbb{Q})$ by linearity, so that $||\cdot||_1 = 2||\cdot||_{GT}$ holds on rational classes. Since $H_2(X, \mathbb{Q})$ is finite dimensional, the convex function $||\cdot||_{GT}$ is continuous with respect to the ordinary l^1-norm. Hence as $||\cdot||_{GT}$ is continuous on the dense subset of rational points it has a unique continuous extension to $H_2(X, \mathbb{R})$. By continuity this function will also satisfy the properties of a (semi-)norm and the equality $||\cdot||_1 = 2||\cdot||_{GT}$ still holds. □

By considering the natural pairing between homology and cohomology, one obtains a norm on cohomology that is dual to the l^1-norm. This norm agrees with the l^∞-norm on cohomology as introduced by Gromov (cf. [Gro]). We then say that a cohomology class is *bounded*, if it is bounded with respect to the l^∞-norm. We may now prove the boundedness of the vertical Euler class.

Proposition 3.1.7 (Morita)**.** *Let $h \geq 2$, then the vertical Euler class $e \in H^2(\Gamma_{h,1})$ is bounded and $||e||_\infty = \frac{1}{2}$.*

Proof. For any natural number n suppose that $n\alpha \in H_2(\Gamma_{h,1})$ can be represented as the image of the fundamental class under a map $\Sigma_{n\alpha} \to B\Gamma_{h,1}$ and assume that $\Sigma_{n\alpha}$ is genus minimising. This in turn corresponds to a surface bundle with a section $S_{n\alpha}$ and by Proposition 3.1.2 we have that

$$n\,|e(\alpha)| = |e(n\alpha)| = |[S_{n\alpha}]^2| \leq 2g(\Sigma_{n\alpha}) - 2 = 2g_{min}(n\alpha) - 2 \tag{3.1}$$

and thus

$$|e(\alpha)| \leq \lim_{n\to\infty} \frac{2g_{min}(n\alpha) - 2}{n} = ||\alpha||_{GT}.$$

Hence for integral classes $\alpha \in H_2(X, \mathbb{Z})$ Lemma 3.1.5 yields

$$\frac{|e(\alpha)|}{||\alpha||_1} \leq \frac{1}{2}.$$

Now since the group $\Gamma_{h,1}$ is finitely presented (cf. [Iva1], Theorem 4.3 D), it follows that $H_2(\Gamma_{h,1})$ is finitely generated. Thus we may extend the Gromov-Thurston norm to real cohomology by Lemma 3.1.6 and

$$\sup_{\alpha\neq 0} \frac{|e(\alpha)|}{||\alpha||_1} \leq \frac{1}{2}.$$

It follows that e is a bounded class with $||e||_\infty \leq \frac{1}{2}$. We will show that this bound is sharp.

Let $h \geq 2$ and let S be the diagonal section of $\Sigma_h \times \Sigma_h$. This section has self-intersection $2 - 2h$ and we denote the corresponding class $\alpha \in H_2(\Gamma_{h,1})$. By taking coverings we obtain sections $S_{n\alpha}$ so that

$$|[S_{n\alpha}]^2| = |e(n\alpha)| = n(2h-2) = 2g(S_{n\alpha}) - 2.$$

Thus it follows that inequality (3.1) is an equality for this α and hence

$$||e||_\infty = \sup_{\alpha \neq 0} \frac{|e(\alpha)|}{||\alpha||_1} \geq \frac{1}{2}.$$

Combining this with our previous estimate gives the result. □

One may also give another proof of the boundedness of e using results on stable commutator lengths of Dehn twists as proven in [EK]. We recall that the commutator length $c_G(g)$ of an element g in a group G is defined as the smallest natural number so that g can be written as a product of $c_G(g)$ commutators. The *stable commutator length* is then defined as

$$||g||_{com} = \lim_{n \to \infty} \frac{c_G(g^n)}{n}.$$

An important property of the stable commutator length is that it is homogeneous in the sense that for any integer k the following holds

$$||g^k||_{com} = |k|\, ||g||_{com}\,.$$

The relationship between commutator lengths and the l^∞-norms of elements in the group cohomology $H^2(G)$ is given in the following lemma.

Lemma 3.1.8. *Let ϕ be an element in $H^2(G)$, which corresponds to a central $\mathbb{Z}$-extension*

$$1 \to \mathbb{Z} \to \hat{G} \to G \to 1.$$

Further, let Δ denote a generator of the kernel. Then the boundedness of ϕ implies that the stable commutator length of Δ is positive. If, in addition, $H_2(G)$ is finitely generated, then the converse also holds and

$$||\phi||_\infty \leq \frac{1}{4||\Delta||_{com}}.$$

Proof. Let $\alpha \in H_2(G)$ be represented by a map $\Sigma_\alpha \to K(G,1)$, which in turn determines a representation $\pi_1(\Sigma_\alpha) \xrightarrow{\rho} G$. We let a_i, b_i be the standard generators of $\pi_1(\Sigma_\alpha)$ and choose lifts α_i, β_i of $\rho(a_i), \rho(b_i)$ in $\hat{G}$. Then $\phi(\alpha)$ is the value of m such that the following holds

$$\prod_{i=1}^{g_\alpha} [\alpha_i, \beta_i] = \Delta^m\,.$$

Conversely, any elements α_i, β_i in $\hat{G}$ that satisfy the above commutator equation define an element $\alpha \in H_2(G)$ that can be represented by a surface of genus g_α with $\phi(\alpha) = m$.

Now suppose by way of contradiction that ϕ is bounded and $||\Delta||_{com} = 0$. Thus, for any $\epsilon > 0$ some power Δ^m can be written as a product of at most $m\epsilon$ commutators. We let α be the element in $H_2(G)$ defined by such a factorisation, which can then be represented by a surface of genus $g_\alpha \leq m\epsilon$. This implies that $||\alpha||_{GT} \leq 2m\epsilon$ and applying Lemma 3.1.5 we conclude that

$$\frac{|\phi(\alpha)|}{||\alpha||_1} \geq \epsilon^{-1}.$$

Since ϵ was arbitrary it follows that ϕ is unbounded and this contradiction proves the first part of the lemma.

Conversely, for any $\alpha \in H_2(G)$ and any natural number n the class $n\alpha$ can be represented by a surface of genus $g_{min}(n\alpha)$. As explained above this implies that Δ^{nk} can be written as a product of $g_{min}(n\alpha)$ commutators, where $k = \phi(\alpha)$, and hence

$$g_{min}(n\alpha) \geq ||\Delta^{nk}||_{com} = nk||\Delta||_{com} = |\phi(n\alpha)|\,||\Delta||_{com}.$$

Thus, we conclude that

$$|\phi(\alpha)| \leq \frac{g_{min}(n\alpha)}{n||\Delta||_{com}}.$$

By taking limits and applying Lemma 3.1.5 we obtain

$$||\Delta||_{com}\,|\phi(\alpha)| \leq \lim_{n\to\infty} \frac{g_{min}(n\alpha)}{n} = \frac{1}{2}\lim_{n\to\infty} \frac{2g_{min}(n\alpha) - 2}{n} = \frac{1}{2}||\alpha||_{GT} = \frac{1}{4}||\alpha||_1$$

so that for integral classes

$$\frac{|\phi(\alpha)|}{||\alpha||_1} \leq \frac{1}{4||\Delta||_{com}}.$$

It follows immediately that the same inequality holds for all elements in $H_2(G, \mathbb{Q})$. Since $H_2(G)$ is finitely generated, both $\phi(\alpha)$ and $||\alpha||_1$ are continuous (cf. Lemma 3.1.6) and hence

$$||\phi||_\infty = \sup_{\alpha\neq 0} \frac{|\phi(\alpha)|}{||\alpha||_1} \leq \frac{1}{4||\Delta||_{com}},$$

proving the second part of the lemma. □

In order to deduce the boundedness of e from Lemma 3.1.8 we consider its defining central $\mathbb{Z}$-extension

$$1 \to \mathbb{Z} \to \Gamma_h^1 \to \Gamma_{h,1} \to 1.$$

The kernel of this extension is generated by a Dehn twist ϕ_C around a curve that is parallel to $C = \partial\Sigma_h^1$. Now the group Γ_h^1 embeds into Γ_{h+1} and the image of ϕ_C is a Dehn twist around a homotopically non-trivial, separating curve in Σ_{g+1}. By [EK] there is a positive lower bound on the stable commutator length of ϕ_C considered as an element in Γ_{h+1} and hence the same holds in Γ_h^1. Since $H_2(\Gamma_{h,1})$ is finitely generated Lemma 3.1.8 implies that e is bounded. However, we cannot compute the exact norm of e in this manner, since the lower bound on $||\phi_C||_{com}$ given in [EK] is smaller than $\frac{1}{2}$.

3.1.2 Bounds on self-intersection numbers of multisections

The self-intersection of a k-multisection in a surface bundle gives a characteristic class $e_k^v \in H^2(\Gamma_{h,k})$. Mitsumatsu and Vogt have posed the question of whether there is a universal bound on the self-intersection numbers of such multisections depending only on the genera g and h of the base and fibre respectively (cf. [MV], Problem 8.11). That is if there exists a constant $C(h, g)$ so that

$$\sup_{k\in\mathbb{N}} |e_k^v(E)| \leq C(h, g) < \infty,$$

where the supremum is taken over all bundles with a k-multisection for fixed h and g.

An initial observation is that the classes e_k^v are bounded and that there is an obvious upper bound for $||e_k^v||_\infty$.

Proposition 3.1.9. *Let $h \geq 2$, then the classes $e_k^v \in H^2(\Gamma_{h,k})$ are bounded and $||e_k^v||_\infty \leq \frac{k}{2}$.*

Proof. Since $\Gamma_{h,k}$ is finitely presented (cf. [Iva1], Theorem 4.3 D) we may apply the first part of the proof of Proposition 3.1.7 *mutatis mutandis* to conclude that

$$||e_k^v||_\infty \leq \frac{k}{2}.$$ □

If one considers only multisections that are *pure* in the sense that the multisection in question consists of k disjoint sections, then one obtains a universal bound $C(h, g)$. A bundle that has a pure k-multisection is given by a holonomy map $\pi_1(\Sigma_g) \to P\Gamma_{h,k}$, where the P means that the holonomy maps fix the marked points pointwise, then it follows from Proposition 3.1.2 that for fixed g and h the self-intersection number of any pure k-multisection is bounded.

Proposition 3.1.10. *Let $\mathcal{C}_{pure}$ denote the class of all bundles with a pure multisection and let g and h be the genus of the base and fibre of E respectively with $h \geq 2$. Then the following holds*

$$\sup_{k\in\mathbb{N}, E\in\mathcal{C}_{pure}} |e_k^v(E)| \leq (2g - 2)(4gh + 2).$$

Proof. First note that the number of sections that can have non-zero self-intersection is bounded by $b_2(E) \leq 4gh + 2$ and this does not depend on k. By Proposition 3.1.2 each section has self-intersection at most $2g - 2$. Thus if the holonomy of a bundle is in $P\Gamma_{h,k}$ we conclude that

$$\sup_{k\in\mathbb{N}, E\in\mathcal{C}_{pure}} |e_k^v(E)| \leq (2g - 2)(4gh + 2).$$ □

Unfortunately the bound obtained in Proposition 3.1.10 grows quadratically in g and thus gives no bound on the norms $||e_k^v||_\infty$, which would in particular yield an affirmative answer to the question posed at the beginning of this section. For given any bundle with a k-multisection and holonomy map $\pi_1(\Sigma_g) \xrightarrow{\rho} \Gamma_{h,k}$ there is a finite cover $\Sigma_{\bar{g}} \xrightarrow{\tau} \Sigma_g$ of degree $N = k!$ and a lift $\bar{\rho}$ so that the following diagram commutes

$$\begin{tikzcd} P\Gamma_{h,k} \arrow[r] & \Gamma_{h,k} \\ \pi_1(\Sigma_{\bar{g}}) \arrow[u, "\bar{\rho}"] \arrow[r, "\tau"] & \pi_1(\Sigma_g). \arrow[u, "\rho"'] \end{tikzcd}$$

Then if $E, \bar{E}$ denote the bundles associated to $\rho, \bar{\rho}$ respectively, we have

$$|Ne_k^v(E)| = |e_k^v(\bar{E})| \leq (2\bar{g}-2)(4\bar{g}h+2) = N(2g-2)(4(N(g-1)+1)h+2).$$

Thus, the bound we obtain on $e_k^v(E)$ in this way depends on N which in turn grows with k.

Instead of asking for a bound on the self-intersection numbers of multisections in surface bundles of fixed Euler characteristic one may ask for a universal bound on the norms $||e_k^v||_\infty$ that is independent of k, as opposed to the bound given by Proposition 3.1.9, which grows linearly with k. This question may in turn be viewed as a stable version of the original question of Mitsumatsu-Vogt. Of course, it is strictly stronger than their original question and is in fact false. In order to show this we will translate the problem into a statement about the vertical Euler class of a certain stable group.

To this end we consider the sequence of inclusions of mapping class groups given by adding an annulus with one marked point to a genus h surface with one boundary component and n marked points:

$$... \to \Gamma^1_{h,n-1} \to \Gamma^1_{h,n} \to \Gamma^1_{h,n+1} \to ...$$

The injective limit of this sequence will be denoted by $\Gamma^1_{h,\infty}$ and the vertical Euler classes on $\Gamma^1_{h,n}$ defines a vertical Euler class e^v_∞ in $H^2(\Gamma^1_{h,\infty})$.

Lemma 3.1.11. *Suppose that the norms $||e_n^v||_\infty$ are bounded independently of n. Then the class e^v_∞ is bounded.*

Proof. We let $C = \sup_n ||e_n^v||_\infty$. This also serves as a universal bound on the associated classes on $\Gamma^1_{h,n}$ and hence for any homology class $\alpha \in H_2(\Gamma^1_{h,\infty})$ one has

$$\begin{aligned} |e^v_\infty(\alpha)| = |e^v_n(\alpha_n)| &\leq ||e^v_n||_\infty \, ||\alpha_n||_1 \\ &\leq C||\alpha_n||_1, \end{aligned}$$

where α_n is any class in $H^2(\Gamma^1_{h,n})$ which projects to α in the limit. By choosing the α_n appropriately we may assume that $||\alpha||_1 = \lim_{n\to\infty} ||\alpha_n||_1$. Thus after applying limits to the inequality above we conclude that $|e^v_\infty(\alpha)| \leq C||\alpha||_1$ and hence that e^v_∞ is a bounded class. $\square$

In order to show that the class e^v_∞ is not bounded, it will be important to have an explicit description of the classes e^v_n on the level of group cohomology. To this end we let $\Sigma_h^{n,1}$ be a surface with $n+1$ boundary components and we choose identifications $h_k : S^1 \to \partial\Sigma_h^{n,1}$ of the k-th boundary component with the circle for $1 \leq k \leq n$. We let $Homeo^+(\Sigma_h^{n,1})$ denote the group of all orientation preserving homeomorphisms of Σ_h^{n+1} that fix the $(n+1)$-st boundary component pointwise and let σ denote the natural map $Homeo^+(\Sigma_h^{n,1}) \to S_n$ given by the action on the boundary components. We next consider the subgroup $\widehat{Homeo}^+(\Sigma_h^{n,1})$ of $Homeo^+(\Sigma_h^{n,1})$ defined by taking those elements ϕ that have the additional property that $\phi \circ h_k = h_{\sigma(k)}$ for all $1 \leq k \leq n$. The group of components of $\widehat{Homeo}^+(\Sigma_h^{n,1})$ will be denoted $\hat{\Gamma}^1_{h,n}$. This group fits into the following exact sequence, where the second map is given by coning off the boundary components and the kernel is generated by Dehn twists around parallels of the boundary components:

$$1 \to \mathbb{Z}^n \to \hat{\Gamma}^1_{h,n} \to \Gamma^1_{h,n} \to 1.$$

Moreover, the conjugation action of $\Gamma^1_{h,n}$ on $\mathbb{Z}^n$ is given by the natural action of the symmetric group. Thus, this extension gives an element $\hat{e}^v_n$ in cohomology with twisted coefficients $H^2(\Gamma^1_{h,n}, \mathbb{Z}^n)$, where the action is given via the natural map to S_n. The map $\mathbb{Z}^n \to \mathbb{Z}$ sending each basis vector to 1 is S_n-equivariant and induces a map to ordinary cohomology and the image of $\hat{e}^v_n$ is $e^v_n \in H^2(\Gamma^1_{h,n}, \mathbb{Z})$. The associated central $\mathbb{Z}$-extension will be denoted by $\overline{\Gamma}^1_{h,n}$. Moreover, the groups $\hat{\Gamma}^1_{h,n}$ give the following commuting diagram of inclusions

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
\cdots \arrow[r] & \mathbb{Z}^{n-1} \arrow[r] \arrow[d] & \mathbb{Z}^n \arrow[r] \arrow[d] & \mathbb{Z}^{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \hat{\Gamma}^1_{h,n-1} \arrow[r] \arrow[d] & \hat{\Gamma}^1_{h,n} \arrow[r] \arrow[d] & \hat{\Gamma}^1_{h,n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Gamma^1_{h,n-1} \arrow[r] \arrow[d] & \Gamma^1_{h,n} \arrow[r] \arrow[d] & \Gamma^1_{h,n+1} \arrow[r] \arrow[d] & \cdots \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Thus, by considering the injective limits of the groups $\hat{\Gamma}^1_{h,n}$, we obtain a group $\hat{\Gamma}^1_{h,\infty}$ that fits into the following short exact sequence

$$1 \to \mathbb{Z}^\infty \to \hat{\Gamma}^1_{h,\infty} \to \Gamma^1_{h,\infty} \to 1.$$

This extension corresponds to an element $\hat{e}^v_\infty \in H^2(\Gamma^1_{h,\infty}, \mathbb{Z}^\infty)$, which denotes the cohomology group taken with twisted $\mathbb{Z}^\infty$-coefficients. As above, there is an S_∞-equivariant map $\mathbb{Z}^\infty \to \mathbb{Z}$ sending each basis vector to 1 that induces a map to ordinary cohomology and the image of $\hat{e}^v_\infty$ is the class e^v_∞ defined above. The class e^v_∞ then determines a central $\mathbb{Z}$-extension

$$1 \to \mathbb{Z} \to \overline{\Gamma}^1_{h,\infty} \to \Gamma^1_{h,\infty} \to 1.$$

Furthermore, the group $\overline{\Gamma}^1_{h,\infty}$ fits into the following exact sequence

$$1 \to K \to \hat{\Gamma}^1_{h,\infty} \to \overline{\Gamma}^1_{h,\infty} \to 1,$$

where K is the kernel of the map $\mathbb{Z}^\infty \to \mathbb{Z}$.

The group $\hat{\Gamma}^1_{h,\infty}$ may also be interpreted as the mapping class group of a certain non-compact surface with infinitely many boundary components. More precisely, we let $\Sigma_h^{\infty,1}$ be a genus h surface having an infinite end with infinitely many open discs removed. We identify the k-th boundary component of $\Sigma_h^{\infty,1}$ with S^1 via a map h_k. Furthermore, we let $Homeo_c(\Sigma_h^{\infty,1})$ denote the group of compactly supported homeomorphisms of $\Sigma_h^{\infty,1}$ and we let σ denote the natural map $Homeo_c(\Sigma_h^{\infty,1}) \to S_\infty$ given by the induced action on the boundary components. We define $\widehat{Homeo}_c(\Sigma_h^{\infty,1})$ to be the group of compactly supported homeomorphisms such that $\phi \circ h_k = h_{\sigma(k)}$. Then $\hat{\Gamma}^1_{h,\infty}$ is the group of components of the group $\widehat{Homeo}_c(\Sigma_h^{\infty,1})$.

There is a relationship between the boundedness of cohomology classes and stable commutator lengths (cf. Lemma 3.1.8). In order to prove that the class e^v_∞ is unbounded, we shall also need to exploit the relationship between stable commutator lengths and quasi-homomorphisms. We recall that a real valued map ϕ on a group G is a quasi-homomorphism if the following supremum is finite

$$D(\phi) = \sup |\phi(gh) - \phi(g) - \phi(h)|.$$

The number $D(\phi)$ is called the *defect* of ϕ. A quasi-homomorphism is homogeneous if $\phi(g^n) = n\phi(g)$ for all integers n. For a group G we will let $\widetilde{QH}(G)$ denote the group of homogeneous quasi-homomorphisms. With this notation we have the following fundamental result of Bavard.

Theorem 3.1.12 ([Bav]). *For any element g in a group G the following holds*

$$||g||_{com} = \sup |\frac{\phi(g)}{2D(\phi)}|\,,$$

where the supremum is taken over all $\phi \in \widetilde{QH}(G)$ and $D(\phi)$ is the defect of ϕ.

We shall also need the fact that any homogeneous quasi-homomorphisms on $\hat{\Gamma}^1_{0,\infty}$ is in fact a homomorphism. For any $n \in \mathbb{N} \cup \{\infty\}$ the group $\hat{\Gamma}^1_{0,n} = \hat{B}_n$ is referred to as the extended braid group since it fits into the following extension

$$1 \to \mathbb{Z}^n \to \hat{\Gamma}^1_{0,n} \to \Gamma^1_{0,n} = B_n \to 1,$$

where B_n denotes the ordinary braid group on n-strands. Kotschick has proven that the only homogeneous quasi-homomorphisms on B_∞ are homomorphisms and the proof of the following proposition is almost identical to that of Theorem 3.5 in [Kot3].

Proposition 3.1.13. *Any homogeneous quasi-homomorphism on the group $\hat{B}_\infty$ is a homomorphism.*

Proof. By considering disjoint embeddings $\Sigma_0^{n,1} \hookrightarrow \Sigma_0^{\infty,1}$ one obtains infinitely many embeddings

$$\hat{B}_n \hookrightarrow \hat{B}_\infty$$

that are conjugate in $\hat{B}_\infty$ and such that any two elements that lie in distinct embeddings commute in $\hat{B}_\infty$. Thus, by Proposition 2.2 in [Kot3] any homogeneous quasi-homomorphism ϕ on $\hat{B}_\infty$ restricts to a homomorphism on $\hat{B}_n$. Furthermore, any element in $\hat{B}_\infty$ lies in some $\hat{B}_n$ and, hence, ϕ is in fact a homomorphism on the whole group $\hat{B}_\infty$. $\square$

With these preliminaries we are ready to prove that the class e^v_∞ is not bounded.

Theorem 3.1.14. *The class $e^v_\infty \in H^2(\hat{\Gamma}^1_{h,\infty})$ is unbounded.*

Proof. We consider the central $\mathbb{Z}$-extension associated to e^v_∞

$$1 \to \mathbb{Z} \to \overline{\Gamma}^1_{h,\infty} \to \Gamma^1_{h,\infty} \to 1,$$

whose kernel is generated by some Δ. By Lemma 3.1.8, if e^v_∞ is bounded, then the stable commutator length of Δ is positive. Hence, in order to show that the class e^v_∞ is unbounded, it will suffice to show that the stable commutator length of Δ is zero.

The element Δ is the image of a Dehn twist around a boundary parallel curve in $\Sigma_h^{\infty,1}$ under the map $\hat{\Gamma}^1_{h,\infty} \to \overline{\Gamma}^1_{h,\infty}$. More generally, let γ_k be a simple closed curve in $\Sigma_h^{\infty,1}$ that bounds k boundary components and let Δ_k be a Dehn twist about γ_k. The stable commutator length of Δ_k depends only on k, since any two curves γ_k and γ'_k are conjugate by a homeomorphism in $\widehat{Homeo}_c(\Sigma_h^{\infty,1})$ and, hence, the corresponding Dehn twists are conjugate in $\hat{\Gamma}^1_{h,\infty}$. In this notation Δ is the image of Δ_1 under the map $\hat{\Gamma}^1_{h,\infty} \to \overline{\Gamma}^1_{h,\infty}$ and it follows that

$$||\Delta||_{com} \leq ||\Delta_1||_{com}.$$

So it will be sufficient to show that $||\Delta_1||_{com} = 0$.

We assume to the contrary that the stable commutator length of Δ_1 is positive. We let $D_4 \hookrightarrow \Sigma_h^{\infty,1}$ be a disc with four smaller discs removed, the boundaries of which are mapped to boundary components of $\Sigma_h^{\infty,1}$. This inclusion gives a map of the compactly supported mapping class group of D_4 into $\hat{\Gamma}^1_{h,\infty}$. We let $C_1, .., C_4$ be the interior boundary components of D_4. We also let $\Delta(i_1, .., i_m)$ denote the Dehn twist around an embedded closed curve containing $C_{i_1}, .., C_{i_m}$. Then the following lantern relation holds in the compactly supported mapping class group of D_4 and, thus, also in $\hat{\Gamma}^1_{h,\infty}$:

$$\Delta(12)\Delta(23)\Delta(13) = \Delta(1)\Delta(2)\Delta(3)\Delta(123).$$

We let ϕ be a homogeneous quasi-homomorphism on $\hat{\Gamma}^1_{h,\infty}$. An inclusion $\Sigma_0^{\infty,1} \hookrightarrow \Sigma_h^{\infty,1}$ induces an embedding $\hat{B}_\infty \hookrightarrow \hat{\Gamma}^1_{h,\infty}$ and by Proposition 3.1.13 any homogeneous quasi-homomorphism restricts to a homomorphism on $\hat{B}_\infty$. In particular, ϕ is a homomorphism on the normal subgroup generated by the Δ_k. Then since a homogeneous quasi-homomorphism is constant on conjugacy classes the lantern relation implies that

$$\phi(\Delta_3) = 3\phi(\Delta_2) - 3\phi(\Delta_1).$$

Similarly one has

$$\Delta(123)\Delta(234)\Delta(34) = \Delta(12)\Delta(3)\Delta(4)\Delta(1234)$$

and by the same reasoning as above it follows that

$$\phi(\Delta_4) = 6\phi(\Delta_2) - 8\phi(\Delta_1).$$

Furthermore, the embedding $\Sigma_h^{\infty,1} \to \Sigma_h^{\infty,1}$ given by attaching a k-punctured disc to each boundary component induces a map on $\hat{\Gamma}^1_{h,\infty}$ that sends a Dehn twist around a boundary component to a Dehn twist about some γ_k and, hence,

$$||\Delta_k||_{com} \leq ||\Delta_1||_{com}. \tag{3.2}$$

By Theorem 3.1.12 and the assumption that $||\Delta_1||_{com}$ is positive, we may choose a homogeneous quasi-homomorphism ϕ such that

$$\frac{\phi(\Delta_1)}{2D(\phi)} \geq \frac{15}{16}||\Delta_1||_{com} > 0.$$

Since the stable commutator length of Δ_2 is bounded from above by the stable commutator length of Δ_1, by applying Theorem 3.1.12 once more we deduce that

$$||\Delta_1||_{com} \geq \frac{\phi(\Delta_1)}{2D(\phi)} \geq \frac{15}{16}||\Delta_1||_{com} \geq \frac{15}{16}\frac{\phi(\Delta_2)}{2D(\phi)}. \tag{3.3}$$

Using inequalities (3.2) and (3.3) we then compute

$$\begin{aligned}
||\Delta_1||_{com} \geq ||\Delta_4||_{com} \geq & |\frac{\phi(\Delta_4)}{2D(\phi)}| \\
&= |6\frac{\phi(\Delta_2)}{2D(\phi)} - 8\frac{\phi(\Delta_1)}{2D(\phi)}| \\
&= |\Big(2\frac{\phi(\Delta_1)}{2D(\phi)} - \frac{6}{16}\frac{\phi(\Delta_2)}{2D(\phi)}\Big) + 6\Big(\frac{\phi(\Delta_1)}{2D(\phi)} - \frac{15}{16}\frac{\phi(\Delta_2)}{2D(\phi)}\Big)| \\
&\geq 2\frac{\phi(\Delta_1)}{2D(\phi)} - \frac{6}{16}\frac{\phi(\Delta_2)}{2D(\phi)} \\
&\geq \frac{24}{16}||\Delta_1||_{com} > ||\Delta_1||_{com},
\end{aligned}$$

which yields a contradiction. □

As an immediate consequence of Lemma 3.1.11 we have the following corollary.

Corollary 3.1.15. *The sequence* $||e^v_n||_\infty$ *is unbounded.*

In fact by Proposition 3.1.9 we know that this sequence can grow at most linearly with n. It would be interesting to know the precise growth rate of the sequence $||e^v_n||_\infty$, in particular whether it is linear or not.

3.2 MMM-classes vanish on amenable groups

There is a family of characteristic classes of oriented surface bundles that can be defined using the vertical Euler class. These are the so-called Mumford-Miller-Morita (MMM) classes. The k-th MMM-class is defined as $e_k = \pi_! e^{k+1}$, where e is the vertical Euler class of an oriented surface bundle and $\pi_!$ denotes integration along the fibre.

It was conjectured by Morita in [Mor4] that all the MMM-classes $e_k \in H^*(\Gamma_h)$ are bounded in the sense of Gromov. It is known that e_k is bounded for k odd ([Mor4], Remark 7.2) so it remains to deal with the case where k is even. A necessary condition that e_k is bounded is that it vanishes on amenable groups and as evidence for his conjecture Morita showed that all the e_k do indeed have this property ([Mor4], Theorem 7.1). We shall give an alternate proof of this result which arises by considering the simplicial volume of surface bundles over bases with amenable fundamental groups. In particular, if $\Sigma_h \to E \to B$ is a surface bundle over a base manifold with amenable fundamental group, we will show that the simplicial volume of the total space vanishes, unless the base is S^1. The latter condition is necessary, as Thurston's Hyperbolisation Theorem implies that any mapping torus with pseudo-Anosov monodromy is hyperbolic and hence has non-vanishing simplicial volume.

The main tool we shall need is the theory of reduction systems for subgroups of the mapping class group (see [BLM], [Iva2]). It will be convenient to use slightly different notation in this section. Namely, we consider a compact, orientable surface Σ with possibly non-empty boundary and define

$$MCG(\Sigma) = PDiff^+(\Sigma)/Diff_0(\Sigma).$$

Here $PDiff^+(\Sigma)$ is the group of orientation preserving diffeomorphisms of Σ that *do not* permute boundary components and $Diff_0(\Sigma)$ denotes those diffeomorphisms that are isotopic to the identity where the isotopy need not fix the boundary. We will also want to consider the group of diffeomorphisms that fix the boundary up to isotopy which we denote by

$$MCG(\Sigma, \partial\Sigma) = PDiff^+(\Sigma, \partial\Sigma)/Diff_0(\Sigma, \partial\Sigma).$$

A subgroup $G \subset MCG(\Sigma)$ is called *reducible* if there exists a homotopically non-trivial, embedded 1-dimensional submanifold $C \subset \Sigma$ which is *componentwise* fixed by every element in G up to isotopy. If Σ has boundary we require that no component of C is isotopic into the boundary. Such a submanifold is called a *reduction system* for G. If no such C exists, then we say that G is *irreducible.*

Next we consider a reducible subgroup $G \subset MCG(\Sigma)$. This then gives a map of G to $MCG(\Sigma\backslash C)$ since the identity component of the group of diffeomorphisms $Diff_0(\Sigma, C)$ that preserve the components of C is contractible (see [Iva1], [Iva2]). We let $MCG(\Sigma, C)$ denote the subgroup of the mapping class group that fixes each component of C up to isotopy. If we let Q_i denote the closure of the components of $\Sigma \setminus C$, then there is a natural map $MCG(\Sigma, C) \to \prod_i MCG(Q_i)$, whose kernel is the abelian group generated by Dehn twists along the components of C. An important fact is that after taking a finite index subgroup $G' \subset G$, there is always a so-called *maximal* reduction system C_{max} so that the image of G' in each $MCG(Q_i)$ is irreducible or trivial ([Iva2], Cor. 7.18).

Moreover, any *irreducible* subgroup either contains a free group on two generators or is virtually cyclic ([Iva2], Cor. 8.6 and Th. 8.9). Thus if G is amenable it contains no free groups on two generators and hence the image H_i of G' in $MCG(Q_i)$ is virtually cyclic. After taking finite index subgroups one may assume that each H_i is either infinite cyclic, or trivial. The analogous result for solvable groups is older and goes back to Birman-Lubotzky-McCarthy in [BLM].

We summarise this discussion in the following theorem.

Theorem 3.2.1 ([Iva2])**.** *Let Σ be any compact surface and let $G \subset MCG(\Sigma)$ be amenable. Then G is virtually abelian. Moreover, there exists a finite index subgroup $G' \subset G$ and a reduction system C so that the images of G' in $MCG(Q_i)$ are infinite cyclic or trivial.*

We are now able to state and prove the following theorem.

Theorem 3.2.2. *Let $\Sigma \to E \to B$ be a surface bundle over a closed manifold of dimension $dim(B) \geq 2$. If $\pi_1(B)$ is amenable, then the simplicial volume $||E||$ vanishes.*

Proof. Since the vanishing of the simplicial volume is unchanged under finite covers, we may assume that the image of the holonomy map of E is free abelian by Theorem 3.2.1. That is E is obtained as the pullback of some bundle E' over a torus T^N and we have the following commuting diagram:

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & T^N. \end{tikzcd}$$

Since the kernel of the induced map $\pi_1(E) \to \pi_1(E')$ is amenable, Gromov's Mapping Theorem implies that $||[E]||_1 = ||f_*[E]||_1$ (cf. [Gro], p. 40). Moreover, by applying the transfer homomorphism in homology to the above diagram we have

$$\begin{tikzcd} H_{n+2}(E) \arrow[r, "\bar{f}_*"] & H_{n+2}(E') \\ H_n(B) \arrow[r, "f_*"] \arrow[u, "\pi^!"] & H_n(T^N). \arrow[u, "(\pi')^!"'] \end{tikzcd}$$

Since every class in $H_*(T^N)$ can be represented as a sum of tori, the class $f_*([B])$ can be represented as a sum of tori. The commutativity of the above diagram implies that the class $\bar{f}_*[E]$ is representable by a sum of the fundamental classes of several Σ-bundles over tori of dimension $n = dim(B)$. Thus it suffices to prove the theorem under the assumption that the base B is a torus of dimension $n \geq 2$ and from now on we shall assume this.

We let C_{max} be a maximal reduction system for the holonomy of E which gives fibrewise embedded S^1-bundles $\xi_i \subset E$ for each component of C_{max}. These S^1-bundles are π_1-injective and have amenable fundamental group. Thus Gromov's Cutting-off Theorem (cf. [Gro], p. 58) implies

$$||E|| = ||E \setminus \bigcup_i \xi_i||.$$

Since the holonomy group of each component Q_i of $\Sigma \setminus C_{max}$ is either infinite cyclic or trivial, we see that each component of $E \setminus \bigcup_i \xi_i$ is diffeomorphic to $M_i \times T^{n-1}$, where M_i is a mapping torus with fibre $Int(Q_i)$. The manifold $M_i \times T^{n-1}$ admits proper self-maps of arbitrary degree, since $n \geq 2$. Hence the simplicial volume is either zero or infinite. As E is closed we know that $||E|| < \infty$ and we conclude that

$$||E|| = ||E \setminus \bigcup_i \xi_i|| = \sum_i ||M_i \times T^{n-1}|| = 0. \qquad \square$$

As a consequence of Theorem 3.2.2 and the boundedness of the vertical Euler class (cf. Proposition 3.1.7) we will show that all MMM-classes vanish on amenable subgroups of $\Gamma_h = MCG(\Sigma_h)$.

Theorem 3.2.3 (Morita, [Mor4])**.** *The images of the MMM-classes in $H^*(G, \mathbb{Q})$ are trivial for amenable $G \subset \Gamma_h$.*

Proof. After taking a finite index subgroup we may assume that $G = \mathbb{Z}^N$ is free abelian. This subgroup corresponds to a surface bundle $\Sigma_h \to E \to T^N$ over the N-torus. Moreover, by Proposition 3.1.7 the vertical Euler class is bounded with $||e||_\infty = \frac{1}{2}$ and thus $e^{k+1} \in H^{2k+2}(\Gamma_{h,1})$ is bounded with

$$||e^{k+1}||_\infty \le (\frac{1}{2})^{k+1}.$$

The group $H_{2k}(T^N)$ has a basis consisting of (embedded) tori $T_j \hookrightarrow T^N$ and the theorem will follow if we show that e_k is trivial on each T_j.

We let E_j denote the restriction of E to T_j and compute

$$\begin{aligned} |\langle e_k(E), [T_j]\rangle| &= |\langle \pi_! e^{k+1}, [T_j]\rangle| \\ &= |\langle e^{k+1}, \pi^![T_j]\rangle| \\ &= |\langle e^{k+1}, [E_j]\rangle| \\ &\le ||e^{k+1}||_\infty \, ||E_j|| \le (\frac{1}{2})^{k+1}||E_j||. \end{aligned}$$

Since the dimension of the torus T_j is $2k$, Theorem 3.2.2 implies that $||E_j|| = 0$ and the result follows. □

Motivated by [KL] we will next investigate which classes in $H_*(\Gamma_h)$ can be represented as the image of the fundamental class of a non-trivial product $N = M_1 \times M_2$ of closed manifolds. In particular, we will show that if $m = \max\{dim(M_1), dim(M_2)\}$, then $e_k([N]) = 0$ for all $k > \frac{m}{2}$. This means that e_k vanishes for any bundle over a non-trivial product N of dimension $dim(N) = 2k$. We begin by proving the following lemma.

Lemma 3.2.4. *Let Σ be a closed, connected surface and let C be a disjoint collection of embedded circles on Σ. We let Q_j be the components of $\Sigma \setminus C$ and let $\overline{Q}_j$ be the closed surface obtained from Q_j by identifying each boundary component to a point. We further let $\bar{\rho}_j$ be the natural map $MCG(\Sigma, C) \to MCG(\overline{Q}_j)$ and $\bar{e}_k$ the k-th MMM-class on $MCG(\overline{Q}_j)$. Then the k-th MMM-class on $MCG(\Sigma, C)$ satisfies*

$$e_k = \sum_{j=1}^{n} \bar{\rho}_j^* \bar{e}_k.$$

Proof. For simplicity let $G_j = MCG(Q_j)$ and $G_C = MCG(\Sigma, C)$. We also let $\overline{G}_j = MCG(\overline{Q}_j)$ be the mapping class group $\overline{Q}_j$.

By definition, the universal bundle over BG_C has a natural decomposition

$$E = \bigcup_{j=1}^{n} E_j,$$

where E_j is a bundle with fibre Q_j. Moreover, the vertical bundle is trivial over ∂E_j with a trivialisation given by taking vectors tangent to the boundary. We let ξ_C denote the union of the S^1-bundles corresponding to ∂E_j. Then the vertical vector bundle on E descends to a bundle on the quotient space $E^* = E/\xi_C$. Similarly the vertical bundle on E_j descends to

$E_j^* = E_j/\partial E_j$ and we note that $E^* = \bigvee_{j=1}^n E_j^*$. Since E^* is a wedge sum and the Euler class is natural under pullbacks, we compute

$$e^{k+1}(E) = \sum_{j=1}^{n} e^{k+1}(E_j^*). \tag{3.4}$$

We let $\overline{E}_j$ denote the bundle obtained from E_j by fibrewise identifying each boundary component of Q_j to a point and we let $BG_C \xrightarrow{\bar{\rho}_j} B\overline{G}_j$ be the classifying map of this bundle. In this way we obtain the following commuting diagram, where each vertical arrow is a fibration:

$$\begin{tikzcd}
 & E_j^* & & \\
E \arrow[ur] \arrow[r] \arrow[d, "\pi"] & \overline{E} \arrow[u] \arrow[d] & \overline{E}_j \arrow[ul] \arrow[l] \arrow[r] \arrow[d] & E\overline{G}_j \arrow[d] \\
BG_C \arrow[r, "Id"] & BG_C & BG_C \arrow[l, "Id"'] \arrow[r, "\bar{\rho}_j"] & B\overline{G}_j.
\end{tikzcd}$$

Using the naturality of the transfer map we conclude that $\pi_!(e^{k+1}(E_j^*)) = \bar{\rho}_j^* \bar{e}_k$ and the lemma follows by equation (3.4). $\square$

This leads us to the following theorem, whose proof is similar to that of Proposition 3.8 in [KL].

Theorem 3.2.5. *Let $\Sigma \to E \to B$ be a surface bundle over a base $B = M_1 \times M_2$ that is a non-trivial product. If $m = max\,\{dim(M_1), dim(M_2)\}$, then $e_k(E) = 0$ for all $k > \frac{m}{2}$.*

Proof. We let $G_i = \pi_1(M_i)$ and $G = G_1 \times G_2 \xrightarrow{\rho} MCG(\Sigma)$ denote the holonomy map of E. If the image of G lies in the kernel of the map Φ_3 given by the composition

$$MCG(\Sigma) \to Aut(H_1(\Sigma, \mathbb{Z})) \to Aut(H_1(\Sigma, \mathbb{Z}_3)),$$

then the existence of a maximal reduction system is guaranteed by ([Iva2], Cor. 7.18), since the kernel of this map consists of *pure* elements of $MCG(\Sigma)$. Thus after taking finite index subgroups of the G_i we may assume that this is the case and without loss of generality we have a maximal reduction system C_{max} so that the images of $G \xrightarrow{\rho_i} MCG(Q_i)$ are trivial or irreducible. If $\rho_i(G)$ is non-trivial it must contain a pseudo-Anosov element $\phi = \rho_i(a, b)$. Since the subgroup generated by $\alpha = \rho_i(a, e)$ and $\beta = \rho_i(e, b)$ is irreducible, abelian and consists of elements in the kernel of the map Φ_3, it must be infinite cyclic and is generated by a pseudo-Anosov element ψ ([Iva2], Cor. 7.14 and Cor. 8.6). In particular, α and β are pseudo-Anosov.

Without loss of generality we assume that the α defined above is non-trivial. Then since the subgroup G_2 commutes with (a, e) we conclude that $\rho_i(G_2)$ lies in the centraliser of α. However, the centraliser of a pseudo-Anosov element is infinite cyclic and is generated by a pseudo-Anosov element ([Iva2], Lemma 8.13). Hence $\rho_i(G_2)$ is also cyclic with a pseudo-Anosov generator or it is trivial. If it is non-trivial then the fact that G_1 commutes with G_2

implies via the same argument that the image $\rho_i(G)$ is also cyclic. Thus we conclude that either $\rho_i(G_j)$ is trivial, in which case ρ_i factors through one of the projections $G \xrightarrow{\pi_j} G_j$, or that the image $\rho_i(G)$ is cyclic (or trivial).

We let Σ_1 denote the subsurface of Σ on which $\rho(G_1)$ is non-trivial but $\rho(G_2)$ is trivial. Similarly, we let Σ_2 be the subsurface where $\rho(G_2)$ is non-trivial but $\rho(G_1)$ is trivial. We finally let Σ_3 be the subsurface on which the holonomy is cyclic. We let $\bar{\rho}_j$ denote the induced maps to $MCG(\overline{\Sigma}_j)$ and by applying Lemma 3.2.4 we conclude that

$$e_k = \sum_{j=1}^{3} \bar{\rho}_j^* \bar{e}_k.$$

The first two summands vanish for dimension reasons if $k > \frac{m}{2}$ and since the image of $\bar{\rho}_3$ is abelian the third vanishes by Theorem 3.2.3. □

We contrast the above result with those of Morita in [Mor1]. In particular, Morita showed that for sufficiently large genus any MMM-class is detected by an iterated surface bundle given by what is now called the Morita m-construction. Repeated application of Theorem 3.2.5 implies that the only MMM-classes that are possibly non-trivial over a base that is a product of surfaces are of the form e_1^k. Moreover, it can be shown that these classes can be detected by products of Riemann surfaces if the genus of the fibre satisfies $g \geq 3k$. In fact, the proof of Theorem 3.2.5 means that this bound is sharp for such bundles, since e_1 is trivial for bundles with fibre of genus $g \leq 2$ (cf. [KM1]).

3.3 MMM-classes are hyperbolic

As previously mentioned Morita has conjectured that the MMM-classes e_k have representatives that are bounded in the sense of Gromov. In particular, he showed that e_k vanishes on amenable groups (cf. Theorem 3.2.3). There is a weaker notion than that of boundedness, so-called *hyperbolicity.* We will extend Morita's original argument to show that the MMM-classes are hyperbolic.

We shall first recall the definition of hyperbolicity for simplicial complexes following [BrK]. To this end we need to consider metrics and differential forms on simplicial complexes. Recall that a metric on a simplicial complex is given by a metric g_σ on each simplex σ so that when $\tau \subset \sigma$ is a face, one has $g_\sigma|_\tau = g_\tau$. Similarly, a differential form is a collection of forms on each simplex compatible with restriction to faces. One can then define the exterior derivative on each simplex and the resulting cohomology is isomorphic to ordinary cohomology for simplicial complexes (cf. [Swa], [Whit]).

Theorem 3.3.1 (Simplicial de Rham Theorem). *Let X be a simplicial complex. Then there is a natural isomorphism $H^k_{dR}(X) \xrightarrow{\Psi} H^k_\Delta(X, \mathbb{R})$ from de Rham cohomology to simplicial cohomology given by integration over chains.*

The de Rham isomorphism Ψ has a natural inverse on the chain level. This is defined as follows: let σ be an oriented simplex of X and let μ_i denote the baracentric coordinate map defined by the the i-th vertex v_i of σ. That is for a simplex τ of X we define $\mu_i|_\tau$ to be zero if v_i is not a vertex of τ, otherwise we let $\mu_i|_\tau(p)$ be the coefficient of v_i given by writing p as a convex combination of the vertices of τ. The functions μ_i are well-defined elements in $\Omega^0(X)$. Then for any oriented cosimplex $\sigma \in C^k_\Delta(X,\mathbb{R})$ we define

$$\Phi_\sigma = k! \sum_{i=0}^{k} (-1)^i \mu_i d\mu_0 \wedge ... \wedge \widehat{d\mu_i} \wedge ... d\mu_k.$$

This is a so-called elementary k-form and has support in the the set $st(\sigma)$. For an arbitrary simplicial cochain c that we think of as a sum $\sum \lambda_\sigma \sigma$ we set

$$\Phi(c) = \sum \lambda_\sigma \Phi_\sigma$$

and this map is the desired inverse of Ψ (cf. [Whit], p. 229 ff).

Recall that a k-form $\alpha \in \Omega^k(M)$ on a manifold is bounded if

$$||\alpha||_g = \sup_{x\in M} |\alpha_x(e_1, ...e_k)| < \infty,$$

where $e_1, ..., e_k$ is any k-tuple of orthonormal vectors in T_xM. For a simplicial complex a form is bounded if there is a universal bound over all simplices.

For a simplicial cochain $c \in C^k_\Delta(X,\mathbb{R})$ one also has a notion of boundedness. Indeed, one has the L^∞-norm

$$||c|| = \sup_\sigma |c(\sigma)|,$$

where the supremum is taken over all k-simplices σ of X. If this number is finite then c is said to be bounded. Moreover, the set of bounded simplicial cochains is a subcomplex of $C^*_\Delta(X,\mathbb{R})$ that we denote by $\hat{C}^*_\Delta(X,\mathbb{R})$. Under certain fairly natural assumptions the de Rham isomorphism and its inverse preserve boundedness:

Proposition 3.3.2. *Let X be a simplicial complex and let g be a metric so that for all k-simplices $Vol(\sigma, g) \le C_k$. Then the map $\Omega^k(X) \xrightarrow{\Phi} C^k_\Delta(X,\mathbb{R})$ given by integration over chains preserves boundedness.*

Conversely, assume that the star of each simplex of X contains a bounded number of k-simplices for some universal constant S_k and that g is a metric on X so that the 1-forms $d\mu_i$ given by baracentric coordinates are uniformly bounded. Then the inverse of the de Rham isomorphism preserves boundedness.

In particular, if $X \xrightarrow{p} Y$ is a (simplicial) covering map and Y is finite, then X endowed with the pullback metric satisfies the hypotheses above.

Proof. We let Ψ denote the de Rham isomorphism, given by integration over chains. For the first statement note that for any k-form ω and any k-simplex σ

$$|\Psi(\omega)(\sigma)| = |\int_\sigma \omega| \le Vol(\sigma, g)||\omega||_g \le C_k||\omega||_g$$

and hence $\Psi(\omega)$ is a bounded simplicial cochain.

Conversely, let c be a bounded simplicial cochain that we write as $c = \sum \lambda_\sigma \sigma$. Then

$$\Phi(c) = \sum \lambda_\sigma \Phi_\sigma$$

and the λ_σ are bounded by definition. Moreover, the Φ_σ are elementary k-forms and as the forms $d\mu_i$ are uniformly bounded the same holds for Φ_σ. Moreover, every point $p \in X$ lies in the interior of a unique simplex τ_p and $\Phi_\sigma(p)$ is necessarily zero unless σ lies in $st(\tau_p)$. Thus $\Phi_\sigma(p)$ is non-zero for at most S_k simplices and it follows that $\Phi(c)$ is uniformly bounded.

Finally, if $X \xrightarrow{p} Y$ is a covering of a finite complex and $g_X = p^*g$ is the pullback metric on X, then the volumes of simplices are the same as their images in Y and these are uniformly bounded by the assumption that Y is finite. The same holds for the 1-forms $d\mu_i$, since these are locally pullbacks of the corresponding forms on Y. Moreover, the star of each simplex in X contains at most S_k simplices, where S_k denotes the number k-simplices in Y and this is finite by assumption, thus proving the final claim. □

We now consider a finite simplicial complex X with a metric g and let $\tilde{g}$ denote the pullback metric on the universal cover $\tilde{X} \xrightarrow{p} X$. With this notation we have the following definition of hyperbolicity of cohomology classes on finite complexes.

Definition 3.3.3. A class $\alpha \in H^k(X, \mathbb{R})$ is called *hyperbolic* if there exists a de Rham representative $\eta \in \Omega^k(X)$ of α, so that the $p^*\eta$ has a bounded primitive with respect to the metric $\tilde{g}$.

By Proposition 3.3.2 this is equivalent to the statement that the simplicial cochain $p^*\Psi(\eta) = \Psi(\pi^*\eta) \in \hat{C}^*_\Delta(\tilde{X}, \mathbb{R})$ is exact as a bounded simplicial cochain. Hence it is clear that the definition is independent of the metric and the chosen representative η. Furthermore, since any continuous map can be approximated by a simplicial map, hyperbolicity is natural under maps between finite complexes.

More generally, if Y is any topological space, then we make the following definition.

Definition 3.3.4. Let Y be a topological space. A class $\alpha \in H^k(Y, \mathbb{R})$ is hyperbolic if $f^*\alpha$ is hyperbolic for every continuous map $X \xrightarrow{f} Y$ of a finite complex X to Y.

As in the case of bounded classes, all hyperbolic classes are trivial if $\pi_1(X)$ is amenable. For 2-dimensional classes this fact was proved by Kędra (cf. [Kęd]) and in full generality it is due to Brunnbauer and Kotschick, whose proof uses certain isoperimetric inequalities (cf. [BrK]). One can however give a more direct proof that follows Gromov's original argument in the bounded case.

Theorem 3.3.5. *Let X be a finite simplicial complex with amenable fundamental group. Then all hyperbolic classes are trivial.*

Proof. We let $\tilde{X} \xrightarrow{p} X$ denote the universal cover and let $G = \pi_1(X)$. Then G acts on $\tilde{X}$ by (simplicial) deck transformations that we denote by T_g. Now assume that $\alpha \in H^k(X, \mathbb{R})$ is a hyperbolic class. By Proposition 3.3.2 this means that for any simplicial representative

a of α the cochain $p^*a \in \hat{C}^*_\Delta(\tilde{X}, \mathbb{R})$ is exact. We let $b \in \hat{C}^{k-1}_\Delta(\tilde{X}, \mathbb{R})$ be a primitive. Then since G is amenable there is an averaging operator on bounded *singular* chains

$$C^k_b(\tilde{X}, \mathbb{R}) \xrightarrow{A} C^k_G(\tilde{X}, \mathbb{R})$$

that maps an arbitrary bounded cochain to a G-equivariant one. This map is defined as follows: let $\mu : L^\infty(G) \to \mathbb{R}$ be a left-invariant mean, which exists since G is amenable. Let $c \in C^k_b(\tilde{X}, \mathbb{R})$ and let σ be any k-simplex, we define a function $\phi_{c,\sigma} : G \to \mathbb{R}$ by

$$\phi_{c,\sigma}(g) = c((T_{g^{-1}})_* \sigma).$$

We then set

$$A(c)(\sigma) = \mu(\phi_{c,\sigma}).$$

Since μ was left invariant $A(c)$ is a G-equivariant cochain on $\tilde{X}$, that is $A(c) = p^*c'$ for a unique cochain in $C^k(X, \mathbb{R})$. One also checks that A is a chain map. Finally, as the deck transformations are simplicial A induces a well-defined map on bounded simplicial cochains. If we let $b' \in C_\Delta(X, \mathbb{R})$ be such that $A(b) = p^*b'$ we compute

$$p^*\delta b' = \delta p^* b' = \delta A(b) = A(p^*a) = p^*a.$$

Thus $\delta b' = a$ since p^* is injective and the class $\alpha \in H^k(X, \mathbb{R})$ is trivial. $\square$

In order to show that the MMM-classes are hyperbolic, we shall need two technical lemmata, the first of which is in essence Theorem 2.1 in [Kędra]. Kędra considered only the universal cover of a manifold, however our assumption that $p^*\alpha$ is exact in bounded cohomology ensures that his proof goes through.

Lemma 3.3.6. *Let $\bar{X} \xrightarrow{p} X$ be a covering of simplicial complexes, with X finite. Let $\alpha \in H^k_b(X, \mathbb{R})$ be a bounded cohomology class such that $p^*\alpha$ is trivial in $H^k_b(\bar{X}, \mathbb{R})$. Then there is a de Rham representative Φ_α of α and a bounded $(k-1)$-form Φ_β with*

$$d\Phi_\beta = p^*\Phi_\alpha.$$

Proof. We first lift the simplices of X to $\bar{X}$ and let $\bar{g}$ be the lifted metric. Now let β be a bounded singular $(k-1)$-cochain on $\bar{X}$ so that $\delta\beta = p^*\alpha$. By restricting to the simplicial cochain complex, we obtain a simplicial cochain

$$\beta_s = \sum \lambda_{\bar{\sigma}} \bar{\sigma},$$

where the $\lambda_{\bar{\sigma}}$ are bounded and $\delta\beta_s = p^*\alpha_s$ as simplicial cochains. Applying the inverse of the de Rham isomorphism to α_s, β_s we obtain forms Φ_α, Φ_β such that $d\Phi_\beta = p^*\Phi_\alpha$ and by Proposition 3.3.2 the form Φ_β is bounded. $\square$

The next lemma gives sufficient conditions under which integration along the fibre maps bounded forms to bounded forms.

Lemma 3.3.7. *Let $F \to E \xrightarrow{\pi} B$ be a smooth fibre bundle over a manifold B, whose fibre is a closed manifold of dimension m. Let g_B be a metric on B, g_E a submersion metric on E and let Ω_v denote the fibrewise volume form induced by g_E. Then the map*

$$\pi_! : \Omega^{k+m}(E) \to \Omega^k(B)$$

maps bounded forms to bounded forms provided that $\pi_!\Omega_v$ is bounded.

Proof. Let ϕ be a bounded $(k+m)$-form with respect to the metric g_E. Let U be an open neighbourhood of $p \in B$ and choose a trivialisation $V = \pi^{-1}(U) \cong U \times F$. We let $\{e_1, ..., e_n\}$ be a local orthonormal frame on U with respect to g_B and $\{e^1, .., e^n\}$ its dual. On V we may decompose ϕ as

$$\phi = \sum_I (f_I \Omega_v) \wedge (\pi^* e^I) + \psi,$$

where I is a multi-index of length k and ψ vanishes on k-tuples of vertical vectors. Choose a local orthonormal frame $\{f_1, ... f_k\}$ about a point (p, x) in V and let $\{\bar{e}_1, ... \bar{e}_n\}$ be a lift of this frame to E that is guaranteed by the assumption that g_E is a submersion metric. Then $\{f_1, ... f_k, \bar{e}_1, ... \bar{e}_n\}$ is an orthonormal frame and, hence, as ϕ is bounded

$$|\phi(f_1, ... f_k, \bar{e}_1, ... \bar{e}_n)(p, x)| = |f_I(p, x)| < C.$$

Thus we conclude locally

$$\pi_!\phi(p) = \sum_I \int_{F_p} (f_I \Omega_v) e^I$$

and

$$|\int_{F_p} f_I \Omega_v| \leq C |\pi_! \Omega_v(p)|$$

so $\pi_!\phi$ is bounded if $\pi_!\Omega_v$ is. □

With the aid of these results we will prove the hyperbolicity of the MMM-classes.

Theorem 3.3.8. *The MMM-classes e_k are hyperbolic.*

Proof. We let e denote the vertical Euler class and $X \xrightarrow{f} B\Gamma_h$ be the classifying map of a bundle E over a finite simplicial complex X. We first assume that $X = B$ is a smooth, compact manifold (possibly with boundary). We let $\tilde{E}$ be the pullback bundle over the universal cover of B:

$$\begin{tikzcd}
\tilde{E} \arrow[r, "p"] \arrow[d, "\pi"] & E \arrow[r] \arrow[d, "\pi"] & E\Gamma_h \arrow[d] \\
\tilde{B} \arrow[r, "p"] & B \arrow[r, "f"] & B\Gamma_h .
\end{tikzcd}$$

We further let $\Sigma_h \xrightarrow{\iota} \tilde{E}$ be the inclusion of a fibre. Morita has shown that $\iota^* e^2$ is trivial in *bounded cohomology* (cf. [Mor4], Section 6). Thus the same holds for $\iota^* e^{k+1}$. Moreover, since $\pi_1(\tilde{E}) \cong \pi_1(\Sigma_g)$ this inclusion induces an isomorphism on bounded cohomology. We may thus choose a bounded chain $b_k \in C_b^{2k-1}(\tilde{E}, \mathbb{R})$ with $p^* e^{k+1} = \delta b_k$.

Then by Lemma 3.3.6 there is a form Φ_k on $\tilde{E}$ which is bounded with respect to the pullback metric and a form Ψ_{k+1} on E that is a representative of e^{k+1} so that $p^*\Psi_{k+1} = d\Phi_k$. Since integration along the fibre is natural and commutes with the exterior derivative, we compute

$$\begin{aligned} p^*e_k &= p^*\pi_!\Psi_{k+1} = \pi_!p^*\Psi_{k+1} \\ &= \pi_!d\Phi_k = d(\pi_!\Phi_k) \end{aligned}$$

We finally need to check that $\pi_!\Phi_k$ is bounded with respect to the pullback metric on $\tilde{B}$. Let g_B be any metric on the base and let g_E be a submersion metric on E. The pullback metric $\tilde{g} = p^*g_E$ is a submersion metric for $g_{\tilde{B}} = p^*g_B$ and the vertical volume form $\tilde{\Omega}_v$ on $\tilde{E}$ is the pullback of the vertical volume form Ω_v on E and thus

$$\pi_!\tilde{\Omega}_v = \pi_!p^*\Omega_v = p^*\pi_!\Omega_v$$

is a pullback of a form on B and is hence bounded. Now Φ_k is bounded with respect to the metric $\tilde{g}$ and thus by Lemma 3.3.7 it follows that $\pi_!\Phi_k$ is bounded and $e_k \in H^{2k}(B)$ is hyperbolic.

In the general case let X be any arbitrary simplicial complex. We may embed X in $\mathbb{R}^N$ for some sufficiently large N. We then let $B = \nu(X)$ be a (compact) regular neighbourhood of X in $\mathbb{R}^N$. Since $\nu(X)$ deformation retracts onto X there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & \nu(X) \arrow[d, "\bar{f}"] \\ & B\Gamma_h. \end{tikzcd}$$

Then by the argument above, $\bar{f}^*e_k$ is hyperbolic and by naturality so is f^*e_k. $\square$

3.4 Holomorphic surface bundles

An interesting special class of surface bundles are those that carry a complex structure. If the base is two dimensional and the genus of fibre and base are greater than 2, then it has been shown by Kotschick that one may assume that there is a complex structure on the base so that the projection is holomorphic (see [Kot2]). In general a complex structure on the total space need not imply the existence of a holomorphic projection to the base. However, if one assumes that π is holomorphic, then there are restrictions on the topology of such bundles. In particular, we have the following result, the first part of which was sketched in an unpublished note of Reznikov (cf. [Rez]). For this we note that if the projection is holomorphic, the fibres are Riemann surfaces and the map $\Phi : B \to \mathcal{M}_g$ that sends a point b to the conformal class of the fibre $\pi^{-1}(b)$ in the moduli space of genus g Riemann surfaces is a holomorphic map.

Theorem 3.4.1. *Let $\Sigma_g \to E \xrightarrow{\pi} B$ be a holomorphic bundle over a complex manifold with holomorphic projection map and $g \geq 3$. If the induced map $B \xrightarrow{\Phi} \mathcal{M}_g$ has maximal rank* *generically, then the simplicial volume* $||B||$ *is positive. If, in addition, the base is Kähler, then either the first MMM-class* e_1 *does not vanish, or there is a fibrewise holonomy-invariant complex structure on the fibres.*

Proof. Since the compactification of the moduli space is an algebraic variety, there is an embedding $\mathcal{M}_g \subset \mathbb{C}P^N$ for some large N. The Fubini-Study form then defines a cohomology class ω on the moduli space. Since the rational cohomologies of $\mathcal{M}_g$ and Γ_g are isomorphic and $H^2(\Gamma_g, \mathbb{Q}) = \mathbb{Q}$ for $g \geq 3$ (cf. [Iva2]), we conclude that the class ω is a multiple of the first MMM-class e_1. Since the map Φ is holomorphic, the Proper Mapping Theorem (cf. [GrH]) implies that $\Phi(B)$ is a subvariety of $\mathbb{C}P^N$. If k denotes the complex dimension of $\Phi(B)$, then ω^k is generically positive on the image of Φ and, hence, ω^k is non-trivial on $\Phi(B)$.

If Φ is generically of maximal rank, then $n = dim(B) = dim(\Phi(B))$ and ω^n pulls back to a generically positive volume form on B. Since ω was a multiple of e_1, the class $\Phi^* e_1^n = e_1^n(E)$ is then non-zero and as this class is bounded we conclude that

$$||e_1||_\infty^n \, ||B|| \geq |e_1^n(E)| > 0,$$

proving the first part of the lemma.

In general, either $\Phi(B)$ is of positive dimension or Φ is constant and there is a fibrewise holonomy-invariant complex structure on the fibres. We assume that the dimension of $\Phi(B)$ is strictly positive. We saw above that this implies that ω^k is non-trivial on $\Phi(B)$. Thus, the Poincaré dual of $\Phi^*\omega^k$ is given by the preimage of a regular point of $\Phi(B)$, which is then an analytic subvariety of B. Since B is Kähler, the Poincaré dual of any subvariety is non-trivial and, hence, $\Phi^*\omega^k$ is non-trivial. Since ω was a multiple of e_1, this implies in particular that $\Phi^* e_1 = e_1(E)$ is non-trivial and the second part of the lemma is proven. □

This leads to the following corollary.

Corollary 3.4.2. *Let* $\Sigma_g \to E \xrightarrow{\pi} B$ *be a holomorphic surface bundle over a compact Kähler manifold with holomorphic projection and let* $\rho : \pi_1(B) \to \Gamma_g$ *denote the associated holonomy map. If* $\pi_1(B)$ *is amenable, then the holonomy group* $\rho(\pi_1(B))$ *is finite, so that* E *has a finite cover that is a product.*

Proof. By Theorem 3.4.1 there is either a holonomy-invariant complex structure on the fibres or the first MMM-class is non-trivial, which would contradict Theorem 3.2.3. Since the automorphism group of a Riemann surface is finite, the result follows. □

Surface bundles with holomorphic structures have also been studied by Harris in [Har], where he considered non-degenerate families of curves. This means that the induced map to the moduli space of Riemann surfaces considered above is assumed to be *finite*. Under this assumption he was able to show that the vertical line bundle is ample and, hence, that the total space is automatically algebraic ([Har], Th. 1). Moreover, any non-trivial monomial in the MMM-classes whose cohomological degree is smaller than the dimension of the base does not vanish ([Har], Cor. 5). There are certain cases of holomorphic bundles where one can ensure that the natural map to the moduli space must be finite, which implies the non-existence of certain types of holomorphic structures.

Proposition 3.4.3. *Let $\Sigma_g \to E \xrightarrow{\pi} B$ be a surface bundle and suppose that $B = \Sigma_1 \times \Sigma_2$ decomposes as a direct product of Riemann surfaces. Then if e_1^2 does not vanish, there is no complex structure on the total space making the projection holomorphic.*

Proof. Since by assumption some MMM-class in top degree is non-trivial, the map $\Phi : B \to \mathcal{M}_g$ must have generically maximal rank. Suppose that the pre-image of some point, say $\Phi^{-1}(p_0)$, has dimension greater than 0. Thus, some irreducible component V of this pre-image must have complex dimension 1.

We let $S_1 = \Sigma_1 \times pt$ and $S_2 = pt \times \Sigma_2$ and we denote by E_1 resp. E_2 the restriction of E to these two embedded curves. Since the first MMM-class vanishes on tori by Theorem 3.2.3 we conclude that

$$e_1(E) = e_1(E_2)[S_1] + e_1(E_1)[S_2].$$

Furthermore, since $e_1^2(E) \neq 0$ and the bundles E_1 and E_2 are holomorphic, both coefficients are in fact positive. We note further that both $[S_1].[V]$ and $[S_2].[V]$ are non-negative and at least one of these numbers is strictly positive. Using the naturality of e_1 we compute that

$$e_1(E|_V) = e_1(E_2)[S_1].[V] + e_1(E_1)[S_2].[V] > 0.$$

Hence, the bundle $E|_V$ has non-trivial first MMM-class. We let $\tilde{V} \to V$ be a desingularisation of V and note that the pullback of E to $\tilde{V}$ also has non-trivial first MMM-class. But this contradicts the fact that the induced map to the moduli space is constant on V and, hence, on $\tilde{V}$. Thus, the map Φ is finite and by the results of Harris e_2 is non-trivial, which contradicts Theorem 3.2.5. $\square$

Although we have only proven Proposition 3.4.3 for the product of two Riemann surfaces, the same proof works for an arbitrary product of Riemann surfaces so that any bundle over a product of Riemann surfaces with $e_1^2 \neq 0$ admits no complex structure making the projection holomorphic.

Chapter 4

Flat surface bundles

In this chapter we shall consider characteristic classes of flat bundles that have closed leaves. In particular, we show that there exist flat bundles with closed leaves that have non-trivial self-intersection numbers. Moreover, given any bundle with a section, we prove that this section can be made a leaf of a foliation after stabilisation under a certain divisibility assumption (Theorem 4.1.7). We also show that there exist flat bundles with symplectic holonomy that have arbitrarily many leaves with prescribed self-intersection numbers.

The holonomy group of a flat bundle with a closed leaf lies in $Diff^+(\Sigma_{h,k})$ and the main result of this chapter is that the abelianisation of this group is $\mathbb{R}^+ \times \mathbb{Z}_2$, if $h \geq 3$ and k is at least 2. We also compute the abelianisation of the group of compactly supported diffeomorphisms on $\mathbb{R}^2$ fixing the origin, which is a special case of a result originally proved by Fukui in all dimensions. Our argument, which uses Sternberg Linearisation, is independent of that given in [Fu], where the argument given seems to be incomplete in the two dimensional case.

4.1 Closed leaves and horizontal foliations

Interesting examples of foliations on 4-manifolds come from considering the horizontal foliations of flat surface bundles. In this section we will focus on the closed leaves of such foliations. Let us first recall the definition of a flat bundle.

Definition 4.1.1. A surface bundle E is *flat* if and only if its holonomy map ρ admits a lift to $Diff^+(\Sigma_h)$

$$\begin{tikzcd} & Diff^+(\Sigma_h) \arrow[d] \\ \pi_1(B) \arrow[ur, dashed, "\bar{\rho}"] \arrow[r, "\rho"] & \Gamma_h \end{tikzcd}$$

If B is a manifold, then this is equivalent to the existence of a foliation that is complementary to the fibres. Such a foliation wil be called a *horizontal* foliation.

Proposition 4.1.2. *Let $\Sigma_h \to E \to B$ be a surface bundle over a manifold B. Then E is flat if and only if it admits a foliation that is complementary to the fibres.*

Proof. First suppose that E is flat, with holonomy $\bar{\rho} : \pi_1(B) \to Diff^+(\Sigma_h)$. Let $\tilde{B} \xrightarrow{p} B$ be the universal cover of B. Then the pullback bundle $p^*E = \Sigma_h \times \tilde{B}$ is trivial. Moreover, E is isomorphic to the quotient of $\Sigma_h \times \tilde{B}$ by the diagonal action of $\pi_1(B)$, where the action on the first factor is given by deck transformations and on the second by $\bar{\rho}$. The horizontal foliation on $\Sigma_h \times \tilde{B}$ descends to a horizontal foliation on the quotient so that E admits a foliation that is complementary to the fibres.

Conversely, let $\mathcal{F}$ be a horizontal foliation. We choose a base point $p \in B$ and consider a loop $\gamma(t)$ in B based at p. Let x be a point in the fibre F_p over p. There is a unique lift $\tilde{\gamma}_x(t)$ of γ that is tangent to $\mathcal{F}$ and such that $\tilde{\gamma}_x(0) = x$. We define a diffeomorphism $\rho(\gamma) : F_p \to F_p$ by the formula

$$\rho(\gamma)(x) = \tilde{\gamma}_x(1).$$

Since $\mathcal{F}$ is a foliation this map only depends on the homotopy class of γ and the choice of base point p. Thus we obtain a map $\bar{\rho} : \pi_1(B) \to Diff^+(\Sigma_h)$. Moreover, $\bar{\rho}$ is a homomorphism and is a lift of the (topological) holonomy map ρ. We conclude that E is a flat bundle. $\square$

For a flat bundle E over a manifold a closed leaf of the horizontal foliation yields k marked points in each fibre. The existence of such a foliation is thus equivalent to a lift of the holonomy map ρ to the group of diffeomorphisms fixing k marked points $Diff^+(\Sigma_{h,k})$

$$\begin{tikzcd} & Diff^+(\Sigma_{h,k}) \arrow[d] \\ \pi_1(B) \arrow[ur, dashed, "\bar{\rho}"] \arrow[r, "\rho"] & \Gamma_h. \end{tikzcd}$$

By taking pullbacks under a suitable finite cover of the base, one obtains a horizontal foliation with a leaf S that intersects each fibre exactly once, in which case the holonomy of E lies in $Diff^+(\Sigma_{h,1})$. Moreover, the horizontal foliation induces a flat structure on the normal bundle ν_S of S, which is given by composing $\bar{\rho}$ with the derivative map at p:

$$Diff^+(\Sigma_{h,1}) \xrightarrow{D_p} GL^+(T_p\Sigma_g) = GL^+(2, \mathbb{R}).$$

In [Mil], Milnor constructed flat bundles with non-trivial Euler class over oriented surfaces and, hence, the image of the Euler class in $H^2(GL^+_\delta(2, \mathbb{R}))$ is non-trivial. Moreover, the self-intersection of S in E is given by $D_p^* e$. In view of this, to show that there are flat bundles with horizontal leaves of non-zero self-intersection it will suffice to show that the map D_p induces an injection $H^2(GL^+_\delta(2, \mathbb{R})) \to H^2(Diff^+_\delta(\Sigma_{h,1}))$.

To this end, we let $\mathcal{G}_p$ be the (discrete) group of smooth diffeomorphism germs that fix the marked point p. We then define

$$\begin{tikzcd} Diff^+(\Sigma_{h,1}) \arrow[r, "\pi"] & \mathcal{G}_p \arrow[r, "\bar{D}_p"] & GL^+(T_p\Sigma_g) \arrow[l, bend left, "s"] \end{tikzcd},$$

where π is the map taking a diffeomorphism fixing p to its germ at p and $\bar{D}_p$ maps a germ to its linear part. The kernel of the map π consists of diffeomorphisms with support disjoint from the marked point p and will be denoted by $Diff^c(\Sigma_{h,1})$. The final map has an obvious section given by considering a linear map as an element of $\mathcal{G}_p$. Thus, to show that the map $H^2(GL^+_\delta(2,\mathbb{R})) \to H^2(Diff^+_\delta(\Sigma_{h,1}))$ is injective, it will be sufficient to show that $H^2(\mathcal{G}_p) \to H^2(Diff^+_\delta(\Sigma_{h,1}))$ is injective. We first prove the following lemma.

Lemma 4.1.3. *The following sequence of groups is exact*

$$1 \to Diff^c(\Sigma_{h,1}) \to Diff^+(\Sigma_{h,1}) \to \mathcal{G}_p \to 1.$$

Proof. The only point to be checked here is that the final map is surjective. Let $\phi \in \mathcal{G}_p$ be a germ and assume that ϕ is defined on a convex neighbourhood U of p. We define

$$\Phi_t(x) = \begin{cases} \frac{1}{t}\phi(tx), & 0 < t \leq 1 \\ D_p\phi(x), & t = 0. \end{cases}$$

Since $GL^+(2,\mathbb{R})$ is connected we may take some path A_t between $D_p\phi$ and Id. The composition of Φ_t and A_t, which we denote by Ψ_t, then gives a path of diffeomorphisms from ϕ to the identity, and without loss of generality we may take this path to be smooth. The path Ψ_t then corresponds to a local flow about p which is generated by some time-dependent vector field X_t. Since $X_t(p) = 0$ for all times $0 \leq t \leq 1$, this local flow is defined on a neighbourhood $W \subset U$ of p at time $t = 1$. We choose a cutoff function $\beta : W \to [0,1]$, with $\beta \equiv 1$ on some neighbourhood of p and $\mathrm{Supp}(\beta) \subset W$ and set $\widetilde{X}_t = \beta\, X_t$. This vector field can be extended as zero on the complement of W in Σ_h and we let $\widetilde{\Psi}_t$ be the flow generated by this vector field. Then by construction $\widetilde{\Psi}_1 = \phi$ as germs. Hence, the right most map in the sequence above is indeed surjective. □

We may now prove the existence of horizontal foliations that have compact leaves with non-trivial self-intersection numbers.

Proposition 4.1.4. *If $h \geq 3$, then there exist flat surface bundles $\Sigma_h \to E \to \Sigma_g$ with horizontal foliations that have leaves of non-zero self-intersection.*

Proof. We consider the last three terms of the five-term exact sequence in cohomology associated to the extension of Lemma 4.1.3:

$$H^1(Diff^c_\delta(\Sigma_{h,1}))^{\mathcal{G}_p} \to H^2(\mathcal{G}_p) \xrightarrow{\pi^*} H^2(Diff^+_\delta(\Sigma_{h,1})).$$

From our discussion above it is sufficient to show that the map π^* is injective or by exactness that $H^1(Diff^c_\delta(\Sigma_{h,1}))^{\mathcal{G}_p} = 0$. We claim that $H_1(Diff^c_\delta(\Sigma_{h,1}))$ is in fact trivial and by the Universal Coefficient Theorem the same holds in cohomology.

Let $\Sigma^\epsilon_h = \Sigma_h \setminus D_\epsilon$ denote Σ_h with a disc of radius ϵ removed. We note that $Diff^c(\Sigma_{h,1})$ is isomorphic to the direct limit of the groups $Diff^c(\Sigma^\epsilon_h)$. By the stability result of Harer $H_1(\Gamma^1_h) = H_1(\Gamma_h)$ for $h \geq 3$ (cf. [Iva1]). Moreover, by [Pow] Γ_h is perfect for $h \geq 3$. Finally, by the classical result of Thurston the identity component of $Diff^c(\Sigma^\epsilon_h)$ is also perfect (see [Th1]). The five-term sequence in homology then implies that $H_1(Diff^c_\delta(\Sigma^\epsilon_h)) = 0$. Hence, each of the groups $H_1(Diff^c_\delta(\Sigma^\epsilon_h))$ is trivial and we conclude that $H_1(Diff^c_\delta(\Sigma_{h,1}))$ also vanishes. □

One may interpret the proof of Proposition 4.1.4 in a more geometric fashion, which gives a sharper result, and we note this in the following proposition.

Proposition 4.1.5. *If $h \geq 3$ and $k \in \mathbb{Z}$, then there exist flat surface bundles $\Sigma_h \to E \to \Sigma_g$ with horizontal foliations that have closed leaves of self-intersection k.*

Proof. Let $a_i, b_i \in \pi_1(\Sigma_g)$ denote the standard generators of the fundamental group and let ξ_k be a flat $GL^+(2,\mathbb{R})$-bundle over Σ_g that has Euler class $k \leq g-1$ as provided by [Mil]. This corresponds to a holonomy representation

$$a_i \mapsto A_i$$

$$b_i \mapsto B_i$$

for $A_i, B_i \in GL^+(2,\mathbb{R})$. Then by performing the extension trick of Lemma 4.1.3 we obtain diffeomorphisms ϕ_i, ψ_i which agree with A_i, B_i in a small neighbourhood of p so that the product $\eta = \prod_{i=1}^{g}[\phi_i, \psi_i]$ has support disjoint from p, that is $\eta \in Diff^c(\Sigma_{h,1})$. This group is perfect and, thus, we may write $\eta^{-1} = \prod_{i=1}^{g'}[\alpha_i, \beta_i]$ where $\alpha_i, \beta_i \in Diff^c(\Sigma_{h,1})$. We define a flat bundle over $\Sigma_{g+g'}$ by the holonomy representation

$$a_i \mapsto \phi_i, \quad b_i \mapsto \psi_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto \alpha_j, \quad b_{g+j} \mapsto \beta_j \text{ for } 1 \leq j \leq g',$$

which we denote by ρ. This bundle then has a compact leaf S corresponding to the marked point p. The Euler class of the normal bundle to S is computed from the induced holonomy representation $D_p\rho$ on ν_S:

$$a_i \mapsto D_p(\phi_i) = A_i, \quad b_i \mapsto D_p(\psi_i) = B_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto D_p(\alpha_j) = Id, \quad b_{g+j} \mapsto D_p(\beta_j) = Id \text{ for } 1 \leq j \leq g'.$$

In view of formula (2.4) in Section 2.4, it is clear that $e(\nu_S) = k$ and thus $[S]^2 = k$ □

With the geometric construction of Proposition 4.1.5 we are now able to say when a section S of a bundle E can become a leaf of a foliation after *stabilisation.*

Definition 4.1.6. A surface bundle E' over a surface is called a *stabilisation* of a bundle E, if it is the fibre sum of E with a trivial bundle $\Sigma_h \times \Sigma_{g'}$. This is then a bundle over the connected sum $\Sigma_g \# \Sigma_{g'} = \Sigma_{g+g'}$ that is trivial over the second factor.

We will show that under certain conditions any bundle E with a section S of self-intersection k can be stabilised to a bundle E' that admits a horizontal foliation with a closed leaf S' that agrees with S on $E'|_{\Sigma_g}$.

If the bundle E is trivial then after stabilisation it remains trivial. For a trivial bundle the vertical Euler class $e(E)$ is divisible by $2h-2$ and, hence, the same is true for the self-intersection of S and its stabilisation S'. Thus, the condition that $e(E)$ is divisible by $2h-2$ is, in general, necessary for the existence of a stabilisation of the desired form. It is, however, also sufficient and this is the content of the following theorem.

Theorem 4.1.7. *Let $\Sigma_h \to E \to \Sigma_g$ be a surface bundle that has a section of self-intersection k, where k is divisible by $2h-2$. Then after stabilisation E admits a flat structure whose horizontal foliation has a closed leaf of self-intersection k.*

Proof. We first stabilise E until the Milnor-Wood equality is satisfied for S. We let $\bar{g} = g + g'$ denote the genus of the base of the stabilisation and let $\bar{\rho} : \pi_1(\Sigma_{\bar{g}}) \to \Gamma_{h,1}$ be its holonomy representation. Since the Milnor-Wood inequality is satisfied for S, it has a tubular neighbourhood ν_S that is diffeomorphic to a flat $GL^+(2,\mathbb{R})$-bundle. We let ξ denote the corresponding horizontal foliation on ν_S. We extend ξ to a horizontal distribution ξ' that agrees with ξ on a (possibly smaller) neighbourhood of S. We choose curves a_i, b_i representing the standard generators of $\pi_1(\Sigma_{\bar{g}})$ and let $\phi_i, \psi_i \in Diff^+(\Sigma_h)$ be the holonomy maps induced by ξ', so that $[\phi_i] = \bar{\rho}(a_i)$ and $[\psi_i] = \bar{\rho}(b_i)$ in Γ_h. Note that these diffeomorphisms depend on the choice of curves and not just their homotopy classes.

By construction the distribution ξ' is a foliation in a neighbourhood of S. Hence the product of commutators $\eta = \prod_{i=1}^{\bar{g}} [\phi_i, \psi_i]$ has compact support disjoint from the marked point corresponding to the section S, and is thus an element in $Diff^c(\Sigma_h^1)$. We next consider the following diagram that relates the mapping class groups $\Gamma_h^1, \Gamma_{h,1}$ and Γ_h

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
 & \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[d] & & \\
1 \arrow[r] & \pi_1(T_1\Sigma_h) \arrow[r] \arrow[d] & \Gamma_h^1 \arrow[r] \arrow[d] & \Gamma_h \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_h) \arrow[r] \arrow[d] & \Gamma_{h,1} \arrow[r] \arrow[d] & \Gamma_h \arrow[r] & 1 \\
 & 1 & 1 & & .
\end{tikzcd}$$

Here $\mathbb{Z}$ is generated by a positive Dehn twist Δ_∂ along an embedded curve parallel to the boundary of Σ_h^1 and $T_1\Sigma_h$ denotes the unit tangent bundle of Σ_h.

Now the image of η in $\Gamma_{h,1}$ is trivial. Thus $\eta = \Delta_\partial^k$ where k is the self-intersection of S. This is because the second column is the central extension corresponding to the vertical Euler class as a characteristic class in the group cohomology of $\Gamma_{h,1}$. By assumption k is divisible by $2h-2$ and hence $\eta \in H_1(\pi_1(T_1\Sigma_h)) = H_1(T_1\Sigma_h)$ is trivial. Again this is because the left most column is the central extension corresponding to the Euler class of the unit tangent bundle over Σ_h and $[\eta]$ is a multiple of the fibre class of this S^1-bundle that is divisible by $2h-2$. Hence $\eta^{-1} = \prod_{j=1}^N [\alpha_j, \beta_j]$ is a product of commutators in Γ_h^1 each of which lie in the kernel of the natural map to Γ_h.

We let $\phi_j, \psi_j \in Diff^+(\Sigma_h^1)$ be representatives of the mapping classes α_j, β_j respectively, and consider the product $\gamma = \eta. \prod_{j=1}^N [\phi_j, \psi_j]$ in $Diff_0^c(\Sigma_h^1)$. Since this group is perfect we may write $\gamma^{-1} = \prod_{l=1}^M [\gamma_l, \delta_l]$. Letting a_i, b_i be standard the generators for $\pi_1(\Sigma_{\bar{g}+N+M})$, we define a flat bundle via the holonomy map

$$a_i \mapsto \phi_i, \quad b_i \mapsto \psi_i \text{ for } 1 \leq i \leq \bar{g}$$

$$a_{\bar{g}+j} \mapsto \alpha_j, \quad b_{\bar{g}+j} \mapsto \beta_j \text{ for } 1 \leq j \leq N$$

$$a_{\bar{g}+N+l} \mapsto \gamma_l, \quad b_{\bar{g}+N+l} \mapsto \delta_l \text{ for } 1 \leq l \leq M.$$

This gives a horizontal foliation with a closed leaf of self-intersection k on a stabilisation of E. That the bundle is a stabilisation of the original bundle follows since the mapping classes represented by $\alpha_j, \beta_j, \gamma_l, , \delta_l$ are trivial in Γ_h. □

In [BCS] Bestvina, Church and Souto show the non-existence of certain lifts of bundles with sections to the diffeomorphism group with marked points, using the bounds on the Euler class given by the Milnor-Wood inequality. In particular, the diagonal section in the product of two genus g surfaces provides such an example. However, by Theorem 4.1.7, these examples do possess lifts after stabilisation.

4.1.1 Computation of $H_1(Diff^+_\delta(\Sigma_{h,k}))$

Our discussion above will enable us to calculate the first group homology of $Diff^+(\Sigma_{h,1})$ and, in particular, we will show that this group is not perfect. In fact it is clear that the group $Diff^+(\Sigma_{h,1})$ is not perfect, as there is a surjection to $GL^+(2,\mathbb{R})$ given by the derivative map and

$$H_1(GL^+_\delta(2,\mathbb{R})) = H_1((SL(2,\mathbb{R}) \times \mathbb{R}^+)_\delta) = \mathbb{R}^+,$$

since $SL(2,\mathbb{R})$ is a perfect group. But this is the only contribution to $H_1(Diff^+_\delta(\Sigma_{h,1}))$ if $h \geq 3$. The proof of this fact is based on the following result due to Sternberg.

Theorem 4.1.8 (Sternberg's Linearisation Theorem, [Ster])**.** *Let ϕ be a smooth diffeomorphism defined in a neighbourhood U of the origin in $\mathbb{R}^n$ and let $\phi(0) = 0$. Further, let $s_1, ...s_n \in \mathbb{C}$ denote the eigenvalues (counted with multiplicities) of the Jacobian $D_0(\phi)$ at the origin and assume that*

$$s_i \neq s_1^{m_1} ... s_n^{m_n},$$

for all non-negative integers $m_1, ..., m_n$ with $\sum m_i > 1$. Then there is a change of coordinates ψ that fixes the origin so that on a possibly smaller neighbourhood $W \subset U$ the following holds

$$\psi\phi\psi^{-1} = D_0(\phi).$$

Remark 4.1.9. We note that the hypotheses of Theorem 4.1.8 hold, in particular, if $D_0(\phi) = \lambda\, Id$ for $\lambda \neq 0, 1$. Sternberg's Theorem may also be interpreted in terms of germs of diffeomorphisms, i.e. if the hypotheses of the theorem are satisfied for a germ $\phi \in \mathcal{G}_p$, then ϕ is conjugate to the germ represented by $D_p(\phi)$.

Proposition 4.1.10. *Let Σ_h be a surface of genus $h \geq 3$. Then $H_1(Diff^+_\delta(\Sigma_{h,1})) = \mathbb{R}^+$.*

Proof. We consider the extension given in Lemma 4.1.3

$$1 \to Diff^c(\Sigma_{h,1}) \to Diff^+(\Sigma_{h,1}) \xrightarrow{\pi} \mathcal{G}_p \to 1.$$

Since the group $Diff^c(\Sigma_{h,1})$ is perfect (cf. Proposition 4.1.4), the associated five-term exact sequence yields $H_1(Diff^+_\delta(\Sigma_{h,1})) = H_1(\mathcal{G}_p)$. Next we consider the exact sequence

$$1 \to \mathcal{G}_{p,Id} \to \mathcal{G}_p \xrightarrow{D_p} GL^+(2,\mathbb{R}) \to 1,$$

where $\mathcal{G}_{p,Id}$ is the set of germs whose linear part is the identity. By Remark 4.1.9 above, if $\phi \in \mathcal{G}_p$ and $D_p(\phi) = \lambda\, Id$ for some $\lambda > 1$, then there is a $\psi \in \mathcal{G}_p$ so that $\psi\phi\psi^{-1} = \lambda\, Id$. We set $A_\lambda = \lambda\, Id$, then for any $\phi \in \mathcal{G}_{p,Id}$, there is a germ ψ such that

$$A_\lambda = \psi(A_\lambda\phi)\psi^{-1}.$$

Since A_λ is central in $GL^+(2,\mathbb{R})$, we may assume that $\psi \in \mathcal{G}_{p,Id}$ after conjugating the above equation with the element $D_p\psi$. Thus $\phi = \psi^{-1}A_\lambda^{-1}\psi A_\lambda = [\psi^{-1}, A_\lambda^{-1}]$ and we have shown that $H_1(\mathcal{G}_{p,Id})_{GL^+(2,\mathbb{R})} = 0$.

In view of this, the five-term exact sequence gives

$$0 = H_1(G_{p,Id})_{GL^+(2,\mathbb{R})} \to H_1(\mathcal{G}_p) \to H_1(GL^+_\delta(2,\mathbb{R})) \to 0$$

and, hence, $H_1(Diff^+_\delta(\Sigma_{h,1})) = H_1(GL^+_\delta(2,\mathbb{R})) = \mathbb{R}^+$. □

We let $PDiff^+(\Sigma_{h,k})$ denote the group of pure orientation preserving diffeomorphisms, i.e. an element $\phi \in PDiff^+(\Sigma_{h,k})$ is a diffeomorphism of Σ_h, which fixes a set of k marked points *pointwise*. With exactly the same argument as in Proposition 4.1.10 we obtain the following.

Proposition 4.1.11. *Let Σ_h be a surface of genus $h \geq 3$. Then $H_1(PDiff^+_\delta(\Sigma_{h,k})) = (\mathbb{R}^+)^k$.*

Using Proposition 4.1.11 it is now possible to compute the first homology of the full diffeomorphism group $Diff^+(\Sigma_{h,k})$. For this we need the following lemma.

Lemma 4.1.12. *Let G be a finite group. Then all elements in $H_2(G)$ are torsion of order at most $|G|$.*

Proof. Let α be an element in $H_2(G)$ given by a G-representation ρ of the fundamental group of some orientable surface $\pi_1(\Sigma_g) \xrightarrow{\rho} G$. Since G is a finite group, the homomorphism ρ has an associated $|G|$-fold covering map $\Sigma_{\bar{g}} \xrightarrow{p} \Sigma_g$. The composition $f_\rho \circ p$ then represents $|G|\alpha$, and since the induced map on fundamental group is trivial so is $|G|\alpha$. We conclude that $H_2(G)$ consists of torsion elements of order at most $|G|$. □

Proposition 4.1.13. *Let Σ_h be a surface of genus $h \geq 3$ and let $k \geq 2$. Then*

$$H_1(Diff^+_\delta(\Sigma_{h,k})) = \mathbb{R}^+ \times \mathbb{Z}_2.$$

Proof. By considering the action on the marked points induced by $Diff^+(\Sigma_{h,k})$, we obtain the following extension of groups

$$1 \to PDiff^+(\Sigma_{h,k}) \to Diff(\Sigma_{h,k}) \to S_k \to 1.$$

The five-term exact sequence for group homology then gives

$$H_2(Diff^+_\delta(\Sigma_{h,k})) \to H_2(S_k) \xrightarrow{\partial} H_1(PDiff^+_\delta(\Sigma_{h,k}))_{S_k} \to H_1(Diff^+_\delta(\Sigma_{h,k})) \to H_1(S_k) \to 0.$$

By Proposition 4.1.11 we have that $H_1(PDiff^+_\delta(\Sigma_{h,k})) = (\mathbb{R}^+)^k$, which, in particular, implies that $H_1(PDiff^+_\delta(\Sigma_{h,k}))_{S_k} = \mathbb{R}^+$. By Lemma 4.1.12 the group $H_2(S_k)$ consists of torsion, hence as $\mathbb{R}^+$ is torsion free the connecting homomorphism ∂ is trivial and we obtain the following short exact sequence:

$$0 \to \mathbb{R}^+ \to H_1(Diff^+_\delta(\Sigma_{h,k})) \to H_1(S_k) = \mathbb{Z}_2 \to 0.$$

Finally, since in $\mathbb{R}^+$ every element has a square root, this extension has a section and we conclude that

$$H_1(Diff^+_\delta(\Sigma_{h,k})) = \mathbb{R}^+ \times \mathbb{Z}_2.$$

□

4.1.2 Computation of $H_1(Diff^c_\delta(\mathbb{R}^2, 0))$

The proof of Proposition 4.1.10 above will allow us to calculate the first group homology of $Diff^c(\mathbb{R}^2, 0)$, which here denotes the group of diffeomorphisms of the plane that have compact support and fix the origin. This fact was stated in a more general form by Fukui in [Fu], however his argument appears to be incomplete.

Fukui argues as follows (see [Fu], p. 485). Let $\phi \in Diff^c(\mathbb{R}^n, 0)$ have $D_0\phi = Id$, then there is a product of commutators so that $\eta = \phi \prod_{i=1}^{g'} [\alpha_i, \beta_i]$ is the identity on some neighbourhood of 0. He then claims that by Thurston's result on the perfectness of the identity component the group of diffeomorphisms, we may write η as a product of commutators of elements in $Diff^c(\mathbb{R}^n \setminus \{0\})$, which denotes the group of compactly supported diffeomorphisms of $\mathbb{R}^n \setminus 0$. In order to apply the result of Thurston, one must have that η is isotopic to the identity through diffeomorphisms with compact support away from the origin. However, it is not clear that η is isotopic to the identity through diffeomorphisms with support disjoint from the origin. In fact for $n = 2$ the mapping class group of compactly supported diffeomorphisms on $\mathbb{R}^2 \setminus \{0\}$ is isomorphic to $\mathbb{Z}$ (cf. [Iva1], Cor. 2.7 E). As a corollary of the results we have obtained thus far we shall be able to give a complete proof of the theorem stated by Fukui in the case $n = 2$.

Theorem 4.1.14. $H_1(Diff^c_\delta(\mathbb{R}^2, 0)) = \mathbb{R}^+$.

Proof. We have the following exact sequences

$$1 \to Diff^c(\mathbb{R}^2 \setminus \{0\}) \to Diff^c(\mathbb{R}^2, 0) \xrightarrow{\pi} \mathcal{G}_p \to 1$$

and

$$1 \to Diff^c_0(\mathbb{R}^2 \setminus \{0\}) \to Diff^c(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{Z} \to 1.$$

We consider the five-term sequence in cohomology associated to the first exact sequence above

$$0 \to H^1(\mathcal{G}_p) \to H^1(Diff^c_\delta(\mathbb{R}^2, 0)) \to H^1(Diff^c_\delta(\mathbb{R}^2 \backslash \{0\}))^{\mathcal{G}_p} \xrightarrow{\delta} H^2(\mathcal{G}_p) \to H^2(Diff^c_\delta(\mathbb{R}^2, 0)).$$

By Thurston's result $Diff^c_0(\mathbb{R}^2 \setminus \{0\})$ is perfect and applying the five-term exact sequence to the second exact sequence above implies that $H^1(Diff^c_\delta(\mathbb{R}^2 \setminus \{0\})) = \mathbb{Z}$.

Next we consider the sequence of classifying spaces

$$BDiff^c_\delta(\mathbb{R}^2, 0) \longrightarrow B\mathcal{G}_p \longrightarrow BGL^+_\delta(2, \mathbb{R}) \longrightarrow BGL^+(2, \mathbb{R}).$$

The Euler class is a generator of $H^2(BGL^+(2, \mathbb{R}))$ and the pullback to $H^2(\mathcal{G}_p)$ is non-zero and primitive, as one sees by evaluating this class on a flat $GL^+_\delta(2, \mathbb{R})$-bundle with Euler class 1, thought of as an element $H_2(\mathcal{G}_p)$. Moreover, a flat bundle with holonomy in $Diff^c(\mathbb{R}^2, 0)$ is topologically trivial, since it admits a section with vanishing self-intersection number. Hence the pullback of e to $H^2(Diff^c_\delta(\mathbb{R}^2, 0))$ is zero. By exactness of the five-term sequence above $e = \delta(f)$ for some $f \in H^1(Diff^c_\delta(\mathbb{R}^2 \setminus \{0\}))^{\mathcal{G}_p} \subset \mathbb{Z}$. Hence as e is a primitive, non-torsion class, the connecting homomorphism for the five-term exact sequence in *homology* must be surjective. Thus by exactness

$$H_1(Diff^c_\delta(\mathbb{R}^2, 0)) = H_1(\mathcal{G}_p) = \mathbb{R}^+,$$

where the second equality was shown in the proof of Proposition 4.1.10. □

4.2 Closed leaves of flat bundles with symplectic holonomy

One may consider flat bundles with additional structure. In the context of surface bundles with horizontal foliations it is natural to consider bundles whose horizontal foliations are transversally symplectic. This is equivalent to the existence of a fibrewise symplectic form that is holonomy invariant, and such a bundle will be called *symplectically flat*. In this way, a flat bundle $\Sigma_h \to E \to B$ with a transversal symplectic structure is equivalent to a holonomy representation $\pi_1(B) \xrightarrow{\rho} Symp(\Sigma_h, \omega)$, where ω is the symplectic form restricted to a fibre. We shall for the most part suppress any explicit reference to the symplectic form, since by Moser stability any two symplectic forms on Σ_h are equivalent after rescaling.

We shall first investigate the possible compact leaves of the horizontal foliation of a symplectically flat bundle. To this end we denote by $Symp(\Sigma_{h,1})$ the group of symplectomorphisms fixing a marked point $p \in \Sigma_h$. Furthermore, we shall denote by $\mathcal{G}^{Symp}_p$ the (discrete) group of symplectomorphism germs that fix the marked point p.

Let E be a flat bundle that is given by a holonomy homomorphism $\pi_1(\Sigma_g) \xrightarrow{\rho} Symp(\Sigma_{h,1})$. We note that such a flat structure induces a flat structure on the normal bundle of the leaf S corresponding to the marked point, which is given by composing ρ with the map $Symp(\Sigma_{h,1}) \to Sp(T_p\Sigma_h)$ that maps ϕ to its derivative at the marked point $p \in \Sigma_h$. The self-intersection of S can then be calculated directly from the holonomy representation $D_p\rho$

(see formula (2.4), Section 2.4). Moreover, by Darboux's Theorem we may assume that the symplectic form ω is standard in a neighbourhood of the marked point p. Thus we have the following sequence

$$\begin{tikzcd} Symp(\Sigma_{h,1}) \arrow[r, "\pi"] & \mathcal{G}_p^{Symp} \arrow[r, "\bar{D}_p"] & Sp(T_p\Sigma_g) \arrow[l, bend left, "s"] \end{tikzcd} = SL(2,\mathbb{R}),$$

where π is the map taking a symplectomorphism fixing p to its germ at p and $\bar{D}_p$ maps a germ to its linear part. The kernel of the map π will be denoted by $Symp^c(\Sigma_{h,1})$ and consists of symplectomorphisms whose supports are disjoint from p. The final map has an obvious section given by considering an element of $SL(2,\mathbb{R})$ as an element in $\mathcal{G}_p^{Symp}$.

By arguing exactly as in the smooth case, we see that the existence of symplectically flat bundles that have closed horizontal leaves of non-zero self-intersection reduces to showing that the map $H^2(\mathcal{G}_p^{Symp}) \xrightarrow{\pi^*} H^2(Symp^c_\delta(\Sigma_{h,1}))$ is injective. As a first step we have the analogue of Proposition 4.1.3.

Proposition 4.2.1. *The following sequence of groups is exact*

$$1 \to Symp^c(\Sigma_{h,1}) \to Symp(\Sigma_{h,1}) \xrightarrow{\pi} \mathcal{G}_p^{Symp} \to 1.$$

Proof. We only need to show that the final map is surjective. Let $\phi \in \mathcal{G}_p^{Symp}$ be a germ. We assume that ϕ is defined on a convex neighbourhood U of p and define

$$\Phi_t(x) = \begin{cases} \frac{1}{t}\phi(tx), & 0 < t \leq 1 \\ D_p\phi(x), & t = 0, \end{cases}$$

which is a smooth path of symplectomorphisms. Since $SL(2,\mathbb{R})$ is connected there is a path A_t between $D_p\phi$ and Id. The composition of Φ_t and A_t, which we denote by Ψ_t, then gives a path of symplectomorphisms from ϕ to the identity and without loss of generality we may assume that Ψ_t is smooth. This path then corresponds to a local flow about p, which is generated by some time-dependent vector field X_t. Since $X_t(p) = 0$ for all times $0 \leq t \leq 1$, we may assume that X_t is defined on a contractible neighbourhood $W \subset U$ of p for all times $0 \leq t \leq 1$.

By the Cartan formula Ψ_t is a path of symplectomorphisms if and only if $0 = d\iota_{X_t}\omega$, and since W is contractible, there is a smooth family of Hamiltonians H_t for X_t on W. We then choose a cutoff function $\beta : W \to [0,1]$, with $\mathrm{Supp}(\beta) \subset W$ and $\beta \equiv 1$ on some neighbourhood of p and set $\widetilde{H}_t = \beta\, H_t$. This function can be extended as zero on the complement of W in Σ_h and the Hamiltonian flow $\widetilde{\Psi}_t$ generated by $\widetilde{H}_t$ is defined in such a way that $\widetilde{\Psi}_1 = \phi$ as germs. Hence, the right most map in the sequence above is indeed surjective. □

We shall next discuss the abelianisation of $Symp^c_\delta(\Sigma_{h,1})$. In order to be able to compute $H_1(Symp^c_\delta(\Sigma_{h,1}))$ we will need to recall several natural homomorphisms on the group of symplectomorphisms of an arbitrary symplectic manifold. We shall let $\widetilde{Symp^c_0}(M)$ denote the universal cover of the identity component of the group of symplectomorphisms with compact support. This may in turn be identified with homotopy classes of paths $[\psi_t]$ in $Symp^c_0(M)$ based at the identity.

Definition 4.2.2 (Flux homomorphism)**.** Let (M, ω) be a symplectic manifold. We define $Flux : \widetilde{Symp_0^c}(M) \to H_c^1(M, \mathbb{R})$ by

$$Flux([\psi_t]) = \int_0^1 \iota_{\dot{\psi}_t}\omega \in H_c^1(M, \mathbb{R}).$$

This map is a well-defined, surjective homomorphism (cf. [McS1]). In the case where $M = \Sigma_h^k$ is a surface of genus $h \geq 2$ with k boundary components, the identity component of the symplectomorphism group is simply-connected and, hence, $Flux$ descends to a well-defined homomorphism on $Symp_0^c(\Sigma_h^k)$. Furthermore, this Flux homomorphism satisfies the following equivariance property, where $\psi \in Symp_0^c(\Sigma_h^k)$ and $\phi \in Symp^c(\Sigma_h^k)$ is arbitrary (cf. [KM1], Lemma 6):

$$Flux(\phi\psi\phi^{-1}) = (\phi^{-1})^* Flux(\psi). \tag{4.1}$$

We next recall the definition of the Calabi map, which can be defined on any exact symplectic manifold.

Definition 4.2.3 (Calabi homomorphism)**.** Let (M, ω) be an exact symplectic manifold and let $Symp_0^c(M)$ denote the group of symplectomorphisms with compact support on M. We choose a λ so that $\omega = -d\lambda$ and define $Cal : Symp_0^c(M) \to \mathbb{R}$ by

$$Cal(\phi) = -\frac{1}{n+1}\int_M \phi^*\lambda \wedge \lambda \wedge \omega^{n-1}.$$

One can show that this map is surjective and that it is independent of the choice of λ on the kernel of $Flux$ (cf. [McS1]) . For any two elements $\phi, \psi \in Symp_0^c(M)$ the Calabi map satisfies the following two properties:

$$Cal(\phi\psi) = Cal(\phi) + Cal(\psi) + \frac{1}{n+1}\int_M Flux(\phi) \wedge Flux(\psi) \wedge \omega^{n-1} \tag{4.2}$$

and for a commutator of elements

$$\begin{aligned}
Cal([\phi, \psi]) &= Cal(\phi\psi) + Cal(\phi^{-1}\psi^{-1}) + \frac{1}{n+1}\int_M Flux(\phi\psi) \wedge Flux(\phi^{-1}\psi^{-1}) \wedge \omega^{n-1} \\
&= Cal(\phi) + Cal(\psi) + Cal(\phi^{-1}) + Cal(\psi^{-1}) \\
&\quad + \frac{1}{n+1}\int_M Flux(\phi) \wedge Flux(\psi) \wedge \omega^{n-1} \\
&\quad + \frac{1}{n+1}\int_M Flux(\phi^{-1}) \wedge Flux(\psi^{-1}) \wedge \omega^{n-1} \\
&= \frac{2}{n+1}\int_M Flux(\phi) \wedge Flux(\psi) \wedge \omega^{n-1},
\end{aligned} \tag{4.3}$$

where we have used the fact that $Flux$ is a homomorphism and $Cal(\phi^{-1}) = -Cal(\phi)$. The subgroup $Ham^c(M)$ of Hamiltonian diffeomorphisms with compact support corresponds to the kernel of the Flux homomorphism and the Calabi map restricts to homomorphism on this group. We then have the following classical result of Banyaga.

Theorem 4.2.4 (Banyaga, [Ban]). *The kernel of the map $Cal : Ham^c(M) \to \mathbb{R}$ is a simple, and hence perfect, group.*

By combining Theorem 4.2.4 and formula (4.1) we deduce the following lemma.

Lemma 4.2.5. *For $h \geq 2$, the Flux homomorphism induces a Γ_h^k-equivariant isomorphism*

$$H_1(Symp_{0,\delta}^c(\Sigma_h^1)) \to H_c^1(\Sigma_h^k, \mathbb{R}),$$

where Γ_h^k acts on $H_1(Symp_{0,\delta}^c(\Sigma_h^1))$ via conjugation and on $H_c^1(\Sigma_h^k, \mathbb{R})$ via $\phi.\alpha = (\phi^{-1})^*\alpha$.

As a final preliminary we need to understand the group of coinvariants $H_c^1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k}$.

Lemma 4.2.6. *For any $h \geq 1$, we have $H_c^1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k} = 0$.*

Proof. By Poincaré duality the group $H_c^1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k}$ is isomorphic to $H_1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k}$, where the mapping class group acts via $\phi.a = \phi_* a$. Let $\{a_i, b_i, c_j\}$ be a basis of $H_1(\Sigma_h^k)$ such that $\{a_i, b_i\}$ is a standard symplectic basis for the homology of the closed surface obtained by filling the boundary components and the c_j are boundary components. Furthermore, let d_j denote an embedded curve so that a_1, d_j and $c_1, ...c_j$ bound a $(j+2)$-punctured sphere in Σ_h^k. We let $\phi_{a_i}, \phi_{b_i}, \phi_{d_j}$ denote positive Dehn twists about these curves and compute

$$\begin{aligned}
[(\phi_{a_i})_* - Id](b_i) &= a_i \\
[(\phi_{b_i})_* - Id](a_i) &= b_i \\
[(\phi_{d_j})_* - Id](b_1) &= -a_1 - c_1 - ... - c_j,
\end{aligned}$$

whence $H_1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k} = 0$. □

We are now in a position to prove that the entire symplectomorphism group $Symp^c(\Sigma_h^k)$ is perfect for sufficiently large genus.

Lemma 4.2.7. *The group $Symp^c(\Sigma_h^k)$ is perfect for $h \geq 3$.*

Proof. We first consider the inclusion $Symp_0^c(\Sigma_h^k) \hookrightarrow Diff_0^c(\Sigma_h^k)$. By Moser stability this map is a weak homotopy equivalence and, hence, we have the following exact sequence

$$1 \to Symp_0^c(\Sigma_h^k) \to Symp^c(\Sigma_h^k) \to \Gamma_h^k \to 1,$$

where Γ_h^k denotes the mapping class group of diffeomorphisms with compact support in the interior of Σ_h^k. If we consider the associated five-term exact sequence in homology we have

$$H_1(Symp_{0,\delta}^c(\Sigma_h^k))_{\Gamma_h^k} \to H_1(Symp_\delta^c(\Sigma_h^k)) \to H_1(\Gamma_h^k) \to 0.$$

By Lemma 4.2.5 the group $H_1(Symp_{0,\delta}^c(\Sigma_h^k))_{\Gamma_h^k}$ is isomorphic to $H_c^1(\Sigma_h^k, \mathbb{R})_{\Gamma_h^k}$, which vanishes by Lemma 4.2.6. Furthermore, $H_1(\Gamma_h^k) = 0$ for $h \geq 3$ (cf. Proposition 4.1.4) and we conclude that $H_1(Symp_\delta^c(\Sigma_h^1))$ vanishes, that is $Symp^c(\Sigma_h^1)$ is perfect. □

With these facts we will now be able to prove the analogue of Proposition 4.1.4 for bundles with symplectic holonomy.

Proposition 4.2.8. *If $h \geq 3$, then there exist symplectically flat surface bundles with fibre Σ_h, whose horizontal foliations have closed leaves of non-zero self-intersection.*

Proof. It suffices to show that the map $H^2(\mathcal{G}_p^{Symp}) \xrightarrow{\pi^*} H^2(Symp_\delta(\Sigma_{h,1}))$ is injective (cf. Proposition 4.1.4). Now if $\Sigma_h^\epsilon = \Sigma_h \setminus D_\epsilon$ denotes Σ_h with a disc of radius epsilon removed, then $Symp^c(\Sigma_{h,1})$ is isomorphic to the injective limit of the groups $Symp^c(\Sigma_h^\epsilon)$. By Lemma 4.2.7 each of the groups $H_1(Symp_\delta^c(\Sigma_h^\epsilon))$ is trivial and hence $H_1(Symp_\delta^c(\Sigma_{h,1}))$ vanishes.

We then take the five-term exact sequence associated to the exact sequence of Proposition 4.2.1 to obtain

$$0 = H^1(Symp_\delta^c(\Sigma_{h,1}))^{\mathcal{G}_p^{Symp}} \to H^2(\mathcal{G}_p^{Symp}) \xrightarrow{\pi^*} H^2(Symp_\delta(\Sigma_{h,1})),$$

and the result follows by exactness. □

There is also a geometric construction of symplectically flat bundles, whose horizontal foliations have leaves with non-trivial self-intersection numbers. This will be somewhat more general than that given in the smooth case and will be recorded in the following proposition.

Proposition 4.2.9. *For $h \geq 3$, there exist flat bundles $\Sigma_h \to E \to \Sigma_g$ that have symplectic holonomy and whose horizontal foliations have arbitrarily many closed leaves S_k of prescribed self-intersection $[S_k]^2 = m_k \leq h-1$. In particular, if $m_k = 0$ we may assume that the horizontal foliation in some neighbourhood ν_k of S_k is given by the kernel of a projection $\nu_k = \Sigma_h \times D^2 \to D^2$.*

Proof. We let $a_i, b_i \in \pi_1(\Sigma_g)$ denote the standard generators of the fundamental group and let ξ_{m_k} be a flat $SL(2,\mathbb{R})$-bundle over Σ_g with $m_k \leq g-1$ as provided by [Mil] (cf. Theorem 2.3.2). This corresponds to a holonomy representation

$$a_i \mapsto A_{i,k}$$

$$b_i \mapsto B_{i,k},$$

with $A_{i,k}, B_{i,k} \in SL(2,\mathbb{R})$. Then by performing the extension trick of Proposition 4.2.1 we obtain symplectomorphisms ϕ_i, ψ_i which agree with $A_{i,k}, B_{i,k}$ in a small neighbourhood of each marked point $p_k \in \Sigma_h$. Moreover the product $\eta = \prod_{i=1}^{g}[\phi_i, \psi_i]$ has support disjoint from the marked points p_k. That is $\eta \in Symp^c(\Sigma_h^\epsilon)$, where $\Sigma_h^\epsilon = \Sigma_h \setminus (\cup_k D_\epsilon(p_k))$ denotes the surface Σ_h with a union of small ϵ-neighbourhoods of the marked points p_k deleted.

By Lemma 4.2.7 this group is perfect, and we may write $\eta^{-1} = \prod_{i=1}^{g'}[\alpha_i, \beta_i]$, where $\alpha_i, \beta_i \in Symp^c(\Sigma_h^\epsilon) \subset Symp^c(\Sigma_{h,k})$. We define a flat bundle over $\Sigma_{g+g'}$ by the holonomy representation

$$a_i \mapsto \phi_i, \quad b_i \mapsto \psi_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto \alpha_j, \quad b_{g+j} \mapsto \beta_j \text{ for } 1 \leq j \leq g',$$

which we denote by ρ. This bundle then has compact leaves S_k corresponding to the marked points p_k. The Euler class of the normal bundle to S_k is computed from the induced holonomy representation $D_{p_k}\rho$:

$$a_i \mapsto D_p(\phi_i) = A_{i,k}, \quad b_i \mapsto D_p(\psi_i) = B_{i,k} \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto D_p(\alpha_j) = Id, \quad b_{g+j} \mapsto D_p(\beta_j) = Id \text{ for } 1 \leq j \leq g'.$$

In view of formula (2.4) in Section 2.4 above, it follows that $e(\nu_{S_k}) = m_k$. If $m_k = 0$, we may take the trivial flat structure on ξ_0 and, hence, on some neighbourhood $\nu_k = \Sigma_h \times D^2$ of S_k the horizontal foliation will be given as the kernel of the projection to the D^2-factor. □

We may now give examples of manifolds with symplectic pairs, both of whose kernel foliations have closed leaves of non-zero self-intersection.

Corollary 4.2.10. *There exist 4-manifolds that admit symplectic pairs (ω_1, ω_2) both of whose kernel foliations $\mathcal{F}_1, \mathcal{F}_2$ have closed leaves L_1, L_2 with $[L_i]^2 \neq 0$.*

Proof. We let E_1 be a flat symplectic bundle with a section s_1 such that $[s_1]^2 \neq 0$. We further let E_2 be a flat symplectic bundle with two sections s_2, t_2, the first of which has non-trivial self-intersection and the second of which is projectable (cf. Definition 2.5.5). The existence of these bundles is guaranteed by Proposition 4.2.9. By a suitable choice of E_1 we may assume that the genus of its fibre $g(F_1)$ to be arbitrarily large. After stabilisation of E_2, we may also assume that the genus of the base of E_2 is $g(F_1)$. By Proposition 2.5.6 the Gompf sum

$$X = E_1 \#_{F_1 = t_2} E_2$$

admits a symplectic pair, with kernel foliations that we denote $\widetilde{\mathcal{F}}_i$. Then by construction the connect sums

$$\sigma_1 = s_1 \# F_2$$

$$\sigma_2 = s_2 \# F_1$$

are leaves of $\widetilde{\mathcal{F}}_i$ and $[\sigma_i]^2 = [s_i]^2 \neq 0$. □

For smooth diffeomorphisms we computed the abelianisation of $Diff^+(\Sigma_{h,k})$. It is then natural to try to determine the abelianisation of $Symp(\Sigma_{h,k})$, however one cannot mimic the proof of Proposition 4.1.10 used above. The first step is still valid and thus one has that $H_1(Symp_\delta(\Sigma_{h,1})) = H_1(\mathcal{G}_p^{Symp})$. There is also a version of the Sternberg Linearisation Theorem for symplectic germs, but the normal form that it yields is not linear, and thus the computation of $H_1(Symp_\delta(\Sigma_{h,1}))$ remains an open question.

4.2.1 The case of genus 0

So far the results that we have obtained have been for bundles whose fibre Σ_h has been of genus at least 3. We shall now consider the case of genus 0, where one can give a fairly precise description of the possible compact leaves of a (symplectically) flat bundle.

Examples of sphere bundles with horizontal foliations that have closed leaves of arbitrary self-intersection have been given by Mitsumatsu (cf. [Mit1]). We shall summarise his construction here. Let $\mathbb{R}^2 \to \xi_k \to \Sigma_g$ be a flat bundle of Euler class $k \leq g-1$ as given by [Mil]. Then the sphere bundle $S_k = S(\xi_k \oplus \mathbb{R})$ is flat and has two sections $L_\pm$ corresponding to the north and south poles of the fibre and $[L_\pm]^2 = \pm k$.

We would of course like to have similar examples for flat bundles with symplectic holonomy. The flat structures that one obtains via the construction of Mitsumatsu cannot have symplectic holonomy. For if so, then one would have a vertical symplectic form ω_v that is positive on each fibre, i.e. $\omega_v([F]) \neq 0$, and vanishes identically on the leaves of the horizontal foliation. But the set $\{L_-, L_+\}$ generates $H_2(S_k, \mathbb{R})$, which is a contradiction. Thus in order to produce horizontal foliations of sphere bundles with symplectic holonomy we will have to adapt the argument of Proposition 4.2.8. Again we let ξ_k be a flat $SL(2, \mathbb{R})$-bundle over Σ_g as above. We let $a_i, b_i \in \pi_1(\Sigma_g)$ denote the standard generators of the fundamental group so that holonomy homomorphism for ξ_k is given by

$$a_i \mapsto A_i$$

$$b_i \mapsto B_i.$$

Then by performing the extension trick of Proposition 4.2.1 we obtain *Hamiltonian* diffeomorphisms ϕ_i, ψ_i, which have compact support inside some disc $D^2_- \subset \mathbb{R}^2$. Moreover, the product $\eta = \prod_{i=1}^{g}[\phi_i, \psi_i]$ has $Cal(\eta) = 0$ and is the identity in some neighbourhood of the origin. So in fact η has support in some annulus $A \subset D^2_-$. If we consider $D^2_- \subset S^2$ as the southern hemisphere of the 2-sphere we see that η may equally well be thought of as a diffeomorphism acting on the upper hemisphere, that is as an element in $Ham^c(D^2_+)$. Then, since $Cal(-\eta) = 0$, Banyaga's Theorem implies that $\eta^{-1} = \prod_{j=1}^{g'}[\alpha_j, \beta_j]$, with $\alpha_j, \beta_j \in Ham^c(D^2_+)$.

We then define a flat S^2-bundle E_k over $\Sigma_{g+g'}$ with symplectic monodromies as follows:

$$a_i \mapsto \phi_i, b_i \mapsto \psi_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto \alpha_j, b_{g+j} \mapsto \beta_j \text{ for } 1 \leq j \leq g',$$

and by construction there is a leaf L corresponding to the south pole that has $[L]^2 = k$. Interestingly, this can be the only compact leaf of the horizontal foliation. For if L' were any other leaf then $\{L, L'\}$ would generate $H_2(E_k, \mathbb{R})$ and this would contradict the existence of a vertical symplectic form.

Chapter 5

Surface bundles and extended Hamiltonian groups

Motivated by the problem of extending flat structures on the boundaries of surface bundles to their interiors, we show that an arbitrary circle bundle over a surface can be filled by a flat surface bundle after stabilisition. We may even assume that the horizontal foliation of such a bundle is symplectic if the genus of the fibre is large enough. The condition that the genus of the fibre is non-zero in the symplectic case is necessary by a result of Tsuboi, which expresses the Euler class of the boundary circle bundle in terms of the Calabi invariant of certain Hamiltonian extensions of the holonomies on the boundary to the disc.

We shall extend this result to the case of arbitrary Riemann surfaces. In the course of generalising Tsuboi's result we are naturally lead to consider extended Hamiltonian groups as introduced by Kotschick and Morita. By considering the extended Hamiltonian groups as extensions of the mapping class group of punctured surfaces, we also obtain a formula that relates the Calabi invariant to the first MMM-class of surface bundles with boundary. Finally, we resolve a special case of a question posed in [KM1] by showing that the second MMM-class vanishes for surface bundles with holonomy in the extended Hamiltonian group.

5.1 Filling flat S^1-bundles

Given an arbitrary manifold M it is natural to ask what sort of manifolds bound M. If M is an S^1-bundle then the natural class of null-cobordisms to consider are surface bundles, whose fibre is a punctured surface Σ_h^1. If M is in addition a flat S^1-bundle, then one would like to know when M bounds a flat surface bundle. To answer the latter question in full generality is a subtle matter. However for bundles over compact surfaces we will show that after *stabilisation* any flat S^1-bundle can be filled in by a flat Σ_h^1-bundle. We first make precise what we mean by a stabilisation in this context.

Definition 5.1.1. Let E be a flat S^1-bundle over Σ_g with holonomy representation given by $\pi_1(\Sigma_g) \xrightarrow{\rho} Diff_0(S^1)$. Let $\Sigma_{g+g'} \to \Sigma_g$ be the map that collapses $\Sigma_{g'}$ in the decomposition $\Sigma_{g+g'} = \Sigma_g \# \Sigma_{g'}$. Then the *stabilisation* of ρ is the flat bundle associated to the composition of ρ with this collapsing map.

If we allow the monodromies to be arbitrary, then it is an easy matter to show the following.

Proposition 5.1.2. *Let $h \geq 3$ or $h = 0$ and let M be a flat S^1-bundle. Then there is a flat bundle $\Sigma_h^1 \to E \to \Sigma_g$, whose boundary is a stabilisation of M. In particular, there exist flat Σ_h^1-bundles whose boundaries have non-trivial Euler class.*

Proof. Let $a_i, b_i \in \pi_1(\Sigma_g)$ denote the standard generators of the fundamental group and let $\phi_i = \rho(a_i)$ and $\psi_i = \rho(b_i)$ be the images of these generators in $Diff_0(S^1)$ under the monodromy homomorphism ρ. Since ϕ_i, ψ_i are isotopic to the identity, we may extend them to diffeomorphisms $\bar{\phi}_i, \bar{\psi}_i$ on a collar of the boundary $[0,1] \times S^1$ in such a way that

$$\bar{\phi}_i(t,x) = (t, \phi_i(x)) \ , \ \bar{\psi}_i(t,x) = (t, \psi_i(x)) \text{ for } 0 \leq t < \epsilon$$

and

$$\bar{\phi}_i(t,x) = \bar{\psi}_i(t,x) = Id \text{ for } 1 - \epsilon < t \leq 1.$$

We then extend by the identity to obtain $\bar{\phi}_i, \bar{\psi}_i \in Diff^+(\Sigma_h^1)$ such that $\eta = \prod_{i=1}^g [\bar{\phi}_i, \bar{\psi}_i]$ lies in $Diff^c(\Sigma_h^1)$.

In the proof of Proposition 4.1.4 we saw that for $h \geq 3$ the group $Diff^c(\Sigma_h^1)$ is perfect and for $h = 0$ this is the classical result of Thurston (cf. [Th1]). Thus we may write $\eta^{-1} = \prod_{i=1}^{g'} [\alpha_i, \beta_i]$, where $\alpha_i, \beta_i \in Diff^c(\Sigma_h^1)$. We define a flat bundle E over $\Sigma_{g+g'}$ by the holonomy representation

$$a_i \mapsto \bar{\phi}_i, \quad b_i \mapsto \bar{\psi}_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto \alpha_j, \quad b_{g+j} \mapsto \beta_j \text{ for } 1 \leq j \leq g'.$$

The boundary of E is a flat S^1-bundle and by construction it has the following holonomy representation:

$$a_i \mapsto \phi_i, \quad b_i \mapsto \psi_i \text{ for } 1 \leq i \leq g$$

$$a_{g+j} \mapsto Id, \quad b_{g+j} \mapsto Id \text{ for } 1 \leq j \leq g'$$

so that ∂E is a stabilisation of M as required.

The second statement follows from the existence of flat S^1-bundles with non-trivial Euler classes (cf. [Mil]). □

Proposition 5.1.2 implies that any flat circle bundle can be filled in by a flat disc bundle after a suitable stabilisation. On the other hand, if we require that the bundle have symplectic holonomy, then this is no longer true (cf. Theorem 5.2.1 below). However, if the fibre has genus $h \geq 3$, then one can indeed find a filling by a symplectically flat bundle after a suitable stabilisation. To this end we shall need an analogue of the extension trick of Proposition 5.1.2 in the symplectic case.

Proposition 5.1.3. *Let $\pi_1(\Sigma_g) \xrightarrow{\rho} Diff_0(S^1)$ be a flat structure on an S^1-bundle M and let ϕ_i, ψ_i denote $\rho(a_i), \rho(b_i)$ respectively. Then there are symplectic extensions $\tilde{\phi}_i, \tilde{\psi}_i$ on the annulus $A = S^1 \times [0,1]$ that are the identity in a neighbourhood of $S^1 \times \{1\}$ such that $\prod_i^g [\tilde{\phi}_i, \tilde{\psi}_i]$ has support in the interior of A.*

Proof. Let $\mathcal{F}$ be the horizontal foliation given by the flat structure on M and let $\alpha \in \Omega^1(M)$ be a defining 1-form for $\mathcal{F}$. We choose a function ϕ on $[0,1]$, which is equal to t on a neighbourhood of 0 and is identically zero for all t in a neighbourhood of 1. We set $\omega = dt \wedge \alpha + \phi(t)d\alpha$ on $E = M \times [0,1]$ and let $\frac{\partial}{\partial\theta}$ denote a vector field that is tangent to the fibres of M. Then

$$\omega(\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta}) = \alpha(\frac{\partial}{\partial\theta}) \neq 0,$$

since $\mathcal{F}$ is transverse to the fibres of M and, thus, ω is a nowhere vanishing 2-form on E. Furthermore, since $\mathcal{F}$ is a foliation we compute:

$$\begin{aligned}\omega^2 &= (dt \wedge \alpha + \phi(t)d\alpha)^2 \\ &= 2\phi(t)dt \wedge \alpha \wedge d\alpha = 0.\end{aligned}$$

Thus $\mathcal{F}_\omega = Ker(\omega)$ is a well-defined distribution that is transverse to the (annular) fibres of $E \to \Sigma_g$. Moreover, since $\omega = d(t\alpha)$ in a neighbourhood of $M \times \{0\}$ this distribution is integrable and transversally symplectic on this neighbourhood, and restricts to $\mathcal{F}$ on $M \times \{0\}$. On a neighbourhood of $M \times \{1\}$ the form ω reduces to $dt \wedge \alpha$ and again the kernel distribution is integrable and agrees with $\mathcal{F}$ on this neighbourhood.

We choose a base point $x_0 \in \Sigma_g$ and embedded representatives a_i, b_i for the standard generators of $\pi_1(\Sigma_g, x_0)$. We let $\bar{\phi}_i, \bar{\psi}_i$ be the holonomies of the curves a_i, b_i given by the distribution $\mathcal{F}_\omega$. Then on $S^1 \times \{0\}$ and near $S^1 \times \{1\}$ these diffeomorphisms are given by $\phi_i \times Id$ and $\psi_i \times Id$ respectively, where ϕ_i, ψ_i are the images of the standard basis under the holonomy representation of M. Since ϕ_i, ψ_i lie in $Diff_0(S^1)$, we may alter the maps $\bar{\phi}_i, \bar{\psi}_i$ near $S^1 \times \{1\}$ so that they restrict to the identity in a neighbourhood of $S^1 \times \{1\}$. We shall continue to denote these altered maps by $\bar{\phi}_i, \bar{\psi}_i$.

We let Ω be the restriction of ω to the annular fibre over x_0. Then the forms $\bar{\phi}_i^*\Omega - \Omega$ and $\bar{\psi}_i^*\Omega - \Omega$ are trivial in compactly supported cohomology, since the holonomies $\bar{\phi}_i, \bar{\psi}_i$ have support in $S^1 \times [0,1)$ and the distribution defining them was transversally symplectic in a neighbourhood of $M \times \{0\}$. By applying a Moser isotopy, which will have support in the interior of $S^1 \times [0,1]$, we obtain symplectomorphisms $\tilde{\phi}_i, \tilde{\psi}_i$ that are symplectic extensions of ϕ_i, ψ_i respectively, and by construction $\prod_i^g [\tilde{\phi}_i, \tilde{\psi}_i]$ has support in the interior of A. □

Proposition 5.1.3 is the main step in extending flat structures symplectically and the following result follows from this and the perfectness of $Symp^c(\Sigma_h^1)$.

Theorem 5.1.4. *Let M be a flat S^1-bundle and assume that $h \geq 3$. Then some stabilisation of M bounds a flat Σ_h^1-bundle with symplectic holonomy.*

Proof. Let $\pi_1(\Sigma_g) \xrightarrow{\rho} Diff_0(S^1)$ be the holonomy representation associated to M and let $\tilde{\phi}_i, \tilde{\psi}_i \in Symp(A)$ be the extensions given by Proposition 5.1.3. After a suitable choice of symplectic form on Σ_h^1, we may symplectically embed $A = S^1 \times [0,1]$ in Σ_h^1 so that $S^1 \times \{0\}$ maps to $\partial\Sigma_h^1$. We then consider $\eta = \prod_i^g [\tilde{\phi}_i, \tilde{\psi}_i]$ as an element in $Symp^c(\Sigma_h^1)$. This group is perfect by Lemma 4.2.7 and, thus, we may write η^{-1} as a product of g' commutators. We then define the associated flat bundle E' over $\Sigma_{g+g'}$ as in the proof of Proposition 5.1.2, and by construction the boundary $\partial E'$ is a stabilisation of M. □

Theorem 5.1.4 can be interpreted in terms of the five-term exact sequence of a certain extension of groups. For this we let $Symp(\Sigma_h^1)$ as usual denote the group of symplectomorphisms of Σ_h^1. We further let $Symp(\Sigma_h^1, \partial\Sigma_h^1)$ denote those symplectomorphisms that restrict trivially to the boundary. Then as a consequence of Proposition 5.1.3 the following sequence, which is given by restriction to $\partial\Sigma_h^1$, is exact:

$$1 \to Symp(\Sigma_h^1, \partial\Sigma_h^1) \to Symp(\Sigma_h^1) \to Diff^+(\partial\Sigma_h^1) = Diff_0(S^1) \to 1.$$

With this notation we have the following proposition.

Proposition 5.1.5. *For $h \geq 3$ the connecting homomorphism in the five-term exact sequence in real cohomology associated to the following exact sequence is trivial:*

$$1 \to Symp(\Sigma_h^1, \partial\Sigma_h^1) \to Symp(\Sigma_h^1) \to Diff^+(\partial\Sigma_h^1) = Diff_0(S^1) \to 1.$$

Proof. By the Universal Coefficient Theorem it suffices to show that the map

$$H_2(Symp_\delta(\Sigma_h^1)) \to H_2(Diff_{0,\delta}(S^1))$$

is surjective on integral cohomology. This follows immediately from Theorem 5.1.4, since any flat S^1-bundle extends after stabilisation and this does not change the homology class represented by this bundle in $H_2(Diff_{0,\delta}(S^1))$. □

The Godbillon-Vey class of the horizontal foliation of a flat S^1-bundle M defines an element GV in $H^2(Diff_{0,\delta}(S^1), \mathbb{R})$, which is non-trivial by the work of Thurston (cf. [Bott]). It is possible that the Godbillon-Vey class provides an obstruction to the existence of a flat symplectic bundle E that bounds M. However, by Proposition 5.1.5 the image of the class GV in $H^2(Symp_\delta(\Sigma_h^1), \mathbb{R})$ is non-trivial. Geometrically, this means that after stabilisation the horizontal foliation of any S^1-bundle extends to a transversally symplectic foliation on some surface bundle E with fibre Σ_h^1. In particular, the Godbillon-Vey class is not an obstruction to finding a null-cobordism that extends the horizontal foliation of M to the interior of E symplectically.

5.2 Flat bundles and the extended Hamiltonian group

We have seen that after a stabilisation any flat circle bundle over a surface can be filled by a flat disc bundle with smooth holonomy. However, as was shown in [Tsu], it is not in general possible to fill in a flat circle bundle by a flat disc bundle that has symplectic holonomy, since the existence of such a filling implies that the Euler class of the circle bundle vanishes. More specifically Tsuboi proved the following theorem.

Theorem 5.2.1 ([Tsu])**.** *Let $\psi : \pi_1(\Sigma_g) \to Diff_0(S^1)$ be a homomorphism and let a_i, b_i be standard generators of $\pi_1(\Sigma_g)$. Furthermore, let $f_i, h_i \in Symp(D^2)$ be extensions of $\psi(a_i), \psi(b_i)$ respectively. Then $Cal([f_1, h_1]...[f_g, h_g])$ is a non-zero multiple of the Euler class of the associated S^1-bundle evaluated on the fundamental class $[\Sigma_g]$.*

In the case of a disc, a diffeomorphism is symplectic if and only if it is Hamiltonian. For bundles with fibres of higher genus we shall generalise Tsuboi's result under the assumption that the holonomies are Hamiltonian. As a first step, following [Tsu] we will show that it is always possible to extend diffeomorphisms on the boundary of a manifold M to its interior in a volume preserving manner. It will suffice to consider the case $M = [0, 1) \times X$ and to show that any diffeomorphism of the boundary extends to a volume preserving diffeomorphism on M that has compact support. The following is a slight refinement of Lemma 2.2 in [Tsu].

Lemma 5.2.2. *Let $M = [0, 1) \times X$ and let Ω be a volume form of finite total volume. Then the map $Diff^c_\Omega(M) \to Diff_0(\partial M)$ given by restriction is surjective.*

Proof. Let (t, x) be coordinates on M. We choose a volume form $\Omega_{\partial M}$ on $\partial M = X$ so that $\Omega|_{\partial M} = dt \wedge \Omega_{\partial M}$ and set $\widetilde{\Omega} = dt \wedge \Omega_{\partial M}$ on M.

Let λ be a step function such that $\lambda(x) = 0$ in a neighbourhood of 0 and $\lambda(x) = 1$ in a neighbourhood of 1, and define a volume form on M by $\bar{\Omega} = \lambda\,\tilde{\Omega} + (1-\lambda)\Omega$. After multiplying $\bar{\Omega}$ by an appropriate positive function, we may assume that $\bar{\Omega}$ has the same total volume as Ω and that it agrees with the original form near the boundary of M. We then consider the family of forms $\Omega_s = s\bar{\Omega} + (1 - s)\Omega$, which has the property $\frac{d}{ds}[\Omega_s] = 0$. Moreover, by construction $\Omega_s = \Omega$ near $\{0\} \times \partial M$ and $\Omega_s = \widetilde{\Omega}$ on $\{1\} \times \partial M$ for all s.

Applying the Moser trick we obtain a compactly supported isotopy ϕ_s such that $\phi_s^*\Omega_s = \Omega_0$ and $\phi_s|_{\partial M} = Id$ for all s. Thus, without loss of generality, we shall assume that Ω is $dt \wedge \Omega_{\partial M}$ for some volume form $\Omega_{\partial M}$ on the boundary. Let $h \in Diff_0(\partial M)$ and choose an isotopy h_s joining $h_0 = Id$ to $h_1 = h$. We let ξ_s be the time dependent vector field generating h_s and define the divergence of ξ_s by

$$div(\xi_s)\Omega_{\partial M} = d\iota_{\xi_s}\Omega_{\partial M}.$$

We then define a vector field on M by $X_s = \xi_s + t\,div(\xi_s)\frac{\partial}{\partial t}$. By the Cartan formula $L_{X_s}\Omega = 0$. Moreover, the form $\iota_{X_s}\Omega$ is identically zero when restricted to ∂M, hence $\iota_{X_s}\Omega$ is exact for all s and there are forms α_{X_s} that depend smoothly on s so that $\iota_{X_s}\Omega = d\alpha_{X_s}$. Finally, by multiplying with a cut off function λ as defined above, the Hamiltonian flow generated by $\widetilde{\alpha}_{X_s} = \lambda\,\alpha_{X_s}$ will have compact support and, hence, provides the desired extension to M. $\square$

As an immediate corollary we obtain an analogous exact sequence for arbitrary manifolds with non-empty boundary. Here the group $Diff_0^\Omega(M, \partial M)$ denotes those volume preserving diffeomorphisms that are isotopic to the identity and restrict to the identity on the boundary.

Corollary 5.2.3. *The following sequence is exact*

$$1 \to Diff_0^\Omega(M, \partial M) \to Diff_0^\Omega(M) \to Diff_0(\partial M) \to 1.$$

We shall now restrict our attention to the case of symplectic surfaces. To this end we choose a symplectic form ω on Σ_h^1 and consider the group of symplectomorphisms $Symp(\Sigma_h^1)$ of (Σ_h^1, ω). We let $Symp(\Sigma_h^1, \partial\Sigma_h^1)$ denote those symplectomorphisms that restrict to the identity on $\partial\Sigma_h^1$. In this case Corollary 5.2.3 implies the exactness of the following sequence:

$$1 \to Symp(\Sigma_h^1, \partial\Sigma_h^1) \to Symp(\Sigma_h^1) \to Diff^+(\partial\Sigma_h^1) = Diff_0(S^1) \to 1. \quad (5.1)$$

We claim that there is also a similar sequence for Hamiltonian diffeomorphisms, where as usual a symplectomorphism $\psi \in Symp_0(\Sigma^1_h)$ will be called Hamiltonian if it is isotopic to the identity via an isotopy ψ_t such that $\iota_{\dot{\psi}_t}\omega = dH_t$ for $0 \le t \le 1$. This is immediate, as given any diffeomorphism f of the boundary of Σ^1_h, the proof of Lemma 5.2.2 gives a Hamiltonian extension of f to Σ^1_h. So one has an exact sequence:

$$1 \to Ham(\Sigma^1_h, \partial\Sigma^1_h) \to Ham(\Sigma^1_h) \to Diff^+(\partial\Sigma^1_h) = Diff_0(S^1) \to 1, \tag{5.2}$$

where $Ham(\Sigma^1_h, \partial\Sigma^1_h)$ denotes the intersection $Symp(\Sigma^1_h, \partial\Sigma^1_h) \cap Ham(\Sigma^1_h)$. We saw in Section 4.2 that the compactly supported Hamiltonian group may be thought of as the kernel of a $Flux$ homomorphism (cf. Definition 4.2.2). For surfaces with boundary one may also define a Flux homomorphism $Symp_0(\Sigma^1_h) \to H^1(\Sigma^1_h, \mathbb{R})$ via the formula

$$Flux(\psi) = \int_0^1 \iota_{\dot{\psi}_t}\omega \, dt,$$

where ψ_t is an isotopy joining ψ to the identity. As in the compactly supported case, one can show that $Flux(\psi) = [\lambda - \psi^*\lambda]$ for any primitive λ such that $\omega = -d\lambda$ (cf. [McS1], Lemma 10.14). Hence, $Flux$ is well-defined independently of the choice of isotopy ψ_t and primitive λ, and $Ker(Flux) = Ham(\Sigma^1_h)$ as the following lemma shows.

Lemma 5.2.4. *Let Σ^1_h be a surface with one boundary component. Then $Ham(\Sigma^1_h) = Ker(Flux)$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd} Symp_0^c(\Sigma^1_h) \arrow[r, shift left, "Flux"] \arrow[d] & H^1_c(\Sigma^1_h, \mathbb{R}) \arrow[l, shift left, "s"] \arrow[d, "\cong"] \\ Symp_0(\Sigma^1_h) \arrow[r, "Flux"] & H^1(\Sigma^1_h, \mathbb{R}), \end{tikzcd}$$

where the upper $Flux$ map has a continuous (set-theoretic) section s (cf. [McS1], p. 318). Since $H^1_c(\Sigma^1_h, \mathbb{R}) \cong H^1(\Sigma^1_h, \mathbb{R})$ for a surface with one boundary component, the bottom $Flux$ map also has such a section and, thus, defines a fibration:

$$Ker(Flux) \to Symp_0(\Sigma^1_h) \stackrel{Flux}{\to} H^1(\Sigma^1_h, \mathbb{R}).$$

Hence, the inclusion $Ker(Flux) \hookrightarrow Symp_0(\Sigma^1_h)$ is a weak homotopy equivalence, and if $\psi \in Ker(Flux)$, then there is an isotopy ψ_t joining ψ to the identity that is wholly contained in $Ker(Flux)$. For such an isotopy

$$\int_0^t \iota_{\dot{\psi}_s}\omega \, ds \text{ is exact for all } t$$

and it follows that

$$\frac{d}{dt}\int_0^t \iota_{\dot{\psi}_s}\omega \, ds = \iota_{\dot{\psi}_t}\omega \text{ is exact for all } t$$

and, hence, ψ is Hamiltonian. Conversely, if ψ_t is a Hamiltonian isotopy then

$$Flux(\psi) = \int_0^1 \iota_{\dot{\psi}_s}\omega \, ds = \int_0^1 dH_s \, ds = 0 \in H^1(\Sigma^1_h, \mathbb{R}). \qquad \square$$

One may also define a Calabi homomorphism Cal on $Ham(\Sigma_h^1, \partial\Sigma_h^1)$. For this one chooses a primitive λ so that $\omega = -d\lambda$ and defines

$$Cal(\phi) = -\frac{1}{3}\int_{\Sigma_h^1} \phi^*\lambda \wedge \lambda.$$

Again this definition is independent of the choice of λ (see [McS1]).

We will extend Tsuboi's result to bundles with fibre Σ_h^1, and then to the so-called extended Hamiltonian group. In order to do this we shall need to reinterpret Theorem 5.2.1 in terms of the five-term exact sequence associated to the exact sequence (5.2) for cohomology groups taken with *real* coefficients. Now the map Cal is an element of $H^1(Ham_\delta(D^2, \partial D^2))$ and we claim that it is invariant under the conjugation action of $Ham(D^2)$. For let $\psi \in Ham(D^2)$ and $\phi \in Ham(D^2, \partial D^2)$, and let λ be a primitive such that $\omega = -d\lambda$. Then $\psi^*\lambda$ is also a primitive for ω and we have

$$\int_{D^2} \phi^*\lambda \wedge \lambda = \int_{D^2} \phi^*(\psi^*\lambda) \wedge (\psi^*\lambda) = \int_{D^2} (\psi\phi\psi^{-1})^*\lambda \wedge \lambda. \tag{5.3}$$

Thus, in fact, $Cal \in H^1(Ham_\delta(D^2, \partial D^2))^{Diff_0(S^1)}$ and we claim that Tsuboi's result, and its extension, can be interpreted as saying that the image of Cal under the connecting homomorphism in the five-term exact sequence is a non-zero multiple of the Euler class e considered as an element in the real group cohomology of $Diff_0(S^1)$.

For this we will need an explicit cocycle description of the connecting homomorphism in the five-term exact sequence. This along with several important facts about the five-term exact sequence have been gathered in Appendix A.

Theorem 5.2.5. *Consider the extension of groups*

$$1 \to Ham(D^2, \partial D^2) \to Ham(D^2) \to Diff_0(S^1) \to 1,$$

and let δ denote the connecting homomorphism in the five-term exact sequence in real cohomology. Then $\delta[Cal]$ is a non-zero multiple of the Euler class in $H^2(Diff_0(S^1), \mathbb{R})$.

Proof. In order to verify the equality $\delta Cal = \mu\, e$ in real cohomology, it suffices to evaluate both sides on 2-cycles Z in $H_2(Diff_0(S^1), \mathbb{Z})$. Such a cycle may be thought of as the image of the fundamental class under the map induced by a representation of a surface group $\pi_1(\Sigma_g) \xrightarrow{\psi} Diff_0(S^1)$. If we let a_i, b_i be standard generators of $\pi_1(\Sigma_g)$, then a generator of $H_2(\pi_1(\Sigma_g))$ may be described by the group 2-cycle

$$\begin{aligned} z = {} & (a_1, b_1) + (a_1 b_1, a_1^{-1}) + ... + (a_1 b_1 ... b_{g-1} a_g^{-1}, b_g^{-1}) \\ & - (2g+1)(e,e) - \sum_{i=1}^{g} (a_i, a_i^{-1}) + (b_i, b_i^{-1}). \end{aligned}$$

Since $[a_1, b_1]...[a_g, b_g] = e$ in $\pi_1(\Sigma_g)$, we compute that

$$\begin{aligned} \partial z = {} & \sum_{i=1}^{g} (a_i) + (a_i^{-1}) + (b_i) + (b_i^{-1}) - \sum_{i=1}^{g} [(a_i) - (e) + (a_i^{-1}) + (b_i) - (e) + (b_i^{-1})] - 2g(e) \\ & + ([a_1, b_1]...[a_g, b_g]) - (e) = 0. \end{aligned}$$

We let f_i, h_i denote representatives of $\psi(a_i), \psi(b_i)$ in $Diff_0(S^1)$ considered as a quotient group, and let $\tilde{z}$ be the associated lift of the fundamental cycle above. Then we compute

$$\partial\tilde{z} = ([f_1, h_1]...[f_g, h_g]) - (e).$$

Thus, by Lemmas A.1 and A.3, there is a set-theoretic extension Cal_S of Cal to $Ham(D^2)$ such that

$$\begin{aligned}\pm\delta Cal(Z) = \bar{\delta}Cal(Z) &= \delta Cal_S(\psi_*[\Sigma_g]) = Cal_S(\partial\tilde{z}) \\ &= Cal_S(([f_1, h_1]...[f_g, h_g]) - (e)) = Cal([f_1, h_1]...[f_g, h_g]),\end{aligned}$$

and by Proposition 5.2.6 this is a multiple of the Euler class. Thus we conclude that $\delta Cal = \mu\, e$ for some non-zero $\mu \in \mathbb{R}$. □

With this formulation it is an easy matter to extend Tsuboi's result to surfaces of higher genus.

Proposition 5.2.6. *Let $\psi : \pi_1(\Sigma_g) \to Diff_0(S^1)$ be a homomorphism and let a_i, b_i be standard generators of $\pi_1(\Sigma_g)$. Let $f_i, h_i \in Ham(\Sigma_h^1)$ be any extensions of $\psi(a_i), \psi(b_i)$ respectively. Then $Cal([f_1, h_1]...[f_g, h_g])$ is a non-zero multiple of the Euler class of the associated S^1-bundle evaluated on the fundamental class $[\Sigma_g]$.*

Proof. By Lemma 5.2.2 we may assume that the extensions f_i, h_i are Hamiltonian and have support in a collar $K = [0, 1) \times S^1$ of the boundary. We may then consider $K \subset D^2$ with an appropriately chosen area form Ω on D^2 and f_i, h_i as elements in $Diff_\Omega(D^2)$. We then compute

$$\psi^*\delta Cal([\Sigma_g]) = Cal^{\Sigma_h^1}([f_1, h_1]...[f_g, h_g]) = Cal^{D^2}([f_1, h_1]...[f_g, h_g]), \tag{5.4}$$

where the first equality follows as in Theorem 5.2.5 and the second follows from our choice of extensions. The latter value is $\mu\, e([\Sigma_g])$ by Theorem 5.2.5. Thus since the left hand side of equation (5.4) is independent of any choices we conclude that for *any* extensions f_i, h_i

$$Cal^{\Sigma_h^1}([f_1, h_1]...[f_g, h_g]) = \mu\, e([\Sigma_g]).$$

□

In particular, it follows by the exactness of the five-term sequence that the boundary of any flat bundle with holonomy in $Ham(\Sigma_h^1)$ is trivial as an S^1-bundle. Furthermore, with our interpretation of Tsuboi's result we may extend our discussion to the extended Hamiltonian group. To this end we first define an extended version of the Flux homomorphism, which will in general only be a crossed homomorphism (cf. [KM1]).

Definition 5.2.7 (Extended Flux)**.** We define $\widetilde{Flux}_\lambda : Symp(\Sigma_h^1) \to H^1(\Sigma_h^1, \mathbb{R})$ by $\widetilde{Flux}_\lambda(\phi) = [(\phi^{-1})^*\lambda - \lambda]$ for some *fixed* primitive $-d\lambda = \omega$.

This definition depends in an essential way on the choice of primitive λ. For if λ' is another primitive, then $\lambda - \lambda' = \alpha$ is closed and

$$\widetilde{Flux}_\lambda(\phi) = \widetilde{Flux}_{\lambda'}(\phi) + [(\phi^{-1})^*\alpha - \alpha]. \tag{5.5}$$

In terms of group cohomology this means that $\widetilde{Flux}_\lambda$ and $\widetilde{Flux}_{\lambda'}$ are cohomologous, when considered as elements in $H^1(Symp(\Sigma_h^1), H^1(\Sigma_h^1, \mathbb{R}))$. The extended Hamiltonian group $\widetilde{Ham}(\Sigma_h^1)$ is defined as the kernel of $\widetilde{Flux}_\lambda$, which is a subgroup since $\widetilde{Flux}_\lambda$ is a crossed homomorphism. The group $Ham(\Sigma_h^1)$ is contained in $\widetilde{Ham}(\Sigma_h^1)$ and we may extend the Calabi homomorphism to a map $\widetilde{Cal}_\lambda$ on the group

$$\widetilde{Ham}(\Sigma_h^1, \partial\Sigma_h^1) = Symp(\Sigma_h^1, \partial\Sigma_h^1) \cap \widetilde{Ham}(\Sigma_h^1)$$

by defining

$$\widetilde{Cal}_\lambda(\phi) = -\frac{1}{3}\int_{\Sigma_h^1} \phi^*\lambda \wedge \lambda = \frac{1}{3}\int_{\Sigma_h^1} (\phi^{-1})^*\lambda \wedge \lambda,$$

where λ is the primitive chosen in the definition of $\widetilde{Flux}_\lambda$. This is a homomorphism on $\widetilde{Ham}(\Sigma_h^1, \partial\Sigma_h^1)$, since the following holds on $Symp(\Sigma_h^1, \partial\Sigma_h^1)$ (cf. [KM2], Prop. 19)

$$\widetilde{Cal}_\lambda(\phi\psi) = \widetilde{Cal}_\lambda(\phi) + \widetilde{Cal}_\lambda(\psi) + \frac{1}{3}\widetilde{Flux}_\lambda(\phi) \wedge (\phi^{-1})^*\widetilde{Flux}_\lambda(\psi). \tag{5.6}$$

Again the definition of $\widetilde{Cal}_\lambda$ depends on the choice of primitive λ. However, we do have the following technical lemma, which will be important in showing the equivariance of $\widetilde{Cal}_\lambda$.

Lemma 5.2.8. *Let $\phi \in \widetilde{Ham}_\lambda(\Sigma_h^1, \partial\Sigma_h^1) \cap \widetilde{Ham}_{\lambda'}(\Sigma_h^1, \partial\Sigma_h^1)$ for two different primitives λ, λ' and set $\alpha = \lambda - \lambda'$, further assume that $\phi^*\alpha - \alpha = dH_\phi$ is exact. Then $\widetilde{Cal}_\lambda(\phi) = \widetilde{Cal}_{\lambda'}(\phi)$.*

Proof. By assumption $\lambda - (\phi^{-1})^*\lambda$ is exact and hence $\phi^*\lambda - \lambda = dF_\phi$ is also exact. Since the boundary of Σ_h^1 is connected we may further assume that $F_\phi = 0$ on $\partial\Sigma_h^1$. We compute

$$\begin{aligned}
\widetilde{Cal}_\lambda(\phi) &= -\frac{1}{3}\int \phi^*\lambda \wedge \lambda \\
&= -\frac{1}{3}\int (\phi^*\lambda - \lambda) \wedge \lambda \\
&= -\frac{1}{3}\int dF_\phi \wedge \lambda \\
&= -\frac{1}{3}\int d(F_\phi\lambda) - F_\phi \wedge d\lambda \\
&= -\frac{1}{3}\int F_\phi\omega.
\end{aligned}$$

Similarly one has

$$\widetilde{Cal}_{\lambda'}(\phi) = -\frac{1}{3}\int F_\phi\omega - \frac{1}{3}\int H_\phi\omega,$$

where $\phi^*\alpha - \alpha = dH_\phi$ and again we assume that $H_\phi = 0$ on $\partial\Sigma_h^1$. We next compute

$$\begin{aligned} -\int H_\phi \omega &= \int H_\phi d\lambda \\ &= \int d(H_\phi \lambda) - dH_\phi \wedge \lambda \\ &= \int (\alpha - \phi^*\alpha) \wedge \lambda \\ &= \int \alpha \wedge \lambda - \phi^*\alpha \wedge \lambda \\ &= \int \phi^*\alpha \wedge \phi^*\lambda - \phi^*\alpha \wedge \lambda \\ &= \int \phi^*\alpha \wedge (\phi^*\lambda - \lambda) \\ &= 0 \end{aligned}$$

since $\lambda - \phi^*\lambda = dF_\phi$, $d\alpha = 0$ and $F_\phi|_{\partial\Sigma_h^1} = 0$. □

Lemma 5.2.8 then implies that $\widetilde{Cal}_\lambda$ is equivariant under the conjugation action of $\widetilde{Ham}(\Sigma_h^1)$.

Corollary 5.2.9. *Let $\psi \in \widetilde{Ham}_\lambda(\Sigma_h^1)$, then $\widetilde{Cal}_\lambda = \widetilde{Cal}_{\psi^*\lambda}$. In particular, $\widetilde{Cal}_\lambda$ is equivariant under the action of $\widetilde{Ham}_\lambda(\Sigma_h^1)$.*

Proof. Let $\lambda' = \psi^*\lambda$, then λ' is also a primitive with $-d\lambda' = \omega$ and $\alpha = \lambda' - \lambda = dH_\psi$ is exact since $\psi \in \widetilde{Ham}_\lambda(\Sigma_h^1)$. By formula (5.5), it follows that $\widetilde{Flux}_\lambda = \widetilde{Flux}_{\lambda'}$ and hence

$$\widetilde{Ham}_\lambda(\Sigma_h^1, \partial\Sigma_h^1) = \widetilde{Ham}_{\lambda'}(\Sigma_h^1, \partial\Sigma_h^1).$$

By applying Lemma 5.2.8 we conclude that $\widetilde{Cal}_\lambda = \widetilde{Cal}_{\psi^*\lambda}$. Furthermore this implies

$$\begin{aligned} \widetilde{Cal}_\lambda(\phi) &= -\frac{1}{3}\int \phi^*\lambda \wedge \lambda \\ &= -\frac{1}{3}\int \phi^*(\psi^*\lambda) \wedge \psi^*(\lambda) \\ &= -\frac{1}{3}\int (\psi\phi\psi^{-1})^*\lambda \wedge \lambda \\ &= \widetilde{Cal}_\lambda(\psi\phi\psi^{-1}), \end{aligned}$$

which is the desired equivariance. □

We may now prove an analogue of Tsuboi's result for the extended Hamiltonian group. For the sake of notational expediency, we shall drop any explicit references to λ.

Theorem 5.2.10. *Let $\psi : \pi_1(\Sigma_g) \to Diff_0(S^1)$ be a homomorphism and let a_i, b_i be standard generators of $\pi_1(\Sigma_g)$. Let $f_i, h_i \in \widetilde{Ham}(\Sigma_h^1)$ be any extensions of $\psi(a_i), \psi(b_i)$ respectively.*

Then $\widetilde{Cal}([f_1,h_1]...[f_g,h_g])$ *is a non-zero multiple of the Euler class of the associated* S^1*-bundle evaluated on the fundamental class* $[\Sigma_g]$*. In particular, if* $\Sigma_h^1 \to E \to \Sigma_g$ *is a flat bundle with holonomy in the extended Hamiltonian group, then the boundary is a trivial bundle.*

Proof. We consider the commuting diagram

$$\begin{tikzcd} 1 \arrow[r] & Ham(\Sigma_h^1, \partial\Sigma_h^1) \arrow[r] \arrow[d] & Ham(\Sigma_h^1) \arrow[r] \arrow[d] & Diff_0(S^1) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \widetilde{Ham}(\Sigma_h^1, \partial\Sigma_h^1) \arrow[r] & \widetilde{Ham}(\Sigma_h^1) \arrow[r] & Diff_0(S^1) \arrow[r] & 1. \end{tikzcd}$$

The five-term sequence then gives the following commuting triangle

$$\begin{tikzcd} H^1(Ham_\delta(\Sigma_h^1, \partial\Sigma_h^1), \mathbb{R})^{Diff_0(S^1)} \arrow[r, "\delta"] & H^2(Diff_{0,\delta}(S^1), \mathbb{R}) \, . \\ H^1(\widetilde{Ham}_\delta(\Sigma_h^1, \partial\Sigma_h^1), \mathbb{R})^{Diff_0(S^1)} \arrow[u] \arrow[ur, "\delta"'] & \end{tikzcd}$$

By Corollary 5.2.9 we see that $\widetilde{Cal} \in H^1(\widetilde{Ham}_\delta(\Sigma_h^1, \partial\Sigma_h^1), \mathbb{R})^{Diff_0(S^1)}$. If ι denotes the inclusion of $Ham(\Sigma_h^1, \partial\Sigma_h^1)$ into $\widetilde{Ham}(\Sigma_h^1, \partial\Sigma_h^1)$, then $\iota^*\widetilde{Cal} = Cal$ and, hence, $\delta(\widetilde{Cal}) = \delta(Cal)$ is also a multiple of the Euler class by Proposition 5.2.6.

The second statement follows from the exactness of the five-term sequence. □

A comparison of Theorem 5.1.4 and Theorem 5.2.10 exhibits a stark difference between the two groups $Symp(\Sigma_h^1)$ and $\widetilde{Ham}(\Sigma_h^1)$.

5.3 The Calabi map and the first MMM-class

We have seen that the Euler class of the boundary of a surface bundle with one boundary component can be interpreted as the image of the Calabi map under the connecting homomorphism of a certain five-term exact sequence. We shall give a similar construction for the first Mumford-Miller-Morita (MMM) class e_1, which represents a generator of $H^2(\Gamma_h^1, \mathbb{R}) \cong \mathbb{R}$ for $h \geq 3$. Recall that the first MMM-Class is up to a constant just the signature of a surface bundle with fibre Σ_h^1 (cf. [Mor1]).

For this purpose we will consider extended Flux homomorphisms $\widetilde{Flux} : Symp^c(\Sigma_h^1) \to H_c^1(\Sigma_h^1)$, which are by definition crossed homomorphisms that restrict to the ordinary flux map on $Symp_0^c(\Sigma_h^1)$. Any such crossed homomorphism defines an element in $H^1(Symp^c(\Sigma_h^1), H_c^1(\Sigma_h^1, \mathbb{R}))$ and this group has been computed by Kotschick and Morita (cf. [KM1], [KM2]).

When one has such a map one may define the extended Hamiltonian group $\widetilde{Ham^c}(\Sigma_h^1)$ as the kernel of $\widetilde{Flux}$ and one obtains the following extension of groups

$$1 \to Ham^c(\Sigma_h^1) \to \widetilde{Ham^c}(\Sigma_h^1) \xrightarrow{p} \Gamma_h^1 \to 1. \tag{5.7}$$

The number of such extensions is, in a sense, rather small, as we have the following lemma, which can be found in [KM2].

Lemma 5.3.1. *Let $\widetilde{Flux}_1, \widetilde{Flux}_2$ be two extended Flux homomorphisms, then as classes in $H^1(Symp^c(\Sigma^1_h), H^1_c(\Sigma^1_h, \mathbb{R}))$*

$$[\widetilde{Flux}_1] - [\widetilde{Flux}_2] = [a.p^* k_{\mathbb{R}}],$$

where $k_{\mathbb{R}} \in H^1(\Gamma^1_h, H^1_c(\Sigma^1_h, \mathbb{R})) \cong \mathbb{R}$ is the generator defined by the extended Johnson homomorphism of Morita.

Proof. We let $\Delta = \widetilde{Flux}_1 - \widetilde{Flux}_2$. For $\phi \in Symp^c_0(\Sigma^1_h)$ and $\psi \in Symp^c(\Sigma^1_h)$ we see that on the level of cochains

$$\begin{aligned}\Delta(\phi.\psi) = [\widetilde{Flux}_1 - \widetilde{Flux}_2](\phi.\psi) &= [\widetilde{Flux}_1(\phi) - \widetilde{Flux}_2(\phi)] + (\phi^{-1})^*[\widetilde{Flux}_1(\psi) - \widetilde{Flux}_2(\psi)]\\ &= [Flux(\phi) - Flux(\phi)] + [\widetilde{Flux}_1(\psi) - \widetilde{Flux}_2(\psi)]\\ &= [\widetilde{Flux}_1(\psi) - \widetilde{Flux}_2(\psi)] = \Delta(\psi).\end{aligned}$$

Moreover, Δ vanishes on $Symp^c_0(\Sigma^1_h)$ by definition and is coclosed, hence $[\Delta] = p^*[\beta]$ for some $[\beta] \in H^1(\Gamma^1_h, H_1(\Sigma^1_h, \mathbb{R}))$. Now this group is isomorphic to $\mathbb{R}$ and is generated by the extended Johnson homomorphism of Morita (cf. [KM2] and [Mor2]). □

Next we define a pairing

$$H^1(G, H^1_c(\Sigma^1_h, \mathbb{R})) \times H^1(G, H^1_c(\Sigma^1_h, \mathbb{R})) \to H^2(G, \mathbb{R}),$$

which we denote $[\alpha.\beta]$ for classes $[\alpha], [\beta] \in H^1(G, H^1_c(\Sigma^1_h, \mathbb{R}))$. This is defined via the following formula

$$\alpha.\beta(\phi, \psi) = \alpha(\phi) \wedge (\phi^{-1})^* \beta(\psi).$$

The induced map on cohomology is well-defined independently of the chosen representatives, and is natural with respect to pullbacks (cf. [KM2], Lemma 18).

We shall let $\widetilde{Flux}_c$ denote $\widetilde{Flux}_\lambda$ for a fixed primitive λ so that in general

$$[\widetilde{Flux}] = [\widetilde{Flux}_c] + a\, p^* k_{\mathbb{R}},$$

for some $a \in \mathbb{R}$. We may now prove the following theorem.

Theorem 5.3.2. *Let $h \geq 2$ and let*

$$1 \to Ham^c(\Sigma^1_h) \to \widetilde{Ham}^c_a(\Sigma^1_h) \xrightarrow{p} \Gamma^1_h \to 1$$

be the extension associated to the extended Hamiltonian group defined by the extended flux map $\widetilde{Flux}_a = \widetilde{Flux}_c + a\, p^* k_{\mathbb{R}}$. Then the image of $[Cal] \in H^1(Ham^c(\Sigma^1_h), \mathbb{R})^{\Gamma^1_h}$ under the connecting homomorphism of the five-term exact sequence is $\pm\frac{1}{3}a^2 e_1$. In particular, if a is non-zero, then any flat bundle with holonomy in $\widetilde{Ham}^c_a(\Sigma^1_h)$ has signature zero.

Proof. We consider the following part of the five-term exact sequence associated to the extended Hamiltonian group

$$H^1(\widetilde{Ham^c_a}(\Sigma^1_h),\mathbb{R}) \to H^1(Ham^c(\Sigma^1_h),\mathbb{R})^{\Gamma^1_h} \xrightarrow{\delta} H^2(\Gamma^1_h,\mathbb{R}).$$

We first claim that the Calabi map lies in the invariant part of $H^1(Ham^c(\Sigma^1_h),\mathbb{R})$. For if λ is a primitive for the symplectic form ω on Σ^1_h, then so is $\psi^*\lambda$ for any $\psi \in \widetilde{Ham^c_a}(\Sigma^1_h)$. Thus, since the Calabi map is independent of the choice of primitive for any $\phi \in Ham^c(\Sigma^1_h)$, we compute that

$$Cal(\phi) = -\frac{1}{3}\int_{\Sigma^1_h} \phi^*\lambda \wedge \lambda = -\frac{1}{3}\int_{\Sigma^1_h} \phi^*(\psi^*\lambda)\wedge(\psi^*\lambda) = -\frac{1}{3}\int_{\Sigma^1_h} (\psi\phi\psi^{-1})^*\lambda\wedge\lambda = Cal(\psi\phi\psi^{-1}),$$

and $[Cal]$ lies in $H^1(Ham^c(\Sigma^1_h),\mathbb{R})^{\Gamma^1_h}$ as claimed.

If i denotes the inclusion $\widetilde{Ham^c_a}(\Sigma^1_h) \hookrightarrow Symp^c(\Sigma^1_h)$, then by definition $\widetilde{Flux_a}$ vanishes on $\widetilde{Ham^c_a}(\Sigma^1_h)$ and we see that

$$i^*[\widetilde{Flux_c}] = -a.i^*(p^*[k_{\mathbb{R}}]).$$

Let $\overline{f} = \widetilde{Cal}_\lambda$ and note that by formula (5.6) this map satisfies the hypotheses of Lemma A.4. Thus we have an explicit description of the connecting homomorphism in terms of $\widetilde{Cal}_\lambda$. More precisely, let $\overline{\phi},\overline{\psi} \in \Gamma^1_h$ be considered as elements of the quotient, then we compute

$$\begin{aligned}\delta(Cal)(\overline{\phi},\overline{\psi}) = \widetilde{Cal}_\lambda(\phi) + \widetilde{Cal}_\lambda(\psi) - \widetilde{Cal}_\lambda(\phi.\psi) &= -\frac{1}{3}\widetilde{Flux_c}(\phi)\wedge(\phi^{-1})^*\widetilde{Flux_c}(\psi)\\ &= -\frac{1}{3}a.i^*(p^*k_{\mathbb{R}})(\phi)\wedge(\phi^{-1})^*a.i^*(p^*k_{\mathbb{R}})(\psi)\\ &= -\frac{1}{3}a^2k_{\mathbb{R}}(\overline{\phi})\wedge(\phi^{-1})^*k_{\mathbb{R}}(\overline{\psi}).\end{aligned}$$

Thus we have shown that

$$[\delta(Cal)] = \pm\frac{1}{3}a^2[k_{\mathbb{R}}.k_{\mathbb{R}}],$$

where the sign ambiguity is a consequence of Lemma A.3. Now we know by [Mor3] that $[k_{\mathbb{R}}.k_{\mathbb{R}}] = -e_1$ where e_1 is the first MMM-class. Thus $\delta[Cal] = \pm\frac{1}{3}a^2e_1$. The second claim follows by the exactness of the five-term exact sequence. □

We may now give an interpretation of the signature of certain surface bundles in terms of the Calabi map of commutators lying in the kernel of an extended Flux homomorphism $\widetilde{Flux}$. Specifically, we let $\widetilde{Flux}$ be the pullback of the extended flux map on $Symp(\Sigma_h)$ under the inclusion $Symp^c(\Sigma^1_h) \hookrightarrow Symp(\Sigma_h)$. By Theorem 12 of [KM2] we know that $[\widetilde{Flux}] = [\widetilde{Flux_c}] + p^*[k_{\mathbb{R}}]$. Then as a consequence of Theorem 5.3.2 and the calculations in the proof of Theorem 5.2.5 we obtain the following corollary.

Corollary 5.3.3. *For $h \geq 2$ let $\Sigma^1_h \to E \to \Sigma_g$ be a bundle with holonomy representation $\pi_1(\Sigma_g) \xrightarrow{\rho} \Gamma^1_h$. We let a_i, b_i be a standard basis for $\pi_1(\Sigma_g)$ and let $\alpha_i = \rho(a_i)$ and $\beta_i = \rho(b_i)$. Then for any lifts $\phi_i,\psi_i \in \widetilde{Ham^c_1}(\Sigma^1_g)$ of α_i,β_i the signature satisfies*

$$\sigma(E) = \frac{1}{3}e_1(E) = \pm Cal([\phi_1,\psi_1]...[\phi_g,\psi_g]).$$

5.4 The second MMM-class vanishes on $\widetilde{Ham}$

In [KM1] it was shown that the k-th power of the first MMM-class is non-trivial in the group $H^*(Symp_\delta(\Sigma_h))$ if $h \geq 3k$ (cf. [KM1], Theorem 3). On the other hand, the Bott vanishing Theorem (cf. [Bott], p. 35) implies that the higher MMM-classes e_k vanish for all flat surface bundles if $k \geq 3$. Thus the question remains open as to the non-triviality of e_2 for flat surface bundles (cf. [KM1], Problem 4). Motivated by this, we will show that the higher MMM-classes as well as all the higher powers of e_1 vanish on the extended Hamiltonian group $\widetilde{Ham}(\Sigma_h) = Ker(\widetilde{Flux})$, where $\widetilde{Flux}$ is the extended Flux map for closed surfaces as defined in [KM1].

As a first step we show that the first MMM-class itself is non-trivial on $\widetilde{Ham}(\Sigma_h)$ for $h \geq 3$.

Proposition 5.4.1. *The image of e_1 under the map induced by the projection $\widetilde{Ham}(\Sigma_h) \to \Gamma_h$ is non-trivial for $h \geq 3$.*

Proof. Applying the five-term exact sequence to the extension

$$1 \to Ham(\Sigma_h) \to \widetilde{Ham}(\Sigma_h) \to \Gamma_h \to 1,$$

exactness and the perfectness of $Ham(\Sigma_h)$ allow us to conclude that the map

$$H^2(\Gamma_h) \to H^2(\widetilde{Ham}_\delta(\Sigma_h))$$

is injective. Since $e_1 \in H^2(\Gamma_h)$ is non-trivial for $h \geq 3$ (cf. [Iva1]), the result follows. □

In the remainder of this section we will show that the classes $e_1^k, e_2 \in H^*(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$ are trivial. To this end we shall need the following two facts. The first is a proposition due to Morita.

Proposition 5.4.2 ([Mor1], Prop. 3.1)**.** *Let $\Sigma_h \to E \xrightarrow{\pi} B$ be any oriented surface bundle. Then the Serre spectral sequence for the fibration collapses on the second page for cohomology with real or rational coefficients. In particular, the map π^* is injective on cohomology.*

Next, in accordance with [KM1] we let $v \in H^2(Symp_\delta(\Sigma_h), \mathbb{R})$ denote the vertical symplectic class, normalised so that $v(F) = 2h - 2$ on the fibre. We further let e denote the vertical Euler class. With this notation we may state the following theorem.

Theorem 5.4.3 ([KM1], Th. 2)**.** *The projection of $e+v$ to the $E_{1,1}^\infty$-term in the Serre spectral sequence of the fibration $\Sigma_h \to ESymp_\delta(\Sigma_h) \to BSymp_\delta(\Sigma_h)$ is the cohomology class of the extended flux homomorphism*

$$[\widetilde{Flux}] \in E_{1,1}^\infty \subset H^1(Symp_\delta(\Sigma_h), H^1(\Sigma_h, \mathbb{R})).$$

We are now ready to prove the main result of this section.

Theorem 5.4.4. *The second MMM-class e_2 vanishes in $H^4(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$ for $h \geq 2$. Moreover, for all $k \geq 2$ the powers $e_1^k \in H^{2k}(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$ are also trivial.*

Proof. Let

$$\Sigma_h \to E\widetilde{Ham}_\delta(\Sigma_h) \xrightarrow{\pi} B\widetilde{Ham}_\delta(\Sigma_h)$$

be the total space of the universal bundle over $B\widetilde{Ham}_\delta(\Sigma_h)$ and let ι denote the inclusion $\widetilde{Ham}(\Sigma_h) \hookrightarrow Symp(\Sigma_h)$. Then by the naturality of the Serre spectral sequence combined with Theorem 5.4.3, we see that the projection of $\iota^*(e+v)$ to the $E_{1,1}^\infty$-term is $\iota^*[\widetilde{Flux}]$, which is trivial by the definition of $\widetilde{Ham}(\Sigma_h)$ as the kernel of the extended Flux map. Thus by Proposition 5.4.2

$$\iota^*(e+v) \in E_{2,0}^\infty = E_{2,0}^2 = \pi^* H^2(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$$

and we conclude that $\iota^*(e+v) = \pi^*\beta$ for some class $\beta \in H^2(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$. We rewrite the equation for e in $H^2(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$ given above, dropping the ι^* for notational convenience

$$e = -v + \pi^*\beta. \tag{5.8}$$

Since $v^2 = 0$, we compute that

$$e_1 = \pi_! e^2 = -2(2h-2)\beta. \tag{5.9}$$

By Bott vanishing, $v \smile e^2 = 0$ in $H^*(Symp_\delta(\Sigma_h), \mathbb{R})$ (cf. [KM2], Section 7), and thus $e^3 = (e+v)^3$. We conclude that

$$e_2 = \pi_! e^3 = \pi_!(e+v)^3 = \frac{1}{(4-4h)^3}\pi_!\pi^* e_1^3 = 0,$$

and, hence, e_2 vanishes in $H^4(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$.

On the other hand equations (5.8) and (5.9) imply that

$$e^3 = \frac{-3}{(4-4h)^2} v \smile \pi^* e_1^2 + \frac{1}{(4-4h)^3}\pi^* e_1^3.$$

Thus applying the transfer map we obtain

$$0 = e_2 = \pi_! e^3 = \frac{-3(2h-2)}{(4-4h)^2} e_1^2$$

so that $e_1^2 = 0$ in $H^4(\widetilde{Ham}_\delta(\Sigma_h), \mathbb{R})$. Finally, the fact that e_1^2 vanishes implies that e_1^k also vanishes for all $k \geq 2$. $\square$

Chapter 6

Characteristic classes of symplectic foliations

In [KM3] Kotschick and Morita defined foliated characteristic classes of transversally symplectic foliations in terms of factorisations of ordinary characteristic classes. Motivated by this we give a geometric construction for defining foliated characteristic classes as factorisations of certain Pontryagin classes. In contrast to [KM3] our construction yields foliated characteristic classes for foliations that are only transversally volume preserving rather than transversally symplectic.

For the special case of codimension 2 foliations, the factorisation of the first Pontryagin class gives a foliated characteristic class γ_1. A similar class, that we denote by γ_{KM}, was defined in [KM3] and we show that the classes γ_1 and γ_{KM} coincide under the assumption that the normal bundle to the foliation is trivial.

In general, any foliated cohomology class defines a genuine characteristic class by restricting to a leaf. We construct transversally symplectic foliations with closed leaves L such that the restriction of γ_{KM} to L is non-trivial. Moreover, these foliations have trivial normal bundles so we deduce that the classes γ_1 and γ_{KM} carry information that is not purely topological.

6.1 Factorisation of Pontryagin classes

In this section we describe factorisations of certain polynomials in the Pontryagin classes of the normal bundles of transversally volume preserving foliations, which are analogous to factorisations obtained in [KM3]. We shall use Chern-Weil theory, rather than Gelfand-Fuks cohomology as in [KM3], to obtain factorisations of polynomials of total degree $4q$ in the Pontryagin classes of the normal bundle of any codimension $2q$, transversally volume preserving foliation $\mathcal{F}$. If P denotes a polynomial of the correct degree and ω is a transverse volume form for $\mathcal{F}$, then these factorisations are of the form

$$P(\Omega) = \omega \wedge \gamma_P.$$

The main benefit of our approach is that the classes γ_P are canonically defined in terms of the foliation $\mathcal{F}$ and the form ω, whereas those given in Theorem 4 of [KM3] are not.

In order to obtain the factorisations mentioned above, we shall need an explicit description of integration along the fibre for the case of $M \times [0,1] \xrightarrow{\pi} M$. We let $\alpha \in \Omega^k(M \times [0,1])$ be a k-form, which may be uniquely written as

$$\alpha = \rho + \sigma \wedge dt,$$

where ρ has no dt component. Then integration along the fibre

$$\pi_! : \Omega^k(M \times [0,1]) \longrightarrow \Omega^{k-1}(M)$$

is given explicitly via the following formula

$$\pi_! \alpha = \int_0^1 \sigma \, dt.$$

Furthermore, we have the following lemma.

Lemma 6.1.1. *Let $\alpha \in \Omega^k(M \times [0,1])$ be a k-form and let ι_0, ι_1 be the inclusions of $M \times \{0\}$ resp. $M \times \{1\}$ in $M \times [0,1]$. Then the following relation holds*

$$\pi_! \, d\alpha - d \, \pi_! \alpha = \iota_1^* \alpha - \iota_0^* \alpha.$$

Proof. We write

$$\alpha = \rho + \sigma \wedge dt.$$

If d_M denotes the exterior derivative on M, then we have

$$d_M \pi_! \alpha = d_M \int_0^1 \sigma \, dt = \int_0^1 d_M \sigma \, dt.$$

Moreover

$$d\alpha = d\rho + d(\sigma \wedge dt) = d_M \rho + \frac{\partial \rho}{\partial t} \wedge dt + d_M \sigma \wedge dt$$

and thus

$$\begin{aligned} \pi_! \, d\alpha - d \, \pi_! \alpha &= \int_0^1 (\frac{\partial \rho}{\partial t} + d_M \sigma) \, dt - \int_0^1 d_M \sigma \, dt \\ &= \int_0^1 \frac{\partial \rho}{\partial t} \, dt \\ &= \iota_1^* \alpha - \iota_0^* \alpha, \end{aligned}$$

where we have used the Fundamental Theorem of Calculus to obtain the final equality. □

We shall next recall the definition of foliated cohomology. To this end we let $I^*(\mathcal{F})$ denote the ideal of forms that vanish on $\mathcal{F}$. The Frobenius Theorem implies that $I^*(\mathcal{F})$ is closed under the exterior differential and, thus, d descends to a differential $d_{\mathcal{F}}$ on the quotient complex $\Omega^*(M)/I^*(\mathcal{F})$. We define the foliated cohomology as the cohomology of this quotient complex:

$$H^*_{\mathcal{F}}(M) = H^*(\Omega^*(M)/I^*(\mathcal{F}), d_{\mathcal{F}}).$$

Another important notion is that of a Bott connection on a foliated manifold.

Definition 6.1.2. Let $\mathcal{F}$ be a foliation on a manifold M. A *Bott connection* on the normal bundle $\nu_{\mathcal{F}} = TM/T\mathcal{F}$ is a connection ∇ such that for $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(\nu_{\mathcal{F}})$

$$\nabla_X Y = [X, \tilde{Y}],$$

where $\tilde{Y}$ denotes any lift of Y to TM.

The most important properties of Bott connections are that they are flat when restricted to the leaves of $\mathcal{F}$ (cf. [Bott]) and that they are canonically defined along any leaf by the formula in Definition 6.1.2. Conversely, to define a Bott connection for a given foliation one chooses a splitting

$$TM \cong T\mathcal{F} \oplus \nu_{\mathcal{F}}.$$

If $X = X_{\mathcal{F}} + X_\nu$ is the decomposition of a vector X with respect to this splitting, then after choosing a connection $\overline{\nabla}$ on $\nu_{\mathcal{F}}$, one may define a Bott connection ∇ as follows:

$$\nabla_X Y = [X_{\mathcal{F}}, \tilde{Y}] + \overline{\nabla}_{X_\nu} Y.$$

There is an alternate formulation of the Bott condition in terms of connection matrices. For this we let $S_1, ..., S_q$ be a local frame for $\nu_{\mathcal{F}}$ and choose lifts $\tilde{S}_1, ..., \tilde{S}_q$ to TM. We let θ_{ij} denote the connection matrix of ∇ with respect to the frame $S_1, ..., S_q$ so that

$$\nabla S_i = \sum_{j=1}^{q} \theta_{ij} \otimes S_j.$$

We further let $\theta_1, ..., \theta_q$ be a dual basis for $S_1, ..., S_q$, which means that θ_i vanishes on $T\mathcal{F}$ and

$$\theta_i(\tilde{S}_j) = \delta_{ij}.$$

To check the Bott condition it suffices to verify that the following holds for all $X \in T\mathcal{F}$ and all S_i:

$$\nabla_X S_i = [X, \tilde{S}_i],$$

or equivalently that

$$\theta_j(\nabla_X S_i) = \theta_{ij}(X) = \theta_j([X, \tilde{S}_i])$$

for all $1 \leq i, j \leq q$. We compute that

$$\begin{aligned} d\theta_j(X, \tilde{S}_i) &= L_X \theta_j(\tilde{S}_i) - L_{\tilde{S}_i} \theta_j(X) - \theta_j([X, \tilde{S}_i]) \\ &= L_X \delta_{ji} - 0 - \theta_j([X, \tilde{S}_i]) = -\theta_j([X, \tilde{S}_i]) \end{aligned}$$

and conclude that ∇ is a Bott connection if and only the connection 1-forms satisfy

$$d\theta_j(X, \tilde{S}_i) = -\theta_{ij}(X).$$

In particular, if $\theta_1, ..., \theta_q$ is any local basis for $I^1(\mathcal{F})$, then by the Frobenius Theorem there exist 1-forms θ_{ij} such that

$$d\theta_j = \sum_{i=1}^{q} \theta_i \wedge \theta_{ij}. \tag{6.1}$$

The matrix of 1-forms (θ_{ij}) then defines a local Bott connection. This description of Bott connections will be useful in our discussion of Gelfand-Fuks cohomology in Section 6.2 below.

We now come the main result of this section.

Proposition 6.1.3. *Let $\mathcal{F}$ be a transversally volume preserving foliation of codimension $2q$ on a manifold M with defining form ω and let P be any polynomial of total degree $4q$ in the Pontryagin classes of the normal bundle. Then there is a factorisation*

$$P(\Omega) = \omega \wedge \gamma_P$$

for a well-defined foliated class $\gamma_P \in H^{2q}_{\mathcal{F}}(M)$.

Proof. Let ∇ be a Bott connection on the normal bundle $\nu_{\mathcal{F}}$ of $\mathcal{F}$. Since a Bott connection is flat along leaves, the components Ω_{ij} of the curvature matrix vanish on $\mathcal{F}$. We choose a local basis $\theta_1, ..., \theta_{2q}$ of $I^1(\mathcal{F})$ such that

$$\theta_1 \wedge \theta_2 ... \wedge \theta_{2q} = \omega.$$

With respect to this basis the curvature forms $\Omega_{ij} \in I^2(\mathcal{F})$ can locally be expressed as

$$\Omega_{ij} = \sum_{k=1}^{2q} \theta_k \wedge \alpha_k.$$

Since the Chern-Weil representative for P is given by a symmetric polynomial of degree $2q$ in the entries of Ω, the following holds locally:

$$P(\Omega) = \theta_1 \wedge \theta_2 ... \wedge \theta_{2q} \wedge \gamma_P = \omega \wedge \gamma_P.$$

Let $Ann(\omega)$ be the subbundle of $2q$-forms annihilated by ω and let $Ann(\omega)^\perp \subset \Lambda^{2q}(M)$ be a complement to $Ann(\omega)$. Then on the level of forms the equation

$$P(\Omega) = \omega \wedge \gamma_P$$

has a unique global solution $\gamma_P \in \Gamma(Ann(\omega)^\perp)$. The form γ_P is well-defined modulo elements in $\Gamma(Ann(\omega)) = I^{2q}(\mathcal{F})$, so we obtain a class $[\gamma_P] \in \Omega^{2q}(M)/I^{2q}(\mathcal{F})$. Next, since ω and $P(\Omega)$ are closed we compute

$$0 = d(\omega \wedge \gamma) = \omega \wedge d\gamma_P$$

and $d\gamma_P \in I^*(\mathcal{F})$. Thus we have a well-defined class $[\gamma_P] \in H^{2q}_{\mathcal{F}}(M)$.

We finally need to show that the class we obtain in foliated cohomology does not depend on the choice of Bott connection. Let ∇^0, ∇^1 be two Bott connections on $\nu_{\mathcal{F}}$ and let π denote the projection $M \times [0,1] \to M$. We then define a connection on $\pi^*\nu_{\mathcal{F}}$ by setting

$$\nabla = t\,\pi^*\nabla^1 + (1-t)\pi^*\nabla^0.$$

This connection is then a Bott connection for the foliation $\pi^*\mathcal{F}$ that is obtained as the preimage of $\mathcal{F}$ under the projection π. Now $\pi^*\omega$ is a defining form for $\pi^*\mathcal{F}$ and, as above, after the choice of a splitting $\Lambda^{2q}(M \times [0,1]) \cong Ann(\pi^*\omega) \oplus Ann(\pi^*\omega)^\perp$, there is a unique form $\gamma_P \in \Gamma(Ann(\pi^*\omega)^\perp)$ so that

$$P(\Omega) = \pi^*\omega \wedge \gamma_P.$$

Since the form $P(\Omega)$ is closed, Lemma 6.1.1 yields

$$\begin{aligned} -d\,\pi_! P(\Omega) &= \iota_1^* P(\Omega) - \iota_0^* P(\Omega) \\ &= P(\Omega^1) - P(\Omega^0). \end{aligned}$$

Hence, one has

$$\omega \wedge (-d\,\pi_! \gamma_P) = \omega \wedge \gamma_P^1 - \omega \wedge \gamma_P^0$$

or equivalently

$$\gamma_P^1 - \gamma_P^0 \cong -d(\pi_! \gamma_P) \bmod I^*(\mathcal{F}).$$

Thus $[\gamma_P^1] = [\gamma_P^0]$ as classes in $H_{\mathcal{F}}^{2q}(M)$. □

If one assumes further that the foliation $\mathcal{F}$ is transversally symplectic with defining form ω, then one obtains a similar factorisation for any polynomial of the form $\omega^k P(\Omega)$, where P is a polynomial in the Pontryagin classes of total degree $4q - 2k$. We note this in the following proposition, whose proof is almost identical to that of Proposition 6.1.3.

Proposition 6.1.4. *Let $\mathcal{F}$ be a transversally symplectic foliation of codimension $2q$ on a manifold M with defining form ω and let P be any polynomial of total degree $4q - 2k$ in the Pontryagin classes of the normal bundle. Then there is a factorisation*

$$\omega^k P(\Omega) = \omega^{2q} \wedge \gamma_P$$

for a well-defined foliated class $\gamma_P \in H_{\mathcal{F}}^{2q}(M)$.

For certain polynomials it is easy to show that the class $P(\Omega) = \omega^{2q} \wedge \gamma_P$ is non-trivial and, hence, that the class γ_P is non-trivial in foliated cohomology. This was shown in [KM3] for polynomials of the form p_1^q. A similar argument covers the other extreme case when $P = p_q$ is the q-th Pontryagin class. For simplicity we write γ_q for the corresponding foliated cohomology class.

Proposition 6.1.5. *There exist foliations $\mathcal{F}_q$ for which the classes $\gamma_q \in H_{\mathcal{F}}^{2q}(M)$ do not vanish.*

Proof. We let $\mathcal{F}$ be a codimension 2, transversally symplectic foliation on a 4-manifold such that $p_1(\nu_{\mathcal{F}})$ is non-trivial (cf. [KM1]). We define $\mathcal{F}_q$ as the product foliation on the q-fold product M^q induced by $\mathcal{F}$ and by π_i the i-th projection. These foliations are transversally volume preserving, they are even transversally symplectic. Finally, using the Whitney sum formula and the naturality of Pontryagin classes we obtain:

$$p_q(\nu_{\mathcal{F}_q}) = p_q\Big(\bigoplus_{i=1}^{q} \pi_i^* \nu_{\mathcal{F}}\Big) = \prod_{i=1}^{q} \pi_i^* p_1(\nu_{\mathcal{F}}) \neq 0.$$ □

Although Proposition 6.1.5 shows that γ_q is non-trivial, we cannot conclude that the classes γ_q and $\gamma_{p_1^q}$ are linearly independent.

6.2 Gelfand-Fuks cohomology

In this section we clarify the relationship between the foliated class $\gamma_1 = \gamma_{p_1}$ given by Proposition 6.1.3 and the Kotschick-Morita class γ_{KM} defined in [KM3]. In particular, we show that both classes agree under the assumption that the normal bundle of $\mathcal{F}$ is trivial.

We begin by recalling the construction of Gelfand-Fuks cohomology for the Lie algebra of formal Hamiltonian vector fields (cf. [GKF], [KM3]). We let $H^{2n}_{\mathbb{R}}$ denote the standard $Sp(2n)$-representation and consider the Lie algebra of polynomials on $H^{2n}_{\mathbb{R}}$ with trivial constant term endowed with the Poisson bracket

$$\mathbb{R}[X_1, ..., X_n, Y_1, ..., Y_n]/\mathbb{R} = \oplus_{k=1}^{\infty} Sym^k H^{2n}_{\mathbb{R}}.$$

The completion of this Lie algebra is the ring of formal power series with the induced Poisson bracket, which we denote by $\mathfrak{ham}_{2n}$. The Gelfand-Fuks cochain complex is then

$$C^*_{GF}(\mathfrak{ham}_{2n}) = \oplus_{k=1}^{\infty} \Lambda^* Sym^k(H^{2n}_{\mathbb{R}}),$$

where the differential is defined in terms of the Poisson bracket in the usual fashion. For any subalgebra $\mathfrak{g}$, the relative complex is by definition the complex of $\mathfrak{g}$-basic forms.

The Lie algebra $\mathfrak{ham}_{2n}$ has a natural filtration

$$\mathfrak{ham}_{2n} \supset \mathfrak{ham}^0_{2n} \supset \mathfrak{ham}^1_{2n} ... \supset \mathfrak{ham}^k_{2n} \supset ... ,$$

where $\mathfrak{ham}^k_{2n}$ denotes those power series that are trivial up to order $k+1$. The subalgebras $\mathfrak{ham}^k_{2n}$ are actually ideals in $\mathfrak{ham}^0_{2n}$ and the quotient $\mathfrak{g}^k_{ham} = \mathfrak{ham}^0_{2n}/\mathfrak{ham}^k_{2n}$ is the Lie algebra of the group of k-jets of Hamiltonian maps that fix 0, which will be denoted by J^k_{ham}. Moreover, the Lie algebra $\mathfrak{sp}_{2n}$ embeds naturally in $\mathfrak{ham}_{2n}$.

For any manifold M with a transversally symplectic foliation of codimension $2n$ there is a natural map $H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{u}_n) \to H^*(M)$. We shall recall the construction of this map following Bott and Haefliger (see [BH]). We let Γ^{ham}_{2n} denote the Lie pseudogroup of Hamiltonian diffeomorphisms of open sets in $\mathbb{R}^{2n}$. We define $P^k_{ham}(\mathcal{F})$ to be the principal J^k_{ham}-bundle whose fibre at x consists of the k-jets of local $\mathcal{F}$-projections that preserve the transverse structure and map x to 0. We further let $P^k(\Gamma^{ham}_{2n})$ denote the principal J^k_{ham}-bundle of k-jets of elements in Γ^{ham}_{2n} at 0 and note that the pseudogroup itself acts transitively on the left of this bundle.

An element γ in $H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{u}_n)$ determines a Γ^{ham}_{2n}-equivariant differential form Φ^{loc}_{γ} on $P^k(\Gamma^{ham}_{2n})$ for some sufficiently large k. Since the bundle $P^k_{ham}(\mathcal{F})$ is locally the pullback of the bundle $P^k(\Gamma^{ham}_{2n})$ under an $\mathcal{F}$-projection and the form Φ^{loc}_{γ} is Γ^{ham}_{2n}-equivariant, the pullbacks of these local forms glue together to give a well-defined form Φ_{γ} on the total space. The form Φ_{γ} may also be defined in terms of the tautological 1-forms, which we discuss in detail in the next paragraph. If γ was $\mathfrak{u}_n$-basic, then the differential form Φ_{γ} descends to the quotient $P^k_{ham}(\mathcal{F})/U(n)$. This is then a bundle with contractible fibre and, hence, we obtain a class in $H^*(M) \cong H^*_{dR}(P^k_{ham}(\mathcal{F})/U(n))$, where the isomorphism is induced by the bundle projection.

An analogue of the map $H^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{u}_n) \to H^*(M)$ described above can also be defined for the Gelfand-Fuks cohomology of the Lie algebra of formal smooth vector fields $\mathfrak{a}_{2n}$, in which case one obtains a map $H^*_{GF}(\mathfrak{a}_{2n}, \mathfrak{o}_{2n}) \to H^*(M)$. The definition of this map uses the principal bundles $P^k(\mathcal{F})$ of k-jets of $\mathcal{F}$-projections and is formally identical to the construction described in the previous paragraph. Moreover, there is a collection of $(2n)^k$ tautological 1-forms $\delta^{i_1}_{i_2 \dots i_k}$ on $P^k(\mathcal{F})$ (cf. [Pit], [Bott3]). These forms have an important interpretation for small k. In particular, the forms δ^i define a trivialisation of the bundle of 1-forms that vanish on the foliation $\mathcal{F}_1$ which is obtained as the preimage of $\mathcal{F}$ under the bundle projection $P(\mathcal{F}) \to M$. They also satisfy the following equation on $P^2(\mathcal{F})$

$$d\,\delta^j = \sum_{i=1}^{2n} \delta^i \wedge \delta^i_j,$$

where by abuse of notation we again denote by δ^i the pullbacks of these forms under the map $P^2(\mathcal{F}) \to P(\mathcal{F})$. Thus, we see that the matrix (δ^i_j) defines a (universal) Bott connection for the foliation $\mathcal{F}_2$ on $P^2(\mathcal{F})$ (cf. equation (6.1) above). By taking pullbacks the same holds on $P^k(\mathcal{F})$ with respect to the pullback foliation $\mathcal{F}_k$ and the curvature forms for this universal Bott connection are by definition

$$\Omega^i_j = d\,\delta^i_j - \sum_{k=1}^{2n} \delta^i_k \wedge \delta^k_j.$$

There is also a natural map from the Gelfand-Fuks group $H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}_{2n})$ to the foliated cohomology of M. In order to define this we let γ be a representative of a given class and consider the pullback of γ to the chain complex $C^*_{GF}(\mathfrak{ham}_{2n}, \mathfrak{sp}_{2n})$ induced by the projection $\mathfrak{ham}_{2n} \xrightarrow{p} \mathfrak{ham}^0_{2n}$, whose kernel consists of linear elements $H^{2n}_{\mathbb{R}}$. We note that this map is *not* a map of Lie algebras. However, if we begin with an $\mathfrak{sp}_{2n}$-basic element, then $p^*\gamma$ is closed modulo elements in the differential ideal $I((H^{2n}_{\mathbb{R}})^*)$ generated by linear elements. Thus, using the construction described above, we obtain a differential form Φ_γ on $P^k_{ham}(\mathcal{F})$. Moreover, the form Φ_γ is closed modulo elements in the differential ideal generated by the tautological 1-forms δ^i. Such an element then descends to a form $\hat{\Phi}_\gamma$ on $\hat{P}^k_{ham}(\mathcal{F}) = P^k_{ham}(\mathcal{F})/U(n)$, which is a bundle over M with contractible fibre. Moreover, $\hat{\Phi}_\gamma$ is an element in the foliated cohomology of the pullback foliation $H^k_{\hat{\pi}^*_k\mathcal{F}}(\hat{P}^k_{ham}(\mathcal{F}))$ and the projection induces an isomorphism to $H^k_{\mathcal{F}}(M)$, thus giving the desired element in the foliated cohomology of M.

Now the class γ_{KM} defines a factorisation of the first Pontryagin class in Gelfand-Fuks cohomology by ([KM3], Theorem 4). Using the map described in the previous paragraph, this decomposition then becomes a factorisation of the first Pontryagin class of the universal Bott connection on $P^3_{ham}(\mathcal{F})$. More specifically, if ω is a closed defining form for $\mathcal{F}$ and π_3 is the bundle projection of $P^3_{ham}(\mathcal{F})$, then $\pi_3^*\omega$ is a defining form for $\mathcal{F}_3 = \pi_3^*\mathcal{F}$ and

$$p_1(\Omega) = \gamma_{KM}(\mathcal{F}_3) \wedge \pi_3^*\,\omega.$$

Now supposing that the bundle $P^3_{ham}(\mathcal{F}) \to M$ has a section s, then by naturality the matrix $(s^*\delta^i_j)$ is the connection matrix of a global Bott connection for $\mathcal{F} = s^*(\mathcal{F}_3)$. Moreover,

$$p_1(s^*\Omega) = s^*p_1(\Omega) = s^*\gamma_{KM}(\mathcal{F}_3) \wedge s^*\pi_3^*\,\omega = \gamma_{KM}(s^*(\mathcal{F}_3)) \wedge \omega = \gamma_{KM}(\mathcal{F}) \wedge \omega,$$

and it follows that $\gamma_{KM}(\mathcal{F}) = \gamma_1(\mathcal{F})$ in $H^*_{\mathcal{F}}(M)$ if the normal bundle of $\mathcal{F}$ is trivial. We note this in the following proposition.

Proposition 6.2.1. *Let $\mathcal{F}$ be a transversally volume preserving foliation of codimension* 2 *and assume that its normal bundle is trivial. Then $\gamma_{KM}(\mathcal{F}) = \gamma_1(\mathcal{F})$.*

6.3 Characteristic classes of leaves

In Section 6.1 we defined characteristic classes of transversally volume preserving foliations in foliated cohomology. The restriction of such a class to a leaf of a foliation defines a cohomology class in ordinary cohomology and, thus, determines a characteristic class of the individual leaves. By using results on normal forms for germs of Hamiltonian diffeomorphisms we will show that there exist symplectically foliated $\mathbb{R}^2$-bundles with holonomy in the group $Symp^k(\mathbb{R}^2, 0)$ of C^k-symplectomorphisms that fix the origin, for which the Kotschick-Morita class γ_{KM} restricts non-trivially on the central leaf that corresponds to the origin. Moreover, these bundles may be chosen to be topological trivial, which means that the same holds for γ_1 by Proposition 6.2.1. These examples show that the classes γ_1 and γ_{KM} carry information that is sensitive to the geometry of the foliation and not just to the homotopy class of the underlying distribution as is the case for p_1.

We let $P^k_{ham}(\mathcal{F})$ be the principal J^k_{ham}-bundle of k-jets of local $\mathcal{F}$-projections that preserve the transverse symplectic structure. Now if L is the leaf of a foliation, then the restriction of the bundle $P^k_{ham}(\mathcal{F})$ to L is a flat principal J^k_{ham}-bundle, which is then determined by its holonomy representation $\pi_1(L) \to J^k_{ham}$. An analogous construction to that given in Section 6.2, associates to any class in $H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{u}_n)$, a class in $H^*(L)$. Moreover, by composing with the natural map

$$H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}_{2n}) \to H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{u}_n)$$

one obtains the following commutative diagram where the rightmost arrow is given by restriction

$$\begin{tikzcd}
H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}_{2n}) \arrow[r] \arrow[d] & H^*_{\mathcal{F}}(M) \arrow[d] \\
H^*_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{u}_n) \arrow[r] & H^*(L).
\end{tikzcd}$$

We remark that these observations still hold for the cohomology of the truncated Lie algebra $\mathfrak{g}^{k-1}_{ham} = \mathfrak{ham}^0_{2n}/\mathfrak{ham}^{k-1}_{2n}$, in the case that the foliation $\mathcal{F}$ is only of class C^k.

In general, any element in the relative Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{k})$ of a pair (G, K) defines a characteristic class of flat principal G-bundles via the construction described above, provided that G/K is contractible. In the context of flat G-bundles this is often referred to as the *Reinhard construction*. It is clear that these classes are natural with respect to pullbacks under both maps between manifolds and between pairs of Lie groups. Furthermore, Reinhard's construction may be used to define elements in the group cohomology of G considered as a discrete group. For in order to define an element $\phi \in H^k(G_\delta, \mathbb{R})$ it suffices to define how ϕ evaluates on integral homology classes. For homology classes that are representable by (singular) manifolds we may use the Reinhard construction directly. Moreover, any two singular manifolds that are homologous are cobordant after possibly taking multiples. Hence, by Stokes' Theorem the map ϕ is well-defined on the subgroup $H^k_{man} \subset H^k(G_\delta)$ of elements that are representable by manifolds. It then follows from Thom's Representability

Theorem that the quotient of these two groups consists of torsion and, thus, any homomorphism $H^k_{man} \to \mathbb{R}$ extends uniquely to a homomorphism $H_k(G_\delta) \to \mathbb{R}$ yielding a well-defined element in $H^k(G_\delta, \mathbb{R})$.

We will restrict ourselves to the special case $G = J^k_{ham}$. In particular, we consider the class $\gamma = \gamma_{KM}$ in $H^2_{GF}(\mathfrak{ham}^0_{2n}, \mathfrak{sp}_2)$ that was defined in [KM3] via the explicit cocycle representative

$$(X^3 \wedge Y^3 - 3X^2Y \wedge XY^2)^*.$$

This cocycle is the pullback of the corresponding element γ_{jet} in the Lie algebra chain group $C^2(\mathfrak{g}^2_{ham}, \mathfrak{sp}_2)$. Via the Reinhard construction the class γ_{jet} defines a cohomology class in $H^2((J^2_{ham})_\delta, \mathbb{R})$, which will also be denoted by γ_{jet}, and we claim that γ_{jet} is non-trivial. In fact, the restriction of γ_{jet} to the subgroup $J^2_{ham,Id} \subset J^2_{ham}$ is non-trivial, where $J^2_{ham,Id}$ is the subgroup of 2-jets with linear part equal to the identity. To see this we take the trivial bundle

$$E = J^2_{ham,Id} \times T^2 \to T^2$$

and let $\tilde{E} \to \mathbb{R}^2$ be its universal cover. Since $J^2_{ham,Id}$ is a simply connected abelian Lie group, it is isomorphic to its Lie algebra $Sym^3 H^2_{\mathbb{R}}$. We let $x_1, ..., x_4$ be coordinates on the fibre with respect to the standard basis of $Sym^3 H^2_{\mathbb{R}}$ and choose coordinates y_1, y_2 on $\mathbb{R}^2$. We then consider the horizontal foliation on $\tilde{E}$ given by the projection

$$(x_1, ..., x_4, y_1, y_2) \to (x_1 - y_1, x_2, x_3, x_4 - y_2).$$

This foliation descends to E and by inspecting the definition of γ_{jet}, or more precisely its restriction to $Sym^3 H^2_{\mathbb{R}}$, it is clear that this class evaluates non-trivially on the (foliated) principal bundle E.

Remark 6.3.1. We note that γ_{jet} and the pullback of the Euler class under the map $J^2_{ham} \to J^1_{ham} = SL(2, \mathbb{R})$ are linearly independent, since the restriction of the Euler class to $J^2_{ham,Id}$ is trivial.

We will further show that the image of γ_{jet} in $H^2((J^m_{ham})_\delta, \mathbb{R})$ is non-trivial for arbitrary finite m, which is an immediate consequence of the following lemma.

Proposition 6.3.2. *The natural map $J^m_{ham} \to J^2_{ham}$ induces an injection on second real cohomology for all $m \geq 2$.*

Proof. We first note that the map factors as

$$J^m_{ham} \to J^{m-1}_{ham} \to ... \to J^2_{ham}.$$

Thus it suffices to show that the map $J^{m-1}_{ham} \to J^{m-2}_{ham}$ induces an injection on cohomology for all $m \geq 4$. We note that the kernel of this map is a simply connected abelian Lie group of rank $m+1$, which we denote by K_{m-1}. As such it is isomorphic to its Lie algebra, which in this case is $Sym^m H^2_{\mathbb{R}}$ with the trivial Lie bracket and an isomorphism is induced by the exponential map. Restricted to the subalgebra $Sym^m H^2_{\mathbb{R}}$ the exponential map is given by considering the $(m-1)$-jet at the origin of the Hamiltonian flow generated by an element in $Sym^m H^2_{\mathbb{R}}$.

By the Universal Coefficient Theorem, injectivity in real cohomology, will follow from the surjectivity of the induced map in integral *homology*. By considering the five-term exact sequence associated to the extension

$$1 \to K_{m-1} \to J^{m-1}_{ham} \to J^{m-2}_{ham} \to 1,$$

it will suffice to show that the group of coinvariants $H_1((K_{m-1})_\delta)_{J^{m-2}_{ham}}$ is trivial.

First recall that if Φ^t_H is the Hamiltonian flow generated by H, then for any symplectomorphism ψ the flow $\psi\,\Phi^t_H\psi^{-1}$ is the Hamiltonian flow generated by $(\psi^{-1})^*H$. Thus under our identification of K_{m-1} with its Lie algebra, conjugation by an element $\psi \in J^{m-1}_{ham}$ corresponds to precomposition in the Lie algebra of K_{m-1} by ψ^{-1}, where K_{m-1} has been identified with the additive group of homogeneous polynomials of degree m. Now if we let

$$u(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

then for any monomial $p(X,Y) = X^kY^{m-k}$, we compute

$$p - p \circ u^{-1}(\lambda) = (1 - \lambda^{m-2k})X^kY^{m-k}.$$

If $m - 2k \neq 0$, we may choose $\lambda > 0$ such that

$$1 - \lambda^{m-2k} = \frac{1}{2}.$$

Thus we conclude that $\frac{1}{2}p(X,Y)$ is trivial in $H_1((K_{m-1})_\delta)_{J^{m-2}_{ham}}$ and, hence, the same holds for $p(X,Y)$.

If $2k = m$, let

$$\rho(\theta) = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}$$

denote the rotation by θ. The coefficient of the X^mY^m-term in $p \circ \rho(\theta)$ is

$$\begin{aligned} c_{m,m}(\theta) &= \sum_{l+q=m} (-1)^{m-l}\binom{m}{l}\binom{m}{q} cos^l(\theta) sin^{m-l}(\theta) sin^q(\theta) cos^{m-q}(\theta) \\ &= \sum_{l+q=m} (-1)^q \binom{m}{l}\binom{m}{q} cos^{2l}(\theta)(1 - cos^2(\theta))^q. \end{aligned}$$

This is a polynomial of degree $2m$ in $cos(\theta)$, whose leading term has coefficient

$$\sum_{l+q=m} (-1)^{2q}\binom{m}{l}\binom{m}{q} = \sum_{l+q=m} \binom{m}{q}^2 \neq 0.$$

Since $c_{m,m}(0) = 1$, there is some θ_0 so that $c_{m,m}(\theta_0) = 1 \pm \epsilon$ for any sufficiently small ϵ. Hence, the coefficient of the X^mY^m-term in $p - p \circ \rho(\theta_0)$ is $\pm\epsilon$. Combining this with the case $m \neq 2k$ considered above, we conclude that $\pm\epsilon X^mY^m$ is trivial in $H_1((K_{m-1})_\delta)_{J^{m-2}_{ham}}$. Since the group generated by such elements includes all monomials aX^mY^m, the group $H_1((K_{m-1})_\delta)_{J^{m-2}_{ham}}$ vanishes and the desired claim follows by the exactness of the five-term sequence. $\square$

In order to extend our results from jets to actual germs we shall need a normal form theorem for Hamiltonian germs. This is provided by a result of Banyaga, de la Llave and Wayne in [BLW]. In order to state their result we need to define several constants.

Let M be a real linear map and assume that there are constants

$$0 < \lambda_-^{-1} < \lambda_+ < 1 < \mu_-^{-1} < \mu_+$$

so that the (complex) eigenvalues λ of M satisfy

$$\lambda_-^{-1} \leq |\lambda| \leq \lambda_+ \text{ or } \mu_-^{-1} \leq |\lambda| \leq \mu_+.$$

We shall call such a a linear map *hyperbolic*. For any such constants we define

$$A = \frac{|\ln\lambda_+|}{(\ln\mu_+ + |\ln\lambda_+|)} \frac{|\ln\mu_-|}{(\ln\lambda_- + |\ln\mu_-|)}$$

and

$$B = 1 - 2A.$$

With these definitions we may now state the following theorem that gives hypotheses under which a Hamiltonian diffeomorphism germ of class C^r can be linearised.

Theorem 6.3.3 ([BLW], Th. 1.1)**.** *Let ϕ be the germ of a C^r-Hamiltonian diffeomorphism fixing the origin, whose $(k-2)$-jet is linear and hyperbolic. Then there exists a C^l-Hamiltonian germ ψ with $D_0\psi = Id$ so that*

$$\psi^{-1}\phi\psi = D_0\phi,$$

provided that $1 \leq l \leq kA - B$ for a suitable choice of $\lambda_\pm, \mu_\pm$ and $r > 2k+4$.

This is the main tool to show that the class γ induces a non-trivial class in the cohomology of Hamiltonian germs of class C^k at the origin.

Theorem 6.3.4. *The class γ is non-trivial in $H^2(\mathcal{G}^k_{ham})$ for all $2 \leq k < \infty$.*

Proof. By Proposition 6.3.2 the image of γ_{jet} in $H^2((J^m_{ham})_\delta, \mathbb{R})$ is non-trivial for all m. We let σ be an integral homology class in $H_2((J^m_{ham})_\delta)$ that pairs non-trivially with γ_{jet}. Such a class is equivalent to a representation of some surface group $\pi_1(\Sigma_g) \to J^m_{ham}$. We let a_i, b_i be standard generators for $\pi_1(\Sigma_g)$ and let α_i, β_i denote their images in J^m_{ham}. Then if we consider the m-jets α_i, β_i as (smooth) germs in the natural way, we see that the m-jet of

$$\phi = \prod_{i=1}^{g} [\alpha_i, \beta_i]$$

is the identity. Thus after multiplying ϕ with a hyperbolic element $M \in SL(2, \mathbb{R})$, Theorem 6.3.3 implies that we may linearise the resulting germ by a Hamiltonian diffeomorphism germ ψ of class C^l. Since the value of l grows linearly with m, after taking m large enough we may assume that $l = k$ and, hence, the following holds in $\mathcal{G}^k_{ham}$

$$\psi \prod_{i=1}^{g} [\alpha_i, \beta_i] M \psi^{-1} = M.$$

In other words

$$\prod_{i=1}^{g}[\psi\alpha_i\psi^{-1},\psi\beta_i\psi^{-1}][\psi,M]=1.$$

The image of the class in $H_2((J^k_{ham})_\delta)$ that is associated to the representation of the surface group $\pi_1(\Sigma_{g+1})$ given by the above relation decomposes as $\sigma+\tau_{\psi,M}$, where σ is our original class and $\tau_{\psi,M}$ is the class associated to the representation of the fundamental group of the 2-torus defined by the k-jets of the elements ψ and M. We note that γ_{jet} depends only on 2-jets and we have the following split exact sequence given by taking a 2-jet to its linear part

$$1\to J^2_{ham,Id}\to J^2_{ham}\to SL(2,\mathbb{R})\to 1.$$

Thus any element $\psi\in J^2_{ham}$ can be written uniquely as $\psi=\psi_{Id}\circ D_0\psi$, where the linear part of ψ_{Id} is the identity. As in the proof of Proposition 6.3.2, we may identify $J^2_{ham,Id}$ with $Sym^3H^2_{\mathbb{R}}$ via the exponential map.

Now after a linear change of coordinates, we may assume that the hyperbolic element M is diagonal and thus M commutes with an element ψ if and only if it commutes with ψ_{Id} and $D_0\psi$. Since $J^2_{ham,Id}$ is just the group of homogeneous polynomials of degree 3, the calculation of Proposition 6.3.2 shows that if ψ and M commute, then ψ_{Id} must be trivial. Furthermore, since the linear part of ψ can be chosen to be the identity, Proposition 6.3.3 implies that the 2-jet of ψ is trivial and, hence, that the homology class $\tau_{\psi,M}$ is trivial in $H_2((J^2_{ham})_\delta)$. Thus we have

$$\gamma(\sigma')=\gamma_{jet}(\sigma+\tau_{\psi,M})=\gamma_{jet}(\sigma)\neq 0$$

and the claim follows. □

The final step is to extend classes given by germs to classes defined by actual C^k-symplectomorphisms. This extension will rely on the following lemma, which mimics the original proof of Mather (cf. [Math1], [LeR]) that the group of compactly support homeomorphisms is perfect.

Lemma 6.3.5. *Let ϕ be an element in the group $Ham^k_c(\mathbb{R}^{2n}\setminus\{0\})$ that consists of Hamiltonian diffeomorphisms of class C^k with compact support. Then ϕ can be written as the product of commutators of elements in $Symp^k(\mathbb{R}^{2n}\setminus\{0\})$, whose supports are disjoint from* 0.

Proof. First of all since ϕ can be connected to the identity by a path of Hamiltonian diffeomorphisms with compact support, the standard fragmentation argument given in the smooth case applies and we may assume that ϕ is a product of elements with support in a ball (cf. [Ban], p. 110). Thus it suffices to consider the case where ϕ has support in a ball B. Now we let B_i be disjoint translates of B and let C_i be larger balls with the property that C_i contains both B_i and B_{i+1} and C_i is disjoint from C_{i+2}. We further assume that the C_i are chosen so that there is a symplectomorphism h_i with support in C_i that interchanges B_i and B_{i+1}.

We then inductively define $g_1 = \phi$ and $g_{i+1} = h_i g_i h_i^{-1}$. The elements g_i commute as they have disjoint supports and the same holds for h_i, h_j and g_i, h_j, provided that $|i - j| \geq 2$. We may then define the following infinite products

$$A = g_2 g_4 g_6 ..., \quad B = h_2 h_4 h_6 ..., \quad C = g_1 g_3 g_5 ..., \quad D = h_1 h_3 h_5 ...$$

and one computes that

$$\phi = [A, B][C, D].$$

Finally, by choosing the balls B_i appropriately one may assume that the supports of A, B, C and D are disjoint from 0. □

We will also need an analogue of Theorem 4.1.14, which in essence gives an explicit description of the Euler class in $H^2(\mathcal{G}_p)$. First of all we note that since the inclusion $Symp_c^k(\mathbb{R}^2 \setminus \{0\})$ in $Diff_c^k(\mathbb{R}^2 \setminus \{0\})$ is a weak homotopy equivalence, the group of compactly supported symplectomorphisms of $\mathbb{R}^2 \setminus \{0\}$ modulo isotopy is isomorphic to the compactly supported mapping class $MCG^c(\mathbb{R}^2 \setminus \{0\})$, which is in turn $\mathbb{Z}$. With the aid of this observation we prove the following proposition.

Proposition 6.3.6. *Let α be the natural homomorphism from $Symp_c^k(\mathbb{R}^2 \setminus \{0\})$ to the compactly supported mapping class group of $\mathbb{R}^2 \setminus \{0\}$ and consider the following extension of groups*

$$1 \to Symp_c^k(\mathbb{R}^2 \setminus \{0\}) \to Symp_c^k(\mathbb{R}^2, 0) \to \mathcal{G}_{ham}^k \to 1.$$

Then the Euler class in $H^2(\mathcal{G}_{ham}^k)$ is, up to sign, the image of α in the five-term exact sequence associated to the above extension.

Proof. We consider the following commutative diagram of group extensions

$$\begin{tikzcd}
1 \arrow[r] & Diff_c^1(\mathbb{R}^2 \setminus \{0\}) \arrow[r] & Diff_c^1(\mathbb{R}^2, 0) \arrow[r] & GL^+(2, \mathbb{R}) \arrow[r] & 1 \\
1 \arrow[r] & Symp_c^k(\mathbb{R}^2 \setminus \{0\}) \arrow[r] \arrow[u] & Symp_c^k(\mathbb{R}^2, 0) \arrow[r] \arrow[u] & \mathcal{G}_{ham}^k \arrow[r] \arrow[u] & 1,
\end{tikzcd}$$

where the map on the right is given by taking a germ to its linear part. We note that the Euler class e in $H^2(\mathcal{G}_{ham}^k)$ is just the pullback of the Euler class in $H^2(GL_\delta^+(2, \mathbb{R}))$ under this map. Now the image of the Euler class in $H^2(Diff_{c,\delta}^1(\mathbb{R}^2, 0))$ is torsion since a flat bundle with holonomy in the group $Diff_c^1(\mathbb{R}^2, 0)$ has a section and is thus topologically trivial.

Since the group $Diff_{c,0}^1(\mathbb{R}^2 \setminus \{0\})$ is perfect by the classic result of Mather (cf. [Math2]), we have that $H^1(Diff_{c,\delta}^1(\mathbb{R}^2 \setminus \{0\})) = \mathbb{Z}$ and an isomorphism is given via the map α to $MCG^c(\mathbb{R}^2 \setminus \{0\})$. Moreover, this map is equivariant with respect to conjugation by $GL^+(2, \mathbb{R})$, thus the invariant part $H^1(Diff_{c,\delta}^1(\mathbb{R}^2 \setminus \{0\}))^{GL^+(2,\mathbb{R})}$ is isomorphic to $\mathbb{Z}$. Similarly the group of coinvariants $H_1(Diff_{c,\delta}^1(\mathbb{R}^2 \setminus \{0\}))_{GL^+(2,\mathbb{R})}$ is also isomorphic to $\mathbb{Z}$. Thus, in particular, $H_1(Diff_{c,\delta}^1(\mathbb{R}^2, 0))$ is torsion free. The Universal Coefficient Theorem then implies that $H^2(Diff_{c,\delta}^1(\mathbb{R}^2, 0))$ is torsion free and we conclude that the image of the Euler class is in fact trivial.

Now since the Euler class e is a primitive class whose image in $H^2(Diff_{c,\delta}^1(\mathbb{R}^2, 0))$ is trivial and $H^1(Diff_{c,\delta}^1(\mathbb{R}^2 \setminus \{0\}))^{GL^+(2,\mathbb{R})} = \mathbb{Z}$, the exactness of the five-term sequence means that $e = \pm\delta\alpha$. By the naturality of the five-term exact sequence the same then holds for the extension involving $Symp^k$. □

We may now prove the main result of this section that the class γ is non-trivial in $H^2(Symp^k_\delta(\mathbb{R}^2, 0))$.

Theorem 6.3.7. *For any $2 \leq k < \infty$ there exist foliated $\mathbb{R}^2$-bundles over a surface with holonomy in $Symp^k(\mathbb{R}^2, 0)$ for which the characteristic class γ is non-trivial. Moreover, we may assume that such a bundle is topologically trivial.*

Proof. By Theorem 6.3.4 there exists a representation of some surface group $\pi_1(\Sigma_g) \to \mathcal{G}^k_{ham}$ for which γ is non-trivial. As usual we let a_i, b_i be the standard basis of $\pi_1(\Sigma_g)$ and let α_i, β_i denote the images of this basis in $\mathcal{G}^k_{ham}$. We further let $\tilde{\alpha}_i, \tilde{\beta}_i$ be representatives of these germs that have compact support in $\mathbb{R}^2$. Then by construction

$$\phi = \prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i]$$

is an element in $Symp^k_c(\mathbb{R}^2 \backslash \{0\})$. By Proposition 6.3.6 and the explicit form of the connecting homomorphism in the five-term exact sequence given by Lemma A.1, the Euler class of the bundle given by the original representation into $\mathcal{G}^k_{ham}$ is given (up to sign) by ϕ considered as an element in $MCG^c(\mathbb{R}^2 \setminus \{0\})$ (see also the proof of Theorem 5.2.5). As the Euler class can be chosen to be trivial (see Remark 6.3.1), we may assume that ϕ lies in the identity component.

Moreover, ϕ lies in $Ham^k_c(\mathbb{R}^2 \setminus \{0\})$ since

$$Flux^{\mathbb{R}^2 \backslash \{0\}}(\phi) = Flux^{\mathbb{R}^2}(\phi) = Flux^{\mathbb{R}^2}(\prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i]) = 0.$$

Thus by Lemma 6.3.5 we my write ϕ^{-1} as a product of N commutators of elements that have support disjoint from 0. We may then define a representation $\pi_1(\Sigma_{g+N}) \to Symp^k(\mathbb{R}^2, 0)$ on which the class γ is non-trivial. Moreover, by construction the associated foliated bundle is topologically trivial. □

Chapter 7

Symplectic cobordism and transverse knots

We introduce the notion of symplectic cobordism for transverse links in contact 3-manifolds considered as the boundary of certain symplectic fillings. This notion is analogous to the Lagrangian cobordism relation for Legendrian links as introduced by Chantraine. As in the case of closed symplectic surfaces in closed manifolds, we show that a symplectic null-bordism is genus minimising under certain assumptions on the topology of the symplectic filling. If the ambient cobordism is the symplectisation of a contact manifold, then the symplectic cobordism relation is transitive and reflexive on the set of isotopy classes of transverse links. However, the genus minimising property of symplectic null-bordisms means that the symplectic cobordism relation is not symmetric and, hence, does not define an equivalence relation. Thus we are led to consider symplectic concordance, for which the minimal slicing genus is no longer an obstruction to symmetry. But this relation too fails define an equivalence relation and we exhibit an infinite family of examples, for which symmetry fails.

7.1 Contact manifolds and symplectic cobordisms

We first review some basic facts about contact manifolds and the symplectic manifolds they bound. For our purposes a contact structure ξ on an oriented 3-manifold M will always be a co-oriented distribution of hyperplanes ξ such that $\lambda \wedge d\lambda > 0$ for any 1-form defining ξ. Since the fundamental work of Giroux it is now common to study contact manifolds in terms of open books.

Definition 7.1.1 (Open Book)**.** Let M be an oriented 3-manifold. An open book (M, π, B) consists of an oriented link B and a fibration $\pi : M \setminus B \to S^1$ so that in a neighbourhood $D^2 \times B$ of the binding π has the form $\pi(re^{i\theta}, x) = e^{i\theta}$. We call the fibres of the projection the *pages* and B the *binding* of the open book. If (M, ξ) is a contact manifold then we say that the open book is *adapted* to ξ if there exists a defining contact form λ that is positive on the binding and so that $d\lambda$ restricts to a positive area form on the pages.

The Thurston-Winkelnkemper construction (cf. [ThW]) shows that given an open book one can always find contact structures adapted to it. Moreover, by a result of Giroux this contact structure is unique up to isotopy (cf. [Gir]). An open book (M, π, B) is determined by the monodromy of the fibration π and this allows one to study contact manifolds by means of the mapping class groups of punctured surfaces.

We are interested in the way contact manifolds can bound symplectic manifolds. When one considers contact manifolds as boundaries of symplectic manifolds there is a certain compatibility of the two structures that is usually required. The precise meaning of this compatibility is contained in the following definition of a symplectic cobordism.

Definition 7.1.2 (Symplectic cobordism)**.** A symplectic cobordism between two contact manifolds (M_+, ξ_+) and (M_-, ξ_-) is a symplectic manifold (X, ω) whose boundary is a disjoint union $\partial X = M_+ \sqcup \overline{M}_-$ such that ξ_+ and ξ_- are positive/negative contact structures on M_+ and $\overline{M}_-$ respectively. Moreover, we require that the symplectic form be positive on the contact planes, i.e. $\omega|_{\xi_\pm} > 0$.

If $M_- = \emptyset$, then X will be called a weak convex filling of $M = M_+$. Similarly, if $M_+ = \emptyset$, then X will be called a weak concave filling of $M = M_-$. Finally, if a weak convex filling exists for a given contact manifold (M, ξ), we will say that M is weakly fillable.

Remark 7.1.3. We shall wish to consider contact structures up to isotopy. If $f_t^\pm$ is an isotopy from $\xi_0^\pm$ to $\xi_1^\pm$, then any symplectic cobordism from ξ_0^- to ξ_0^+ can be modified to give one from ξ_1^- to ξ_1^+. One first adds small collars to the ends of the symplectic cobordism X and extends the symplectic form to some $\hat{\omega}$, which again yields a symplectic cobordism from ξ_0^- to ξ_0^+. One then applies the map $g_t = (t, f_t^\pm)$ on the collars and the form $g_t^* \hat{\omega}$ defines the desired cobordism from ξ_1^- to ξ_1^+.

The canonical example of a symplectic filling is S^3 considered as the boundary of the 4-ball $B^4 \subset \mathbb{C}^2$ with the standard symplectic structure, where the contact structure ξ_{st} on S^3 is defined by the set of complex tangencies in TS^3. There are strong restrictions on weakly symplectically fillable contact manifolds as provided by the following fundamental theorem of Eliashberg and Gromov.

Theorem 7.1.4 ([Eli1])**.** *Let (M, ξ) be a weakly symplectically fillable contact manifold, then ξ is tight.*

There are more restrictive classes of fillings, the first of which are the so-called strong fillings.

Definition 7.1.5 (Strong symplectic filling)**.** A contact manifold is *strongly symplectically fillable* if there is a weak filling (X, ω) and an outward pointing symplectic dilation on ∂X. Here a symplectic dilation is a vector field V defined on a neighbourhood of ∂X that points out of X along the boundary and has the property that $L_V \omega = \omega$.

Such a filling is often referred to simply as a *convex* filling. One can equally well define a *concave* filling as one with an inward pointing symplectic dilation, or equivalently an outward pointing *contraction*, that is a V so that $L_V \omega = -\omega$. One can show that (X, ω) is a strong filling if and only if X is a weak filling and ω is exact on a neighbourhood of ∂X. This is a consequence of the following lemma of Eliashberg, which we note for future reference.

Lemma 7.1.6 ([Eli2], Prop. 4.1)**.** *Let (X, ω) be a weak filling of a contact manifold (M, ξ). Suppose that ω is exact on an open set $U \subset M$ that contains some compact set K. Then there is a symplectic form $\tilde{\omega}$ that fills (M, ξ) and*

$$\tilde{\omega}|_K = C\, d(e^t \lambda)$$

for some large constant $C > 0$ and λ a contact form for ξ. In particular, if $\omega = d\lambda$ in a neighbourhood of the boundary then $(X, \tilde{\omega})$ defines a strong filling of (M, ξ).

In the setting of the contact topology of 3-manifolds there are two special classes of links that one considers. Namely the Legendrian links, that are everywhere tangent to the contact structure, and transverse links that are everywhere transverse to it. When one has a symplectic cobordism (X, ω) it is natural to study cobordisms of such links. In the case of Legendrian links one may consider Lagrangian cobordisms, as defined in [Cha], and in the transverse case one can consider symplectic cobordisms. For technical reasons it will be convenient to define the notion of transverse link cobordism for links that lie in the ends of cobordisms with strongly convex/concave ends. In this case one may attach positive and negative ends to X to obtain an open symplectic manifold

$$\hat{X} = (M_- \times (-\infty, 0]) \cup X \cup (M_+ \times [0, \infty)),$$

where $M_+ \times [0, \infty)$, $M_- \times (-\infty, 0]$ are given the symplectic structures $\omega_\pm = d(e^t \lambda_\pm)$ for contact forms $\lambda_\pm$. We go to the trouble of adding these ends to make gluing cobordisms well-defined, and so that transversely isotopic knots are cobordant in the symplectisation of M. With these preliminaries we make the following definition.

Definition 7.1.7 (Symplectic cobordism of links)**.** Let (X, ω) be a strong symplectic cobordism and let $L_\pm \subset M_\pm$ be transverse links, oriented by the co-orientations of the contact structures. Then we will say the L_- is symplectically cobordant to L_+ if there is a properly embedded symplectic surface $\Sigma \subset \hat{X}$ such that for some $R > 0$

$$\Sigma \cap (M_+ \times [R, \infty)) = L_+ \times [R, \infty)$$

and

$$\Sigma \cap (M_- \times (-\infty, -R]) = L_- \times (-\infty, -R].$$

If Σ_R denotes the truncated surface given by deleting the ends $M_+ \times (R, \infty)$ and $M_- \times (-\infty, -R)$, we require that for $\partial \Sigma_R = L_+ \sqcup \overline{L}_-$, where Σ_R is given the symplectic orientation.

To show that this definition is a sensible one, we first show that two transversely isotopic links are cobordant in the symplectisation of (M, ξ).

Lemma 7.1.8. *Let L, L' be isotopic transverse links in (M, ξ), then they are cobordant in $(M \times [0,1], d(e^t \lambda))$.*

Proof. We assume for simplicity that L is a knot. Let $f(\theta, t) : S^1 \times \mathbb{R} \to M$ be an isotopy from L' to L through transverse knots, which we assume to be independent of t for $|t| > 1$. For any $\epsilon > 0$ we define a map $F_\epsilon : S^1 \times \mathbb{R} \to M \times \mathbb{R}$ by

$$F_\epsilon(\theta, t) = (f(\theta, \epsilon t), t).$$

This is clearly an embedding and has the correct asymptotic properties for a cobordism. Thus we need only to see that it defines a symplectic annulus. To this end we calculate

$$\begin{aligned}\omega(\frac{\partial F_\epsilon}{dt},\frac{\partial F_\epsilon}{d\theta}) &= (e^t dt\wedge\lambda)(\frac{\partial F_\epsilon}{dt},\frac{\partial F_\epsilon}{d\theta}) + e^t d\lambda(\frac{\partial F_\epsilon}{dt},\frac{\partial F_\epsilon}{d\theta})\\ &= (e^t dt\wedge\lambda)(\frac{\partial F_\epsilon}{dt},\frac{\partial F_\epsilon}{d\theta}) + \epsilon\, e^t d\lambda(\frac{\partial f}{\partial t},\frac{\partial F_\epsilon}{d\theta})\\ &= e^t[\lambda(\frac{\partial f}{d\theta}(\epsilon t)) + \epsilon\, d\lambda(\frac{\partial f}{\partial t},\frac{\partial F_\epsilon}{d\theta})].\end{aligned}$$

As all knots are positive transverse, the first term is strictly positive for all t. Thus for ϵ sufficiently small the annulus we obtain is indeed a symplectic cobordism from L' to L. For disconnected links the exact same argument holds if we replace S^1 by a disjoint union of circles. □

Another natural construction is that of gluing together two cobordisms.

Lemma 7.1.9 (Gluing cobordisms). *Let $\Sigma_1 \subset X_1$ and $\Sigma_2 \subset X_2$ be symplectic cobordisms between L_1 and L and L and L_2 respectively, where the positive end (M_1, ξ_1) of X_1 is contactomorphic to the negative end (M_2, ξ_2) of X_2. Then we may glue the cobordisms to obtain a symplectic cobordism $\Sigma = \Sigma_1 \cup \Sigma_2$ from L_1 to L_2 in $X = X_1 \cup X_2$.*

Proof. Let $\hat{X}_1$ and $\hat{X}_2$ denote the manifolds obtained by attaching half infinite symplectic ends. The symplectic form on the positive end of $\hat{X}_1$ has the form $d(e^t\lambda_1)$ and $d(e^t\lambda_2)$ on the negative end of $\hat{X}_2$. Let $R > 0$ be such that $\Sigma_1 \cap (M_1 \times [R, \infty))$ and $\Sigma_2 \cap (M_2 \times (-\infty, -R])$ are products as prescribed by Definition 7.1.7. Next we take ψ to be a contactomorphism between (M_1, ξ_1) and (M_2, ξ_2), so that $\psi^*\lambda_2 = g\lambda_1$ for some strictly positive function g. We map a piece of the positive neck $M_1 \times (R+\epsilon, R+3\epsilon)$ to a piece of the negative neck $M_2 \times (-R-3\epsilon, -R-\epsilon)$ by defining

$$\Psi(x,t) = (\psi(x), t - (R+3\epsilon))$$

so that $\Psi^*(d(e^t\lambda_2)) = d(e^{-(R+3\epsilon)} g e^t \lambda_1)$. To compensate for the multiplicative factor $e^{-(R+3\epsilon)}g$ we will need to multiply the symplectic form ω_2 on $\hat{X}_2$ by a large positive constant K in order to perform an inflation. We take this constant to be so large that the function $g_K = Ke^{-(R+3\epsilon)}g$ is greater than 1 and, thus, has a positive logarithm, which we denote by $h = log(g_K)$. We next consider the map on $M_1 \times [R, \infty)$ given by

$$\Phi(x,t) = (x, t + \gamma\, h(x)),$$

where γ is a non-decreasing cut off function that is identically 0 for $t \leq R + \frac{3}{2}\epsilon$ and is identically 1 for $t \geq R + 2\epsilon$. By construction $\Phi^* d(e^t\lambda_1) = d(e^{-(R+3\epsilon)} g_K e^t\, \lambda_1)$ for $t \geq R+2\epsilon$. Thus the map $\Psi \circ \Phi^{-1}$ allows us to glue the cobordisms together in a manner compatible with the symplectic structures. Moreover, all the maps given preserve the product structure of the symplectic ends and, thus, this gluing defines a symplectic cobordism between the links L_1 and L_2. □

For two transverse links $L_\pm$ in a contact manifold, let us write $L_- \prec_s L_+$ if they are symplectically cobordant in the symplectisation of M, where $L_\pm$ correspond to the positive resp. negative ends of Σ. In the case of a symplectisation Lemma 7.1.9 is significantly easier and the ambient cobordism obtained in the gluing construction can be assumed to be the symplectisation of (M, ξ) itself, albeit with its symplectic form scaled by some positive constant. As this does not affect symplectic cobordism we obtain the following proposition as a consequence of the two previous lemmas.

Proposition 7.1.10. *The relation $\prec_s$ gives a transitive relation on the set of isotopy classes of transverse links.*

It is particularly interesting to consider symplectic null-cobordisms, in which case a given transverse knot or link bounds a symplectic surface in the filling X. We shall call such links *symplectic* and the bounding surface a *symplectic spanning surface*. The main examples of such links are provided by $\mathbb{C}$-links in the 3-sphere.

Example 7.1.11 ($\mathbb{C}$-links). We let L be defined as the intersection of a smooth algebraic curve Σ with the boundary of a polydisc $D^2 \times D^2 \subset \mathbb{C}^2$, where one assumes that this intersection is transverse and is contained in the interior of $\partial D^2 \times D^2$. Following [Rud1], such a surface is called *quasipositive*. Since Σ is complex, it is symplectic for the standard symplectic structure on $\mathbb{C}^2$. By rescaling the first factor by a large, positive constant we may assume that the tangent spaces of Σ are almost horizontal near the boundary. One may then add an infinite end to Σ and since Σ was almost horizontal this can be smoothed in such a manner that the resulting surface $\hat{\Sigma}$ is a symplectic spanning surface for L.

7.2 Symplectic spanning surfaces and slice genus

In the symplectic topology of 4-manifolds an embedded symplectic surface minimises genus in its homology class. This is known as the symplectic Thom conjecture and was finally proven in complete generality by Ozváth and Szabó.

Theorem 7.2.1 ([OS])**.** *A closed symplectic surface Σ in a closed symplectic 4-manifold (X, ω) is genus minimising in its homology class.*

We would like to say the same for symplectic spanning surfaces. For this we shall introduce the notion of the minimal (negative) Euler characteristic of a spanning surface of a null-homologous knot or link sitting in the boundary of a 4-manifold.

Definition 7.2.2. Let $L \subset M = \partial X$ be an oriented null-homologous link. We define the minimal Euler characteristic of L in X to be

$$\chi^X_{min}(L) = min\{-\chi(\Sigma) \mid \partial\Sigma = L,\ \Sigma \text{ a properly embedded, connected surface in } X\}.$$

Under the assumption $H_2(X) = 0$ we will prove that $-\chi(\Sigma) = \chi^X_{min}(L)$ for symplectic spanning surfaces. This is the symplectic analogue of a similar statement in [Cha] for the case of a Lagrangian spanning surfaces bounding Legendrian links in B^4. For the case of a symplectic spanning surface in B^4 this result is due to Boileau and Orevkov (cf. [BO]), whose proof relies on J-holomorphic curves and the local Thom Conjecture. Instead we shall use the construction of Gay to add symplectic handles to X in such a way as to cap off Σ to a closed symplectic surface in a symplectic manifold with convex boundary. Then a result of Eliashberg will allow us to embed this manifold symplectically in a closed symplectic manifold and the result will follow from the Symplectic Thom Conjecture. In this way we obtain a slight strengthening of the result of Orevkov and Boileau.

Theorem 7.2.3. *Let (X, ω) be a convex symplectic filling with $H_2(X) = 0$ and $L \subset \partial X$ a transverse link with symplectic spanning surface Σ. Then $\chi^X_{min}(L) = -\chi(\Sigma)$.*

Before we give a proof of Theorem 7.2.3 we shall need to review the construction for attaching symplectic handles along transverse knots in the boundaries of convex symplectic fillings following [Gay]. We first consider a standard symplectic 2-handle, which is defined as a subset of $H \subset \mathbb{R}^4$. We use polar coordinates $x = (r_1, \theta_1, r_2, \theta_2)$ and set $f = -r_1^2 + r_2^2$, then H is defined as the locus of points with $H = \{x \mid \epsilon_1 \leq f(x) \leq \epsilon_2\}$ for some small $\epsilon_1 < 0 < \epsilon_2$. We let

$$\omega_0 = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$$

be the standard symplectic form on $\mathbb{R}^4$ and

$$V = \frac{1}{2}[(r_1 - \frac{1}{r_1})\frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2}].$$

This vector field is an inward pointing symplectic dilation, that is a *symplectic contraction.*

We set $H_1 = f^{-1}(\epsilon_1)$ and take the attaching circle to be $\partial_1 H = H_1 \cap \{r_2 = 0\}$, which is a transverse knot for the contact form on H_1 given by

$$\alpha_1 = \iota_V \omega_0 = \frac{1}{2}[(r_1^2 - 1)d\theta_1 + r_2^2 d\theta_2].$$

Since V is inward pointing along H_1, the induced orientation will agree with that given by the coordinates $r = r_2, \mu = \theta_2, \lambda = -\theta_1$. In these coordinates α_1 becomes

$$\frac{1}{2}[r^2 d\mu - (r^2 - \epsilon_1 - 1)d\lambda]$$

so that ∂H_1 is a *positive* transverse knot with respect to this contact form.

Now let K be a transverse knot in the boundary of a strong symplectic filling X with symplectic form $d(e^t\alpha)$ near the boundary and symplectic dilation $\frac{\partial}{\partial t}$. By the Darboux Theorem for transverse knots there is a diffeomorphism ϕ between a neighbourhood U of K and W of ∂H_1 so that $\phi^*\alpha_1 = g\alpha$ for some positive function g. By multiplying ω_0 by a large constant we may assume that $g > 1$ and we let $h = log(g)$. The final step will be to inflate X so that the induced contact form on ∂X is $e^h\alpha = g\alpha$, which we record in the following lemma (see also Lemma 7.1.9).

Lemma 7.2.4 (Inflation Lemma)**.** *Let $\omega_0 = d(e^t\alpha)$ be a symplectic form on the symplectisation X of a contact manifold $(M, ker(\alpha))$ and let $g > 1$ be a smooth function on M. Then there is a diffeomorphism $\phi(x, t) = (x, log(g) + t)$ so that $\phi^*(\omega_0) = d(e^t g\alpha)$.*

In order to apply Lemma 7.2.4 we cut off h so that it has support in the chosen neighbourhood U of K and is equal to h on some possibly smaller neighbourhood. By abuse of notation we continue to call this function h. We then add a long neck to X and consider the subset

$$X_h = X \cup \{(p,t) \mid 0 \leq t \leq h(p)\}.$$

This is again a strong symplectic filling of M with symplectic dilation $\frac{\partial}{\partial t}$. We next identify ∂X with $\partial X_h = \{(p, h(p))\}$ by sending $p \mapsto (p, h(p))$. Under this identification the pullback of the contact structure is exactly $e^h \alpha$.

Hence, by using a Darboux chart to identify U with W and matching inward and outward pointing symplectic dilations, we may smoothly attach H to X in a way that is compatible with both symplectic structures at the expense of multiplying ω_0 by some large constant. Moreover, if $K \times (-\epsilon, 0]$ were a product piece of a symplectic spanning surface, then the extension of this to X_h is again symplectic with transverse boundary and under the gluing map described above such a surface will attach smoothly to the symplectic core disc $r_2 = 0$ in H. Thus, if Σ was a symplectic spanning surface with boundary K, then we have described how to cap it off to get a closed symplectic surface $\hat{\Sigma} = \Sigma \cup D^2$ in the interior of a symplectic manifold $\hat{X}$. Furthermore, by ([Gay], Theorem 1.1) we may assume that $\hat{X}$ is a convex symplectic filling. With this construction we may now prove Theorem 7.2.3.

Proof of Theorem 7.2.3. We let L be a transverse link. Then by attaching handles along each component as above we obtain an embedded symplectic surface

$$\hat{\Sigma} = \Sigma \cup_{i=1}^{k} D_i^2$$

in the interior of some convex filling $\hat{X}$. By Theorem 1.3 in [Eli2] we may symplectically embed $\hat{X}$ in a closed symplectic manifold Y so that the Symplectic Thom conjecture implies that $\hat{\Sigma}$ minimises genus in its homology class in Y. Now let Σ' be any other spanning surface of L, then we can close this up in $\hat{X}$ to a surface $\hat{\Sigma}'$, which we then embed in Y. If $H_2(X)$ is trivial, then $[\hat{\Sigma}] = [\hat{\Sigma}']$ as homology classes in Y. Thus, as $\hat{\Sigma}$ minimises genus in its homology class, we have

$$-(\chi(\Sigma) + k) = -\chi(\hat{\Sigma}) \leq -\chi(\hat{\Sigma}') = -(\chi(\Sigma') + k).$$

We conclude that $-\chi(\Sigma) \leq -\chi(\Sigma')$ for all spanning surfaces of L and hence

$$-\chi(\Sigma) \leq \chi^X_{min}(L).$$

The opposite inequality is obvious by the definition of χ^X_{min} and hence $\chi^X_{min}(L) = -\chi(\Sigma)$. □

The main examples for which the hypotheses of Theorem 7.2.3 hold are S^3 and $S^2 \times S^1$ with their Stein fillings B^4 and $B^3 \times S^1$ respectively.

7.3 Symplectic links are quasipositive in S^3

In this section we shall consider the special case of symplectic links in (S^3, ξ_{st}) considered as the boundary of B^4 with the standard symplectic structure. The question as to which links are symplectic has been considered in [BO], where it is shown that the symplectic links are precisely those that are *quasipositive*. Before stating this result we need to recall the notion of a braid and Rudolph's definition of quasipositivity.

The braid group on n-strands B_n is defined as the fundamental group of the configuration space $C_n = (\mathbb{C}^n \setminus \Delta_n)/S_n$, where Δ_n is the diagonal subvariety consisting of all vectors $(z_1, z_2, \dots, z_n)$ for which at least two distinct entries are equal and S_n acts by permutations on the coordinate vectors. Inherent in this definition is the choice of base point, which can be solved by considering free homotopy classes of loops, or equivalently conjugacy classes of braids. Now the braid group is generated by positive half twists $\sigma_1, ..., \sigma_{n-1}$ and has the following presentation

$$B_n = \langle \sigma_1, ..., \sigma_{n-1} \mid \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1 \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2\rangle.$$

An important fact that we shall need is that the braid group is isomorphic to the mapping class group $MCG^c(D^2, \{p_1, p_2..., p_n\})$ of compactly supported diffeomorphisms of D^2 that fix n marked points $\{p_1, p_2..., p_n\}$ as a set (cf. [Bir]). Given an element $\beta \in B_n$ we can construct its closure in $S^1 \times D^2$ that we denote by $\hat{\beta}$. This is obtained by taking the associated element $\phi \in MCG^c(D^2, \{p_1, p_2..., p_n\})$ and defining $\hat{\beta}$ to be the image of $[0, 1] \times \{p_1, p_2..., p_n\}$ in the mapping torus defined by ϕ.

Definition 7.3.1 (Quasipositivity, [Rud1])**.** A braid β is called *quasipositive* if it has a factorisation of the form

$$\beta = \prod_{i=1}^{k} Q_i\sigma_1Q_i^{-1},$$

where $Q_i \in B_n$ are arbitrary braids.

Any link in S^3 is isotopic to a link in braided position (cf. [Bir]), thus when we speak of a quasipositive link we will mean that it is quasipositive after being braided. With this notion of quasipositivity we may state the following result.

Theorem 7.3.2 ([BO])**.** *A link $L \subset S^3$ is symplectic if and only if it is quasipositive.*

Remark 7.3.3. In fact Boileau and Orevkov show that if Σ is any symplectic spanning surface in B^4, then the pair (B^4, Σ) is diffeomorphic to (B^4, Σ_{alg}), where Σ_{alg} is a piece of algebraic curve that is quasipositive (cf. Example 7.1.11). We also note that the proof of Theorem 7.3.2 in [BO] requires only that the link in question is positively transverse with respect to the contact structure without any additional assumption on the asymptotics of the spanning surface.

Thus, when studying transverse links from the perspective of symplectic cobordism, it is reasonable to single out the set of null-bordant links, which in view of Theorem 7.3.2 is just the set of quasipositive links and we will denote this set by $\mathcal{QP}$. There is an obvious map on $\mathcal{QP}$ given by $\chi_{min} = \chi_{min}^{B^4}$, which by Theorem 7.2.3 is order preserving.

Proposition 7.3.4. *Let $L_1, L_2 \in \mathcal{QP}$ and assume that $L_1 \prec_s L_2$ via a connected symplectic cobordism Σ, then*

$$\chi_{min}(L_1) = -\chi(\Sigma) + \chi_{min}(L_2).$$

In particular, $\chi_{min}(L_1) \leq \chi_{min}(L_2)$.

This corollary exhibits the asymmetry of the symplectic cobordism relation. To see this we note that any quasipositive link L can be isotoped to lie on an algebraic curve C_d of degree d in $\mathbb{C}P^2$. By perturbing this curve we may assume that it intersects the line at infinity transversally. This curve then defines a symplectic cobordism $K \prec_s \Delta_d^2$, where $\Delta_d^2 \in B_d$ denotes the braid given by a full twist on all d strands. In the case of the right-handed trefoil K we have $d = 3$. Furthermore, $\chi_{min}(K) = \chi_{min}(\Delta_3^2) = 1$ and, thus, Proposition 7.3.4 implies that any symplectic cobordism Σ from Δ_d^2 to K must have $\chi(\Sigma) = 0$. But *any* cobordism Σ between Δ_d^2 and K must have at least $3 + 1 = 4$ boundary components, that is $\chi(\Sigma) < 0$ and we arrive at a contradiction. Thus, we conclude that Δ_d^2 is not symplectically cobordant to K.

We next note that there are obstructions for a transverse knot or link to be quasipositive and, hence, symplectic. The first of these is given by the self-linking number of the closure of a braid.

Definition 7.3.5. Let $\beta \in B_d$ be a braid given by a factorisation

$$\beta = Q_1 \sigma_{i_1}^{\epsilon_1} (Q_1)^{-1} ... Q_k \sigma_{i_k}^{\epsilon_i} (Q_k)^{-1},$$

where $\epsilon_i = \pm 1$. We let n_-, n_+ be the number of ϵ_i that are $+1, -1$ respectively. We then define the self-linking number

$$sl(\beta) = n_+ - n_- - d.$$

This of course corresponds to the classical self-linking number of a transverse knot in S^3 (cf. [Etn]). A classical result of Rudolph is the following.

Theorem 7.3.6 (Rudolph, [Rud2])**.** *Let β be a quasipositive braid, then $sl(\beta) = \chi_{min}(\hat{\beta})$.*

As an immediate corollary we have our first obstruction to a transverse link being quasipositive.

Proposition 7.3.7. *Let L be a quasipositive link, then $sl(L) > -1$.*

7.4 Branched covers and symplectic spanning surfaces

We have seen that there are topological obstructions to transverse links in S^3 having symplectic spanning surfaces in B^4. In this section we shall show that there are also contact topological obstructions to the existence of a symplectic spanning surface Σ bounding a transverse knot or link in an arbitrary symplectic filling X. These obstructions arise when one takes branched coverings over symplectic spanning surfaces. If L is a transverse link in a contact 3-manifold that spans a symplectic spanning surface, then one can take a branched cover $\widetilde{M}$ with branching locus L. This will bound the cover of X branched over Σ, which is then a symplectic filling of $\widetilde{M}$ so that the induced contact structure on $\widetilde{M}$ is tight. Of course one must assume such a cover exists, but this is always true for example in the case $X = B^4$.

As a first step we shall explain how one obtains a contact structure on a cover branched over a transverse link L, which will be assume to be a knot for simplicity (cf. [Pla]). We choose coordinates in a neighbourhood of the branching knot K so that the covering map $\widetilde{M} \xrightarrow{\pi} M$ has the form $(z, w) \mapsto (z, w^d)$, where z corresponds to the core circle of a tubular neighbourhood $S^1 \times D^2$ of K. By the Darboux Theorem we may assume that the contact distribution is $Ker(dz + r^2 d\theta)$, where we take polar coordinates on D^2. Then the pullback under the projection of this form is $dz + dr^{2d} d\theta$, which is a contact form away from the core circle $r = 0$. To fix this we define a function $f_\delta(r)$ which interpolates between r^2 and dr^{2d}, has positive derivative and is equal to dr^{2d} for $r > \delta$.

The form $dz + f_\delta d\theta$ is a contact form on $S^1 \times D^2$ and is isotopic to the standard contact form on the δ-neighbourhood U_δ for delta sufficiently small. By defining $\tilde{\alpha}$ as $\pi^*\alpha$ outside of U_δ we obtain the desired contact form on $\widetilde{M}$.

It is shown in [Pla] that for sufficiently small δ the contact structure thus defined is unique up to isotopy. We shall next show that the symplectic form on a branched cover that is branched over a symplectic spanning surface may be chosen in a fashion compatible with the construction of the contact structure on the boundary.

Proposition 7.4.1. *Let X be a weak symplectic filling and let $\widetilde{X} \xrightarrow{\pi} X$ be a branched covering branched over a symplectic spanning surface Σ. Then $\widetilde{X}$ is a weak symplectic filling of $\widetilde{M}$. In particular, the induced contact structure is tight.*

Proof. It suffices to consider the case of a knot. The condition of being a weak filling is that $\omega \wedge \alpha > 0$ on M, for a contact form α. We first choose a trivialisation $D^2 \times \Sigma \to N$ of a tubular neighbourhood of Σ such that on the boundary the induced trivialisation is a Darboux chart of the contact form in a neighbourhood of $K = \partial\Sigma$. Furthermore, we assume that π is of the form $(p, z) \mapsto (p, z^d)$ in these coordinates.

By our assumption on the boundary trivialisation there is a contact form such that $\alpha = (dz + r^2 d\theta)$ on a δ-neighbourhood $U_\delta = D^2 \times \partial\Sigma$. We let $\tilde{f}_\delta(r) d\theta = \pi^*\alpha - \tilde{\alpha}$ on U_δ and note that $\tilde{f}_\delta$ has support in U_δ. Moreover, for all sufficiently small r there is a positive constant C so that

$$|\tilde{f}_\delta(r)| \leq C\, r^2. \tag{7.1}$$

We choose a 2-form τ on D^2 of the form $\phi\, dx \wedge dy$ for some non-negative bump function ϕ with support in a δ-ball about the origin. We set $\tilde{\omega} = \pi^*\omega + \epsilon\tau$ as in Proposition 2.5.11 and note that this is symplectic for all sufficiently small ϵ. By definition τ is positive on planes that intersect Σ transversally. Thus, by our orientation convention, this is true for positive transversals to K and $\tau \wedge \pi^*\alpha > 0$ on K. Moreover, the following holds on the neighbourhood N:

$$\begin{aligned}
\tilde{\omega} \wedge \tilde{\alpha} &= (\pi^*\omega + \epsilon\tau) \wedge (\pi^*\alpha - \tilde{f}_\delta d\theta) \\
&= \pi^*(\omega \wedge \alpha) + \epsilon(\tau \wedge \pi^*\alpha) - \tilde{f}_\delta[(\pi^*\omega + \epsilon\tau) \wedge d\theta].
\end{aligned}$$

Hence, as the first term is positive away from K and the second is positive on K, by taking ϵ sufficiently small we conclude that the sum of the first two terms is strictly positive on $\widetilde{M}$. By equation (7.1) we may assume (independent of our choice of ϵ) that the last term is arbitrarily small and hence $\tilde{\omega} \wedge \tilde{\alpha}$ is strictly positive on $\widetilde{M}$ as desired. □

As an application of Proposition 7.4.1 we note the following corollary.

Corollary 7.4.2. *There exist infinitely many transverse knots of arbitrary self-linking number that cannot bound symplectic spanning surfaces in* $S^3 = \partial B^4$*.*

Proof. We may always take a cyclic 2-fold branched cover over any properly embedded surface Σ in B^4. If Σ is symplectic then the induced contact structure on the boundary of this branched cover is fillable and, hence, tight by Proposition 7.4.1. However the negative stabilisation K^- of any transverse knot has an overtwisted branched cover by Proposition 4.2 of [HKP]. Thus K^- cannot bound any positive symplectic surface.

Finally, if K has $sl(K) = N$, then the self-linking number of the negative stabilisation is $sl(K^-) = N - 2$. Thus for an appropriate choice of K we may assume that $sl(K^-)$ is arbitrarily large. □

7.5 Symplectic concordance

A natural specialisation of the relation of symplectic cobordism on the set of transverse knots is *symplectic concordance.*

Definition 7.5.1 (Symplectic concordance of knots in S^3)**.** Let K_1 and K_2 be transverse knots in S^3. We say that K_1 is symplectically concordant to K_2 if they are symplectically cobordant in the symplectisation of S^3 via an annulus whose negative end is K_1 and whose positive end is K_2. In this case we write $K_1 \prec_c K_2$.

We have already seen that the notion of symplectic cobordism is much stronger than that of ordinary smooth cobordism on the set of quasipositive links, as shown for example by Proposition 7.3.4, which shows that the symplectic cobordism relation is far from being symmetric unlike its smooth counterpart. However, this asymmetry was detected by the slice genus of the respective knots, so it is not *a priori* clear whether the same holds for the symplectic concordance relation. In this section we will consider examples for which the symplectic concordance relation fails to be symmetric as well.

To this end we let K_n be the quasipositive knots given as the closure of the following family of braids in B_3:

$$\beta_n = \sigma_1 \sigma_2^n \sigma_1 \sigma_2^{-n}.$$

It follows from Theorem 7.3.6 that K_n is a slice quasipositive knot since $sl(\beta_n) = -1$. We let $\Sigma(\emptyset, K)$ be a holomorphic slicing disc in B^4 for $K = K_n$ (cf. Remark 7.3.3) and we let K_0 denote the transverse unknot. Then by taking the complement of a small ball about some $x \in \Sigma(\emptyset, K)$, we conclude that $K_0 \prec_c K$.

We will show that the opposite relation does not hold. As a first step we will show that if a quasipositive slice knot is symplectically concordant to the unknot, then it is in fact *doubly slice.* Recall that a knot is called (smoothly) doubly slice if it is obtained as the transverse intersection of an embedded 3-sphere in S^4 with an unknotted, embedded 2-sphere $S^2 \hookrightarrow S^4$. Such knots have been considered by many authors and there are well-known obstructions to a given knot being doubly slice.

Proposition 7.5.2. *Let K be a slice quasipositive knot. If K is symplectically concordant to the unknot, then K is doubly slice.*

Proof. We let $\Sigma(K, K_0)$ be a symplectic concordance from K to K_0. After gluing the concordance $\Sigma(K, K_0)$ together with the holomorphic slicing disc $\Sigma(\emptyset, K)$ we obtain a symplectic disc Δ in a large ball B^4_N of radius N, which in turn embeds into $\mathbb{C}P^2$ with the Fubini-Study form ω_{FS}. This disc may then be capped off at infinity by a standard holomorphic (and hence symplectic) linear disc D^2 to obtain a symplectically embedded sphere $S = \Delta \cup D^2 \subset \mathbb{C}P^2$.

The sphere S may then be made J-holomorphic for some J that is tamed by ω_{FS} and as the sphere is a linear disc at infinity we may assume that the almost complex structure J agrees with the standard, integrable almost complex structure J_0 in a neighbourhood of the line at infinity $\mathbb{C}P^1_\infty$.

Let $y = S \cap \mathbb{C}P^1_\infty$ and let x be a point in the interior of the disc Δ. By a result of Gromov, any two symplectic spheres S_1, S_2 representing a generator of $H_2(\mathbb{C}P^2)$ are isotopic through symplectic spheres (cf. [McS2]). In order to construct an isotopy between S_1 and S_2 one chooses ω_{FS}-tamed almost complex structures J_1, J_2 that make S_1 and S_2 almost complex. If J_t is a path of tamed complex structures joining J_1 to J_2, then there is a unique J_t-holomorphic line containing x and y that we denote by $L_{x,y}(t)$ (cf. [McS2], Cor. 9.4.5), and this family of symplectic spheres provides the desired isotopy.

In our case we have $J_1 = J_0$ and $J_1 = J$ and by construction we may assume that the path J_t interpolating the two almost complex structures in standard in a neighbourhood of infinity. Hence the line at infinity $\mathbb{C}P^1_\infty$ is J_t-holomorphic for all t and by positivity of intersections $L_{x,y}(t)$ intersects $\mathbb{C}P^1_\infty$ transversely for all t. Thus we may embed this isotopy in an ambient isotopy, which fixes the point x and $\mathbb{C}P^1_\infty$ as a set. By performing a further isotopy in a neighbourhood of $\mathbb{C}P^1_\infty$, we may assume that our isotopy has compact support disjoint from the line at infinity. Thus the affine part of S, that we denote by $\hat{S} = S \setminus y$, is isotopic through compactly supported diffeomorphisms to a complex line and hence embeds into S^4 as an *unknotted* sphere that we continue to denote by S. Since the alterations that were made occurred in a neighbourhood of the line at infinity, the intersection with $S^3 \subset S^4$ is still K, and hence K is doubly slice. □

There are many obstructions to a knot being doubly slice. In particular, the torsion numbers of the 2-fold cover of S^3 branched over K come in pairs.

Lemma 7.5.3 ([Sum], Cor. 2.6)**.** *Let K be a doubly slice knot and let M_K be the 2-fold cover of S^3 branched over K. Then $H_1(M_K)$ is finite and the torsion numbers come in pairs, that is $H_1(M_K) \cong A \oplus A$, for some finite group A.*

We are now ready to show that K_n is not symplectically concordant to the unknot for certain values of n.

Theorem 7.5.4. *The symplectic concordance relation is not symmetric. In fact, there are infinitely many quasipositive slice knots that are not symplectically concordant to the unknot.*

Proof. We let $K = K_n$ be as above and we assume that n is odd. It follows from Proposition 7.5.2 that if K_n is symplectically concordant to the unknot then it is doubly slice and we will show that this is not the case.

To this end we let M_K denote the 2-fold cover of S^3 branched over K. The standard open book on S^3, whose binding is a Hopf circle, pulls back under this branched double cover to give an open book decomposition on M_K whose page P is a once punctured torus and whose monodromy A is given by lifting the word $\beta \in B_3$ to the mapping class group Γ_1^1. This group fits into the exact sequence

$$1 \to \mathbb{Z} \to \Gamma_1^1 \to \Gamma_1 = SL(2,\mathbb{Z}) \to 1.$$

We let $\bar{A}$ denote the image of the monodromy in $SL(2,\mathbb{Z})$. The images of the lifts of the generators are

$$\bar{\sigma}_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\bar{\sigma}_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and we compute that

$$\bar{A} = \begin{pmatrix} 1-n & n^2 \\ n-2 & -n^2+n+1 \end{pmatrix}.$$

Thus M_K is the union of the mapping torus $P_{\bar{A}}$ with monodromy $\bar{A}$ and a solid torus whose meridian is glued to a longitude on the boundary. The Wang sequence gives the following exact sequence

$$0 \to H_1(P) \stackrel{\bar{A}_*-Id_*}{\to} H_1(P) \to H_1(P_{\bar{A}}) \to H_0(P) \to 0,$$

where the first map is given by the matrix

$$\bar{A} - Id = \begin{pmatrix} -n & n^2 \\ n-2 & -n^2+n \end{pmatrix}.$$

Hence $H_1(P_{\bar{A}}) \cong \mathbb{Z}_{n^2} \oplus \mathbb{Z}$ if n is odd. Then attaching the meridian of a solid torus along a longitude will kill the second factor and we deduce that $H_1(M_K) \cong \mathbb{Z}_{n^2}$. This, however, contradicts Lemma 7.5.3, since $\mathbb{Z}_{n^2}$ does not split as a sum of groups of equal order and we conclude that K_n is not symplectically concordant to the unknot for n odd. $\square$

It is in fact difficult to imagine that any non-trivial knot is symplectically concordant to the unknot and this motivates the following conjecture.

Conjecture 7.5.5. *Let K be a transverse knot. Then $K_0 \prec_c K$ and $K \prec_c K_0$ if and only if $K = K_0$.*

Of course if every quasipositive slice knot were not doubly slice then the conjecture would be trivially true, however for example the 9_{46} knot is doubly slice by [Sum] and quasipositive. An explicit quasipositive braiding is given by the following representation in the braid group on 4 strands B_4:

$$(\sigma_1\sigma_2\sigma_2\sigma_3\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_2\sigma_1\sigma_2^{-1})(\sigma_2\sigma_3\sigma_2^{-1}).$$

A special case of Conjecture 7.5.5 is that the 9_{46} knot is not symplectically concordant to the unknot. For Legendrian knots and Lagrangian concordance this has been shown by Chantraine (cf. [Cha]) for a particular Legendrian representative, and one would expect that the same should hold for symplectic concordance.

We conclude with some remarks concerning the proof of Theorem 7.5.4.

Remark 7.5.6. The argument of Theorem 7.5.4 is in essence topological and the additional fact that we are considering symplectic concordance was only used to deduce that the knots under consideration are doubly slice. Indeed our initial approach to this question was the following: by taking the 2-fold cyclic branched coverings over a symplectic concordance from K to the unknot one obtains a strong symplectic cobordism from the 2-fold branched cover M_K to S^3. If the contact structure on M_K were a perturbation of a taut foliation, then one can easily derive a contradiction using certain cut and paste arguments and the uniqueness of symplectic fillings of S^3. However, a criterion developed by Honda-Kazez-Matic in [HKM] for determining when a particular open book with pseudo-Anosov monodromy supports a contact structure obtained as a perturbation of a taut foliation is unfortunately not decisive in this special case. For even though the monodromy is pseudo-Anosov, the fractional Dehn twist coefficient c can be computed by an algorithm due to Davie to be $\frac{1}{2}$ (cf. [Dav]) and this is precisely the case not covered by Theorem 4.2 of [HKM]. Thus this approach fails as stated, even though it is still possible that ξ_k is a perturbation of a taut foliation for other reasons.

Remark 7.5.7. If one considers the stronger relation of *J-tamed* concordance, then there is a more elementary proof of Theorem 7.5.4. Here a symplectic concordance Σ is J-tamed if there is a non-singular J-convex function ϕ on $S^3 \times [0,1]$ so that $\phi(S^3 \times 0) = 0$ and $\phi(S^3 \times 1) = 1$ for some almost complex J with respect to which Σ is J-holomorphic. If such a cobordism exists, then the branched double cover over Σ is a cobordism from M_K to S^3 and contains only 1- and 2-handles in equal number. From this one sees immediately that M_K must have trivial first homology. We saw in the proof of Theorem 7.5.4 that $H_1(M_{K_n})$ is non-trivial and, hence, there can exist no J-tamed concordance form K_n to K_0. In fact, such a concordance gives a ribbon concordance as defined by Gordon and there are severe restrictions on the existence of such concordances (cf. [Gor]).

Appendix A

Five-term exact sequences

To any extension of groups $1 \to N \to G \to Q \to 1$ one may associate a five-term exact sequence is group cohomology of the following form:

$$1 \longrightarrow H^1(Q,R) \longrightarrow H^1(G,R) \longrightarrow H^1(N,R)^Q \xrightarrow{\delta} H^2(Q,R) \longrightarrow H^2(G,R)\ .$$

This exact sequence is generally derived by means of the Serre spectral sequence, but we choose to give an alternate description in order to obtain an explicit description of the connecting homomorphism. This account is based on the analogy with the definition of Euler classes in the case of central extensions (cf. [Bro]).

Lemma A.1. *Let $1 \to N \to G \xrightarrow{\pi} Q \to 1$ be an extension of groups and let S denote a normalised set-theoretic section of the final map so that $s(e.N) = e$. Further let $\phi \in H^1(N,R)^Q$ lie in the invariant part of $H^1(N,R)$ for any coefficient ring R. Define*

$$\phi_S(g) = \phi(n_g) + f(s(N.g))$$

where $n_g \in N$ is the unique element such that $g = n_g.s(N.g)$ and f is a function on the set of coset representatives determined by s.

Then the map $\bar{\delta} : H^1(N,R) \to H^2(Q,R)$ defined by $\bar{\delta}\phi = \overline{\delta\phi_S}$ is well-defined and the five-term sequence is exact with $\bar{\delta}$ as the connecting homomorphism. Furthermore if $\frac{1}{2} \in R$, then we may assume that $\phi_S(g^{-1}) = -\phi_S(g)$.

Proof. First we note by definition

$$\phi_S(n.g) = \phi_S(n) + \phi_S(g) - f(s(N.e))$$

for all $n \in N,\ g \in G$. Thus using bracket notation for inhomogeneous chains (cf. [Bro], p. 36) we calculate

$$\begin{aligned}(\delta\phi_S)(na,b) &= \phi_S(na) - \phi_S(nab) + \phi_S(b)\\ &= \phi_S(n) - f(s(N.e)) + \phi_S(a) - \phi_S(n) + f(s(N.e)) - \phi_S(ab) + \phi_S(b)\\ &= (\delta\phi_S)(a,b).\end{aligned}$$

Similarly we compute

$$\begin{aligned}(\delta\phi_S)(a,nb) &= \phi_S(a) - \phi_S(anb) + \phi_S(nb)\\ &= \phi_S(a) - \phi_S(ana^{-1}) + f(s(N.e)) - \phi_S(ab) + \phi_S(n) - f(s(N.e)) + \phi_S(b)\\ &= (\delta\phi_S)(a,b),\end{aligned}$$

where we have used the fact that ϕ lies in the invariant part of the first cohomology to deduce the last equality. Thus $\delta\phi_S$ does not depend on the coset representative and, hence, descends to a well-defined cochain $\overline{\delta\phi_S} \in C^2(Q,R)$. This cochain is by construction closed and we claim that it is independent of the choice of section. For given another section S', we have

$$(\phi_S - \phi_{S'})(ng) = (\phi_S - \phi_{S'})(g),$$

so this difference descends to a cochain $\psi = \overline{\phi_S - \phi_{S'}} \in C^1(Q,R)$ and

$$\overline{\delta\phi_S} - \overline{\delta\phi'_S} = \delta\psi.$$

Similarly, if we choose different functions f, f' in the definition of $\phi_{S,f}, \phi_{S,f'}$ respectively, then we see that

$$\phi_{S,f} - \phi_{S,f'} = f - f' \in C^1(Q,R),$$

and thus $\overline{\delta\phi_{S,f}} - \overline{\delta\phi_{S,f'}}$ is exact in $H^2(Q,R)$.

Next we claim that $[\overline{\delta\phi_S}] = 0$ if and only if ϕ extends to a homomorphism on G. If ϕ extends to $\tilde{\phi}$ this is clear as we set $f = \tilde{\phi}$ in the definition of ϕ_S so that $\phi_S = \tilde{\phi}$ and $\delta\phi_S = 0$.

Conversely, assume that $\overline{\delta\phi_S} = \delta\psi$ for $\psi \in C^1(Q,R)$. We set $f = \pi^*\psi$ in the definition of ϕ_S and define $\tilde{\phi} = \phi_S - \pi^*\psi$. This is a set-theoretic extension of ϕ and we compute

$$\delta\tilde{\phi} = \delta\phi_S - \pi^*\delta\psi = \delta\phi_S - \pi^*\overline{\delta\phi_S} = 0.$$

Thus ϕ extends to a homomorphism on G and we have shown exactness at $H^1(N,R)^Q$.

By construction the composition

$$H^1(N,R)^Q \xrightarrow{\bar{\delta}} H^2(Q,R) \xrightarrow{\pi^*} H^2(G,R)$$

is zero. Next we need to check exactness at $H^2(Q,R)$. To this end let $\alpha \in C^2(Q,R)$ and assume $\pi^*\alpha = \delta\beta$. Then $\pi^*\alpha(n,g) = \pi^*\alpha(e,g)$ if and only if

$$\beta(n) - \beta(ng) + \beta(g) = \beta(e) - \beta(g) + \beta(g) = \beta(e)$$

and by adding an *exact* constant cochain to $\pi^*\alpha$, we may assume that $\beta(e) = 0$. So in fact $\beta(ng) = \beta(n) + \beta(g)$ for all $n \in N,\ g \in G$. In particular β is a homomorphism on N. Similarly, we deduce from the equation $\pi^*\alpha(g,n) = \pi^*\alpha(g,e)$ that $\beta(gn) = \beta(g) + \beta(n)$ for all $n \in N,\ g \in G$. Then using the fact that $\pi^*\alpha(gng^{-1},g) = \pi^*\alpha(e,g)$ we see that $\beta(gng^{-1}) = \beta(n)$. Thus setting $f = \beta$ and $\phi = \beta|_N$ we obtain $\phi_S = \beta$ for any section and $[\alpha] = \bar{\delta}\phi$.

Finally, given any normalised section S we obtain another S^{-1} by inverting elementwise. We set $f = 0$ and define

$$\phi_{Sym}(g) = \frac{1}{2}(\phi_S(g) - \phi_{S^{-1}}(g^{-1})),$$

to obtain a cochain that is antisymmetric under inversion and such that $\delta\phi_{Sym} = \delta\phi$. □

This alternative connecting homomorphism is natural in the following sense.

Lemma A.2. *Consider the following commuting diagramm of group extenstions*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & Q \arrow[r] & 1 \\ 1 \arrow[r] & N' \arrow[r] \arrow[u, "\beta"] & G' \arrow[r] \arrow[u] & Q' \arrow[r] \arrow[u, "\gamma"] & 1. \end{tikzcd}$$

This induces a commutative diagram of five-term exact sequences

$$\begin{tikzcd} 1 \arrow[r] & H^1(Q,R) \arrow[r] \arrow[d, "\gamma^*"] & H^1(G,R) \arrow[r] \arrow[d] & H^1(N,R)^Q \arrow[r, "\bar{\delta}"] \arrow[d, "\beta^*"] & H^2(Q,R) \arrow[r] \arrow[d, "\gamma^*"] & H^2(G,R) \arrow[d] \\ 1 \arrow[r] & H^1(Q',R) \arrow[r] & H^1(G',R) \arrow[r] & H^1(N',R)^{Q'} \arrow[r, "\bar{\delta}"] & H^2(Q',R) \arrow[r] & H^2(G',R). \end{tikzcd}$$

Proof. Choose normalised set-theoretic sections s, s' so that the following diagramm commutes:

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & Q \arrow[r] \arrow[l, bend left, "s"] & 1 \\ 1 \arrow[r] & N \arrow[r] \arrow[u, "\beta"] & G' \arrow[r] \arrow[u] & Q' \arrow[r] \arrow[u, "\gamma"] \arrow[l, bend left, "s'"] & 1. \end{tikzcd}$$

We set $f = 0$ in the definition of $\bar{\delta}$ given in Lemma A.1. Then on the level of cochains we have

$$\gamma^*\bar{\delta}\phi = \gamma^*\delta\phi_S = \delta\gamma^*\phi_S = \delta(\gamma^*\phi)_{S'} = \bar{\delta}(\gamma^*\phi).$$

As commutativity is clear for all other squares the result follows. □

As a consequence of Lemma A.2 we see that $\bar{\delta}$ agrees with the ordinary connecting homomorphism δ up to sign if we consider cohomology with *real coefficients*.

Lemma A.3. *Let δ be the connecting homomorphism in the five-term exact sequence and let $\bar{\delta}$ be the map defined in Lemma* A.1. *Then $\bar{\delta} = \pm\delta$ on cohomology with* real coefficients.

Proof. Consider a group extension

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

and let $\phi \in H^1(N,\mathbb{Z})^Q$. Further let $\sigma \in H_2(Q,\mathbb{Z})$ be a homology class represented by a map $h : \Sigma \to K(Q,1)$ from some closed, oriented surface. By taking the pullback of the above extension under h and applying Lemma A.2 we have a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1(Q,\mathbb{Z}) \arrow[r] \arrow[d, "\gamma^*"] & H^1(G,\mathbb{Z}) \arrow[r] \arrow[d] & H^1(N,\mathbb{Z})^Q \arrow[r, "\bar{\delta}"] \arrow[d] & H^2(Q,\mathbb{Z}) \arrow[r] \arrow[d, "\gamma^*"] & H^2(G,\mathbb{Z}) \arrow[d] \\ 0 \arrow[r] & H^1(Q',\mathbb{Z}) \arrow[r] & H^1(G',\mathbb{Z}) \arrow[r] & H^1(N,\mathbb{Z})^{Q'} \arrow[r, "\bar{\delta}"] & H^2(Q',\mathbb{Z}) = \mathbb{Z} \arrow[r] & H^2(G',\mathbb{Z}). \end{tikzcd}$$

By exactness the maps $\delta, \bar{\delta} : H^1(N, \mathbb{Z})^{Q'} \to H^2(Q', \mathbb{Z}) = \mathbb{Z}$ have the same kernel and image. Any two homomorphisms from a group to $\mathbb{Z}$ with this property must agree up to sign. Since this holds for *all* elements of $H_2(Q, \mathbb{Z})$, we conclude that $\delta\phi = \pm\bar{\delta}\phi$ as elements in $Map(H_2(Q, \mathbb{Z}), \mathbb{Z})$ and by the Universal Coefficient Theorem this also holds in real cohomology. $\square$

It will be convenient to give a slightly different formulation of Lemma A.1 for performing calculations.

Lemma A.4. *Let $1 \to N \to G \to Q \to 1$ be an extension of groups and let $f \in H^1(N, R)^Q$. Further let $\bar{f}$ be an extension of f to G such that $\overline{f}(n.g) = \overline{f}(n) + \overline{f}(g)$ for all $n \in N$ and $g \in G$. Then for any $[g_1], [g_2] \in Q$ in the quotient we have a representative cocycle for $\bar{\delta}\phi$ such that*

$$\bar{\delta} f([g_1], [g_2]) = \overline{f}(g_1) + \overline{f}(g_2) - \overline{f}(g_1.g_2).$$

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