Hamiltonian unknottedness of certain monotone Lagrangian tori in $S^2 \times S^2$

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Zusammenfassung

Die Klassifikation von Lagrange Untermannigfaltigkeiten in symplektischen Mannigfaltigkeiten bis auf Isotopie (Lagrange, Symplektische und Hamiltonsche) ist eine schwere und interessante Frage. Bekanntes in diesem Gebiet beschränkt sich vornehmlich auf das Problem der Lagrange Isotopie, da unter den genannten Typen der Isotopie dieses das einfachste ist.

In der vorliegenden Arbeit beweisen wir die Klassifikation von monotonen Lagrange Tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ bis auf Hamiltonsche Isotopie für eine besondere Klasse von monotonen Lagrange Tori unter einer zusätzlichen Annahme.

Die Klasse von monotonen Lagrange Tori, die wir betrachten sind die sogenannten gefaserten monotonen Lagrange Tori. Ein Lagrange Torus L in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ heißt gefasert, falls es eine Blätterung $\mathcal{F}$ von $S^2 \times S^2$ durch symplektische 2-Sphären in der Homologieklasse $[pt \times S^2]$ und eine kompakte, symplektische Untermannigfaltigkeit Σ' (Schnitt der Blätterung) in der Klasse $[S^2 \times pt]$ mit den folgenden Eigenschaften gibt:

- Σ' ist transversal zu den Blättern von $\mathcal{F}$ und Σ' ist disjunkt zu L;
- $\mathcal{F}$ induziert eine Blätterung von L durch Kreise (die Blätter von $\mathcal{F}$ schneiden L in Kreisen).

Die Motivation, diese Klasse von monotonen Lagrange Tori zu betrachten, kommt aus der Doktorarbeit von A. Ivrii [12], in welcher er unter anderem beweist, dass jeder Lagrange Torus in $S^2 \times S^2$ gefasert ist.

Wir beweisen in der vorliegenden Arbeit den Satz 2.5.1, dass ein gefaserter monotoner Lagrange Torus zu dem es noch einen zweiten symplektischen Schnitt Σ mit bestimmten Eigenschaften gibt, Hamiltonsch isotop zum Standard Torus ist. Der Standard Torus ist der monotone Lagrange Torus, der aus den Äquatoren in den beiden kartesischen Faktoren gebildet wird. Es ist bekannt [28],[24],[25],[26],[27] und [23], dass es in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ exotische monotone Lagrange Tori gibt. Es folgt deshalb sofort, dass es den zweiten symplektischen Schnitt wie in unserem Satz gefordert für diese Tori nicht geben kann.

Ausblickend in die Zukunft kann man deshalb hoffen, dass die Klassifikation von monotonen Lagrange Tori in $S^2 \times S^2$ in den Bereich des Möglichen gelangt, falls man die Bedingungen versteht, unter denen der zweite symplektische Schnitt existiert.

Abstract

The classification of Lagrangian submanifolds in symplectic manifolds up to isotopy (Lagrangian, symplectic and Hamiltonian) is a hard and interesting question. Known results in this area concern mainly the problem of Lagrangian isotopy, since among the types of isotopy mentioned above this is the easiest case.
In the following thesis, we prove the classification of monotone Lagrangian tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ up to Hamiltonian isotopy for a special class of monotone Lagrangian tori under an additional assumption.
The class of monotone Lagrangian tori considered in this thesis are *fibered* monotone Lagrangian tori. A Lagrangian torus L in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ is called fibered if there exists a foliation $\mathcal{F}$ of $S^2 \times S^2$ by symplectic 2-spheres in the homology class $[pt \times S^2]$ and a compact symplectic submanifold Σ' in class $[S^2 \times pt]$ with the following properties:

- Σ' is transverse to the leaves of $\mathcal{F}$ and is disjoint from L;
- $\mathcal{F}$ induces a foliation of L by circles (the leaves of $\mathcal{F}$ intersect L in circles).

The motivation to consider this class of monotone Lagrangian tori is A. Ivrii's PhD thesis [12] in which he proves among other things that any Lagrangian torus in $S^2 \times S^2$ is fibered. The theorem we prove in this thesis (Theorem 2.5.1) states that a fibered monotone Lagrangian torus for which there exists a second symplectic section Σ with certain properties, is Hamiltonian isotopic to the standard torus L_{std}.
L_{std} is the monotone Lagrangian torus made up of the equators in both cartesian factors. It is known [28],[24],[25],[26],[27] and [23] that there exist exotic monotone Lagrangian tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$. Consequently, the second symplectic section as described above cannot exist for these tori.
As an outlook, one can hope that the classification of monontone Lagrangian tori in $S^2 \times S^2$ comes within reach if we understand the conditions under which the second symplectic section exists.

Contents

Chapter 1

Introduction

The classification of Lagrangian submanifolds in symplectic manifolds up to Lagrangian, symplectic or Hamiltonian isotopy is an interesting problem. Mainly, the known results concern Lagrangian spheres or tori in symplectic manifolds of dimension four. Results in the area are R. Hinds classification up to Hamiltonian isotopy of Lagrangian 2-spheres in $S^2 \times S^2$ with the standard product symplectic form in 2004 [22] and very recently, J. Evans paper [21] on the Lagrangian unknottedness of Lagrangian spheres in certain Del-Pezzo surfaces.
Another result which is of importance for this thesis is A. Ivriis PhD thesis about the Lagrangian unknottedness of Lagrangian tori in $\mathbb{R}^4$, $S^2 \times S^2$, $\mathbb{CP}^2, T^*\mathbb{T}^2$ with the standard symplectic forms in 2003 [12].
We address the question of Hamiltonian (un-)knottedness of a certain class of monotone Lagrangian tori in $S^2 \times S^2$.
A Lagrangian torus is called monotone if the symplectic area of any relative 2-cycle with boundary on the Lagrangian torus is a fixed multiple of its Maslov index. The cartesian product of the equators in each S^2-factor in $S^2 \times S^2$ is called the standard Lagrangian torus L_{std} (or the Clifford torus). This torus is monotone for the symplectic form $\omega_{std} \oplus \omega_{std}$. It is known by results of Chekanov-Schlenk [28], Entov-Polterovich [24], Biran-Cornea [25], Fukaya-Oh-Ohta-Ono [26], Albers-Frauenfelder [27] and Yau [23], that there exist monotone Lagrangian tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ which are not Hamiltonian isotopic.
Two such are the Clifford torus and the Chekanov-Schlenk Torus L_{CS} (one of the tori constructed in [28]).
In this thesis, we consider L a monotone Lagrangian torus in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$.
Ivrii's result motivates the following definition. A monotone Lagrangian torus L in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ is called fibered if there exists a foliation $\mathcal{F}$ of $S^2 \times S^2$ by symplectic 2-spheres in the homology class $[pt \times S^2]$ and a symplectic submanifold Σ' in class $[S^2 \times pt]$ with the following properties:

- Σ' is transverse to the leaves of $\mathcal{F}$ and is disjoint from L;

- $\mathcal{F}$ induces a foliation of L by circles (the leaves of $\mathcal{F}$ intersect L in circles).

We will also say that L is fibered by $\mathcal{F}$ and Σ' if we want to name the foliation and the section explicitly in the definition.
Note that Σ' singles out a disk in each leaf of $\mathcal{F}$ which intersects the torus L. These disks together form a solid torus T with $\partial T = L$.
One part of Ivrii's result now says that any monotone Lagrangian torus in $S^2 \times S^2$ is fibered.
We prove the following

Theorem (Main Theorem)**.** *Let $L \subset (S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ be a monotone Lagrangian torus which is fibered by $\mathcal{F}$ and Σ'.*
Assume in addition, that there exists a symplectic submanifold Σ in homology class $[S^2 \times pt]$ which is transverse to the leaves of $\mathcal{F}$ and which is disjoint from Σ' and T. Then L is Hamiltonian isotopic to the standard torus L_{std}.

As an immediate consequence, for the torus L_{CS} constructed by Y.Chekanov and F.Schlenk, there cannot exist the additional section Σ. This instantly rises the question whether the classification of monotone Lagrangian tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ up to Hamiltonian isotopy comes within reach if we understand the rôle of the second section Σ.
Now we turn towards outlining the proof of the Main Theorem. By a *relative symplectic fibration* on $S^2 \times S^2$, we mean a quintuple of the form

$$(\mathcal{F}, \omega, L, \Sigma, \Sigma'),$$

where L is a monotone Lagrangian torus for the symplectic form ω and L is fibered by $\mathcal{F}$ and Σ'. Further Σ is an additional symplectic section with properties as required for the Main Theorem.
Now note that the triple $(\mathcal{F}, \Sigma, \Sigma')$ is diffeomorphic to $(\mathcal{F}_{std}, S^2 \times \{N\}, S^2 \times \{S\})$ where $\mathcal{F}_{std}$ denotes the standard foliation on $S^2 \times S^2$ given by the fibers of the projection $p_1 \colon S^2 \times S^2 \to S^2$; $(x_1, x_2) \mapsto x_1$. On the other hand ω is always diffeomorphic to $\omega_{std} \oplus \omega_{std}$ (see [3]). So each of ω and $\mathcal{F}$ on its own is not interesting, but the pair of them carries interesting structure. Important in the following is the symplectic curvature of the symplectic connection defined by $(\mathcal{F}, \omega, \Sigma')$. The symplectic connection on a symplectic fiber bundle is given by the symplectic orthogonal complements to the tangent spaces to the fibers. A priori, the symplectic curvature is a two-form on the base with values in the vertical symplectic vector fields. But as the curvature identity tells us, the curvature vector field is Hamiltonian, so that we can regard the symplectic curvature as a two form on the base with values in the functions (the Hamiltonians) on the fiber.
The main step in the proof of the main theorem is to find a deformation of a given relative symplectic fibration to one with vanishing symplectic curvature. If a relative symplectic fibration has vanishing symplectic curvature, then using symplectic parallel transport, we can write down explicitly a symplectomorphism which maps the foliation to the standard foliation, the Lagrangian torus to the standard torus, and which is the identity on homology. By a theorem of Gromov, there exists a symplectic isotopy from the identity to ϕ.

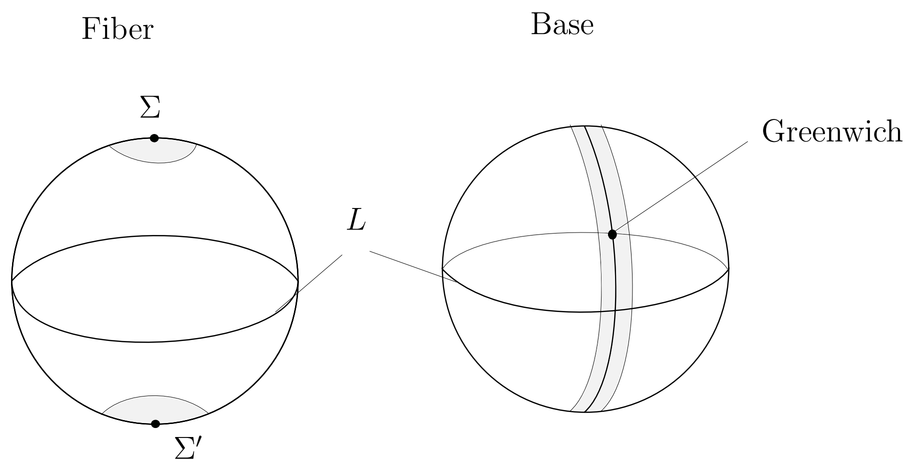

Figure 1.1: Where the symplectic curvature vanishes after the first step.

Since $S^2 \times S^2$ is simply-connected, this symplectic isotopy is Hamiltonian.
Thus, most of the work will go into showing the existence of a deformation of relative symplectic fibrations to one with vanishing symplectic curvature. In the first step, which makes up Chapter 2, we kill the symplectic curvature near the two sections Σ, Σ' and near a leaf of $\mathcal{F}$ intersecting L.
After applying the diffeomorphism described before, we can therefore assume that $(\mathcal{F}, \Sigma, \Sigma')$ is the standard fibration p_1 and Σ, Σ' are the constant sections at the north and southpole. Moreover we can assume that the symplectic curvature vanishes near the two sections and near the fibers over the line of longitude through Greenwich in the base.

Let (λ, μ) be spherical polar coordinates on S^2 where λ denotes the latitude and μ denotes the longitude.
In step 2 (first part of chapter 3) we kill the monodromy along all circles of latitude C^λ. Observe that after Step 1 the monodromy maps ϕ^λ along C^λ give a loop in $Ham(A, \partial A, \omega_{std})$, the group of Hamiltonian symplectomorphisms of the annulus which are fixed in some neighbourhood of the boundary. Since the fundamental group of $Ham(A, \partial A, \omega_{std})$ vanishes, we can contract the loop $\psi^\lambda = (\phi^\lambda)^{-1}$ and obtain a family of Hamiltonians H^λ_μ which generate the contraction.
Now consider the closed two form

$$\Omega = \omega + d(H^\lambda_\mu d\mu).$$

This form gives a symplectic connection whose monodromies along C^λ are the identity. However Ω need not be symplectic if $\frac{\partial H^\lambda_\mu}{\partial \lambda}$ is large. This can be remedied by the inflation procedure due to McDuff and Lalonde [3]. To keep the Lagrangian torus monotone in the inflation, we have to make some modifications in the procedure. Consider a leaf $F \cong S^2$ intersecting L in the equator. Monotonicity of L forces the upper hemisphere D_{uh} and the lower hemisphere D_{lh} in F to have symplectic area $\frac{1}{2}$. Then if we alter the symplectic form in the inflation procedure and we want to keep L fixed and monotone, we have to make sure that both D_{uh} and D_{lh} keep their symplectic area. This "symmetrical" inflation is the reason why we have to use two symplectic sections, whereas usually only

one submanifold suffices. After this step, we can assume that the monodromy along the circles of latitude is the identity.
In step 3 (the second part of chapter 3) we write down a explicit deformation of the symplectic form to obtain vanishing symplectic curvature. This finishes the outline of the proof.

Chapter 2

Setup

2.1 Framework

In the following M will be a compact smooth 4-manifold which is diffeomorphic to $S^2 \times S^2$. Further by $L \cong T^2$ we mean an embedded two torus in M.
The second homology $H_2(M;\mathbb{R}) \cong \mathbb{R}^2$. Fix a diffeomorphism $\theta\colon S^2 \times S^2 \to M$. Then we define $A := [\theta(S^2 \times \{pt\})]$ and $B := [\theta(\{pt\} \times S^2)]$. Since θ_* is an isomorphism A, B span $H_2(M)$. Let

$$p_i\colon S^2 \times S^2 \to S^2$$

$$(x_1, x_2) \mapsto x_i.$$

Definition 2.1.1. *A symplectic form ω on $S^2 \times S^2$ which is of the form*

$$\omega = p_1^*\omega_1 + p_2^*\omega_2$$

for symplectic forms $\omega_1, \omega_2 \in \Omega^2(S^2)$ is called split.

In Appendix A the symplectic form ω_{std} on S^2 with area 1 is defined.

Definition 2.1.2. *The symplectic form*

$$\omega_0 = p_1^*\omega_{std} + p_2^*\omega_{std}$$

on $S^2 \times S^2$ is called the standard symplectic form *on $S^2 \times S^2$.*

Definition 2.1.3. *Consider $S^2 \times S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ in the standard way with standard coordinates $((x,y,z),(u,v,w))$ on $\mathbb{R}^3 \times \mathbb{R}^3$. Then let*

$$L_{std} := \big\{((x,y,z),(u,v,w)) \in S^2 \times S^2 | z = 0 \ ; w = 0\big\}$$

be the cartesian product of the two equators in S^2. We will call L_{std} the Clifford torus in $S^2 \times S^2$.

Remark
We will call $\theta_*\omega_0$ the standard symplectic form on M and for convenience we will also denote it by ω_0.
In the sequel we will consider M with various symplectic forms ω. The standard form ω_0 will often be used as a "reference" symplectic form.
We will also call $\theta(L_{std})$ the Clifford torus in M and denote it by L_{std}.
Next we will call $p_i \circ \theta^{-1} \colon M \to S^2$ the standard projections on M and denote them by p_i.

2.2 Symplectic foliations, fibrations and the symplectic connection

2.2.1 Foliations

Let X be a smooth manifold of dimension n.

Definition 2.2.1. *A foliation $\mathcal{F}$ of dimension k on X is given by an open covering $\{U_\alpha\}_{\alpha\in A}$ and charts $\phi_\alpha \colon V_\alpha \to U_\alpha$ with $V_\alpha \subset \mathbb{R}^n$ such that for all α, β the transition function*

$$\phi_{\alpha\beta} \colon \phi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

$$x \mapsto \phi_\beta \circ \phi_\alpha(x)$$

maps (subsets of) the fibers of the standard fibration $p \colon \mathbb{R}^n \to \mathbb{R}^{n-k}$; $(x_1, ..., x_n) \mapsto (x_1, ..., x_{n-k})$ to (subsets of) fibers of p. Then $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ as above is called a foliating atlas of X for $\mathcal{F}$.

Remark
Let X be a smooth manifold and $\phi \colon \mathbb{R}^n \to U \subset X$ a chart, then $\phi(\{x_1, .., x_{n-k}\} \times \mathbb{R}^k)$ defines a family of k-dimensional submanifolds on X. A foliation on X is then to say that these families of submanifolds defined by different charts match up.
Example
Consider the 2-torus T^2 as $\mathbb{R}^2/\mathbb{Z}^2$. Then a line of irrational slope through the origin defines a foliation of T^2.
Another example are the fibers of any smooth surjective submersion $\pi \colon X \to B$. Foliations of this type are called simple foliations.

Definition 2.2.2. *Let $\mathcal{F}$ be a foliation of dimension k on X. Then let $x \in X$ and consider a chart (U_α, ϕ_α) around x of a foliating atlas for $\mathcal{F}$. Then this chart defines a submanifold S_x of dimension k through x. Now we define the leaf $\mathcal{F}_x$ of $\mathcal{F}$ through the point x as the set of points on X which can be connected to x by paths lying entirely in S_x or its continuations by other foliating charts.*

Remark
If we speak of a *foliation of X by 2-spheres* we mean a foliation $\mathcal{F}$ of X whose leaves are all diffeomorphic to the 2-sphere.

We want to see how we can alter a foliation.

Definition 2.2.3. *A smooth family of foliations* $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ *of dimension* k *on the manifold* X *is defined to be a foliation* $\mathcal{F}$ *of dimension* $k+1$ *on* $X \times \mathbb{R}$ *such that* $\mathcal{F}$ *restricts to* $\mathcal{F}_t$ *on* $X \times \{t\}$.

Remark
Let ϕ be a diffeomorphism of X and $\mathcal{F}_t$ a smooth family of foliations on X then $\phi(\mathcal{F}_t)$ is also a smooth family of foliations on X. Consider the diffeomorphism $\Phi = \phi \times id$ of $X \times \mathbb{R}$ then $\Phi(\mathcal{F})$ is the desired foliation of $X \times \mathbb{R}$.

Theorem 2.2.4. *Let* $\mathcal{F}_0$ *be a foliation of dimension* k *of the* n*-dimensional manifold* X*. Assume that there exists an embedding* $G \colon U \times F \to X$ *with* U *open in* $\mathbb{R}^{n-k}$ *such that* G *maps* $\{x\} \times F$ *diffeomorphically onto a leaf of* $\mathcal{F}$ *for all* $x \in U$*. Let* $V \subset U$ *be an open set in* $\mathbb{R}^{n-k}$ *with* $\bar{V} \subset U$ *and assume that*

$$G_s \colon U \times F \to X$$

for $s \in \mathbb{R}$ *is a smooth family of embeddings such that* $G_s|_{(U\setminus\bar{V})\times F} = G|_{(U\setminus\bar{V})\times F}$ *then the embeddings* G_s *define a smooth family of foliations* $\mathcal{F}_s$ *on* X*.*

Proof. A detailed proof will be given in the appendix. Note first that all the embeddings G_s are local diffeomorphisms for dimensional reasons and thus $G(U \times F)$ is an open set of X. Since the embeddings G_s agree on $(U \setminus \bar{V}) \times F$ it follows that the image $G_s(U \times F) = G(U \times F)$ is fixed for all s. We write $Z = G(\bar{V} \times F)$ and $Y = G(U \times F)$. Then Z is closed and Y is open in X. First we want to show that using the embeddings G_s we can define foliating charts for all s, so that indeed we get a foliation $\mathcal{F}_s$ on X for every s.
If $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ is a foliating atlas for $\mathcal{F}_0$ then we can define a new atlas by restricting the old atlas to $(\tilde{U}_\alpha = U_\alpha \cap (X \setminus Z), \tilde{\phi}_\alpha = \phi_\alpha|_{\tilde{U}_\alpha})$. And choosing as new foliating charts on Y,

$$(U \times W'_\beta, G_s \circ (id \times \psi_\beta))$$

where $\psi_\beta \colon W'_\beta \to W_\beta \subset F$ are charts of F and $U \times W'_\beta \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ is open. By construction, the restricted charts coming from the old foliation match up with the new ones on the overlaps. Also clearly the new foliating charts satisfy the foliation condition. Thus indeed this defines a foliation $\mathcal{F}_s$ on X.
We are left to show that the foliations $\mathcal{F}_s$ form a smooth family. Therefore we only have to choose a smooth way to group leaves of the foliations $\mathcal{F}_s$ together to form a leaf of $\mathcal{F}$. Write $\bar{G} \colon U \times F \times \mathbb{R} \to X \times \mathbb{R}$; $(x, s) \mapsto (G_s(x), s)$. Then $\bar{G}$ is an embedding. Thus we define the foliation $\mathcal{F}$ on $X \times \mathbb{R}$ by specifying its leaves through any point:

$$\mathcal{F}_{x,t} = \begin{cases} (\mathcal{F}_0)_x \times \mathbb{R} \text{ for } x \notin Z \\ \bar{G}(\{x\} \times F \times \mathbb{R}) \text{ for } x \in Y \end{cases}$$

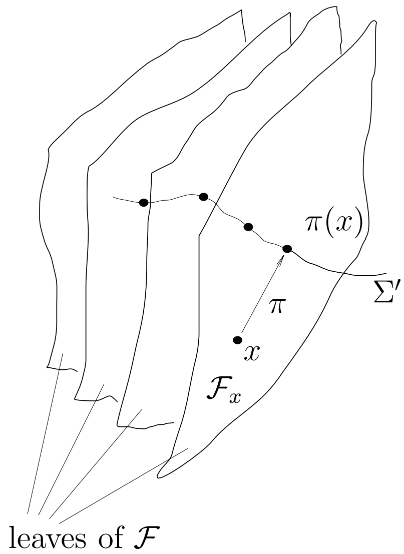

Figure 2.1: The map π

Then on $Y - Z$ the two definitions agree due to the fact that G_s is fixed to G there. Clearly $\mathcal{F}$ restricts to $\mathcal{F}_s$ on $X \times \{s\}$. This proves the theorem. $\square$

The following theorem will be used frequently in the sequel.

Theorem 2.2.5. *Let $\mathcal{F}$ be a foliation of M by symplectic 2-spheres. Further let Σ be a submanifold of M which is transverse to $\mathcal{F}_q$ for all $q \in M$. Then Σ is diffeomorphic to S^2, Σ intersects every leaf of $\mathcal{F}$ in a single point and the map*

$$\pi\colon M \to \Sigma$$

$$q \in \mathcal{F}_q \mapsto \mathcal{F}_q \cap \Sigma$$

is a surjective submersion. Moreover there exist diffeomorphisms $\phi\colon M \to S^2 \times S^2$ and $u\colon \Sigma \to S^2$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & S^2 \times S^2 \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle p_1} \\ \Sigma & \xrightarrow{u} & S^2 \end{array} \tag{2.1}$$

Proof. A detailed proof is given in Appendix B. Let $\mathcal{B} = M/\sim$ be the set of equivalence classes of the equivalence relation $\sim$ on the set of points of M which is defined by

$$p \sim q \quad iff \quad q \in \mathcal{F}_p$$

i.e. if both points belong to the same leaf of $\mathcal{F}$. For a foliation $\mathcal{F}$ of a connected manifold M whose leaves are all simply connected and compact, it follows that $p\colon M \to \mathcal{B};\ q \mapsto [q]$

given by the projection to the leaf space is a smooth fibration. Consequently, the leaf-space $\mathcal{B}$ is a closed orientable 2-manifold. Further it follows that the Euler-Characteristics of the spaces involved satisfy

$$\chi(M) = \chi(S^2)\chi(\mathcal{B}).$$

Thus $\chi(\mathcal{B}) = 2$ and so $\mathcal{B}$ is diffeomorphic to S^2. Let $u\colon \mathcal{B} \to S^2$ be a diffeomorphism then $u \circ p\colon M \to S^2$ is a S^2-bundle over S^2. But there are only two such bundles, the trivial one and a non-trivial one. Note that the intersection forms of the total spaces of the two S^2-bundles differ. But M is diffeomorphic to $S^2 \times S^2$ which is the trivial S^2-bundle over S^2. Hence it has the intersection form of the trivial S^2-bundle and consequently $u \circ p$ is the trivial S^2-bundle over S^2. Hence there exists a trivialisation:

$$\begin{array}{ccc} M & \xrightarrow{\tau} & S^2 \times S^2 \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p_1} \\ \mathcal{B} & \xrightarrow{u} & S^2 \end{array}$$

Now push Σ forward under τ. Then $\tau(\Sigma)$ is transverse to $\{q\} \times S^2$ for all $q \in S^2$. This implies however that $p_1|_{\tau(\Sigma)}\colon \tau(\Sigma) \to S^2$ is a covering and so by simply-connectedness of S^2, $p_1|_{\tau(\Sigma)}$ is a diffeomorphism. Hence $\tau(\Sigma)$ is the image of a section σ of p_1. But then $\sigma' = \tau^{-1} \circ \sigma \circ u$ defines a section of p with image Σ and $\pi = \sigma' \circ p$. From this it is clear that π is a smooth surjective submersion and that Σ is diffeomorphic to S^2. The existence of the trivialisation can be deduced as above for the space of leaves $\mathcal{B}$. This proves the theorem. □

Definition 2.2.6. *Let $\mathcal{F}$ be a smooth foliation of M. Then $\mathcal{F}$ is called a symplectic foliation if every leaf of $\mathcal{F}$ is symplectic.*

Remark

We will exclusively study symplectic foliations $\mathcal{F}$ on M whose leaves are symplectic 2-spheres and for which there exists a section Σ as in 2.2.5. Then by the theorem, the symplectic foliations in our setup can always be thought of as being symplectic fibrations. The foliation obtained by the fibers of the standard projection p_1 is called the standard foliation and is denoted by $\mathcal{F}_{std}$.

2.2.2 Symplectic fibrations and symplectic vector bundles

Definition 2.2.7. *A fibration $p\colon M \to B$ is called a* symplectic fibration *if $(p^{-1}(b), \omega|_{p^{-1}(b)})$ is a symplectic submanifold of (M, ω) for all $b \in B$.*

Definition 2.2.8. *Let $p\colon M \to B$ be a symplectic fibration with fiber diffeomorphic to S^2, then p is called a* symplectic ruling *of M.*

In particular if $\pi\colon M \to B$ is a symplectic ruling, then the proof of 2.2.5 shows that B is diffeomorphic to S^2 and that π is the trivial S^2-bundle over S^2.

Symplectic vector bundles

Definition 2.2.9. *Let $\pi\colon E \to M$ be a smooth vector bundle. Then (π, E, M, ω) is a symplectic vector bundle if ω is a non-degenerate 2-form on E, i.e. a smooth section of the bundle*

$$E^* \wedge E^* \to M$$

which restricts to a non-degenerate form on every fiber.

Remark
If (X, ω) is a symplectic manifold and $\pi\colon TX \to X;\ v \in T_xX \to x$ then (π, TX, X, ω) is a symplectic vector bundle.

2.2.3 The symplectic connection and its curvature

The symplectic connection and some properties

Definition 2.2.10. *Let (X, ω) be a symplectic manifold and $p\colon X \to B$ be a symplectic fibration. Then the symplectic form defines a connection on p via a splitting*

$$TX = H \oplus \ker dp$$

given by the symplectic orthogonal complements to the tangent spaces of the fibers of p

$$H_x = (\ker dp_x)^{\perp_\omega}.$$

This is called the symplectic connection *induced by (p, ω) on X.*

Then we have

Proposition 2.2.11. *Let (X, ω) be a symplectic manifold and $p\colon X \to B$ be a symplectic fibration. Then the parallel transport $P_\gamma\colon p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$ for the symplectic connection induced by (p, ω) on X along the path $\gamma \subset B$ satisfies*

$$P_\gamma^*\omega_{\gamma(1)} = \omega_{\gamma(0)}$$

where ω_x denotes the symplectic form $\omega|_{p^{-1}(x)}$.

Proof. Let Z be the vector field on X defined by the horizontal lift of a vector field Y on B. Let ϕ_t denote the flow of Z and ψ_t denote the flow of Y. Obviously ϕ_t preserves the fibration since a flowline of Z is the horizontal lift of a flow-line of Y. Then $\delta_t := \phi_t|_{p^{-1}(x)}\colon p^{-1}(x) \to p^{-1}(\psi_t(x))$ is a diffeomorphism between the fibers. Consider the restriction of $\phi_t^*\omega$ to the fiber $p^{-1}(x)$ over x. This is just the pull-back $\delta_t^*\omega_{\psi_t(x)}$ of the symplectic form $\omega_{\psi_t(x)}$ under δ_t. If $\delta_t^*\omega_{\psi_t(x)}$ is independent of t then it follows immediately that $\delta_t^*\omega_{\psi_t(x)} = \omega_x$ showing the statement of the proposition that symplectic parallel transport is through fiberwise symplectomorphisms.
Thus consider

$$\frac{d}{dt}\phi_t^*\omega = L_Z\omega = d\iota_Z\omega + \iota_Z d\omega = d\iota_Z\omega.$$

Now let $\iota_x\colon p^{-1}(x) \hookrightarrow X$ be the inclusion of the fiber $p^{-1}(x)$ into X. Then

$$\iota_x^* \frac{d}{dt}\phi_t^*\omega = \frac{d}{dt}\delta_t^*\omega_{\psi_t(x)}.$$

Thus to show that $\delta_t^*\omega_{\psi_t(x)}$ is independent of t it is enough to show that the restriction of $\frac{d}{dt}\phi_t^*\omega$ to the fiber $p^{-1}(x)$ vanishes.
For any 1-form α we have the following identity:

$$d\alpha(v,w) = L_v\alpha(w) - L_w\alpha(v) - \alpha([v,w]).$$

Thus

$$d\iota_Z\omega(v,w) = L_v\iota_Z\omega(w) - L_w\iota_Z\omega(v) - \iota_Z\omega([v,w])$$

and since we only have to consider vertical vectors v, w, by the definition of the symplectic connection, it follows that $d\iota_Z\omega$ vanishes as desired. This shows the proposition. $\square$

Proposition 2.2.12 (Symplectic trivialisation by symplectic parallel transport)**.** *Let (X,ω) be a symplectic manifold and let $p\colon X \to \mathbb{C}$ be a symplectic fibration. Let $F_z = p^{-1}(z)$, then there exists a diffeomorphism $\phi\colon \mathbb{C}\times F_0 \to X$*

$$\begin{array}{ccc} \mathbb{C}\times F_0 & \xrightarrow{\phi} & X \\ \big\downarrow{\scriptstyle p_1} & & \big\downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

such that $\phi^*\omega$ restricted to $\{z\}\times F_0$ equals $\omega(0)$ for all $z \in \mathbb{C}$. And where $\omega(z)$ denotes the restriction of ω to F_z. Moreover ϕ is the identity on F_0.

Proof. By Proposition 2.2.11 the symplectic paralleltransport satisfies

$$P_\gamma^*\omega(\gamma(1)) = \omega(\gamma(0))$$

for any path γ in the base. Thus we are going to prove this proposition by symplectic parallel transport over the specific set of paths $\gamma_{x,y}$ given by first going along the x-axis from the origin to $(x,0)$ and then going to (x,y) along the parallel to the y-axis through $(x,0)$.
Let

$$Z = \widetilde{\frac{\partial}{\partial x}}$$

denote the vector field on X defined by the horizontal lift of the vector field $\frac{\partial}{\partial x}$ on the base. Similarly Y denotes the horizontal lift of $\frac{\partial}{\partial y}$. Let ϕ_s^Z, ϕ_s^Y denote the flows of Z, Y. Now we define

$$\phi\colon \mathbb{C}\times F_0 \to X;\ (t+is, w) \mapsto \phi_s^Y(\phi_t^Z(w)).$$

Then the pullback form $\phi^*\omega$ restricts to $\omega(0)$ on $\{z\}\times F_0$. Moreover it is clear that ϕ is the identity on F_0. This proves the proposition. $\square$

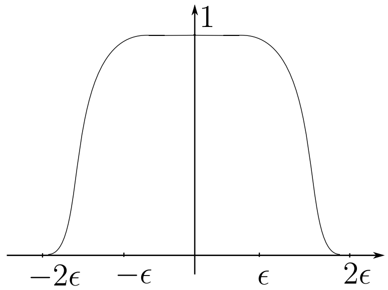

Figure 2.2: The bump function ρ

Remark

In the case when $p\colon \mathbb{C} \times F \to \mathbb{C};\ (z, w) \mapsto z$ and only the symplectic form varies from fiber to fiber, then in the construction above we can write $\phi_s^Z(x + iy, w) = (x + s + iy, P_s^Z(x, y)(w))$ where $P_s^Z(x, y)\colon F \to F$ is a diffeomorphism of F which depends smoothly on s, x, y. Note that $P_0^Z(x, y) = id$ for all x, y. Similarly we define $P_s^Y(x, y)$. Note that

$$\phi_s^Y \circ \phi_t^Z(0, w) = (t + is, P_s^Y(t, 0) \circ P_t^Z(0, 0)(w)).$$

We now consider a cut-off function ρ (see figure 2.2).

and define

$$\tau\colon \mathbb{C} \times F \to \mathbb{C} \times F;\ (t + is, w) \mapsto (t + is, P^Y_{\rho(s^2+t^2)s}(t, 0) \circ P^Z_{\rho(s^2+t^2)t}(0, 0)(w)).$$

Then by construction $\tau^*\omega$ restricts to $\omega(0)$ on the fibers in a neighbourhood of F and τ is the identity outside some bigger neighbourhood.

The symplectic curvature and some properties

Definition 2.2.13. *Let (X, ω) be a symplectic manifold and $p\colon X \to B$ a symplectic fibration then we define the vertical tangent-bundle associated to (X, p) to be the subbundle $VTX := \ker dp$ of TX.*

Let $\Gamma(p\colon E \to X)$ denote the vector space of sections of the vector bundle $p\colon E \to X$. $\Gamma(X)$ is shorthand for the vector fields on X.

Definition 2.2.14. *The curvature of a connection H (distribution of horizontal subspaces) on $\pi\colon X \to B$ is defined to be a map*

$$\Omega_H\colon \Gamma(B) \times \Gamma(B) \to \Gamma(\widetilde{\pi}\colon VTX \to X)$$

$$\Omega_H(Y, Z) = [\tilde{Y}, \tilde{Z}]^{vert}$$

for any two vector fields Y, Z in $\Gamma(B)$ where $\tilde{.}$ denotes the horizontal lift and $.^{vert}$ denotes the projection to the vertical subbundle VTX of TX along H.

Remark
By the Frobenius theorem it is clear that the curvature as defined above measures the non-integrability of the horizontal distribution.

Note that $\Omega_H(v_1, v_2)$ is linear over functions: write $\tilde{f} = \pi^* f$

$$\Omega_H(fv_1, gv_2) = [\tilde{f}\tilde{v}_1, \tilde{g}\tilde{v}_2]^{vert} = (\tilde{f}\tilde{v}_1(\tilde{g})\tilde{v}_2 - \tilde{g}\tilde{v}_2(\tilde{f})\tilde{v}_1 + \tilde{f}\tilde{g}[\tilde{v}_1, \tilde{v}_2])^{vert}$$

but the first two terms vanish because the vector fields are horizontal, thus

$$\Omega_H(fv_1, gv_2) = \pi^* f \pi^* g [\tilde{v}_1, \tilde{v}_2]^{vert} = fg\Omega_H(v_1, v_2)$$

so the value of $\Omega_H(v_1, v_2)$ at $b \in B$ depends only on $v_1(b), v_2(b)$ in the base not on a neighbourhood of b. Moreover it is skew symmetric and bilinear, so Ω_H is a 2-form on the base with values in the vertical vector fields on the fibers.

Remark
Since we only consider 2-dimensional horizontal distributions it is clearly enough to consider two linearly independent vectors in the base to determine the integrability of the horizontal distribution at some point.

Proposition 2.2.15 (The curvature identity)**.** *Let (X, ω) be a symplectic manifold and $\pi\colon X \to B$ be a symplectic fibration then the following holds:*

$$d\iota_{\tilde{v}_2}\iota_{\tilde{v}_1}\omega = \iota_{[\tilde{v}_1, \tilde{v}_2]}\omega$$

when restricted to VTX for all vectors v_1, v_2 tangent to the base.

Proof of the curvature identity. For any 2-form α and vector fields Y_0, Y_1, Y_2 we have the following identity:

$$d\alpha(Y_0, Y_1, Y_2) = L_{Y_0}(\alpha(Y_1, Y_2)) - L_{Y_1}(\alpha(Y_0, Y_2)) + L_{Y_2}(\alpha(Y_0, Y_1))$$
$$-\alpha([Y_0, Y_1], Y_2) + \alpha([Y_0, Y_2], Y_1) - \alpha([Y_1, Y_2], Y_0)$$

hence for $\alpha = \omega$, $Y_0 = \tilde{v}_1$, $Y_1 = \tilde{v}_2$ and $Y_2 = v \in VTX$ this gives:

$$d\omega(\tilde{v}_1, \tilde{v}_2, v) = L_{\tilde{v}_1}\omega(\tilde{v}_2, v) - L_{\tilde{v}_2}\omega(\tilde{v}_1, v) + L_v\omega(\tilde{v}_1, \tilde{v}_2) - \omega([\tilde{v}_1, \tilde{v}_2], v) + \omega([\tilde{v}_1, v], \tilde{v}_2) - \omega([\tilde{v}_2, v], \tilde{v}_1)$$

But $[\tilde{v}, w]$ is vertical for any vertical w. To see this note that the flow $\tilde{\phi}_t$ of $\tilde{v}$ is the horizontal lift of the flow ϕ_t of v on the base. Hence $\tilde{\phi}_t$ restricted to the fiber $\pi^{-1}(x)$ equals the parallel transport map $P_t\colon \pi^{-1}(x) \to \pi^{-1}(\phi_t(x))$ of the symplectic connection for the path $\{\phi_{st}(x)\}_{s\in[0,1]}$ in the base. Therefore if ψ_s denotes the flow of w then for fixed t,

$$\tilde{\phi}_t \circ \psi_s \circ (\tilde{\phi}_t)^{-1}(z) = P_t \circ \psi_s \circ P_t^{-1}(z)$$

remains in the fiber $\pi(z)$ for all s. Consequently the vector field $(\tilde{\phi}_t)_* w$ is vertical for all t. This shows that the Lie-bracket $[\tilde{v}, w]$ is also vertical (cf. Remark 6.26 in [13] and the discussion before). Then the above reduces to

$$d\omega(\tilde{v}_1, \tilde{v}_2, v) = L_v(\omega(\tilde{v}_1, \tilde{v}_2)) - \omega([\tilde{v}_1, \tilde{v}_2], v)$$

for all vertical v. Thus

$$d\iota_{\tilde{v}_2}\iota_{\tilde{v}_1}\omega - \iota_{\tilde{v}_2}\iota_{\tilde{v}_1}d\omega = \iota_{[\tilde{v}_1,\tilde{v}_2]}\omega \text{ on } VTX.$$

But ω is closed, hence it reduces to the required form. □

Thus given any two vector fields v_1, v_2 on the base, the curvature vector field

$$\Omega(v_1, v_2) \in \Gamma(VTX)$$

is Hamiltonian since

$$\iota_{\Omega(v_1,v_2)}\omega = d(\omega(\tilde{v}_1, \tilde{v}_2)).$$

Hence the Hamiltonian $H := \omega(\tilde{v}_1, \tilde{v}_2)$. The vector field determines the Hamiltonian up to a constant. Fixing this constant by requiring that $\int_{p^{-1}(x)} H\omega_{std} = 0$, we can thus view the symplectic curvature as a two form on the base with values in the functions on the fibers, i.e.

$$\Omega(v_1, v_2) = \omega(\tilde{v}_1, \tilde{v}_2) - c_\omega$$

where c_ω is the fiberwise constant normalising the Hamiltonian as required above. Indeed if $\omega(\frac{\tilde{\partial}}{\partial x}, \frac{\tilde{\partial}}{\partial y})$ is constantly equal to c, then $c_\omega = c$ and the curvature vanishes.

Remark

Let (X, ω) be symplectic and $p\colon X \to B$ be a symplectic fibration. Let Ω denote the curvature of the connection induced by (p, ω). Vanishing curvature implies that the Lie-bracket for any two horizontally lifted vector fields v, w has vanishing vertical part, thus it is entirely horizontal. But then the Frobenius theorem implies that the horizontal distribution is integrable, since it is spanned by such vector fields. Thus there exist integral submanifolds whose tangent distribution equals the horizontal distribution.
Observe that if the base B is simply connected, then vanishing curvature implies that the monodromy map around any closed curve in the base is the identity (if B is not simply connected, a integral submanifold could be a cover of B in which case the monodromy need not be the identity). In particular, parallel transport is independent of the path and depends only on endpoints. In this case we say that (p, ω) has trivial monodromy.

Example

Consider $(S^2 \times S^2, \omega)$ with ω split and the standard projection p_1. Since the submanifolds $S^2 \times pt$ are horizontal, they are integral submanifolds for the symplectic connection induced by (p_1, ω) on $S^2 \times S^2$ and so by the Frobenius theorem the symplectic connection has vanishing symplectic curvature. In particular, symplectic parallel transport is the identity for all paths in the base.

Definition 2.2.16. *Let $\phi\colon X \to Y$ and $u\colon B_X \to B_Y$ be diffeomorphisms such that*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ B_X & \xrightarrow{u} & B_Y \end{array}$$

with (Y, ω) symplectic and p_Y a symplectic fibration. Then the pullback symplectic connection $\phi^*(p_Y, \omega)$ on p_Y is defined to be the symplectic connection induced by $(p_X, \phi^*\omega)$ on X.

Proposition 2.2.17. *Let (Y, ω) be symplectic and $p_Y : Y \to B_Y$ a symplectic fibration such that the symplectic connection induced by (p_Y, ω) has vanishing symplectic curvature. Let $\phi\colon X \to Y$ and $u\colon B_X \to B_Y$ be diffeomorphisms such that*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ B_X & \xrightarrow{u} & B_Y \end{array}$$

then the pull-back symplectic connection has also vanishing symplectic curvature. Moreover ϕ maps the integral submanifolds for $\phi^*(p_Y, \omega)$ to the integral submanifolds for (p_Y, ω).

Proof. From above we know that vanishing symplectic curvature implies integrability of the horizontal distribution. Thus let S be an integral submanifold for the horizontal distribution. Now consider $S' = \phi^{-1}S$, then $d\phi_x w \in T_{\phi(x)}S$ if $w \in T_x S'$. Thus

$$\omega_{\phi(x)}(d\phi_x w, v) = 0$$

for all $v \in V_{\phi(x)}TY$ by definition of the symplectic connection induced by (p_Y, ω). Now ϕ maps the fibers of p_X diffemorphically onto the fibers of p_Y, thus $v = d\phi_x \bar{v}$ for $\bar{v} \in V_x TX$ and so

$$0 = \omega_{\phi(x)}(d\phi_x w, d\phi_x \bar{v}) = (\phi^*\omega)_x(w, \bar{v})$$

for all $\bar{v} \in V_x X$. Thus indeed $T_x S'$ is horizontal for the pull-back symplectic connection and S' is an integral submanifold. Again by the Frobenius theorem, integrability of the horizontal distribution implies vanishing symplectic curvature. This proves the proposition. □

Proposition 2.2.18. *Let (X, ω) be a symplectic four manifold and let $p\colon X \to \mathbb{C}$ be a symplectic fibration. Assume that the symplectic curvature of the symplectic connection induced by (p, ω) vanishes, then there exists a diffeomorphism $\phi\colon \mathbb{C} \times F_0 \to X$ such that*

$$\begin{array}{ccc} \mathbb{C} \times F_0 & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p} \\ \mathbb{C} & \xrightarrow{id} & \mathbb{C} \end{array}$$

and such that $\phi^*\omega$ is split.

Proof. Consider the diffeomorphism ϕ constructed in proposition 2.2.12 by symplectic parallel transport. If we fix a point $w \in F_0$ then $\phi(\gamma_{x,y} \times \{w\})$ is the horizontal lift of

$\gamma_{x,y} \subset \mathbb{C}$ starting at w. Since (p, ω) has vanishing symplectic curvature, this horizontal lift is contained in the integral submanifold S through the point w. This is true for all x, y in $\mathbb{C}$. Hence, since ϕ is a diffeomorphism $\phi(\mathbb{C} \times \{w\}) = S$. But we have seen before that ϕ maps the integral submanifolds for the pull-back symplectic connection to those of the original connection. Consequently $\mathbb{C} \times \{w\}$ are the integral submanifolds for the symplectic connection induced by $(p_1, \phi^*\omega)$ on $\mathbb{C} \times F_0$. Then we can write

$$\phi^*\omega = \omega(0) + \alpha_z \wedge dx + \beta_z \wedge dy + f dy \wedge dx$$

where α_z, β_z are 1-forms on $\{z\} \times F_0$ which are allowed to vary with the foot-point $z = (x, y)$ in $\mathbb{C}$. By construction, the vertical part of $\phi^*\omega$ is fixed to $\omega(0)$. But now note that the horizontal lifts of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ at (z, w) are $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ itself since $\mathbb{C} \times \{w\}$ is horizontal. Hence

$$\phi^*\omega(\frac{\partial}{\partial x}, v) = 0$$

for all vertical tangent vectors $v \in T_{z,w}\{z\} \times F_0$. Thus

$$-\alpha(v) = 0 \ \forall v$$

and similarly for β. $\alpha = \beta = 0$ and hence $\phi^*\omega = \omega(0) + f dx \wedge dy$. Moreover f needs to be constant in the vertical direction since otherwise $\phi^*\omega$ would not be closed.
Thus we can write

$$\phi^*\omega = \omega(0) + p_1^* g dx \wedge dy$$

for a function $g \colon \mathbb{C} \to \mathbb{R}$. This proves the proposition. $\square$

Remark
As before, if $p = p_1 \colon \mathbb{C} \times F \to \mathbb{C}$, then by introducing a suitable cut-off function, the assertion in proposition 2.2.18 can also be realised in a neighbourhood of the origin and with ϕ being fixed outside some bigger neighbourhood.

2.3 Monotonicity of Lagrangian submanifolds

Recall that M is diffeomorphic to $S^2 \times S^2$ and ω is a symplectic form on M which is cohomologous to ω_0.

Definition 2.3.1. *A submanifold L of dimension 2 in (M, ω) is called Lagrangian if $\omega|_L \equiv 0$.*

2.3.1 The Maslov index

Now we will define the maslov index

$$\mu \colon \pi_2(M, L) \to \mathbb{Z}$$

for a Lagrangian submanifold L in the symplectic manifold (M, ω). We will rely on the Maslov index $Maslov\colon \pi_1(\mathcal{L}) \to \mathbb{Z}$ defined in Appendix G.
First we consider a continuous map $u\colon \mathbb{E} \to M$ where $\mathbb{E}$ denotes the closed unit disk in $\mathbb{C}$ such that $u(\partial\mathbb{E}) \subset L$. Since the tangent bundle $TM|_{u(\mathbb{E})}$ is a trivial symplectic vector bundle, there exists a bundle map

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{R}^4 & \xrightarrow{\tau} & TM|_{u(\mathbb{E})} \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{E} & \xrightarrow{u} & u(\mathbb{E}) \end{array}$$

such that $\tau_x^* \omega_{u(x)} = \Omega_0 = dx \wedge dy + du \wedge dv$ where $\tau_x\colon \{x\} \times \mathbb{R}^4 \to T_{u(x)}M$ is the isomorphism given by restricting τ to the fiber over $x \in \mathbb{E}$. Then

$$L_\theta := \tau_{e^{i\theta}}^{-1}(T_{u(e^{i\theta})}L)$$

defines a loop of Lagrangian subspaces in $(\mathbb{R}^4, \Omega_0)$. The map $Maslov$ from Appendix G assigns an integer to this loop, so we define

$$\mu(u) = Maslov(L_\theta).$$

First we have to check that this is well-defined. The choice we made was in the symplectic trivialisation τ. But any two such τ, τ' differ by a map $\phi\colon \mathbb{E} \to Sp(4)$ into the group of symplectic matrices of $(\mathbb{R}^4, \Omega_0)$. More precisely let

$$\tilde{\phi}\colon \mathbb{E} \times \mathbb{R}^4 \to \mathbb{E} \times \mathbb{R}^4$$

$$(z, v) \mapsto \tilde{\phi}(z, v) = \phi(z)(v)$$

then

$$\tau' = \tau \circ \tilde{\phi}.$$

Now $\Phi(\theta) = \phi(e^{i\theta})^{-1}$ is a contractible loop of symplectomorphism since it extends to the disk. The loop of Lagrangians with respect to the trivialisation $\tau \circ \phi$ instead of τ is given by

$$L'_\theta = \phi(e^{i\theta})^{-1}(L_\theta) = \Phi(\theta) L_\theta.$$

Then by Appendix G

$$Maslov(L'_\theta) = Maslov(L_\theta) + 2Maslov(\Phi) = Maslov(L_\theta)$$

and the Maslov index of u is well-defined.
Further if $u_t\colon \mathbb{E} \to M$ with $u_t(\partial\mathbb{E}) \subset L$ is a continuous family then the maslov index $\mu(u_t)$ depends continuously on t. Since μ is integer-valued $\mu(u_t)$ is constant. This implies the homotopy invariance of μ.

If we write the group operation in $\pi_2(M, L)$ as $+$, then any map $u\colon \mathbb{E} \to M$ with $u(\partial\mathbb{E}) \subset L$ in the relative homology class $[u] = [a] + [b]$ is homotopic to the concatenation (see App. F) of a and b so $\mu([a]+[b]) = Maslov(L^a * L^b) = Maslov(L^a) + Maslov(L^b) = \mu(a) + \mu(b)$ where L^a, L^b are the loops of Lagrangians defined by the maps $a, b\colon \mathbb{E} \to M$ with boundary on L.

Definition 2.3.2. *Let L be a Lagrangian submanifold in (M, ω). Then the Maslov index $\mu\colon \pi_2(M, L) \to \mathbb{Z}$ is the homomorphism which assigns to a relative cycle the Maslov index of the corresponding loop of Lagrangians.*

2.3.2 Monotonicity

Definition 2.3.3. *Let L be a Lagrangian submanifold in (M, ω), then L is called* monotone *with* monotonicity constant λ *if there exists a constant $\lambda \in \mathbb{R}_+$ such that*

$$\mu([u]) = \lambda \int_{u(\mathbb{E})} \omega$$

for all relative cycles $[u] \in \pi_2(M, L)$.

Remark We have to show first that this definition makes sense, i.e. that the symplectic area of a relative cycle is well-defined. This is true since if $U\colon \mathbb{E} \times I \to M$ with $u_t(\mathbb{E}) = U(\partial\mathbb{E} \times \{t\}) \subset L$ for all t is smooth, then

$$0 = \int_{U(\mathbb{E}\times I)} d\omega = \int_{\partial U(\mathbb{E}\times I)} \omega = \int_{u_1(\mathbb{E})} \omega - \int_{u_0(\mathbb{E})} \omega - \int_{U(\partial\mathbb{E}\times I)} \omega.$$

Now $U(\partial\mathbb{E} \times I) \subset L$ thus the last term vanishes. Since by smooth approximations we can always assume smoothness, this shows that symplectic area of relative cycles is indeed well-defined.
By the appendix $\pi_2(M, L) \cong H_2(M, L)$ so that we can also view μ as a linear map from $H_2(M, L)$ to $\mathbb{Z}$. Thus to check monotonicity, it suffices to check the evaluation of ω on a set generating $H_2(M, L)$ with known Maslov indices.
Also if ω is a symplectic form such that L is monotone Lagrangian, then L is monotone Lagrangian for ω' if L is ω' Lagrangian and $\int_D \omega = \int_D \omega'$ for all relative cycles $D \in H_2(M, L)$. This amounts to say that ω, ω' are relative cohomologous.
In order to show that the Clifford torus L_{std} in $(S^2 \times S^2, \omega_0)$ is monotone we need to show that the Maslov index of the relative cycles given by the section $S^2 \times \{z_0\}$ and the fiber $\{z_0\} \times S^2$ is 4. Therefore we prove

Theorem 2.3.4. *Let $f\colon (S^2, z_0) \to (M, x_0)$ be an embedding with $f(z_0) = x_0 \in L$ with trivial normal bundle, then $\mu(u) = 4$ for $u = f \circ \phi\colon \mathbb{E} \to M$. In the definition of u, via stereographic projection from z_0, we have identified $\mathbb{E} \subset \mathbb{C}$ with the western hemisphere in S^2. Further $\phi\colon S^2 \to S^2$ collapses the eastern hemisphere to z_0 and has mapping degree 1.*

Proof. The proof is given in Appendix B. □

Remark
The western and eastern hemispheres in $S^2 \subset \mathbb{R}^3$ are defined by $x \lesseqgtr 0$ respectively.

Proposition 2.3.5. *The Clifford torus L_{std} in $(S^2 \times S^2, \omega_0)$ is monotone Lagrangian.*

Proof. It is clear that L_{std} is Lagrangian for ω_0. By Appendix F, we know that $H_2(S^2 \times S^2, L_{std})$ is generated by

$$[S^2 \times \{pt\}], [\{pt\} \times S^2], [D_{lh} \times \{pt\}], [\{pt\} \times D_{lh}]$$

where D_{lh} denotes the closed lower hemisphere in S^2.
Now let $u, v \colon (S^2, z_0) \to (S^2 \times S^2, (z_0, z_0))$ be given by $u(z) = (z, z_0)$ and $v(z) = (z_0, z)$. Then u, v are the standard parametrisations of $S^2 \times \{z_0\}, \{z_0\} \times S^2$. But u, v satisfy the conditions of Theorem 2.3.4 so that $\mu(u \circ \phi) = \mu(v \circ \phi) = 4$. From Appendix A, we know that $\int_{S^2} \omega_{std} = 1$ so that the monotonicity constant (if it exists) is fixed to $\lambda = \frac{1}{4}$.
On the other hand in the proof of Proposition 2.4.3, it is shown that $\mu(D_{lh} \times \{pt\}) = \mu(\{pt\} \times D_{lh}) = 2$. Using stereographic projection from N on S^2, the disk D_{lh} is mapped to the closed unit disk $\mathbb{E}$ in $\mathbb{C}$. Further from Appendix A, we know that ω_{std} is pushed forward to $\frac{r}{\pi(1+r^2)^2} dr \wedge d\theta$ on $\mathbb{C}$ under stereographic projection from N. Then

$$\int_{\mathbb{E}} \frac{r}{\pi(1+r^2)^2} dr \wedge d\theta = \int_{D_{lh}} \omega_{std}.$$

This indeed equals $\frac{1}{2}$ showing the monotonicity of L_{std}. $\square$

Proposition 2.3.6. *Let $\phi \in Diff^+(M)$ and L a monotone Lagrangian torus in (M, ω) then $\phi(L)$ is monotone Lagrangian for the symplectic form $\phi_* \omega$.*

Proof. First note that $\phi(L)$ is Lagrangian for the push-forward symplectic form. Next assume that $u \colon (\mathbb{E}, \partial\mathbb{E}) \to (M, L)$ is smooth, then $\mu(u) = \lambda \int_{u(\mathbb{E})} \omega$. Thus for $\phi \circ u \colon (\mathbb{E}, \partial\mathbb{E}) \to (M, \phi(L))$ we have

$$\int_{\phi \circ u(\mathbb{E})} \phi_* \omega = \int_{u(\mathbb{E})} \omega.$$

We are left to show that $\mu(\phi \circ u) = \mu(u)$. Let

$$\begin{array}{ccc} \mathbb{E} \times \mathbb{R}^4 & \xrightarrow{\tau} & TM|_{u(\mathbb{E})} \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{E} & \xrightarrow{u} & u(\mathbb{E}) \end{array}$$

such that $\tau_x^* \omega_{u(x)} = \Omega_0$, be a bundle map as in the definition of the Maslov index of u, with

$$L_\theta = \tau_{u(e^{i\theta})}^{-1}(T_{u(e^{i\theta})} L).$$

Then

$$\begin{array}{ccc} \mathbb{E}\times\mathbb{R}^4 & \xrightarrow{\tilde{\tau}=d\phi\circ\tau} & TM|_{\phi\circ u(\mathbb{E})} \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{E} & \xrightarrow{\phi\circ u} & \phi\circ u(\mathbb{E}) \end{array}$$

is a bundle map, since ϕ is a diffeomorphism so that $d\phi$ is an isomorphism on tangent spaces. Further

$$\tilde{\tau}_x^*(\phi_*\omega)_{\phi\circ u(x)} = \Omega_0.$$

First note that the tangent space to the Lagrangian torus $\phi(L)$ at $\phi\circ u(e^{i\theta})$ is given by

$$T_{\phi\circ u(e^{i\theta})}\phi(L) = d\phi_{u(e^{i\theta})}T_{u(e^{i\theta})}L.$$

Now the loop of Lagrangians defined by the trivialisation $\tilde{\tau}$ equals

$$\widetilde{L}_\theta = \tilde{\tau}_{e^{i\theta}}^{-1}T_{\phi\circ u(e^{i\theta})}\phi(L) = \tau_{u(e^{i\theta})}^{-1}(d\phi_{u(e^{i\theta})})^{-1}d\phi_{u(e^{i\theta})}(T_{u(e^{i\theta})}L) = \tau_{u(e^{i\theta})}^{-1}(T_{u(e^{i\theta})}L) = L_\theta.$$

Thus using the trivialisation $\tilde{\tau}$ we see straight away that

$$\mu(u) = \mu(\phi\circ u).$$

By smooth approximations we can always assume the smoothness of u, this proves the proposition. □

Remark
This shows that L_{std} in M is monotone Lagrangian for ω_0 since it is the push forward under θ (cf. section 2.1) of the Clifford torus in $S^2\times S^2$. Hence we also call $L_{std}\subset M$ the Clifford torus in M.

2.4 Monotone Lagrangian tori lying nicely in symplectic fibrations

2.4.1 Fibered tori and some properties

Recall that M is diffeomorphic to $S^2\times S^2$.

Definition 2.4.1. *Let $p\colon M\to B$ be a smooth fibration over the (real) surface B and let $L\subset M$ be an embedded 2-torus then we say that L* is fibered by *p if*

- *$\gamma := p(L)$ is an immersed loop with transverse self-intersections which are at most double points;*
- *$p^{-1}(\gamma(t))\cap L$ is diffeomorphic to S^1 if $\gamma(t)$ is not a double point and to two disjoint S^1's if $\gamma(t)$ is a double point;*

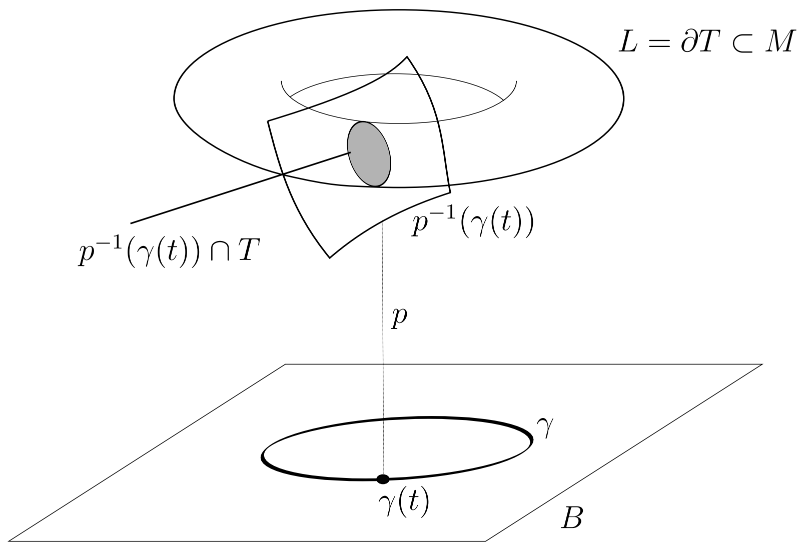

Figure 2.3: L fibered by p

- *in each of the S^1's in $p^{-1}(\gamma(t)) \cap L$ we can fill in an embedded disk $D \subset p^{-1}(\gamma(t))$ in the fiber such that the two disks at a double point are disjoint and all the disks form a solid torus $T \cong S^1 \times D^2$ with L as its boundary.*

Remark

Compare fig. 2.3. For example, the Clifford torus L_{std}, is fibered by either standard projection p_i. This is also what one shall have in mind when thinking of tori fibered by some fibration p.

As it is stated, the definition above is of topological nature. We will mainly have the situation that $p\colon M \to B$ is a symplectic fibration and L is a monotone Lagrangian torus. In this case, we have the following two important results on which most of the sequel is based. In fig. 2.4, the cylinder formed by the lifted paths is then part of the Lagrangian torus L by the following proposition:

Proposition 2.4.2. *Let $L \subset M$ be an embedded Lagrangian torus which is fibered by the symplectic fibration $\pi\colon M \to B$. Then L is given by parallel transport of the symplectic connection induced by (π, ω) on M of the S^1 in the fiber over a non-double point along the projection curve $\gamma = \pi(L)$ in B.*

Proof. Fix a parametrisation $\gamma(t)$ of γ such that $\gamma'(t) \neq 0$ for all t (from the definition of being fibered, it follows that γ is an immersed curve). Then parallel transport along γ is defined by integrating the vectorfield X on $p^{-1}(\gamma)$ given by the unique horizontal lifts in H_x of $\gamma'(t)$ at all points x over $\gamma(t)$. So it suffices to prove that the tangent space to the Lagrangian torus L at x is spanned by $X(x)$ and a non-zero vector v which is tangent to the S^1 in the fiber given in the definition of being fibered by π.

Clearly $v \in T_xL$. Since π is a symplectic fibration we can choose a linear independent tangent vector w in $T_xp^{-1}(\gamma(t))$ such that $\omega(v, w) = 1$. Since M is four dimensional, the

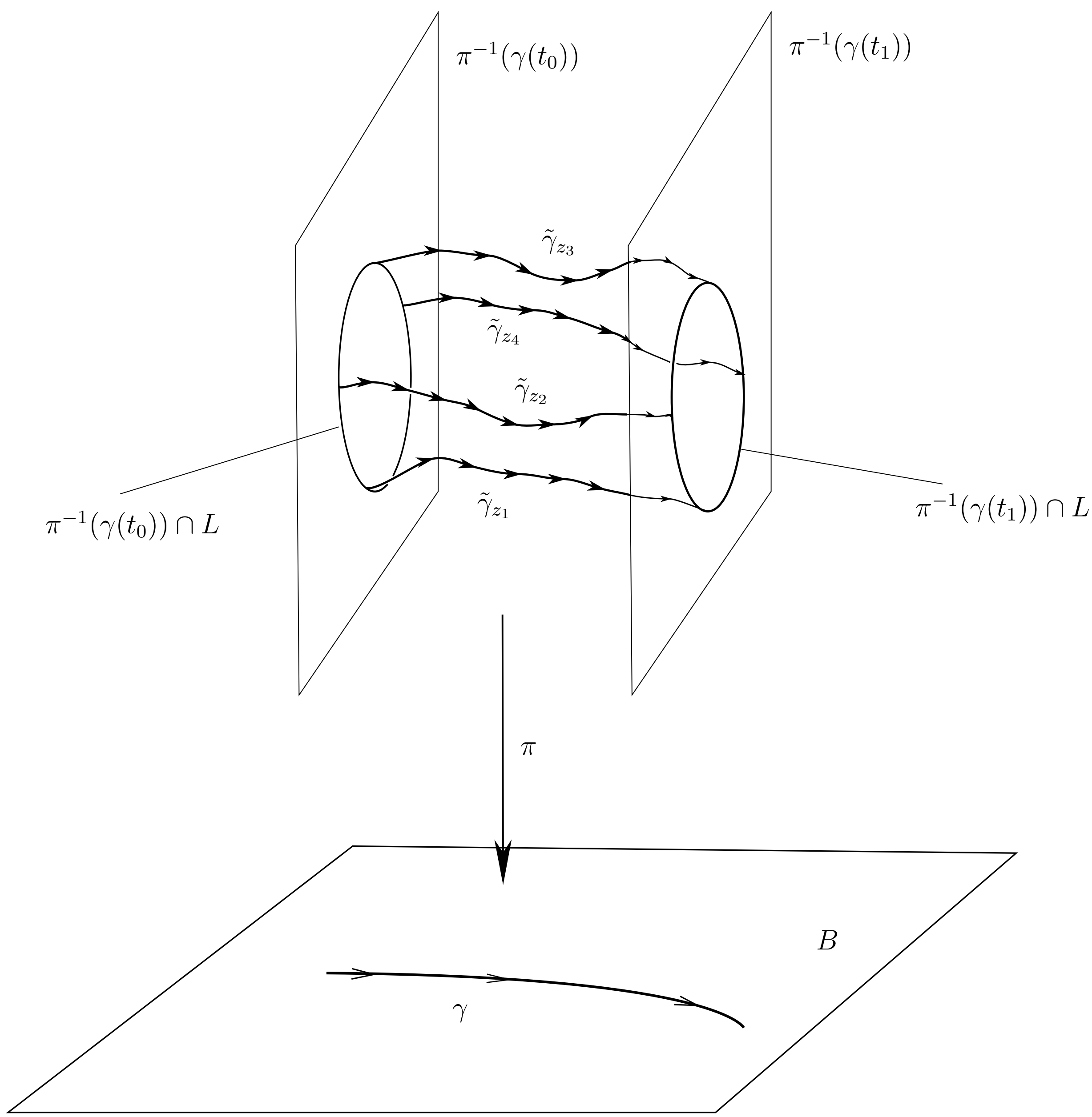

Figure 2.4: L is generated by parallel transport

fibers of π are two dimensional and v, w span $T_xp^{-1}(\gamma(t))$.
Further $X(x) \in H_x$ lies in the symplectic orthogonal complement to $T_xp^{-1}(\gamma(t))$. This is also a symplectic subspace so we may choose another linear independent vector $u \in H_x$ such that $\omega(X(x), u) = 1$. Thus $\{v, w, X(x), u\}$ forms a symplectic basis of T_xM.
Now let $\tilde{w} \in T_xL$ then

$$d\pi(\tilde{w}) = \lambda\gamma'(t)$$

for some $\lambda \in \mathbb{R}$ since $\pi(L) = \gamma$. But then both $\lambda X(x)$ and $\tilde{w}$ project under $d\pi$ to the same vector in the tangent space of the base. Consequently they can only differ by a vector in the kernel of $d\pi$ which means by a vector tangent to the fiber

$$\tilde{w} = \lambda X(x) + \mu_1 v + \mu_2 w.$$

But L is Lagrangian, hence

$$0 = \omega(v, \tilde{w}) = \mu_2$$

and so all vectors tangent to L at x are of the form $\lambda X(x) + \mu v$. But T_xL is two dimensional so that there exists at least one vector $\tilde{w}$ in T_xL with $\lambda \neq 0$ and thus $X(x) = \frac{1}{\lambda}(\tilde{w} - \mu v) \in T_xL$ as claimed. □

Remark
Let L in M be fibered by p. Let $N = p^{-1}\gamma$ be the 3-dimensional submanifold of M formed by the fibers in which the torus L sits. In the topological definition of being fibered by p we didn't require the torus to be transverse to the fibers of p in N. If however L is Lagrangian and p is symplectic, Proposition 2.4.2 shows that we get this property for free. Also important is the following proposition, which shows that in the case that L is also monotone, then the curve γ in the base must be embedded.

Lemma 2.4.3. *Let $L \subset (M, \omega)$ be a monotone Lagrangian torus with ω cohomologous to ω_0. Further let $p\colon M \to B$ be a symplectic fibration over the (real) surface B such that L is fibered by p. Then the loop $\gamma := p(L)$ is an embedded curve,i.e. has no double points.*

Proof. The idea is as follows. First note, that because of Theorem 2.3.4 and the cohomology assumption on ω, it follows that the monotonicity constant equals $\frac{1}{4}$.
Assume now the contrary and consider a double point $q \in \gamma$. We get two disjoint embedded disks in $p^{-1}(q)$. Since the Maslov index of such a disk is 2, each of these disks has symplectic area $\frac{1}{2}$. Consequently the total symplectic area of the fiber is bigger than 1. The desired contradiction. We are left to show that the maslov index of such a disk is 2. We may assume that the fibration p is the trivial fibration, the base is equal to $\mathbb{C}$ with standard coordinates x, y such that $\gamma'(q)$ is in direction $\frac{\partial}{\partial x}$. Let D be one of the disks in $p^{-1}(q)$.
Let $u\colon \mathbb{E} \to M$ be an embedding with image D where $\mathbb{E}$ denotes the closed unit disk in $\mathbb{C}$ with standard coordinates $u+iv$. On $E := TM|_D$ we can define a smooth almost complex structure J which is compatible with ω as follows.
Write

$$T_xM = T_xD \oplus H_x$$

for all $x \in D$ and define J by
$$J(v) = du(idu^{-1}(v))$$
for $v \in TD$ and by
$$J\widetilde{\frac{\partial}{\partial x}} = \widetilde{\frac{\partial}{\partial y}};\ J\widetilde{\frac{\partial}{\partial y}} = -\widetilde{\frac{\partial}{\partial x}}$$
where $\widetilde{\frac{\partial}{\partial x}}, \widetilde{\frac{\partial}{\partial y}}$ denote the horizontal lifts of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ spanning H_x. This defines a ω-compatible smooth almost complex structure on E. Moreover a trivialisation of the footpoint map
$$\pi\colon E \to D; v \in T_xE \mapsto x$$
is given by
$$\left\{\widetilde{\frac{\partial}{\partial x}}, J\widetilde{\frac{\partial}{\partial x}}, du(\frac{\partial}{\partial u}), Jdu(\frac{\partial}{\partial u})\right\}.$$
For calculating the Maslov index of a disk D with boundary on a Lagrangian submanifold we first fix a trivialisation of the tangent bundle restricted to the disk D. Using the trivialisation we can fix a standard Lagrangian subspace L^{std} in each tangent space T_xM. Now we want to use Lemma G.0.40 to calculate the Maslov index, so we seek a loop of unitary matrices $A(\theta)$ which map the standard Lagrangian L^{std} to $L_{e^{i\theta}}$ the tangent space to the Lagrangian L at $u(e^{i\theta})$. Then according to Lemma G.0.40, the winding number of the loop given by $\det A^2(\theta)$ is then the Maslov index of D.
We fix the standard Lagrangian to $L^{std} := span_{\mathbb{R}}\left\{\widetilde{\frac{\partial}{\partial x}}, du(\frac{\partial}{\partial u})\right\}$. Then by Proposition 2.4.2
$$L_{e^{i\theta}} = span_{\mathbb{R}}\left\{\widetilde{\frac{\partial}{\partial x}}, du((-\sin(\theta) + i\cos(\theta))\frac{\partial}{\partial u}) = -\sin(\theta)du(\frac{\partial}{\partial u}) + \cos(\theta)J(du(\frac{\partial}{\partial u}))\right\}$$
If we consider T_xM to be a complex vector space with complex scalar multiplication defined by $(\alpha + i\beta)v := \alpha v + \beta Jv$ then we can write
$$L_{e^{i\theta}} = span_{\mathbb{R}}\left\{\widetilde{\frac{\partial}{\partial x}}, ie^{i\theta}du(\frac{\partial}{\partial u})\right\}.$$
Hence with respect to the complex basis $\widetilde{\frac{\partial}{\partial x}}, du(\frac{\partial}{\partial u})$ the loop of unitary matrices $A(\theta)$ is given by
$$A(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & ie^{i\theta} \end{pmatrix}.$$
Thus
$$\det A^2(\theta) = -e^{2i\theta}$$
and the winding number is 2 as claimed. $\square$

Remark

In particular, if L is a monotone embedded Lagrangian torus in (M,ω) which is fibred by $p\colon M\to B$, by Lemma 2.4.3, there exists a unique disk $D_q = T\cap p^{-1}(q)$ in the fiber $p^{-1}(q)$ for all points $q\in\gamma := p(L)$.
We will also need the following:

Proposition 2.4.4. *Let $(S^2\times S^2,\omega)$ be such that p_1 is a symplectic fibration, and such that the Clifford torus L_{std} in $S^2\times S^2$ is Lagrangian for ω. Further ω and ω_0 induce the same orientation on $S^2\times S^2$. Then the relative cycle $\widetilde{D} = D_{uh}\times\{z_0\}$ with $z_0=(1,0,0)$ and boundary on L_{std} has Maslov index 2.*

Proof. Consider stereographic projection from S in the base and consider coordinates $z=x+iy$ on $S^2\setminus\{S\}$ in the base. Then in these coordinates $D_{uh}=\{|z|\le 1\}$. Let $\pi\colon T(S^2\times S^2)|_{\widetilde{D}}\to\widetilde{D};\ v\in T_p(S^2\times S^2)\mapsto p$.
Consider the equator $E=\{z=0\}\subset S^2$ and the 0-meridian $m=\{y=0\}\subset S^2$. Then both E,m go through the point z_0 in S^2. Let v and w be tangent vectors to E respectively m in $T_{z_0}S^2$ such that $\{v,w\}$ is positively oriented. Consider $v,w\in T_{z,z_0}(\{z\}\times S^2)$ for all $z\in D_{uh}$. Now let $\alpha(z) := \omega_{(z,z_0)}(v,w)$. Then since v,w are positively oriented, $\alpha(z)>0$ for all $z\in D_{uh}$.
Since the fibers of p_1 are symplectic, so are the horizontal complements $H_q=(\ker d(p_1)_q)^\perp$. Let $\frac{\partial}{\partial x}^h(q), \frac{\partial}{\partial y}^h(q)$ denote the horizontal lifts of $\frac{\partial}{\partial x},\frac{\partial}{\partial y}$ at q. Then $\frac{\partial}{\partial x}^h(q), \frac{\partial}{\partial y}^h(q)$ are linearly independent and span H_q for all $q\in(S^2\setminus\{S\})\times S^2$. So define $\beta(z) := \omega_{(z,z_0)}\left(\frac{\partial}{\partial x}^h,\frac{\partial}{\partial y}^h\right) > 0$ for $z\in D_{uh}$ (this is greater than zero by the assumption that ω_0,ω induce the same orientation). Thus

$$\left\{\frac{1}{\sqrt{\alpha(z)}}v, \frac{1}{\sqrt{\alpha(z)}}w, \frac{1}{\sqrt{\beta(z)}}\frac{\partial}{\partial x}^h, \frac{1}{\sqrt{\beta(z)}}\frac{\partial}{\partial y}^h\right\}$$

is a symplectic basis of $T_{(z,z_0)}(S^2\times S^2)$ for all $(z,z_0)\in\widetilde{D}$. This defines a symplectic trivialisation of π. Further we define a compatible almost complex structure J on π by defining

$$Jv=w;\ Jw=-v;\ J\frac{\partial}{\partial x}^h = \frac{\partial}{\partial y}^h;\ J\frac{\partial}{\partial y}^h = -\frac{\partial}{\partial x}^h.$$

Via complex multiplication $(a+ib)v = av+bJv$, we can consider π to be a complex vector bundle. Now ω and J define a hermitian structure $h(u_1,u_2)=\omega(u_1,Ju_2)+i\omega(u_1,u_2)$ on every tangent space $T_{(z,z_0)}(S^2\times S^2)$. With respect to h,

$$\left\{\frac{1}{\sqrt{\alpha(z)}}v, \frac{1}{\sqrt{\beta(z)}}\frac{\partial}{\partial x}^h\right\}$$

is a unitary basis of $T_{(z,z_0)}(S^2\times S^2)$ (or unitary trivialisation of π). Now we define the standard Lagrangian L_0 in every tangent space $T_{(z,z_0)}(S^2\times S^2)$ to be

$$L_0 :=< v, \frac{\partial}{\partial x}^h >_{\mathbb{R}}.$$

By construction v is tangent to L_{std} along all points of $\partial\widetilde{D}$. Further by Proposition 2.4.2 and the fact that L_{std} is fibered by p_1, it follows that the horizontal lift $\left(-\sin(t)\frac{\partial}{\partial x}+\cos(t)\frac{\partial}{\partial y}\right)^h$ at $(\cos(t)+i\sin(t),z_0)$ of the tangent vector $-\sin(t)\frac{\partial}{\partial x}+\cos(t)\frac{\partial}{\partial y}$ to the projection curve $p_1(L_{std})$ at $(\cos(t)+i\sin(t))$ is also tangent to L_{std}. Hence

$$T_{(\cos(t)+i\sin(t),z_0)}L_{std} =< v, \left(-\sin(t)\frac{\partial}{\partial x}+\cos(t)\frac{\partial}{\partial y}\right)^h > .$$

But since

$$\left(-\sin(t)\frac{\partial}{\partial x}+\cos(t)\frac{\partial}{\partial y}\right)^h = -\sin(t)\frac{\partial}{\partial x}^h + \cos(t)\frac{\partial}{\partial y}^h = -\sin(t)\frac{\partial}{\partial x}^h + \cos(t)J\frac{\partial}{\partial x}^h = ie^{it}\frac{\partial}{\partial x}^h,$$

we see that

$$T_{(e^{it},z_0)}L_{std} =< v, ie^{it}\frac{\partial}{\partial x}^h > .$$

Hence

$$A(t) = \begin{pmatrix} 1 & 0 \\ 0 & ie^{it} \end{pmatrix}$$

is a loop of unitary matrices which maps L_0 to $T_{(e^{it},z_0)}L_{std}$ along the boundary $\partial\widetilde{D}$ of $\widetilde{D}$. By Lemma G.0.40, it follows that

$$\mu(\widetilde{D}) = wind(\det A^2(t)).$$

But $\det A^2(t) = -e^{2it}$ and the winding number is 2. This proves the Proposition. □

2.4.2 Relative symplectic fibrations and their properties

Recall that M is diffeomorphic to $S^2\times S^2$ via the diffeomorphism $\theta\colon S^2\times S^2 \to M$ and $A=[\theta(S^2\times pt)]$ and $B=[\theta(pt\times S^2)]$ span $H_2(M)$.

Definition 2.4.5. *A quintuple of the form $(\mathcal{F},\omega,L,\Sigma,\Sigma')$ is called a* relative symplectic fibration *on M if*

- *$\mathcal{F}$ is a smooth foliation of M by 2-spheres in homology class B;*
- *ω is a symplectic form on M making the leaves of $\mathcal{F}$ symplectic with $\omega(A)=\omega(B)=1$;*
- *Σ,Σ' are disjoint symplectic submanifolds in class A which are transverse to all the leaves of $\mathcal{F}$, so by Theorem 2.2.5 $\pi\colon M\to\Sigma';\ x\in\mathcal{F}_x\mapsto\Sigma'\cap\mathcal{F}_x$ is a symplectic ruling;*
- *$L\subset M$ is an embedded monotone Lagrangian torus fibered by π;*

- *Σ is disjoint from the solid torus T ($\partial T = L$ in Def. 2.4.1);*
- *Σ' intersects each of the fibers $\pi^{-1}(\gamma(t))$ in the interior of the disk $T \cap \pi^{-1}(\gamma(t))$.*

Now we define what we mean by a homotopy of relative symplectic fibrations.

Definition 2.4.6. *By a homotopy of relative symplectic fibrations, we mean a smooth 1-parameter family*

$$(\mathcal{F}_t, \omega_t, L_t, \Sigma_t, \Sigma'_t)_{t\in[0,1]}$$

of relative symplectic fibrations where L_t, Σ_t, Σ'_t are smooth isotopies of submanifolds, ω_t is a smooth family of symplectic forms and $\mathcal{F}_t$ is a smooth family of foliations on M.

Definition 2.4.7. *Two relative symplectic fibrations $(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma'_1)$ and $(\mathcal{F}_2, \omega_2, L_2, \Sigma_2, \Sigma'_2)$ are* diffeomorphic *if there exists a diffeomorphism ϕ of M such that*

$$\phi(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma'_1) = (\phi(\mathcal{F}_1), \phi_*\omega_1, \phi(L_1), \phi(\Sigma_1), \phi(\Sigma'_1)) = (\mathcal{F}_2, \omega_2, L_2, \Sigma_2, \Sigma'_2).$$

Remark

By the definition of a relative symplectic fibrations, it follows that a diffeomorphism ϕ which makes two relative symplectic fibrations diffeomorphic induces the identity on the second homology group $H_2(M)$. Conversely, the push-forward $(\phi(\mathcal{F}), \phi_*\omega, \phi(L), \phi(\Sigma), \phi(\Sigma'))$ of a relative symplectic fibration $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ under the diffeomorphism ϕ which induces the identity on $H_2(M)$ is again a relative symplectic fibration. In particular the image of a relative symplectic fibration under $\phi \in Diff_0(M)$ is again a relative symplectic fibration. Note further that the symplectic connection induced by $(\phi(\mathcal{F}), \phi_*\omega, \phi(\Sigma'))$ is the push-forward symplectic connection of $(\mathcal{F}, \omega, \Sigma)$.

Proposition 2.4.8. *Let $p\colon M \to B$ be a symplectic fibration over $u\colon B \cong S^2$ by 2-spheres in homology class $B = [\theta(pt \times S^2)]$. Consider the symplectic foliation $\mathcal{F}$ given by the fibers of p and assume, that $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ is a relative symplectic fibration on M. Further let ϕ be a diffeomorphism such that*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & S^2 \times S^2 \\ \downarrow p & & \downarrow p_1 \\ B & \xrightarrow{u} & S^2 \end{array} \tag{2.2}$$

commutes. Then $\phi(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ is a relative symplectic fibration on $S^2 \times S^2$.

Proof. All we have to show is, that $\phi_*\colon H_2(M) \to H_2(S^2 \times S^2)$ induces the identity, where we identify $H_2(M)$ with $H_2(S^2 \times S^2)$ via θ. By the Remark above, it then follows, that

$$\phi(\mathcal{F}, \omega, L, \Sigma, \Sigma')$$

defines a relative symplectic fibration.

Since ϕ maps the fibers of p to the fibers of p_1 which both lie in homology class B, we

have $\phi_* B = B$. Now Σ' is a section of p, hence $\phi(\Sigma')$ is a section of p_1. But a section of p_1 is homologically of the form $A + nB$ for some n. Indeed let $\sigma\colon S^2 \to S^2 \times S^2$ be a section of p_1 and assume that $[\sigma(S^2)] = mA + nB$, then $p_1 \circ \sigma = id$ and so $[S^2] = (p_1)_*\sigma_*[S^2] = (p_1)_* mA + nB = m[S^2]$. Hence $m = 1$ as claimed.
But ϕ_* is a ring homomorphism for the intersection product and $A.A = 0$ hence

$$0 = \phi_*(A.A) = \phi_*(A).\phi_*(A) = (A + nB).(A + nB) = 2n.$$

This shows that $\phi_* A = A$ and so ϕ_* is the identity on $H_2(M)$. $\square$

Definition 2.4.9. *Two relative symplectic fibrations $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$, $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma'})$ on M are said to be equivalent if there exists a sequence $(\mathcal{F}_i, \omega_i, L_i, \Sigma_i, \Sigma_i')$, $i = 1, .., N$ of relative symplectic fibrations such that*

$$(\mathcal{F}, \omega, L, \Sigma, \Sigma') = (\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$$

and

$$(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma'}) = (\mathcal{F}_N, \omega_N, L_N, \Sigma_N, \Sigma_N').$$

We require, that any two consecutive relative symplectic fibrations in the sequence are either diffeomorphic or the endpoints of a homotopy of relative symplectic fibrations.

Remark
Note that in the above definitions nothing has been said about the isotopy class of the diffeomorphisms ϕ_i, so that it is unknown if equivalent relative symplectic fibrations are homotopic in general!

Theorem 2.4.10. *Let $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ and $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}')$ be equivalent relative symplectic fibrations on M where $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}')$ has vanishing symplectic curvature and the sections $\overline{\Sigma}, \overline{\Sigma}'$ are horizontal. Then there exists a homotopy of relative symplectic fibrations $(\mathcal{F}_s, \omega_s, L_s, \Sigma_s, \Sigma_s')$ with $(\mathcal{F}_0, \omega_0, L_0, \Sigma_0, \Sigma_0') = (\mathcal{F}, \omega, L, \Sigma, \Sigma')$ such that*

$$(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$$

has vanishing symplectic curvature and the sections Σ_1, Σ_1' are horizontal.

Proof. Let $(\mathcal{F}^i, \omega^i, L^i, \Sigma^i, \Sigma^{i\prime})$, $i = 1, .., N$ be the sequence of relative symplectic fibrations such that

$$(\mathcal{F}, \omega, L, \Sigma, \Sigma') = (\mathcal{F}^1, \omega^1, L^1, \Sigma^1, \Sigma^{1\prime})$$

and

$$(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma'}) = (\mathcal{F}^N, \omega^N, L^N, \Sigma^N, \Sigma^{N\prime})$$

as in Definition 2.4.9. We can assume without loss of generality that every other step consists of a homotopy of relative symplectic fibrations and the first step is given by a diffeomorphism. Indeed, by composition and concatenation we may replace several consecutive steps by diffeomorphisms or homotopies by a single one. Also if the first step

is a homotopy, we may start with its second endpoint. Further, we can assume that the last step is a homotopy. If this is not the case, we use the fact that vanishing symplectic curvature is carried along by diffeomorphisms (cf. Proposition 2.2.17), to conclude that the endpoint of the last homotopy also has vanishing symplectic curvature. Hence the even steps are homotopies and the odd steps are diffeomorphisms.
Hence, let ϕ_{2i-1}, $i = 1, ..k$ denote the diffeomorphisms and

$$(\mathcal{F}_s^{2i}, \omega_s^{2i}, L_s^{2i}, \Sigma_s^{2i}, (\Sigma_s')^{2i})$$

with $i = 1, .., k$ the homotopies. By definition,

$$(\mathcal{F}_0^{2i}, \omega_0^{2i}, L_0^{2i}, \Sigma_0^{2i}, (\Sigma_0')^{2i}) = \phi_{2i-1}(\mathcal{F}_1^{2i-2}, \omega_1^{2i-2}, L_1^{2i-2}, \Sigma_1^{2i-2}, (\Sigma_1')^{2i-2})$$

for all $i = 2, .., k$. Thus start with $i = k$, then

$$\phi_{2k-1}^{-1}(\mathcal{F}_s^{2k}, \omega_s^{2k}, L_s^{2k}, \Sigma_s^{2k}, (\Sigma_s')^{2k})$$

is a homotopy of relative symplectic fibrations which starts at

$$(\mathcal{F}_1^{2k-2}, \omega_1^{2k-2}, L_1^{2k-2}, \Sigma_1^{2k-2}, (\Sigma_1')^{2k-2})$$

and ends at

$$\phi_{2k-1}^{-1}(\mathcal{F}_1^{2k}, \omega_1^{2k}, L_1^{2k}, \Sigma_1^{2k}, (\Sigma_1')^{2k}).$$

Now concatenate the two homotopies

$$(\mathcal{F}_s^{2k-2}, \omega_s^{2k-2}, L_s^{2k-2}, \Sigma_s^{2k-2}, (\Sigma_s')^{2k-2})$$

and

$$\phi_{2k-1}^{-1}(\mathcal{F}_s^{2k}, \omega_s^{2k}, L_s^{2k}, \Sigma_s^{2k}, (\Sigma_s')^{2k}).$$

Iterate this process. Then, we obtain a homotopy of relative symplectic fibrations

$$(\mathcal{F}_s, \omega_s, L_s, \Sigma_s, \Sigma_s')$$

which starts at $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ and ends at

$$(\phi_1)^{-1} \circ .. \circ \phi_{2k-1}^{-1}(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}') = (\phi_{2k-1} \circ ..\phi_1)^{-1}(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}').$$

Since $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}')$ has vanishing symplectic curvature and is diffeomorphic to

$$(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1'),$$

we find that

$$(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$$

has vanishing symplectic curvature (cf. Proposition 2.2.17). Further, since $\overline{\Sigma}, \overline{\Sigma}'$ are horizontal, and horizontal submanifolds are mapped to horizontal submanifolds, Σ_1, Σ_1' are horizontal and the theorem follows. $\square$

Remark

As was already said in the introduction, we need to deform a given relative symplectic fibration to have vanishing symplectic curvature. In chapters 3 and 4 we will construct an equivalent relative symplectic fibration with vanishing symplectic curvature. Then Theorem 2.4.10 gives us the required homotopy to a relative symplectic fibration with vanishing symplectic curvature.

Lemma 2.4.11. *Let $(\mathcal{F}_t, \omega_t, L_t, \Sigma_t, \Sigma_t')$ be a homotopy of relative symplectic fibrations. Then there exists a homotopy of relative symplectic fibrations $(\widetilde{\mathcal{F}}_t, \omega_0, \widetilde{L}_t, \widetilde{\Sigma}_s, \widetilde{\Sigma}_t')$ such that $(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$ and $(\widetilde{\mathcal{F}}_1, \omega_0, \widetilde{L}_1, \widetilde{\Sigma}_1, \widetilde{\Sigma}_1')$ are diffeomorphic and the Lagrangian isotopy $\widetilde{L}_t$ can be realised by a Hamiltonian isotopy ψ_t of (M, ω_0).*

Proof. ω_t is a family of cohomologous symplectic forms and so Moser's theorem gives an isotopy ϕ_t of diffeomorphisms of M, such that $\phi_t^*\omega_t = \omega_0$. Then we claim that

$$\phi_t^{-1}(\mathcal{F}_t, \omega_t, L_t, \Sigma_t, \Sigma_t') = (\phi_t^{-1}\mathcal{F}_t, \omega_0, \phi_t^{-1}(L_t), \phi_t^{-1}(\Sigma_t), \phi_t^{-1}(\Sigma_t'))$$

is the desired homotopy of relative symplectic fibrations.
To show this, first note that L_t is monotone for ω_t for all t by assumption. Now

$$(\widetilde{L}_t = \phi_t^{-1} L_t, \omega_0 = (\phi_t^{-1}{}_*\omega_t))$$

is the push-forward data of (L_t, ω_t) under the diffeomorphism ϕ_t^{-1}, hence by Proposition 2.3.6, it follows that $\widetilde{L}_t$ is monotone for ω_0. The other conditions are trivial to check. To show that the Lagrangian isotopy can be realised by a Hamiltonian isotopy we use Banyaga's isotopy extension theorem. Let $l_t \colon L_0 \to M$ be a isotopy realising $L_t = l_t(L_0)$, then $\widetilde{L}_t = \phi_t^{-1} \circ l_t(L_0)$ realises $\widetilde{L}_t$.
First find a symplectic extension $\hat{\psi}_t \colon U \to M$ of the Lagrangian isotopy $\phi_t^{-1} \circ l_t \colon L_0 \to M$ with U a neighbourhood of L_0 in M. Now extend this to a smooth diffeotopy ρ_t of M by the isotopy extension theorem and consider the symplectic form $\rho_t^*\omega_0$. Again by Proposition 2.3.6, it follows that the push-forward torus

$$\rho_t^{-1}(\widetilde{L}_t)$$

is monotone for

$$\rho_{t*}^{-1}\omega_0 = \rho_t^*\omega_0.$$

But ρ_t extends $\phi_t^{-1} \circ l_t$ thus

$$\rho_t^{-1}\widetilde{L}_t = L_0$$

for all t. Hence L_0 is monotone for $\rho_t^*\omega_0$ for all t. This implies however that the relative cohomology class of $\rho_t^*\omega_0$ is constant in t and so Banyaga's isotopy extension theorem implies that the smooth isotopy ρ_t can be altered to a symplectic isotopy ψ_t which extends $\phi_t^{-1} \circ l_t$. But then the symplectic isotopy ψ_t is actually Hamiltonian since M is simply connected. Thus $\psi_t(L) = \widetilde{L}_t$ is the desired Hamiltonian isotopy. This proves the Lemma. □

2.5 The main result

The aim of this thesis is to prove the following result:

Theorem 2.5.1. *Let $L \subset (M, \omega_0)$ be an embedded monotone Lagrangian torus fibered by a symplectic ruling $\pi\colon M \to S$, the fibers of which are in the homology class B. Let Σ, Σ' be two disjoint symplectic sections of π in the homology class A such that $\Sigma \cap T = \emptyset$ where T is the solid torus in Def. 2.4.1 with $\partial T = L$. Further, for all $q \in \gamma := \pi(L)$, Σ' intersects the unique disk $D_q = T \cap \pi^{-1}(q)$ in its interior.*
Then L is Hamiltonian isotopic to the Clifford torus L_{std}.

The strategy to prove this will be the following. The conditions in the theorem give rise to a relative symplectic fibration $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ where the foliation $\mathcal{F}$ is given by the fibers of π. Note that the symplectic connection on M induced by (π, ω_0) need not have vanishing symplectic curvature in general. If, however, it has vanishing curvature then, using symplectic paralleltransport, we can write down explicitly a symplectomorphism ϕ of (M, ω_0) which induces the identity on homology, maps the fibers of π to the fibers of the standard projection p_1 and which maps L to the Clifford torus L_{std}. By a theorem of Gromov there exists a symplectic isotopy ϕ_t between id and ϕ. Since M is simply connected this symplectic isotopy will also be Hamiltonian.
Thus most of our work will go into showing that starting with any relative symplectic fibration, there exists a homotopy of relative symplectic fibrations to one with vanishing symplectic curvature.
In chapter 3 we will bring our data in a particularly nice form. In chapter 4 we will do the actual work of changing the symplectic connection to have vanishing symplectic curvature. In chapter 5 we will then discuss how the results of chapter 4 are translated in the setting of homotopies of relative symplectic fibrations.

Chapter 3

The standardisation

In this chapter and the next chapter, we will assume that L is an embedded monotone Lagrangian torus in $(M = S^2 \times S^2, \omega)$ fibered by the standard projection p_1. ω is some symplectic form on M such that p_1 is a symplectic fibration. Further we assume that Σ, Σ' are two disjoint, symplectic sections of p_1 in the homology class $A = [S^2 \times pt]$ such that Σ is disjoint from the solid torus T ($\partial T = L$) and Σ' intersects the unique closed disk $T \cap \pi^{-1}(q)$ over a point $q \in p_1(L)$ in its interior. Let $\mathcal{F}_{std}$ denote the foliation of $S^2 \times S^2$ by the fibers of p_1. Then, in other words, we assume in this chapter, that $(\mathcal{F}_{std}, \omega, L, \Sigma, \Sigma')$ is a relative symplectic fibration on $S^2 \times S^2$.

Remark

In the following we will consider several results to make the setup nicer (these will give rise to equivalent relative symplectic fibrations). Most of them will be of topological nature and only a few will have symplectic content. To help the reader distinguish between a result where nothing really happens and we just look at the problem from a different angle, we indicate this by a label T. If however there is something symplectically important happening, we indicate this by a label S.

3.1 Conveniently fibered Lagrangian tori

We now want to adjust the Lagrangian torus which is already fibred by p_1 in a particular nice way. We want it to lie over the equator in the base and to intersect the fiber over the point $z_0 = (1, 0, 0)$ in the equator.

Consider $S^2 \subset \mathbb{R}^3$ in the standard way. Let $z_0 = (1, 0, 0)$, the northpole $N = (0, 0, 1)$ and the southpole $S = (0, 0, -1)$. Moreover let $E = \{z = 0\} \subset S^2$ denote the equator. Let $\sigma^p \colon S^2 \setminus \{p\} \to \mathbb{C}$ denote stereographic projection (see Appendix A) from $p \in S^2$. Let $F_x = p_1^{-1}(x)$ and let $F := p_1^{-1}(z_0)$.

Definition 3.1.1 (T)**.** *Let $p \colon S^2 \times S^2 \to S^2$ be a fibration such that $p^{-1}(z_0) = p_1^{-1}(z_0)$ i.e. that the fibers over z_0 of p and p_1 agree. Then an embedded torus $L \subset S^2 \times S^2$ is* conveniently fibered *by p if*

- *L is fibred by p;*
- *$p(L) = E \subset S^2$ the equator in the base;*
- *$p^{-1}(z_0) \cap L = E \subset S^2$ the equator in the fiber over z_0;*

After these preliminary remarks we can now phrase:

Proposition 3.1.2 (T)**.** *There exists a diffeomorphism τ of $M = S^2 \times S^2$ such that τ preserves the standard fibration p_1 and such that $\tau^{-1}(L)$ is conveniently fibered by p_1.*

Proof. A detailed proof can be found in Appendix B. Here is an outline. We will only show the proposition for the base curve since the proof for the curve in the fiber is precisely the same. Let $p(L) = \gamma_L$ be the closed embedded curve in the base. By the Jordan curve theorem the complement in S^2 of γ_L are two simply connected disks. Thus take the union of the curve and one of them and use (an extension to the boundary of) the Riemann mapping theorem to find a diffeomorphism θ to the closed upper half disk in S^2. Then θ^{-1} and the inclusion of the upper half disk are two embeddings of the closed disk in S^2. Hence they are isotopic (any two embeddings of the closed disk in a manifold are isotopic) and by the isotopy extension theorem there exists a diffeomorphism H of S^2 which maps one to the other. Pulling all the data back by the diffeomorphism $\tau = H \times id$ gives the required properties. □

For future reference, we summarize Theorem 2.2.5, Proposition 2.4.8 and Proposition 3.1.2 by:

Corollary 3.1.3. *Let $M, L, \pi, \Sigma, \Sigma'$ as in Theorem 2.5.1. Then $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ with $\mathcal{F}$ given by the fibers of π is the relative symplectic fibration on M. Then there exist diffeomorphisms $\tau\colon M \to S^2 \times S^2$ and $u\colon B \to S^2$ such that the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{\tau} & S^2 \times S^2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ B & \xrightarrow{u} & S^2 \end{array} \tag{3.1}$$

and such that $\tau(L)$ is an embedded monotone Lagrangian torus for $\tau_*\omega_0$, which is conveniently fibered by p_1. Further, $\tau(\mathcal{F}, \omega_0, L, \Sigma, \Sigma') = (\mathcal{F}_{std}, \tau_*\omega, \tau(L), \tau(\Sigma), \tau(\Sigma'))$ is a relative symplectic fibration on $S^2 \times S^2$.

3.2 Standardisation of the symplectic fibration near a fiber

Note that the standardisation can't be done by diffeomorphisms preserving p_1 alone due to a possibly non-vanishing curvature of the symplectic connection (a diffeomorphism can't map a non-integrable distribution to an integrable one). Indeed by the example (p_1, ω_0) on $S^2 \times S^2$ has vanishing curvature. Thus at some point we will have to alter the form and the fibration independent (not related by diffeomorphisms) of each other.

3.2.1 Standardisation of the symplectic form

Before we start with the actual standardisation of the symplectic form we want to make the symplectic form standard in the vertical direction, therefore we have the following two results.

Proposition 3.2.1 (T)**.** *Let ω, ω_{std} be symplectic cohomologous forms on $S^2 \subset \mathbb{R}^3$. Let E denote the equator and D the upper hemi-sphere. Let $\int_D \omega = \frac{1}{2}$. Then there exists a diffeomorphism $h \in Diff^+(S^2)$ such that*

- $h(E) = E$
- $h^*\omega = \omega_{std}$

Proof. The linear interpolation between the two symplectic forms gives rise to a Moser isotopy ϕ_t. Consider the Lagrangian isotopy given by $\psi_t \colon E \to S^2; e \mapsto \phi_t^{-1}(e)$. Then extend this to a symplectic isotopy $\hat{\psi}_t$ of a neighbourhood U of E in S^2. Extend this by the isotopy extension theorem to a diffeotopy ρ_t of S^2 and check that by the conditions on the two symplectic forms the extension ρ_t satisfies the conditions of the Banyaga isotopy extension theorem. Hence there exists actually a symplectic extension α_t of ψ_t. Now consider the diffeomorphism

$$h = \phi_1 \circ \alpha_1.$$

Then

$$h^*\omega = \alpha_1^*\phi_1^*\omega_1 = \alpha_1^*\omega_{std} = \omega_{std}$$

and

$$h(E) = \phi_1 \circ \alpha_1(E) = \phi_1 \circ \psi_1(E) = \phi_1 \circ \phi_1^{-1}(E) = E.$$

□

Thus by Proposition 3.2.1, we can assume that the symplectic form restricted to the fiber over z_0 is the standard form ω_{std}.

Proposition 3.2.2 (T)**.** *Let M, p_1, L as above and moreover let $\omega|_F = \omega_{std}$, then there exists a p_1-preserving diffeomorphism τ of M such that*

- *$\tau^{-1}(L)$ is monotone Lagrangian for $\tau^*\omega$ and conveniently fibered by p_1;*
- *$\tau^*\omega = \omega_{std}$ when restricted to F_z for all z in a neighbourhood V of z_0;*
- *$\tau = id$ outside a neighbourhood $U \times S^2$ where U is a neighbourhood of z_0 and $V \subset U$.*

Proof. This is just Proposition 2.2.12, trivialising by symplectic parallel transport, and the Remark after the proof. We only note that τ preserves the fibers of p_1 and is the identity on F, thus it follows that $\tau^{-1}(L)$ is a monotone Lagrangian torus for the pull-back symplectic form $\tau^*\omega$ which is conveniently fibered by p_1. This proves the proposition. □

Theorem 3.2.3 (T)**.** *There exists a diffeomorphism $\hat{\tau} \in Diff_0^+(M)$ and neighbourhoods $\hat{V} \subset \hat{W}$ of F in M such that*

- *$supp(\hat{\tau}) \subset \hat{W}$;*
- *$\hat{\tau}^*\omega$ restricted to $\hat{V}$ is ω_0;*
- *$\hat{\tau}$ is the identity on $p_1^{-1}(E) = E \times S^2$;*

Proof. Choosing local coordinates on the base, centered at z_0, we can assume that $p_1 \colon \mathbb{C} \times S^2 \to \mathbb{C}; (z, w) \mapsto z$ and L projects to the real line.
First we find a diffeomorphism χ of $\mathbb{C} \times S^2$ with compact support which pulls the symplectic form back to a form which agrees with the standard form on $TM|_{Q_{2\epsilon}}$ where $Q_\epsilon := p_1^{-1}((-\epsilon, \epsilon))$. ϵ is chosen so small that $(-2\epsilon, 2\epsilon) \subset V$ from Proposition 3.2.2. Then we alter the obtained form by a Moser type argument such that it agrees with the standard form in a neighbourhood of F.
After applying Proposition 3.2.2, the horizontal lifts $\tilde{\frac{\partial}{\partial x}}, \tilde{\frac{\partial}{\partial y}}$ of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ with respect to ω are just $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ for points in $Q_{2\epsilon}$ (cf. the proof of 2.2.12). Thus the symplectic forms ω and ω_0 agree on $TQ_{2\epsilon}$ and have the same orthogonal complements to $\ker dp_1$, the span of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. It's just the evaluation on $\tilde{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x}, \tilde{\frac{\partial}{\partial y}} = \frac{\partial}{\partial y}$ which might still differ.
Consider the non-zero vector field

$$Y = \frac{1}{\omega(\tilde{\frac{\partial}{\partial x}}, \tilde{\frac{\partial}{\partial y}})} \frac{\partial}{\partial y}$$

and let ψ_s be its flow, then

$$\chi \colon D(0, \epsilon) \times S^2 \to \mathbb{C} \times S^2$$

$$(t + is, w) \mapsto \psi_s(t + i0, w)$$

is an embedding of a neighbourhood of F into $\mathbb{C} \times S^2$. Indeed Y is non-zero, so $\psi_{-s}(t + i0, w)$ is a smooth inverse. It is the identity on $\mathbb{R} \times S^2$ where it is defined and it distorts the fibers of p_1 in y-direction.
Extend χ to a diffeomorphism of all of $S^2 \times S^2$ which is the identity outside a bigger neighbourhood $\hat{W} \subset D(0, 2\epsilon) \times S^2$ of F. Pulling back the data by χ finishes the first step, i.e. we can assume that ω agrees with the standard form ω_0 on $TM|_{Q_\epsilon}$. Further since $\chi = id$ on the part of $p_1^{-1}(\mathbb{R})$ where it is defined, we can assume that the extension is also the identity on $p_1^{-1}(E)$. So $\chi^{-1}L$ is still conveniently fibered by $p := p_1 \circ \chi^{-1}$.
Since symplecticity is open and ω, ω_0 agree on $TM|_{Q_\epsilon}$, there exists a neighbourhood of Q_ϵ on which the linear interpolation of the two forms is symplectic. There exists a primitive σ of $\omega - \omega_0$ which is defined in a neighbourhood of Q_ϵ and which vanishes for points in Q_ϵ.

Lemma 3.2.4. *There exists a 1-form $\sigma \in \Omega^1(U)$ defined on a neighbourhood U of Q_ϵ such that*

$$\omega - \omega_0 = d\sigma$$

and $\sigma_x = 0$ for all $x \in Q_\epsilon$

Proof. For a proof compare Lemma 3.14 page 94 in [13] and Appendix C. □

Hence the Moser isotopy for the form σ is the identity on Q_ϵ and by cutting off suitably we obtain a diffeomorphism $\hat{\tau}$ which pulls the form ω back to the standard form ω_0 in a neighbourhood $\hat{V}$ of F as desired and which has support in $\hat{W}$. By construction, $\hat{\tau}$ is isotopic to the identity. This proves the theorem. □

3.2.2 Standardisation of the symplectic foliation near the fiber

$\hat{\tau}$ was not p_1-fibre-preserving (except over the equator in the base) and did alter the standard foliation $\mathcal{F}_{std}$. This will be fixed in the next theorem in a smaller neighbourhood of the fibre F. We will describe how we can see the bent fibers close to F as a smooth family of graphs of functions $\bar{f}^\lambda\colon F \to \mathbb{C}$, parametrised by their intersection point (λ, N) with the disk $D(z_0, \epsilon) \times \{N\}$ (see the fig. 3.1 on the following page). Recall that $z_0 = (1, 0, 0)$.

Let F^λ denote the leaf of the foliation $\hat{\tau}^{-1}\mathcal{F}_{std}$ through the point (λ, N).

Proposition 3.2.5 (T). *There exists an $\epsilon > 0$ and a smooth family of smooth functions $f^\lambda\colon S^2 \to \mathbb{C};\ \lambda \in D(z_0, \epsilon)$ such that*

$$F^\lambda = \left\{(z, w) \in \mathbb{C} \times S^2 | z = \lambda + f^\lambda(w)\right\}$$

with $f^\lambda = 0$ for λ real and such that $f^\lambda(N) = 0$ for all λ.

Proof. Again we think of the base being $\mathbb{C}$, F to be the fiber over the origin and L along the real line. Then for λ real, by construction $F^\lambda = \{\lambda\} \times S^2$ and so F^λ is transverse to all fibers $S_q = p_2^{-1}(q)$ of the standard projection p_2. Since transversality is generic there exists a neighbourhood U of $0 \subset \mathbb{C}$ for which all leaves F^λ with $\lambda \in U$ are transverse to all the S_q. The following topological lemma proves the proposition.

Lemma 3.2.6. *Let X, Y be smooth manifolds with X compact and simply connected. Let $S \subset X \times Y$ be a smooth, compact submanifold of the same dimension as X and such that S is transverse to $\{x\} \times Y$ for all $x \in X$. Then S can be written as the graph of a unique function $f\colon X \to Y$*

Proof. Let $\pi\colon X \times Y \to Y;\ (x, y) \mapsto x$ be the standard projection. Then by transversality if $(x, y) \in S$ then $d\pi_{(x,y)}\colon T_{(x,y)}S \to T_xX$ is an isomorphism. Indeed it's injective since if $d\pi_{(x,y)}(v) = 0$ then $v \in \ker d\pi$. Thus $v \in T_{(x,y)}\{x\} \times Y$ so that by transversality $v = 0$. Hence $\pi|_S$ is a local diffeomorphism. But since S is compact any point in $x \in X$ can have at most finitely many preimages in S under π. So that $\pi|_S\colon S \to X$ is a covering map. But X is simply connected, hence it has no non-trivial cover. It follows that π is a diffeomorphism. Let $\varphi = (\pi|_S)^{-1}$ then $s = \iota_S \circ \varphi\colon X \to X \times Y$ is a section of π with image S. But in a trivial fibration $\pi\colon X \times Y \to X$ any section s is of the form $s(x) = (x, f(x))$ for some unique function $f\colon X \to Y$ hence $S = \Gamma(f)$ the graph of f. □

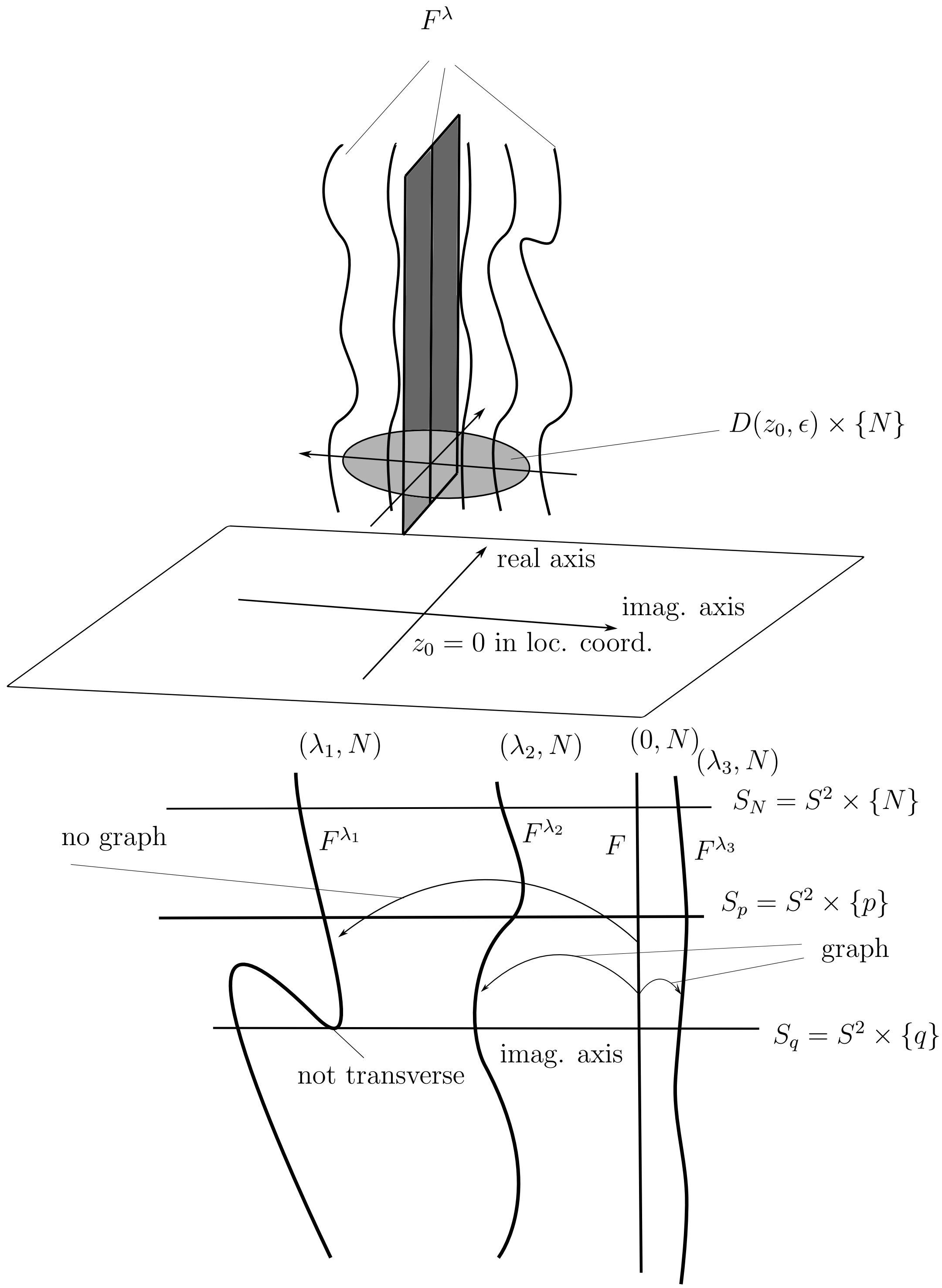

Figure 3.1: The bent fibers over the imaginary axis

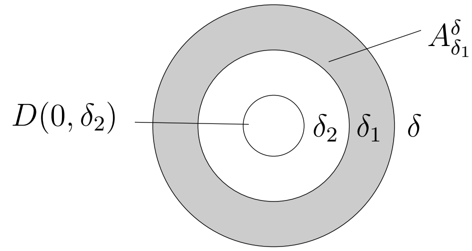

Figure 3.2: where the alteration of G_s takes place in the base

$\square$

Remark-1
The smooth map

$$G\colon D(0,\delta) \times S^2 \to \mathbb{C} \times S^2$$

$$(\lambda, w) \mapsto (\lambda + f^{\lambda}(w), w)$$

is an embedding for $\delta < \epsilon$ small enough (see C.0.14 in Appendix C). By construction

$$G(\{\lambda\} \times S^2) = F^{\lambda}$$

the leaf of the foliation $\mathcal{F} = \hat{\tau}^{-1}\mathcal{F}_{std}$ through the point $(\lambda, N) \in \mathbb{C} \times S^2$. Recall that $\hat{\tau}$ is the identity over the real line (the equator in the base), hence $f^{\lambda} = 0$ for real λ and hence $G(\{\lambda\} \times S^2) = \{\lambda\} \times S^2$ for real λ.
Let $A_r^R := \{z \in \mathbb{C} | r < |z| < R\}$. In the following we will alter $G(\lambda, w) = (\lambda + f^{\lambda}(w), w)$ through embeddings $G_s(\lambda, w) = (\lambda + f_s^{\lambda}(w), w)$ $(G_1 = G)$ being fixed on $A_{\delta_1}^{\delta} \times S^2$ for some $\delta_1 < \delta$ to an embedding G_0 being the identity on $D(0, \delta_2) \times S^2$ for $\delta_2 < \delta_1$ (see fig 3.2). Observe, that being the identity for $|\lambda| < \delta_2$ implies that the functions f_1^{λ} vanish for $|\lambda| < \delta_2$. But the deformation G_s will be chosen such that $f_s^{\lambda} = 0$ for λ real for all s, consequently, the images $G_s(\{\lambda\} \times S^2) = \{\lambda\} \times S^2$ for real λ and all s.

Further, the family f_s^{λ} (and thus the embeddings G_s) will be chosen such that the images under G_s of the sets of the form $\{\lambda\} \times S^2$ will be symplectic and transverse to the symplectic sections Σ, Σ' for all s. Then the family G_s of embeddings satisfies all the conditions of Theorem 2.2.4. Thus these images will form the leaves of a family of symplectic foliations $\mathcal{F}_s$ which standardises the symplectic fibration near F. Moreover, the leaves over the real line in which L sits have not been altered at all since $G_s(\{\lambda\} \times S^2) = \{\lambda\} \times S^2$ for real λ and all s. Hence the quintuple $(\mathcal{F}_s, \omega, L, \Sigma, \Sigma')$ is a relative symplectic fibration for all s. Indeed after the alteration on $D(0, \delta_2) \times S^2$, the foliation $\mathcal{F}_0$ as well as the symplectic form $\omega = \omega_0$ are standard. In particular there, near F, we have vanishing symplectic curvature.

Remark

- Note that by construction $\omega = \omega_0$ on $G(D(0, \delta) \times S^2)$.

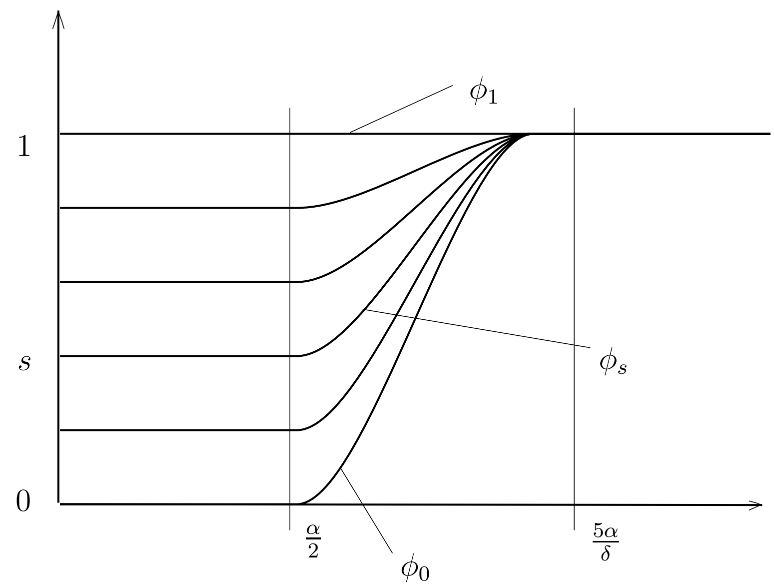

Figure 3.3: The family ϕ_s of cut-off functions

In Proposition 3.2.9 below we need two families of cut-off functions whose properties are fixed in the following two propositions (the proofs are postphoned to Appendix C). Fig. 3.3 describes pictorially both families of the cut-off functions.

Proposition 3.2.7. *For every $1 > \delta > 0,\ \alpha > 0$ there exists a smooth family of non-decreasing functions $\phi_s \colon [0, \infty) \to [0, 1], s \in [0, 1]$ satisfying*

$$0 \leq r\phi_s'(r) + \phi_s(r) < \frac{1}{1-\delta} \tag{3.2}$$

such that $\phi_s(r) = s$ for $r \leq \frac{\alpha}{2}$, $\phi_s(r) = 1$ for $r \geq \frac{5\alpha}{\delta}$ and $\phi_1 \equiv 1$.

Proposition 3.2.8. *There exists a constant $\mathcal{C} > 0$ such that for all $\epsilon > 0$ there exists a smooth family of functions $\phi_s^\epsilon \colon [0, \infty) \to [0, 1], s \in [0, 1]$ such that*

- *$\phi_s^\epsilon(r) = s$ for all $r \leq \frac{\epsilon}{20}$*
- *$\phi_1^\epsilon(r) \equiv 1$*
- *$\phi_s^\epsilon(r) = 1$ for all $r \geq \epsilon$, for all s*
- *$\max_{r \in [0,\infty)} |\phi_s'^\epsilon(r)| \leq \frac{1}{\epsilon}\mathcal{C}$ for all s.*

The following proposition is prepatory for the theorem below.

Proposition 3.2.9 (S)**.** *Let $G \colon D(0, \delta) \times S^2 \to \mathbb{C} \times S^2;\ (\lambda, w) \mapsto (\lambda + f^\lambda(w), w)$ be the embedding from **Remark-1** on the previous page. Hence in particular $f^{x+i0}(w) \equiv 0$ and so $G(x + i0, w) = (x + i0, w)$. Then there exist positive real numbers $0 < \delta_2 < \delta_1 < \delta$ and a smooth family of embeddings $G_s \colon D(0, \delta) \times S^2 \to \mathbb{C} \times S^2;\ (\lambda, w) \mapsto (\lambda + f_s^\lambda(w), w)$ for $s \in \mathbb{R}$, with $G_s = G_1 = G$ for $s \geq 1$ and $G_s = G_0$ for $s \leq 0$ such that*

- $G_s|_{A^{\delta}_{\delta_1}\times S^2} = G|_{A^{\delta}_{\delta_1}\times S^2}$;
- $G_0|_{D(0,\delta_2)\times S^2} = id$;
- $G_s(x+i0,w) = (x+i0,w)$ *for all* x, w;
- $F^{\lambda}_s := G_s(\{\lambda\}\times S^2)$ *is symplectic for all* s, λ;
- F^{λ}_s *is transverse to* Σ, Σ' *for all* s, λ;

Proof. We will show how to construct the family G_s in three steps, first without taking care of the last two assertions. Once we have seen this we will show in an explicit calculation that the remaining two assertions can also be realised.

Linearization We write $f(\lambda,w) = f^{\lambda}(w)$ and consider the first order Taylor expansion of f with respect to λ with remainder (here we regard λ as a vector in $\mathbb{R}^2$)

$$f(\lambda,w) = f(0,w) + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \lambda + f_2(\lambda,w)$$

with $|f_2(\lambda,w)| \leq C_1|\lambda|^2$ for some constant C_1 for all w and $|\lambda| \leq \frac{\delta}{2}$. Now we multiply the remainder by the smooth family of cut-off functions ϕ^{ϵ}_s from Proposition 3.2.8. Then we can show that for $\epsilon < \frac{\delta}{4}$ small enough, the smooth maps

$$G_s\colon D(0,2\epsilon)\times S^2 \to \mathbb{C}\times S^2$$

$$G_s(\lambda,w) = \left(\lambda + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \lambda + \phi^{\epsilon}_s(|\lambda|)f_2(\lambda,w), w\right)$$

are embeddings. It is enough to show the injectivity and the immersion property.
To show injectivity we will proceed in all three steps as follows: Assume first that

$$G_s(\lambda,w) = G_s(\lambda',w')$$

then clearly $w = w'$ and

$$\lambda + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \lambda + \phi^{\epsilon}_s(|\lambda|)f_2(\lambda,w) = \lambda' + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \lambda' + \phi^{\epsilon}_s(|\lambda'|)f_2(\lambda',w).$$

We rewrite this equation as

$$\left(Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix}\right)(\lambda-\lambda') = \phi^{\epsilon}_s(|\lambda'|)f_2(\lambda',w) - \phi^{\epsilon}_s(|\lambda|)f_2(\lambda,w).$$

Since G is an embedding, its differential

$$dG = \begin{pmatrix} Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} & B_s \\ 0 & Id \end{pmatrix}$$

is non-degenerate and we can find a constant μ such that

$$0 < \mu \leq \inf_{|v|=1; w \in S^2} \left| \left(Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \right) v \right|.$$

Let

$$H_s^w(\lambda) = \phi_s^\epsilon(|\lambda|) f_2(\lambda, w)$$

then if the operator norm of the differential $\|(DH_s^w)_\lambda\| < \mu$ for all w and $|\lambda| \leq 2\epsilon$, then

$$\mu|\lambda - \lambda'| \leq \left| \left(Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \right) (\lambda - \lambda') \right| = |H_s^w(\lambda) - H_s^w(\lambda')| \leq$$

$$\leq \|(DH_s^w)_\lambda\|_{\bar{D}(0,2\epsilon)\times S^2} |\lambda - \lambda'| < \mu|\lambda - \lambda'|$$

showing the injectivity of G_s. Furthermore G_s is immersive if its differential

$$dG_s = \begin{pmatrix} Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} + DH_s^w & B_s \\ 0 & Id \end{pmatrix}$$

is non-degenerate. This is equivalent to

$$Id + \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} + DH_s^w$$

being non-degenerate. Again this is true if

$$\|(DH_s^w)_\lambda\| < \mu$$

on $\bar{D}(0, 2\epsilon) \times S^2$.
Now estimate the operator norm of $(DH_s^w)_\lambda$. For

$$|\lambda| \leq 2\epsilon$$

$$\|(DH_s^w)_\lambda\| \le |(\phi_s^\epsilon)'||f_2(\lambda,w)| + |\phi_s^\epsilon|\|Df_2\| \le \mathcal{C}\frac{1}{\epsilon}C_1 4\epsilon^2 + 2C_2\epsilon \le 4(\mathcal{C}C_1 + C_2)\epsilon.$$

Where $\|Df_2\| \le C_2|\lambda|$ for all w and $|\lambda| \le \frac{\delta}{2}$ and where $\mathcal{C}$ is the constant from Proposition 3.2.8. Choose

$$\epsilon < \min\left\{\frac{\delta}{4}, \frac{\mu}{4(\mathcal{C}C_1 + C_2)}\right\}$$

then by construction

$$G_s\colon D(0,2\epsilon)\times S^2 \to \mathbb{C}\times S^2$$

is a smooth family of embeddings.
By the properties of ϕ_s^ϵ, G is only changed for $|\lambda| \le \epsilon$. Thus the alteration takes place in $G(D(0,\epsilon)\times S^2)$, G is not at all altered on $(-2\epsilon, 2\epsilon)\times S^2$ and G_0 is linear in λ on $D(0,\frac{\epsilon}{20})\times S^2$ (compare Proposition 3.2.8).
Since $G_s = G$ on $(D(0,2\epsilon)\setminus \bar{D}(0,\epsilon))\times S^2$ it follows as in the proof of Theorem 2.2.4 in Appendix C, that $G_s(D(0,2\epsilon)\times S^2) = G(D(0,2\epsilon)\times S^2)$ for all s. Hence we may extend the embeddings G_s by G to embeddings defined on all of $D(0,\delta)\times S^2$.
This proves the linearisation.

Standardisation By the linearisation we may assume that

$$f(\lambda,w) = \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix}\lambda$$

is linear in λ. But $f(\lambda,w) = 0$ for real λ and so

$$f(\lambda,w) = \begin{pmatrix} 0 & a(w) \\ 0 & b(w) \end{pmatrix}\lambda.$$

Thus

$$G\colon D(0,\delta)\times S^2 \to M$$
$$(\lambda_1,\lambda_2,w) \mapsto (\lambda_1 + a(w)\lambda_2, \lambda_2(1+b(w)), w).$$

A priori there is no bound on a except that $a(N) = 0$. Thus depending on the foliation given by G, $a(w) - a(N)$ can be arbitrarily large. Run along the imaginary axis starting from the origin and look what happens to the leaves. First note that the northpole is always fixed, i.e. $G(0,\lambda_2,N) = (0,\lambda_2,N)$ but $G(0,\lambda_2,w) = (a(w)\lambda_2, \lambda_2(1+b(w)), w)$ is arbitrarily far along the λ_1-axis depending on the maximum of $|a(w)|$. We would like to continue, by using a family of cut-off functions as in the linearisation, to kill the remaining terms which cause G to differ from the identity in a neighbourhood of $\{0\}\times S^2$.
Although it is probably not intrinsic to the problem, an explicit calculation shows that this method forces us to do this in two steps. First we kill the term involving $a(w)$ and then the term involving $b(w)$.
We cut off by a family of functions ϕ_s^ϵ as before, but which have now support on an ellipse

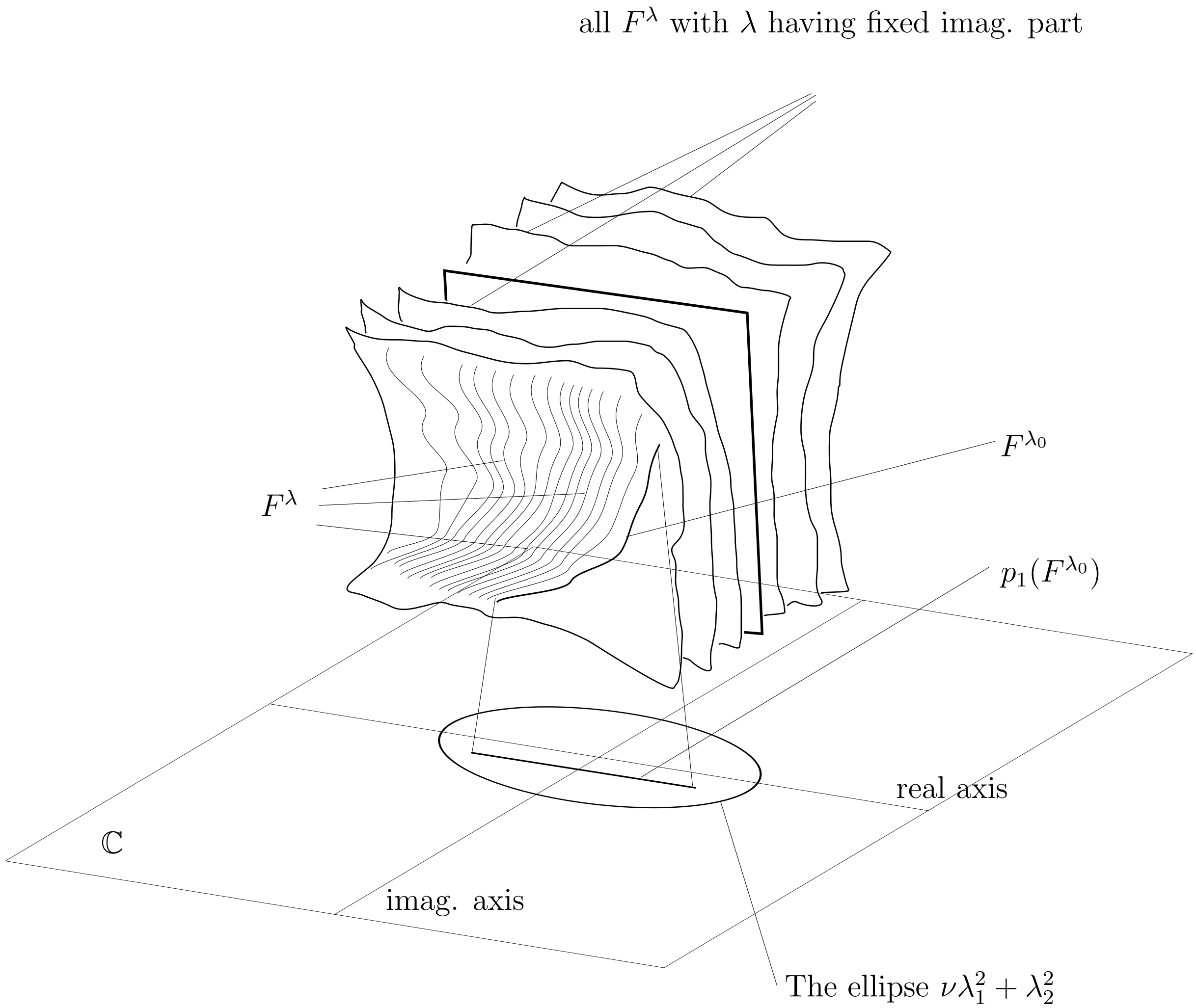

Figure 3.4: The figure shows a possible big distorsion of F^{λ_0} along the real axis

with excentricity related to the maximum of $|a(w)|$.
Let $c := \max_{w \in S^2} |a(w)|$ and choose positive real numbers ν, ϵ such that

$$\nu < \min\left\{\frac{1}{4\mathcal{C}^2 c^2}, 1\right\}$$

$$\epsilon < \nu\delta^2$$

($\mathcal{C}$ is the constant from Proposition 3.2.8). Again let ϕ_s^ϵ be the family of functions from Proposition 3.2.8 and consider the map

$$G_s \colon D(0,\delta) \times S^2 \to \mathbb{C} \times S^2$$

$$(\lambda_1, \lambda_2, w) \mapsto (\lambda_1 + \phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2)a(w)\lambda_2, (1 + b(w))\lambda_2, w).$$

With the choice of ν, ϵ we can show similarly as before that G_s is an embedding for all s.
Injectivity
Assume

$$G_s(\lambda, w) = G_s(\lambda', w')$$

then

1. $\lambda_1 + \phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2)a(w)\lambda_2 = \lambda_1' + \phi_s^\epsilon(\nu{\lambda_1'}^2 + {\lambda_2'}^2)a(w')\lambda_2'$
2. $(1 + b(w))\lambda_2 = (1 + b(w'))\lambda_2'$
3. $w = w'$

From G being immersive, it follows that $(1 + b(w)) > 0$. So from 2.,3. it follows that $w = w'$ and $\lambda_2 = \lambda_2'$. We can thus write

$$\lambda_1 - \lambda_1' = -a(w)\lambda_2 \left(\phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2) - \phi_s^\epsilon(\nu\lambda_1'^2 + \lambda_2^2)\right).$$

Let

$$H_s(\lambda_1) = \phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2)$$

then by the mean value theorem

$$H_s(\lambda_1) - H_s(\lambda_1') = H_s'(\zeta)(\lambda_1 - \lambda_1')$$

for $\zeta \in (\lambda_1', \lambda_1)$. By construction we have

$$H_s'(\zeta) = \phi_s^{\epsilon\prime}(\nu\zeta^2 + \lambda_2^2)2\nu\zeta$$

so that

$$|\lambda_1 - \lambda_1'| = |-a(w)||\lambda_2||H_s(\lambda_1) - H_s(\lambda_1')| = |-a(w)||\lambda_2||\phi_s^{\epsilon\prime}(\nu\zeta^2 + \lambda_2^2)|2\nu|\zeta||\lambda_1 - \lambda_1'|.$$

Note that $\phi_s^{\epsilon\prime}(r) = 0$ for $r > \epsilon$ since there $\phi_s^\epsilon(r)$ constantly equals 1. Hence for $\nu\zeta^2 + \lambda_2^2 \geq \epsilon$, it follows that $\lambda_1 = \lambda_1'$. Thus let

$$\nu\zeta^2 + \lambda_2^2 \leq \epsilon$$

then $|\zeta| \leq \frac{\sqrt{\epsilon}}{\sqrt{\nu}}$ and $|\lambda_2| \leq \sqrt{\epsilon}$. Hence

$$|-a(w)||\lambda_2||\phi_s^{\epsilon}{}'(\nu\zeta^2+\lambda_2^2)|2\nu|\zeta| \leq c\sqrt{\epsilon}\frac{\mathcal{C}}{\epsilon}2\nu\frac{\sqrt{\epsilon}}{\sqrt{\nu}} = c\mathcal{C}2\sqrt{\nu} < 1.$$

As desired this implies that $\lambda_1 = \lambda_1'$, showing the injectivity.
The immersion property
Note that dG_s is a block matrix of the form.

$$dG_s = \begin{pmatrix} A_s & B_s \\ 0 & Id \end{pmatrix}.$$

Thus $\det dG_s = \det(A_s)$ with

$$A_s = \begin{pmatrix} 1 + a(w)\phi_s^{\epsilon}{}'(\nu\lambda_1^2+\lambda_2^2)2\nu\lambda_1\lambda_2 & a(w)\left(\phi_s^{\epsilon}(\nu\lambda_1^2+\lambda_2^2)+\phi_s^{\epsilon}{}'(\nu\lambda_1^2+\lambda_2^2)2\lambda_2^2\right) \\ 0 & 1+b(w) \end{pmatrix}.$$

Thus $\det(dG_s) = (1+a(w)\phi^{\epsilon}{}'_s(\nu\lambda_1^2+\lambda_2^2)2\nu\lambda_1\lambda_2)(1+b(w))$ and $(1+b(w)) > 0$. But as for the injectivity by the choice of ν, ϵ

$$|a(w)||\phi_s^{\epsilon}{}'(\nu\lambda_1^2+\lambda_2^2)|2\nu|\lambda_1||\lambda_2| < 1$$

and the immersion property follows.
Now comes the last step.
We may assume that

$$G\colon D(0,\delta)\times S^2 \to \mathbb{C}\times S^2$$
$$(\lambda_1,\lambda_2,w) \mapsto (\lambda_1,(1+b(w))\lambda_2,w).$$

Since G is immersive $(1+b(w)) > 0$ for all $w \in S^2$. Let $1 > \xi > 0$ be a real number such that

$$1-\xi > -b(w) \quad \forall w \in S^2.$$

Let $\epsilon > 0$ be so small that

$$\frac{5\epsilon}{\xi} < \frac{\delta}{2}$$

and let $\phi_s\colon [0,\infty) \to [0,1]$ be the family of functions from Proposition 3.2.7 for $\alpha = \epsilon, \delta = \xi$. Now consider the smooth map

$$G_s\colon D(0,\delta)\times S^2 \to \mathbb{C}\times S^2$$

$$(\lambda_1,\lambda_2,w) \mapsto (\lambda_1,(1+\phi_s(|\lambda|)b(w))\lambda_2,w).$$

Then as claimed

$$G_0(\lambda,w) = (\lambda,w) \quad \text{for } |\lambda| < \frac{\epsilon}{2}.$$

By the special choice of the cut-off function we can show as before that G_s is an embedding.
Injectivity
Assume

$$G_s(\lambda, w) = G_s(\lambda', w')$$

then

1. $\lambda_1 = \lambda_1'$
2. $(1 + \phi_s(|\lambda|)b(w))\lambda_2 = (1 + \phi_s(|\lambda'|)b(w'))\lambda_2'$
3. $w = w'$

By 1., 3. the second equation can be written as

$$\lambda_2 - \lambda_2' = -b(w)\left(\phi_s(|\lambda|)\lambda_2 - \phi_s(|\lambda'|)\lambda_2'\right)$$

Write

$$H_s(\lambda_2) = \phi_s\left(\sqrt{\lambda_1^2 + \lambda_2^2}\right)\lambda_2$$

then by the mean value theorem

$$H_s(\lambda_2) - H_s(\lambda_2') = H_s'(\zeta)(\lambda_2 - \lambda_2')$$

for $\zeta \in (\lambda_2', \lambda_2)$. By construction

$$H_s'(\zeta) = \phi_s'(|\lambda|)\frac{\zeta^2}{|\lambda|} + \phi_s(|\lambda|)$$

with $\lambda = (\lambda_1, \zeta)$. But $|\zeta| \le |\lambda|$ so that

$$H_s'(\zeta) \le \phi_s'(r)r + \phi_s(r)$$

for $r = |\lambda|$. Hence

$$\lambda_2 - \lambda_2' = -b(w)H_s'(\zeta)(\lambda_2 - \lambda_2') \le -b(w)(\phi_s'(r)r + \phi_s(r))(\lambda_2 - \lambda_2') <$$

$$< (1-\xi)\frac{1}{1-\xi}(\lambda_2 - \lambda_2') < (\lambda_2 - \lambda_2').$$

This shows the injectivity of G_s.
The immersion property
Again dG_s is a block matrix of the form.

$$dG_s = \begin{pmatrix} A_s & B_s \\ 0 & \mathbf{1} \end{pmatrix}.$$

Thus $\det dG_s = \det(A_s)$ with

$$A_s = \begin{pmatrix} 1 & 0 \\ \\ \phi_s'(|\lambda|)\frac{\lambda_1}{|\lambda|}b(w)\lambda_2 & 1 + \phi_s'(|\lambda|)\frac{\lambda_2^2}{|\lambda|}b(w) + \phi_s(|\lambda|)b(w) \end{pmatrix}.$$

Then

$$\det(dG_s) = 1 + b(w)\left(\phi_s'(|\lambda|)\frac{\lambda_2^2}{|\lambda|} + \phi_s(|\lambda|)\right).$$

But

$$-b(w)(\phi_s'(|\lambda|)\frac{\lambda_2^2}{|\lambda|} + \phi_s(|\lambda|)) < 1$$

as before, and the immersion property follows.

We are left to show the assertion, that we can assure that all submanifolds

$$F_s^\lambda = G_s(\{\lambda\} \times S^2)$$

are symplectic and that the two submanifolds Σ, Σ' remain transverse to all the F_s^λ in all three deformations.

Transversality of Σ, Σ'

Let $G\colon D(0,\delta) \times S^2 \to \mathbb{C} \times S^2$ be an embedding as before such that Σ is transverse to F^λ for all λ. Consider the submanifold $\widetilde{\Sigma} := G^{-1}(\Sigma)$ of $D(0,\delta) \times S^2$ which is transverse to the standard fibers $\{\lambda\} \times S^2$.

Similarly to Proposition 3.2.6, transversality implies that there exists a section $s(z) = (z, g(z))$ of $p_1\colon D(0,\delta) \times S^2 \to D(0,\delta); (z,w) \mapsto z$ with image $\widetilde{\Sigma}$. Thus $\sigma := G \circ s$ is a parametrisation of $\Sigma \cap Im(G)$. Now let G_s be a smooth family of embeddings as before which alters the foliation in $D(0,\epsilon) \times S^2$ and such that $G_1 = G$.

Then transversality of F_s^λ to Σ means precisely, that the map

$$p_1 \circ G_s^{-1} \circ \sigma$$

is a submersion (or a local diffeomorphism for dimensional reasons).

As before dG_s is a block matrix of the form

$$dG_s = \begin{pmatrix} A_s & B_s \\ \\ 0 & Id \end{pmatrix}.$$

We write $A := A_1$, $B := B_1$ and $G = G_1$. By elementary facts about block matrices it follows (see C.0.16 for details) that

$$d(p_1 \circ G_s^{-1} \circ \sigma) = A_s^{-1}\left(A + (B - B_s) \circ dg\right).$$

Due to the immersion property of G_s,

$$\det A_s^{-1} > 0,$$

and the transversality of Σ boils down to show that the matrix

$$A + (B - B_s) \circ dg$$

is invertible for all s. Observe that $(B - B_s)(\lambda, w)$ is non-zero only for $|\lambda| \le \epsilon$ by the assumptions on the family G_s.
Let

$$0 < \mu := \inf_{|v|=1, w \in S^2, |\lambda| \le \frac{\delta}{2}} |A(\lambda, w)v| .$$

Then if

$$\|dg\| \|B - B_s\|_{\bar{D}(0,\epsilon) \times S^2} < \mu$$

for all s, the matrix

$$A + (B - B_s) \circ dg$$

is invertible and Σ is indeed transverse to all the F_s^λ as claimed. So we have to check whether we can make $\|B - B_s\|_{\bar{D}(0,\epsilon) \times S^2}$ arbitrary small for all three deformations above. We will do this after we have shown that for the symplecticity we need the same condition (possibly with other constants).

Symplecticity

Let G_s be a family of embeddings with $G_1 = G$ as before, which alters the foliation in $D(0, \epsilon) \times S^2$. We are required to show that the leaves F_s^λ are symplectic for the symplectic form ω_0 for all λ, s. This is equivalent to the condition that $G_s^* \omega_0 = \omega_s$ restricts to a symplectic form on any fiber $\{\lambda\} \times S^2$.
Since dG_s is a block matrix of the form

$$dG_s = \begin{pmatrix} A_s & B_s \\ & \\ 0 & Id \end{pmatrix},$$

a vertical vector $(0, v) \in T_z D(0, \delta) \times T_w S^2 = T_{(z,w)} (D(0, \delta) \times S^2)$ is mapped by dG_s to

$$dG_s(0, v) = \begin{pmatrix} A_s & B_s \\ & \\ 0 & Id \end{pmatrix} \begin{pmatrix} 0 \\ \\ v \end{pmatrix} = \begin{pmatrix} B_s v \\ \\ v \end{pmatrix}.$$

Thus

$$\omega_s((0, v), (0, w)) = G_s^* \omega_0((0, v), (0, w)) = \omega_{std}(v, w) + \omega_{std}(B_s v, B_s w).$$

Every two form α on S^2 is of the form $\alpha = f \omega_{std}$ for a function $f \colon S^2 \to \mathbb{R}$. Thus write $\omega_s^\lambda := \omega_s|_{\{\lambda\} \times S^2}$ and $f_s^\lambda \colon S^2 \to \mathbb{R}$ for the function satisfying

$$\omega_s^\lambda = f_s^\lambda \omega_{std}.$$

Symplecticity of F_s^λ is then equivalent to $f_s^\lambda > 0$ for all λ, s.
We write $f^\lambda := f_1^\lambda$ because of $G = G_1$. Since $\omega := \omega_1 = G^*\omega_0$ restricts to symplectic forms on $\{\lambda\} \times S^2$ by assumption, $f^\lambda > 0$ for all $\lambda \in D(0,\delta)$. We define real numbers

$$\nu := \inf_{\lambda \in \bar{D}(0,\frac{\delta}{2}), w \in S^2} |f^\lambda(w)| > 0$$

and

$$N = \sup_{(\lambda,w) \in \bar{D}(0,\frac{\delta}{2}) \times S^2} \|B(\lambda, w)\|.$$

Claim: *If*

$$\sup_{(\lambda,w) \in \bar{D}(0,\frac{\delta}{2}) \times S^2} \|B_s(\lambda, w) - B(\lambda, w)\| < \min\left\{\frac{\nu}{6N}, N\right\}$$

for all s, then the fibres F_s^λ are symplectic.
To see this, let v, w be a symplectic basis of $T_{(\lambda,p)}(\{\lambda\} \times S^2)$ for ω_{std}. Then

$$f_s^\lambda(p) = f_s^\lambda(p)\omega_{std}(v, w) = \omega_s^\lambda(p)(v, w)$$

where

$$\begin{gathered}\omega_s^\lambda(v, w) = \omega_{std}(v, w) + \omega_{std}(B_s v, B_s w) = \\ = \omega_{std}(v, w) + \omega_{std}(Bv, Bw) + \omega_{std}((B_s - B)v, B_s w) + \omega_{std}(Bv, (B_s - B)w) = \\ = \omega(v, w) + \omega_{std}((B_s - B)v, B_s w) + \omega_{std}(Bv, (B_s - B)w).\end{gathered}$$

But then

$$f_s^\lambda(p) \geq \nu - |\omega_{std}((B_s - B)v, B_s w) + \omega_{std}(Bv, (B_s - B)w)|.$$

Let g be the Riemannian metric on S^2 given by $g(v, w) = \omega_{std}(v, iw)$ and define $|v| := \sqrt{g(v, v)}$ for $v \in TS^2$. Let

$$u := \frac{(B_s - B)v}{|(B_s - B)v|},\ \tilde{u} := \frac{(B_s - B)w}{|(B_s - B)w|}, z := \frac{B_s w}{|B_s w|},\ \tilde{z} := \frac{Bv}{|Bv|}$$

if these vectors are defined.
So we estimate the last term:

$$\begin{gathered}|\omega_{std}((B_s - B)v, B_s w) + \omega_{std}(Bv, (B_s - B)w)| \leq \\ \leq |(B_s - B)v||B_s w||\omega_{std}(u, z)| + |Bv||(B_s - B)w||\omega_{std}(\tilde{z}, \tilde{u})|\end{gathered}$$

Since $u, \tilde{u}, z, \tilde{z}$ have norm 1 and g is compatible with ω_{std} it follows that $|\omega_{std}(u, z)|, |\omega_{std}(\tilde{z}, \tilde{u})| \leq 1$. Hence

$$|\omega_{std}((B_s - B)v, B_s w) + \omega_{std}(Bv, (B_s - B)w)| \leq \frac{\nu}{6N} 2N + N\frac{\nu}{6N} \leq \frac{\nu}{2}.$$

Thus

$$f_s^\lambda(p) \geq \nu - \frac{\nu}{2} = \frac{\nu}{2} > 0$$

and indeed F_s^λ is symplectic for all λ, s.
Estimation of $\|B_s - B\|$ in the deformations

In the first deformation we had

$$f_s(\lambda, w) = \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix} \lambda + \phi_s^\epsilon(|\lambda|) f_2(\lambda, w).$$

Thus

$$f(\lambda, w) - f_s(\lambda, w) = (1 - \phi_s^\epsilon(|\lambda|)) f_2(\lambda, w).$$

Consequently

$$(B - B_s)_{\lambda,w} = (1 - \phi_s^\epsilon(|\lambda|)) d(f_2^\lambda)_w.$$

where $f_2^\lambda(w) = f_2(\lambda, w)$ indicates that we regard λ as a parameter and differentiate with respect to w. But $\|(df_2^\lambda)_w\| \leq C_2|\lambda|$ for $|\lambda| < \frac{\delta}{2}$ since df_2^λ is part of the differential df_2 with $\|df_{2(\lambda,w)}\| \leq C_2|\lambda|$ for $|\lambda| < \frac{\delta}{2}$. Thus

$$\|B - B_s\|_{\bar{D}(0,\frac{\delta}{2}) \times S^2} < C_2 \epsilon$$

since $(1 - \phi_s^\epsilon)$ has support in $D(0, \epsilon)$. So indeed we can make $\|B_s - B\|$ arbitrarily small. In the second deformation we have

$$(B - B_s)_{\lambda,w} = \begin{pmatrix} \lambda_2 da_w - \lambda_2 \phi^\epsilon(\nu\lambda_1^2 + \lambda_2^2)_s da_w \\ \lambda_2 db_w - \lambda_2 db_w \end{pmatrix} = \begin{pmatrix} \lambda_2(1 - \phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2)) da_w \\ 0 \end{pmatrix}$$

Note that $1 - \phi_s^\epsilon(\nu\lambda_1^2 + \lambda_2^2) = 0$ for $\nu\lambda_1^2 + \lambda_2^2 \geq \epsilon$, thus we may assume that $\lambda_2^2 < \epsilon$ or equivalently that $|\lambda_2| < \sqrt{\epsilon}$. Let $C_4 = \|da_w\|_{w \in S^2}$ with respect to the metric g on S^2, then

$$\|B_s - B\|_{\bar{D}(0,\frac{\delta}{2}) \times S^2} < C_4 \sqrt{\epsilon}.$$

This can also be made arbitrarily small.
Finally in the last deformation we find

$$(B - B_s)_{\lambda,w} = \begin{pmatrix} 0 \\ (1 - \phi_s(|\lambda|)) \lambda_2 db_w \end{pmatrix}$$

define $C_5 := \|db_w\|_{w \in S^2}$ with respect to g, then

$$\|B_s - B\|_{\bar{D}(0,\frac{\delta}{2}) \times S^2} < C_5 \frac{5\epsilon}{\xi}$$

if ϕ_s is the family of functions from Proposition 3.2.7 for $\delta = \xi, \alpha = \epsilon$, since $(1 - \phi_s(|\lambda|))$ has support in $D(0, \frac{5\epsilon}{\xi})$. This proves the proposition. $\square$

Theorem 3.2.10. *(S) Let $\mathcal{F}$ be a foliation of $(M = S^2 \times S^2, \omega_0)$ by symplectic 2-spheres such that $\mathcal{F}$ agrees with the standard foliation $\mathcal{F}_{std}$ over the equator E in the base, i.e. $\mathcal{F}_x = p_1^{-1}(p_1(x))$ for all $x \in p_1^{-1}(E)$. Further Σ, Σ' are two disjoint symplectic submanifolds transverse to the leaves of $\mathcal{F}$. Then there exists a family of symplectic foliations $\mathcal{F}_s$ on M such that $\mathcal{F}_1 = \mathcal{F}$ with the following properties:*

- *$(\mathcal{F}_s)_x = p_1^{-1}(p_1(x))$ for any $x \in p_1^{-1}(E)$ for all s;*
- *Σ, Σ' are transverse to all the leaves of $\mathcal{F}_s$ for all s;*
- *$\mathcal{F}_0 = \mathcal{F}_{std}$ on a neighbourhood of $F = p_1^{-1}(z_0)$.*

Proof. By Proposition 3.2.5 and Proposition 3.2.9 we obtain a family of embeddings G_s which satisfies all the conditions of Theorem 2.2.4 in chapter 2. Thus the family of embeddings G_s gives rise to a smooth family of foliations $\mathcal{F}_s$ on M. All the desired properties follow now from the corresponding properties of the isotopy G_s. This proves the theorem. □

From the theorem we obtain a smooth family of foliation $\mathcal{F}_s$ of M by symplectic 2-spheres which are transverse to Σ, Σ' and such that the symplectic connection induced by $(\mathcal{F}_0, \omega)$ has vanishing symplectic curvature near F. Further throughout the deformation, the leaves of $\mathcal{F}_s$ through points in $p_1^{-1}(e)$ agree with $p_1^{-1}(e)$ for all $e \in E$ the equator in the base. Consequently, the torus L is still monotone Lagrangian for ω and it is fibered by the symplectic fibrations

$$\pi_s \colon M \to \Sigma';\ x \in (\mathcal{F}_s)_x \mapsto (\mathcal{F}_s)_x \cap \Sigma'.$$

Thus $(\mathcal{F}_s, \omega, L, \Sigma, \Sigma')$ is a homotopy of relative symplectic fibrations. Now we can apply corollary 3.1.3 to get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\tau} & M \\ \big\downarrow{\scriptstyle \pi_0} & & \big\downarrow{\scriptstyle p_1} \\ \Sigma' & \xrightarrow{u} & S^2 \end{array} \tag{3.3}$$

such that $\tau(L)$ is conveniently fibered by p_1 and monotone Lagrangian for the push-forward symplectic form. Since the diffeomorphism group of a closed manifold (Σ') is transitive on points, we can assume that $\tau(F) = p_1^{-1}(z_0)$. Further $\tau(\mathcal{F}_0, \omega, L, \Sigma, \Sigma')$ is a relative symplectic fibration.

Proposition 3.2.11. *(T) There exists a p_1-fiberpreserving diffeomorphism ϑ of M such that $\vartheta(L)$ is conveniently fibered by p_1 and $\vartheta_*\omega$ is the standard form on a neighbourhood of $F := p_1^{-1}(z_0)$.*

Proof. By Proposition 3.2.1 we may assume that the symplectic form on the fiber F is ω_{std}. Since the symplectic connection induced by $(\mathcal{F}_0, \omega)$ has vanishing symplectic curvature near the fiber F and vanishing symplectic curvature is preserved under push-forward by diffeomorphisms, the symplectic connection induced by $(p_1, \tau_*\omega)$ has vanishing symplectic curvature near F. By Proposition 2.2.18 and the Remark after its proof, there exists a p_1-fiberpreserving diffeomorphism ϕ of M, which pulls the symplectic form back to a split form $\omega = p_1^*\omega_1 + p_2^*\omega_{std}$ near F. We seek a diffeomorphism of the base S^2 which preserves the equator pointwise, and which pulls ω_1 back to the standard form on U, a neighbourhood of z_0. Hence the following lemma proves the proposition.

Lemma 3.2.12. *Given symplectic forms ω, ω' on $\mathbb{C}$, then there exists a diffeomorphism ϕ of $\mathbb{C}$ with compact support which*

- *is the identity on the real line*
- *$\phi^*\omega = \omega'$ on U a neighbourhood of the origin*

Proof. The proof is trivial, but can be found in Appendix C. □

□

This finishes the standardisation near a fiber.

3.3 Standardisation of the symplectic fibration near the sections

3.3.1 Topological Standardisation of the sections

We seek a p_1-fiberpreserving diffeomorphism of M which maps the symplectic sections Σ, Σ' to the constant sections at N the north- and S the southpole. The idea is first to use the transitivity of the symplectomorphism group on points and then to use the isotopy extension theorem.
Let $S_\infty := S^2 \times \{N\}$ and $S_0 := S^2 \times \{S\}$. By the following lemma, we may assume that Σ, Σ' go through N, respectively S over the point z_0.

Lemma 3.3.1. *Let $p \in S^2$ be a point in the open upper hemi-sphere D_{uh}. Then there exists a $\phi \in Symp_0(S^2, \omega)$ such that $\phi(N) = p$ with support in D_{uh}.*

Proof. The proof is trivial, but nevertheless given in Appendix C. □

Remark
Let σ, σ' be parametrisations of Σ, Σ'.
In Proposition 3.3.8 we prove how to locally alter a symplectic section through symplectic sections to be the constant section (actually, there, we do much more). Doing this here we can assume that the sections σ, σ' are constantly N, S in a neighbourhood of F. Observe, that this gives rise to a homotopy of relative symplectic fibrations with $\mathcal{F}_{std}, \omega, L$ fixed.

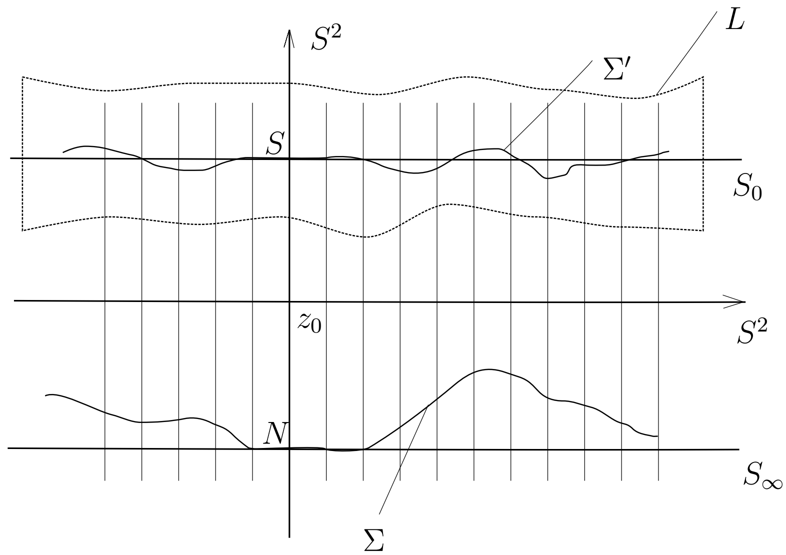

Figure 3.5: The effect of the diffeomorphism τ in Thm. 3.3.2

Theorem 3.3.2. *(T) Let $M, \omega, L, p_1, \Sigma, \Sigma'$ be as above. Then there exists a p_1-fiber-preserving diffeomorphism τ of M which is the identity on a neighbourhood of F such that $\tau(S_\infty) = \Sigma$, $\tau(S_0) = \Sigma'$.*

Proof. Write $\sigma(z) = (z, f(z))$. Then by Hurewicz's Theorem, $H_2(M) \equiv \pi_2(M)$. Since $[\Sigma] = A$ we find that f is nullhomotopic. By smooth approximations, we can find a smooth contraction f_t of f to N, and furthermore, we can assume that f_t maps a neighbourhood V' of z_0 to N for all t (cf. C.0.17). Then the family of sections $\sigma_t(z) = (z, f_t(z))$ is constantly N in V'. We want to use the isotopy extension theorem to get the desired diffeomorphism, but since we have the additional requirement that the resulting diffeomorphism is fiber preserving we have to review the proof of the isotopy extension theorem and make the necessary modifications. By a reparametrisation we assume that the track

$$F\colon S^2 \times \mathbb{R} \to M \times \mathbb{R};\ (x, t) \mapsto (\sigma_t(x), t)$$

of the isotopy σ_t is defined for all $t \in \mathbb{R}$ and is constant for $t < \frac{1}{3}$ and $t > \frac{2}{3}$. The vector field $X = dF(\frac{\partial}{\partial t}) = X_t + \frac{\partial}{\partial t}$ on $Im(F)$ generates the isotopy σ_t. By construction X_t is vertical (tangent to the fibers of p). To obtain a fiber-preserving diffeotopy of M, we need to extend the vertical vector field X on $Im(F)$ to a global vertical vector field $Z = Z_t + \frac{\partial}{\partial t}$ on $M \times \mathbb{R}$. First note that this condition is convex so that we can use a partition of unity to reduce the problem to the neighbourhood of a point in $M \times \mathbb{R}$. Let $C := F(S^2 \times [0, 1])$. Then for points outside the closed set C, on a neighbourhood not meeting C, we simply define the vector field Z to be $\frac{\partial}{\partial t}$.
Let $x = (z_0, f_s(z_0), s)$ be a point in C. Choose local coordinates (z, w, t) on $M \times \mathbb{R}$ around x via stereographic projection from $-z_0, -f_s(z_0)$ on each factor of $S^2 \times S^2$. Using this chart on the range and the chart of S^2 given by stereographic projection from $-z_0$ on the

domain, we may assume that

$$F\colon U\times\mathbb{R}\to\mathbb{C}\times\mathbb{C}\times\mathbb{R}$$
$$F(z,t)=(z,f_t(z),t)$$

with $U\subset\mathbb{C}$ open, $0\in U$ and $f_s(0)=0$.
Now we define a new chart of $M\times\mathbb{R}$ around x by the smooth map

$$\alpha\colon U\times\mathbb{C}\times\mathbb{R}\to\mathbb{C}\times\mathbb{C}\times\mathbb{R}$$
$$(z,w,t)\mapsto(z,f_t(z)+w,t).$$

Indeed, since $d\alpha_{0,0,s}$ has full rank, α is a local diffeomorphism and $\alpha(z,0,t)=F(z,t)$. Geometrically, it foliates a neighbourhood of $x\in M$ by translating the image of F in the fiber direction w. We have to shrink the domain of the local diffeomorphism α if necessary in order to avoid the rest of the image of F. Then a suitable vertical extension of the vector field X near x is given by

$$Z=d\alpha\left(\frac{\partial}{\partial t}\right).$$

By construction, the obtained diffeomorphism is fiberpreserving and the identity in a neighbourhood of the fiber F. Thus the image of the torus is still conveniently fibered by p_1. Now do the same for Σ'. Note that we have to choose the domains of the maps α in such a way that they do not interfere with $\Sigma=S_\infty$. This proves the theorem. $\square$

Hence pulling back the data by τ, we have a monotone Lagrangian torus for the pull-back symplectic form which is conveniently fibered by p_1. Moreover the solid torus T is disjoint from the symplectic section S_∞ and the symplectic section S_0 intersects $\mathring{T}$ in each fibre over the equator.

3.3.2 Standardisation of the symplectic form near the sections

We are now going to make the symplectic form split near the sections S_0,S_∞. Why do we not want to make it equal to the standard symplectic form which is also split? The reason is that for proceeding in the next chapter, we require the Lagrangian torus to be conveniently fibered by p_1. It is important to notice that we cannot assume that the curve $p_1(L)=\gamma$ encloses a disk of area $\frac{1}{2}$ in the base (note that we identify the base with the section S_0). If L is conveniently fibered by p_1 and we have the standard form ω_0 near S_0, then indeed $p_1(L)=\gamma$ encloses a disk of area $\frac{1}{2}$.
We will do this in 3 steps. In step 1 we will make the symplectic form constant on the sets $S^2\times\{p\}$ for p in a neighbourhood $S^2\times U_\infty\cup U_0$ of the sections S_∞,S_0. In step 2 we will construct (as in the symplectic neighbourhood theorem) an isotopy ϕ_t, with support in a possibly smaller neighbourhood, such that the pull-back form $\phi_1^*\omega$ agrees with some split form on $TM|_{S_0},TM|_{S_\infty}$. In the third step we consider a Moser isotopy ψ_t given by the linear interpolation between the split form and the form obtained in step 2. Then the pull-back form $\psi_1^*\phi_1^*\omega$ is split near the sections S_∞,S_0.

Step I

Let $\omega(p)$ denote the restriction of ω to $S^2 \times \{p\}$ (by openness of symplecticity there is a neighbourhood of $S_0 \cup S_\infty$ where these are symplectic forms). With the following proposition, we are going to make ω equal to $\omega(S)$, the restriction of ω to S_0, on the sets $S^2 \times \{p\}$, for p in a neighbourhood of N and S. Note that we consider S_0 to be the base, so that changing the symplectic form on S_0 results in a rearrangement of the fibers and therefore in changing the projection of L. Thus L would not be conveniently fibered anymore. Avoiding this is the reason why we make the symplectic form on the sets $S^2 \times \{p\}$ in neighbourhoods of S_∞ and S_0 equal to its restriction to S_0.

Proposition 3.3.3. *(T) There exists a $\tau \in Diff_0(M)$ with support in a neighbourhood of $S_\infty \cup S_0$ away from the Lagrangian torus L such that the pull-back form $\tau^*\omega$ restricts to $\omega(S)$ on the fibers $S^2 \times \{p\}$ of p_2 for p in a smaller neighbourhood of $S_\infty \cup S_0$. Moreover $\tau = id$ on a neighbourhood of the fiber F.*

Proof. The idea is to construct a diffeomorphism of M by using Moser isotopies ϕ_t^p obtained from the linear interpolation $\omega_t(p) = (1-t)\omega(S) + t\omega(p)$ of symplectic forms on $S^2 \times \{p\}$ (for $p \in U_\infty \cup U_0$ a neighbourhood of $N \cup S$). The issue here is to find a smooth family of primitives $\sigma(p)$ such that $\tau(p) = \omega(p) - \omega(S) = d\sigma(p)$ and such that σ vanishes in a neighbourhood of z_0. $\tau(p)$ is a closed form which vanishes on V, a neighbourhood of z_0, for all $p \in U_\infty \cup U_0$ and which is trivial in cohomology. So for all p, we can view $\tau(p)$ as an element of $\Omega_c^2(\mathbb{R}^2)$, the compactly supported 2-forms on $\mathbb{R}^2$ with support in $B(0,R)$ for some $R > 0$. Now we have

Lemma 3.3.4. *Let $\tau \in \Omega_c^2(\mathbb{R}^2)$ be closed with support in $D(0,1)$ and such that*

$$\int_{\mathbb{R}^2} \tau = 0.$$

Then there exists a canonical choice of $\sigma \in \Omega_c^1(\mathbb{R}^2)$ such that $d\sigma = \tau$

Proof. We will do this by altering the (non-compactly supported !) primitive obtained from the Poincare Lemma to one with compact support. See Appendix C for a proof. □

Applying a suitable diffeomorphism (scaling) on $\mathbb{R}^2$ we may assume that $\tau(p)$ has support in $D(0,1)$ for all p. So by the lemma we obtain a smooth family (in p) of 1-forms $\sigma(p)$ which we extend by zero to 1-forms $\beta(p) \in \Omega^1(S^2)$. These vanish at points where $\omega(S)$ and $\omega(p)$ agree and which satisfy $\omega(p) - \omega(S) = d\beta(p)$. Let ϕ_t^p denote the Moser isotopy corresponding to the 1-form $\beta(p)$ for $p \in U_\infty \cup U_0$. These families depend smoothly on p by construction. Now choose a suitable cut-off function $\rho\colon \mathbb{R} \to \mathbb{R}$ with support in $[-2\epsilon, 2\epsilon]$ and being 1 on $[-\epsilon, \epsilon]$. Consider the diffeomorphism

$$\tau(z,w) = (\phi^w_{\rho(|w|)}(z), w).$$

Choosing ϵ small enough this diffeomorphism τ doesn't alter the torus L. Near F we have $\phi_t^p = id$ by construction (the Moser vector fields vanish), so the standard fibration is unaltered in the region where we standardised the symplectic form near the fiber F. Furthermore by construction $\tau = id$ on S_0. The pull-back symplectic form (via τ) is equal to $\omega(S)$ on the sets $S^2 \times \{p\}$ in a (smaller) neighbourhood of $S_0 \cup S_\infty$. By construction, τ is isotopic to the identity. This finishes step 1. □

Step II

As in the standardisation of the symplectic form near a fiber, we first show the existence of a diffeomorphism ϕ of M which has support near $S_\infty \cup S_0$ and which pulls the symplectic form back to a symplectic form which agrees with $\omega_1 = p_1^*\omega(S) + p_2^*\omega_{std}$ on $TM|_{S_\infty}$ and on $TM|_{S_0}$. Further ϕ needs to be the identity near the fiber F. This is of course nothing else than the first part of the proof of the symplectic neighbourhood theorem, but because of the last condition we will write down a proof but with the necessary modifications. Since we can easily split the problem in two, it is enough to find such a diffeomorphism for S_∞.

Lemma 3.3.5 (T). *There exists a smooth isotopy $\phi_t \colon \tilde{V} \to M$ such that*

- *$\tilde{V}$ is a neighbourhood of S_∞ in M;*
- *$\phi_t|_{S_\infty} = id$ for all t;*
- *$\phi_0|_{\tilde{V}} = id$;*
- *$\phi_t = id$ for all t in a neighbourhood of F;*
- *$\phi_1^*\omega$ agrees with ω_1 on $TM|_{S_\infty}$.*

Proof. The idea is to write down a smooth family of bundle maps A^t of the vector bundle $\pi \colon TM|_{S_\infty} \to S_\infty;\ v \in T_xM \mapsto x$ starting at the identity, such that A^1 maps a symplectic basis for ω_1 to one for ω. Now use the exponential map of some metric on TM to get from the family A^t an isotopy ϕ_t of a neighbourhood of S_∞ which is the identity on S_∞ and $\phi_1^*\omega = \omega_1$ on $TM|_{S_\infty}$. This is just the first part of the symplectic neighbourhood theorem.
We only have to show, how to construct A^t such that ϕ_t is the identity in a neighbourhood of F.
We think of the second factor in $M = S^2 \times S^2$ as being $\mathbb{C}$ and $S_\infty = S^2 \times \{0\}$. Since symplecticity is an open condition, there exists a neighbourhood $S^2 \times D(0,\epsilon)$ such that all the fibres of $p_2 \colon S^2 \times D(0,\epsilon) \to D(0,\epsilon);\ (z,w) \mapsto w$ are symplectic for the symplectic form ω. Consequently there exists a symplectic connection on p_2 given via a distribution of horizontal subspaces H defined by the symplectic orthogonal complements to $\ker dp_2$. Consider the standard coordinates $u + iv$ on $\mathbb{C}$ and lets denote by $\widetilde{\frac{\partial}{\partial u}}(z,w), \widetilde{\frac{\partial}{\partial v}}(z,w)$ the horizontal lifts of $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ with respect to H at (z,w). Moreover let $K(z,w) = \omega_{(z,w)}(\widetilde{\frac{\partial}{\partial u}}, \widetilde{\frac{\partial}{\partial v}})$

and $c = \omega_{std}(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) > 0$.
For all $x \in S_\infty$, any tangent space T_xM has two splittings:

$$T_xM = T_xS_\infty \oplus (T_xS_\infty)_1^\perp$$

$$T_xM = T_xS_\infty \oplus (T_xS_\infty)^\perp.$$

where $()_1^\perp$ is the symplectic orthogonal complement with respect to ω_1 and $()^\perp$ is the symplectic orthogonal complement with respect to ω. Note that $(T_xS_\infty)_1^\perp$ is spanned by $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ whereas H_x is spanned by $\widetilde{\frac{\partial}{\partial u}}(x), \widetilde{\frac{\partial}{\partial v}}(x)$. Moreover let e_1, f_1 be a symplectic basis for T_xS_∞ for both ω_1 and ω (both forms agree on fibers of p_2 because of step I). Then $e_1, f_1, \frac{1}{\sqrt{c}}\frac{\partial}{\partial u}, \frac{1}{\sqrt{c}}\frac{\partial}{\partial v}$ is a symplectic basis of T_xM for ω_1 whereas $e_1, f_1, \frac{1}{\sqrt{K}}\widetilde{\frac{\partial}{\partial u}}(x), \frac{1}{\sqrt{K}}\widetilde{\frac{\partial}{\partial v}}(x)$ is a symplectic basis of T_xM for ω.
Then

$$A^t \colon TM|_{S_\infty} \to TM|_{S_\infty}$$

with $t \in [0, 1]$ is given by

- $A^t_x|_{T_xS_\infty} = id$ for all $x \in S_\infty$
- $A^t_x(\frac{1}{\sqrt{c}}\frac{\partial}{\partial u}) = (1-t)\frac{1}{\sqrt{c}}\frac{\partial}{\partial u} + \frac{t}{\sqrt{K(x)}}\widetilde{\frac{\partial}{\partial u}}(x)$ for all $x \in S_\infty$
- $A^t_x(\frac{1}{\sqrt{c}}\frac{\partial}{\partial v}) = (1-t)\frac{1}{\sqrt{c}}\frac{\partial}{\partial v} + \frac{t}{\sqrt{K(x)}}\widetilde{\frac{\partial}{\partial v}}(x)$ for all $x \in S_\infty$

In a neighbourhood of F, we have $\omega = \omega_1$ thus $\widetilde{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u}$, $\widetilde{\frac{\partial}{\partial v}} = \frac{\partial}{\partial v}$ and $K(x) = c$. Thus $A^t = id$ there. This proves the lemma. □

Choosing $\tilde{V}$ small enough we can assure that the image of the isotopy stays away from the Lagrangian torus L. By the isotopy extension theorem there exists a diffeomorphism H^N of M which extends ϕ_1 and which is supported away from the torus and S_0. Doing the same for S_0, we obtain a diffeomorphism H^S with similar properties. After pulling back ω by $\tau = H^N \circ H^S$ we may hence assume that ω agrees with ω_1 on $TM|_{S_\infty}, TM|_{S_0}$. Obviously, by construction, τ is isotopic to the identity. This finishes Step 2.

Step III

Proposition 3.3.6 (T). *There exists $\tau \in Diff_0(M)$ which preserves $S_\infty \cup S_0$ pointwise, is supported in a neighbourhood of $S_\infty \cup S_0$ away from the Lagrangian torus L and which pulls the symplectic form ω back to a form which agrees with the symplectic form $\omega_1 = p_1^*\omega(S) + p_2^*\omega_{std}$ in a smaller neighbourhood of $S_\infty \cup S_0$. Moreover, $\tau = id$ on a neighbourhood of F.*

Proof. Without loss of generality, it suffices to do this for S_∞. Consider the linear interpolation between ω and ω_1. Since they both agree on $TM|_{S_\infty}$ and symplecticity is an open condition, there exists a neighbourhood $S^2 \times U_\infty$ of S_∞ on which the family of closed 2-forms

$$\omega_t = (1-t)\omega_1 + t\omega$$

is symplectic. As in Lemma 3.2.4, there exists a 1-form β which vanishes on S_∞ and on a neighbourhood of F such that $\omega - \omega_1 = d\beta$. Then the vector fields X_t generating the Moser isotopy ϕ_t vanish on S_∞ so that $\phi_t = id$ there. Thus for all t, any neighbourhood of S_∞ is mapped by ϕ_t to a neighbourhood of S_∞. Choosing the neighbourhood small enough, call it $S^2 \times U'$ say, we can assure that it is mapped into $S^2 \times U_\infty$ by all ϕ_t. This defines an isotopy of $S^2 \times U'$ into M starting at the inclusion (since $\phi_0 = id$). Hence by the isotopy extension theorem there exists a diffeomorphism μ of M which extends $(\phi_1)|_{S^2 \times U'}$. We can choose this diffeomorphism to have support in $S^2 \times U_\infty$ (by shrinking U' even further if necessary) and to be disjoint from the Lagrangian torus L and S_0. Hence the pull-back $\mu^*\omega = \omega_1$ on $S^2 \times U'$. Moreover by the above $\beta \equiv 0$ on a neighbourhood of F thus there $X_t = 0$ and hence $\phi_t = id$ as claimed. Let $\tau := \mu$ with support in $S^2 \times U_\infty$. By construction τ is isotopic to the identity. This proves Step 3. □

Hence the symplectic form is now split in $S^2 \times U_\infty \cup U_0$ a neighbourhood of $S_\infty \cup S_0$ and in $V \times S^2$ a neighbourhood of F as claimed. Note that the fibration is not standard yet in a neighbourhood of $S_\infty \cup S_0$, only the symplectic form is!

3.3.3 Standardisation of the symplectic foliation near the sections

The diffeomorphisms used so far preserved S_∞, were the identity on S_0, but did alter the standard foliation near S_0, S_∞. In this step we want to deform the foliation back to the standard foliation in a smaller neighbourhood of $S_\infty \cup S_0$. This deformation should be through symplectic foliations. Again it suffices to study one of S_0, S_∞.
Up to now, all diffeomorphisms used in steps $1-3$ were the identity on a neighbourhood $V \times S^2$ of F, hence the standard foliation wasn't altered in $V \times S^2$ and we can use Proposition 3.3.8 to alter the symplectic foliation to the standard foliation in a smaller neighbourhood. Let σ^p denote stereographic projection from p on S^2 and consider the following diffeomorphism

$$\tau\colon S^2 \setminus \{z_0\} \times S^2 \setminus \{S\} \to \mathbb{R}^4$$
$$(z, w) \mapsto (\sigma^{z_0}(z), \sigma^S(w)).$$

By construction $\tau(S^2 \setminus \{z_0\} \times U_\infty)$ is a neighbourhood of $\mathbb{R}^2 \times \{0\}$ on which the push-forward symplectic form $\tau_*\omega$ is split but not yet $\Omega_0 = dx \wedge dy + du \wedge dv$ ($\mathbb{R}^4$ with standard coordinates x, y, u, v and $z = x + iy$, $w = u + iv$).
By applying Lemma 3.2.12 on each factor we can assume that we indeed have the standard form $dx \wedge dy + du \wedge dv$ on $\tau(S^2 \setminus \{z_0\} \times U_\infty)$. The smooth foliation we obtain on $\tau(S^2 \setminus \{z_0\} \times U_\infty)$ by pushing the foliation on M forward via τ is standard, i.e. the leaves are of the form $\{z\} \times \mathbb{R}^2$ for $|z| > \frac{R}{2}$ for some $R > 0$.
Preliminary discussion to Proposition 3.3.8
Regard any smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, then the pullback $F^*\Omega_0$ of the symplectic form Ω_0 under the embedding

$$F\colon \mathbb{R}^2 \to \mathbb{R}^4$$

$$z \mapsto (z, f(z))$$

is given by

$$(1 + detDf)dx \wedge dy.$$

Hence the graph $\Gamma(f) \subset \mathbb{R}^4$ of f is symplectic if and only if $1 + detDf > 0$ and if it inherits its orientation from the projection to the z-plane.
Now assume that $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and that $\phi\colon \mathbb{R} \to \mathbb{R}$ is a smooth function. Then a short calculation shows that for $f(z) = \phi(|z|)Az$

$$detDf = detA(\phi^2(r) + r\phi(r)\phi'(r))$$

with $r = |z|$. This proves

Lemma 3.3.7. *Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear with $detA \geq -1 + \epsilon$. Let $\phi\colon [0, \infty) \to \mathbb{R}$ be a smooth function. Then the graph of $f(z) = \phi(|z|)A(z)$ is symplectic provided that for all $r > 0$,*

$$0 \leq \phi(r)^2 + r\phi(r)\phi'(r) < \frac{1}{1-\epsilon}.$$

Proposition 3.3.8 (S)**.** *Let $\Lambda := \overline{D}(0, R) \subset \mathbb{R}^2$ be a closed disk and $\left(S^\lambda\right)_{\lambda \in \Lambda}$ be a smooth foliation of a region in $(\mathbb{R}^4, \Omega_0)$ by symplectic hypersurfaces S^λ intersecting $\mathbb{R}^2 \times \{0\}$ transversely in $(\lambda, 0)$. Then for every neighbourhood $W \subset \mathbb{R}^4$ of Λ there exists a neighbourhood $U \subset W$ of Λ and a family of foliations $\left(S_s^\lambda\right)_{s \in [0,1], \lambda \in \Lambda}$ with the following properties.*

- $S_0^\lambda = S^\lambda$;
- S_s^λ *is symplectic and intersects* $\mathbb{R}^2 \times \{0\}$ *transversely in* $(0, \lambda)$;
- $S_s^\lambda = S^\lambda$ *outside* W;
- $S_1^\lambda = \{\lambda\} \times \mathbb{R}^2$ *in* U. *Moreover, for every* λ *with* $S^\lambda = \{\lambda\} \times \mathbb{R}^2$ *in* W *we have* $S_s^\lambda = S^\lambda$ *for all* s.

Proof. The proof is in spirit similar to the proof of Proposition 3.2.9. After shrinking W, we may assume that in W each surface can be written as a graph $S^\lambda = \left\{z = \lambda + f^\lambda(w)\right\}$ over the w-plane with $f^\lambda(0) = 0$ (cf. the discussion before Proposition 3.2.9). After a C^1-small perturbation we may assume that the f^λ are linear functions A^λ. We do this as in the linearization in Proposition 3.2.9 by cutting off the Taylor expansion of f^λ in w after the linear term.
Symplecticity implies $detA^\lambda > -1$. Since we may choose W compact, there exists a $\delta > 0$ with $detA^\lambda \geq -1 + \delta$ in W for all λ. Moreover, we may assume that the δ-neighbourhood of Λ is contained in W. Pick $\alpha > 0$ so small that $\frac{5\alpha}{\delta} < \delta$. Let ϕ_s be the family of functions

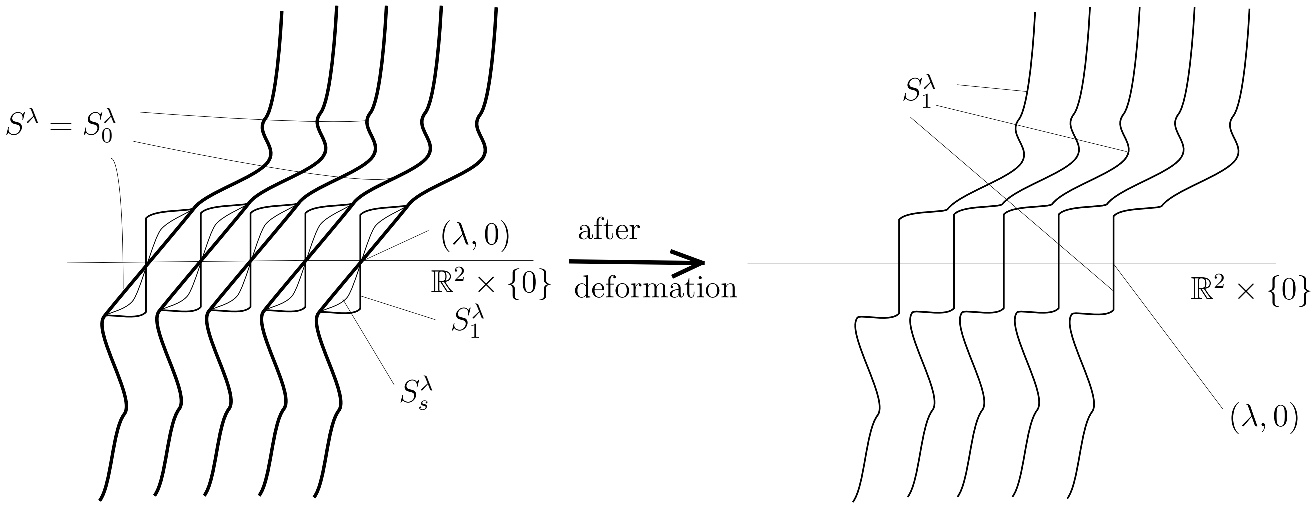

Figure 3.6: The family of foliations S_s^λ

from Lemma 3.2.7 for α, δ. Then since $0 \leq \phi_s(r) + r\phi_s'(r) < \frac{1}{1-\delta}$ it also follows that $0 \leq \phi_s^2(r) + r\phi_s(r)\phi_s'(r) < \frac{1}{1-\delta}$. Now define

$$f_s^\lambda(w) = \phi_{1-s}(|w|)f^\lambda(w).$$

Then by Lemma 3.3.7, the graph S_s^λ of $\lambda + f_s^\lambda$ satisfies all the conditions of the proposition, where U is the $\frac{\alpha}{2}$-neighbourhood of Λ. Note that if $S^\lambda = \{\lambda\} \times \mathbb{R}^2$ for some λ, then $f^\lambda(z) = 0$ and thus $S_s^\lambda = S^\lambda$ for all s.
It only remains to verify that the surfaces $(S_s^\lambda)_{\lambda\in\Lambda}$ form a foliation for each s or equivalently, that the map

$$F_s \colon \Lambda \times D(0, \delta) \to \mathbb{R}^4$$

$$(\lambda, w) \mapsto (\lambda + f_s^\lambda(w), w) = (\lambda + \phi_{1-s}(|w|)A^\lambda w, w)$$

is an embedding.
To show injectivity suppose that $F_s(\lambda, w) = F_s(\lambda', w')$. Then $w = w'$ and

$$\lambda - \lambda' = -\phi_{1-s}(|w|)(A^\lambda - A^{\lambda'})w.$$

This implies

$$|\lambda - \lambda'| \leq \|A^\lambda - A^{\lambda'}\||w| \leq \|A^\lambda - A^{\lambda'}\|\delta.$$

Since f^λ depends smoothly on λ, there exists a constant C such that $\|A^\lambda - A^{\lambda'}\| \leq C|\lambda - \lambda'|$. For $\delta < \frac{1}{C}$ this implies $\lambda = \lambda'$. For the immersion property, consider the differential

$$DF_s(\lambda, w) = \begin{pmatrix} \mathbf{1} + B_s & \frac{\partial f_s^\lambda}{\partial w} \\ 0 & \mathbf{1} \end{pmatrix}, \quad B_s = \frac{\partial f_s^\lambda}{\partial \lambda}.$$

This is an embedding iff the matrix

$$\mathbf{1} + B_s = \mathbf{1} + \phi_{1-s}(|w|)\frac{\partial A^\lambda}{\partial \lambda}w$$

is invertible. By smoothness in λ, there exists a constant C with $\|\frac{\partial A^\lambda}{\partial \lambda}w\| \leq C|w|$. Then for $\delta < \frac{1}{C}$ we get

$$\|\phi_{1-s}(|w|)\frac{\partial A^\lambda}{\partial \lambda}w\| \leq C|w| \leq C\delta < 1,$$

which implies invertibility of $\mathbf{1} + B_s$. $\square$

As in the standardisation of the symplectic fibration near a fiber, Proposition 3.3.8 provides a smooth family of foliations $\mathcal{F}_s$ on $S^2 \times S^2$ such that $(\mathcal{F}_1, \omega)$ has vanishing symplectic curvature near the sections S_∞, S_0 and near F. We need to show:

Proposition 3.3.9. $(\mathcal{F}_s, \omega, L, S_\infty, S_0)$ *is a homotopy of relative symplectic fibrations.*

Proof. We need to show, that L is fibered by $\pi_s\colon x \in (\mathcal{F}_s)_x \mapsto (\mathcal{F}_s)_x \cap S_0$ for all s. Consider the embeddings F_s in the proof of Proposition 3.3.8 and note, that for all s, they have the same open set $\widetilde{W}$ as image in $\mathbb{R}^4$ and agree in a neighbourhood of $\partial\widetilde{W}$. Thus $F_s \circ F_0^{-1}\colon \tilde{W} \to \tilde{W} \subset M$ (we identify $\mathbb{R}^4$ with a subset of M) is an isotopy which is the identity near $\partial\widetilde{W}$ for all s. Extend this isotopy by the identity to all of M and denote it by ψ_s. Hence the smooth family of symplectic foliations on M is given by $\mathcal{F}_s = \psi_s(\mathcal{F}_0)$. Further note, since $F_s(\lambda, 0) = (\lambda + \phi_{1-s}(|0|)A^\lambda 0, 0) = (\lambda, 0)$, that ψ_s preserves S_∞, S_0 pointwise. Now consider

$$\pi_s\colon x \in (\mathcal{F}_s)_x \mapsto (\mathcal{F}_s)_x \cap S_0.$$

Since $(\mathcal{F}_s)_x = \psi_s((\mathcal{F}_0)_{\psi_s^{-1}(x)})$ and ψ_s preserves S_0 pointwise, it follows, that $\pi_s(x) = \pi_0(\psi_s^{-1}(x))$. Also $\psi_s = id$ outside a neighbourhood of $S_\infty \cup S_0$, so in particular $\psi_s(L) = L$ pointwise for all s. Thus $\pi_s(L) = \pi_0(\psi_s^{-1}L) = \pi_0(L) = \gamma$ is an embedded curve and $\pi_s^{-1}(\gamma(t)) \cap L$ is an embedded circle. Further let T be the solid torus for the relative symplectic fibration $(\mathcal{F}_0, \omega, L, S_\infty, S_0)$, then the solid torus $T_s = \psi_s(T)$ is made up of symplectic embedded disks in each leaf of $\mathcal{F}_s$ which bound the embedded circles from before, so that T_s bounds L. This shows, that L is fibered by π_s for all s. To check the other properties that $(\mathcal{F}_s, \omega, L, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations is trivial. $\square$

Hence we can assume that there exists a neighbourhood $S^2 \times (U_\infty \cup U_0)$ of $S_\infty \cup S_0$ on which both the foliation $\mathcal{F} := \mathcal{F}_1$ as well as the symplectic form are standard and which is away from the Lagrangian torus L. Observe that, by construction, L still lies above the equator in $S_0 \cong S^2$.

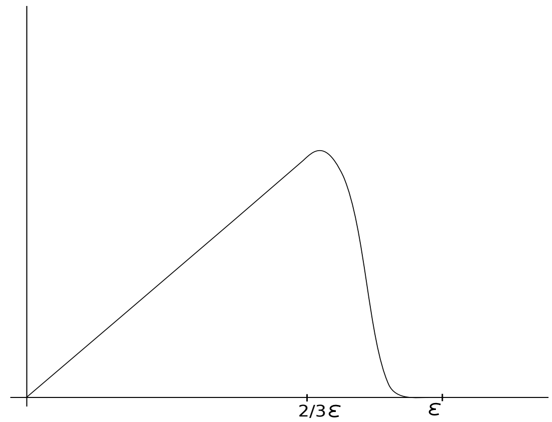

Figure 3.7: The cut-off function ρ

3.3.4 Trivialising the fibration

By a suitable diffeomorphism we want to obtain the standard foliation again.
Consider the map $\pi = \pi_1 \colon M \to S^2$ given by $(z, w) \in \mathcal{F}_x \mapsto x$ where $\mathcal{F}_x$ denotes the leaf of the altered foliation from Proposition 3.3.8 through the point $(x, S) \in S^2 = S_0$.
As before it is the trivial S^2-bundle over S^2, hence there exits a trivialisation ϕ, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ S^2 & \xrightarrow{id} & S^2 \end{array} \tag{3.4}$$

Since ϕ covers the identity, the fibers over the equator in $S^2 \cong S_0$ are mapped to the fibers over the equator. Thus $\phi(L)$ is a monotone Lagrangian torus for $\phi_* \omega$ which is fibered by p_1 and which lies above the equator in the base. It is however not conveniently fibered since ϕ might have messed up the region near the fiber F. Also the sections S_0, S_∞ are not a priori preserved by ϕ. The following lemma however shows that we can assume without loss of generality that $\phi(L)$ is conveniently fibered and ϕ preserves the sections S_0, S_∞.

Lemma 3.3.10 (T)**.** *There exists a diffeomorphism τ of M, such that the diagram above commutes, and such that $\tau = id$ on a neighbourhood $S^2 \times (U_\infty \cup U_0)$ of $S_\infty \cup S_0$ and on $V \times S^2$ a neighbourhood of F.*

Proof. Clearly it suffices to show the lemma for one of S_∞, S_0. First we show that we can

assume that ϕ is the identity in $V \times S^2$, the neighbourhood of F where the foliation has not been altered. We do this by applying a suitable fiber preserving diffeomorphism ψ of M which undoes ϕ on $V \times S^2$.
Therefore note that $\phi_z = \phi|_{p_1^{-1}(z)}$ is a diffeomorphism of $S^2 \cong p_1^{-1}(z)$ for all $z \in V$. Consider the cut-off function ρ in the fig. 3.7 and define $u \colon D(z_0, \epsilon) \to Diff(S^2)$ by $u(z) = \phi^{-1}_{\rho(|z|)\frac{z}{|z|}}$ where z are local coordinates around z_0. Extending u by ϕ_0^{-1} to all of S^2, we found the desired diffeomorphism of $S^2 \times S^2$ by setting $\psi(z, w) = u(z)(w)$. Then indeed $\psi \circ \phi$ is the identity in a neighbourhood of z_0.
Then using Theorem 3.3.2, we may assume that ϕ preserves the sections S_∞ and S_0. Next $\pi = p_1$ on a small neighbourhood $S^2 \times U_\infty$ of S_∞ (this is what we have obtained in Proposition 3.3.8). Fix $z \in S^2$, then ϕ maps a small disk $D(N, \epsilon) \subset p_1^{-1}(z)$ into a neighbourhood of N in $p_1^{-1}(z) \cong S^2$. Denote this embedding of the disk $D := D(N, \epsilon)$ into S^2 by τ_z. These embeddings are the identity for points near z_0, so we can regard $\tau_z = \tau_{t+is}$ as a two parameter family of embeddings of the disk into S^2 which is the identity for $t^2 + s^2$ large. By the isotopy extension theorem with parameters we find a two parameter family ψ_z of diffeomorphisms of S^2 which agree with τ_z on D and which have support in an arbitrarily small neighbourhood $\bar{V}$ of N with $D \subset \bar{V}$. From this we can construct a diffeomorphism of $S^2 \times S^2$ by setting $\hat{\psi}(z, w) = \psi_z^{-1}(w)$ such that $\tau = \hat{\psi} \circ \phi$ has the desired properties. □

$\tau(L)$ is monotone Lagrangian for $\tau_*\omega$ and is conveniently fibered by p_1. This proves the standardisation near the two sections S_∞, S_0.

3.4 Topological standardisation of the torus

In this section we construct a p_1-fiberpreserving diffeomorphism τ of M which maps the Lagrangian torus L to the Clifford Torus L_{std}.
The definitions of A, $Diff(A, \partial A)$ and $Ham(A, \partial A, \omega_{std})$ used in the sequel can be found in Appendix E and chapter 4.
We construct τ, by first finding a loop of diffeomorphisms of the annulus A, which realises the torus L over the equator (see below what we mean by this). Then we use the fact that the fundamental group of $Diff(A, \partial A)$ is trivial so that we can find a contraction of this loop of diffeomorphisms. We then use this contraction to explicitly construct a fiber-preserving diffeomorphism of M which maps the Clifford torus L_{std} to L and which preserves S_∞ and S_0.

Proposition 3.4.1 (T). *Let $(S^2 \times S^2, \omega)$ be symplectic such that p_1 is a symplectic fibration and let L be a monotone Lagrangian torus conveniently fibered by p_1. Further ω is of the form $\omega = \omega_0$ on $\bar{V} \times S^2$ a neighbourhood of F and $\omega = \omega_1$ on $S^2 \times \bar{U}_\infty \cup \bar{U}_0$ a neighbourhood of $S_\infty \cup S_0$. Then there exists a p_1-fiber-preserving diffeomorphism τ of $S^2 \times S^2$ which maps L_{std} to L and which is the identity on a neighbourhood $S^2 \times (U_\infty \cup U_0)$ of $S_\infty \cup S_0$ and on a neighbourhood $V \times S^2$ of F.*

Proof. The Lagrangian torus L is given by symplectic parallel transport of the equator $E \subset F$ around the equator in the base by Proposition 2.4.2.
Since the symplectic connection is flat near S_∞, S_0, the diffeomorphisms of the fibers given by symplectic parallel transport

$$P_t \colon F \to F_t$$
$$(z_0, w) \to ((\cos(2\pi t), \sin(2\pi t), 0), \phi_t(w))$$

fixes neighbourhoods of the north- and the southpole pointwise (Note that the fibers are smoothly identified but not symplectically so!). Hence we can view ϕ_t as living in $Diff(A, \partial A)$ the group of diffeomorphisms of the annulus which are fixed near the boundary (again we smoothly identify the annuli in all fibers). Define $E_t := \phi_t(E) = p_1^{-1}((\cos(t), \sin(t), 0)) \cap L$. Furthermore symplectic parallel transport is the identity in $V \times S^2$ near F. Hence ϕ_t defines a smooth path in $Diff(A, \partial A)$ which is stationary near its ends. It is a well-know fact that $\pi_i(Diff(A, \partial A)) = id$ for all $i \geq 1$ (see Appendix E). From Theorem 4.2.10 in chapter 4 we know that the monodromy map ϕ_1 is hamiltonian. Since L is a torus generated by symplectic parallel transport, it follows that ϕ_1 is hamiltonian with $\phi_1(E) = E$. We have the following lemma:

Lemma 3.4.2. *Let $\psi \in Ham(A, \partial A, \omega_{std})$ (for a definition see 4.2.5) such that $\psi(E) = E$, i.e. that ψ fixes the equator, then there exists a smooth path $\psi_t \in Ham(A, \partial A, \omega_{std})$ from the identity to ψ such that $\psi_t(E) = E$ for all t.*

Proof. This is proved in Appendix D, Lemma D.0.24. □

Apply Lemma 3.4.2 to $\psi = \phi_1^{-1}$. Then let ψ_t be the path given by the Lemma and reparametrise in t such that ψ_t is constant in t near its ends. Now consider the path

$$\theta_t = \phi_t \circ \psi_t$$

and note that, by construction, it is still true that $\theta_t(E) = E_t$. But by construction θ_t is a loop since $\phi_1 \circ \psi_1 = id$ and it is also smooth by construction ($\theta_t = id$ near its ends).
But $\pi_1(Diff_0(A, \partial A)) = id$ so that the loop θ_t is contractible. By smooth approximations we may assume that θ_t^s is a smooth contraction of θ_t, i.e. $\theta_t^1 = \theta_t$; $\theta_t^0 = id$; $\theta_0^s = \theta_1^s = id$. Again by a reparametrisation in s, we may assume that θ_t^s is constant in s near its ends. We can extend the diffeomorphism θ_t^s by the identity to all of S^2.

Choosing cylindrical coordinates $z, \mu \in (-1, 1) \times \mathbb{R}/2\pi\mathbb{Z}$ on $S^2 \setminus \{N, S\}$ we can define a global diffeomorphism of $S^2 \times S^2$ by

$$\tau \colon S^2 \times S^2 \to S^2 \times S^2; \quad ((z, \mu), w) \mapsto ((z, \mu), \theta_{\frac{\mu}{2\pi}}^{1-z^2}(w))$$

on $S^2 \setminus \{N, S\} \times S^2$ and by the identity on $\{N, S\} \times S^2$. This is a fiber-preserving diffeomorphism of $S^2 \times S^2$ by construction and it maps the standard torus L_{std} onto L. Trivially, τ fixes a neighbourhood $S^2 \times (U_\infty \cup U_0)$ of $S_\infty \cup S_0$ by construction. Moreover, since θ_t^s is constant in s near its ends and $\theta_t^1 = id \in Diff^+(S^2)$ for t near $0, 1$, it follows that $\tau = id$ in a neighbourhood $V \times S^2$ of F. This proves the proposition. □

Pulling back all data by τ gives the required standardisation theorem of the torus. Hence ω is a symplectic form on $S^2 \times S^2$ which makes the fibers of p_1 symplectic, such that $\omega = \omega_0$ on $V \times S^2$, $\omega = \omega_1$ on $S^2 \times (U_\infty \cup U_0)$ and such that L_{std} is monotone Lagrangian for ω. This finishes the standardisation.

3.5 Summary

Recall that $\mathcal{F}_{std}$ is the foliation on $S^2 \times S^2$ given by the fibers of p_1, $S_\infty = S^2 \times \{N\}$, $S_0 = S^2 \times \{S\}$ and $F = p_1^{-1}((1,0,0))$.

For convenience, we summarize the results in this chapter by Theorem 3.5.1. Observe, that all the diffeomorphisms in this chapter to simplify the setup induce the identity on $H_2(M)$. This follows either since they are isotopic to the identity or by Proposition 2.4.8. Hence, all these diffeomorphisms map relative symplectic fibrations to relative symplectic fibrations. On the other hand, those steps in this chapter whose output is not a diffeomorphism giving diffeomorphic relative symplectic fibrations, where shown to be homotopies of relative symplectic fibrations. This proves

Theorem 3.5.1 (Standardisation). *Assume that*

$$(\mathcal{F}_{std}, \omega, L, \Sigma, \Sigma')$$

is a relative symplectic fibration. Then $(\mathcal{F}_{std}, \omega, L, \Sigma, \Sigma')$ *is equivalent to*

$$(\mathcal{F}_{std}, \overline{\omega}, L_{std}, S_\infty, S_0)$$

in the sense of Definition 2.4.9 such that $\overline{\omega} = p_1^*\widetilde{\omega} + p_2^*\omega_{std}$ *on the set* $W = (V \times S^2) \cup (S^2 \times (U_\infty \cup U_0))$. *Where* $V \times S^2$ *is a neighbourhood of* F *and* $S^2 \times (U_\infty \cup U_0)$ *is a neighbourhood of* $S_\infty \cup S_0$. *So in particular,* S_∞, S_0 *are horizontal. Furthermore,* $\widetilde{\omega} = \omega_{std}$ *near* z_0.

Chapter 4

Killing the monodromy

Recall that $\mathcal{F}_{std}$ is the foliation on $S^2 \times S^2$ given by the fibers of p_1, $S_\infty = S^2 \times \{N\}$, $S_0 = S^2 \times \{S\}$ and $F = p_1^{-1}((1,0,0))$. In this chapter, we assume, that $(\mathcal{F}_{std}, \omega, L_{std}, S_\infty, S_0)$ is a relative symplectic fibration, such that $\omega = p_1^*\widetilde{\omega} + p_2^*\omega_{std}$ on $W = (V \times S^2) \cup (S^2 \times (U_\infty \cup U_0))$. In particular S_∞, S_0 are horizontal for the symplectic connection. Moreover note, that $\pi\colon x \in (\mathcal{F}_{std})_x \mapsto (\mathcal{F}_{std})_x \cap S_0$ is just given by $\pi(z,w) = (z,S)$ for all $(z,w) \in S^2 \times S^2$, and so π is basically p_1. In the following, we identify S_0 with S^2.

4.1 Suitable coordinates on the base

As before we consider $S^2 \subset \mathbb{R}^3$ and $z_0 = (1,0,0)$. Let $\phi_t \in Diff(S^2)$ be the gradient flow of the height function $h\colon S^2 \to \mathbb{R};\ (x,y,z) \mapsto x$. Then $\phi_t(E) = E$ for all t. Let

$$B_b := \left\{(x,y,z) \in S^2 | x \geq b\right\} = h^{-1}([b,\infty)).$$

By shrinking, we can assume that $V = D(z_0, \epsilon)$ is the neighbourhood of z_0 over which the symplectic form is standard after the standardisation in chapter 3. Now take $t = T$ so big, that $\phi := \phi_T(B_{-\frac{1}{\sqrt{2}}}) \subset D(z_0,\epsilon)$. Pull back all the data by the diffeomorphism $\phi \times id$.

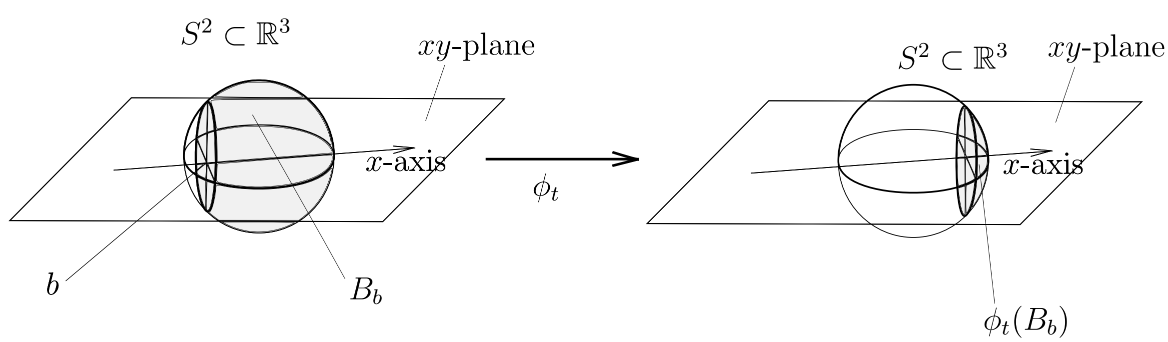

Figure 4.1: The gradient flow of h and its effect on B_b

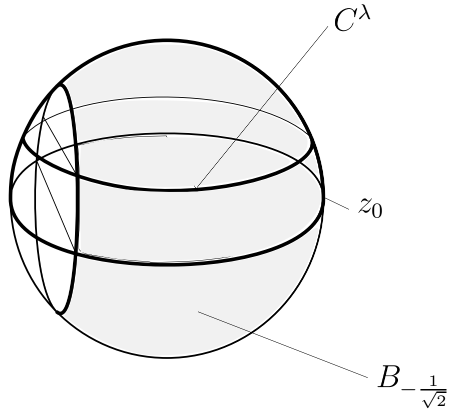

Figure 4.2: Circles of latitude and the set $B_{-\frac{1}{\sqrt{2}}}$

Clearly the symplectic form is not the original form over $B_{-\frac{1}{\sqrt{2}}}$ anymore, it is however still split of the form $\omega_1 := p_1^*\widehat{\omega} + p_2^*\omega_{std}$. Consider the usual spherical polar coodinates $\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2}), \mu \in [0, 2\pi]$ on the base $S^2 \setminus \{N, S\}$ centered at z_0.

Denote by C^λ the circle of latitude λ in the base and by ϕ^λ the symplectic parallel transport around C^λ parametrised in the obvious way. Since the starting and end point of the parametrisation of C^λ is contained in $B_{-\frac{1}{\sqrt{2}}}$ for all λ and the symplectic form ω equals $p_1^*\widehat{\omega} + p_2^*\omega_{std}$ over $B_{-\frac{1}{\sqrt{2}}}$, we can regard ϕ^λ as living in $Symp(S^2, \omega_{std})$ for all λ. In the following proposition, we want to fix two properties of ϕ^λ which are due to the standardisation in chapter 3:

Proposition 4.1.1. *1. $\phi^\lambda = id$ for $|\lambda| \geq \frac{\pi}{4}$;*

2. ϕ^λ restricts to the identity on $D(N, 2\epsilon), D(S, 2\epsilon)$ for some $\epsilon > 0$;

Proof. $C^\lambda \subset B_{-\frac{1}{\sqrt{2}}}$ for all $|\lambda| \geq \frac{\pi}{4}$. Since the symplectic form is split in $B_{-\frac{1}{\sqrt{2}}}$, it follows that symplectic parallel transport ϕ^λ around those C^λ is the identity.
Further, by the Standardisation 3.5.1, the symplectic form is split in a neighbourhood of the symplectic sections S_∞, S_0 so that there exists $\epsilon > 0$ and neighbourhoods $D(N, 2\epsilon)$, $D(S, 2\epsilon)$ of N, S in S^2 such that $\phi^\lambda|_{D(N,2\epsilon)} = \phi^\lambda|_{D(S,2\epsilon)} = id$. □

By stereographic projection σ^N from N the set $S^2 \setminus (D(N, \epsilon) \cup D(S, \epsilon))$ is a closed annulus A in $\mathbb{C}$ which is centerd at the origin. For a simpler notation we will assume without loss of generality that

$$A = \left\{ z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 2 \right\}$$

and by Appendix A, $\omega_{std} = \frac{r}{\pi(1+r^2)^2} dr \wedge d\theta$ on $\mathbb{C}$ via stereographic projection. Further let $\lambda_{std} = \frac{-1}{2(1+r^2)\pi} d\theta$ be the standard primitive of ω_{std}. Thus we may assume that for all λ we have $\phi^\lambda \in Symp(A, \partial A, \omega_{std})$ (see the definition in Appendix E). Further by the first conclusion in proposition 4.1.1, $\lambda \mapsto \phi^\lambda$ defines a loop in $Symp_0(A, \partial A, \omega_{std})$ which is constant near its ends (for $|\lambda| \geq \frac{\pi}{4}$). Consider the loop $\psi^\lambda = (\phi^\lambda)^{-1}$.

4.2 Monodromy is Hamiltonian

In this section we show that the monodromy maps ϕ^λ and their inverses ψ^λ are Hamiltonian.

Lemma 4.2.1. *Let $\phi \in Symp(A, \partial A, \omega_{std})$. Then there exists a smooth function $F\colon A \to \mathbb{R}$ which is constant in a neighbourhood of the boundary (not necessarily the same constant near the two boundary components !) such that*

$$\phi^*\lambda_{std} - \lambda_{std} = dF.$$

Proof. Trivial (Appendix D). □

Now we can define:

Definition 4.2.2. *Let $\phi \in Symp_0(A, \partial A, \omega_{std})$, the identity component of $Symp(A, \partial A, \omega_{std})$. Then $Flux(\phi) \in \mathbb{R}$, the* Flux of ϕ, *is defined to be*

$$Flux(\phi) = F(2) - F(\frac{1}{2})$$

where $F\colon A \to \mathbb{R}$ is the smooth function from Lemma 4.2.1 which satisfies

$$dF = \phi^*\lambda_{std} - \lambda_{std}.$$

Remark If F, F' are two functions such that $dF = \phi^*\lambda_{std} - \lambda_{std} = dF'$ then $F' = F + c$ for some constant c and obviously $Flux(\phi) = F(2) - F(\frac{1}{2}) = F'(2) - F'(\frac{1}{2})$. Hence $Flux$ is well-defined.
Also note that $Flux(\phi)$ for $\phi \in Symp_0(A, \partial A, \omega_{std})$ is independent of the primitive of ω_{std}. Since this is important in the sequel, we phrase it as a lemma:

Lemma 4.2.3. *If $Flux^\lambda(\phi)$ denotes the Flux of $\phi \in Symp_0(A, \partial A, \omega_{std})$ defined with respect to λ with $d\lambda = \omega_{std}$ instead of λ_{std}, then*

$$Flux(\phi) = Flux^\lambda(\phi).$$

Proof. Trivial (Appendix D). □

Lemma 4.2.4. *Let $\phi, \psi \in Symp_0(A, \partial A, \omega_{std})$ then $Flux(\phi \circ \psi) = Flux(\phi) + Flux(\psi)$ and $Flux(Id) = 0$.*

Proof. Trivial (Appendix D). Follows also from Lemma 4.2.3. □

Now we can define

Definition 4.2.5. *The group $Ham(A, \partial A, \omega_{std})$ of Hamiltonian symplectomorphisms of the annulus A which are fixed in some neighbourhood of the boundary is defined to be*

$$Ham(A, \partial A, \omega_{std}) := \left\{\phi \in Symp_0(A, \partial A, \omega_{std}) | Flux(\phi) = 0\right\}.$$

Note that Lemma 4.2.4 above shows that $Ham(A, \partial A, \omega_{std})$ forms a group under composition. We proceed by showing two Lemmata which will be useful in the sequel.

Lemma 4.2.6. *Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Let ϕ_t be a symplectic isotopy starting at $\phi_0 = id$. Let ϕ_t be generated by the time-dependent vector field X_t, i.e*

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Then $\iota_{X_t}\omega = dH_t$ for a smooth family of functions $H_t \colon M \to \mathbb{R}$ if and only if $\phi_t^*\lambda - \lambda = dF_t$ for a smooth family of functions $F_t \colon M \to \mathbb{R}$. Moreover F_t and H_t are related by the equations

$$F_t = \int_0^t (H_s + \iota_{X_s}\lambda) \circ \phi_s ds$$

$$H_s = \left(\frac{d}{dt}\,|_{t=s}\, F_t\right) \circ \phi_s^{-1} - \iota_{X_s}\lambda.$$

Proof. see Appendix D. ☐

Lemma 4.2.7. *Given any real number a, there exists a canonical symplectomorphism $\phi^a \in Symp_0(A, \partial A, \omega_{std})$ such that*

$$Flux(\phi^a) = a$$

and $\phi^0 = id$. Further ϕ^a depends smoothly on a.

Proof. Trivial (Appendix D). ☐

Now we can show

Lemma 4.2.8. *If $\phi \in Ham(A, \partial A, \omega_{std})$, then there exists a smooth family of functions $H_t \colon A \to \mathbb{R}$ which have support away from the boundary, such that ϕ is the time-one map of the isotopy ϕ_t of A, generated by the time-dependent vector field X_t, defined by $\iota_{X_t}\omega_{std} = dH_t$. In particular the group $Ham(A, \partial A, \omega_{std})$ is path-connected.*

Proof. Since $\phi \in Ham(A, \partial A, \omega_{std}) \subset Symp_0(A, \partial A, \omega_{std})$ there exists a smooth path $\phi_t \in Symp_0(A, \partial A, \omega_{std})$ from id to ϕ. Define $A(t) := Flux(\phi_t)$. By smoothness of the path, $A(t)$ is smooth in t. Hence consider the smooth path $\psi_t = \phi^{-A(t)} \circ \phi_t$ between id and ϕ (from above $\phi^0 = id$). Since $Flux(\phi^{-A(t)} \circ \phi_t) = Flux(\phi^{-A(t)}) + Flux(\phi_t) = 0$ (cf. Proposition 4.2.4), this is a path in $Ham(A, \partial A)$ which connects id to ϕ. Then $\psi_t^*\lambda_{std} - \lambda_{std} = dF_t$ with $F_t(2) = F_t(\frac{1}{2}) = 0$ and $H_t = ((\frac{d}{ds}{}_{s=t}F_s) \circ \phi_t^{-1} - \iota_{X_t}\lambda)$ has support away from the boundary and the desired properties by Lemma 4.2.6. This shows also that $Ham(A, \partial A, \omega_{std})$ is path-connected. ☐

We have the following proposition:

Proposition 4.2.9. *Let $M := \mathbb{C} \times A$ and Ω a symplectic form on M such that the fibers of $p_1 \colon \mathbb{C} \times A \to \mathbb{C};\ (z, w) \mapsto z$ are symplectic. Moreover let $\Omega = p_1^*\alpha + p_2^*\omega_{std}$ near $\mathbb{C} \times \partial A$ be split.*
Let $\delta \colon [0,1] \to \mathbb{C}$ be a closed curve in the base such that $\Omega|_{p_1^{-1}(\delta(0))} = \omega_{std}$. Then the monodromy map ϕ of the symplectic connection around δ is Hamiltonian.

Proof. Since $H^2(M) = 0$ it follows that $\Omega = d\Lambda$ is exact. Let $A := p^{-1}(\delta(0))$ and let $\Lambda|_A = \lambda$.
Since Ω is split near ∂A there exists a neighbourhood U of ∂A such that $\phi|_U = id$. Symplectic parallel transport is symplectic and the loop δ is contractible, hence it follows, that $\phi \in Symp_0(A, \partial A, \omega_{std})$ (indeed, the monodromy maps around the loops in a contraction with fixed endpoints define a symplectic path to the identity). Thus by Lemma 4.2.1, we have

$$\phi^*\lambda - \lambda = dF$$

for some function F and we can define $Flux^\lambda(\phi)$. By Lemma 4.2.3 $Flux(\phi)$ is independent of the primitive λ and we surpress λ in the notation for $Flux$.
Now if $\gamma \colon [0,1] \to A$ traces out the straight line element $A \cap \mathbb{R}_+$ between the two boundary components, then by the Fundamental Theorem of Calculus,

$$Flux(\phi) = F(\gamma(1)) - F(\gamma(0)) = \int_\gamma dF = \int_\gamma \phi^*\lambda - \lambda.$$

Define

$$P_s \colon A \to p^{-1}(\delta(s))$$

the parallel transport map for the path $\delta_s \colon [0, s] \to \mathbb{C};\ \ t \mapsto \delta(t)$ and

$$\Phi \colon [0,1] \times [0,1] \to M;\quad (s,t) \mapsto P_s(\gamma(t)).$$

Let $C := \Phi([0,1] \times [0,1])$. Observe that $P_0 = id$ and $P_1 = \phi$. Now

$$\int_C \Omega = \int_0^1 \int_0^1 \Omega\left(\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial t}\right) dsdt$$

Note that

$$\beta_t \colon [0,1] \to M;\ s \mapsto \Phi(s,t) = P_s(\gamma(t))$$

is by definition the horizontal lift of δ starting at $\gamma(t)$ and thus

$$\frac{\partial \Phi}{\partial s}(t,s) = \dot{\beta}_t(s)$$

is horizontal. But

$$\frac{\partial \Phi}{\partial t}(t,s) = dP_s(\gamma'(t))$$

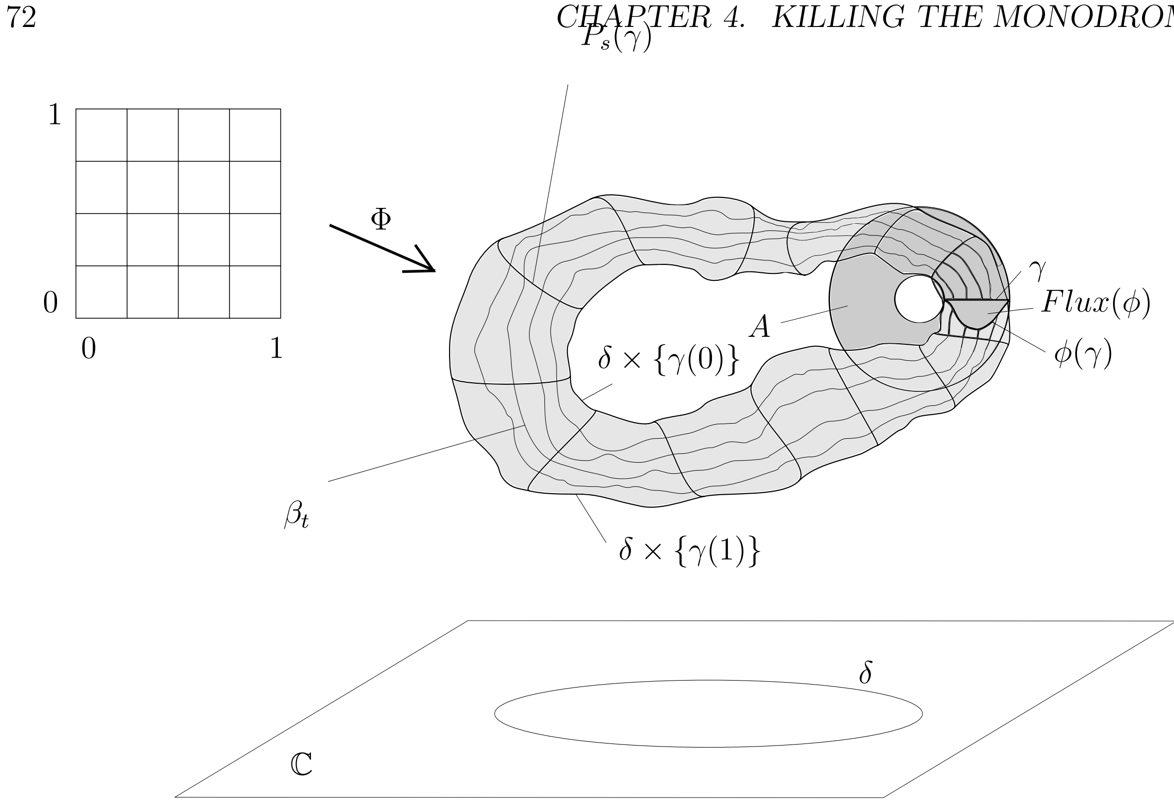

Figure 4.3: The map Φ and its image

is vertical. Hence by definition of the symplectic connection

$$\Omega\left(\frac{\partial \Phi}{\partial s}(s,t), \frac{\partial \Phi}{\partial t}(s,t)\right) = 0$$

and so

$$\int_C \Omega = 0.$$

On the other hand

$$\int_C \Omega = \int_C d\Lambda = \int_{\partial C} \Lambda = \int_{\Phi(\{1\}\times[0,1])} \Lambda - \int_{\Phi(\{0\}\times[0,1])} \Lambda + \int_{\Phi([0,1]\times\{1\})} \Lambda - \int_{\Phi([0,1]\times\{0\})} \Lambda =$$

$$= \int_{\phi\gamma} \lambda - \int_{\gamma} \lambda + \int_{\delta\times\{\gamma(1)\}} \Lambda - \int_{\delta\times\{\gamma(0)\}} \Lambda = \mathrm{Flux}(\phi) + \int_{\delta\times\{\gamma(1)\}} \Lambda - \int_{\delta\times\{\gamma(0)\}} \Lambda.$$

Let $i_w \colon \mathbb{C} \to M;\quad z \mapsto (z,w)$ denote the inclusion and let $\lambda_i := i^*_{\gamma(i)}\Lambda$. Since near $\gamma(i)$, $\Omega = p_1^*\alpha + p_2^*\omega_{std}$ and $p_1 \circ i_w = id$ and $p_2 \circ i_w \equiv w$, it follows that $i^*_{\gamma(i)}\Omega = \alpha$ and consequently $d\lambda_0 = \alpha = d\lambda_1$. Thus $\lambda_0 = \lambda_1 + df$ for a function $f \colon \mathbb{C} \to \mathbb{R}$. But then

$$\int_{\delta\times\{\gamma(i)\}} \Lambda = \int_{i_{\gamma(i)}(\delta)} \Lambda = \int_{\delta} \lambda_i$$

and so, since δ is closed

$$\int_\delta \lambda_0 = \int_\delta \lambda_1.$$

This implies by the equation above that

$$\int_{\delta\times\{\gamma(1)\}} \Lambda - \int_{\delta\times\{\gamma(0)\}} \Lambda = 0$$

and thus

$$\mathrm{Flux}(\phi) = 0$$

as claimed. □

The following theorem is a simple corollary of Proposition 4.2.9

Theorem 4.2.10. *The monodromy map ϕ^λ of the symplectic connection defined by (p_1, ω) around the circle of latitude λ is an element of $Ham(A, \partial A, \omega_{std})$.*

Proof. Via stereographic projection from the northpole N applied to the base, we may assume that we meet the conditions from Proposition 4.2.9. Then C^λ is just a closed circle around the origin. Hence $\phi^\lambda \in Ham(A, \partial A, \omega_{std})$ as claimed. □

Since $\psi^\lambda = (\phi^\lambda)^{-1}$, it follows straight away that also $\psi^\lambda \in Ham(A, \partial A, \omega_{std})$.

Theorem 4.2.11. *$Ham(A, \partial A, \omega_{std})$ is simply connected.*

Proof. This is proved in Proposition E.0.31 in Appendix E. □

Thus the loop ψ^λ is contractible.

4.3 A special contraction

In the previous section, we have shown that the loop ψ^λ is contractible. In order for the inflation procedure to work in the next section we need a special contraction ψ_s^λ with $\lambda \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, $s \in [0,1]$ and such that $\psi_s^0(E) = E$ where E is the equator in S^2. We will also call $E = \{|z| = 1\} \subset A$ the equator.

Theorem 4.3.1. *There exists a smooth contraction ψ_s^λ of ψ^λ, with $\lambda \in [\frac{-\pi}{2}, \frac{\pi}{2}], s \in [0,1]$, such that:*

- *$\psi_1^\lambda = \psi^\lambda$;*
- *$\psi_0^\lambda \equiv id$;*
- *$\psi_s^\lambda = id$ for $|\lambda| \geq \frac{\pi}{4}$ for all s;*
- *ψ_s^λ is constant in s near its ends;*

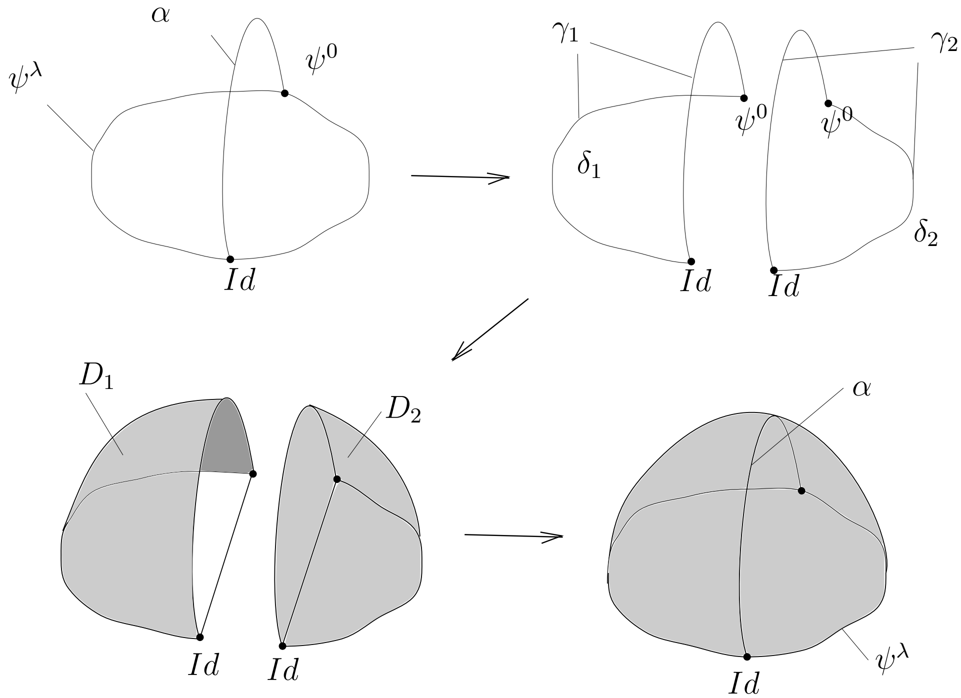

Figure 4.4: The idea to construct the special contraction ψ_s^λ

- $\psi_s^0(E) = E$ *for all* s.

Proof. We want to use Lemma 3.4.2 and the fact that $Ham(A, \partial A, \omega_{std})$ is simply connected to find the desired special contraction. Here is the idea (cf. fig 4.4): Since ψ^0 is the monodromy map around the equator in the base, it follows that it preserves E (recall that a fibered Lagrangian torus is generated by symplectic parallel transport by Lemma 2.4.2). Thus by Lemma 3.4.2 there exists a path $\alpha(t) \in Ham(A, \partial A, \omega_{std})$ between the identity and ψ^0 which preserves E for all t. Now split the loop ψ^λ in two paths $\delta_1 := \left\{\psi^\lambda\right\}_{\lambda\in[-\frac{\pi}{2},0]}$ and $\delta_2 := \left\{\psi^\lambda\right\}_{\lambda\in[0,\frac{\pi}{2}]}$. Form two loops $\gamma_1 := \delta_1 * \bar{\alpha}$ and $\gamma_2 := \alpha * \bar{\delta_2}$ where $*$ means concatenation of paths and $\bar{\gamma}$ means travelling through γ in opposite direction. Since these loops are contractible in $Ham(A, \partial A, \omega_{std})$, we can fill in disks D_1, D_2 which will agree along α. Gluing them suitably together along α gives the desired contraction which contains α. The issue here is that we have to do this in a smooth way. So the actual proof looks somewhat different.

First we apply a smooth homotopy (a smooth family of reparametrisations) such that ψ^λ is constantly $\phi = \psi^0$ for λ near 0. Now we define the paths

$$\delta_1 \colon [0, \frac{\pi}{2}] \to Ham(A, \partial A, \omega_{std});\ t \mapsto \psi^{t-\frac{\pi}{2}}$$

$$\delta_2 \colon [0, \frac{\pi}{2}] \to Ham(A, \partial A, \omega_{std});\ t \mapsto \psi^{\frac{\pi}{2}-t}$$

and these paths are now constant near the ends due to the homotopy above.
Apply Lemma 3.4.2 to $\phi = \psi^0$ and reparametrize the path $\alpha(t)$ in t such that $t \in [0, \frac{\pi}{2}]$

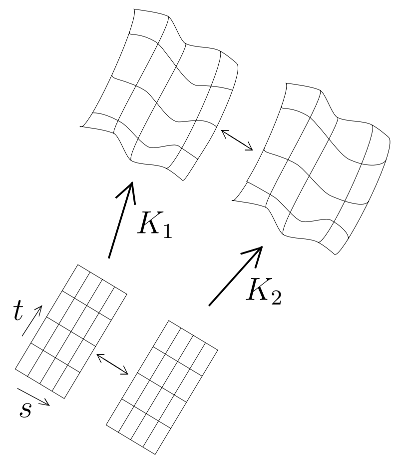

Figure 4.5: The construction of K

and such that it is constant near its ends.
In a simply connected space, all path with the same endpoints are homotopic through such paths. Thus we get two continuous maps

$$K_1, K_2\colon [0,1] \times [0, \frac{\pi}{2}] \to Ham(A, \partial A, \omega_{std})$$

with $K_1(s,t)$ a homotopy between the paths δ_1 and α and $K_2(s,t)$ a homotopy between α and δ_2. We can assume that K_i is constant in t near $0, \frac{\pi}{2}$ (say for t being 4ϵ close). Now reparametrise in s such that $K_i(s,t) = K_i(0,t)$ for $s < 4\epsilon$ and $K_i(s,t) = K_i(1,t)$ for $s > 1-4\epsilon$. Since $\delta_1, \delta_2, \alpha$ are smooth paths, it follows that K_i is smooth in a 4ϵ-neighbourhood of the boundary by construction.
We concatenate the two homotopies above by concatenating the paths $K_i^t(s) = K_i(s,t)$ for fixed t to obtain a homotopy

$$K\colon [0,1] \times [0, \frac{\pi}{2}] \to Ham(A, \partial A, \omega)$$

$$K^t = K_2^t * K_1^t$$

By construction K is a homotopy between δ_1 and δ_2 such that $K(\frac{1}{2}, t) = \alpha(t)$. It is smooth on the 2ϵ neighbourhood of its boundary and it is constant in s for $s \in [\frac{1}{2} - 2\epsilon, \frac{1}{2} + 2\epsilon]$.
Next consider the continuous map (cf. fig. 4.6)

$$\tau\colon [0,1] \times [0,3] \to [-1,1] \times [0,1]$$

given by

$$\tau(x,t) = \begin{cases} (-x, tx) \text{ for } 0 \le t \le 1 \\ (-x + 2(t-1)x, x) \text{ for } 1 \le t \le 2 \\ (x, x-(t-2)x) \text{ for } 2 \le t \le 3 \end{cases} .$$

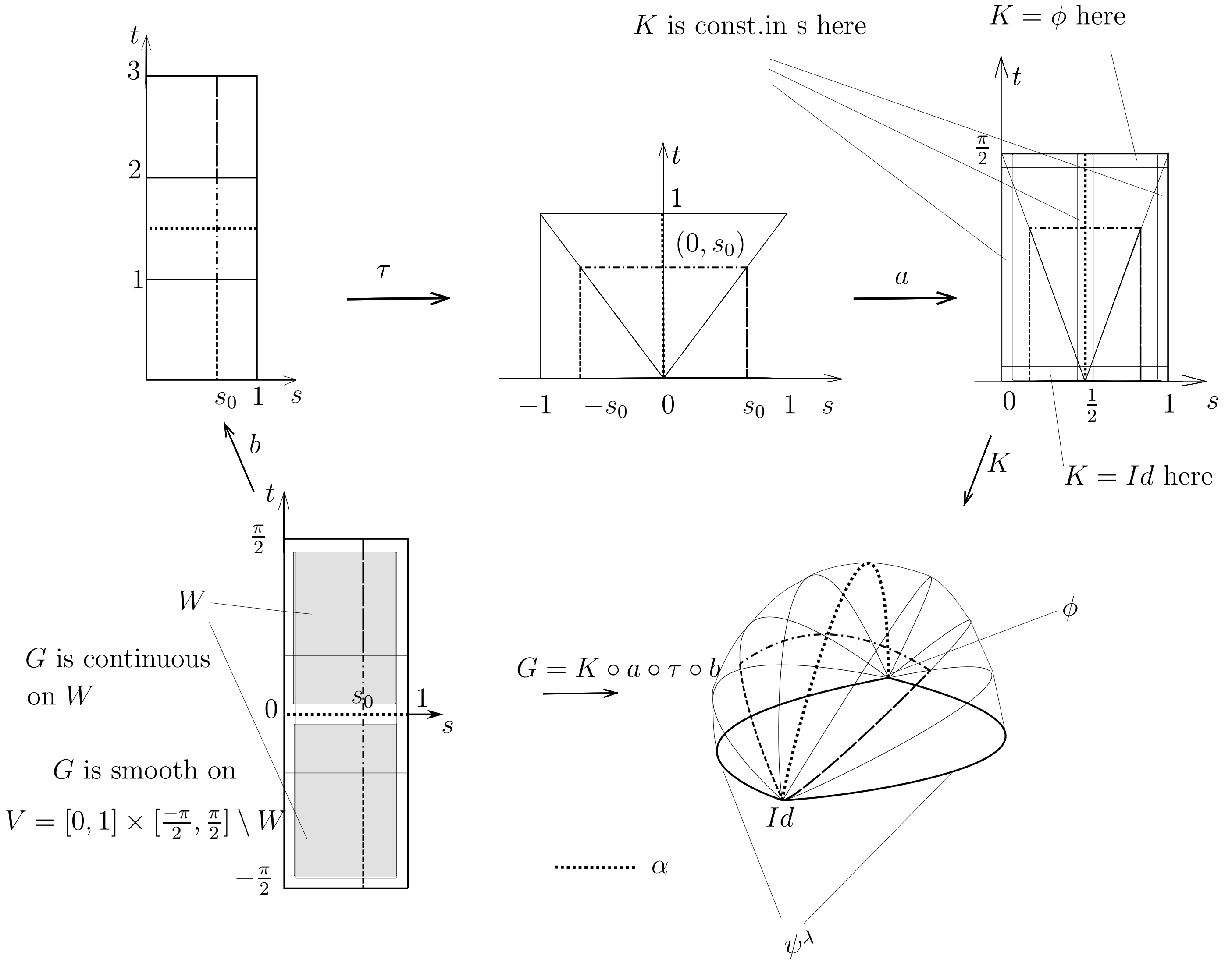

Figure 4.6: The construction of the map G

The following two smooth maps

$$a\colon [-1,1]\times[0,1] \to [0,1]\times[0,\frac{\pi}{2}];\ (x,y)\mapsto (\frac{1}{2}(x+1),\frac{\pi}{2}y)$$

$$b\colon [0,1]\times[-\frac{\pi}{2},\frac{\pi}{2}] \to [0,1]\times[0,3];\ (x,y)\mapsto (x,\frac{3}{\pi}(y+\frac{\pi}{2}))$$

are only used to adjusted the domain and the range of τ to those of K. Then

$$G\colon [0,1]\times[-\frac{\pi}{2},\frac{\pi}{2}] \mapsto Ham(A,\partial A,\omega_{std})$$

$$G(s,t) = K\circ a\circ\tau\circ b(s,t)$$

is a continuous contraction of the loop $\delta_1 * c_\phi * \bar{\delta}_2$ where c_ϕ denotes the constant path at ϕ.
Note that $K(s,t)$ is constantly ϕ for $t > \frac{\pi}{2} - 2\epsilon$ and constantly id for $t < 2\epsilon$ (cf. fig. 4.6). Thus $G(s,t) = id$ for $s < \epsilon$ and $G(s,t)$ is a reparametrisation of $\delta_1 * \bar{\delta}_2$ for $s > 1-\epsilon$ which is constantly ϕ while t runs through the non-smooth points of τ. Hence G is smooth for s near $0,1$. Furthermore, since $\tau(s,t)$ is smooth near $t = \frac{3}{2}, s > 0$ and $K(s,t)$ is constant in s near $s = \frac{1}{2}$, it follows that $G(s,t)$ is smooth near $t = 0$ with

$$G(s,0) = K(\frac{1}{2},\frac{\pi}{2}s) = \alpha(\frac{\pi}{2}s).$$

Hence G is a continuous contraction (of a reparametrisation) of $\delta_1 * \bar{\delta}_2$ which contains α and which is smooth on the set V (see fig. 4.6). Now consider the following lemma which will be proved in Appendix D.

Lemma 4.3.2. *There exists a smooth approximation* $\widetilde{G}\colon [0,1]\times[-\frac{\pi}{2},\frac{\pi}{2}] \to Ham(A,\partial A,\omega_{std})$ *such that* $\widetilde{G} = G$ *on* $\widetilde{V}\subset V$ *an open set which is a neighbourhood of the boundary, such that* $\widetilde{G}(s,0) = \alpha(\frac{\pi}{2}s)$.

Thus by the lemma there exists a smooth map $\widetilde{G}$ which is a contraction of a reparametrisation of $\delta_1 * \bar{\delta}_2$ in $Ham(A,\partial A,\omega_{std})$ and which contains the path α. Reparametrising suitably in s,t gives the required contraction $\psi_s^\lambda = \tilde{G}(s,\lambda)$ with $\psi_s^0 = \alpha(\frac{\pi}{2}s)$. This proves the theorem. □

4.4 Construction of a suitable Hamiltonian function

ψ_s^λ, the special contraction from Theorem 4.3.1, is Hamiltonian for all λ, s. It follows that $(\psi_s^\lambda)^*\lambda_{std} - \lambda_{std} = dF_s^\lambda$ for a function F_s^λ being constant near the boundary of A and such that $F_s^\lambda(2) - F_s^\lambda(\frac{1}{2}) = 0$. But F_s^λ is determined by the equation above up to a constant, so we choose the constant such that F_s^λ vanishes near the boundary. Because of this canonical choice and the smoothness of the contraction ψ_s^λ in λ, s, we obtain a smooth family $F_s^\lambda\colon A\to\mathbb{R}$ of smooth functions. By Lemma 4.2.6, the family F_s^λ is related to

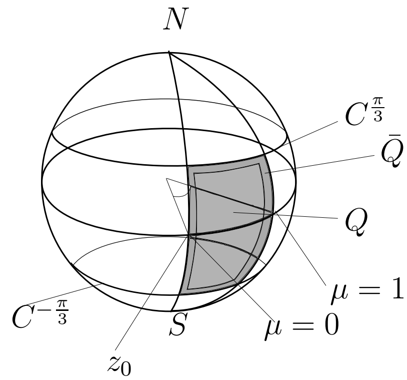

Figure 4.7: Q and $\bar{Q}$

the family of Hamiltonians H_s^λ generating the Hamiltonian isotopy ψ_s^λ for fixed λ, by the formula

$$H_s^\lambda = \frac{dF_s^\lambda}{ds} \circ (\psi_s^\lambda)^{-1} - \iota_{X_s^\lambda}\lambda_{std}$$

where

$$\frac{d}{dt}\psi_t^\lambda = X_t^\lambda \circ \psi_t^\lambda$$

and thus, H_s^λ is also a smooth family of functions in λ, s. Note that, since ψ_s^0 preserves the equator, the Hamiltonian vector field X_s^0 is tangent to E for all s, so that $H_s^0|_E$ is constant for all s. Further, since ψ_s^λ is constant near its ends in both s and λ, H_s^λ vanishes near its ends. Hence we have $H_s^\lambda(a) = 0$ for $|\lambda| \geq \frac{\pi}{4}$, $s < 2\epsilon$; $s > 1 - 2\epsilon$ or a near the boundary of A.

We define

$$Q := \left\{(\mu, \lambda) \in S^2 \setminus \{N, S\} \,|\epsilon \leq \mu \leq 1 - \epsilon; |\lambda| \leq \frac{\pi}{4}\right\}$$

$$\bar{Q} := \left\{(\mu, \lambda) \in S^2 \setminus \{N, S\} \,|\mu \leq 1;\ |\lambda| \leq \frac{\pi}{3}\right\}.$$

Define a smooth function

$$H \colon \bar{Q} \times A \to \mathbb{R}$$

$$(\lambda, \mu, a) \mapsto H_\mu^\lambda(a).$$

Now extend H to all of $S^2 \times S^2$ by zero and denote the resulting function also by H. Then by construction H has support in $Q \times A$ and $H(0, \mu)|_E$ is constant for all μ ($H(\lambda, \mu) = H|_{p_1^{-1}(\lambda,\mu)}$). Now we can evoke the inflation procedure.

4.5 Inflation

Recall, that in this chapter, we assume, that $(\mathcal{F}_{std}, \omega, L_{std}, S_\infty, S_0)$ is a relative symplectic fibration. Further the symplectic form ω is split of the form $\omega = p_1^*\widehat{\omega} + p_2^*\omega_{std}$ on

$$W := (S^2 \times (U_\infty \cup U_0)) \cup (B_{\frac{-1}{\sqrt{2}}} \times S^2)$$

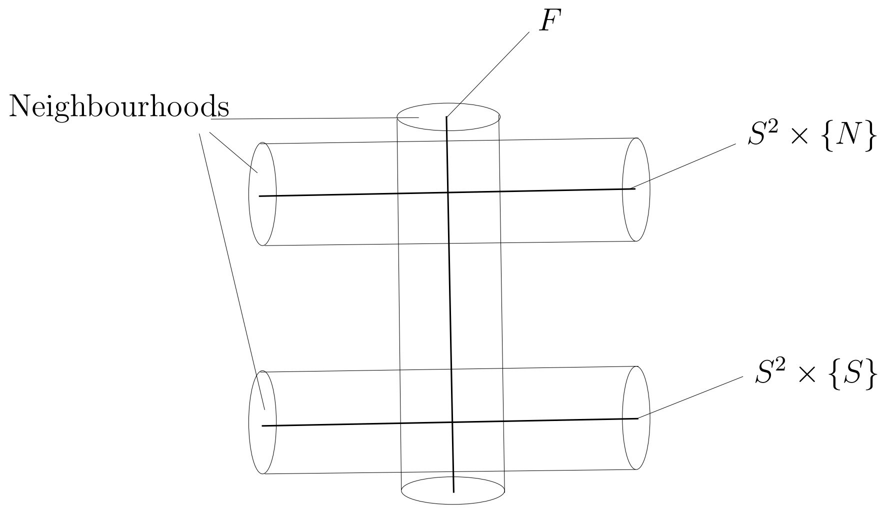

Figure 4.8: Where ω is split.

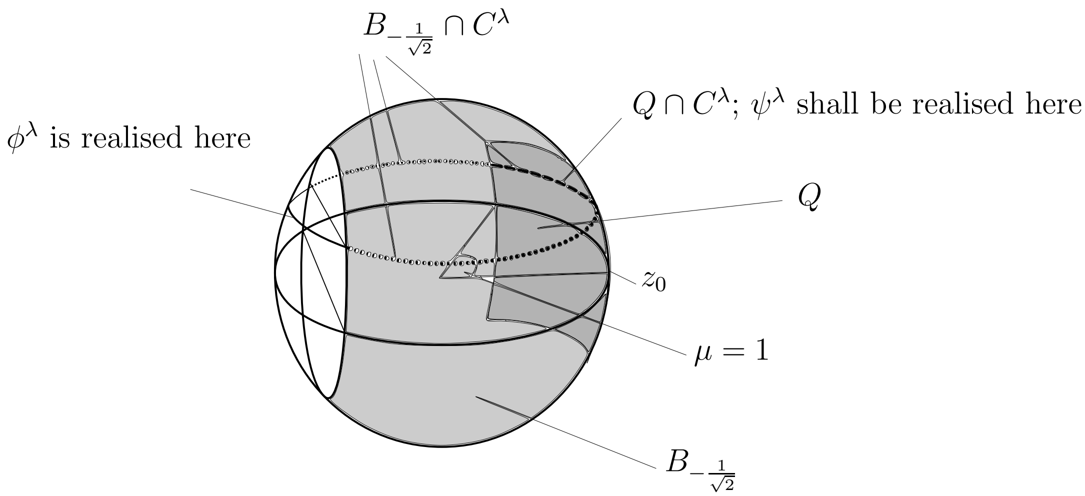

Figure 4.9: The path $Q \cap C^\lambda$ and $B_{-\frac{1}{\sqrt{2}}} \cap C^\lambda$

where $S^2 \times (U_\infty \cup U_0)$ is a neighbourhood of the symplectic sections $S_\infty = S^2 \times \{N\}$ and $S_0 = S^2 \times \{S\}$ (see figure 4.2 for the definition of $B_{\frac{-1}{\sqrt{2}}}$ and figure 4.8).

In this section, we are going to show, how to change the symplectic form ω, in its relative cohomology class in $H^2(S^2 \times S^2, L_{std}; \mathbb{R})$, to a form which has trivial monodromy around the circles of latitude. To explain the idea, fix the circle of latitude C^λ(cf. fig. 4.9). Then symplectic parallel transport equals the identity along the part of C^λ lying within $B_{-\frac{1}{\sqrt{2}}}$ since the symplectic form is split over $B_{-\frac{1}{\sqrt{2}}}$. It also follows that the monodromy ϕ^λ will be realised by travelling along the part of C^λ not lying within $B_{-\frac{1}{\sqrt{2}}}$.

The idea is now to smoothly change the symplectic form ω to a relative cohomologous symplectic form ω', such that the symplectic connection of ω' outside $Q \times S^2$ agrees with

that of ω and such that its symplectic connection realises the inverse monodromy ψ^λ along $C^\lambda \cap Q$ for all λ.
We define

Definition 4.5.1. *A smooth function $G\colon S^2 \times S^2 \to \mathbb{R}$ is called* admissible *if*

- $Supp(G) \subset int(Q) \times int(A)$
- $G|_{p_1^{-1}(e) \cap L_{std}} =$ *constant for all $e \in E$, where E is the equator in the base.*

The following diffeomorphism

$$\varphi\colon [\frac{-1}{2}, \frac{1}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}] \subset \mathbb{C} \to \bar{Q} \subset S^2$$

$$x + iy \mapsto (\lambda = y, \mu = x + \frac{1}{2})$$

provides local coordinates x, y around the point $(\lambda, \mu) = (0, \frac{1}{2})$ in S^2. Further $\bar{Q}$ is covered by these local coordinates.

In these coordinates, the admissible function G has support in $[\frac{-1}{2}, \frac{1}{2}] \times [-\frac{\pi}{3}, \frac{\pi}{3}] \times A$. Furthermore $G|_{\{x+i0\} \times E}$ is constant for all x. Let G be an admissible function. Then consider the closed form

$$\Omega_G = \omega + dG \wedge dx.$$

Since Ω_G is vertically non-degenerate, the Ω_G-orthogonal complements to the tangent spaces of the fibers of p_1 induce a symplectic connection on $S^2 \times S^2$.

Definition 4.5.2. *The symplectic connection on p_1 induced by the form Ω_G for the admissible function G will be denoted by ∇^G_{symp} and will be called* the connection induced by G*.*

First notice that Ω_G is obviously closed. It will be symplectic, if it is non-degenerate. This is equivalent for the form $\Omega_G \wedge \Omega_G$ to be a volume form on $S^2 \times S^2$. This statement is local and trivially satisfied outside $Supp(G)$. On $Supp(G)$,

$$\omega = p_1^*\widehat{\omega} + p_2^*\omega_{std}.$$

We work in the chosen coordinates x, y and write for the form $\widehat{\omega}$ on the base $\widehat{\omega} = f dx \wedge dy$ for some function $f > 0$. Hence

$$\Omega_G \wedge \Omega_G = \left(1 - \frac{1}{f}\frac{\partial G}{\partial y}\right) \omega \wedge \omega.$$

So Ω_G will be symplectic iff $1 - \frac{1}{f}\frac{\partial G}{\partial y} > 0$ everywhere. This need not be true for general G.

We are now interested in the following question: Given any admissible function G, can we find a deformation ω_t of ω through symplectic forms such that L_{std} remains monotone Lagrangian throughout the deformation and such that ω_1 has as symplectic connection ∇^G_{symp}?

Since $\Omega_G = \omega$ outside $Supp(G)$, the induced connection of G is the same as the induced connection of ω outside $Supp(G)$. Within $Supp(G)$, $\omega = p_1^*\widehat{\omega} + p_2^*\omega_{std}$ is split, so that its induced connection is flat. Thus the horizontal spaces of the induced connection of Ω_G are spanned by the horizontal lifts of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. These can be easily seen to be $\frac{\partial}{\partial x} + X_{G_{x,y}}$ and $\frac{\partial}{\partial y}$. Here $X_{G_{x,y}}$ is the Hamiltonian vector field on $(p_1^{-1}(x,y), \omega_{std})$ of the Hamiltonian function $G_{x,y}(a) = G(x,y,a)$. Indeed let $\tilde{\frac{\partial}{\partial x}} = \frac{\partial}{\partial x} + v_x$ be the horizontal lift of $\frac{\partial}{\partial x}$ with v_x vertical, then by definition

$$0 = \Omega_G(\widetilde{\frac{\partial}{\partial x}}, v) = \Omega_G(\frac{\partial}{\partial x}, v) + \Omega_G(v_x, v) = -dG(v) + \omega_{std}(v_x, v)$$

for all vertical v. Hence

$$dG = \iota_{v_x}\omega_{std}$$

which implies that $v_x = X_{G_{x,y}}$. The same calculation shows that $v_y = 0$.

4.5.1 The inflation procedure

Now let $f_\sigma, \bar{f}_\tau$ be two smooth non-negative bump-functions on S^2 where we think of f_σ as living on the fiber $F = \{z_0\} \times S^2$ and of $\bar{f}_\tau$ as living on the base. Let f_σ be such that $Supp(f_\sigma) \subset D(S,\epsilon) \cup D(N,\epsilon)$ where $D(N,\epsilon) \subset U_\infty; D(S,\epsilon) \subset U_0$ are neighbourhoods of N, S. In particular let f_σ be such that

$$\int_{D(N,\epsilon)} f_\sigma \omega_{std} = \int_{D(S,\epsilon)} f_\sigma \omega_{std} = \frac{1}{2}.$$

Now let $\bar{f}_\tau$ be a bump-function with support in $\bar{Q}$ so that $\bar{f}_\tau(x,y) = \bar{f}_\tau(x,-y)$ and that $\bar{f}_\tau|_Q = 1$. Let

$$a := \int_{\bar{Q}} \frac{\bar{f}_\tau}{f}\widehat{\omega} = \int_{\bar{Q}} \bar{f}_\tau dx \wedge dy$$

and define

$$f_\tau := \frac{\bar{f}_\tau}{af}.$$

Then

$$\int_{\bar{Q}} f_\tau \widehat{\omega} = \frac{1}{a}\int_{\bar{Q}} \bar{f}_\tau dx \wedge dy = 1$$

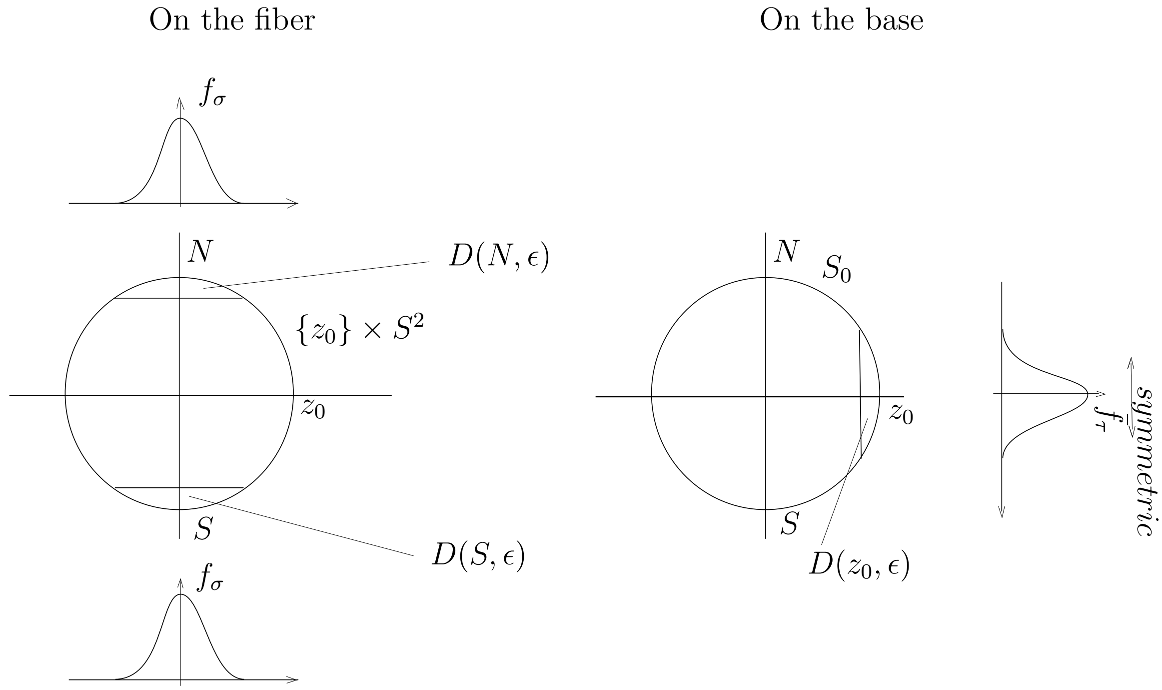

Figure 4.10: The functions $f_\sigma, \bar{f}_\tau$

and by the symmetry of $\bar{f}_\tau$

$$\int_{\bar{Q}\cap\{y\leq 0\}} f_\tau \widehat{\omega} = \frac{1}{a}\int_{\bar{Q}\cap\{y\leq 0\}} \bar{f}_\tau dx\wedge dy = \frac{1}{2}.$$

Let σ, τ be the two non-negative 2-forms $\sigma = f_\sigma \omega_{std}$ and $\tau = f_\tau \widehat{\omega}$ and consider the family of 2-forms

$$\omega_c = \frac{1}{c+1}\left(\omega + cp_1^*\tau + cp_2^*\sigma\right).$$

Then on $W = (S^2 \times (U_\infty \cup U_0)) \cup (B_{\frac{-1}{\sqrt{2}}} \times S^2)$,

$$\omega_c = \frac{1}{c+1}((1+cp_1^* f_\tau)p_1^*\widehat{\omega} + (1+cp_2^* f_\sigma)p_2^*\omega_{std}).$$

ω_c is obviously closed for all $c \geq 0$, $\omega_0 = \omega$ and ω_c is non-degenerate on W if

$$\omega_c \wedge \omega_c = \frac{1}{(c+1)^2}(1+cp_1^* f_\tau)(1+cp_2^* f_\sigma)\omega\wedge\omega > 0$$

everywhere. But this is obviously true by the choice of c, f_τ, f_σ. Furthermore, on $(S^2 \times S^2) \setminus W$

$$\omega_c = \frac{1}{c+1}\omega$$

which is also symplectic. Thus ω_c is symplectic for all $c \geq 0$.
It is important to observe that the symplectic connections on p_1 defined with respect to

ω and ω_c coincide for all $c \geq 0$. Indeed, on W the form ω_c is split as is ω, hence here both symplectic connections are flat and the horizontal subspaces are the tangent spaces to the other cartesian factor. Note that by construction on $(S^2 \times S^2) \setminus W$

$$\omega_c = \frac{1}{c+1}\omega,$$

and the symplectic horizontal complements H_x to $\ker dp_x$ are not affected by scaling of the symplectic form. Hence the symplectic connections of both ω, ω_c do agree for all $c \geq 0$ as claimed.
But ω_c is also relative cohomologous to ω_{std} in $H^2(S^2 \times S^2, L_{std}; \mathbb{R})$ for all $c \geq 0$. This follows by checking that both forms vanish on L_{std} and that they evalute equally on a basis of $H_2(S^2 \times S^2, L_{std}; \mathbb{R})$. We take as a basis $[F], [S_0], [D], [\Sigma]$ where $D = T \cap F$ and $\Sigma = D_{lh} \times \{z_0\}$ denotes the image of the constant section at z_0 restricted to the lower hemisphere D_{lh}. Firstly, it is clear that L_{std} is Lagrangian for all c. Secondly note that L_{std} is monotone for ω and both D and Σ have Maslov index 2 (Lemma 2.4.3 and Proposition 2.4.4 in chaper 2 show this for D respectively Σ). The monotonicity constant is $\frac{1}{4}$ because of the normalisation condition $\int_F \omega_0 = 1$ and $\mu(F) = 4$ by Theorem 2.3.4, so it follows that $\int_D \omega = \int_\Sigma \omega = \frac{1}{2}$. But now

- $\int_F \omega_c = \frac{1}{c+1}(\int_F \omega + cp_2^*\sigma) = \frac{1}{c+1}(1+c) = 1$
- $\int_{S_0} \omega_c = \frac{1}{c+1}(\int_{S_0} \omega + cp_1^*\tau) = \frac{1}{c+1}(1+c) = 1$
- $\int_D \omega_c = \frac{1}{c+1}(\int_D \omega + cp_2^*\sigma) = \frac{1}{c+1}(\frac{1}{2} + c\frac{1}{2}) = \frac{1}{2}$
- $\int_\Sigma \omega_c = \frac{1}{c+1}(\int_\Sigma \omega + cp_1^*\tau) = \frac{1}{c+1}(\frac{1}{2} + c\frac{1}{2}) = \frac{1}{2}$

for all $c \geq 0$. To see the last equation, notice that $p_1(\Sigma) \cap \bar{Q} = \{y \leq 0\} \cap \bar{Q}$. Hence by the discussion above for $f_\tau = \frac{\bar{f}_\tau}{af}$ with $\widehat{\omega} = f dx \wedge dy$ over $\bar{Q}$ it follows that $\int_{\{y \leq 0\} \cap \bar{Q}} f_\tau \widehat{\omega} = \frac{1}{2}$. In particular, L_{std} is still monotone for all $c \geq 0$. Since $Supp(G)$ and $Supp(p_2^* f_\sigma)$ are disjoint, $\omega_c = \frac{1}{c+1}(\omega + cp_1^*\tau)$ on $Supp(G)$. Hence if we restrict ω_c to the fibers, on $supp(G)$, it is just the standard form ω_{std} scaled by $\frac{1}{c+1}$. Now we want to find an admissible function $\widetilde{G}$, which induces the connection ∇^G_{symp} for the form ω_c. By this we mean that the symplectic connection $\nabla^{\widetilde{G}}_{symp_c}$ induced by the form

$$\Omega^c_{\widetilde{G}} = \omega_c + d\widetilde{G} \wedge dx$$

should agree with ∇^G_{symp}. For this to be true, all horizontal lifts have to agree. Since the horizontal lifts of $\frac{\partial}{\partial y}$ are trivial anyway, it boils down to comparing the horizontal lifts of $\frac{\partial}{\partial x}$. In particular the fiberwise Hamiltonian vector fields $X_{G_{x,y}}$ of G with respect to the symplectic form ω_{std} on the fiber and the fiberwise Hamiltonian vector fields $X_{\widetilde{G}_{x,y}}$ of $\widetilde{G}$ with respect to the symplectic form $\frac{1}{c+1}\omega_{std}$ on the fiber have to agree. Hence

$$d\widetilde{G}_{x,y} = \iota_{X_{\widetilde{G}_{x,y}}} \frac{1}{c+1}\omega_{std} = \frac{1}{c+1}\iota_{X_{G_{x,y}}}\omega_{std} = \frac{1}{c+1} dG_{x,y}$$

thus

$$\widetilde{G} = \frac{1}{c+1} G$$

Hence $\Omega^c_{\frac{1}{c+1}G}$ induces the connection ∇^G_{symp}. For c big enough the forms $\Omega^c_{\frac{1}{c+1}G}$ will be symplectic (see below) but not yet relative cohomologous to ω. Hence we have to modify these forms further to obtain this property as well.

To do this, we denote $K(x) := G_{x,0}(l)$ with $l \in E$ the equator in the fiber. Hence $K(x)$ is the constant which G equals on the $E = S^1$ that L cuts out of $p_1^{-1}(x, 0)$. Let ρ be a smooth symmetric cut-off function in y such that $\rho(0) = 1$ and $Supp(\rho(y)K(x)) \subset Q$. Now consider the family of 2-forms

$$\Omega^c_t = \omega_c + td\frac{1}{c+1}(G - \rho(y)K(x)) \wedge dx$$

and note that

$$\Omega^c_1 = \Omega^c_{\frac{1}{c+1}G} + p_1^*\alpha$$

for

$$\alpha = \frac{1}{c+1}\rho'(y)K(x)dx \wedge dy = \frac{\rho'(y)K(x)}{(c+1)f}\widehat{\omega}.$$

But a pull-back form from the base does not change the symplectic connection, hence Ω^c_1 still induces the symplectic connection ∇^G_{symp}.

4.5.2 Symplecticity

On $(S^2 \setminus Q) \times S^2$ the form Ω^c_t is just ω_c which we know is symplectic. Now on $Q \times S^2 \subset W$

$$\omega_c \wedge \omega_c = \frac{1}{(c+1)^2}(1 + cp_1^* f_\tau)(1 + cp_2^* f_\sigma)\omega \wedge \omega$$

while

$$\omega_c \wedge td\frac{1}{c+1}(G - \rho K) \wedge dx = \frac{t}{2f(c+1)^2}\left(\rho' K - \frac{\partial G}{\partial y}\right)(1 + cp_2^* f_\sigma)\,\omega \wedge \omega.$$

Hence we find

$$\Omega^c_t \wedge \Omega^c_t = \omega_c \wedge \omega_c + 2\left(\omega_c \wedge td\frac{1}{c+1}(G - \rho K) \wedge dx\right) =$$

$$= \frac{1}{(c+1)^2}\left(1 + cp_1^* f_\tau + \frac{t}{f}\left(\rho' K - \frac{\partial G}{\partial y}\right)\right)(1 + cp_2^* f_\sigma)\,\omega \wedge \omega$$

on $Q \times S^2$. $(1 + cp_2^* f_\sigma)$ is greater than zero for all c anyway as before. But the function $\rho' K - \frac{\partial G}{\partial y}$ is supported in the compact set $Q \times S^2$ hence there exists a constant $M \geq |\rho' K - \frac{\partial G}{\partial y}|$ on $Q \times S^2$. Now on $Q \times S^2$, the function $p_1^* f_\tau = \frac{1}{af}$ and so for $c > Ma$

$$\left(1 + \frac{c}{a} + \frac{t}{f}\left(\rho' K - \frac{\partial G}{\partial y}\right)\right) > 0 \text{ on } Q \times S^2.$$

Hence for c big enough, the family Ω_t^c is symplectic, starts at ω_c and Ω_1^c has as symplectic connection ∇_{symp}^G as desired. So we are left to show that the deformation is through relative cohomologous forms.

4.5.3 Lagrangian monotonicity

We first have to check that L_{std} is Lagrangian for Ω_t^c for all t. But $T_x L_{std}$ is spanned by $\frac{\partial}{\partial x}$ and a vector v tangent to the $S^1 = p_1^{-1}(p_1(x)) \cap L_{std}$ in the fiber. But by assumption G is constant on this S^1. Hence so is $G - \rho K$, since ρK is a function of the base. It follows that $d(G-\rho K) \wedge dx(\frac{\partial}{\partial x}, v) = 0$ and L_{std} is Lagrangian for Ω_t^c for all t. Furthermore since $d(G-\rho K)\wedge dx = d((G-\rho K)dx)$ is exact, the integrals of Ω_t^c over F, S_0 are independent of t. Also $\int_D \Omega_t^c = \int_D \omega_c = \frac{1}{2}$ and

$$\int_\Sigma \Omega_t^c = \int_\Sigma \omega_c + \frac{t}{c+1}\int_{\partial\Sigma}(G-\rho K)dx = \frac{1}{2} + \frac{t}{c+1}\int_{\partial\Sigma} 0dx = \frac{1}{2}$$

To see this, note that Σ has boundary on L_{std} which projects to the equator on the base. This means that $\rho(\partial\Sigma)) = 1$ (where it is defined) and so by definition of K it follows that $G(\partial\Sigma) - \rho(\partial\Sigma)K(\partial\Sigma) = 0$. Thus the above implies that Ω_t^c is relative cohomologous to ω_c as desired.

Indeed, for any admissible function G, we can find a deformation

$$\omega_t$$

of symplectic forms through relative cohomologous forms in $H^2(S^2 \times S^2, L_{std}; \mathbb{R})$, such that one end of the deformation is ω and the other end has as symplectic connection the connection induced by G: first consider the deformation ω_c with $c \in [0, C]$ and C so big that

$$\left(1 + \frac{C}{fa} + \frac{t}{f}\left(\rho' K - \frac{\partial G}{\partial y}\right)\right) > 0.$$

Then we consider the deformation

$$\Omega_t^C$$

with $t \in [0,1]$ which starts at ω_C. Then the form Ω_1^C has ∇_{symp}^G as its symplectic connection as desired.

Hence by construction, $(\mathcal{F}_{std}, \omega_t, L_{std}, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations. Note that the symplectic forms ω_t are still split near the sections S_∞, S_0.

4.6 Killing the monodromy

4.6.1 Killing the monodromy along circles of latitude

Putting the last two sections together, we can now prove:

Proposition 4.6.1. *Let $(\mathcal{F}_{std}, \omega, L_{std}, S_\infty, S_0)$ be a relative symplectic fibration with $\omega = p_1^*\widehat{\omega} + p_2^*\omega_{std}$ on*

$$W := (S^2 \times (U_\infty \cup U_0)) \cup (B_{\frac{-1}{\sqrt{2}}} \times S^2).$$

Then there exists a deformation ω_t of the symplectic form ω such that $(\mathcal{F}_{std}, \omega_t, L_{std}, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations and the monodromy maps ϕ^λ along the circles of latitude λ with respect to the symplectic connection induced by ω_1 are the identity for all λ. Further, ω_1 is still split near the sections S_∞, S_0

Proof. This follows directly from the inflation procedure applied to the admissible function $H\colon S^2 \times S^2 \to \mathbb{R}$ obtained from the special contraction in Theorem 4.3.1. Indeed by construction the function H is admissible. Hence by the inflation procedure, we can find a deformation ω_t of the symplectic form ω through symplectic forms, such that $(\mathcal{F}_{std}, \omega_t, L_{std}, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations, such that the symplectic connection induced by ω_1 equals that of ω outside $Q \times S^2$ and realises the symplectic connection induced by $\Omega = \omega + dH \wedge dx$ over $Q \times S^2$. Further, ω_1 is split near the sections S_∞, S_0 by construction. We are left to show that the monodromy of the symplectic connection induced by Ω along $Q \cap C^\lambda$ precisely equals $\psi^\lambda = (\phi^\lambda)^{-1}$.
By the choice of the local coordinates (x, y), $\bar{Q} \cap C^\lambda = (x, \lambda)$ with $x \in [\frac{-1}{2}, \frac{1}{2}]$. Thus we have to calculate the horizontal lift of $\frac{\partial}{\partial x}$ which we know by the inflation procedure equals $\frac{\partial}{\partial x} + X_{H_{x,\lambda}}$ over the point (x, λ). Here $X_{H_{x,\lambda}}$ denotes the Hamiltonian vector field on the fiber $p_1^{-1}(x, \lambda)$ of the function $H_{x,\lambda} = H|_{p_1^{-1}(x,\lambda)}$. But by construction of H in section 4.4 $H_{x,\lambda} = H^\lambda_{x+\frac{1}{2}} = H^\lambda_s$ for $s \in [0, 1]$ with $s = x + \frac{1}{2}$. Recall that H^λ_s in 4.4 was the family of Hamiltonians generating the special contraction ψ^λ_s. Consequently the monodromy map equals indeed ψ^λ once we have travelled along the path $\bar{Q} \cap C^\lambda = \{(x, \lambda)\}_{x \in [\frac{-1}{2}, \frac{1}{2}]}$. This finishes the proof of the proposition. □

Thus by Proposition 4.6.1 we can assume that the monodromy maps ϕ^λ along the circles of latitude are trivial (the identity).

4.6.2 Killing all the monodromy

We alter the symplectic form in its relative cohomology class such that the resulting form has trivial monodromy along any closed curve in the base. We need the following lemma.

Lemma 4.6.2. *Let ω, ω' be linear symplectic forms on $\mathbb{R}^4$ which define the same orientation and agree on a real hyperplane H. Then $\omega_t := (1 - t)\omega + t\omega'$ is symplectic for all $t \in [0, 1]$. Further, ω_t agrees with either form on H.*

Proof. Take a symplectic basis e_1, f_1, e_2, f_2 for ω such that e_1, f_1, e_2 is a basis of H. Take a vector $f_2' = a_1e_1 + b_1f_1 + a_2e_2 + b_2f_2$ such that e_1, f_1, e_2, f_2' is a symplectic basis for ω'. Since ω, ω' induce the same orientation, we have $b_2 > 0$, and therefore

$$\omega(e_2, f_2') = b_2 > 0, \quad \omega'(e_2, f_2) = \frac{1}{b_2} > 0.$$

For $\omega_t := (1-t)\omega + t\omega'$ we find

$$\omega_t \wedge \omega_t = (1-t)^2\omega^2 + 2t(1-t)\omega \wedge \omega' + t^2(\omega')^2,$$

and therefore

$$\begin{gathered}\omega_t \wedge \omega_t(e_1, f_1, e_2, f_2') = \\ = 2(1-t)^2\omega(e_1, f_1)\omega(e_2, f_2') + 2t^2\omega'(e_1, f_1)\omega'(e_2, f_2') + \\ +2t(1-t)\left(\omega(e_1, f_1)\omega'(e_2, f_2') + \omega(e_2, f_2')\omega'(e_1, f_1)\right) > 0\end{gathered}$$

□

Proposition 4.6.3. *Let $(\mathcal{F}_{std}, \Omega, L_{std}, S_\infty, S_0)$ be a relative symplectic fibration, such that the monodromy ϕ^λ around all circles of latitude $C^\lambda \subset S_0$ is the identity and $\Omega = p_1^*\varpi_1 + p_2^*\varpi_2$ is split near S_∞, S_0. Then there exists a deformation Ω_t of symplectic forms starting at Ω, such that $(\mathcal{F}_{std}, \Omega_t, L_{std}, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations and $(\mathcal{F}_{std}, \Omega_1, L_{std}, S_\infty, S_0)$ has trivial monodromy.*

Proof. Outline: Using the triviality of the monodromy around the circles of latitude we write down a p_1-fiber-preserving diffeomorphism ϕ of $M = S^2 \times S^2$ which pulls the symplectic form Ω back to a form which agrees with the standard form ω_0 on $T(C^\lambda \times S^2)$ for all λ. Then the linear deformation between $\phi^*\Omega$ and ω_0 will be shown to be a symplectic relative cohomologous deformation and obviously ω_0 has trivial monodromy. The desired deformation will then be the family, obtained by pushing forward the linear interpolation between the forms $\phi^*\Omega$ and ω_0 by ϕ.

Denote the 0-meridian by $m_0 := \{(\lambda, \mu) \in S^2 | \mu = 0\}$. And let

$$P_\lambda \colon \{N\} \times S^2 \to \{(\lambda, 0)\} \times S^2$$

the parallel transport map of the symplectic connection defined by Ω on p_1 along m_0 from N to $(\lambda, 0)$.

Moreover denote by

$$P_\mu^\lambda \colon \{(\lambda, 0)\} \times S^2 \to \{(\lambda, \mu)\} \times S^2$$

the parallel transport map with respect to the symplectic connection defined by Ω along the path $\gamma_\mu^\lambda(t) = (\lambda, t\mu)$ (this is the path along the circle of latitude λ from $(\lambda, 0)$ to (λ, μ)).

Observe, that due to the fact that Ω is split near S_∞, S_0, $P^\lambda(w) = id = P_\mu^\lambda(w)$ for w near N, S.

Further we will denote the restriction of the symplectic form Ω to the fiber $\{z\} \times S^2$ by Ω_z.

We construct ϕ by parallel transport on the left sphere in figure 4.11 with respect to the symplectic connection defined by ω_0 by first going backwards along the circle of latitude

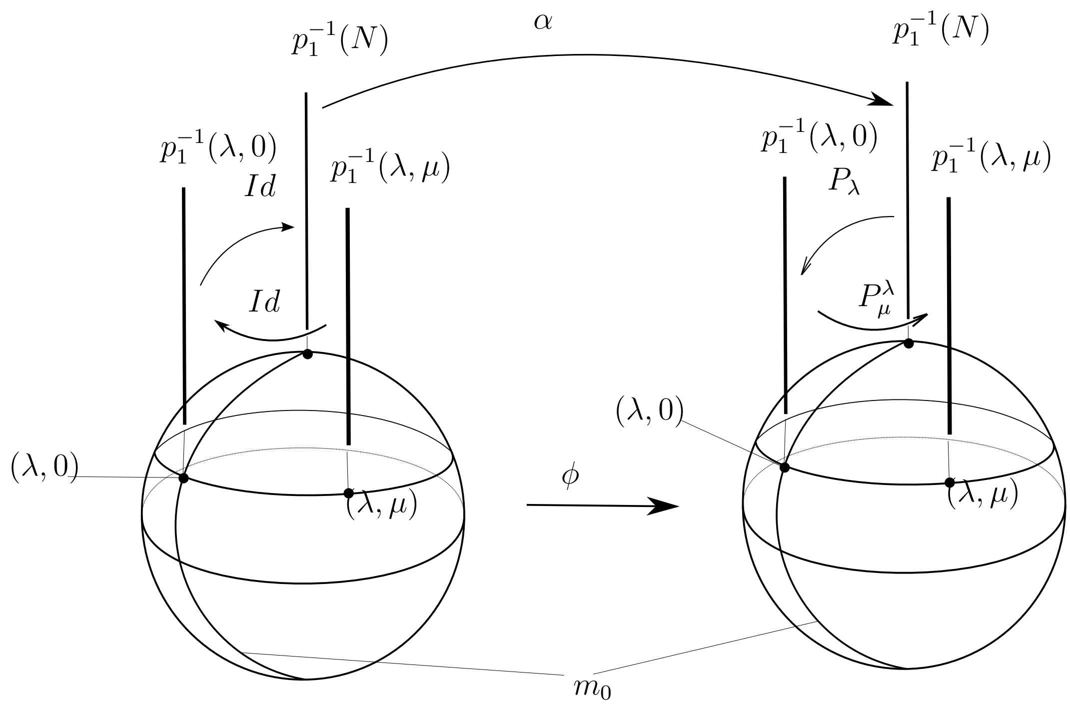

Figure 4.11: The maps P_λ, P_μ^λ and the construction of ϕ.

until we hit the meridian m_0 and then upwards along m_0 until we hit the north pole N. Then via a symplectomorphism

$$\alpha\colon (\{N\} \times S^2, \omega_{std}) \to (\{N\} \times S^2, \Omega_N)$$

we symplectically identify the fibers over the northpole. Finally we use symplectic parallel transport for the induced symplectic connection by Ω along m_0 first and then along the circle of latitude to land in the fiber over the original point. The fact that the monodromies along the circles of latitude of the symplectic connection induced by Ω are trivial assures that the construction of ϕ is well-defined.

Since parallel transport with respect to the symplectic connection defined by ω_0 is the identity for all paths, we neglect this first part of the construction above.

We require the diffeomorphism ϕ to preserve the Clifford torus L_{std}. Therefore it is necessary to choose a symplectomorphism α with the property that $P_0 \circ \alpha\colon (\{N\} \times S^2, \omega_{std}) \to (\{(0,0)\} \times S^2, \Omega_{z_0})$ preserves the equator (Note that in the description of the construction of ϕ above, this should actually be written $P_0 \circ \alpha \circ Id$).

To obtain this, note that ω_{std} and Ω_{z_0} are two cohomologous symplectic forms, which give the upper hemi-sphere symplectic area $\frac{1}{2}$. Thus, by Proposition 3.2.1 in the standardisation, there exists a diffeomorphism h of S^2 such that $h(E) = E$ and $h^*\Omega_{z_0} = \omega_{std}$. Let $\alpha := (P_0)^{-1} \circ h$ then α has the desired properties.

Now define ϕ (cf. fig. 4.11) by

$$\phi\colon S^2 \times S^2 \to S^2 \times S^2$$

$$((\lambda, \mu), w) \mapsto P_\mu^\lambda(P_\lambda(\alpha(N, w))).$$

Note that P_μ^λ, P_λ are diffeomorphisms which are smooth in λ, μ and $\bar{\phi}\colon S^2 \times S^2 \to S^2 \times S^2; ((\lambda, \mu), u) \mapsto ((\lambda, \mu), \alpha^{-1}(\bar{P}_\lambda(\bar{P}_\mu^\lambda(u)))$ is a smooth inverse where $\bar{P}_\gamma$ indicates parallel transport along $\bar{\gamma}$. Thus ϕ is a diffeomorphism. It preserves the Clifford Torus L_{std} because

$$\phi(\{(0, \mu)\} \times E) = P_\mu^0(P_0(\alpha(\{N\} \times E))) = P_\mu^0(\{(0, 0)\} \times E) = \{(0, \mu)\} \times E$$

since $P_0 \circ \alpha = h$ and the Clifford torus L_{std} is given by parallel transport of the equator in the fiber around the equator in the base. Further, since $P^\lambda, P_\mu^\lambda = id$ for points near N, S, it follows that $\phi(z, w) = (z, \alpha(N, w))$ for points $(z, w) \in S^2 \times S^2$ near $\phi^{-1}(S_\infty), \phi^{-1}(S_0)$. Now consider the pull-back form $\bar{\Omega} := \phi^*\Omega$. Note first, that near $\phi^{-1}(S_\infty), \phi^{-1}(S_0)$,

$$\bar{\Omega} = p_1^*\varpi_1 + p_2^*\alpha^*\varpi_2$$

is split. By construction, $\bar{\Omega}$ restricts to ω_{std} on every fiber and moreover the horizontal lifts of vectors tangent to circles of latitude with respect to ω_0 and $\bar{\Omega}$ agree. Accordingly, ω_0 and $\bar{\Omega}$ agree on the 3-dimensional subspaces $T_{((\lambda,\mu),z)}(C^\lambda \times S^2)$ in $T_{((\lambda,\mu),z)}(S^2 \times S^2)$ for all $((\lambda, \mu), z) \in S^2 \times S^2$. Thus by Lemma 4.6.2, the linear interpolation $\bar{\Omega}_t := (1 - t)\bar{\Omega} + t\omega_0$ is through symplectic forms which are invariant on the 3-dimensional subspaces $T_{((\lambda,\mu),z)}(C^\lambda \times S^2)$.
By Proposition 2.4.8, ϕ induces the identity on H_2, thus $\bar{\Omega}_t$ is a family of cohomologous symplectic forms. Moreover near $\phi^{-1}(S_\infty), \phi^{-1}(S_0)$,

$$\bar{\Omega}_t := p_1^*((1 - t)\varpi_1 + t\omega_{std}) + p_2^*((1 - t)\alpha^*\varpi_2 + t\omega_{std})$$

is split. Hence $\phi^{-1}(S_\infty), \phi^{-1}(S_0)$ remain symplectic and horizontal throughout the deformation. Next,

Proposition 4.6.4. *L_{std} is monotone Lagrangian for $\bar{\Omega}_t$ for all t.*

Proof. First we show that L_{std} is Lagrangian for $\bar{\Omega}_t$ for all t. This follows immediately since $\bar{\Omega}_t = \omega_0$ on $T(C^\lambda \times S^2)$ for all λ and all t and the Clifford torus is contained in $E \times S^2 = C^0 \times S^2$.
We are left to show that L_{std} is monotone for $\bar{\Omega}_t$, for all t.
Let D be a relative cycle, then

$$\int_D \bar{\Omega}_t = \int_D \bar{\Omega} + t \int_D (\omega_0 - \bar{\Omega}).$$

Hence we are done if we can show that L_{std} is monotone for both forms $\omega_0, \bar{\Omega}$. For $\bar{\Omega}$ this follows from the fact that L_{std} was monotone for the symplectic form Ω and by Proposition 2.3.6: By construction $L_{std} = \phi^{-1}(L_{std})$ and $\bar{\Omega} = \phi^*\Omega$ is the pull-back data under ϕ. For ω_0 this follows from Proposition 2.3.5. ∎

Now consider the deformation $\Omega_t = \phi_* \bar{\Omega}_t$. This is the desired deformation. It is invariant on the 3-dimensional subspaces $T_{((\lambda,\mu),z)}(C^\lambda \times S^2)$ since $\bar{\Omega}_t$ is.
Further, since ϕ preserves L_{std}, it follows straight away that L_{std} is Lagrangian for Ω_t. Then note that

$$\Omega_t = (1-t)\Omega + t\phi_*\omega_0$$

since ϕ_* is linear. Now L_{std} is monotone Lagrangian for both Ω and $\phi_*\omega_0$. For Ω this is clear by assumption and for $\phi_*\omega_0$ this follows since $(\phi(L_{std}) = L_{std}, \phi_*\omega_0)$ is the push-forward data under ϕ of (L_{std}, ω_0) which is monotone by Proposition 2.3.5. Hence along the lines of the proof of Proposition 4.6.4 it follows that L_{std} is indeed monotone Lagrangian for Ω_t for all t. Next, near S_∞, S_0,

$$\Omega_t = p_1^*((1-t)\varpi_1 + t\omega_{std}) + p_2^*((1-t)\varpi_2 + t\alpha_*\omega_{std})$$

is split. Hence S_∞, S_0 remain symplectic and horizontal throughout the deformation.
To see that Ω_1 has trivial mondromy, note that (p_1, ω_0) has vanishing symplectic curvature and $(p_1, \phi_*\omega_0)$ is the push-forward symplectic connection which, by Proposition 2.2.17, also has vanishing symplectic curvature. Then symplectic parallel transport is the identity for all loops in the base S^2. By construction, $(\mathcal{F}_{std}, \Omega_t, L_{std}, S_\infty, S_0)$ is a homotopy of relative symplectic fibrations. This proves the proposition. □

4.7 Summary

We summarize this chapter by Theorem 4.7.1. Again all steps in this chapter either give rise to diffeomorphic relativ symplectic fibrations or to homotopies of relative symplectic fibrations, this proves

Theorem 4.7.1. *Assume that $(\mathcal{F}_{std}, \omega, L_{std}, S_\infty, S_0)$ is a relative symplectic fibration, such that $\omega = p_1^*\widetilde{\omega} + p_2^*\omega_{std}$ on $W = (V \times S^2) \cup (S^2 \times (U_\infty \cup U_0))$. In particular S_∞, S_0 are horizontal for the symplectic connection. Then $(\mathcal{F}_{std}, \omega, L_{std}, S_\infty, S_0)$ is equivalent to $(\mathcal{F}_{std}, \omega', L_{std}, S_\infty, S_0)$ in the sense of Definition 2.4.9 such that $(\mathcal{F}_{std}, \omega', L_{std}, S_\infty, S_0)$ has vanishing symplectic curvature. Further S_∞, S_0 are still horizontal.*

Chapter 5

Hamiltonian isotopy of fibered monotone Lagrangian tori

In this chapter we show, how the results, which we have obtained in chapters 3 and 4 to kill the monodromy, can be applied to the original foliation and symplectic form. First we will deform these to have trivial monodromy. Then by Moser's theorem we will fix the symplectic form to be standard all the way through the deformation process. In the next step we construct a symplectomorphism ϕ of (M, ω_0) by parallel transport, which maps the Lagrangian torus L to the Clifford torus L_{std}. Moreover ϕ will induce the identity on homology. So, by a theorem of Gromov, there exists a symplectic isotopy ϕ_t from the identity to ϕ. Since M is simply-connected, this isotopy is Hamiltonian. This proves Theorem 2.5.1.

5.1 Killing the monodromy by a homotopy of relative symplectic fibrations

This is the main section of the thesis. Let $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ be a relative symplectic fibration on M as given by the assumptions in Theorem 2.5.1.

Theorem 5.1.1. *There exists a homotopy $(\mathcal{F}_t, \omega_0, L_t, \Sigma_t, \Sigma'_t)$ of relative symplectic fibrations with*

$$(\mathcal{F}_0, \omega_0, L_0, \Sigma_0, \Sigma'_0) = (\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$$

such that

$$(\mathcal{F}_1, \omega_0, L_1, \Sigma_1, \Sigma'_1)$$

has trivial monodromy. Further, the symplectic sections Σ_1, Σ'_1 are horizontal and the isotopy L_t can be realised by a Hamiltonian isotopy.

Proof. We start with the relative symplectic fibration $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ on M. Corollary 3.1.3 implies that there exists a diffeomorphism $\tau\colon M \to S^2 \times S^2$, such that $\tau(\mathcal{F}, \omega_0, L, \Sigma, \Sigma') =$

$(\mathcal{F}_{std}, \phi_*\omega_0, \phi(L), \phi(\Sigma), \phi(\Sigma'))$ is a relative symplectic fibration on $S^2 \times S^2$. Now, by combining Theorem 3.5.1 and Theorem 4.7.1 we can assume that $(\mathcal{F}_{std}, \phi_*\omega_0, \phi(L), \phi(\Sigma), \phi(\Sigma'))$ is equivalent to a relative symplectic fibration $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}')$ with vanishing symplectic curvature such that $\overline{\Sigma}, \overline{\Sigma}$ are horizontal. But then also $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ and $(\overline{\mathcal{F}}, \overline{\omega}, \overline{L}, \overline{\Sigma}, \overline{\Sigma}')$ are equivalent. Thus Theorem 2.4.10 gives a homotopy of relative symplectic fibrations $(\mathcal{F}_t, \omega_t, L_t, \Sigma_t, \Sigma_t')$ on M, starting at $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ and ending at $(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$ with vanishing symplectic curvature such that Σ_1, Σ_1' are horizontal. Since Σ' is simply connected the monodromy map around any closed loop in Σ' is the identity.
Finally, by Lemma 2.4.11, there exists a homotopy $(\widetilde{\mathcal{F}}_t, \omega_0, \widetilde{L}_t, \widetilde{\Sigma}_t, \widetilde{\Sigma}_t')$ of relative symplectic fibrations, starting at $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$, which keeps the symplectic form fixed and whose endpoint is diffeomorphic to $(\mathcal{F}_1, \omega_1, L_1, \Sigma_1, \Sigma_1')$. Further, the Lagrangian isotopy $\widetilde{L}_t$ can be realised by a Hamiltonian isotopy. Since the endpoints are diffeomorphic, $(\widetilde{\mathcal{F}}_1, \omega_0, \widetilde{L}_1, \widetilde{\Sigma}_1, \widetilde{\Sigma}_1')$ has trivial monodromy and the sections Σ, Σ' are horizontal (cf. Proposition 2.2.17). This proves Theorem 5.1.1. □

The purpose of the following proposition is to show, that if we have a relative symplectic fibration with vanishing symplectic curvature, then the symplectic area of a disk D in Σ' and a horizontal lift $\tilde{D}$ of D have invariant symplectic area. This is necessary in the sequel to show that the torus L can be mapped onto L_{std}.

Proposition 5.1.2. *Let $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ be a relative symplectic fibration with vanishing symplectic curvature. Let $D \subset \Sigma'$ be a disk enclosed by $\gamma = \pi(L)$ where $\pi\colon x \in \mathcal{F}_x \mapsto \mathcal{F}_x \cap \Sigma'$. Further let $\tilde{D}$ denote a horizontal lift of D. Then*

$$\int_{\tilde{D}} \omega = \int_D \omega.$$

Proof. Let $z_0 = \gamma(0)$, $z_1 \in \mathcal{F}_{z_0} \cap \tilde{D}$ and let $\delta\colon [0,1] \to \mathcal{F}_{z_0}$ be a path such that $\delta(0) = z_0$ and $\delta(1) = z_1$(cf. fig. 5.1). Since the symplectic curvature vanishes, symplectic parallel transport depends only on the endpoints of a path, not on the path itself. Thus let γ_z denote a path in D which connects z_0 to z and let P_z denote the parallel transport along γ_z. Now we define

$$\Phi\colon D \times [0,1] \to M;\ (z,t) \mapsto P_z(\delta(t)).$$

This is a smooth map (actually an embedding) and we define

$$C := \Phi(D \times [0,1]).$$

Then

$$0 = \int_C d\omega = \int_{\partial C} \omega = \int_{\tilde{D}} \omega - \int_D \omega - \int_{\Phi(\partial D \times [0,1])} \omega.$$

But the last integral vanishes since $\Phi(\partial D \times [0,1])$ denotes the surface traced out by parallel transport of the path δ. Thus any tangent space is spanned by a vertical vector and a horizontal vector and so the symplectic area of $\Phi(\partial D \times [0,1])$ vanishes. This proves the proposition. □

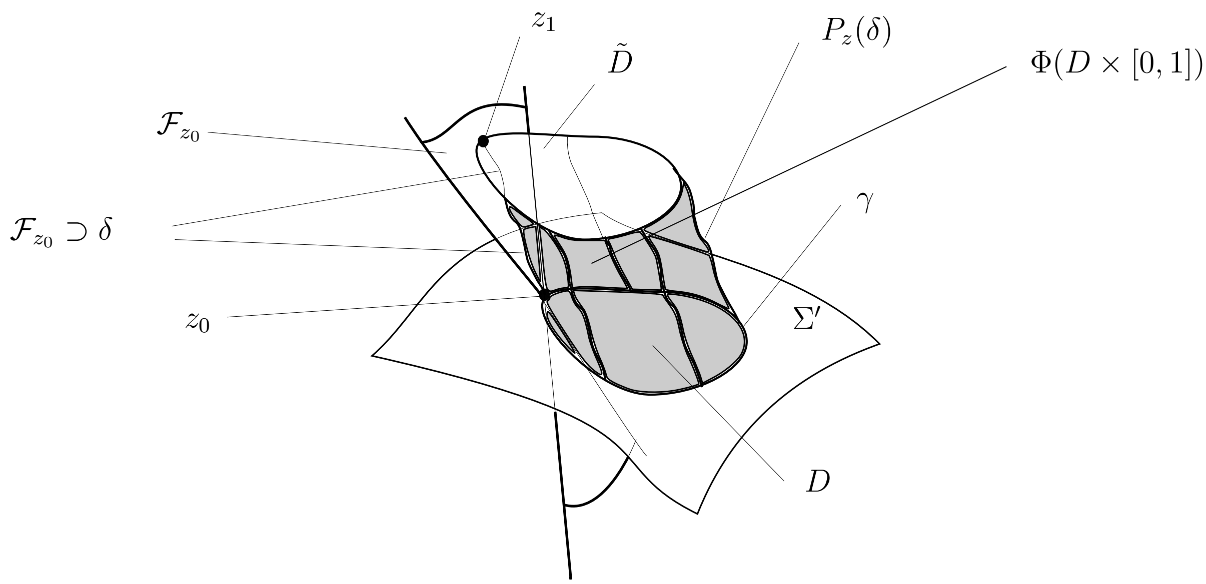

Figure 5.1: The path δ, the disk D and a horizontal lift $\tilde{D}$

Proposition 5.1.3. *Let $(\mathcal{F}, \omega, L, \Sigma, \Sigma')$ be a relative symplectic fibration with vanishing symplectic curvature. Let $D \subset \Sigma'$ be a disk enclosed by $\gamma = \pi(L)$ where $\pi\colon x \in \mathcal{F}_x \mapsto \mathcal{F}_x \cap \Sigma'$. Then*

$$\int_D \omega = \frac{1}{2}.$$

Proof. Let $z_1 \in \mathcal{F}_{\gamma(0)} \cap L$ and let $\tilde{D}$ be the horizontal lift of D through z_1. Since L is generated by symplectic parallel transport it follows that $\partial\tilde{D} \subset L$. By Proposition 5.1.2

$$\int_{\tilde{D}} \omega = \int_D \omega.$$

L is monotone with monotonicity constant $\frac{1}{4}$, thus

$$\int_{\tilde{D}} \omega = \frac{1}{4}\mu(\tilde{D})$$

with $\mu(\tilde{D}) \in 2\mathbb{Z}$. But γ is an embedded S^1, thus by the Jordan curve theorem its complement in $\Sigma' \cong S^2$ consists of two disks and either of these disks has non-vanishing symplectic area. Since the total symplectic area of Σ' equals 1, it follows that

$$0 < \int_D \omega < 1.$$

Hence $\mu(\tilde{D}) = 2$ and indeed

$$\int_D \omega = \frac{1}{2}.$$

This proves the proposition. Note that one could also calculate the Maslov index of $\tilde{D}$ as in Proposition 2.4.4. $\square$

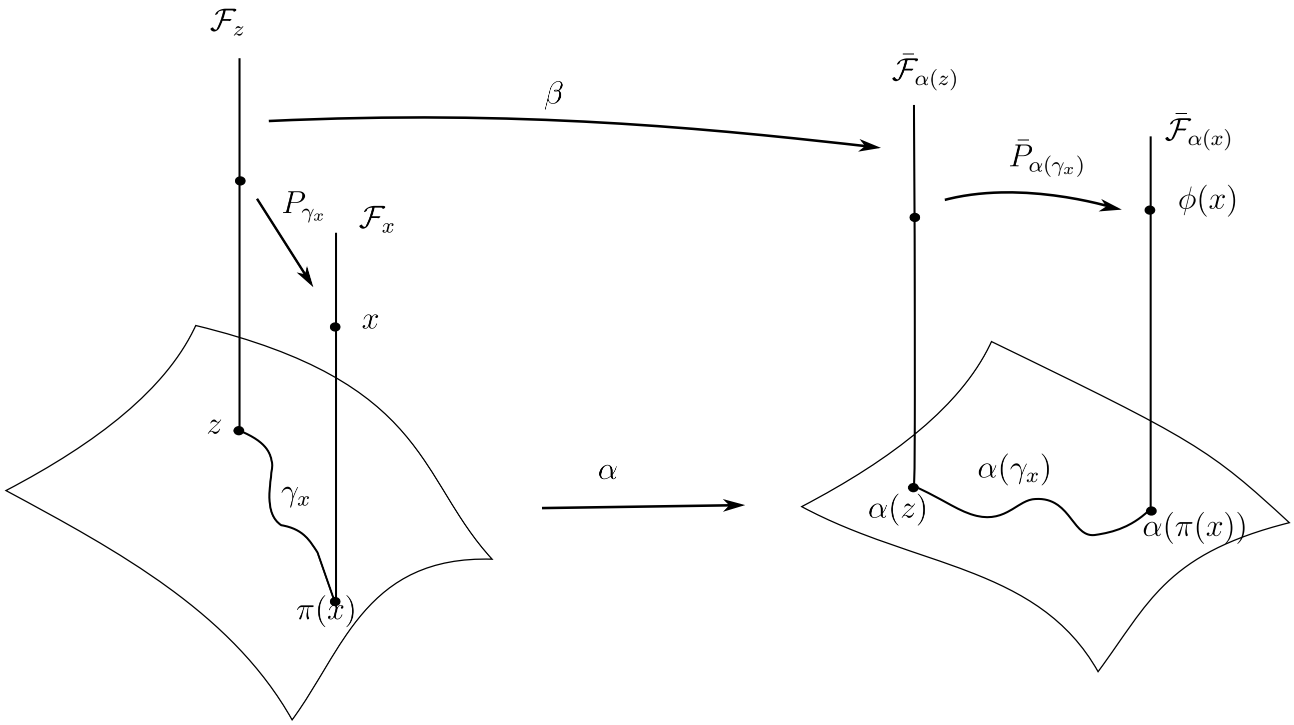

Figure 5.2: The construction of ϕ

5.2 Hamiltonian isotopy to the Clifford torus

With the help of the following Proposition and a theorem of Gromov, we show the existence of the desired Hamiltonian isotopy.
We start with some preliminary remarks. Let $(\mathcal{F}, \omega, L, \Sigma, \Sigma'), (\bar{\mathcal{F}}, \bar{\omega}, \bar{L}, \bar{\Sigma}, \bar{\Sigma}')$ be relative symplectic fibrations and let $\delta \subset \Sigma'$ be a path then let $P_\delta \colon \mathcal{F}_{\delta(0)} \to \mathcal{F}_{\delta(1)}$ denote the parallel transport along δ for the symplectic connection defined by $\mathcal{F}$ and ω. Similarly $\bar{P}_\delta$ denotes symplectic parallel transport along the path $\delta \subset \bar{\Sigma}'$ for the symplectic connection defined by $\bar{\mathcal{F}}$ and $\bar{\omega}$.

Definition 5.2.1. *Let $(\mathcal{F}, \omega, L, \Sigma, \Sigma'), (\bar{\mathcal{F}}, \bar{\omega}, \bar{L}, \bar{\Sigma}, \bar{\Sigma}')$ be relative symplectic fibrations. Then they are said to have* conjugate monodromy *if*

- *$\Sigma', \bar{\Sigma}'$ are horizontal;*
- *there exists a $z \in \Sigma'$ and symplectomorphisms $\alpha \colon (\Sigma', \omega|_{\Sigma'}) \to (\bar{\Sigma}', \bar{\omega}|_{\bar{\Sigma}'})$ and $\beta \colon (\mathcal{F}_z, \omega|_{\mathcal{F}_z}) \to (\bar{\mathcal{F}}_{\alpha(z)}, \bar{\omega}|_{\bar{\mathcal{F}}_{\alpha(z)}})$ with $\alpha(z) = \beta(z)$;*
- $$\bar{P}_{\alpha(\gamma)} = \beta \circ P_\gamma \circ \beta^{-1}$$
  for all closed path γ in Σ' starting and ending at z.

Proposition 5.2.2. *If two relative symplectic fibrations $(\mathcal{F}, \omega, L, \Sigma, \Sigma'), (\bar{\mathcal{F}}, \bar{\omega}, \bar{L}, \bar{\Sigma}, \bar{\Sigma}')$ on M have conjugate monodromy, then there exists a diffeomorphism ϕ of M such that $\phi(\mathcal{F}) = \bar{\mathcal{F}}$ and $\phi^*\bar{\omega} = \omega$.*

Proof. Let $\pi \colon M \to \Sigma'$; $x \in \mathcal{F}_x \mapsto \mathcal{F}_x \cap \Sigma'$. Then given any $x \in M$, pick a path $\gamma_x \colon [0,1] \to \Sigma'$ between z and $\pi(x)$.
Now we define

$$\phi \colon M \to M$$
$$x \mapsto \bar{P}_{\alpha(\gamma_x)} \circ \beta \circ P_{\gamma_x}^{-1}(x).$$

For the construction of ϕ, see figure 5.2. Since the two relative symplectic fibrations have conjugate monodromy, this definition doesn't depend on the choice of the path γ_x and so ϕ is well-defined.
ϕ is smooth and has a obvious smooth inverse (just construct it the other way round), so that ϕ is a diffeomorphism.
By construction leaves of $\mathcal{F}$ are mapped symplectically onto leaves of $\bar{\mathcal{F}}$. Moreover ϕ maps the horizontal distributions to each other and it maps Σ' symplectically onto $\bar{\Sigma}'$ (by the map α). Hence the symplectic form $\tau := \phi^*\bar{\omega}$ has the following properties: $\tau = \omega$ on the leaves of $\mathcal{F}$ and on Σ'. Moreover τ, ω induce the same symplectic connection on π and consequently the same symplectic curvature. But by Proposition 2.2.15, this implies that the evaluation of τ and ω on horizontal vectors can only differ by a fiberwise constant. Thus the two forms τ and ω differ by a pull-back form from the base. So $\tau - \omega = \pi^*\sigma$ for a two-form σ on Σ'. But on Σ' we have seen that $\tau = \omega$ and hence $\sigma = 0$. So indeed, ϕ is a symplectomorphism as claimed, which maps the foliations onto each other. This proves the proposition. □

Remark

Recall from section 1.1 that M is diffeomorphic to $S^2 \times S^2$ via the fixed diffeomorphism θ. Via θ, we define all the "standard" data on M which has been previously defined on $S^2 \times S^2$. For example $p_i \circ \theta^{-1}$ are the standard projections on M, then $\theta_*\omega_0 = (p_1 \circ \theta^{-1})^*\omega_{std} + (p_2 \circ \theta^{-1})^*\omega_{std}$ is the standard symplectic form on M and $\theta^{-1}L_{std}$ is the Clifford torus. We denote the data on M, defined via θ, by the same notation as on $S^2 \times S^2$.

Theorem 5.2.3. *Let $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ be a relative symplectic fibration with vanishing symplectic curvature and such that Σ, Σ' are horizontal. Then there exists a symplectomorphism ϕ of (M, ω_0) which makes the following diagram commute*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\pi"] & M \arrow[d, "p_1"] \\ \Sigma' \arrow[r, "f"] & S^2 \end{tikzcd} \tag{5.1}$$

Furthermore, ϕ maps L to L_{std}, is trivial on homology and $f \colon (\Sigma', \omega_0|_{\Sigma'}) \to (S^2, \omega_{std})$ is a symplectomorphism which maps $\pi(L)$ onto the equator in S^2.

Proof. We need to show that $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ and $(\mathcal{F}_{std}, \omega_0, L_{std}, S^2 \times \{N\}, S^2 \times \{S\})$ have conjugate monodromy. Then by Proposition 5.2.2, there exists the required symplectomorphism.
Let $z \in \gamma := \pi(L)$ and let $D \subset \Sigma'$ be one of the disks enclosed by γ. First we construct symplectomorphisms

$$\alpha\colon (\Sigma', \omega_0|_{\Sigma'}) \to (S^2 \times \{S\}, \omega_{std})$$
$$\beta\colon (\mathcal{F}_z, \omega_0|_{\mathcal{F}_z}) \to (\{(1,0,0)\} \times S^2, \omega_{std})$$

such that $\alpha(\gamma) = E \subset S^2$ with $\alpha(z) = (1,0,0)$ and $\beta(L \cap \mathcal{F}_z) = E \subset S^2$ with $\beta(z) = S$. We only show the construction for α since that of β is along the same lines. Compare fig. 5.3.
From Theorem 2.2.5, we know that Σ' is diffeomorphic to S^2. Since $\omega_0|_{\Sigma'}, \omega_{std}$ integrate to 1 over $\Sigma', S^2 \times \{S\}$ respectively, by Moser's theorem, there exists a symplectomorphism

$$\tilde{\alpha}\colon (\Sigma', \omega_0|_{\Sigma'}) \to (S^2 \times \{S\}, \omega_{std}).$$

As in Proposition 3.1.2, we can find a diffeomorphism k of S^2 such that $k(\tilde{\alpha}(\gamma)) = E$, i.e. a diffeomorphism which maps $\tilde{\alpha}(\gamma)$ to the equator in S^2.
Consider the push-forward symplectic form $\omega := k_* \omega_{std}$.
By Proposition 5.1.3 $\int_D \omega_0 = \frac{1}{2}$ and $k \circ \tilde{\alpha}$ maps D to D_{uh}, thus $\int_{D_{uh}} \omega = \frac{1}{2}$.
Since ω_{std} and ω are cohomologous and give the upper hemi-sphere D_{uh} area $\frac{1}{2}$ it follows by Proposition 3.2.1 that there exists a diffeomorphism h of S^2 such that $h(E) = E$ and $h^*\omega = \omega_{std}$.

Now consider the symplectomorphism

$$\alpha := h^{-1} \circ k \circ \tilde{\alpha}.$$

It maps γ to E, so that $\alpha(z)$ lies on the equator. If necessary, composition with a rotation around the north pole (this is ω_{std} symplectic) will assure that $\alpha(z) = (1,0,0)$.
Σ' and $S^2 \times \{S\}$ are horizontal and both relative symplectic fibrations $(\mathcal{F}, \omega_0, L, \Sigma, \Sigma')$ and $(\mathcal{F}_{std}, \omega_0, L_{std}, S^2 \times \{N\}, S^2 \times \{S\})$ have vanishing symplectic curvature. Hence, the monodromy around any closed curve is the identity. But then, the last condition in Definition 5.2.1 is trivially satisfied and they have indeed conjugate monodromy. Let ϕ be the symplectomorphism from Proposition 5.2.2 and let $f := p_1 \circ \alpha$. Since ϕ_* preserves the classes A, B, it is trivially the identity on homology. This proves the theorem. $\square$

Now we quote Gromov's theorem from [2]

Theorem 5.2.4. *Let $\phi \in Symp(S^2 \times S^2, \omega_0)$ be trivial on homology. Then there exists a symplectic isotopy $\phi_t \in Symp(S^2 \times S^2, \omega_0)$ with $\phi_0 = id$ and $\phi_1 = \phi$.*

Thus there exists a symplectic isotopy ϕ_t which starts at the identity and ends at the symplectomorphism ϕ from Proposition 5.2.3. Since M is simply-connected, this isotopy is Hamiltonian. Hence there exists a Hamiltonian isotopy from L to the Clifford torus L_{std}. This proves Theorem 2.5.1.

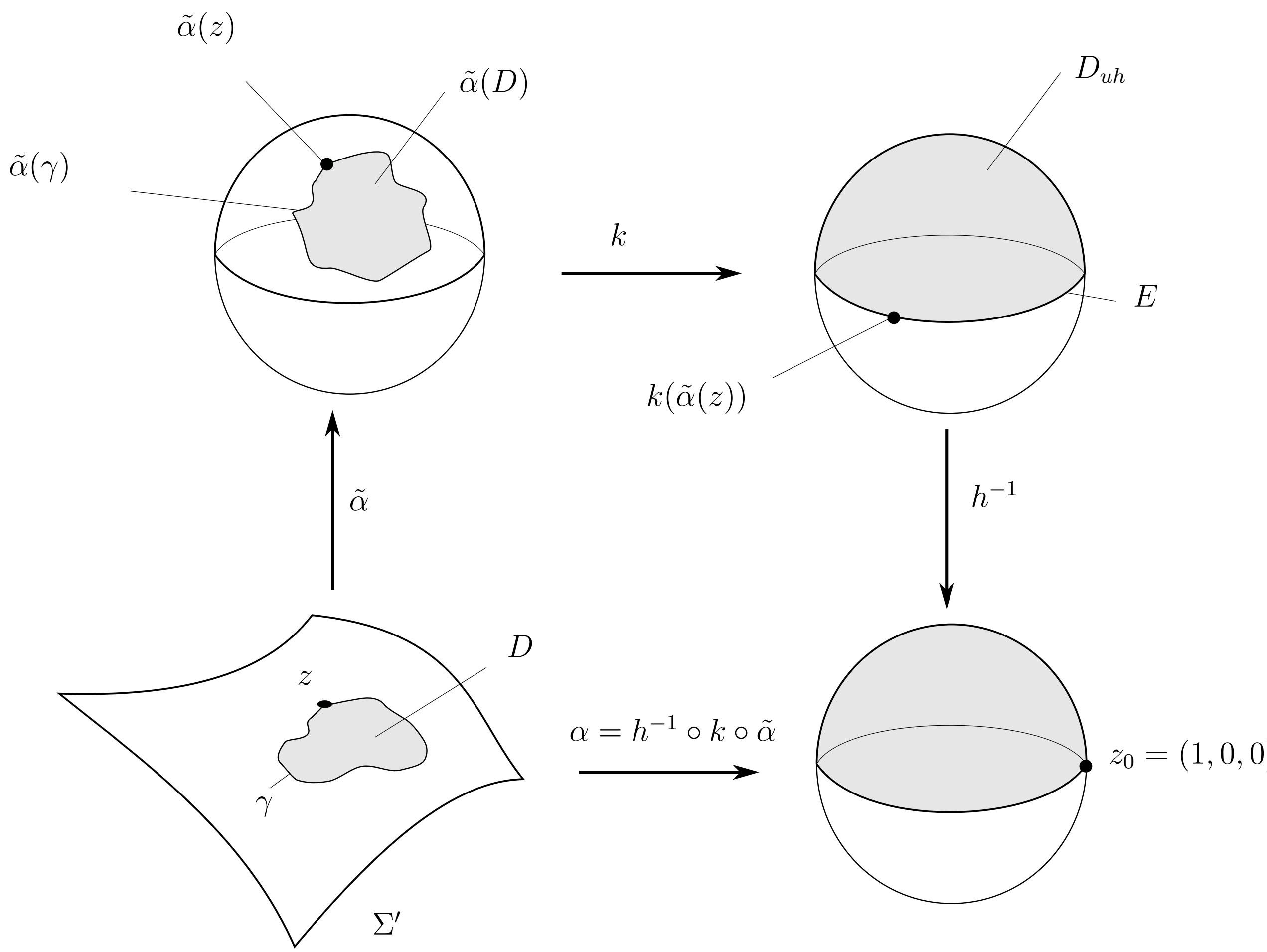

Figure 5.3: The construction of the map α

Chapter 6

A.Ivrii's result and its relation to the Main Theorem

6.1 The Chekanov-Schlenk Torus

In this section, we show the construction of the Chekanov-Schlenk Torus L_{CS} following [28]. They show, that it is a monotone Lagrangian torus in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ which is not Hamiltonian isotopic to the Clifford torus L_{std}.

6.1.1 The construction

We construct L_{CS} in $\mathbb{C}^2$ with the split symplectic form $\omega_0 = \omega_{std} \oplus \omega_{std}$, where $\omega_{std} = \frac{1}{\pi(1+x^2+y^2)^2} dx \wedge dy$ is the push-forward of ω_{std} under stereographic projection $\psi_N \colon S^2 \setminus N \to \mathbb{C}$ (see Appendix A) and show the relevant properties there. Then via $(\psi_N \times \psi_N)^{-1}$ we conclude the existence of L_{CS} in $(S^2 \times S^2, \omega_0)$.
Let $\gamma \colon S^1 \to \mathbb{C}$ be an embedded circle with image in the positive halfplane $x > 0$ (cf. fig.6.1) which encloses a region of symplectic area $\frac{1}{4}$ with respect to ω_{std}. Now we define

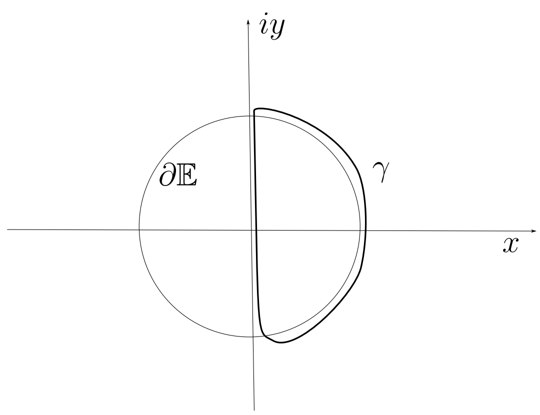

Figure 6.1: The embedded curve γ

$$\phi\colon S^1 \times S^1 \to \mathbb{C}^2$$

$$(t,\alpha) \mapsto (e^{i\alpha}\gamma(t), e^{-i\alpha}\gamma(t)).$$

This defines the Chekanov Schlenk Torus L_{CS} in $\mathbb{C}^2$.

6.1.2 Properties

Embedded

Obviously ϕ is smooth and $S^1 \times S^1$ is compact, thus is suffices to show that ϕ is an injective immersion.

The differential of ϕ at (t,α) can be written as

$$d\phi_{(t,\alpha)} = \begin{pmatrix} e^{i\alpha}\gamma'(t) & ie^{i\alpha}\gamma(t) \\ e^{-i\alpha}\gamma'(t) & -ie^{-i\alpha}\gamma(t) \end{pmatrix}.$$

It's determinant over $\mathbb{C}$ equals

$$\det d\phi_{(t,\alpha)} = -2i\gamma(t)\gamma'(t).$$

Since γ is embedded, $\gamma'(t)$ doesn't vanish. Further γ lies in the positive half-plane $\{x > 0\}$, so that $\gamma(t) \neq 0$. Hence it follows that ϕ is immersive.

To show injectivity, assume, that

$$\phi(t,\alpha) = \phi(t',\alpha'),$$

then we get two equations

$$e^{i\alpha}\gamma(t) = e^{i\alpha'}\gamma(t');\; e^{-i\alpha}\gamma(t) = e^{-i\alpha'}\gamma(t').$$

Multiplying both equations leads to

$$\gamma(t)^2 = \gamma(t')^2.$$

Writing $\gamma(t) = r(t)e^{i\theta(t)}$, this gives

$$r(t) = r(t')$$

and

$$2(\theta(t) - \theta(t')) = 2\pi k$$

for some $k \in \mathbb{Z}$. But since γ lies in the positive half plane, $\theta(t) - \theta(t') \in (-\pi, \pi)$ so that $k = 0$ and $\theta(t) = \theta(t')$. Hence

$$\gamma(t) = r(t)e^{i\theta(t)} = r(t')e^{i\theta(t')} = \gamma(t'),$$

so since γ is an embedding, it follows that $t = t'$. Then obviously $\alpha = \alpha'$ and ϕ is injective.
Lagrange
We have to show, that $\phi^*\omega_0$ vanishes. Therefore note first that

$$(\omega_{std})_z(\alpha, \beta) = \frac{1}{\pi(1+|z|^2)^2}\Im(\overline{\alpha}\beta).$$

Here $\Im(z)$ denotes the imaginary part of the complex number z.

Then

$$\phi^*\omega_0(\frac{\partial}{\partial t}, \frac{\partial}{\partial \alpha}) = (\omega_0)_{\phi(t,\alpha)}\left(d\phi_{(t,\alpha)}\frac{\partial}{\partial t}, d\phi_{(t,\alpha)}\frac{\partial}{\partial \alpha}\right).$$

But

$$d\phi_{(t,\alpha)}\left(\frac{\partial}{\partial t}\right) = (e^{i\alpha}\gamma'(t), e^{-i\alpha}\gamma'(t))$$

and

$$d\phi_{(t,\alpha)}\left(\frac{\partial}{\partial \alpha}\right) = (ie^{i\alpha}\gamma(t), -ie^{-i\alpha}\gamma(t)).$$

Thus

$$(\omega_0)_{\phi(t,\alpha)}\left(\left(e^{i\alpha}\gamma'(t), e^{-i\alpha}\gamma'(t)\right), \left(ie^{i\alpha}\gamma(t), -ie^{-i\alpha}\gamma(t)\right)\right) =$$

$$= (\omega_{std})_{(e^{i\alpha}\gamma(t))}\left(e^{i\alpha}\gamma'(t), ie^{i\alpha}\gamma(t)\right) + (\omega_{std})_{(e^{-i\alpha}\gamma(t))}\left(e^{-i\alpha}\gamma'(t), -ie^{-i\alpha}\gamma(t)\right).$$

So

$$(\phi^*\omega_0)_{(t,\alpha)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \alpha}\right) = \frac{1}{\pi(1+|\gamma(t)|^2)^2}\Im(i\overline{\gamma'(t)}\gamma(t)) + \frac{1}{\pi(1+|\gamma(t)|^2)^2}\Im(-i\overline{\gamma'(t)}\gamma(t)) = 0.$$

Monotonicity
Since we are in a vector space, calculating Maslov indices is easy, since we can use the ambient "trivialisation" of the tangent bundle and write down the loop of Lagrangians straight away. Similarly, by Stokes theorem, the symplectic area of every disk in $\mathbb{C}^2$ is equal to the integral of a primitive of ω_0 around the boundary. Since $\omega_0 = \omega_{std} \oplus \omega_{std}$ and ω_{std} on $\mathbb{C}$ is exact, there exists a primitive of ω_0 of the form $\lambda_0 = \lambda_{std} \oplus \lambda_{std}$.
We only have to check monotonicity for a pair of loops in L_{CS} spanning the homology. Hence as the first loop take $\delta_1(t) = (\gamma(t), \gamma(t))$ and as the second loop take $\delta_2(\alpha) = (e^{i\alpha}\gamma(0), e^{-i\alpha}\gamma(0))$. From the definition of the embedding ϕ it is clear that δ_1, δ_2 span $H_1(L_{CS}, \mathbb{Z})$.
We regard $\mathbb{C}^2$ with the standard complex- and hermitian structure (denoted i, h). Obviously

$$v(t,\alpha) := d\phi_{(t,\alpha)}\frac{\partial}{\partial t}, w(t,\alpha) := d\phi_{(t,\alpha)}\frac{\partial}{\partial \alpha}$$

span $T_{\phi(t,\alpha)}L_{CS}$ over $\mathbb{R}$; over $\mathbb{C}$, $v(t,\alpha), w(t,\alpha)$ span $\mathbb{C}^2$. Let the standard Lagrangian L^{std} be given by $\frac{\partial}{\partial x}, \frac{\partial}{\partial u}$ for standard coordinates $x+iy, u+iv$ on $\mathbb{C}^2$. Note that v, w are orthogonal with respect to h. So normalising them yields a unitary map $A(t,\alpha)$, which maps the standard Lagrangian L^{std} to $T_{\phi(t,\alpha)}L_{CS}$:

$$A(t,\alpha) = \begin{pmatrix} \frac{v(t,\alpha)}{|v(t,\alpha)|} & \frac{w(t,\alpha)}{|w(t,\alpha)|} \end{pmatrix}.$$

Consequently, by Lemma G.0.40, the Maslov index of δ_i is given by

$$\mu(\delta_i) = wind(\det A^2(\delta_i)).$$

But $|v(t,\alpha)| = |\gamma'(t)|$ and $|w(t,\alpha)| = |\gamma(t)|$, thus

$$A(\delta_1(t)) = \begin{pmatrix} \frac{\gamma'(t)}{|\gamma'(t)|} & i\frac{\gamma(t)}{|\gamma(t)|} \\ \\ \frac{\gamma'(t)}{|\gamma'(t)|} & -i\frac{\gamma(t)}{|\gamma(t)|} \end{pmatrix}.$$

$$A(\delta_2(\alpha)) = \begin{pmatrix} e^{i\alpha}\frac{\gamma'(0)}{|\gamma'(0)|} & ie^{i\alpha}\frac{\gamma(0)}{|\gamma(0)|} \\ \\ e^{-i\alpha}\frac{\gamma'(0)}{|\gamma'(0)|} & -ie^{-i\alpha}\frac{\gamma(0)}{|\gamma(0)|} \end{pmatrix}.$$

Hence

$$\det A^2(\delta_1(t)) = -\frac{\gamma'(t)^2}{|\gamma'(t)|^2}\frac{\gamma(t)^2}{|\gamma(t)|^2}$$

and

$$\det A^2(\delta_2(\alpha)) = -\frac{i\gamma'(0)^2\gamma(0)^2}{|\gamma'(0)^2\gamma(0)^2|}.$$

But then, since γ is embedded and lies in the positive half-plane, it follows that

$$\mu(\delta_1) = 2$$

(only γ' contributes once to the winding number) and

$$\mu(\delta_2(\alpha)) = 0$$

since the second loop is constant.

Now we have to calculate the symplectic area of disks spanned into δ_1, δ_2 or alternatively

$$\int_{\delta_1} \lambda_0$$

and

$$\int_{\delta_2} \lambda_0.$$

Thus

$$\int_{\delta_1} \lambda_0 = 2\int_{\gamma} \lambda_{std} = 2\frac{1}{4} = \frac{1}{2}.$$

On the other hand let $\gamma_1(\alpha) = e^{i\alpha}\gamma(0)$ and $\gamma_2(\alpha) = e^{-i\alpha}\gamma(0)$, then $\delta_2(\alpha) = (\gamma_1(\alpha), \gamma_2(\alpha))$. Then

$$\int_{\delta_2} \lambda_0 = \int_0^{2\pi} \lambda_0(\delta_2'(\alpha))d\alpha = \int_0^{2\pi} (\lambda_0)_{(\gamma_1(\alpha),\gamma_2(\alpha))}((\gamma_1'(\alpha), \gamma_2'(\alpha))d\alpha =$$
$$= \int_0^{2\pi} (\lambda_{std})_{\gamma_1(\alpha)}(\gamma_1'(\alpha))d\alpha + \int_0^{2\pi} (\lambda_{std})_{\gamma_2(\alpha)}(\gamma_2'(\alpha))d\alpha =$$
$$= \int_{\gamma_1} \lambda_{std} + \int_{\gamma_2} \lambda_{std}.$$

But $\gamma_2(\alpha) = e^{i(2\pi-\alpha)} = \gamma_1(2\pi - \alpha)$ is a reparametrisation with the opposite orientation, hence $\int_{\gamma_1} \lambda_{std} = -\int_{\gamma_2} \lambda_{std}$ and thus

$$\int_{\delta_2} \lambda_0 = 0.$$

Consequently, L_{CS} is monotone in $\mathbb{C}^2$ with monotonictiy constant $\frac{1}{4}$. Putting L_{CS} into $S^2 \times S^2$ via $(\psi_N \times \psi_N)^{-1}$ gives indeed an embedded, Lagrangian torus. Since $\int_{S^2} \omega_{std} = 1$ and $\mu(S^2 \times pt) = \mu(pt \times S^2) = 4$ by Theorem 2.3.4, it follows that L_{CS} is also monotone in $S^2 \times S^2$.

Theorem 6.1.1 (Chekanov-Schlenk,[28])**.** *L_{CS} and L_{std} are not Hamiltonian isotopic in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$.*

Remark

M.-L. Yau also proved this by completely different methods in [23].

6.2 A.Ivrii's result

We want to describe briefly the methods that A.Ivrii uses to prove that any Lagrangian torus in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ is fibered. The methods are based on Symplectic Field Theory as introduced by Eliashberg, Givental and Hofer in [30].

Definition 6.2.1. *A contact manifold (V, α) is a $2n-1$-dimensional manifold V, such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on V. The $2n-2$ dimensional distribution $\zeta := \ker \alpha$ is called the contact structure and the vector field X_R defined by*

$$\iota_{X_R} d\alpha = 0, \ \alpha(X_R) = 1,$$

is called the Reeb vector field.

Definition 6.2.2. *A contact type hypersurface $V \subset (M, \omega)$ is a contact manifold (V, α) such that $\omega = d(e^s\alpha)$ in a neighbourhood $N \cong (-\epsilon, \epsilon) \times V$ of V in M such that the vector field $\frac{\partial}{\partial s}$ is everywhere transversal to V.*

Definition 6.2.3. *Let* (V, α) *be a contact manifold, then the* symplectization *of* V *is defined to be the symplectic manifold* $(\mathbb{R} \times V, d(e^s\alpha))$*, where* s *denotes the coordinate along the* $\mathbb{R}$*-factor.*

Definition 6.2.4. *An almost complex structure* J *on the symplectization* $(\mathbb{R} \times V, d(e^s\alpha))$ *of* (V, α) *is called* cylindrical *if*

- J *is invariant under translation in the* $\mathbb{R}$*-direction;*
- $J\zeta = \zeta$*;*
- $J\frac{\partial}{\partial s} = X_R$.

By the Weinstein neighbourhood theorem, every Lagrangian submanifold $L \subset (X, \omega)$ has a neighbourhood U in X which is symplectomorphic to a neighbourhood of the zero-section in T^*L with the canonical symplectic form. The boundary $V := \partial D_r^*$ of the disk bundle $D_r^*L = \{(q,p) \in T^*L||p| \leq r\}$ is a contact type hypersurface. We obtain two symplectic manifolds (M_+, ω) and (M_-, ω), each with boundary V and $\frac{\partial}{\partial s}$ pointing outwards of M_- and inwards of M_+ with $M = M_+ \cup M_-$.
Now we define a family (M^τ, ω^τ) of symplectic manifolds by

$$(M^\tau, \omega^\tau) = (M_-, e^{-\tau}\omega) \cup (V \times [-\tau, \tau], d(e^s\alpha)) \cup (M_+, e^\tau\omega).$$

In the limit as $\tau \to \infty$, we can view (M^τ, ω^τ) as a decomposition of (M, ω) into the union of

$$M_-^\infty = M_- \cup ([0, \infty) \times V, d(e^s\alpha))$$

and

$$M_+^\infty = M_+ \cup ((-\infty, 0] \times V, d(e^s\alpha)).$$

In our case, M_+^∞ is symplectomorphic to $(M \setminus L, \omega)$ and M_-^∞ is symplectomorphic to $(T^*L, d\lambda_{can})$.
Start with an almost complex structure J on M, which is compatible with ω. Now it is possible to choose a family of compatible almost complex structures J^τ on (M^τ, ω^τ) such that $J^\tau = J$ on M_+ and M_- and J^τ is cylindrical on $V \times [-\tau, \tau]$. Note that (M^τ, ω^τ) is symplectomorphic to (M, ω) for all finite τ, so that we can actually assume only the almost complex structure J^τ to vary (this can also be done more directly, see [21]). This procedure is called a neck-stretch of J along V.
Now let $(M = S^2 \times S^2, \omega = \omega_{std} \oplus \omega_{std})$ and $L \subset M$ a Lagrangian torus, then for any J^τ (τ finite), there exists a J^τ-holomorphic foliation by spheres in the homology class $[pt \times S^2]$ ($[S^2 \times pt]$). Symplectic field theory provides a compactness theorem for J^τ-holomorphic maps in a splitting as above akin to the Gromov compactness theorem.
Ivrii examines how the J^τ-holomorphic foliations in class $[pt \times S^2]$ degenerate for $\tau \to \infty$ and deduces the existence of a limit foliation $\mathcal{F}$ with the following properties (here the properties are stated only for monotone tori! cf. Lemma 2.4.3):

- a S^1-family of leaves of $\mathcal{F}$ intersects L in an embedded circle (so these leaves break into two embedded disks along L).

- These families of disks together form two solid tori T_1, T_2 with $\partial T_1 = \partial T_2 = L$.

Observe, that the leaves of $\mathcal{F}$ are not necessarily smooth along L. A. Ivrii shows that the foliation $\mathcal{F}$ can be smoothened (see below how he does this) near L such that it remains symplectic with otherwise the same properties.

In Theorem 4.4.1, A. Ivrii shows the existence of a symplectic section Σ' of the foliation $\mathcal{F}$ in the homology class $[S^2 \times pt]$ which is disjoint from L. His proof proceeds as follows:

Let J_0 be a split almost complex structure on $S^2 \times S^2$ and consider the foliation by J_0-holomorphic spheres $S^2 \times pt$. Apply a neck-stretch to J_0 in a neighbourhood V of L in $S^2 \times S^2$. By A. Ivrii's result, the foliations $\mathcal{F}^\tau$ for J_0^τ in the neck-stretch converge to a J_0^∞-holomorphic foliation $\mathcal{F}^\infty$ by symplectic spheres in the homology class $[S^2 \times pt]$ such that a S^1-family of leaves intersects L. $\mathcal{F}^\infty$ is not smooth near the torus L. But following Ivrii (see Theorem 4.1.1), we can smoothen the S^1-family of leaves intersecting L in an arbitrary small neighbourhood $N \subset V$ of L such that they remain symplectic. Further, we can find a compatible almost complex structure J_1 which agrees with J_0^∞ outside N such that the smoothened spheres intersecting L become J_1-holomorphic (see Theorem 4.1.1 in [12]). Now consider the J_1-holomorphic foliation $\mathcal{F}_1$ of $S^2 \times S^2$ by symplectic spheres in the homology class $[S^2 \times pt]$. By construction, a S^1-family of the leaves of $\mathcal{F}_1$ intersects L, hence we can choose a leaf Σ' not intersecting L (the leaves of $\mathcal{F}_1$ are parametrised by a S^2).
Now apply a neck-stretch for J_1 along a neighbourhood V_1 of L which is disjoint from Σ'. Now we are interested in the degeneration of the J_1^τ-holomorphic foliations by spheres in class $[pt \times S^2]$. Again, by Ivrii's result, the foliations in the neck-stretch converge to a J_1^∞-holomorphic foliation $\mathcal{F}_1^\infty$ which is not smooth near L and such that a S^1-family of leaves of $\mathcal{F}_1^\infty$ intersects L. As in the previous step, we can smoothen these spheres in N_2 and find a almost complex structure J_2 which agrees with J_1^∞ outside N_2 and which makes the smoothened spheres J_2-holomorphic. Observe that J_1^∞ agrees with J_1 outside V_1, so that J_2 agrees with J_1 outside V_1. Consider the foliation $\mathcal{F}_2$ given by the J_2-holomorphic spheres in class $[S^2 \times pt]$. Since Σ' is still J_2-holomorphic, positivity of intersections implies that Σ' intersects the leaves of $\mathcal{F}_2$ uniquely and transversely. Hence we can regard Σ' as a section of $\mathcal{F}_2$. This proves the existence of the symplectic section Σ'.
Hence L is fibered by $\mathcal{F}$ and Σ'.

6.3 Relation to the Main Theorem and Outlook

Ivrii's results says that any monotone Lagrangian torus L in $S^2 \times S^2$ is fibered. Hence, there exists a foliation $\mathcal{F}$ and a section Σ' of $\mathcal{F}$ as in the previous section. If there exists a second section Σ meeting the requirements of the Main Theorem 2.5.1, then L is Hamiltonian isotopic to the standard torus L_{std}.

The Chekanov-Schlenk torus L_{CS} is monotone Lagrangian in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ but not Hamiltonian isotopic to L_{std} by Theorem 6.1.1, hence by Ivrii's result it is fibered by $\mathcal{F}$ and Σ' but there cannot exist the second symplectic section Σ.
This instantly rises the question whether the classification of monotone Lagrangian tori in $(S^2 \times S^2, \omega_{std} \oplus \omega_{std})$ up to Hamiltonian isotopy comes within reach if we understand the rôle of the second section Σ.

Appendix A

The standard form ω_{std} and stereographic projection

Consider $S^2 \subset \mathbb{R}^3$ with standard coordinates x, y, z as the submanifold of $\mathbb{R}^3$ given by the zero set of the function $f\colon \mathbb{R}^3 \to \mathbb{R};\ (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$. Let

$$\phi_N\colon S^2 \setminus \{N\} \to \mathbb{C}$$
$$(x, y, z) \mapsto \frac{x}{1-z} + i\frac{y}{1-z}$$

be stereographic projection from the northpole N in S^2 (stereographic projection from a different point $p \in S^2$ is obtained by precomposing ϕ_N by some element in $SO(3)$ which maps p to N). This is a diffeomorphism with inverse given by

$$\psi_N(u, v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

where $u + iv$ are the standard cartesian coordinates on $\mathbb{C}$. Consider the 2-form

$$\sigma = \frac{1}{4\pi}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dz)$$

on $\mathbb{R}^3$. Then if we denote the restriction of σ to S^2 by ω, we have

Lemma A.0.1.

$$\psi_N^* \omega = -\frac{1}{\pi(1+u^2+v^2)^2} du \wedge dv$$

Proof. Denote $\alpha = 1 + u^2 + v^2$ then

$$d\psi_N\left(\frac{\partial}{\partial u}\right) = \frac{1}{\alpha^2}\left(2(1-u^2+v^2)\frac{\partial}{\partial x} - 4uv\frac{\partial}{\partial y} + 4u\frac{\partial}{\partial z}\right)$$

$$d\psi_N\left(\frac{\partial}{\partial v}\right) = \frac{1}{\alpha^2}\left(-4uv\frac{\partial}{\partial x} + 2(1-v^2+u^2)\frac{\partial}{\partial y} + 4v\frac{\partial}{\partial z}\right).$$

Any 2-form β on $\mathbb{C}$ is of the form

$$\beta_z = f(z)du \wedge dv$$

for a smooth function $f\colon \mathbb{C} \to \mathbb{R}$. Hence we write $\psi_N^*\omega = f du \wedge dv$ and so

$$(\psi_N^*\omega)_{u,v}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = f(u+iv).$$

Then

$$\omega_{\psi_N(u,v)}\left(d\psi_N\left(\frac{\partial}{\partial u}\right), d\psi_N\left(\frac{\partial}{\partial v}\right)\right) =$$

$$= \frac{2u}{\alpha}dy \wedge dz - \frac{2v}{\alpha}dx \wedge dz + \frac{u^2+v^2-1}{\alpha}dx \wedge dy\left(d\psi_N\left(\frac{\partial}{\partial u}\right), d\psi_N\left(\frac{\partial}{\partial v}\right)\right) =$$

$$= -\frac{1}{\pi}\frac{\alpha^3}{\alpha^5} = -\frac{1}{\pi(1+u^2+v^2)}.$$

as claimed. □

Note that on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with polar coordinates $r, \theta \in (0,\infty) \times [0, 2\pi)$

$$\psi_N^*\omega = \frac{-1}{\pi(1+r^2)^2} r dr \wedge d\theta.$$

Lemma A.0.2. *The 2-form ω is a volume form on S^2 with total volume -1.*

Proof. This is Exercise 4.3.1 in [10]. We have to show that $\omega = \iota_{S^2}^*\sigma$ is closed, everywhere non-degenerate and that it integrates to -1 over S^2. It is obviously closed, since any 2-form on S^2 is closed for dimensional reasons. To show that it is non-degenerate on $S^2\setminus\{N\}$, note that since ψ_N is a diffeomorphism, this is equivalent to the function f in the proof of Lemma A.0.1 to be non-zero. This is obviously true. Using stereographic projection from the southpole, we can show similarly that the form ω is non-degenerate at N. This shows that ω is a volume form. We are left to show that it has total volume -1. Since the set $\{S, N\}$ has measure zero, we can consider polar coordinates $r, \theta \in (0,\infty) \times [0, 2\pi)$ on $\mathbb{C}^*$ and hence

$$\int_{S^2} \omega = \int_{\mathbb{C}^*} \psi_N^*\omega = \int_0^{2\pi}\int_0^\infty \frac{-1}{\pi(1+r^2)^2} r dr d\theta = \frac{-2\pi}{\pi}\int_0^\infty \frac{r}{(1+r^2)^2}dr.$$

With

$$\int_0^\infty \frac{r}{(1+r^2)^2}dr = \int_0^\infty \frac{d}{dr}\left(\frac{-1}{2(1+r^2)}\right)dr = \lim_{r\to\infty}\frac{-1}{2(1+r^2)} + \frac{1}{2} = \frac{1}{2}.$$

Hence as claimed $\int_{S^2}\omega = -1$. □

Now we define

Definition A.0.3. *The standard symplectic form ω_{std} on S^2 is defined to be*

$$\omega_{std} := -\omega.$$

Remark

The Fubini-Study form ω_{FS} on $\mathbb{CP}^1$ is defined in the standard chart $U_0 = \{z_0 \neq 0\}$ by the formula

$$\frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

Hence indeed $\omega_{std} = \omega_{FS}$ under the usual identification $\mathbb{CP}^1 \cong S^2$.

Appendix B

Proofs of results in chapter 2

The proof of Theorem 2.2.4

We will need the following lemma.

Lemma B.0.4. *Let X be a topological space and consider $[0,1] \subset \mathbb{R}$ with the subspace topology. Let U be an open set in X and let $\gamma\colon [0,1] \to X$ be a continuous path such that $\gamma(0) \in U$ and $\gamma(1) \in X \setminus U$. Then there exists a $t_0 \in (0,1]$ such that $\gamma(t_0) \in \partial U$.*

Proof. Recall the definition of ∂U for U an open set in X:

$$\partial U := \{x \in X \setminus U | \exists x_n \in U \ s.t. \ x_n \to x\}.$$

Consider the non-empty set $T = \gamma^{-1}U$. Since γ is continuous, T is open in $[0,1]$. Since $[0,1]$ is bounded $t_0 = sup(t \in T)$ exists. Now $t_0 \notin T$. To see this assume $t_0 \in T$, then either $t_0 = 1$ with contradicts the assumption that $\gamma(1) \notin U$ or $t_0 < 1$. But T is open, thus there exists an interval $(t_0 - \epsilon, t_0 + \epsilon) \subset T$ $((0, \epsilon))$ for ϵ small enough. But then $t_0 + \frac{\epsilon}{2} \in T$ contradicting that t_0 is the sup in T. Thus $t_0 \notin T$. There exists $t_n \to t_0$ in T, thus by continuity of γ, we have $\gamma(t_n) \to \gamma(t_0)$ with $\gamma(t_n) \in U$ and $\gamma(t_0) \notin U$. Hence $\gamma(t_0) \in \partial U$. This proves the lemma. □

Theorem (2.2.4)**.** *Let $\mathcal{F}_0$ be a k-dimensional foliation of the n-dimensional manifold X. Assume that there exists an embedding $G\colon U \times F \to M$ with U open in $\mathbb{R}^{n-k}$ and F a closed k-dimensional manifold such that G maps $\{x\} \times F$ diffeomorphically onto a leaf of $\mathcal{F}_0$ for all $x \in U$. Let $V \subset W \subset U$ be open sets in $\mathbb{R}^{n-k}$ with $\bar{V} \subset W$ and $\bar{W} \subset U$ and such that W is path-connected. Assume that*

$$G_s\colon U \times F \to M$$

for $s \in \mathbb{R}$ is a smooth family of embeddings such that $G_s|_{(U \setminus \bar{V}) \times F} = G|_{(U \setminus \bar{V}) \times F}$. Then the embeddings G_s define a smooth family of foliations $\mathcal{F}_s$ on M.

Proof. Note first, that all the embeddings G_s are local diffeomorphisms for dimensional reasons. Thus $Y := G(U \times F)$ is an open set of X. We want to show that $G_s(U \times F) = Y$ for all $s \in \mathbb{R}$.
Assume the opposite and let $Z = G(W \times F)$. Then there exists a $s \in [0,1]$ and a point $x \in V \times F$ such that $G_s(x) \notin Y$. Since W is path-connected and $V \subset W$, there exists a continuous path $\beta\colon [0,1] \to W \times F$ such that $\beta(0) \in (W \setminus \bar{V}) \times F$ and $\beta(1) = x$. Thus $\gamma(t) = G_s(\beta(t))$ is a continuous path in X such that $\gamma(0) \in Z$ and $\gamma(1) \notin Z$. By Lemma B.0.4, there exists a $t_0 \in (0,1]$ such that $\gamma(t_0) \in \partial Z$. But $\partial Z \subset (Y \setminus Z)$ and there $G_s = G$ so that

$$\beta(t_0) = G^{-1}G_s(\beta(t_0)) = G^{-1}\gamma(t_0).$$

Since a diffeomorphism ϕ satisfies $\partial\phi = \phi\partial$ it follows that

$$\beta(t_0) \in \partial(W \times F).$$

Hence $\beta(t_0) \notin W \times F$ which contradicts the definition of β and indeed $G_s(U \times F) = G(U \times F)$ for all s.
Consider the map

$$\bar{G}\colon U \times \mathbb{R} \to X \times \mathbb{R};\ (x,s) \mapsto (G_s(x), s).$$

This is an embedding since it is a diffeomorphism onto $Y \times \mathbb{R}$. Indeed, by the discussion above,

$$\tilde{G}\colon Y \times \mathbb{R} \to U \times \mathbb{R}$$

$$(y,s) \to (G_s^{-1}(y), s)$$

is a smooth inverse.

Consider the closed set $P = G(\bar{V} \times F)$ in X. First we want to show how to obtain foliating charts from an embedding $G\colon U \times F \to X$ which has the property that $\{x\} \times F$ is mapped by G onto a leaf of $\mathcal{F}_0$. Let $\left\{\psi_\beta\colon W'_\beta \to W_\beta\right\}_{\beta \in B}$ be an atlas for F. Since $G(\{x\} \times F)$ is a leaf of $\mathcal{F}_0$

$$\delta_\beta = G \circ (Id \times \psi_\beta)\colon U \times W'_\beta \to G(U \times W_\beta) \subset X$$

is a foliating chart for $\mathcal{F}_0$. Indeed, any two such $\delta_{\beta_1}, \delta_{\beta_2}$, satisfy the foliation condition in Definition 2.2.1 for the foliation $\mathcal{F}_0|_Y$ on Y.

Now use the embeddings G_s and the fact that they agree on $(U \setminus \bar{V}) \times F$ to define foliating charts for all s in order to get foliations $\mathcal{F}_s$ on X for every s. Define δ^s_β as δ_β but using G_s instead of G.
Let $\{\phi_\alpha\colon U'_\alpha \to U_\alpha\}_{\alpha \in A}$ be a foliating atlas for $\mathcal{F}_0$. Define a new foliating atlas by restricting the old atlas to $\tilde{U}_\alpha = U_\alpha \cap (X \setminus P)$, $\tilde{\phi}_\alpha = \phi_\alpha|_{\tilde{U}_\alpha}$. And choosing as new foliating charts on Y,

$$(U \times W'_\beta, \delta^s_\beta)$$

where $U \times W'_\beta \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$ is open. Since $G = G_s$ on $Y - P$, there the old and the new foliating charts match up, and on Y, the charts δ^s_β form foliating charts. This foliating atlas defines a foliation $\mathcal{F}_s$ on X. Clearly the leaves of $\mathcal{F}_s$ are the images $G_s(\{x\} \times F)$ for $x \in U$ and agree with the old leaves outside of P.

We are left to show that the foliations $\mathcal{F}_s$ form a smooth family. First note that for a constant family $\mathcal{F}_t = \mathcal{F}_0$ of foliations, the foliating charts of the required foliation $\mathcal{F}$ on $X \times \mathbb{R}$ are given by $(\phi_\alpha \times id, U_\alpha \times \mathbb{R})$. But the map $\bar{G}$ is an embedding, so the construction above shows precisly how to obtain the foliating charts for the required foliation $\mathcal{F}$ on $X \times \mathbb{R}$: Restrict the charts for the constant foliation as above and use the embedding $\bar{G}$ to obtain foliating charts on the remaining open set.

The leaves of the foliation $\mathcal{F}$ on $X \times \mathbb{R}$ are given by

$$\mathcal{F}_{x,t} = \begin{cases} (\mathcal{F}_0)_x \times \mathbb{R} \text{ for } x \notin Z \\ \bar{G}(\{x\} \times F \times \mathbb{R}) \text{ for } x \in Y \end{cases}$$

This proves the theorem. □

The proof of Theorem 2.2.5

Theorem B.0.5. *A foliation whose leaves are all compact and simply connected is simple. Moreover if the ambient manifold X is connected, then the projection to the leaf-space $p\colon X \to X/\sim$ provides X with the structure of a smooth fiber bundle.*

Proof. This is Corollary 8.6 on p. 92 in [11]. Note the difference in the statement. Simply connectedness of the leaves, however, implies trivial leaf holonomy. Compare Definition 7.6 on p.86 in [11]. □

Definition B.0.6. *A cover $\{U_\alpha\}_{\alpha \in A}$ of a smooth n-manifold X by open sets is called a good cover, if all non-empty finite intersections*

$$U_{\alpha_1} \cap U_{\alpha_2} \cap \ldots \cap U_{\alpha_n}$$

for any $n \in \mathbb{N}$ and $\alpha_i \in A$ are diffeomorphic to $\mathbb{R}^n$.

Theorem B.0.7. *Every compact smooth manifold N has a finite good cover.*

Proof. This is Theorem 5.1 in [10] on p.42. □

Proposition B.0.8. *If $\pi\colon X \to B$ is a fiber bundle with fiber F with finite-dimensional cohomology and B admits a finite good cover then $\chi(X) = \chi(F)\chi(B)$*

Proof. This is Exercise 14.37 on p.182 in [10]. □

Theorem B.0.9 (3.11)**.** *Two connected compact surfaces are diffeomorphic if and only if they have the same Euler characteristic and the same number of boundary components, and both are orientable or both are non-orientable.*

Proof. [7], page 207. □

Theorem (2.2.5)**.** *Let $\mathcal{F}$ be a foliation of (M,ω) by symplectic 2-spheres. Further let Σ be a submanifold of M which is transverse to $\mathcal{F}_q$ for all $q \in M$. Then Σ is diffeomorphic to S^2, Σ intersects every leaf of $\mathcal{F}$ in a single point and the map*

$$\pi \colon M \to \Sigma$$

$$q \in \mathcal{F}_q \mapsto \mathcal{F}_q \cap \Sigma$$

is a surjective submersion. Moreover there exist diffeomorphisms $\phi \colon M \to S^2 \times S^2$ and $u \colon \Sigma \to S^2$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & S^2 \times S^2 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p_1} \\ \Sigma & \xrightarrow{u} & S^2 \end{array} \tag{B.1}$$

Proof. Since the leaves are compact, symplectic and simply connected, by Theorem B.0.5 $p \colon M \to M/\sim$ is a smooth symplectic fibration. Consequently, the leaf-space $\mathcal{B}$ is a closed orientable 2-manifold: p is an open map so that all points in $\mathcal{B}$ are interior points and $\mathcal{B}$ has no boundary. An open cover of $\mathcal{B}$ lifts to an open cover of M which by compactness has a finite subcover. The projection of this subcover by p is the required finite subcover of $\mathcal{B}$. Thus $\mathcal{B}$ is a closed manifold. Finally (p,ω) induces a symplectic connection on M. Two linearly independent vectors v, w in H_x the horizontal space with respect to the symplectic connection are positively oriented if $\omega(v,w) > 0$. Since $dp_x \colon H_x \to T_{p(x)}\mathcal{B}$ is an isomorphism, we can use this to put a orientation on $T_{p(x)}\mathcal{B}$. Clearly this definition is well-defined. Hence $\mathcal{B}$ is closed and orientable.
Since $\mathcal{B}$ is compact, by Theorem B.0.7, $\mathcal{B}$ admits a finite good cover. Clearly the homology of the fibers is finite dimensional, thus by Proposition B.0.8, it follows that the Euler-Characteristics of the spaces involved satisfy

$$\chi(M) = \chi(S^2)\chi(\mathcal{B}).$$

Thus $\chi(\mathcal{B}) = 2$ and by Theorem B.0.9, $\mathcal{B}$ is diffeomorphic to S^2. Let $u \colon \mathcal{B} \to S^2$ be a diffeomorphism then $u \circ p \colon M \to S^2$ is a S^2-bundle over S^2. But there are only two such bundles, the trivial one and a non-trivial one. Note that the intersection forms of the total spaces of the two S^2-bundles differ. But M is diffeomorphic to $S^2 \times S^2$ which is the trivial S^2-bundle over S^2. Hence it has the intersection form of the trivial S^2-bundle and consequently $u \circ p$ is the trivial S^2-bundle over S^2. Hence there exists a trivialisation:

$$\begin{array}{ccc} M & \xrightarrow{\tau} & S^2 \times S^2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} \\ \mathcal{B} & \xrightarrow{u} & S^2 \end{array}$$

Now push Σ forward under τ. Then $\tau(\Sigma)$ is transverse to $\{q\} \times S^2$ for all $q \in S^2$. By Proposition 3.2.6, this implies that $\tau(\Sigma)$ is the image of a section σ of p_1. But then $\sigma' = \tau^{-1} \circ \sigma \circ u$ defines a section of p with image Σ and $\pi = \sigma' \circ p$. From this it is clear that π is a smooth surjective submersion and that Σ is diffeomorphic to S^2. The existence of the trivialisation can be deduced as above for the space of leaves $\mathcal{B}$. This proves the theorem. ☐

The proof of theorem 2.3.4

Consider $S^2 \subset \mathbb{R}^3$ in the standard way and let $D_e = \{(x, y, z) \in S^2 | x \geq 0\}$ and $D_w := \{(x, y, z) \in S^2 | x \leq 0\}$ be the closed eastern and the western hemispheres in S^2. Denote by $C = D_e \cap D_w = \{x = 0\}$ the meridian of longitude 0.
In the proof of the theorem, we need a function $\phi \colon S^2 \to S^2$ with degree 1 which collapses the closed eastern hemisphere D_e to z_0. Therefore, let $\rho \colon \mathbb{R} \to \mathbb{R}$ be a smooth decreasing function which is equal to 1 for $r \leq 1$ and 0 for $r > 1 + \epsilon$. Now consider the family of functions $\psi_t(r) := 1 - t\rho(r)$. Via stereographic projection from $-z_0$ we identify $S^2 \setminus \{-z_0\}$ with $\mathbb{C}$ and consider the family of functions $\bar{\phi}_t \colon \mathbb{C} \to \mathbb{C};\ z \mapsto \psi_t(|z|)z$. By construction $\bar{\phi}_t(z) = z$ for $|z| > 1 + \epsilon$, so that we can extend $\bar{\phi}_t$ by the identity to maps ϕ_t of S^2. Note that ϕ_1 collapses D_e to z_0 so we define $\phi := \phi_1$. Since $\phi_0 = id$, ϕ_t is a homotopy from id to ϕ. This shows that id and ϕ have degree 1 as maps from S^2 to S^2.
For completeness, we will now recall the definition of the first chern number of a complex vector bundle as defined in [13] on p. 74:

Theorem (2.69 in [13])**.** *There exists a unique functor c_1, called the **first chern number**, that assigns an integer $c_1(E) \in \mathbb{Z}$ to every symplectic vector bundle E over a compact oriented Riemann surface Σ without boundary and satisfies the following axioms.*

- ***(naturality)*** *Two symplectic vector bundles E and E' over Σ are isomorphic iff they have the same dimension and the same Chern number.*
- ***(functorality)*** *For any smooth map $\phi \colon \Sigma' \to \Sigma$ of oriented Riemann surfaces and any symplectic vector bundle $E \to \Sigma$*

$$c_1(\phi^* E) = deg(\phi) \cdot c_1(E).$$

- ***(additivity)*** *For two symplectic vector bundles $E_j \to \Sigma$ of rank n_j:*

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2),\ c_1(E_1 \otimes E_2) = n_2 c_1(E_1) + n_1 c_1(E_2).$$

- ***(normalization)*** *The chern number of the tangent bundle of Σ is*

$$c_1(T\Sigma) = 2 - 2g$$

  where g is the genus.

We will only consider complex vector bundles, which are in particular symplectic. Let $E \to S^2$ be a complex 1-dimensional vector bundle, then there exist trivialisations over D_e, D_w. The first chern number of E is then given by the degree of the transition map $\Psi\colon S^1 \cong C = D_e \cap D_w \to S^1 = U(1)$ (cf. the discussion on p.75 in [13]).

Theorem (2.3.4)**.** *Let $f\colon (S^2, z_0) \to (M, x_0)$ be an embedding with $f(z_0) = x_0 \in L$ with trivial normal bundle, then $\mu(u) = 4$ for $u = f \circ \phi\colon \mathbb{E} \to M$. In the definition of u, via stereographic projection from z_0, we have identified $\mathbb{E} \subset \mathbb{C}$ with the western hemisphere in S^2.*

Proof. We have

$$f^*TM \cong TN \oplus \nu_f$$

where ν_f denotes the normal bundle of f in M. By assumption ν_f is trivial.
Note that any real two dimensional vector bundle over an orientable surface can be regarded as a complex vector bundle. This follows, since any metric on a 2-dimensional bundle gives rise to a complex structure by counter-clockwise rotation by $\frac{\pi}{2}$. Obviously the Whitney-sum of two complex vector bundles is also a complex vector-bundle.
Now let c_1 denote the first chern number as defined above. By construction $\bar{u}\colon S^2 \to M;\ z \mapsto f \circ \phi(z)$ has image $f(S^2)$ as well and

$$c_1(\bar{u}^*TM) = c_1((f \circ \phi)^*TM) = c_1(\phi^* f^*TM) = c_1(\phi^*(TS^2 \oplus \nu_f)) =$$

$$= deg(\phi) c_1(TS^2 \oplus \nu_f) = c_1(TS^2) = 2.$$

Further it follows that $\bar{u}^*TM$ also decomposes as a Whitney sum $(\bar{u}^*TS^2) \oplus \bar{u}^*\nu_f$. Since $\bar{u}^*\nu_f$ is trivial, it follows that $c_1(\bar{u}^*TS^2) = 2$.
The chern number of a complex 1-dimensional vector bundle over S^2 is the degree of the transition map, hence, the complex two dimensional vector-bundle over S^2, given by $\bar{u}^*TS^2 \oplus \bar{u}^*\nu_f$ has a transition function of the form

$$\delta\colon C \to U(2)$$

$$t \mapsto \begin{pmatrix} e^{i\theta_1(t)} & 0 \\ 0 & e^{i\theta_2(t)} \end{pmatrix}$$

where $\theta_1(2\pi) - \theta_1(0) = 4\pi$ and $\theta_2(2\pi) - \theta_2(0) = 0$.
Now note that $u := \bar{u}|_{D_w}$ can be regarded as a map from $\mathbb{E}$ to M with image $f(S^2)$ and which collapses the boundary of $\partial\mathbb{E}$ to x_0 in M. So we can calculate the Maslov index of u with the definition given in Chapter 2. But note that by construction, the Lagrangians L_z in $\bar{u}^*TM$ over points $z \in D_e$ are constant ($L_z = L^{std}$), due to the fact that $\bar{u}$ collapses D_e to x_0. Thus, if we trivialise the complex vector bundle with respect to u, then the loop of Lagrangians along the boundary $\partial\mathbb{E}$ is precisely the image under the transition

map δ of the standard Lagrangian L^{std}. Hence by Lemma G.0.40, the Maslov index of u is given by the degree of the map

$$t \mapsto det\left(\delta(t)^2\right) = det\left(\begin{pmatrix} e^{i\theta_1(t)} & 0 \\ 0 & e^{i\theta_2(t)} \end{pmatrix}^2\right) = e^{i2(\theta_1(t)+\theta_2(t))}$$

Thus

$$\mu(u) = 2\frac{\theta_1(2\pi) - \theta_1(0))}{2\pi} + 2\frac{\theta_2(2\pi) - \theta_2(0))}{2\pi} = 4.$$

This proves the theorem. $\square$

Appendix C

Proofs of results in chapter 3

A detailed proof of Proposition 3.1.2

In the proof, we need the Jordan curve theorem and an extension of the Riemann mapping theorem to the boundary(in case the boundary is nice). Thus we quote:

Theorem C.0.10 (Jordan curve theorem)**.** *Let H be a simple closed curve in $\mathbb{C}$. Then*

- *$\mathbb{C} \setminus H$ has exactly two connected components one of which is bounded and simply connected (called the interior) and the other one is unbounded (the exterior);*
- *The boundary of every component of $\mathbb{C} \setminus H$ is H;*
- *If $\gamma\colon [0,1] \to \mathbb{C}$ is a simple loop with $\gamma([0,1]) = H$ then $wind(\gamma, x) = 0$ if x lies in the unbounded component of $\mathbb{C} \setminus H$ and $wind(\gamma, x) = \pm 1$ if x lies in the bounded component.*

Proof. see [19] Chapter IX, App. 4.2, page 256 for the statement and its proof and [20] for the proof of the additional statement that the interior is simply connected. □

Theorem C.0.11 (Painlevé/Warschawski)**.** *Let G be a proper (not all of $\mathbb{C}$) simply connected domain in $\mathbb{C}$ such that the boundary curve can be parametrised by a smooth simple closed curve, then there exists a diffeomorphism*

$$\phi\colon \overline{G} \to \overline{D}(0,1)$$

between the closure of G and the closed unit disk.

Proof. The proof is given in [17]. □

Furthermore we need the following theorem about the isotopy type of embeddings of the closed disk into a smooth manifold:

Theorem C.0.12. *Let X be a connected n-manifold and $f, g\colon \overline{D}^k \to X$ embeddings of the closed k-disk, $0 \le k \le n$. If $k = n$ and X is orientable, assume that f, g both preserve, or both reverse, orientation. Then f and g are isotopic. If $f(\overline{D}^k) \cup g(\overline{D}^k) \subset X \setminus \partial X$, an isotopy between them can be realized by a diffeotopy of X having compact support.*

Proof. The statement and the proof can be found in [7], Chapter 8.3, Isotopies of Disks, on page 185. □

Proposition (3.1.2)**.** *Let $L \subset S^2 \times S^2$ be a torus fibered by p_1. Then there exists a diffeomorphism τ of $S^2 \times S^2$ such that τ preserves the standard fibration p_1 and such that $\tau^{-1}(L)$ is conveniently fibered by p_1.*

Proof. Since $\gamma_L = p_1(L)$ has a tubular neighbourhood, there exists a point a in the complement of γ_L in the base. On the base, consider stereographic projection σ^a from a. Then $\sigma^a(\gamma_L)$ is a closed embedded curve in $\mathbb{C}$ (so in particular it is a simple closed curve). Let G be the interior of $\sigma^a(\gamma_L)$. Then by the Jordan curve theorem, G is simply connected but not all of $\mathbb{C}$ and its boundary curve is the simple closed curve $\sigma^a(\gamma_L)$. So by Theorem C.0.11, there exists a diffeomorphism $\phi\colon \overline{G} \to \overline{D}(0,1)$. Then

$$\beta_1 := \iota \circ (\sigma^a)^{-1} \circ \phi^{-1} \colon \overline{D}(0,1) \to S^2$$

$$\beta_0 := \iota \circ (\sigma^N)^{-1} \colon \overline{D}(0,1) \to S^2$$

are two embeddings which satisfy the conditions of Theorem C.0.12 for $X = S^2$. Then there exists a diffeomorphism h of S^2 which satisfies $h \circ \beta_0 = \beta_1$ (the time-1 map of the diffeotopy given by the theorem). Clearly h maps the equator to γ_L by construction. Consider the diffeomorphism $H := h \times id$ of $S^2 \times S^2$.
Analogously we achieve the second part (but now on the other factor) of being conveniently fibered. But it is slightly easier because L being fibered implies already that $F \cap T = V$ is diffeomorphic to a closed disk (so that we don't have to refer to the Jordan curve theorem). Let the diffeomorphism obtained be denoted by $\tilde{H}$.
Now we can define the desired diffeomorphism τ by $\tau := \tilde{H} \circ H$. Since τ respects the product structure of $S^2 \times S^2$, it preserves the standard fibration p_1 and by construction $\tau^{-1}(L)$ is conveniently fibered by p_1. This proves the proposition. □

The proofs of the various little results

Lemma (3.2.4)**.** *There exists a 1-form $\sigma \in \Omega^1(U)$ defined on a neighbourhood of Q such that*

$$\omega - \omega_0 = d\sigma$$

and $\sigma_x = 0$ for all $x \in Q$

Proof. Compare Lemma 3.14 page 94 in [13]. Let $\tau := \omega - \omega_0$. Outline: Choose a Riemannian metric g (a product metric) on M and let exp be the exponential map with respect to g. Let

$$U_\epsilon := \left\{ v \in TQ^{\perp_g} | g(v,v) < \epsilon \right\}.$$

Choosing ϵ small enough (let $\tilde{Q} := E \times S^2$ and choose ϵ such that it works for the compact submanifold $\tilde{Q}$) then

$$exp\colon U_\epsilon \to M$$

is an embedding with $exp(U_\epsilon) = \mathcal{N}_0$. Define $\phi_t\colon \mathcal{N}_0 \to \mathcal{N}_0$ by $\phi_t(exp_q(v)) = exp_q(tv)$. Then ϕ_t is an embedding for all $t > 0$ with $\phi_0(\mathcal{N}_0) \subset Q$, $\phi_1 = id$ and $\phi_t|_Q = id$. Since τ vanishes on Q it follows that $\phi_0^*\tau = 0$.
Let $v_t(\phi_t(x))$ denote the tangent vector to the curve $s \mapsto \phi_s(x)$ at $s = t$. Define a 1-form

$$\sigma := \int_0^1 \phi_t^* \iota_{v_t} \tau dt$$

(One has to prove that the family of forms $\sigma_t := \phi_t^* \iota_{v_t} \tau$ is smooth in t).
Then

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt} \phi_t^* \tau dt =$$

$$= \int_0^1 \phi_t^*(L_{v_t}\tau) dt = \int_0^1 \phi_t^*(\iota_{v_t} d\tau + d\iota_{v_t}\tau) dt = d\sigma.$$

Since τ vanishes on Q so does σ_t and hence also σ. Let $U := \mathcal{N}_0$. $\square$

Let F^λ denote the leaf of the foliation $\hat{\tau}^{-1}(\mathcal{F}_0)$ through the point (λ, N). This is a closed submanifold of $S^2 \times S^2$. Note that $F^\lambda = \{\lambda\} \times S^2$ for λ real and that $F^\lambda = \hat{\tau}^{-1}(\{q(\lambda)\} \times S^2)$ for $q(\lambda) = p_1 \hat{\tau}(\lambda, N)$.

Lemma C.0.13. *dq_0 is an isomorphism.*

Proof. Note that $d(\hat{\tau})_{0,N}$ is a very special isomorphism. If v_1, v_2, w_1, w_2 is a basis of $T_{0,N}M$ with v_1, v_2 spanning $T_{0,N}\, \{0\} \times S^2$ the tangent space to the fiber and w_1, w_2 spanning $T_{0,N}\mathbb{C} \times \{N\}$ the standard orthogonal complement such that w_1 is along the real axis, then

$$d(\hat{\tau})_{0,N}(v_i) = v_i;\ d(\hat{\tau})_{0,N}(w_1) = w_1$$

and $d(\hat{\tau})_{0,N}(w_2)$ is linearly independent of v_1, v_2, w_1. Consequently

$$dq_0(w_1) = d(p_1)_{0,N}(d(\hat{\tau})_{0,N} w_1) = w_1;\ dq_0(w_2) = d(p_1)_{0,N}(d(\hat{\tau})_{0,N}(w_2)) = \lambda w_1 + \mu w_2$$

with $\mu \neq 0$. Otherwise $d(\hat{\tau})_{0,N}(w_2)$ lies in the span of v_1, v_2, w_1 and thus would not be linearly independent. This shows that dq_0 has rank two and is an isomorphism. $\square$

Lemma C.0.14. *G is an embedding (by shrinking δ if necessary).*

Proof. Since $\hat{\tau}$ is a diffeomorphism it follows that G is an embedding if and only if $\hat{\tau} \circ G$ is an embedding. By construction, $G|_{\{\lambda\} \times S^2}$ is a diffeomorphism onto F^λ, but $\hat{\tau}^{-1}|_{\{q(\lambda)\} \times S^2}$ is also a diffeomorphism onto F^λ. Thus we can write

$$\hat{\tau} \circ G(\lambda, w) = (q(\lambda), \phi^\lambda(w))$$

with $\phi^\lambda \in Diff^+(S^2)$. But by Lemma C.0.13, dq_0 is an isomorphism and hence a local diffeomorphism, clearly $d\phi^\lambda$ is also an isomorphism. Since

$$d\hat{\tau} \circ G = \begin{pmatrix} dq & 0 \\ \\ \frac{\partial \phi^\lambda}{\partial \lambda} & d\phi^\lambda \end{pmatrix}$$

it follows that $d(\hat{\tau} \circ G)_{0,w}$ is an isomorphism for all $w \in S^2$. Consequently by compactness of S^2 after shrinking $\delta > 0$ if necessary $\hat{\tau} \circ G \colon D(0,\delta) \times S^2 \to \mathbb{C} \times S^2$ is an embedding. $\square$

Proposition (3.2.7)**.** *For every $1 > \delta > 0$, $\alpha > 0$ there exists a smooth family of non-decreasing functions $\phi_s \colon [0,\infty) \to [0,1], s \in [0,1]$ satisfying*

$$0 \le r\phi_s'(r) + \phi_s(r) < \frac{1}{1-\delta} \tag{C.1}$$

such that $\phi_s(r) = s$ for $r \le \frac{\alpha}{2}$, $\phi_s(r) = 1$ for $r \ge \frac{5\alpha}{\delta}$ and $\phi_1 \equiv 1$.

Proof. For $r > 0$ define $\chi(r)$ by $\chi(r) = r\phi(r)$ then $\chi' = r\phi' + \phi$, so that the condition on ϕ is equivalent to

$$0 \le \chi'(r) < \frac{1}{1-\delta}.$$

This will be the case if χ solves the differential equation

$$\chi'(r) = \frac{1}{1 - \frac{\delta}{4}}$$

then

$$\chi(r) = \frac{r}{1-\frac{\delta}{4}} + c$$

for some constant c. We fix the constant to

$$c = \frac{-\alpha}{1-\frac{\delta}{4}}$$

by requiring that $\chi(\alpha) = 0$.
Then

$$\chi(r) = \frac{1}{1-\frac{\delta}{4}}(r-\alpha).$$

Observe that $\chi(\frac{4\alpha}{\delta}) = \frac{4\alpha}{\delta}$.
Now note, that $\phi(r) = 0$ for $r \le \frac{\alpha}{2}$ is equivalent to $\chi(r) = 0$ for $r \le \frac{\alpha}{2}$ and $\phi(r) = 1$ for $r \ge \frac{5\alpha}{\delta}$ is equivalent to $\chi(r) = r$ for $r \ge \frac{5\alpha}{\delta}$. Consider the continuous function

$$\bar{\theta}(r) = \begin{cases} max(0, \chi(r)) \text{ for } r \le \frac{4\alpha}{\delta} \\ r \text{ for } r \ge \frac{4\alpha}{\delta} \end{cases} .$$

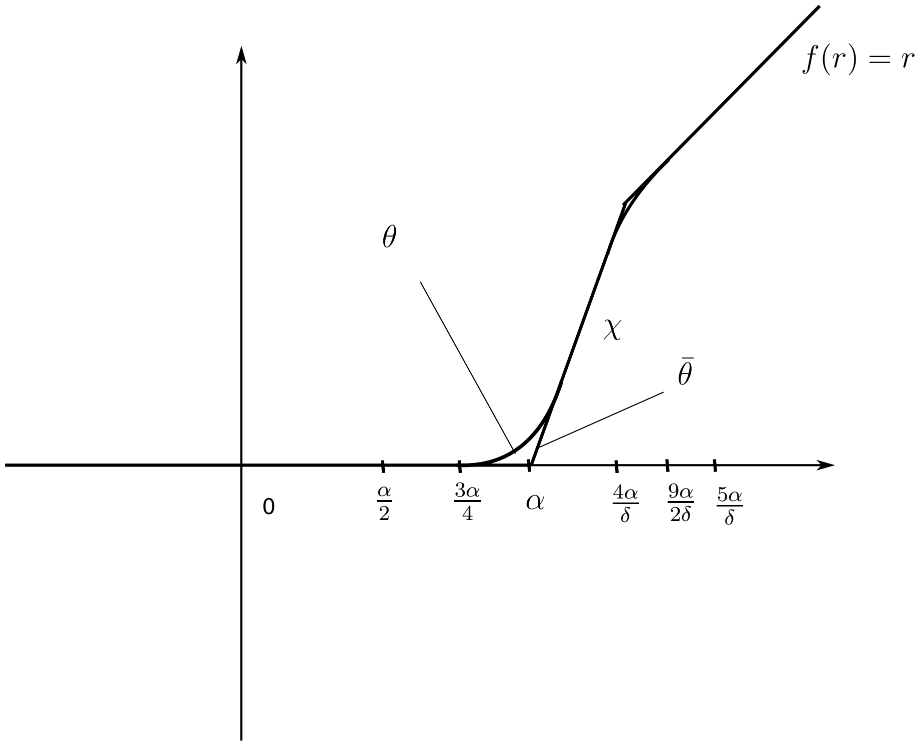

Figure C.1: The functions $\bar{\theta}$ and θ

Let ρ be a smooth non-negative bump-function on $\mathbb{R}$ with support in $[-\epsilon, \epsilon]$, such that

$$\int_{-\infty}^{\infty} \rho(r)dr = 1.$$

Let $\epsilon < \min\left\{\frac{\alpha}{4}, \frac{\alpha}{2\delta}\right\}$. Then the convolution

$$\theta(r) = \rho * \bar{\theta}(r) = \int_{-\infty}^{\infty} \rho(s)\bar{\theta}(r-s)ds$$

is a non decreasing smooth function which satisfies

$$\theta'(r) \leq \frac{1}{1-\frac{\delta}{4}}.$$

Further $\theta(r) = 0$ for $r \leq \frac{3\alpha}{4}$ and $\theta(r) = r$ for $r \geq \frac{5\alpha}{\delta}$. Thus $\phi_0 = \frac{1}{r}\theta(r)$ has all the required properties. Now let

$$\phi_s := \rho * \max(s, \phi_0).$$

This proves the proposition. $\square$

Proposition (3.2.8)**.** *There exists a constant $\mathcal{C} > 0$ such that for all $\epsilon > 0$ there exists a smooth family of functions $\phi_s^\epsilon\colon [0,\infty) \to [0,1]$, $s \in [0,1]$ such that*

- $\phi_s^\epsilon(r) = s$ *for all* $r \leq \frac{\epsilon}{20}$
- $\phi_s^\epsilon(r) = 1$ *for all* $r \geq \epsilon$
- $\max_{r\in[0,\infty)} |\phi_s'^\epsilon(r)| \leq \frac{1}{\epsilon}\mathcal{C}$

for all s.

Proof. Let $\delta = \frac{1}{2}$ and $\alpha = \frac{\epsilon}{10}$. Then by Proposition 3.2.7, there exists a smooth, non negative family of functions $\phi_s \colon [0,\infty) \to [0,1]$ such that $\phi_s'(r)r + \phi_s(r) < 2$ and such that $\phi_s(r) = s$ for $r \leq \frac{\epsilon}{20}$. But then $\phi_s'(r) = 0$ for $r \leq \frac{\epsilon}{20}$. So that

$$\frac{\epsilon}{20}\phi_s'(r) \leq r\phi_s'(r)$$

for all r. Together with ϕ_s being non negative, it follows that

$$\frac{\epsilon}{20}\phi_s'(r) \leq \phi_s'(r)r + \phi_s(r) < 2.$$

Thus $\phi_s'(r) < \frac{40}{\epsilon}$, $\phi_s(r) = s$ for $r \leq \frac{\epsilon}{20}$ and $\phi_s(r) = 1$ for $r \geq \frac{5\alpha}{\delta} = \epsilon$. Let $\phi_s^\epsilon := \phi_s$ and $\mathcal{C} := 40$. This proves the Lemma. □

In the proof of the next lemma we use the following facts from linear algebra for block matrices:

Proposition C.0.15. *1.* $\begin{pmatrix} A & B \\ 0 & Id \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} AC + BD \\ D \end{pmatrix}$

2. $\det \begin{pmatrix} A & B \\ 0 & Id \end{pmatrix} = \det A$

3. If $\begin{pmatrix} A & B \\ 0 & Id \end{pmatrix}$ *is invertible then* $\begin{pmatrix} A & B \\ 0 & Id \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & Id \end{pmatrix}$

In the proof of the transversality of F_s^λ to Σ, we have

Lemma C.0.16.

$$d(p_1 \circ G_s^{-1} \circ \sigma) = A_s^{-1}\left(A + (B - B_s) \circ dg\right)$$

Proof. $ds = \begin{pmatrix} Id \\ dg \end{pmatrix}$, hence by the Proposition C.0.15

$$d\sigma = \begin{pmatrix} A & B \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id \\ dg \end{pmatrix} = \begin{pmatrix} A + B \circ dg \\ dg \end{pmatrix}.$$

Further

$$(dG_s)^{-1} = \begin{pmatrix} A_s^{-1} & -A_s^{-1}B_s \\ & \\ 0 & Id \end{pmatrix}$$

and thus

$$(dG_s)^{-1}d\sigma = \begin{pmatrix} A_s^{-1}(A + (B - B_s) \circ dg) \\ \\ dg \end{pmatrix}$$

by Proposition C.0.15. Hence

$$dp_1 \circ dG_s^{-1} \circ d\sigma = A_s^{-1}\left(A + (B - B_s) \circ dg\right)$$

□

Lemma (3.2.12)**.** *Given symplectic forms ω, ω' on $\mathbb{C}$, then there exists a diffeomorphism ϕ of $\mathbb{C}$ with compact support which*

- *fixes the origin*
- *is the identity on the real line*
- *$\phi^*\omega = \omega'$ on U a neighbourhood of the origin*

Proof. We write $\omega = f dx \wedge dy$ and $\omega' = f' dx \wedge dy$ for positive functions $f, f' \colon \mathbb{C} \to \mathbb{R}$. We do this by an Moser argument for the linear interpolation of forms

$$\omega_t = ((1-t)f + tf')dx \wedge dy.$$

Obviously this is a smooth family of symplectic forms on $\mathbb{C}$ with $\tau = \frac{\partial \omega_t}{\partial t} = (f' - f)dx \wedge dy$. Consider the 1-form

$$\sigma_{x,y} = \left(\int_0^y f(x,s) - f'(x,s)ds \right) dx.$$

Then

$$d\sigma_{x,y} = (f'(x,y) - f(x,y))dx \wedge dy = \tau.$$

Moreover the vector fields X_t defined by

$$\iota_{X_t}\omega_t = \sigma$$

are given by

$$X_t(x,y) = -\frac{\int_0^y (f(x,s) - f'(x,s))ds}{(1-t)f(x,y) + tf'(x,y)} \frac{\partial}{\partial y}.$$

But since $\sigma_{x,0} = \int_0^0 (f(x,s) - f'(x,s))dsdx = 0$, it follows that $X_t(x,0) = 0$ for all t. Consequently, the flow ψ_t of the time-dependent vector field X_t on $\mathbb{C}$ is the identity on the real line.

Choose a non negative cut-off function $\rho_\epsilon \colon \mathbb{R} \to \mathbb{R}$ which has support in $[-2\epsilon, 2\epsilon]$ and which satisfies $\rho(r) = 1$ for $r \in [-\epsilon, \epsilon]$. Now consider the time-dependent vector field $Y_t(x, y) = \rho_\epsilon(\sqrt{x^2 + y^2})X_t(x, y)$ on $\mathbb{C}$ and denote its flow by ϕ_t. Then ϕ_t has compact support and preserves the real line pointwise. In particular it preserves the origin for all t, so that any neighbourhood of 0 is mapped to a neighbourhood of 0. By compactness of $[0, 1]$ there exists a disk $D(0, \mu)$ such that $\phi_t(D(0, \mu)) \subset D(0, \epsilon)$ for all t. But there, $X_t = Y_t$, hence the flow lines of ϕ_t and ψ_t starting at points in $D(0, \mu)$ coincide. Consequently $\phi_1(z) = \psi_1(z)$ for all $z \in D(0, \mu)$. It follows that $\phi_1^*\omega' = \omega$ on $D(0, \mu)$. Choosing $\phi := \phi_1$ and $U := D(0, \mu)$ proves the lemma. □

Remark

By choosing a suitable cut-off function in the proof above, one can also assure that $\phi^*\omega = \omega'$ on a prescribed open set $0 \in U$ and $\phi = id$ on V an open set satisfying $\bar{U} \subset V$.

Lemma (3.3.1). *Let ω be any symplectic form on S^2 and let $p \in S^2$ be a point in the open upper hemi-sphere D_{uh}. Then there exists a $\phi \in Symp_0(S^2, \omega)$ such that $\phi(N) = p$ with support in D_{uh}.*

Proof. Via stereographic projection from S, we identify $S^2 \setminus \{S\}$ with $\mathbb{C}$ and consider the push-forward symplectic form. Choosing coordinates x, y suitably on $\mathbb{C}$, we can assume without loss of generality, that $p = (x_0, 0) \in \mathbb{R}$ with $0 \leq x_0 < 1$. Then the push-forward symplectic form can be written as $f dx \wedge dy$ for a positive function f on $\mathbb{C}$. Let ϵ be so small that $1 - \epsilon > x_0$ and let $\rho \colon \mathbb{R}^+ \to [0, 1]$ be a non increasing cut-off function such that $\rho(r) = 1$ for $r \leq 1 - \epsilon$ and $\rho(r) = 0$ for $r > 1 - \frac{\epsilon}{2}$. Now define a function $H(x, y) = \rho(\sqrt{x^2 + y^2})y$. Then the Hamiltonian vector field X defined by $\iota_X(f dx \wedge dy) = dH$ equals $X = \frac{1}{f}\frac{\partial}{\partial x}$ in $D(0, 1-\epsilon)$. It particular it will never vanish there. It follows that ϕ_t, the Hamiltonian flow of H, will map 0 to p for a sufficiently big time T. Hence $\phi_T(0) = p$ with $supp(\phi) \subset \mathbb{E}$. Going back to S^2 via stereographic projection, and extending ϕ by the identity to all of S^2, we found the desired symplectomorphism $\phi \in Symp(S^2, \omega)$ □

Lemma C.0.17. *For any null-homotopic smooth map $f \colon (S^2, z_0) \to (S^2, N)$ for which there exists a neighbourhood U of z_0 with $f(U) = N$, there exists a smooth contraction $f_t \colon (S^2, z_0) \to (S^2, N)$ with $f_t(U') = N$ for a possibly smaller neighbourhood $U' \subset U$ of z_0 with $f_0 \equiv N$ and $f_1 = f$.*

Proof. Since f is null-homotopic, there exists a continuous family $h_t \colon (S^2, z_0) \to (S^2, N)$ with $t \in [0, 1]$ such that $h_1 = f$; $h_0 \equiv N$; $h_t(z_0) = N$ for all t. By smooth approximations, we may assume that h_t is actually a smooth family through smooth maps, by this we mean that the map $H \colon S^2 \times I \to S^2$; $(x, t) \mapsto h_t(x)$ is smooth.

We construct a smooth map $g \colon S^2 \to S^2$, which maps a neighbourhood U' of z_0, with $\bar{U}' \subset U$ to z_0, and which has support in $U \subset f^{-1}(N)$. Then the smooth family $f_t = h_t \circ g$ has the property that:

$$f_1 = h_1 \circ g = h_1 = f; \quad f_t(U') = N; \quad f_0 \equiv N$$

as claimed. To construct g, consider stereographic projection σ^{-z_0} from $-z_0$. Then $\widetilde{U} = \sigma^{-z_0}(U)$ is a neighbourhood of the origin. Choose an $\epsilon > 0$ such that the disk $D(0, 2\epsilon) \subset \widetilde{U}$. Let $\rho \colon \mathbb{R} \to \mathbb{R}$ be a smooth non decreasing bumpfunction, such that $\rho(r) = 0$ for $r \leq \epsilon$ and $\rho(r) = 1$ for $r \geq 2\epsilon$. Define $\bar{g} \colon \mathbb{C} \to \mathbb{C}$ by

$$\bar{g}(z) = \rho(|z|)z.$$

By construction $\bar{g}$ is smooth, has support in $\widetilde{U}$ and maps $D(0, \epsilon)$ to 0. Extend $\bar{g}$ to $-z_0$ by the identity then the resulting smooth map from S^2 to S^2 is the desired map g. □

Lemma (3.3.4)**.** *Let $\tau \in \Omega^2_c(\mathbb{R}^2)$ be closed, with support in $D(0, 1)$, and such that*

$$\int_{\mathbb{R}^2} \tau = 0.$$

Then there exists a canonical choice of $\sigma \in \Omega^1_c(\mathbb{R}^2)$ such that $d\sigma = \tau$.

Proof. We will do this by altering the (non-compactly supported !) primitive obtained from the Poincare Lemma to one with compact support.
Let x, y be standard coordinates on $\mathbb{C}$ and write $\tau = f dx \wedge dy$ for a function $f \colon \mathbb{R}^2 \to \mathbb{R}$ with support in $D := \{(x, y) \in \mathbb{R}^2 || (x, y)| < 1\}$.
Then from the Poincare Lemma $\eta \in \Omega^1(\mathbb{R}^2)$ defined by

$$\eta_{x,y} = \int_0^1 t f(tx, ty) dt (x dy - y dx)$$

is a primitive of τ.

Away from the origin, in polar coordinates $re^{i\theta}$ we can write $dx \wedge dy = r dr \wedge d\theta$. Now define a function $g \colon S^1 \to \mathbb{R}$ by

$$g(e^{i\theta}) = \int_0^1 t f(te^{i\theta}) dt.$$

Then

$$0 = \int_{\mathbb{R}^2} f dx \wedge dy = \int_0^{2\pi} \int_0^{\infty} r f(re^{i\theta}) dr d\theta =$$

$$= \int_0^{2\pi} \int_0^1 (t f(te^{i\theta})) dt d\theta = \int_0^{2\pi} g(e^{i\theta}) d\theta = \int_{S^1} g d\theta$$

it follows that $g d\theta \in \Omega^1(S^1)$ defines

$$[g d\theta] = 0 \in H^1(S^1).$$

Consequently there exists a function $h \colon S^1 \to \mathbb{R}$ with $dh = g d\theta$. If, however, $h' \colon S^1 \to \mathbb{R}$ is another such function, then $d(h - h') = 0$, which means that the function $h - h'$ is locally constant, and by connectedness of S^1, globally constant. Thus $h' = h + c$ for some constant $c \in \mathbb{R}$. Obviously $d(h + c) = g d\theta$ for any constant c. So there exists a unique function $h\colon S^1 \to \mathbb{R}$ which satisfies $dh = gd\theta$ and $h(1) = 0$. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth, once and for all fixed, non decreasing cut-off function which is equal to 1 for $r > 1 - \epsilon$ and zero for $r < \epsilon$. Now consider the 1-form

$$\sigma := \eta - d(\rho h).$$

We claim that this is the desired compactly supported 1-form.
Trivially $d\sigma = \tau$. On the other hand $\sigma = \eta - h\rho' dr - \rho dh$. So that for $r > 1$,

$$\sigma = \eta - gd\theta.$$

But now, for $|x| = r > 1$, since

$$d\theta = \frac{xdy - ydx}{r^2},$$

we have

$$\eta_{x,y} = \int_0^1 tf(tx, ty)dtr^2 d\theta = \int_0^1 tf(tre^{i\theta})dtr^2 d\theta =$$
$$= \int_0^r \frac{s}{r} f(se^{i\theta})\frac{ds}{r} r^2 d\theta = \int_0^1 sf(se^{i\theta})dsd\theta = gd\theta.$$

We have made the substitution $s = tr$ in the third equality. Hence it follows indeed, that up to the choice of ρ, $\sigma \in \Omega^1_c$ is a canonical primitive as desired. This proves the lemma. $\square$

Appendix D

Proofs of results in chapter 4

Proofs of the various little results

Lemma (4.2.1)**.** *Let $\phi \in Symp(A, \partial A, \omega_{std})$ then there exists a smooth function $F\colon A \to \mathbb{R}$ which is constant in a neighbourhood of the boundary (not necessarily the same constant near the two boundary components !) such that*

$$\phi^*\lambda_{std} - \lambda_{std} = dF.$$

Proof. The lemma is actually true for any closed 1-form $\alpha \in \Omega^1(A)$ which vanishes identically in a neighbourhood U of the boundary. Write $\alpha = \lambda dr + \mu d\theta$ for $\lambda, \mu\colon A \to \mathbb{R}$ smooth functions which vanish on U.

$$F\colon A \to \mathbb{R}$$
$$(r, \theta) \mapsto \int_{\frac{1}{2}}^{r} \lambda(t, \theta)dt$$

is smooth and satisfies $dF = \alpha$. Indeed

$$\frac{\partial F}{\partial r}(r, \theta) = \lambda(r, \theta)$$

and since α is closed

$$\frac{\partial \mu}{\partial r}(r, \theta) = \frac{\partial \lambda}{\partial \theta}(r, \theta)$$

so that

$$\frac{\partial F}{\partial \theta}(r, \theta) = \int_{\frac{1}{2}}^{r} \frac{\partial \lambda}{\partial \theta}(t, \theta)dt = \int_{\frac{1}{2}}^{r} \frac{\partial \mu}{\partial r}(t, \theta)dt = \mu(r, \theta) - \mu(\frac{1}{2}, \theta) = \mu(r, \theta)$$

since $\mu(\frac{1}{2}, \theta) = 0$.
Since $\phi = id$ in a neighbourhood of ∂A, the 1-form $\phi^*\lambda_{std} - \lambda_{std}$ vanishes near the boundary and since ϕ^λ is symplectic it follows that $d(\phi^*\lambda_{std} - \lambda_{std}) = \phi^* d\lambda_{std} - d\lambda_{std} = 0$. $\square$

Lemma (4.2.3)**.** *If $Flux^\lambda(\phi)$ denotes the Flux of $\phi \in Symp_0(A, \partial A, \omega_{std})$, defined with respect to λ with $d\lambda = \omega_{std}$, instead of λ_{std}, then*

$$Flux(\phi) = Flux^\lambda(\phi).$$

Proof. Let $\gamma\colon [0,1] \to A; t \mapsto ((1-t)\frac{1}{2}+2t, 0)$ be the straight line element in $\mathbb{R}_+ \cap A$ which connects the two boundary components, then by the Fundamental theorem of Calculus

$$Flux(\phi) = \int_\gamma \phi^*\lambda_{std} - \lambda_{std}$$

and

$$Flux^\lambda(\phi) = \int_\gamma \phi^*\lambda - \lambda.$$

Since $\phi(2) = 2$ and $\phi(\frac{1}{2}) = \frac{1}{2}$, we can form a loop $\sigma_\phi = \phi(\gamma) * \bar{\gamma}$ where $*$ means concatenation of paths and $\bar{\gamma}(s) = \gamma(1-s)$. Then

$$Flux(\phi) = \int_{\sigma_\phi} \lambda_{std}; \qquad Flux^\lambda(\phi) = \int_{\sigma_\phi} \lambda.$$

Thus

$$Flux^\lambda(\phi) - Flux(\phi) = \int_{\sigma_\phi} \lambda - \lambda_{std}.$$

But

$$d(\lambda - \lambda_{std}) = \omega_{std} - \omega_{std} = 0,$$

and $\beta := \lambda - \lambda_{std}$ defines a cohomology class in A. Since $\phi \in Symp_0(A, \partial A, \omega_{std})$, the identity component of $Symp(A, \partial A, \omega_{std})$, the loop σ_ϕ is null-homotopic, and consequently

$$\int_{\sigma_\phi} \beta = 0,$$

showing that

$$Flux^\lambda(\phi) = Flux(\phi)$$

as claimed. □

Lemma (4.2.4)**.** *Let $\phi, \psi \in Symp_0(A, \partial A, \omega_{std})$ then $Flux(\phi \circ \psi) = Flux(\phi) + Flux(\psi)$ and $Flux(Id) = 0$.*

Proof. Let $F_\phi, F_\psi\colon A \to \mathbb{R}$ be the functions such that

$$\phi^*\lambda_{std} - \lambda_{std} = dF_\phi$$

and

$$\psi^*\lambda_{std} - \lambda_{std} = dF_\psi.$$

Then

$$(\phi \circ \psi)^* \lambda_{std} - \lambda_{std} = \psi^*(\phi^* \lambda_{std} - \lambda_{std}) + \psi^* \lambda_{std} - \lambda_{std} = \psi^* dF_\phi + dF_\psi =$$
$$= d(F_\phi \circ \psi + F_\psi).$$

Hence

$$Flux(\phi \circ \psi) = F_\phi(\psi(2)) + F_\psi(2) - (F_\phi(\psi(\frac{1}{2})) + F_\psi(\frac{1}{2})) =$$
$$= F_\phi(2) - F_\phi(\frac{1}{2}) + F_\psi(2) - F_\psi(\frac{1}{2}) = Flux(\phi) + Flux(\psi).$$

Since $\psi(2) = 2;\ \psi(\frac{1}{2}) = \frac{1}{2}$. The statement for $Flux(Id) = 0$ is clear. $\square$

Lemma (4.2.6)**.** *Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Let ϕ_t be a symplectic isotopy starting at $\phi_0 = id$. Let ϕ_t be generated by the time-dependent vector field X_t, i.e*

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Then $\iota_{X_t}\omega = dH_t$ for a smooth family of functions $H_t\colon M \to \mathbb{R}$ if and only if $\phi_t^*\lambda - \lambda = dF_t$ for a smooth family of functions $F_t\colon M \to \mathbb{R}$. Moreover F_t and H_t are related by the equations

$$F_t = \int_0^t (H_s + \iota_{X_s}\lambda) \circ \phi_s ds$$
$$H_s = \left(\frac{d}{dt} \mid_{t=s} F_t\right) \circ \phi_s^{-1} - \iota_{X_s}\lambda.$$

Proof. Assume first that $\iota_{X_s}\omega = dH_s$ then we have:

$$\phi_t^*\lambda - \lambda = \int_0^t \frac{d}{du} \mid_{u=s} \phi_u^*\lambda ds = \int_0^t \phi_s^* L_{X_s}\lambda ds =$$
$$= \int_0^t \phi_s^*(\iota_{X_s} d\lambda + d\iota_{X_s}\lambda) ds = d\int_0^t (H_s + \iota_{X_s}\lambda) \circ \phi_s ds.$$

For $F_t = \int_0^t (H_s + \iota_{X_s}\lambda) \circ \phi_s ds$ this shows that $\phi_t^*\lambda - \lambda = dF_t$ and the relation stated above. Conversely if $\phi_t^*\lambda - \lambda = dF_t$ then we differentiate the equation to obtain

$$\frac{d}{dt} \mid_{t=s} \phi_t^*\lambda = d\frac{d}{dt} \mid_{t=s} F_t$$

since partial derivatives commute. Then

$$\phi_s^*(L_{X_s}\lambda) = d\frac{d}{dt} \mid_{t=s} F_t$$
$$d\iota_{X_s}\lambda + \iota_{X_s} d\lambda = d\left(\frac{d}{dt} \mid_{t=s} F_t\right) \circ \phi_s^{-1}$$

and hence

$$H_s = \left(\frac{d}{dt} \mid_{t=s} F_t\right) \circ \phi_s^{-1} - \iota_{X_s}\lambda$$

as claimed. $\square$

In order to get a canonical choice for a symplectomorphism with given Flux a, we put the following bump-function before the statement of the lemma: $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth, non decreasing function which equals 0 for $r < \frac{3}{4}$ and 1 for $r > \frac{5}{4}$.

Lemma (4.2.7)**.** *Given any real number a, there exists a canonical symplectomorphism $\phi^a \in Symp_0(A, \partial A, \omega_{std})$ such that*

$$Flux(\phi^a) = a.$$

Further ϕ^a depends smoothly on a.

Proof. Consider the smooth function

$$H_a\colon A \to \mathbb{R}$$

$$(r, \theta) \mapsto a\rho(r).$$

Define a vector field X on A by the equation $\iota_X \omega_{std} = dH_a$. Let ϕ_t denote the flow of X. By Lemma 4.2.1 there exists a function F with $\phi_1^* \lambda - \lambda = dF$ where $F = \int_0^1 (H + \iota_X \lambda) \circ \phi_t dt$. Since $\phi_t(2) = 2$, $\phi_t(\frac{1}{2}) = \frac{1}{2}$ and X vanishes near ∂A we have

$$Flux(\phi_1) = F(2) - F(\frac{1}{2}) = \int_0^1 a\rho(2)ds - \int_0^1 a\rho(\frac{1}{2})ds = \int_0^1 ads = a.$$

Define $\phi^a := \phi_1$. This proves the lemma. Obviously $\phi^0 = id$ and ϕ^a depends smoothly on a.

□

The proofs of the results used in Theorem 4.3.1

The proof of Lemma 3.4.2

Before we can start the proof of Lemma 3.4.2 we have to prove a couple of propositions. Recall that in Chapter 4 we assume for simplicity that

$$A = \left\{ z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 2 \right\}$$

with symplectic form

$$\omega_{std} = \frac{r}{\pi(1 + r^2)^2} dr \wedge d\theta.$$

We abbreviate: $f(r) = \frac{r}{\pi(1+r^2)^2}$. Let $E = \{|z| = 1\} \subset A$ be called the equator.

Proposition D.0.18. *Given $\phi \in Ham(A, \partial A, \omega_{std})$ with $\phi(E) = E$. Then there exists a smooth path $\phi_t \in Ham(A, \partial A, \omega_{std})$ which connects ϕ to $\phi' \in Ham(A, \partial A, \omega_{std})$ such that $\phi'(e) = e$ for all $e \in E$ and such that $\phi_t(E) = E$ for all t.*

Proof. We restrict ϕ to E and denote $\psi := (\phi|_E)^{-1}$. Then ψ is a orientation preserving diffeomorphism of S^1. Since $Diff^+(S^1)$ is path-connected (see [15], Corollary 2.7.B), there exists a smooth path $\psi_t \in Diff^+(S^1)$ with $\psi_0 = id$ and $\psi_1 = \psi$. Let Y_t be the time-dependent vector field on S^1, defined by

$$Y_t \circ \psi_t = \frac{\partial \psi_t}{\partial t}.$$

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function with support in $[\frac{3}{4}, \frac{5}{4}]$, $\rho'(1) = 1$ and $\rho(1) = 0$. Consider on A the smooth family of functions

$$H_t(r, \theta) = -f(r)\rho(r)Y_t(\theta).$$

Then

$$\frac{\partial H_t}{\partial r}(1, \theta) = Y_t(\theta);\ \frac{\partial H_t}{\partial \theta}(1, \theta) = 0.$$

Let X_t be the time-dependent vector field defined by $\iota_{X_t}\omega_{std} = dH_t$ on A, and let its flow be ϕ_t. Then by construction $\phi_t = \psi_t$ on E and thus in particular preserves E. By Proposition 4.2.6, and the fact that the family of functions H_t has support away from the boundary, it follows that ϕ_t is a Hamiltonian isotopy. Then $\phi \circ \phi_t \in Ham(A, \partial A, \omega_{std})$ is a smooth path which starts at ϕ and ends at a Hamiltonian symplectomorphism ϕ' which is the identity on E. This proves the proposition. ☐

To prove the next proposition, we need the following two little Lemmata: By [18] page 15, we have

Lemma D.0.19. *If $f(t, q)$ is a smooth function on $I_\epsilon \times M$ where I_ϵ is an open interval $(-\epsilon, \epsilon)$, such that $f(0, q) = 0$ for all $q \in M$ then there exists a smooth function $g(t, q)$ on $I_\epsilon \times M$ such that*

$$f(t, q) = g(t, q)t.$$

Moreover $g(0, q) = f'(0, q)$, where $f' = \frac{\partial f}{\partial t}$, for $q \in M$

Proof. It is sufficient to define $g(t, q) := \int_0^1 f'(ts, q)ds$. ☐

Lemma D.0.20. *Let W be a neighbourhood of the zero-section in T^*S^1 and let $\alpha \in \Omega^1(W)$ be a closed 1-form which vanishes on the zero-section, then there exists a function $F\colon W \to \mathbb{R}$ with $\alpha = dF$ which also vanishes on the zero-section.*

Proof. In the standard coordinates q, p on T^*S^1 we can write $\alpha = adq + bdp$. Then

$$\frac{\partial a}{\partial p}(q, p) = \frac{\partial b}{\partial q}(q, p)$$

on W, since α is closed, and $a(q, 0) = b(q, 0) = 0$, since α vanishes on the zero-section. Now it suffices to define

$$F(q, p) = \int_0^p b(q, s)ds,$$

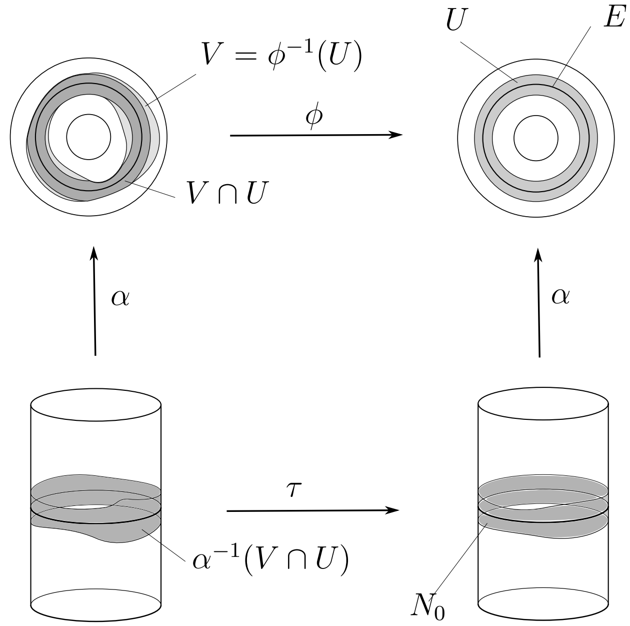

Figure D.1: The construction of τ

since $dF = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial p} dp$ with

$$\frac{\partial F}{\partial p}(q,p) = \frac{d}{dp}\int_0^p b(q,s)ds = b(q,p)$$

and

$$\frac{\partial F}{\partial q}(q,p) = \int_0^p \frac{\partial b}{\partial q}(q,s)ds = \int_0^p \frac{\partial a}{\partial p}(q,s)ds = a(q,p) - a(q,0) = a(q,p).$$

Thus $dF = \alpha$ and $F(q,0) = \int_0^0 b(q,s)ds = 0$ as claimed. ☐

Proposition D.0.21. *Given $\phi \in Ham(A, \partial A, \omega_{std})$ with $\phi|_E = id$, then there exists a smooth path $\phi_t \in Ham(A, \partial A, \omega_{std})$ and a neighbourhood U of E such that $\phi_0 = \phi$, $\phi_1|_U = id$ and $\phi_t|_E = id$ for all t.*

Proof. Since $E \subset A$ is a Lagrangian submanifold, by the Weinstein neighbourhood theorem, there exists a symplectomorphism $\alpha\colon (N_0, d\lambda_{can}) \to (U, \omega_{std})$ between N_0 a neighbourhood of the zero-section in T^*S^1 and U a neighbourhood of $E \subset A$. For simplicity, we assume that $N_0 = D_{\epsilon_0} = \{(q,p) \in T^*S^1 | |p| < \epsilon_0\}$.

Let $V := \phi^{-1}U$ then we can define (see figure D)

$$\tau\colon \alpha^{-1}(U \cap V) \to N_0$$

by

$$\tau = \alpha^{-1} \circ \phi \circ \alpha.$$

Let $(q, p) \in S^1 \times \mathbb{R}$ be global coordinates on T^*S^1 and write

$$\tau(q, p) = (Q(q, p), P(q, p)).$$

From τ being the identity on the zero-section, it follows that its differential at a point $(q, 0)$ on the zero-section is of the form

$$d\tau_{(q,0)} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ & \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} 1 & * \\ & \\ 0 & * \end{pmatrix}.$$

But τ is also a symplectomorphism hence $\tau^* d\lambda_{can} = d\lambda_{can}$ which implies that $\frac{\partial P}{\partial p} = 1$. Now apply Lemma D.0.19 to the function $P\colon S^1 \times (-\epsilon_1, \epsilon_1) \to \mathbb{R}$ with $P(q, 0) = 0$ and $D_{\epsilon_1} \subset \alpha^{-1}(U \cap V)$.

We obtain a smooth function $\tilde{P}\colon S^1 \times (-\epsilon_1, \epsilon_1) \to \mathbb{R}$ such that $P(q, p) = p\tilde{P}(q, p)$ and $\tilde{P}(q, 0) = \frac{\partial P}{\partial p}(q, 0) = 1$.
Since $\tilde{P}$ is continuous and $\tilde{P}(0, q) = 1$, for every $q \in S^1$ there exists an $\epsilon_q > 0$ such that $|\tilde{P}(q, p) - 1| < \frac{1}{2}$ for all $|p| < \epsilon_q$ and by compactness of S^1 there exists a $\epsilon_2 > 0$ such that $|\tilde{P}(q, p) - 1| < \frac{1}{2}$ for all $|p| < \epsilon_2$ and all q.
Further let $0 < \epsilon_3 < \min\left\{\frac{2}{3}\epsilon_0, \epsilon_2\right\}$ then

$$|P(q, p)| = |p\tilde{P}(q, p)| = |p||\tilde{P}(q, p)| < \frac{2}{3}\epsilon_0(1 + \frac{1}{2}) = \epsilon_0$$

on D_{ϵ_3}.
Let M_s be fiberwise multiplication by the real number s:

$$M_s\colon T^*S^1 \to T^*S^1;\ (q, p) \mapsto (q, sp).$$

We define a smooth isotopy

$$\tau_s\colon D_{\epsilon_3} \to N_0$$
$$(q, p) \mapsto M_{\frac{1}{s}} \circ \tau \circ M_s(q, p) = (Q(q, sp), \frac{1}{s}P(q, sp)) = (Q(q, sp), p\tilde{P}(q, sp)).$$

The last equality comes from the equation $P(q, sp) = (sp)\tilde{P}(q, sp)$. Because of the first defining expression and the fact that the M_s are diffeomorphisms for all $s \neq 0$, it follows that the τ_s are embeddings. Because of the choice of ϵ_3, it follows that the images all land in N_0 and by the last expression, it follows that the family is smooth in s.
Observe that by construction

$$\tau_s(q, 0) = (Q(q, 0), \frac{1}{s}P(q, 0)) = (q, 0)$$

is the identity on the zero-section for all s and

$$\tau_0(q, p) = (Q(q, 0), p\tilde{P}(q, 0)) = (q, p)$$

is the identity.
Since τ is symplectic, the 1-form $\tau^*\lambda_{can} - \lambda_{can}$ is closed. Since τ is the identity on the zero-section and $\lambda_{can} = \frac{1}{2}p^2 dq$, it follows that $\tau^*\lambda_{can} - \lambda_{can}$ vanishes on the zero-section. So by Lemma D.0.20, there exists a function $F\colon D_{\epsilon_3} \to \mathbb{R}$ which vanishes on the zero-section with $dF = \tau^*\lambda_{can} - \lambda_{can}$.
But then for $s \neq 0$

$$\tau_s^*\lambda\text{can} - \lambda\text{can} = M_s^*\tau^* M_{\frac{1}{s}}^*\lambda\text{can} - \lambda\text{can} = M_s^*(\tau^* M_{\frac{1}{s}}^*\lambda\text{can} - M_{\frac{1}{s}}^*\lambda\text{can}).$$

The last equation follows from the fact that M_s is a diffeomorphism with inverse $M_{\frac{1}{s}}$ so that $M_s^* \circ M_{\frac{1}{s}}^* = id$. But $dM_s(\frac{\partial}{\partial q}) = \frac{\partial}{\partial q}$; $dM_s(\frac{\partial}{\partial p}) = s\frac{\partial}{\partial p}$ thus

$$(M_s^*\lambda\text{can})_{(q,p)}(\frac{\partial}{\partial q}) = \frac{1}{2}(sp)^2 = s^2\lambda\text{can}_{(q,p)}(\frac{\partial}{\partial q}).$$

Hence

$$M_s^*\lambda\text{can} = s^2\lambda\text{can}$$

and so

$$\tau_s^*\lambda\text{can} - \lambda\text{can} = M_s^*(\frac{1}{s^2}(\tau^*\lambda\text{can} - \lambda\text{can})) = \frac{1}{s^2}M_s^* dF = d\frac{1}{s^2}F \circ M_s.$$

We show that the family of functions $F_s(q,p) := \frac{1}{s^2}F \circ M_s(q,p)$ is smooth in s. By construction, F vanishes on the zero-section and furthermore since $\tau^*\lambda\text{can} - \lambda\text{can} = dF$ and $\tau^*\lambda\text{can} - \lambda\text{can}$ vanishes on the zero-section, it follows that dF also vanishes on the zero-section. In particular $\frac{\partial F}{\partial p}(q,0) = 0$ for all $q \in S^1$. So we can apply Lemma D.0.19 twice and obtain smooth functions $\tilde{F}, \tilde{\tilde{F}}\colon D_{\epsilon_3} \to \mathbb{R}$ such that

$$F(q,p) = p\tilde{F}(q,p) = p^2\tilde{\tilde{F}}(q,p).$$

Thus

$$F_s(q,p) = \frac{1}{s^2}F(q,sp) = \frac{1}{s^2}(sp)^2\tilde{\tilde{F}}(q,sp) = p^2\tilde{\tilde{F}}(q,sp).$$

Hence indeed, F_s is a smooth family of functions, which vanish on the zero-section and such that

$$\tau_s^*\lambda\text{can} - \lambda\text{can} = dF_s.$$

From Proposition 4.2.6, we know that given the functions F_s above, a potential smooth family of functions whose Hamiltonian flow generates τ_s has the form

$$H_s = \left(\frac{d}{dt}\,|_{t=s}\, F_t\right) \circ \tau_s^{-1} - \iota_{X_s}\lambda$$

where $\frac{\partial}{\partial s}\tau_s = X_s \circ \tau_s$. Observe that if they are defined, then dH_s vanishes on the zero-section due to the fact that $\tau_s|_{0-section} = id$ (thus $X_s(q,0) = 0$).

We want to show, that there exists a neighbourhood of the zero-section in T^*S^1, on which τ_s is generated by the Hamiltonians H_s defined by the equation above.
If we write

$$\tau_s(q,p) = (Q_s(q,p), P_s(q,p))$$

then by the choice of ϵ_3

$$|P_s(q,p)| = |p\tilde{P}(q,sp)| \geq \frac{\epsilon_3}{2}\frac{1}{2} = \frac{\epsilon_3}{4}$$

for all $|p| = \frac{\epsilon_3}{2}$ and all q. Indeed $1 - |\tilde{P}(q,p)| \leq |\tilde{P}(q,p) - 1| < \frac{1}{2}$ for all q and all $|p| \leq \epsilon_3$.
Thus $D_{\frac{\epsilon_3}{4}} \subset \tau_s(D_{\epsilon_3})$ for all s.
Thus, X_s is defined on all of $D_{\frac{\epsilon_3}{4}}$ for all s. So is $\iota_{X_s}\lambda_{can}$. Moreover since $\frac{d}{ds}F_s$ is defined on all of D_{ϵ_3} (since F_s is defined on D_{ϵ_3}) it follows that $\frac{d}{ds}F_s \circ \tau_s^{-1}$ is defined on all of $D_{\frac{\epsilon_3}{4}}$ as well.
Thus indeed, on $D_{\frac{\epsilon_3}{4}}$, the smooth family of functions

$$H_s = (\frac{d}{dt}\,|_{t=s}\, F_t) \circ \tau_s^{-1} - \iota_{X_s}\lambda$$

is defined, and by construction, for all points $(q,p) \in D_{\frac{\epsilon_3}{4}}$, it follows that

$$(\iota_{X_s}\omega)_{(q,p)} = (dH_s)_{(q,p)}$$

as desired.
Next we are going to cut-off the functions H_s suitably. Let ρ be a cut-off function $\rho\colon \mathbb{R} \to \mathbb{R}$ with support in $[-\frac{\epsilon_3}{4}, \frac{\epsilon_3}{4}]$ and which satisfies $\rho(r) = 1$ for $r \in [-\frac{\epsilon_3}{8}, \frac{\epsilon_3}{8}]$. Consider the following family of functions

$$K_s(q,p) := \rho(p)H_s(q,p).$$

Since $K_s = H_s$ on $D_{\frac{\epsilon_4}{8}}$ the Hamiltonian vector fields Y_s of K_s agrees with X_s there. Denote the flow of Y_s by ψ_s.
We are left to show, that there exists a neighbourhood D_{ϵ_4} of the zero-section, which is mapped into $D_{\frac{\epsilon_3}{8}}$ by τ_s for all s. Then the whole flow-line of the time-dependent vector field X_s starting at a point in D_{ϵ_4} lies inside $D_{\frac{\epsilon_3}{8}}$, and consequently, $\tau_s = \psi_s$ for all s when restricted to D_{ϵ_4}.
Consider

$$\epsilon_4 := \inf_{s\in[0,1], q\in S^1, |p| = \frac{\epsilon_3}{8}} |\bar{P}_s(q,p)| > 0$$

if $\tau_s^{-1}(q,p) = (\bar{Q}_s(q,p), \bar{P}_s(q,p))$. Then ϵ_4 is greater than zero since the set $[0,1] \times S^1 \times S^0$ over which we take the infinum is compact and τ_s^{-1} is an embedding which maps the zero-section to the zero-section (thus $\bar{P}_s(q,p) \neq 0$ for $p \neq 0$). Then indeed $\tau_s(D_{\epsilon_4}) \subset D_{\frac{\epsilon_3}{8}}$ for all s and consequently

$$\psi_1(q,p) = \tau_1(q,p) = \tau(q,p)$$

for all $(q,p) \in D_{\epsilon_4}$.
By construction, $K_s \circ \alpha^{-1}$ is a family of smooth functions with support in U. Extend by zero to all of A and let G_s denote the resulting family of functions on A. Now consider the flow χ_s of the time-dependent vector field Z_s defined by $\iota_{Z_s}\omega_{std} = dG_s$ on A. Then by construction on $U' := \alpha(D_{\epsilon_4})$, we have $\chi_1 = \phi$, and furthermore, the isotopy χ_s lives in $Ham(A, \partial A, \omega_{std})$. This follows from Proposition 4.2.6 and the fact that the family G_s has support in U away from the boundary. By construction, $\chi_s(E) = E$ pointwise for all s.
Then $\phi \circ \chi_s^{-1}$ is a smooth path of Hamiltonian symplectomorphisms such that $\phi \circ \chi_s^{-1}|_E = id$ for all s, and such that $\phi \circ \chi_1^{-1}$ is the identity on U'. This proves the proposition. □

For the definition of A_i, $Symp(A_i, \partial A_i, \omega_{std})$ and ϕ^D in the following proposition, consult Appendix E.

Proposition D.0.22. *Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth function with support in $[\frac{3}{4}, \frac{5}{4}]$ and which is equal to 1 for $r \in [\frac{4}{5}, \frac{6}{5}]$. Consider the functions $T_k\colon A \to \mathbb{R}$ given by $T_k(re^{i\theta}) = \frac{k}{1+r^2}\rho(r)$. Then the flow ϕ_t^k generated by the vector field X^k which is defined by $\iota_{X^k}\omega_{std} = dT_k$ is Hamiltonian and preserves E for all t. Furthermore $\phi_1^k|_{A_i} \in Symp(A_i, \partial A_i, \omega_{std})$ and is in this group isotopic to $(\phi^D)^k \in Symp(A_i, \partial A_i, \omega_{std})$.*

Proof. The flow ϕ_t^k is Hamiltonian, since by Proposition 4.2.6 $(\phi_t^k)^*\lambda_{std} - \lambda_{std} = dF_t$ with

$$F_t = \int_0^t (T_k + \iota_{X^k}\lambda_{std}) \circ \phi_s^k ds.$$

Now T_k has support in the interior of A, thus X^k vanishes in a neighbourhood of the boundary and $\phi_t^k = id$ in some neighbourhood of the boundary. Hence $Flux(\phi_t) = F_t(2) - F_t(\frac{1}{2}) = 0$ as claimed. Next calculate the Hamiltonian vector field $X^k = \alpha\frac{\partial}{\partial r} + \beta\frac{\partial}{\partial\theta}$.

$$\iota_{X^k}\frac{r}{\pi(1+r^2)^2}dr \wedge d\theta = dT_k = \left(\frac{k}{1+r^2}\rho'(r) - \frac{2kr}{(1+r^2)^2}\rho(r)\right) dr.$$

Thus

$$\beta(r) = 2\pi k\rho(r) - \frac{(1+r^2)k\pi}{r}\rho'(r)$$

and $\alpha = 0$.
Set $\phi_t^k(r,\theta) = (r, \theta + tg^k(r))$ for some yet to be found function $g^k\colon [\frac{1}{2}, 2] \to \mathbb{R}$ and differentiate

$$\frac{d}{dt}\phi_t^k(r,\theta) = g^k(r)\frac{\partial}{\partial\theta} = X^k(\phi_t(r,\theta)) = \beta(r)\frac{\partial}{\partial\theta}.$$

Hence

$$\phi_t^k(r,\theta) = (r, \theta + t\beta(r))$$

is the flow of X^k. Now observe that for $r \in [\frac{4}{5}, \frac{6}{5}]$, $\rho(1) = 1$; $\rho'(r) = 0$ so that there $\beta(r) = 2\pi k$. Hence for $r \in [\frac{4}{5}, \frac{6}{5}]$

$$\phi_t^k(r,\theta) = (r, \theta + t2\pi k).$$

In particular ϕ_t^k preserves E for all t.
Then $\phi_1^k|_{A_i} \in Symp(A_i, \partial A_i, \omega_{std})$ and both $\phi_1^k|_{A_i}$ and $(\phi^D)^k$ satisfy the conditions of Lemma E.0.32: They are isotopic in $Symp(A_i, \partial A_i, \omega_{std})$ as claimed. □

Proposition D.0.23. *Let $\phi \in Ham(A, \partial A, \omega_{std})$ be such that it is the identity in a neighbourhood of E. Further assume that $\phi|_{A_i} \in Symp_0(A_i, \partial A_i, \omega_{std})$, the identity component of $Symp(A_i, \partial A_i, \omega_{std})$. Then, there exists a Hamiltonian path $\phi_t \in Ham(A, \partial A, \omega_{std})$ such that $\phi_0 = \phi$, ϕ_t preserves a neighbourhood of E pointwise and such that $\phi_1|_{A_i} \in Ham(A_i, \partial A_i, \omega_{std})$.*

Proof. Since ϕ is Hamiltonian, there exists a function $F\colon A \to \mathbb{R}$ with

$$\phi^*\lambda_{std} - \lambda_{std} = dF$$

such that $Flux(\phi) = F(2) - F(\frac{1}{2}) = 0$. Since the 1-form $\phi^*\lambda_{std} - \lambda_{std}$ vanishes in a neighbourhood V of E we find that $dF = 0$ in V and consequently F is constant on V. Hence we can define

$$Flux(\phi_o) = F(2) - F(1);\ Flux(\phi_i) = F(1) - F(\frac{1}{2}).$$

Consequently

$$Flux(\phi_o) = -Flux(\phi_i).$$

We need a Hamiltonian path which preserves E, and redistributes Flux between the two annuli A_i and A_o, in order to make ϕ_o, ϕ_i Hamiltonian.
Consider a cut-off function ρ with support in $[\frac{3}{4}, \frac{5}{4}]$ and such that $\rho(r) = 1$ for $r \in [\frac{4}{5}, \frac{6}{5}]$. Let

$$H_a(r, \theta) = +a\rho(r)$$

and let χ_t^a be the flow of the vector field X^a defined by $\iota_{X^a}\omega_{std} = dH_a$. χ_t^a is Hamiltonian as before, since H_a has support away from the boundary, and χ_t^a is the identity near E. But the flux of $\chi_1^a|_{A_i}$ is precisely a. Indeed by Proposition 4.2.6,

$$(\chi_1^a)^*\lambda_{std} - \lambda_{std} = dF^a$$

with

$$F^a = \int_0^1 (H_a + \iota_{X^a}\lambda_{std}) \circ \chi_t^a dt.$$

Then in V, we have $\chi_t^a = id$, $X^a = 0$ and $H_a = a$, thus

$$Flux(\chi_1^a|_{A_o}) = F^a(2) - F^a(1) = 0 - a = -a;\ Flux(\chi_1^a|_{A_i}) = F^a(1) - F^a(\frac{1}{2}) = a - 0 = a.$$

Now consider the isotopy $\phi \circ \chi_t^{-Flux(\phi_i)}$ which preserves E for all t, starts at ϕ and ends at a symplectomorphism with zero Flux on each of the annuli A_o, A_i. This proves the proposition. □

Now we can proof Lemma 3.4.2:

Lemma D.0.24. *Given $\phi \in Ham(A, \partial A, \omega_{std})$ with $\phi(E) = E$, then there exists a smooth path $\phi_t \in Ham(A, \partial A, \omega_{std})$ which connects ϕ to the identity and which satisfies $\phi_t(E) = E$ for all t.*

Proof. By Propositions D.0.18 and D.0.21, we may assume that ϕ is already the identity in a neighbourhood U of E. Thus $\phi_o = \phi|_{A_o}$, $\phi_i = \phi|_{A_i}$ are elements in $Symp(A_o, \partial A_o, \omega_{std})$, $Symp(A_i, \partial A_i, \omega_{std})$ respectively. A priori ϕ_i may not lie in the identity component in case we have chosen the path $\psi_t \colon S^1 \to S^1$ in D.0.18 in the wrong homotopy class.
Since ϕ^D generates $\pi_0(Symp(A_i, \partial A_i, \omega_{std}))$ (see Theorem E.0.29 and the Remark after Proposition E.0.30), there exists a $k \in \mathbb{Z}$ such that ϕ_i and $(\phi^D)^k$ are isotopic in $Symp(A_i, \partial A_i, \omega_{std})$.
Consider the flow $\phi_t^k \in Ham(A, \partial A, \omega_{std})$ of the function T_k given by Proposition D.0.22. Then, the isotopy $\phi \circ (\phi_t^k)^{-1}$ in $Ham(A, \partial A, \omega_{std})$ preserves E for all t and ends at a Hamiltonian symplectomorphism $\phi \circ (\phi_1^k)^{-1}$ which restricts to a symplectomorphism $\phi \circ (\phi_1^k)^{-1}|_{A_i}$ in the identity component $Symp_0(A_i, \partial A_i, \omega_{std})$.
But ϕ itself is in $Symp_0(A, \partial A, \omega_{std})$, this implies in particular that as soon as we killed any Dehn-twist in the inner Annulus, we also killed all Dehn-twists in the outer Annulus (otherwise a net-Dehn-twist would survive making ϕ non-isotopic to id). Hence we may assume that ϕ restricts to symplectomorphisms ϕ_o, ϕ_i in $Symp_0(A_o, \partial A_o, \omega_{std})$, $Symp_0(A_i, \partial A_i, \omega_{std})$ respectively. But then ϕ satisfies the conditions of Proposition D.0.23. Hence we may assume without loss of generality that $\phi_o \in Ham(A_o, \partial A_o, \omega_{std})$ and $\phi_i \in Ham(A_i, \partial A_i, \omega_{std})$. But $Ham(A_o, \partial A_o, \omega_{std})$ and $Ham(A_i, \partial A_i, \omega_{std})$ are path connnected by Lemma 4.2.8, hence there exists Hamiltonian path to the identity on both annuli. In total we found a Hamiltonian isotopy $\phi_t \in Ham(A, \partial A, \omega_{std})$ which connects id to ϕ and such that $\phi_t(E) = E$ for all t. This proves the Lemma. □

The proof of Lemma 4.3.2

Lemma (4.3.2)**.** *There exists a smooth approximation $\widetilde{G} \colon [0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to Ham(A, \partial A, \omega_{std})$ of G such that $\widetilde{G} = G$ on $\widetilde{V} \subset V$, an open set which is a neighbourhood of the boundary, such that $\widetilde{G}(s, 0) = \alpha(\frac{\pi}{2} s)$.*

Proof. Observe that, by construction, G is smooth in a neighbourhood $V = [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] - W$ of the boundary of $[0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}]$ and in a neighbourhood of $[0,1] \times \{0\}$ (cf.fig. 4.6). Let

$$V' = [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] - W'$$

and

$$\widetilde{V} = [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] - \widetilde{W}.$$

Compare figure D.2.

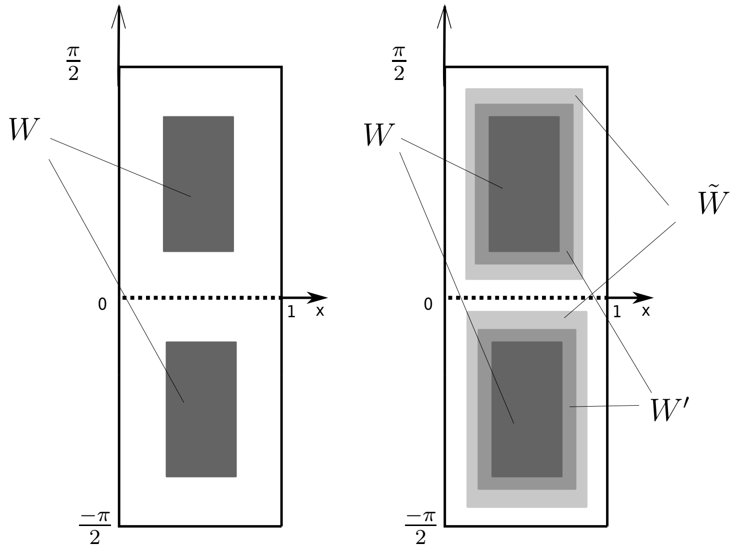

Figure D.2: Where G is smooth

By smooth approximations as in chapter 2 in [7], in particular Theorem 2.4 on p.48, we may assume there exists a smooth map

$$\widetilde{H}\colon [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] \to Diff(A, \partial A)$$

such that $\widetilde{H}$ agrees with G on V'. In particular $\widetilde{H}(s,0) = \alpha(\frac{\pi}{2}s)$.

Next we apply a parametrised version of Moser's theorem, to push this smooth map down into $Symp(A, \partial A, \omega_{std})$. Thus consider the smooth 2-parameter family

$$\omega(s,t) = \tilde{H}(s,t)^*\omega_{std}$$

and its primitive $\lambda(s,t) = \tilde{H}(s,t)^*\lambda_{std}$. Further consider the linear interpolation

$$\omega_u(s,t) = (1-u)\omega_{std} + u\omega(s,t).$$

Then

$$\frac{\partial \omega_u(s,t)}{\partial u} = d(\lambda(s,t) - \lambda_{std}).$$

Consider the vector fields defined by

$$\iota_{X_u(s,t)}\omega_u(s,t) = \lambda(s,t) - \lambda_{std}.$$

If $(s,t) \in V'$ then $\omega_u(s,t) = \omega_{std}$ for all u, further the 1-form $\lambda(s,t) - \lambda$ is closed so that the vector field $X(s,t)$ defined by

$$\iota_{X(s,t)}\omega_{std} = \lambda(s,t) - \lambda_{std}$$

is symplectic. Thus, if $\phi_u(s,t)$ denotes the flow of the time-dependent vector field $X_u(s,t)$, then for $(s,t) \in V'$ it follows that $\phi_u(s,t) \in Symp(A, \partial A, \omega_{std})$. By construction $\phi_u(s,t)$ depends smoothly on s, t.
Let ρ be a smooth cut-off function which is 1 on W' and zero on $\widetilde{V}$. Consider the smooth map

$$K\colon [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] \to Symp(A, \partial A, \omega_{std})$$
$$(s,t) \mapsto \widetilde{H}(s,t) \circ \phi_{\rho(s,t)}(s,t).$$

Then indeed

$$K(s,t)^*\omega_{std} = \omega_{std}$$

for all s, t and $K = G$ on $\widetilde{V}$. In particular $K(s,0) = \alpha(\frac{\pi}{2}s)$.
In the last step we push K down into Ham by using Lemma 4.2.7. Therefore note first that $Flux(K(s,t))$ depends smoothly on s, t. Indeed

$$K(s,t)^*\lambda_{std} - \lambda_{std} = \sigma(s,t),$$

a smooth family of closed 1-forms on A, which vanish near the boundary of A. Hence, there exists functions $F(s,t)$ such that

$$\sigma(s,t) = dF(s,t).$$

Since these functions are determined up to a constant, by choosing $F(s,t)(\frac{1}{2}) = 0$ we have determined these functions uniquely. Consequently the family $F(s,t)$ is smooth in s, t and so it follows that

$$Flux(K(s,t)) = F(s,t)(2) - F(s,t)(\frac{1}{2}) = F(s,t)(2)$$

is indeed smooth. But the family $\phi^{-Flux(K(s,t))}$ in Lemma 4.2.7 depends smoothly on s, t. So we can define

$$\widetilde{G}\colon [0,1] \times [\frac{-\pi}{2}, \frac{\pi}{2}] \to Ham(A, \partial A, \omega_{std})$$
$$(s,t) \mapsto \phi^{-Flux(K(s,t))} \circ K(s,t).$$

Now $K = G$ on $\widetilde{V}$ and $G(s,t) \in Ham(A, \partial A, \omega_{std})$. Since $\phi^0 = id$, it follows that $\phi^{-Flux(K(s,t))} = id$ for $(s,t) \in \widetilde{V}$. Then $\widetilde{G}$ agrees with G on $\widetilde{V}$ and in particular $\widetilde{G}(s,0) = \alpha(\frac{\pi}{2}s)$. This proves the lemma. $\square$

Appendix E

Homotopy groups of some diffeomorphism groups

We discuss the homotopy groups of $Symp(A, \partial A, \omega_{std})$ and $Ham(A, \partial A, \omega_{std})$. First we have to begin with a few definitions. Let $D := \{z \in \mathbb{C} | |z| \leq 2\}$ and $A := \left\{z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 2\right\}$ be equipped with the symplectic form $\omega_{std} = \frac{r}{\pi(1+r^2)^2} dr \wedge d\theta$. Let $\lambda_{std} = \frac{-1}{2\pi(1+r^2)} d\theta$ be the standard primitive of ω_{std} on A. In the following we define some diffeomorphism groups which are equipped with the subspace topology of the C^∞-topology of smooth maps $C^\infty(A; \mathbb{R}^2)$. This topology on $C^\infty(A; \mathbb{R}^2)$ is the topology induced by the metric $d(f, g) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|f-g|_k}{1+|f-g|_k}$ where

$$|f - g|_k = \max_{i, m+n=k} \sup_{x \in A} |\frac{\partial^k f_i}{\partial x_1^m \partial x_2^n}(x) - \frac{\partial^k g_i}{\partial x_1^m \partial x_2^n}(x)|.$$

By a smooth map $f: X \to C^\infty(A, \mathbb{R}^2)$ we shall mean smoothness of the induced map $\bar{f}: X \times A \to \mathbb{R}^2$. The following proposition implies that this gives genuine continuity of the map f.

Proposition E.0.25. *Let X be a topological space and let $F: X \to C^\infty(A, \mathbb{R}^2)$. Then F is continuous iff the induced map $\bar{F}: X \times A \to \mathbb{R}^2$; $(x, a) \mapsto F(x)(a)$ and all its partial derivatives in directions of A are continuous.*

Proof. A function $F: X \to C^\infty(A, \mathbb{R}^n)$ is continuous if for any convergent sequence $(x_n \to x)$ in X, the image sequence $F(x_n) \to F(x)$ converges to $F(x)$. Hence if we start with a continuous function F and (x_n) a convergent sequence, then $F(x_n) \to F(x)$ means that given any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$

$$d(F(x_n), F(x)) < \epsilon$$

in the metric defined above. This means in particular that the partial derivatives of any order are ϵ close for $n \geq N(\epsilon)$.
Then the function $\bar{F}$ is continuous, since for any convergent sequence $(x_n, k_n) \to (x, k)$ we have

$$|\bar{F}(x_n, k_n) - \bar{F}(x, k)| = |F(x_n)(k_n) - F(x)(k)| \leq |F(x_n) - F(x)|_0 + |F(x)(k_n) - F(x)(k)| \leq 2\epsilon$$

for all $n \geq N$ and N so big that the first term is smaller than ϵ (see above) and the second term is smaller than ϵ by the continuity of $F(x)$. The same works for all partial derivatives of $\bar{F}$ in directions of A. Conversely if we have a continuous function $\bar{F}$ as above which is smooth when restricted to the factor A and for which all partial derivatives in directions of A are continuous functions then this defines a continuous function $F\colon X \to (C^\infty(A, \mathbb{R}^n), C^\infty)$. To see this observe that

$$\frac{\partial \bar{F}_i}{\partial y_j}(x,k) = \frac{d}{dt}\bar{F}_i(x, k+ty_j) = \frac{d}{dt}F(x)_i(k+ty_j) = \frac{\partial F(x)_i}{\partial y_j}(k)$$

Hence

$$|F(x_n) - F(x)|_1 = max_{i,j} sup_{k\in A} |\frac{\partial F(x_n)_i}{\partial y_j}(k) - \frac{\partial F(x)_i}{\partial y_j}(k)| =$$

$$= max_{i,j} sup_{k\in K} |\frac{\partial \bar{F}_i}{\partial y_j}(x_n,k) - \frac{\partial \bar{F}_i}{\partial y_j}(x,k)|$$

Consequently, by compactness of A, we can find for any given $\epsilon > 0$, a N such for all $n \geq N$

$$|\frac{\partial \bar{F}_i}{\partial y_j}(x_n,k) - \frac{\partial \bar{F}_i}{\partial y_j}(x,k)| < \epsilon$$

for all $k \in K$ and all i, j. Thus

$$|F(x_n) - F(x)|_1 \leq \epsilon$$

The same can be done for all higher partial derivatives. Thus given any $\epsilon > 0$, fix a $r \in \mathbb{N}$ such that $2^{r-1} < \epsilon$. Now we can find a N such that for all $n \geq N$:

$$|F(x_n) - F(x)|_l < \frac{\epsilon}{4}$$

for all $l = 0..r$. Hence

$$d(F(x_n), F(x)) \leq \sum_{l=0}^{l=r} \frac{1}{2^l}\frac{\epsilon}{4(1+\frac{\epsilon}{4})} + \sum_{l=r+1}^{\infty}\frac{1}{2^l} < \epsilon$$

and so F is continuous. □

Definition E.0.26. *We define:*

- $Diff(D, \partial D)$ *to be the group of diffeomorphisms of the closed disk* D*, such that every element* ϕ *is equal to the identity in some neighbourhood of the boundary.*

- $Diff(A, \partial A)$ *to be the group of diffeomorphisms of the closed annulus* A*, such that every element* ϕ *is equal to the identity in some neighbourhood of the boundary.*

- $Symp(A, \partial A, \omega_{std}) \subset Diff(A, \partial A)$ *to be the subgroup of symplectomorphisms of* (A, ω_{std})*.*

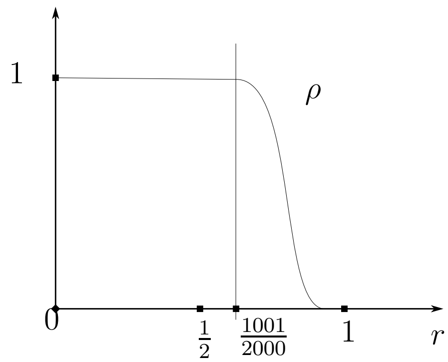

Figure E.1: The bump function ρ

- *$Symp_0(A, \partial A, \omega_{std})$ to be the identity component of $Symp(A, \partial A, \omega_{std})$.*
-
$$Ham(A, \partial A, \omega_{std}) := \{\phi \in Symp_0(A, \partial A, \omega_{std}) | Flux(\phi) = 0\}.$$

Further we need the following definitions. Define the outer annulus

$$A_o = \{z \in \mathbb{C} | 1 \leq |z| \leq 2\}$$

and the inner annulus

$$A_i = \left\{ z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 1 \right\}.$$

Definition E.0.27. *The groups $Diff(A_o, \partial A_o)$, $Diff(A_i, \partial A_i)$, $Symp(A_o, \partial A_o, \omega_{std})$, $Symp(A_i, \partial A_i, \omega_{std})$, $Symp_0(A_o, \partial A_o, \omega_{std})$, $Symp_0(A_i, \partial A_i, \omega_{std})$, $Ham(A_o, \partial A_o, \omega_{std})$, $Ham(A_i, \partial A_i, \omega_{std})$ are defined as the corresponding groups for (A, ω_{std}).*

Now we quote

Theorem E.0.28 (Smale)**.** *The group $Diff(D, \partial D)$ is contractible.*

Proof. This is proved in [4]. □

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function as in fig. E.1

Theorem E.0.29. *$\pi_i(Diff(A, \partial A)) = 0$ for $i > 0$ and $\pi_0(Diff(A, \partial A)) = \mathbb{Z}$ and is generated by the Dehn-twist*

$$\phi^D\colon A \to A$$

$$(re^{i\theta}) \mapsto re^{i(\theta + 2\pi\rho(r))}.$$

Proof. The proof is taken from [15] and goes as follows. Let $Emb(B, \mathring{D})$ be the space of embeddings of the closed disk $B := \bar{D}(0, \frac{1}{2}) \subset \mathbb{R}^2$ into the open disk $\mathring{D} = D(0,2)$. Again the topology is given by the subspace topology of the compact open topology on the corresponding space of smooth functions.
Let UTD denote the unit tangent bundle of the disk D, then firstly the map $f\colon Emb(B, \mathring{D}) \to UTD;\ \phi \mapsto \frac{d\phi_0(\frac{\partial}{\partial x})}{|d\phi_0(\frac{\partial}{\partial x})|}$ is a weak homotopy equivalence. UTD is homotopy equivalent to S^1 and is generated by the fiber. Since f_* is an isomorphism on the homotopy groups, it follows that the loop

$$\gamma\colon S^1 \cong \mathbb{R}\backslash 2\pi\mathbb{Z} \to Emb(B, \mathring{D});\ \theta \mapsto (z \mapsto e^{i\theta} z)$$

is a generator of $\pi_1(Emb(B, \mathring{D}))$ (cf. Theorems 2.6.C and 2.6.D in [15]).
Secondly, a multiparameter version of the isotopy extension theorem shows that the continuous map

$$p\colon Diff(D, \partial D) \to Emb(B, \mathring{D})$$

given by restriction to B is a Serre fibration (has the homotopy lifting property for cubes). Consider the fiber over the inclusion. This is the set of diffeomorphisms in $Diff(D, \partial D)$ which restrict to the identity on B or in other words, the diffeomorphisms of A which are the identity near $\{|z| = 2\}$ and which can be extended to D by the identity. Denote this set by D^{ext}. Since the homotopy long exact sequence applies to Serre fibrations, we obtain that $0 = \pi_{i+1}(S^1) \cong \pi_{i+1}(Emb(B, \mathring{D})) \cong \pi_i(D^{ext}))$ for $i \geq 1$. Further the map $\mathbb{Z} \cong \pi_1(S^1) \cong \pi_1(Emb(B, \mathring{D})) \to \pi_0(D^{ext})$ given by the boundary homomorphism is an isomorphism.
Recall that the boundary homomorphism $\partial\colon \pi_1(B) \to \pi_0(F)$ in the homotopy long exact sequence of a fibration $F \to E \to B$, can be described by the homotopy lifting property. Explicitly let $\gamma\colon [0, 2\pi] \to B$ with $\gamma(0) = \gamma(2\pi)$ be a representative of an element in $\pi_1(B)$ and let $\tilde{\gamma}\colon [0, 2\pi] \to E$ be a lift of γ starting at the basepoint x_0 in F, then the boundary homomorphism is given by

$$\partial([\gamma]) = [\tilde{\gamma}(2\pi)]$$

where $[\tilde{\gamma}(2\pi)]$ denotes the path component in F of the endpoint of the lift $\tilde{\gamma}$ (cf. the discussion on p.209 in [10]).
Hence we are required to find a lift of the loop γ starting at the identity in D^{ext}. Let ρ be the cut-off function from Figure E.1 and consider the extension

$$\tilde{R}_\theta \in Diff(D, \partial D);\ \tilde{R}_\theta(z) = e^{i\rho(|z|)\theta} z$$

of the embedding $R_\theta\colon B \to \mathring{D};\ z \mapsto e^{i\theta} z$. Thus $\tilde{\gamma}\colon [0, 2\pi] \to Diff(D, \partial D);\ \theta \mapsto \tilde{R}_\theta$ is the required lift of γ. Consequently, $\tilde{R}_{2\pi} = \phi^D$ generates $\pi_0(D^{ext})$. Note that by the choice of ρ, actually $\tilde{R}_{2\pi} = \phi^D \in Diff(A, \partial A)$.
Now let $u\colon S^n \to Diff(A, \partial A)$ be a continuous map, which represents an element in $\pi_n(Diff(A, \partial A))$ for $n \geq 1$. Since S^n is compact, there exists a real number $\epsilon > 0$ such that $u(x)$ is the identity in an ϵ-neighbourhood of $\left\{|z| = \frac{1}{2}\right\}$ for all $x \in S^n$. Let $A_\delta = \{z \in \mathbb{C} | \delta \leq |z| \leq 2\}$ for $0 < \delta < \frac{1}{2}$ and let $Diff(A_\delta, \partial A_\delta)$ be defined as $Diff(A, \partial A)$.

Let $\alpha\colon A \to A_{\frac{1}{4}}$ be a diffeomorphism which is the identity outside of the ϵ-neighbourhood of the set $\left\{|z| = \frac{1}{2}\right\}$ in A.
Since any element in $Diff(A, \partial A)$ is extendable to the disk D by the identity, we can regard $u\colon S^n \to D^{ext}$. But then there exists a contraction $u_t\colon S^n \to D^{ext}$ since $\pi_n(D^{ext}) = 0$ for $n \geq 1$.
Obviously $D^{ext} \subset Diff(A_{\frac{1}{4}}, \partial A_{\frac{1}{4}})$. Thus we can regard $u_t\colon S^n \to Diff(A_{\frac{1}{4}}, \partial A_{\frac{1}{4}})$ as a contraction of u in $Diff(A_{\frac{1}{4}}, \partial A_{\frac{1}{4}})$ and

$$v_t\colon S^n \to Diff(A, \partial A);\ x \mapsto \alpha^{-1} \circ u_t(x) \circ \alpha$$

is the desired contraction of u in $Diff(A, \partial A)$. Indeed α is the identity on the support of u. This shows the first assertion of the theorem.
Any $\psi \in D^{ext}$ is isotopic in D^{ext} to $(\phi^D)^i$ for some i. Let $\phi \in Diff(A, \partial A)$, then by the same argument as above, we can find an isotopy in $Diff(A, \partial A)$ between ϕ and $(\phi^D)^i$. This shows that the Dehn-twist ϕ^D generates $\pi_0(Diff(A, \partial A))$. $\square$

Proposition E.0.30. *The groups $Diff(A, \partial A)$ and $Symp(A, \partial A, \omega_{std})$ are weakly homotopy equivalent.*

Proof. Let $\Omega \subset \Omega^2(A)$ be the set of symplectic forms on A. Since every 2-form ω on A can be written as

$$\omega = f dr \wedge d\theta$$

for some function $f\colon A \to \mathbb{R}$, we can identify Ω with a subset of $C^\infty(A, \mathbb{R})$. As above we can define the C^∞-topology on $C^\infty(A, \mathbb{R})$ and endow Ω with the subspace topology.
We are going to show, that the map

$$p\colon Diff(A, \partial A) \to \Omega$$

$$\phi \mapsto \phi^*\omega_{std}$$

is continuous and is a Serre fibration (i.e. p has the homotopy lifting property for all cubes $[0,1]^k$). Further $p^{-1}(\omega_{std})$ equals $Symp(A, \partial A, \omega_{std})$ and Ω is contractible since it is convex. The proposition then follows since the long exact sequence for homotopy applies to Serre-fibrations.
Continuity of p is trivial and the homotopy lifting property for cubes follows by Moser's theorem with parameters. We only show this for the 1-dimensional cube $[0,1]$ since the proof for $[0,1]^k$ works entirely analogously.
Let $\gamma\colon [0,1] \to \Omega$ be a continuous map and $\phi \in p^{-1}(\gamma(0))$, then we seek a continuous map $\tilde{\gamma}\colon [0,1] \to Diff(A, \partial A)$ such that $p\tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \phi$.
First note that for $\omega \in \Omega$ we can define a canonical primitive λ^ω. To do this we write $\omega = f dr \wedge d\theta$ and define

$$(\lambda^\omega)_{r,\theta} = \left(\int_{\frac{1}{2}}^{r} f(s,\theta) ds\right) d\theta.$$

Now consider the linear interpolation of symplectic forms

$$\omega_s(t) = s\omega_{std} + (1-s)\gamma(t).$$

Then

$$\frac{\partial \omega_s(t)}{\partial s} = d(\lambda_{std} - \lambda^{\gamma(t)})$$

and we define

$$\sigma(t) = \lambda_{std} - \lambda^{\gamma(t)}.$$

Now define the Moser vector fields $X_s(t)$ by

$$\iota_{X_s(t)}\omega_s(t) = \sigma(t)$$

and denote their flows by $\phi_s(t)$. Hence

$$\phi_1(t)^*\omega_{std} = \gamma(t)$$

and by continuity of γ and therefore of $\lambda^{\gamma(t)}$, it follows that the path

$$t \mapsto \phi_1(t)$$

is a continuous lift of γ. Consider

$$h = \phi \circ (\phi_1(0))^{-1}.$$

Since $\phi^*\omega_{std} = \phi_1(0)^*\omega_{std}$, it follows that $h^*\omega_{std} = \omega_{std}$ and therefore

$$\tilde{\gamma}(t) = h \circ \phi_1(t)$$

is the desired lift of γ. □

Remark

Note that the Dehn-twist $\phi^D \in Symp(A, \partial A, \omega_{std})$ and thus by Proposition E.0.30 ϕ^D generates $\pi_0(Symp(A, \partial A, \omega_{std}))$.

Proposition E.0.31. *The group $Symp_0(A, \partial A, \omega_{std})$ deformation retracts onto $Ham(A, \partial A, \omega_{std})$.*

Proof. By Proposition 4.2.7, given any real number a there exists a canonical symplectomorphism $\phi^a \in Symp_0(A, \partial A, \omega_{std})$ such that $Flux(\phi^a) = a$. Thus define

$$r_t \colon Symp_0(A, \partial A, \omega_{std}) \to Symp_0(A, \partial A, \omega_{std})$$

$$\phi \mapsto \phi^{-tFlux(\phi)} \circ \phi$$

for $t \in [0,1]$. Since composition $\circ$ in $Symp(A, \partial A, \omega_{std})$ is continuous, to show continuity of r_t, it suffices to show that the map

$$\phi \mapsto \phi^{-tFlux(\phi)}$$

is continuous. But ϕ^a is canonical for $a \in \mathbb{R}$, hence it suffices to show that $\phi \mapsto Flux(\phi)$ is continuous (obviously, then $\phi \mapsto tFlux(\phi)$ is also continuous).

$$Flux(\phi) - Flux(\psi) = Flux(\phi \circ \psi^{-1}),$$

and $\phi \circ \psi^{-1}$ is close to id if and only if ϕ is close to ψ. Hence $Flux$ is continuous, if it is continuous at id.
Write $\phi(r,\theta) = (R(r,\theta), Q(r,\theta))$ for $\phi \in Symp(A, \partial A, \omega_{std})$ being ϵ-close to the identity. Then

$$Flux(\phi) = \int_\gamma \phi^*\lambda_{std} - \lambda_{std}$$

where $\gamma(t) = ((1-t)\frac{1}{2} + 2t, 0)$ is the path along $\mathbb{R}_+$ which connects the two boundary components.
Further

$$(\phi^*\lambda_{std})_{r,\theta} = \frac{-1}{2\pi(1 + R(r,\theta)^2)} \left(\frac{\partial Q}{\partial r}(r,\theta)dr + \frac{\partial Q}{\partial \theta}(r,\theta)d\theta \right)$$

and thus

$$|Flux(\phi)| = \left| \int_\gamma \phi^*\lambda_{std} - \lambda_{std} \right| = \left| \int_0^1 \phi^*\lambda_{std} \left((2 - \frac{1}{2})\frac{\partial}{\partial r} \right) dt \right| \leq$$

$$\leq \frac{3}{2} \max \left| \frac{1}{2\pi(1+R^2)} \frac{\partial Q}{\partial r} \right| \leq \frac{3}{\pi} \max \left| \frac{\partial Q}{\partial r} \right| \leq \max \left| \frac{\partial Q}{\partial r} \right|.$$

But ϕ is ϵ-close to id thus $\left|\frac{\partial Q}{\partial r}\right| < \epsilon$. This shows the continuity of $Flux$ and that of r_t. Now

$$r_1 \colon Symp_0(A, \partial A, \omega_{std}) \to Ham(A, \partial A, \omega_{std})$$

since $Flux(\phi^{-Flux(\phi)} \circ \phi) = 0$.
Finally $r_t|_{Ham(A,\partial A,\omega_{std})} = id$, this follows from the fact that $\phi^a = id$ for $a = 0$. Thus r_t is indeed a deformation retraction as claimed. □

Remark

The corresponding statements for the groups $Diff(A_o, \partial A_o)$, $Diff(A_i, \partial A_i)$, $Symp(A_o, \partial A_o, \omega_{std})$, $Symp(A_i, \partial A_i, \omega_{std})$, $Symp_0(A_o, \partial A_o, \omega_{std})$, $Symp_0(A_i, \partial A_i, \omega_{std})$, $Ham(A_o, \partial A_o, \omega_{std})$, $Ham(A_i, \partial A_i, \omega_{std})$ can be proved in exactly the same way. Thus we can replace the annulus A in any of the results above by either A_o or A_i.

Consider the following lemma, which we need in Appendix D.

Lemma E.0.32. *Let $\phi_0, \phi_1 \in Diff(A, \partial A)$ be of the special form $\phi_i(re^{i\theta}) = re^{i(\theta + f_i(r))}$ with $f_i(\frac{1}{2}) = 2\pi k$ and $f_i(2) = 0$ for $k \in \mathbb{Z}$ then there exists a smooth path $\phi_t \in Diff(A, \partial A)$ which connects ϕ_0 and ϕ_1. Furthermore if $\phi_0, \phi_1 \in Symp(A, \partial A, \omega_{std})$ then the $\phi_t \in Symp(A, \partial A, \omega_{std})$ for all t.*

Proof. It suffices to write down the linear isotopy $\phi_t(re^{i\theta}) = re^{i(\theta+(1-t)f_0(r)+tf_1(r))}$. Then ϕ_t is smooth with smooth inverse $\phi_t^{-1}(re^{i\theta}) = re^{i(\theta-(1-t)f_0(r)-tf_1(r))}$. Clearly ϕ_t restricts to the identity near ∂A due to the fact that $f_0(\frac{1}{2}) = f_1(\frac{1}{2})$ and $f_0(2) = f_1(2)$ near ∂A. Further by the defining equation it is clear that ϕ_t depends smoothly on t. Moreover note that a diffeomorphism of this form is always symplectic. This proves the lemma. $\square$

Appendix F

Homology of (M, L)

Definition of the second relative homotopy group

Let $x_0 \in L$ be the base point and let $I^2 = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square in $\mathbb{R}^2$ with standard coordinates x, y.

Definition F.0.33.

$$\pi_2(M, L) := \left\{ u \colon (I^2, \partial I^2, J) \to (M, L, x_0) | u \quad continuous \right\} / \sim$$

where ∂I^2 is the boundary of I^2 and J the closure of the boundary with the edge $\{0\} \times [0,1]$ removed. Further $u \sim v$ if and only if there exists a continuous family $u_t \colon (I^2, \partial I^2, J) \to (M, L, x_0)$ for $t \in [0,1]$ such that $u_0 = u$ and $u_1 = v$.

This is a group under composition given by concatenation of paths in the x-direction. By this we mean to fix y and then concatenate the two path $u^y, v^y \colon (I, \partial I) \to (M, x_0)$ defined by $u^y(x) = u(x, y)$ and $v^y(x) = v(x, y)$.

For convenience we will usually identify $(I^2, \partial I^2, J)$ with $(\mathbb{E}, \partial \mathbb{E}, 1)$ (its harder to define the group operation here).

Proposition F.0.34. *Let M be diffeomorphic to $S^2 \times S^2$ and L be an embedded T^2, then*

$$\pi_2(M, L) \cong H_2(M, L) \cong H_2(M) \oplus H_1(L)$$

Proof. Follows by the long exact sequence on homology/homotopy for the pair (M, L). $\square$

In particular, if L is the Clifford torus in $S^2 \times S^2$ then

$$\pi_2(M, L)$$

is abelian and is spanned by

$$[S^2 \times \{pt\}], [\{pt\} \times S^2], [D_{uh} \times \{pt\}], [\{pt\} \times D_{uh}]$$

where D_{uh} denotes the closed upper hemisphere in S^2.

Appendix G

Maslov index for the symplectomorphism group and the Lagrangian Grassmanian

The Maslov index for the group of symplectomorphisms $Sp(4)$

Proposition G.0.35. *The unitary group $U(n)$ is a maximal compact subgroup of $Sp(2n)$ and the quotient $Sp(2n)/U(n)$ is contractible.*

Proof. This is Proposition 2.22 in [13] on p.45. □

Proposition G.0.36. *The fundamental group of $U(n)$ is isomorphic to the integers. The determinant map $\det\colon U(n) \to S^1$ induces an isomorphism of fundamental groups.*

Proof. This is Proposition 2.23 in [13] on p.46. □

It follows from the propositions above that the fundamental group of $Sp(n)$ is isomorphic to $\mathbb{Z}$. An explicit isomorphism is given by the Maslov index:

Theorem G.0.37. *There exists a unique functor* Maslov*, called the Maslov index, which assigns an integer $Maslov(\Psi)$ to every loop $\Psi\colon \mathbb{R}/\mathbb{Z} \to Sp(4)$ of symplectic matrices and satisfies the following axioms*

- *(**homotopy**) Two loops in $Sp(4)$ are homotopic if and only if they have the same Maslov index.*
- *(**product**) For any two loops $\Psi_1, \Psi_2\colon \mathbb{R}/\mathbb{Z} \to Sp(4)$ we have*

$$Maslov(\Psi_1\Psi_2) = Maslov(\Psi_1) + Maslov(\Psi_2).$$

In particular, a constant loop $\Psi(t) \equiv Id$ has Maslov index 0.

- *(**direct sum**) Consider $Sp(2) \oplus Sp(2) \subset Sp(4)$ as a subgroup in the obvious way. Then*
$$Maslov(\Psi_1 \oplus \Psi_2) = Maslov(\Psi_1) + Maslov(\Psi_2).$$
- *(**normalization**) The loop $\Psi \colon \mathbb{R}/\mathbb{Z} \to U(1) \subset Sp(2)$ defined by*
$$\Psi(t) = e^{2\pi i t}$$
has Maslov index 1.

Proof. This is Theorem 2.29 in [13] on page 48. The proof can also be found there. □

The Maslov index for the Lagrangian Grassmanian

Consider $(\mathbb{R}^4, \Omega_0)$ where $\Omega_0 = dx \wedge dy + du \wedge dv$ for standard coordinates x, y, u, v on $\mathbb{R}^4$. Then let $\mathcal{L}$ be the set of linear Lagrangian subspaces of $(\mathbb{R}^4, \Omega_0)$. Then we have

Lemma G.0.38. *1. If $\Lambda \in \mathcal{L}$ and $\phi \in Sp(4)$ then $\phi\Lambda \in \mathcal{L}$;*

2. *For any two Lagrangian subspaces $\Lambda, \Lambda' \in \mathcal{L}$ there exists a symplectic matrix $\phi \in U(n)$ such that $\Lambda' = \phi\Lambda$;*

3. *There is a natural isomorphism $\mathcal{L} = U(2)/O(2)$.*

Proof. This is Lemma 2.31 in [13] on page 51. There can also be found the proof. □

From the lemma follows that $\pi_1(\mathcal{L}) = \mathbb{Z}$ and an explicit homomorphism $Maslov \colon \pi_1(\mathcal{L}) \to \mathbb{Z}$ is the Maslov index. Its properties are fixed in

Theorem G.0.39. *There exists a unique functor* Maslov*, called the Maslov index, which assigns an integer $Maslov(\Lambda)$ to every loop $\Lambda \colon \mathbb{R}/\mathbb{Z} \to \mathcal{L}$ of Lagrangian subspaces and satisfies the following axioms*

- *(**homotopy**) Two loops in $\mathcal{L}$ are homotopic if and only if they have the same Maslov index.*
- *(**product**) For any two loops $\Lambda \colon \mathbb{R}/\mathbb{Z} \to \mathcal{L}$ and $\Psi \colon \mathbb{R}/\mathbb{Z} \to Sp(4)$ we have*
$$Maslov(\Psi\Lambda) = Maslov(\Lambda) + 2Maslov(\Psi).$$
In particular, a constant loop $\Lambda(t) \equiv \Lambda_0$ has Maslov index 0.
- *(**direct sum**) Let $\Lambda \colon \mathbb{R}/\mathbb{Z} \to \mathcal{L}$ be a direct sum of Lagrangian subspaces in $\mathbb{C} \cong \mathbb{R}^2$. So $\Lambda(t) = \Lambda_1(t) \oplus \Lambda_2(t)$ with $\Lambda_i \colon \mathbb{R}/\mathbb{Z} \to G(2,1)$ with $G(2,1)$ the Grassmanian of 1-dimensional subspaces in $\mathbb{R}^2$ and were we have identified $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Then*
$$Maslov(\Lambda) = Maslov(\Lambda_1 \oplus \Lambda_0) + Maslov(\Lambda_0 \oplus \Lambda_2)$$
where $\Lambda_0 \colon \mathbb{R}/\mathbb{Z} \to G(2,1)$ denotes the constant loop at the real line.

- ***(normalization)*** *The loop* $\Lambda\colon \mathbb{R}/\mathbb{Z} \to \mathcal{L}$ *defined by*

$$\Lambda(t) =< (\cos \pi t, \sin \pi t, 0, 0), (0, 0, 1, 0) >$$

  has Maslov index 1*.*

Proof. This is Theorem 2.35 in [13] on page 52. The proof can also be found there. ☐

The following will be used to calculate Maslov indices in the text:

Lemma G.0.40. *If a loop of Lagrangian subspaces* $\Lambda\colon \mathbb{R}/\mathbb{Z} \to \mathcal{L}$ *is given by* $\Lambda(t) = U(t)\Lambda(0)$ *for a loop of unitary matrices* $U : \mathbb{R}/\mathbb{Z} \to U(2)$*, then* $Maslov(\Lambda) = wind(\det U^2)$*, the winding number of* $S^1 \to S^1;\ t \mapsto \det(U(t)^2)$*.*

Proof. This follows from the proof of Theorem 2.35 in [13]. ☐

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