Applications of gauge/gravity dualities with charged Anti-de Sitter black holes

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München 2010

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Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
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aus Königs Wusterhausen

München, den 17. Mai 2010

Erstgutachter: Prof. Dr. Dieter Lüst
Zweitgutachter: Priv.-Doz. Dr. Johanna Erdmenger
Tag der mündlichen Prüfung: 21. Juli 2010

To my family

Zusammenfassung

Die vorliegende Doktorarbeit beschäftigt sich mit Anwendungen der Anti-de-Sitter/Konforme-Feldtheorie (AdS/CFT) -Korrespondenz. Die AdS/CFT Korrespondenz ist eine mutmaßliche Dualität in der Superstringtheorie zwischen einer stark gekoppelten vier-dimensionalen $\mathcal{N} = 4$ superkonformen Yang-Mills-Theorie und einer schwach gekoppelten Typ IIB Stringtheorie in einer fünf-dimensionalen AdS-Raumzeit. Diese Dualität liefert eine leistungsfähige Methode um stark gekoppelte vier-dimensionale Systeme im Niederenergiebereich zu untersuchen, in der die nötigen Berechnungen stellvetretend in einer schwach gekoppelten fünf-dimensionalen Supergravitation ausgeführt werden. In dieser Arbeit benutzen wir die AdS/CFT Korrespondenz um drei verschiedene stark gekoppelte Systeme zu erforschen, nämlich eine Branenwelt, die eine stark gekoppelte Feldtheorie beherbergt, ein stark gekoppeltes Fluid, das auf einer Drei-Sphäre propagiert, und eine stark gekoppelte Supraflüssigkeit mit p-Symmetrie. In allen drei Fällen beinhaltet die duale Supergravitationsbeschreibung geladene Schwarze Löcher in der AdS-Raumzeit.

Das erste hier untersuchte System ist eine Randall-Sundrum-artige Branenwelt, die sich im Hintergrund eines fünf-dimensionalen nicht-extremalen Schwarzen Lochs der geeichten $\mathcal{N} = 2$ Supergravitation bewegt. Es stellt sich heraus, dass die Bewegungsgleichungen der Brane den Friedmann-Robertson-Walker (FRW)-Gleichungen für ein geschlossenes Universum entsprechen. Das geschlossene Branenuniversum hat spezielle thermodynamische Eigenschaften. Die Energie der Branenfeldtheorie weist einen subextensiven Casimir-Anteil auf, und die Entropie kann durch eine Cardy-Verlinde-artige Formel ausgedrückt werden. Es wird gezeigt, dass beide Größen in einer Form geschrieben werden können, die jeweils den zwei FRW-Gleichungen ähnelt. Am Ereignishorizont des Schwarzen Lochs verschmelzen diese beiden Sets von Gleichungen sogar miteinander, was auf die Existenz einer gemeinsamen zugrundeliegenden Theorie schließen lassen könnte. Zusätzlich finden wir, als Nebenresultat, dass die nicht-extremalen Schwarze-Loch-Lösungen eine alternative Beschreibung durch Differentialgleichungen erster Ordnung, sogenannter Flussgleichungen, zulassen. Ähnliche Flussgleichungen sind bekannt vom Attraktor-Mechanismus extremaler Schwarzer Löcher in der Stringtheorie.

Das zweite hier zu erforschende System ist ein konformes Fluid, das auf einer Drei-Sphäre propagiert. Durch das endliche Volumen der Drei-Sphäre enthält die Gesamtenergie des Fluids wieder einen subextensiven Casimir-Anteil. Wir untersuchen mögliche Korrekturen zur bekannten Rate aus Scherviskosität und Entropiedichte $\eta/s = \hbar/(4\pi k_B)$ im Falle von Fluiden auf einer Drei-Sphäre. Dazu konstruieren wir verschiedene deformierte Schwarze-Loch-Lösungen im AdS-Raum im Rahmen des STU-Modells in der geeichten $\mathcal{N} = 2$ Supergravitation. Diese neuen Lösungen sind dual zu verschiedenen Fluiden mit einem bestimmten Geschwindigkeitsfeld. Dann berechnen wir die entsprechenden Energie-Impuls-Tensoren der Fluide. Dabei stellt sich heraus, dass die Scherviskosität in der dritten Ordnung der Gradientenentwicklung des Energie-Impuls-Tensors eine positive Korrektur erhält, die proportional zur Krümmung der Drei-Sphäre ist.

Das dritte hier zu untersuchende System ist eine Supraflüssigkeit mit p-Symmetry. Dazu konstruieren wir numerisch die duale nicht-Abelsche Schwarze-Loch-Lösung in der $SU(2)$- Einstein-Yang-Mills-Theorie mit flachem Horizont im AdS-Raum. Dabei berücksichtigen wir die komplette Kopplung der Eichfelder an die Hintergrundgeometrie. Für eine ausreichend niedrige Temperatur entwickelt diese Schwarze-Loch-Lösung Vektor-Haare, was in der dualen Feldtheorie einem Übergang in eine supraflüssige Phase mit spontan gebrochener Rotationssymmetrie entspricht. Die Einstein-Yang-Mills-Theorie hat einen einzigen freien Parameter, den wir mit α bezeichnen, nämlich die Rate aus der fünf-dimensionalen Gravitationskonstante und der Yang-Mills-Kopplungskonstante. Es zeigt sich, dass sich der Phasenübergang für Werte von α über einem kritischen Wert $\alpha_c = 0.365 \pm 0.001$ von einem Übergang zweiter Ordnung zu einem Übergang erster Ordnung ändert.

Diese Doktorarbeit basiert auf der Forschungsarbeit der Autorin am Max-Planck-Institut für Physik in München zwischen März 2007 und Mai 2010. Diese wurde in den Publikationen [1–3] veröffentlicht.

Abstract

In this thesis, we deal with different applications of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. The AdS/CFT correspondence, which is also more generally referred to as gauge/gravity duality, is a conjectured duality in superstring theory between strongly-coupled four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory and weakly-coupled type IIB string theory in five-dimensional AdS spacetime. This duality provides a powerful method to investigate strongly-coupled low-energy systems in four dimensions by substitutionally carrying out calculations in five-dimensional weakly-coupled supergravity. In this work, we use the AdS/CFT correspondence to explore three different strongly-coupled systems, namely a brane world accommodating a strongly-coupled field theory, a strongly-coupled fluid on a three-sphere and a strongly-coupled p-wave superfluid. In all these cases, the dual supergravity descriptions involve charged AdS black holes.

The first system studied here is a Randall-Sundrum brane world moving in the background of a five-dimensional non-extremal black hole of $\mathcal{N} = 2$ gauged supergravity. The equations of motion of the brane are found to be equal to the Friedmann-Robertson-Walker (FRW) equations for a closed universe. The closed brane universe has special thermodynamic properties. The energy of the brane field theory exhibits a subextensive Casimir contribution, and the entropy can be expressed as a Cardy-Verlinde-type formula. We show that the equations for both quantities can take forms that strongly resemble the two FRW equations. At the horizon of the black hole, these two sets of equations are shown to even merge with each other which might suggest the existence of a common underlying theory. In addition, as a by-product result, the non-extremal black hole solutions considered here are found to admit an alternative description in terms of first-order flow equations similar to those which are well-known from the attractor mechanism of extremal black holes in string theory.

The second system to explore here is a conformal fluid propagating on a three-sphere. Due to the finite volume of the three-sphere the total energy again contains a subextensive Casimir contribution. We investigate possible corrections to the famous ratio of shear viscosity to entropy density $\eta/s = \hbar/(4\pi k_B)$ in case of fluids on a three-sphere. For this purpose, we construct different deformed black hole

solutions on the basis of the AdS-STU black holes of $\mathcal{N} = 2$ gauged supergravity. These new black hole solutions are dual to different fluids with a specified fluid flow. Then, we compute the corresponding fluid energy-momentum tensors. It turns out that the shear viscosity receives a positive correction at third order in the derivative expansion of the energy-momentum tensor which is proportional to the curvature of the three-sphere.

The third system, which we investigate, is a p-wave superfluid. For this purpose, we numerically construct the dual non-Abelian AdS black hole solution with a flat horizon in $SU(2)$ Einstein-Yang-Mills theory, taking the full back-reaction of the gauge fields on the geometry into account. For sufficiently low temperature, this black hole solution develops vector hair which in the dual field theory corresponds to a phase transition to a superfluid state with spontaneously broken rotational symmetry. The bulk theory has a single free parameter, the ratio of the five-dimensional gravitational constant to the Yang-Mills coupling constant, which we denote as α. We find that for values of α above a critical value $\alpha_c = 0.365 \pm 0.001$, the transition changes from second to first order.

This thesis is based on work which was carried out by the author between March 2007 and May 2010 at the Max-Planck Institute for Physics in Munich, and which was published in [1–3].

Acknowledgements

First of all, I would like to thank Dieter Lüst for giving me the opportunity to complete my doctoral studies under his supervision in such an inspiring scientific environment. It was a pleasure for me to be a member of his research group in Munich during the last years.

Special thanks go to Gabriel Lopes Cardoso for his continuous supervision during my first two years as well as for his guidance and collaboration on two very interesting projects, and to Johanna Erdmenger for letting me be a member of her nice group during the last year, for her organization of social get-togethers as well as for her collaboration on another very interesting project.

I would like to thank all the remaining people who I enjoyed to collaborate with: Martin Ammon, Gianguido Dall'Agata, Patrick Kerner and Andy O'Bannon.

Furthermore, I want to thank Michael Haack and Amos Yarom for enlightening discussions about the fluid/gravity correspondence.

I also want to thank my former office mates Daniel Härtl, Nitin Rughoonauth and Maximilian Schmidt-Sommerfeld for interesting conversations and a nice atmosphere, and all the members of the AdS/CFT group at the MPI for proof-reading the chapters of this thesis as well as being kind and helpful colleagues.

Finally, I would like to thank my friends, my brother, my sister, my parents and especially my boyfriend Michael for their understanding, support and love.

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Chapter 1

Introduction and overview

The long-term objective of particle physics research is to understand the fundamental building blocks of matter and their interactions. Today, four basic interactions are known: the electromagnetic interaction acting between electrically charged particles, the weak interaction which is responsible for the nuclear beta decay, the strong interaction holding together the constituents of protons and neutrons, and gravity as the attractive force between masses. Through their infinitely long range, the electromagnetic and the gravitational interactions enter everyday life, and have therefore already been known for a long time. In contrast, the weak and the strong interactions are with 10^{-16} cm and 10^{-13} cm very short-ranged and therefore discovered later. Compared to the strong interaction, the electromagnetic interaction is 137 times weaker, the weak interaction is 10^5 times weaker and gravity is on the order of 10^{39} times weaker[1]. Despite their immense differences in range and strength, physicists believe that all interactions might be described by a single underlying theory and that a more complete understanding of the physical laws of nature goes along with the discovery of such a theory.

To date, a verified universal theory treating all interactions in the same manner is still far from being discovered. Nevertheless, there have been a number of important achievements since the first attempt in the 1920'ies when Kaluza and Klein proposed a model to unify the classical theories of gravity and electromagnetism [4, 5]. When quantum mechanics came into play in the 1940'ies, the task turned out to be much more difficult. An intermediate result of this quest for a unified theory is our current fundament of theoretical physics which is formed of the *standard model of particle physics* combining the electromagnetic, the weak and the strong interactions, and Einstein's *general theory of relativity* describing gravity. The main difference between these two is that the standard model is a quantum theory while general relativity is a classical theory which fails to be quantized by means of the usual procedure. This fact strongly suggests that there

[1] The exact strengths depend on the particles and the energies involved.

must exist another more fundamental theory unifying quantum theory with gravity. A promising candidate which seems to accomplish this is *string theory* [6, 7]. So far, string theory is merely a highly developed ansatz which has to be confirmed experimentally in the future to become a complete theory. Despite this, it has afforded several interesting concepts to solve some of the problems of the standard model. Probably the most successful new concept is the conjecture of the *Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence* [8, 9]. The AdS/CFT correspondence has two remarkable features. First, in its strongest form, it is conjectured to be an exact relation between non-perturbative string theory, including gravity, and quantum field theory without gravity at all. Second, in a well-controlled low-energy limit, it relates a strongly-coupled quantum field theory to weakly-coupled classical gravity. Thus, in a strong-coupling regime in which perturbation theory usually fails to be applicable, AdS/CFT provides an effective description of the strong-coupling dynamics. This was not achieved before, and so the feature of the weak/strong coupling duality has already been extensively exploited in various applications of the AdS/CFT correspondence involving strongly-coupled systems. Some of these applications, which are discussed in this thesis, are strongly-coupled thermal field theories such as those which describe the recently-discovered *Quark-Gluon Plasma (QGP)* [10] produced at heavy ion colliders. Another striking field of application are *quantum critical condensed matter systems* [11–13], possibly including high-T_c superconductors[2]. The third application is the theory of *brane worlds* [14] which suggests that our world is a four-dimensional membrane moving in a five-dimensional space.

Before explaining in section 1.4 the aims and results of the research presented in this thesis, the underlying theories and concepts are introduced. In section 1.1 we give a rough idea of the standard model and general relativity and argue why these theories are not fundamental. In section 1.2, we briefly introduce string theory and argue that this might be a promising candidate to unify the standard model and gravity. Section 1.3 gives a rough overview of the AdS/CFT correspondence.

1.1 Standard model and general relativity

The standard model of particle physics is a quantum field theory with gauge group $SU(3) \times SU(2) \times U(1)$. It contains the electroweak theory with gauge group $SU(2) \times U(1)$ unifying the electromagnetic and the weak interaction as well as Quantum Chromodynamics (QCD) with gauge group $SU(3)$ describing the strong interaction. The particle content of the standard model comprises on the one hand the force mediators which are the photon of the electromagnetic force,

[2]Here T_c denotes the critical temperature were the superfluid transition occurs.

the W and Z bosons of the weak force and the gluons of the strong force. These are all spin-1 bosons. On the other hand, there are matter particles including three generations of leptons and quarks. In addition, there is strong evidence for a scalar particle, the Higgs boson, which still lacks experimental confirmation, but might be found in the near future at the Large Hadron Collider (LHC) at CERN. Apart from that, the standard model has been tested to very high precision and proves to be valid for thirty years.

The standard model is supplemented by Einstein's general theory of relativity describing gravity. General relativity is a classical theory unifying special relativity and Newtonian gravity. In general relativity, the gravitational interaction is mediated by the geometry of the spacetime. Thus, masses/energy cause the surrounding spacetime to curve. The geometry is measured in terms of the metric field playing the role of the gravitational field. In a quantized version of general relativity, the quanta of the metric field are represented by a spin-2 gauge boson, the *graviton*, which in analogy to the standard model interactions mediates the gravitational force. Unfortunately, the usual quantization procedure fails to be consistent, since general relativity is non-renormalizable. That is, it cannot be treated within the framework of perturbation theory which is only applicable in an energy regime where the coupling constant is small. Normally, at higher orders of the perturbative expansion of interacting quantum field theories, divergent terms appear which can be absorbed by a finite number of counterterms. However, in case of quantized general relativity, an infinite number of such divergent terms are produced which cannot be canceled by a finite number of counterterms.

General relativity proves to be valid in astrophysical and cosmological observations, such as the *gravitational red-shift* or the *gravitational lens effect*, whereas at very small distances it breaks down, since quantum effects have to be taken into account. In contrast, the standard model governs the physics on nuclear scales, but does not include gravity which becomes important on very large scales. Despite the successes of both theories in their different regimes of validity, it remains unsatisfying that phenomena such as the Big Bang singularity or certain properties of black holes, where both types of interactions simultaneously become important, cannot be exhaustively described. Probably the most prominent puzzle in a quantum mechanical description of black holes is the information loss paradox which states that information forever disappears behind a black hole horizon implying that all states inside a black hole horizon are the same. Connected to this puzzle is the question, whether the black hole entropy, given by the *Bekenstein-Hawking formula*, has an explanation in terms of the number of microstates as in statistical mechanics.

In addition, the standard model has further shortcomings on its own. For instance, the standard model has about twenty free parameters which have to be put in by hand letting the theory appear somehow arbitrary. Moreover, as a perturbative quantum field theory, it relies on small coupling constants. However, the coupling constants are not true constants, but depend on the energy scale. This implies that the standard model only provides correct results at a certain energy scale, namely for which the coupling constant remains small. In particular, the coupling constant of QCD grows large for small energies, leading to confinement of quarks and gluons, and small for very large energies at which the quarks and gluons are asymptotically free. Thus, confinement is difficult to study whereas asymptotic freedom can be studied easily using QCD.

The incompatibility of the standard model and general relativity as well as their own shortcomings point to the possible existence of a more fundamental theory in view of which both theories have to be considered as effective theories at different energy scales.

1.2 String theory

A promising candidate for a theory of everything incorporating gravity and the standard model interactions is string theory. It is named after its most striking feature, which is the fact that particles are represented by vibrational modes of a one-dimensional object, a *string*, with a length on the order of the Planck length, $l_P = 1.6 \times 10^{-33}$ cm. These strings can be open or closed. While closed strings can move everywhere in space, open strings are attached to hyperplanes called *branes* along which they can move freely

To date, string theory is not a verified theory yet, it can rather be regarded as a prototype of a complete theory. Nevertheless, string theory already affords a number of interesting and revolutionary concepts suggesting solutions to some of the problems mentioned in the preceding section.

General relativity is included in string theory as an effective theory arising at low energies and large distances. In string theory, gravity is UV finite such that divergencies at higher orders in the perturbative expansion of gravity do not occur. In addition, string theory contains Yang-Mills gauge theories of the type of the standard model. However, an explanation of why $SU(3) \times SU(2) \times U(1)$ is singled out in nature is still missing. Moreover, in contrast to the standard model, string theory has no adjustable dimensionless free parameters, and is therefore unique in contrast to the standard model.

String theory admits a very promising duality relation, known as AdS/CFT correspondence, which equates string theory with conformal field theory. The AdS/CFT correspondence tries to solve several outstanding problems of quantum theory and gravity. For instance, it was suggested that AdS/CFT could explain the information loss paradox [15]. Moreover, the AdS/CFT correspondence is a weak/strong coupling duality and therefore, in principle, qualifies to describe confinement in QCD.

String theory has still more interesting implications. A major prediction is the existence of *supersymmetry* which is an appealing symmetry relating bosons and fermions, and which is likewise conjectured to exist outside string theory. Furthermore, string theory exhibits *extra dimensions*. Supersymmetric string theories are only consistent in ten dimensions which contradicts experimental experience. Fortunately, at low energies or large distances these extra dimensions can be hidden, such that string theory looks effectively four-dimensional. In this case, the extra dimensions are regarded as compactified on spaces with very special topological properties, such as *Calabi-Yau spaces*. By means of including extra dimensions, the entropy of black holes, which was hitherto given by the Bekenstein-Hawking formula, could be shown to admit a description in terms of microstates [16].

In total, five different supersymmetric string theories are known to be consistent in ten dimensions. These are type I, type IIA/B and two heterotic theories with gauge groups $SO(32)$ and $E_8 \times E_8$, respectively. However, these theories are not independent. It is believed that they arise as different limits of a unique eleven-dimensional theory called *M-theory*.

In summary, string theory or, equivalently, M-theory provides promising suggestions to solve problems in quantum gravity. Nevertheless, it is still incomplete and has to be verified by experiments to become a generally accepted theory in the future. The AdS/CFT correspondence may accelerate the progress in narrowing the gap between string theory and experiments.

1.3 AdS/CFT correspondence

As originally conjectured by Maldacena in 1997 [17], there exists a duality within string theory, relating a CFT to type IIB superstring theory in AdS space. This is known as the AdS/CFT correspondence. The conformal field theory entering the correspondence is given by four-dimensional $\mathcal{N} = 4$ *super Yang-Mills theory* which is a gauge theory including four supersymmetries. Anti-de Sitter space is a five-dimensional maximally symmetric space with constant negative curvature. In its strongest form, the AdS/CFT correspondence claims to constitute an exact relation between non-perturbative string theory including gravity and a quantum field theory without gravity at all. Besides this first remarkable fact, a second remarkable fact is, that in a well-controlled low-energy limit, this correspondence is a duality between classical weakly-coupled supergravity and a strongly-coupled quantum field theory. For this form of the correspondence, an explicit dictionary can be formulated translating between the two dual theories. In this respect, AdS/CFT qualifies to effectively describe quantum field theories at strong coupling similar to the confined phase of QCD. Outside AdS/CFT, effective descriptions of strongly-coupled systems in general do not exist. This is the reason why AdS/CFT is of significance for physicists of different research areas such as high-energy physics as well as condensed matter physics.

The AdS/CFT correspondence can also describe conformal field theory at finite temperature [18] and finite charge density. For that purpose, the five-dimensional AdS space is modified by a black hole sitting in its center which heats the field theory up to finite temperature through Hawking radiation. Finite charge density in the field theory is obtained by assigning conserved charges to the black hole. In this way, the thermodynamic properties of the conformal field theory are related to the thermodynamic properties of the black hole. A prominent example for a realistic strongly-coupled thermal field theory is given by the QGP which is described by QCD near the confinement-deconfinement temperature. This system is similar to the special Yang-Mills theory entering the AdS/CFT and thus might be investigated by means of the AdS/CFT correspondence, at least by drawing parallels between the two theories.

However, despite the similarity between the conformal field theory involved in the AdS/CFT correspondence, $\mathcal{N} = 4$ super Yang-Mills theory, and QCD, there are essential differences between the two theories. In addition, no modification of the correspondence is known to directly include QCD, rendering quantitative statements about QCD impossible. Nevertheless, there seem to exist properties which are universal for a larger class of strongly-coupled theories, including QCD and $\mathcal{N} = 4$ super Yang-Mills theory. Thus, it might be at least possible to qualitatively extract information about QCD by investigating $\mathcal{N} = 4$ super Yang-Mills theory via AdS/CFT. There is strong evidence that an example of such a universal property is the ratio of the shear viscosity of a strongly-coupled plasma to its entropy density. This quantity was measured for the QGP at the Relativistic Heavy Ion Collider (RHIC) and found to be very similar to the value which was predicted by AdS/CFT on the basis of $\mathcal{N} = 4$ super Yang-Mills theory.

Another interesting observation is that the AdS/CFT correspondence can be interpreted as a realization of the holographic principle [19, 20]. AdS space is infinite in extent, but nevertheless exhibits a boundary. It can be argued that the dual field theory can be considered to live on the four-dimensional boundary of the five-dimensional AdS space. Then, by virtue of the correspondence, the five-dimensional physics inside the AdS boundary is captured by the four-dimensional CFT on the boundary.

During the last years, the AdS/CFT correspondence has found many interesting fields of application involving strongly-coupled systems. Some of these are covered in this thesis such as holographic brane worlds, hydrodynamics of strongly-coupled fluids and holographic superconductors. The extensions and numerous applications of AdS/CFT are today often combined under the generic term *gauge/gravity dualities*. These gauge/gravity dualities not only provide interesting new insights into strongly-coupled systems, but also into the physics of black holes. For instance, new black hole solutions are constructed in order to describe special types of dual field theory systems.

Finally, it is worth mentioning, that in view of the similarity between the AdS/CFT prediction and the measurement of η/s for the QGP, we might speculate whether in the future gauge/gravity dualities could provide a channel for experimental verification of the AdS/CFT correspondence or even string theory.

1.4 Overview of the research projects covered in this thesis

The aim of the research presented in this thesis is to be suggestive of the broad range of applications of the AdS/CFT correspondence, to conduct further theoretical tests of the AdS/CFT correspondence and simultaneously obtain new insights into strongly-coupled systems at finite temperature which were hardly accessible before the discovery of such a weak/strong coupling duality. For that purpose, three different applications are considered, namely holographic brane cosmology (see 1.4.2), the fluid/gravity correspondence (see 1.4.3) and holographic superconductors (see 1.4.4). A large part of the work consists in finding black hole solutions as well as studying their properties. This can also lead to interesting results in black hole physics itself. For instance, in investigating special aspects of holographic brane cosmology and the fluid/gravity correspondence we make use of a special background geometry which is given by the *non-extremal charged static black hole solutions of $\mathcal{N} = 2$ gauged supergravity* in five dimensions. Therefore, before dealing with applications of AdS/CFT, we study a special aspect of these black hole solutions, namely whether they can be derived by means of first-order differential equations instead of the usual second-order equations of motion (see 1.4.1).

1.4.1 First-order flow equations for non-extremal black holes

Besides a detailed review of the five-dimensional non-extremal charged static black hole solutions of $\mathcal{N} = 2$ gauged supergravity [21] in chapter 3, we uncover a new convenient feature of these solutions, namely that they admit a description in terms of *first-order differential (flow) equations* [1].

Since the advent of gauge/gravity dualities there has been renewed interest in gauged supergravity theories in various dimensions. In this thesis, we are primarily interested in five-dimensional $\mathcal{N} = 2$ gauged supergravity [22] which has AdS space as a vacuum solution. The five-dimensional supergravity can be derived from the eleven-dimensional supergravity, which is the low-energy limit of M-theory, by compactifying six extra dimensions [23–25]. Moreover, a special truncation of the aforementioned black hole solutions can be embedded into type IIB string theory, for which reason these can be investigated in the context of the AdS/CFT correspondence.

These non-extremal AdS black hole solutions were first constructed in [21] by solving the equations of motion derived from the $\mathcal{N} = 2$ gauged supergravity action. As is shown in chapter 3, they can be alternatively derived by solving a set of first-order differential (flow) equations.

First-order flow equations are well-known from the *attractor mechanism* which is a feature of a class of black hole solutions in supergravity. General black hole solutions of supergravity are supported by neutral scalar fields. In attractive black hole backgrounds, these scalar fields are driven to values which are completely determined by the charges carried by the black hole, regardless of their values at infinity. Here, the radial evolution, from infinity to the horizon of the black hole, follows a set of first-order differential equations, which constitute a gradient flow on the target space of the scalar fields, governed by a generalized *superpotential*. The attractor mechanism was first observed for the special case of supersymmetric black holes [26–29] for which the flow equations are implied by supersymmetry. In this case, the superpotential is expressed through the central charge. In contrast, for non-supersymmetric black holes, the flow equations are no longer guaranteed to exist. For some non-sypersymmetric extremal black holes first-order flow equations were derived in [30, 31]. Non-extremal black holes never exhibit attractor behavior [32, 33] and are therefore a priori not expected to admit a description in terms of first-order flow equations.

However, for the special non-supersymmetric non-extremal black hole solutions we look at, such first-order flow equations exist, even though they are not attractive. The first-order equations can be shown to be consistent with the equations of motion, and thus provide an alternative and, in addition, easier derivation of the non-extremal black hole solutions.

1.4.2 Holographic brane cosmology

In chapter 4, a *Randall-Sundrum-type* brane world [34] with vanishing cosmological constant is considered in the context of the AdS/CFT correspondence to investigate a surprising relation between thermodynamic properties of the brane world and its cosmological evolution equations. Such a brane world is essentially a three-brane moving in the background of a five-dimensional static AdS black hole. In the Randall-Sundrum brane-world picture all the standard model fields and also four-dimensional gravity are confined to the three-brane. It was proposed to be an alternative to compactification in the sense that our four-dimensional world arises from a higher-dimensional theory while the extra dimensions stay large but inaccessible from the four-dimensional world.

It was suggested that this construction can be viewed in light of the AdS/CFT correspondence relating the brane theory to the bulk theory [35]. The main difference to the usual AdS/CFT prescription is that five-dimensional gravity couples to the brane. This enables the bulk black hole to influence the brane metric during its motion. The metric of the spherical brane turns out to take the standard *Friedmann-Robertson-Walker (FRW)* form and its equations of motion are the *Friedmann equations*[3] for a closed universe. Via AdS/CFT, the cosmological evolution can also be regarded as being driven by the dual field theory on the brane which can be imagined as follows. The brane starts to expand inside the black hole and passes the horizon. The expansion goes on up to some turning point at which it starts to recontract, and finally it falls again through the horizon of the black hole.

According to the usual AdS/CFT prescription, the entropy of the CFT on the brane is given by the Bekenstein-Hawking entropy of the black hole [18]. In addition, Verlinde made the interesting proposal that the *Cardy formula* for the entropy of a (1+1)-dimensional CFT can be generalized to arbitrary dimensions [36]. He showed that by choosing a specific normalization, the generalized Cardy formula is equal to the Bekenstein-Hawking entropy. Furthermore, Verlinde found out that, when written as a generalized Cardy formula, the entropy of the CFT takes a form that is similar to the first Friedmann equation, while the Casimir energy of the CFT takes a form that is similar to the second Friedmann equation [37]. Moreover, at the moment when the brane passes the horizon both sets of equations, the thermodynamic equations and the cosmological equations, even coincide. These two moments in the evolution of the brane universe, when it crosses the black hole horizon, seem to be special, since the entropy satisfies a cosmological bound proposed by Verlinde [36]. This merging suggests that there might exist an underlying theory relating the Friedmann equations to the entropy and the Casimir energy. This phenomenon seems to be independent of the equation of state characterizing the matter on the brane, and it was already probed for different types of matter [38–41].

The question which is answered in chapter 4 is, whether the merging of the Friedmann equations with the entropy and the Casimir energy still holds for the exotic brane field theory dual to the non-extremal static electrically charged black hole solution of $\mathcal{N} = 2$ gauged supergravity [1]. For that purpose, we derive the equations of motion of a brane with vanishing cosmological constant in this black hole background, and show that these take the form of the standard Friedmann equations for a closed universe with an energy density exhibiting a complicated scaling behavior. Then, we check that the entropy of such a field theory can be written as a *Cardy-Verlinde-type formula* modified by two functions of the scalar fields supporting the black hole solution. Furthermore, we compute the *Casimir energy* and the extensive energy by using an analog of the Smarr formula. Finally,

[3] In the following, the Friedmann equations are often referred to as FRW equations.

when comparing both sets of equations at the black hole horizon, we find that they again coincide.

Toward the end of chapter 4, we try to further generalize these findings by including higher-derivative curvature terms in the five-dimensional action. However, such a generalization turns out to be difficult, since the notion of a Cardy-Verlinde formula in the context of higher-derivative gravity is unclear.

1.4.3 Fluid/gravity correspondence

In chapter 5, we deal with the *fluid/gravity correspondence* [42, 43] which essentially means the application of the finite-temperature AdS/CFT correspondence to strongly-coupled field theories in the hydrodynamic regime. This regime can be accessed by focussing on near-equilibrium dynamics and restricting to long wavelengths. The energy-momentum tensor of a fluid belongs to a special class of conserved energy-momentum tensors which are determined by only four parameters. Therefore, only a special class of AdS black holes is suitable to model strongly-coupled fluids. These are the *boosted* black hole solutions in *Eddington-Finkelstein coordinates*, which are four-parameter families of solutions that are regular at the black hole horizon. The fluid/gravity correspondence constitutes a concrete relation between fluid dynamics and black holes.

Exploring the fluid/gravity correspondence is of significance for theoretical as well as experimental physics. On the one hand, fluid dynamics provides many interesting and yet unmanageable long-term challenges as for instance the search for globally regular solutions to the Navier-Stokes equations for non-relativistic incompressible viscous fluids, or a detailed understanding of turbulence in the fluid dynamical evolution. A holographic mapping of the fluid dynamical system to classical gravitational dynamics may shed new light on these issues [44–46].

On the other hand, the fluid/gravity correspondence opens a new perspective on the physics of black holes by means of fluid dynamics. It provides an algorithm to systematically construct regular black hole solutions whenever a solution to the fluid equations of motion is given. Moreover, the stability and aspects of the phase structure of black holes may be understood through the fluid model.

The strongest motivation, however, for investigating the fluid/gravity correspondence is its relevance for real systems which can be studied in experiments. In particular, it proves to be useful in describing the dynamics of the QGP. The QGP is a state of matter consisting of quarks and gluons. It is believed that, originally, this kind of matter existed shortly after the Big Bang. Nowadays, it can be produced and studied at heavy ion colliders, for instance at RHIC. There, the QGP was created at a temperature just above the confinement-deconfinement temperature $T_c \approx 170\text{MeV}$. RHIC data reveals that the QGP is strongly coupled and suggests that it behaves like a nearly perfect fluid. Moreover, it will soon be investigated more intensely at the LHC at CERN at a temperature of about

$5T_c$ [47].

A description of the dynamics of the strongly-coupled QGP is difficult, since conventional techniques such as perturbation theory, which only makes sense for weakly- coupled theories, or lattice QCD, which is useful for calculating equilibrium properties, are not applicable. In contrast, the AdS/CFT correspondence as a tool to describe strongly-coupled field theories is more successful. Even though the QGP is qualitatively very different from the field theory described by the AdS/CFT correspondence, it is believed that these differences become less crucial for a temperature $T \approx T_c$, and that there exist universal properties. A quantity that seems to be universal is given by the *shear viscosity to entropy density ratio*, η/s. In [48], it was conjectured by means of the AdS/CFT correspondence that the lowest possible value for η/s is given by that of a strongly-coupled superconformal *Yang-Mills plasma* in the large N, large λ limit, which is

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \approx 0.08 \frac{\hbar}{k_B} \,. \tag{1.1}$$

First calculations based on RHIC data show that the lowest value for the QGP is $\eta/s \approx 0.1(\hbar/k_B)$ [10, 49] which is remarkably close to (1.1). The bound (1.1) was tested under different conditions [50–52]. It turned out that away from the large N limit it can be violated.

So far, the dual fluid was assumed to propagate in a flat spacetime. In chapter 5, we ask the question, how is the ratio η/s corrected, if we slightly move away from the hydrodynamic limit and consider the fluid to propagate on a three-sphere [2]. From chapter 4, we know that the energy of a field theory on a three-sphere has a Casimir contribution. Since the ratio η/s is proportional to the total energy we in fact expect a correction proportional to the Casimir energy. In order to investigate this issue, we construct dynamical black hole solutions and map these to the energy-momentum tensors of fluids by means of the AdS/CFT correspondence. The shear viscosity can be read off from the shear term in the energy-momentum tensor. This analysis is repeated for different special cases of the charged black hole solutions of $\mathcal{N} = 2$ gauged supergravity supported by scalar fields. We find that the corrections to η/s are positive in all cases, and proportional to the Casimir energy in the special cases in which the scalar fields are constant. In case of non-trivial scalar fields the corrections differ from the Casimir energy.

1.4.4 Holographic superconductors

In chapter 6, we deal with the application of the AdS/CFT correspondence to superconductors, a field which is referred to as holographic superconductors. Holographic superconductors can be defined as strongly-coupled field theories which undergo a superconducting phase transition at some critical temperature and which have a gravity dual in the sense of the AdS/CFT correspondence [53–55].

Usually, conventional superconductors are extremely well explained by the *BCS (Bardeen-Cooper-Schrieffer) theory* and its extensions. In these theories a charged composite operator condenses at sufficiently low temperature due to the attraction of two fermionic quasiparticles. These composite operators are called *Cooper pairs* and the attractive force is mediated by phonons. However, BCS theory fails to describe non-conventional superconductors such as high-T_c superconductors. A reason for that is, that such systems are strongly coupled above the critical temperature of the superconducting phase transition, and therefore do not admit a weakly-interacting quasi-particle description. A microscopic description of this type of superconductors is still lacking and the search for it is an active research area in theoretical physics.

Recently, new light on this question was shed by the AdS/CFT correspondence. It was found that AdS/CFT can be used to model some sort of high-T_c superconductors. This is reasonable, since high-T_c superconductors behave similar to quantum critical systems which at zero temperature exhibit spacetime scale invariance. Moreover, quantum critical systems are strongly coupled and therefore very similar to the strongly-coupled CFT entering the AdS/CFT correspondence. The term quantum critical system derives from the fact that such systems undergo a quantum phase transition at zero temperature. These transitions are driven by quantum fluctuations rather than thermal fluctuations. The position in the phase diagram at which a quantum phase transition occurs is referred to as quantum critical point. It is believed that quantum critical points can influence the system at finite temperature.

However, to date, AdS/CFT can only provide an effective description of toy-model superconductors, since AdS/CFT itself is most powerful in a specific low-energy limit, and exact gravity duals to realistic superconductors are not known yet. Nevertheless, it constitutes an attractive method to calculate observables in a strongly-coupled superconducting system which has many properties in common with realistic superconductors. The general hope in investigating such holographic superconductors is to finally get closer to a full microscopic description of non-conventional superconductors which comprises the effective AdS/CFT description as a specific limit, analogous to BCS theory from which the phenomenological Landau-Ginzburg theory of superconductivity can be derived.

In general, a superconductor is a material in which electromagnetic gauge invariance is broken. Within the framework of gauge field theory, this is expressed as the spontaneous breaking of a $U(1)$ gauge symmetry. Spontaneous symmetry breaking occurs, if an operator charged under the $U(1)$ acquires a non-zero expectation value which is referred to as the *operator condensing*. The simplest bulk action that can describe such a transition is Einstein-Maxwell theory coupled to a charged scalar which is zero in the normal phase of the system and non-zero in the superconducting phase. It is worth noting that the field theory dual to the bulk Einstein-Maxwell-scalar system does not have a dynamical photon, thus it is more accurately called a *superfluid*.

AdS/CFT can also describe superfluid states in which the condensing operator is a vector and hence rotational symmetry is broken, that is, *p-wave* superfluid states [56, 57]. Here the CFT has a global $SU(2)$ symmetry and for a sufficiently large charge density for some $U(1)$ subgroup of $SU(2)$, the charge current operator associated to this $U(1)$ condenses. In this case not only the $U(1)$ is broken, but spatial rotational symmetry is also broken to some subgroup.

Previous analyses of holographic p-wave superfluids have employed the probe limit which means that the influence of the gauge fields on the background geometry was neglected. In contrast to that, in chapter 6 we consider an Einstein-Yang-Mills system to construct a holographic p-wave superfluid with full back-reaction of the gauge fields on the bulk background [3]. For that purpose, we numerically find a new non-Abelian AdS black hole solution with a flat horizon which for sufficiently low temperature has a non-zero vector field. The strength of the back-reaction is measured by the ratio $\alpha = \kappa_5/\hat{g}$, where κ_5 is the gravitational constant and $\hat{g}$ is the Yang-Mills coupling constant. Moreover, we investigate the phase structure of these solutions and find that the superfluid transition is second order for some value of α below a critical value α_c. Above this critical value, we find that the phase transition becomes first order.

1.5 Outline of this thesis

The main part of this thesis is structured as follows.

In chapter 2, we give an introduction to the AdS/CFT correspondence in view of the applications discussed in chapter 4, 5 and 6. Therefore, of particular importance is the extension to finite temperature and finite density as well as the correspondence in case of global Anti-de Sitter space. In this case, the dual field theory is considered to live on a three-sphere exhibiting special thermodynamic properties due to the finite volume. We show that on a three-sphere the total energy of the field theory receives a Casimir contribution and the entropy can be written as a Cardy-Verlinde formula.

In chapter 3, we review non-extremal charged static black hole solutions of $\mathcal{N} = 2$ gauged supergravity in five dimensions as well as the emergence of five-dimensional $\mathcal{N} = 2$ gauged supergravity from eleven-dimensional supergravity. In addition, it is shown that the non-extremal black hole solutions in question admit a description in terms of first-order flow equations. This calculation was carried out by the author of this thesis in collaboration with Gabriel Lopes Cardoso and was published in [1]. At the end, we explain how a subset of these black hole solutions can be embedded into type IIB string theory.

Chapter 4 deals with a holographic spherical brane universe which is investigated in the context of the black hole solutions of chapter 3. We show that the cosmological evolution equations of this brane universe are in a mysterious way related to the Casimir energy and the Cardy-Verlinde entropy formula of the brane field theory. This chapter is based on work which was done by the author of this thesis in collaboration with Gabriel Lopes Cardoso and which was published in [1].

In chapter 5, we study the fluid/gravity correspondence in the context of the black hole solutions of chapter 3. In particular, we construct new deformed black hole solutions and analyze the ratio η/s with regard to finite-size and curvature effects in case of the dual fluid living on a three-sphere. This chapter is based on work which was done by the author of this thesis in collaboration with Gabriel Lopes Cardoso and Gianguido Dall'Agata, and which was published in [2].

Chapter 6 deals with a holographic p-wave superfluid. We construct new non-Abelian charged AdS black hole solutions with flat horizons including the full back-reaction of the gauge fields. These solutions are shown to undergo a superconducting phase transition. In addition, we investigate the phase structure of these solutions. This chapter is based on work which was done by the author of this thesis in collaboration with Martin Ammon, Johanna Erdmenger, Patrick Kerner and Andy O'Bannon, and which was published in [3].

Chapter 7 contains a summary of the results found within this thesis and a conclusion.

Chapter 2

The AdS/CFT correspondence

This chapter is intended to review the *Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence* in view of the research projects presented in this thesis. Therefore, the emphasis lies on ingredients which are relevant for the chapters 4, 5 and 6.

The structure of this review, which is mainly based on [8,9], is as follows. After a short overview, we start with introducing the two participating theories. Section 2.2 presents basic facts about $\mathcal{N} = 4$ super Yang-Mills theory [58–60], and 2.3 reviews type IIB superstring theory [6,7] including p-branes, D-branes [61–63] and Anti-de Sitter space. Section 2.4 presents the original Maldacena conjecture which is then made more precise with regard to explicit mappings between the two theories in section 2.5. Section 2.6 deals with important extensions of the AdS/CFT correspondence. In particular, section 2.6.1 explains how finite-temperature CFT can be included. In section 2.6.2, based on [36], we present some special properties of thermal CFTs in finite volume which play a major role in chapter 4 and 5. In particular, the Casimir energy of a thermal CFT and the Cardy-Verlinde entropy formula are introduced. Section 2.6.3 deals with the extension of the correspondence to CFTs at finite charge density. Finally, section 2.7 briefly summarizes the chapter and gives an outlook on the subsequent chapters.

2.1 What is AdS/CFT?

One of the most exciting discoveries in modern theoretical physics is the duality between gravity and gauge theory. The prototype example of such a duality is the correspondence of superconformal Yang-Mills theory in four flat dimensions and type IIB string theory on five-dimensional Anti-de Sitter space commonly known as AdS/CFT correspondence [17]. Similar dualities also exist for other dimensions, however in this thesis we exclusively deal with the correspondence in four and five dimensions.

Besides the remarkable fact that, in its strongest form, the AdS/CFT correspondence claims to relate a quantum theory to gravity, it is simultaneously a *weak/strong coupling duality* meaning that if one of the two dual theories is strongly coupled, the other one is weakly coupled and vice versa. In strongly-coupled theories explicit calculations are difficult, since perturbation theory is not applicable. The key feature of the AdS/CFT correspondence is that it might function as a tool to obtain results in the strongly-coupled theory by performing calculations in the weakly-coupled theory. The case which is of most interest in view of the search for new tools to study strongly-coupled field theories like the confinement in QCD is that of a strongly-coupled CFT dual to weakly-coupled supergravity. In this thesis, we take this perspective to investigate different strongly-coupled systems.

Moreover, the AdS/CFT correspondence can be considered as a realization of the *holographic principle* [19,20]. It becomes obvious in this chapter that just like in conventional holograms in which the information of a system in three spatial dimensions is stored on a two-dimensional plate, the information about a five-dimensional gravitational theory is stored in a CFT on a four-dimensional space.

Since its invention, the AdS/CFT correspondence was further extended and usefully applied to various systems. Some interesting applications are *strongly-coupled fluids* [42, 43] similar to the QGP, quantum critical condensed matter systems which are believed to include, for instance, *non-conventional superconductors* [11, 12], as well as the connection of *brane worlds* with the AdS/CFT correspondence [14]. These applications are dealt with in chapter 4, 5 and 6.

2.2 $\mathcal{N} = 4$ super Yang-Mills theory

One of the two theories participating in the AdS/CFT correspondence is $\mathcal{N} = 4$ *super Yang-Mills theory*. This is a highly symmetric gauge theory with $\mathcal{N} = 4$ supersymmetries relating a particle to its four superpartners, for instance a boson to four fermions. The complete global symmetry is given by the supergroup $PSU(2,2|4)$ containing the bosonic subgroup $SU(2,2) \times SU(4)$. Here $SU(2,2) \cong SO(2,4)$ is the group of conformal transformations which are diffeomorphisms preserving the metric up to an overall coordinate-dependent scale factor as well as all angles,

$$g_{\mu\nu}(x) \to \Omega^2(x) g_{\mu\nu}(x) \ . \tag{2.1}$$

There are two types of conformal transformations, the *scale transformations*

$$x^\mu \to \lambda x^\mu \,, \tag{2.2}$$

and the *special conformal transformations*

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2} \ . \tag{2.3}$$

The second part of the bosonic subgroup is the R-symmetry group $SU(4) \cong SO(6)$ which is an internal symmetry rotating the supercharges into one another. It commutes with all other symmetry generators. In addition, the supergroup contains 32 fermionic generators.

The field content of $\mathcal{N} = 4$ super Yang-Mills theory is given by a gauge multiplet containing a gauge field A_μ, four left Weyl fermions λ^a_α with $a = 1, ..., 4$ as well as six scalars X^i with $i = 1, ..., 6$. Under the $SU(4)$ R-symmetry, A_μ transforms as a singlet, λ^a_α as a **4** and X^i as a rank-two anti-symmetric **6**.

$\mathcal{N} = 4$ super Yang-Mills theory is a conformally invariant theory. This has been proved to be true also at the quantum level and is not only a property of the classical theory. The theory can be defined at all energy scales whereas, however, the coupling constant is not running, thus independent of the energy scale. This implies that, due the absence of scales in CFTs, the coupling constant of $\mathcal{N} = 4$ super Yang-Mills theory can be chosen either such that the theory appears strongly coupled or such that it appears weakly coupled. In section 2.4, it becomes clear that the coupling constant of $\mathcal{N} = 4$ super Yang-Mills theory and that of the dual string theory are inversely related by virtue of the AdS/CFT duality.

Moreover, the trace of the energy-momentum tensor of a CFT vanishes, which follows directly from Noether's theorem.

So far, we have collected the facts about $\mathcal{N} = 4$ super Yang-Mills theory which are necessary for understanding the AdS/CFT correspondence in view of the research work presented in chapter 4, 5 and 6. In the following section, we present the second theory participating in the correspondence in more.

2.3 Type IIB string theory and supergravity

String theory has the remarkable feature that it incorporates quantum theory and gravity. The most simple type of string theory is the *bosonic string*. This theory describes open and closed strings in 26 spacetime dimensions. The closed string spectrum contains, among others, a symmetric traceless state which transforms as a massless spin-2 particle under the 24-dimensional Lorentz group $SO(24)$ and which is identified with the graviton. However, this theory is unsatisfactory in two respects. First, the ground states of both the open and the closed string spectrum are tachyonic. Tachyons have negative $(mass)^2$ and thus constitute an instability of the ground state. The second shortcoming is the absence of fermionic states and thus the absence of supersymmetry. It turned out that both deficiencies can be remedied by imposing supersymmetry as well as the successive application of the *Ramond-Neveu-Schwarz (RNS) formalism* and the *Gliozzi-Scherk-Olive (GSO) projection*. In the RNS formalism the bosonic string action is modified by introducing new world-sheet fermions which are subject to certain boundary conditions. The resulting theory is the RNS string which lives in ten spacetime dimensions as required by causality. Afterwards, the GSO projection truncates the RNS spectrum in a specific way which eliminates the tachyon and simultaneously leads to two different spacetime supersymmetric theories in ten dimensions. These are the non-chiral *type IIA* and the chiral *type IIB string theory* describing closed strings with $\mathcal{N} = 2$ supersymmetry.

In this thesis we draw our attention to the IIB theory, since it is involved in the AdS/CFT correspondence. The IIB string spectrum splits into the bosonic NS-NS sector whose massless fields are the metric $G_{\hat{\mu}\hat{\nu}}$, where $\hat{\mu} = 1, ..., 10$, the scalar dilaton ϕ, a rank-two anti-symmetric tensor field $B_{\hat{\mu}\hat{\nu}}$, the bosonic R-R sector whose massless fields are a scalar C_0, a two-form field C_2 and a four-form field C_4 with a self-dual five-form field strength $F_5 = \star F_5$ as well as the fermionic NS-R and R-NS sectors each containing a spin-3/2 gravitino and a spin-1/2 dilatino. The maximal number of supercharges present in type IIB string theory is 32.

String theory can be considered as a double expansion in two parameters. One parameter is the dimensionless *string coupling constant* g_s which is the expectation value of the exponentiated dilaton. The expansion in g_s corresponds to an expansion in the number of string loops, or equivalently, in the genus of the string world sheet. The second parameter is the *Regge slope* α'. This parameter appears in the string tension per unit length

$$T = \frac{1}{2\pi\alpha'} \,, \tag{2.4}$$

which is also referred to as mass of the string per unit length. Furthermore, α' is related to the string length as $\alpha' = l_s^2$. The expansion in α' can be viewed as an expansion in the *stringiness* about the point-particle limit. Since α' has dimensions of $(length)^2$, dimensionless expansion parameters are $\alpha' E^2$ or α'/L^2 in spaces with an additional characteristic length scale L. The limit of $\alpha' E^2 \ll 1$ or $\alpha'/L^2 \ll 1$ is the *supergravity limit* of string theory. Thus at energies or inverse distances much smaller than the string scale $1/l_s^2$ string theory is well approximated by the dynamics and interactions of its massless modes which represent the field content of *type IIB supergravity.*

The bosonic part of the ten-dimensional type IIB supergravity action reads [6]

$$\begin{aligned} 16\pi G_{10} S_{\mathrm{IIB}} = & \int d^{10}x\sqrt{-G}\mathrm{e}^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ & -\frac{1}{2}\int d^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \\ & -\frac{1}{2}\int C_4 \wedge H_3 \wedge F_3 \,, \end{aligned} \tag{2.5}$$

where $G_{10} = 8\pi^6 g_s^2 (\alpha')^4$ is the ten-dimensional Newton constant and

$$F_1 = dC_0\,, \qquad H_3 = dB\,, \tag{2.6}$$

$$F_3 = dC_2\,, \qquad \tilde{F}_3 = F_3 - CH_3\,, \tag{2.7}$$

$$F_5 = dC_4\,, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3\,. \tag{2.8}$$

In addition the self-duality condition

$$\tilde{F}_5 = \star \tilde{F}_5 \tag{2.9}$$

for the five-form field strength $\tilde{F}_5$ has to be imposed as a constraint which supplements the equations of motion following from the action (2.5).

Type IIB supergravity in a flat background exhibits $\mathcal{N} = 2$ supersymmetry with 32 supercharges as the original superstring theory.

In addition to one-dimensional strings, string theory was discovered to admit solutions corresponding to higher-dimensional extended objects which are called *D-branes*, where D stands for *Dirichlet*. These objects are defined to be higher-dimensional hypersurfaces on which open strings can end, thus satisfying Dirichlet boundary conditions. Hence, type II string theory which was originally defined as a theory of closed strings only, additionally contains open strings. D-branes are believed to be different descriptions of certain supergravity solutions called *p-branes* which are introduced in the following section.

2.3.1 p-branes, D-branes and Anti-de Sitter space

Classical solutions to supergravity with non-trivial $(p+1)$-form A_{p+1} charge are referred to as p-branes. Here p stands for the spatial dimension of the brane. A p-brane can be visualized as the generalization of a two-dimensional membrane to arbitrary dimensions. A p-brane moving in time cuts out a $(p+1)$-dimensional world volume with geometry $\mathbb{R}^{p+1}$.

An anti-symmetric $(p+1)$-form gauge field A_{p+1} naturally couples to the $(p+1)$-dimensional world volume Σ_{p+1} of the p-brane by the diffeomorphism invariant action

$$S_{p+1} = \mu_p \int_{\Sigma_{p+1}} A_{p+1}\,. \tag{2.10}$$

Here μ_p is one unit of the electric charge and given by the electric flux through a $(p+2)$-dimensional sphere S^{p+2} which can be determined by Gauss's law, $\mu_p = \int_{S^{p+2}} F_{p+2}$, where F_{p+2} is the gauge invariant $(p+2)$-form field strength. The flux is conserved by virtue of the Bianchi identity $dF = 0$. In addition, this implies that the field strength F is closed. It has a Poincaré dual $(d-p-2)$-form F_{d-p-2} which is referred to as magnetic dual field strength. It has a $(d-p-3)$-form gauge field

A_{d-p-3} which again couples to a $(d-p-4)$-brane. Therefore, a $(d-p-4)$-brane is said to be the magnetic dual of a p-brane.

In type IIB supergravity, the branes to which the anti-symmetric gauge fields couple are distinguished as follows. The NS-NS sector contains the anti-symmetric two-form gauge field $B_{\mu\nu}$ which couples to a one-brane. This is the *fundamental string F1* and its magnetic dual is the five-brane *NS5*. The branes to which the R-R sector gauge fields couple are generally known as *Dp-branes*. According to the given rules above, the zero-form field C_0 should couple to a D(-1)-brane. This can be interpreted as a D-*instanton*, since it is localized in time and space. Its magnetic dual is a D7-brane. The two-form gauge field C_2 couples electrically to a D1-brane, which is also called D1-string, and magnetically to a D5-brane. The four-form gauge field C_4 couples both electrically and magnetically to a D3-brane. These D3-branes are the same and carry self-dual charge μ_3, since the five-form field strength $F_5 = \star F_5$ is self dual.

The geometry of a p-brane is that of a flat hypersurface with Poincaré invariance group $R^{p+1} \times SO(1,p)$. The transverse space is $(d-p-1)$-dimensional. A p-brane is rotationally $SO(d-p-1)$-symmetric in the transverse space. Thus, p-branes in type IIB supergravity are solutions with symmetry group $R^{p+1} \times SO(1,p) \times SO(d-p-1)$. The line element for the Dp-brane solution expressed in the *string frame* reads [8]

$$ds^2 = f(y)^{-1/2} dx_\mu dx^\mu + f(y)^{1/2} dy_a dy^a \ , \quad \mathrm{e}^\Phi = g_s f(y)^{(3-p)/4} \ , \tag{2.11}$$

where the x^μ with $\mu = 0, ..., p$ are the coordinates on the brane, the y^a with $a = p+1, ..., 10$ are the coordinates in the transverse directions and g_s is the string coupling constant. The function $H(y)$ is given by the harmonic function

$$f(y) = 1 + \frac{L^{7-p}}{|\vec{y}|^{7-p}} \ , \tag{2.12}$$

where L denotes the radius of the Dp-brane which is given by

$$L^{7-p} = \mu_p g_s (4\pi)^{(5-p)/2} \Gamma((7-p)/2) (\alpha')^{(7-p)/2} \, . \tag{2.13}$$

The solution (2.11) with (2.12) has a coordinate singularity at $\vec{y} = 0$ which thus constitutes a horizon of the Dp-brane. For $|\vec{y}| \to \infty$, and thus far away from the brane, (2.12) tends to $f(y) = 1$ and (2.11) becomes the Minkowski-space line element. In addition, instead of a single Dp-brane there are also *multi-center* solutions corresponding to a number i of parallel branes separated by some distance. The total R-R charge of this stack of p-branes is the sum $N = \sum_i \mu_{pi}$ over the single charge units μ_{pi}. In the limit of vanishing separation distance, the branes are said to be coincident.

The p-brane solutions presented here are solutions of classical supergravity. Let us investigate the domain of validity of the p-brane solution for the case of

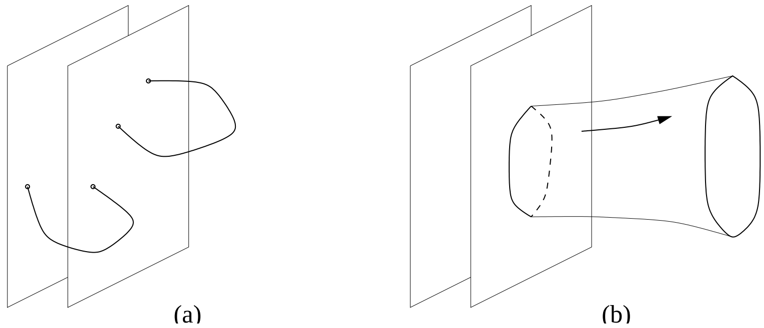


Figure 2.1: (a) Open strings attached to D-branes. (b) D-branes are sources of closed strings. (figure taken from [9])

$p = 3$. The supergravity description is appropriate as long as the curvature of the p-brane geometry is small compared to the string length l_s^2. Since the curvature scale of the D3-brane geometry is set by $L = (4\pi g_s N)^{1/4} l_s$, this requires $L \gg l_s$ which implies $g_s N \gg 1$. In addition, to neglect string loop corrections the effective string coupling e^{ϕ} is required to be small. For the D3-brane, the dilaton is constant and e^{ϕ} can be made small everywhere in the D3-brane geometry by setting $g_s < 1$. Thus combining both conditions leads to

$$1 \ll g_s N < N \,. \tag{2.14}$$

However, it is believed that p-brane solutions can also be generalized to solutions of superstring theory. The involved fields may then be subject to α' corrections as the string length is no longer negligible. Nevertheless, the string coupling g_s may still be weak. In this limit, string perturbation theory is applicable providing a description of the weak-coupling limit of the α'-corrected Dp-branes. In this description, the Dp-branes were originally defined as $(p+1)$-dimensional hypersurfaces in flat ten-dimensional spacetime on which open strings can end (see figure 2.1(a)). The end points are tied to the brane, thus satisfying Dirichlet boundary conditions in directions perpendicular to the brane, but can move freely along the brane, which corresponds to Neumann boundary conditions parallel to the brane. Moreover, D-branes are considered to be sources of closed strings (see figure 2.1(b)), for instance in scattering processes of closed strings and D-branes.

The perturbative string theory and the supergravity description of D-branes are believed to be different descriptions of the same object. This can be seen again in the context of the D3-brane example. Whereas the supergravity description is valid in the supergravity limit (2.14), the string theory description is appropriate in the weak-coupling limit

$$1 \gg g_s N \,, \tag{2.15}$$

which is complementary to the regime (2.14).

p-branes are also called *soliton* solutions. Such solutions are said to be interpolating between different vacua. For instance, the D3-brane interpolates between ten-dimensional Minkowski space at infinity and $AdS_5 \times S^5$ near the horizon as becomes clear in section 2.3.1. Furthermore, the p-brane solution (2.11) is called *extremal*, since its mass satisfies the relation

$$M = \frac{\mu_p}{(2\pi)^p g_s l_s^{p+1}} \,. \tag{2.16}$$

If the mass exceeds the right hand side of (2.16), we speak of *non-extremal black* p-branes which are described in section 2.6.1. Thus, the above mass relation constitutes a lower bound on the mass of p-branes. For Dp-branes in type II supergravity, it coincides with the *Bogomolnyi-Prasad-Sommerfield (BPS)* bound, $M \leq Z$, where Z is the central charge of $\mathcal{N} = 2$ supersymmetry, dictated by supersymmetry. When the mass equals the central charge one half of the supercharges present in the original theory vanishes. Those p-brane solutions which satisfy the bound, thus extremal branes, are also called *BPS branes*. Hence, in the presence of an extremal Dp-brane the number of supercharges is reduced from 32 to 16.

The low-energy effective theory of open strings on the $(p+1)$-dimensional world volume of a Dp-brane is $U(N)$ gauge theory [64]. Thus on a stack of several Dp-branes each with R-R charge unit μ_{pi} located at $\vec{y}_i$, the gauge group is $U(\mu_{p1}) \times U(\mu_{p2}) \times \ldots$. If all these branes coincide, the gauge group is promoted to $U(N)$ with total charge $N = \sum_i \mu_{pi}$. The gauge group $U(N)$ is essentially equivalent to the product group $SU(N) \times U(1)$. The $U(1)$ vector supermultiplet contains six scalars describing the center of mass motion of the branes which, in general, can be consistently set to zero leaving an $SU(N)$ gauge theory on the brane world volume.

D3-branes

From now on, we specify to the case of D3-branes, since these enter the AdS/CFT correspondence. Its solution reads [9]

$$\begin{aligned} ds^2 =& f(r)^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f(r)^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \,, && (2.17)\\ g_s =& \mathrm{e}^{\phi} \,, \quad C = \text{const.} \,, && (2.18)\\ B_{\mu\nu} =& A_{2\mu\nu} = 0 \,, && (2.19)\\ F_5 =& (1+\star) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1} \,. && (2.20) \end{aligned}$$

The metric of the D3-brane is here given in terms of a special coordinate system. It consists of the four-dimensional flat brane part which is written in Cartesian coordinates t, x, y, z and which is multiplied by the inverse of the square root of the harmonic function f, and of the six-dimensional transverse part which is

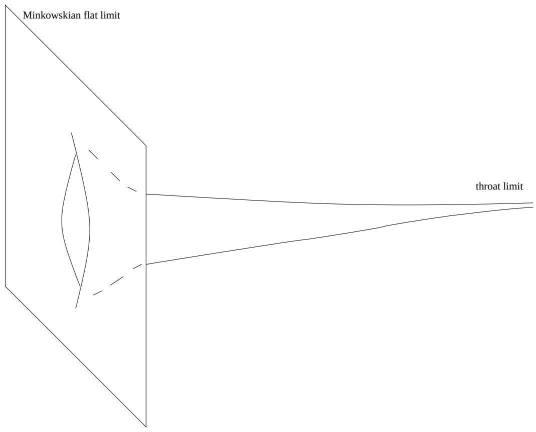


Figure 2.2: Near-horizon geometry of the D3-brane. The directions on the S^5 as well as the time and one Minkowski direction are suppressed. (figure taken from [8])

written in spherical coordinates and which is multiplied by the square root of the harmonic function f. Here, $d\Omega_5$ denotes the volume element of the five-sphere S^5 and r is a radial coordinate. The harmonic function only depends on the radial coordinate r and reads

$$f(r) = 1 + \frac{L^4}{r^4}\,, \tag{2.21}$$

where $L = (4\pi g_s N)^{1/4} l_s$ denotes the radius of the D3-brane solution. In case of a stack of several coincident D3-branes, the solution is the same as (2.17) together with (2.21) and $N = \sum_i \mu_{3i}$.

The solution (2.17) has several features. Its world volume has four-dimensional Poincaré invariance and the scalar fields, axion and dilaton, are constant. Moreover, it has a self-dual field strength implying that it carries both electric as well as magnetic charge.

Let us study the geometry (2.17) in more detail by considering different limits. For $r \gg L$, we recover flat Minkowski spacetime. For $r \ll L$, the line element (2.17) reduces to the product spacetime,

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2\,. \tag{2.22}$$

There are two components, a five-dimensional Anti-de Sitter space AdS_5 whose line element is $r^2/L^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + L^2/r^2 dr^2$ and a five-sphere S^5 with the line element $L^2 d\Omega_5^2$. Therefore, the geometry is denoted by $AdS_5 \times S^5$. Both components have identical radius L. The geometry (2.22) is referred to as the *near-horizon limit*, since it describes the geometry close to the brane at $r \sim 0$. Figure 2.3 depicts the AdS_5 part of the near-horizon geometry. In the following, we elucidate the geometry of Anti-de Sitter space, since its symmetry structure plays an important role in the AdS/CFT correspondence.

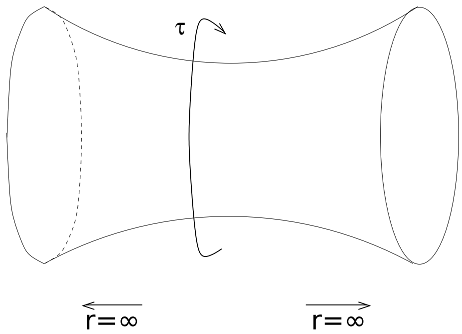


Figure 2.3: In $\mathbb{R}^3$, AdS_2 is represented by a hyperboloid. τ denotes the time direction which has $SO(2)$ symmetry representing closed time-like curves. (figure taken from [8])

The geometry of Anti-de Sitter space

Anti-de Sitter space (AdS) is a maximally symmetric space with constant negative curvature and Lorentz symmetry group $SO(2,4)$. Five-dimensional AdS space can be embedded in six-dimensional Minkowski space through the equation

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = L^2 \,, \tag{2.23}$$

which describes a hyperboloid with radius L. Its metric reads

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2 \,. \tag{2.24}$$

AdS has maximal Lorentz invariance group $SO(2,4)$, and is homogeneous and isotropic. The embedding equation (2.23) can be solved in terms of *global coordinates* of AdS. The circle S^1 is identified with the time direction which leads to closed time-like curves. Unwrapping S^1, thus mapping the time coordinate to the interval $]-\infty,\infty[$, gives the *universal covering* of AdS which is causal. In the following, we always have this space in mind when referring to AdS_5.

A global coordinate system on AdS_5 which covers the whole hyperboloid is given by (t, r, α_i) in which the metric (2.24) becomes

$$ds^2 = -\left(k + \frac{r^2}{L^2}\right) dt^2 + \frac{1}{\left(k + \frac{r^2}{L^2}\right)} dr^2 + r^2 d\Omega_3^2 \,, \tag{2.25}$$

where $k = 1$, α_i with $i = 1, 2, 3$ are the angles parametrizing a unit three-sphere S^3 and $d\Omega_3$ denotes the corresponding volume element. Another set of coordinates on AdS_5 are *Poincaré coordinates* (u, t, x^i) with $i = 1, ..., n$. These coordinates cover one half of the hyperboloid (2.23), in which its metric takes the form

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{L^2}{r^2} dr^2 \,, \tag{2.26}$$

which equals the first part in (2.22).

AdS space has a conformal boundary. The location of the boundary is determined by the second-order pole of the metric. The AdS metric (2.26) has a second-order pole at $r = \infty$, and thus the boundary is located at $r = \infty$. Since (2.26) blows up at $r = \infty$, it does not induce a metric on the boundary itself. Otherwise a metric on the boundary is given by

$$g(x) = z(r)^2 G(r, x)|_{r=\infty} , \tag{2.27}$$

where $z(r)$ is positive in the bulk of AdS, but has a first-order zero at the boundary, and G denotes the AdS metric which is evaluated at the boundary at $r = \infty$. However, g is only defined up to conformal transformations. Thus, the AdS metric induces a conformal structure on the boundary which means a metric up to conformal transformations. In Poincaré coordinates, the boundary consists of Minkowski space, parametrized by $\{t, x, y, z\}$, together with a point added at infinity such that the action of the conformal group $SO(2, 4)$, which is the isometry group of AdS, on the boundary is well-defined.

The isometry group $SO(2, 4)$ of AdS_5 can be promoted to a supergroup. Since AdS preserves as many supersymmetries as flat space, $\mathcal{N} = 2, 4, 6, 8$ supergravities are realizable in AdS_5. Thus, whereas in general the D3-brane background breaks one half of the supersymmetries, in the near-horizon region with topology $AdS_5 \times S^5$ the number of supercharges is preserved. In this thesis we are interested in the case of $\mathcal{N} = 2$ supergravity with 32 supercharges.

Having collected all necessary ingredients entering the AdS/CFT correspondence, we are prepared to discuss the Maldacena conjecture in the following section.

2.4 The Maldacena conjecture

In the preceding two sections, two different theories were presented. One is

- four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in Minkowski space with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its superconformal phase

and the second one is

- type IIB superstring theory with string coupling constant g_s on curved $AdS_5 \times S^5$, with equal radius L, and where the self-dual five-form F_5 has integer flux $N = \int_S^5 F_5$.

The AdS/CFT correspondence now states that these both theories are equivalent [17], or dual, where the following parameters are identified

$$g_{YM}^2 = g_s , \qquad L^4 = 4\pi g_s N(\alpha')^2 . \tag{2.28}$$

How this correspondence comes about can be imagined as follows [17]. Let us consider a single D3-brane separated from a stack of N coincident D3-branes by a distance r. The background geometry is generated by the stack of N D3-branes and the influence of the single D3-brane on this background can be neglected. In addition to the six scalar fields corresponding to the position of the branes in the transverse space, there exist two kinds of perturbative excitations in this geometry. On the one hand, there are closed strings which are excitations of the bulk. On the other hand there are open strings stretching between the single brane and the stack exciting the D3-branes. The masses of the open strings are proportional to $r/\alpha' = r/l_s^2$ which in the field theory gives a Higgs expectation value to one of the scalar fields corresponding to the position of the branes in one of the transverse directions. For low energies, the theory on the D3-branes decouples from the bulk by virtue of the vanishing of the interaction Lagrangian relating bulk and D3-brane. However, here we want to take a limit where the full spectrum on the brane is kept. For this purpose, we apply the limit $\alpha' \to 0$ while keeping all dimensionless parameters, as for instance g_s and N, as well as the energy r/α' fixed. The latter requires to take also $r \to 0$. In doing so, the single brane is brought closer to the stack of branes while the masses, or equivalently the effective tensions, of the strings remain fixed. The resulting theory on the world volume of the D3-branes is four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $U(N)$. This $U(N)$ gauge theory is essentially equivalent to $SU(N) \times U(1)$, at which the $U(1)$ degrees of freedom are the six scalar fields mentioned above. These scalar fields decouple from all other fields and can therefore be excluded leaving an $SU(N)$ super Yang Mills theory.

Instead of considering this system from the brane field theory perspective, we can investigate what happens on the bulk gravitational theory side. Consider the D3-brane solution (2.17). Then the limit $\alpha' \to 0$ and $r \to 0$ with $r/\alpha' = \text{const.}$ coincides with the limit $r \ll L$ discussed in section 2.3.1. In this near-horizon limit with all other dimensionless parameters, including g_s and N, kept fixed, the line element (2.17) takes the form (2.22). String theory on the near-horizon background now decouples from the theory near infinity. Thus, we now have string theory in the background of $AdS_5 \times S^5$ rather than in the full D3-brane background. This brings us to the above conjecture that $\mathcal{N} = 4$ super Yang-Mills in four-dimensional flat space is dual to type IIB superstring theory in $AdS_5 \times S^5$.

How the decoupling of string theory in the near-horizon geometry from the bulk theory is achieved can be visualized as follows. The energy measured by an observer somewhere in the bulk of the geometry (2.17) is red-shifted compared to the energy measured by an observer at infinity. This is because the proper time with respect to which the proper energy is measured changes with r as can be seen from the r-dependent tt-component of the metric (2.17). Thus the same energy $E \sim 1$ measured at infinity arises twofold. The energy of an object close to the horizon appears to be $E \sim rE_p \sim 1$. Close to the horizon r is very small and thus E_p is very large. The energy of an object near infinity is measured to be $E = (1 + L^4/r^4)^{-1/4}E_p \sim 1$. In this case, r is nearly infinite and therefore E_p must be of order one. These two kinds of excitations decouple from one another. The near-horizon modes cannot escape to infinity, since the red-shift constitutes a potential well in the direction to the bulk which they cannot overcome. The bulk modes decouple from the near-horizon modes, since their wavelengths are too large to be absorbed by the D3-branes.

The Maldacena decoupling limit is constructed such that the full spectrum of $\mathcal{N} = 4$ super Yang-Mills theory is captured by type IIB string theory on $AdS_5 \times S^5$. Furthermore, from the red-shift factor $E = (r/\alpha')\sqrt{\alpha'}E_p$ in the near-horizon region we see that keeping the proper energy E_p in the bulk and α' fixed, we obtain higher and higher energies E on the gauge theory side when we tend against the boundary of $AdS_5 \times S^5$. This is known as the *UV-IR relation* [65] which reveals that small distances from the D3-branes correspond to large-scale physics, or small energies E, in the field theory while large distances from the D3-branes correspond to small-scale physics, or large energies E, in the field theory. Thus, the radial bulk coordinate can be interpreted as an energy scale in the gauge theory which in the coordinates of (2.22) becomes $E \sim r$. Moreover, since the full gauge theory spectrum without any truncation to low energies shall be described by string theory in the near-horizon geometry, the gauge theory is considered to be located at the boundary of $AdS_5 \times S^5$. In that sense, the AdS/CFT correspondence constitutes a realization of the *holographic principle* which states that all physics within a volume can be described by the physics on the boundary enclosing this volume.

The above formulation is the *strong form* of the Maldacena conjecture, since no approximation in g_s or N is made. However, in practice this strong form is of not much use, since non-perturbative quantum string theory on a curved background is still out of reach. Thus, it is interesting to consider the following non-trivial limits which lead to weaker, but more tractable forms of the Maldacena conjecture. The *'t Hooft limit* [66] is applied by keeping the 't Hooft coupling $\lambda \equiv g_{YM}^2 N = g_s N$ fixed and taking $N \to \infty$. On the $\mathcal{N} = 4$ super Yang-Mills side, this corresponds to an expansion in Feynman diagrams such that only planar diagrams are taken into account. On the string theory side, this is interpreted as weak-coupling string perturbation theory, since $g_s = \lambda/N \to 0$. A further simplification is achieved by the limit $\lambda \to \infty$ and $N \to \infty$. In this limit, $\mathcal{N} = 4$ super Yang-Mills theory is strongly coupled and cannot be treated by perturbative methods. However, on the string theory side this corresponds to the supergravity approximation, since $\lambda \sim \frac{L^4}{(\alpha')^2} \to \infty$, which means that the curvature of the string theory background is taken to be small compared to the string length $l_s^2 = \alpha'$.

In taking the latter limit, we obtain the weakest form of the AdS/CFT correspondence between strongly-coupled $\mathcal{N}=4$ super Yang-Mills theory with large N $SU(N)$ gauge group in four-dimensional flat space and classical type IIB supergravity on weakly-curved $AdS_5 \times S^5$. In the subsequent chapters of this thesis, we exclusively work in this limit.

In the following section, a working recipe is developed to apply the correspondence to explicit examples.

2.5 A more precise correspondence

The conjectured correspondence between $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory in four-dimensional flat spacetime and type IIB string theory on $AdS_5 \times S^5$ presented in the preceding section is generally imposed and only of qualitative nature. Therefore, in this section, we are interested in more precise relations between the two theories to be able to extract quantitative results from the AdS/CFT correspondence. These explicit relations are often referred to as the *AdS/CFT dictionary*.

First, the various coupling constants of both theories are related as in (2.28). The parameter N was defined to be the flux of the type IIB five-form field strength. In the gauge theory N corresponds to the parameter of the $SU(N)$ gauge group.

Second, the symmetry groups of the two theories agree and are given by the supergroup $PSU(2,2|4)$ with the bosonic subgroup $SU(2,2) \times SU(4)$. As already mentioned in section 2.3.1, on the string theory side AdS_5 has the spacetime symmetry group $SO(2,4) \cong SU(2,2)$ and the S^5 has isometry $SO(6) \cong SU(4)$. Moreover, $AdS_5 \times S^5$ realizes all 32 supercharges of the type IIB string theory. On the gauge theory side, we have a conformally invariant quantum field theory with conformal group $SO(2,4)$. The $SU(4)$ symmetry of the S^5 arises on the gauge theory side as a global $SU(4)$ R-symmetry of the 32 supercharges.

Third, the field content of both theories should be explicitly related in order to perform calculations on the supergravity side and to transfer the results to the strongly-coupled field theory side where calculations are difficult. Such explicit mappings between supergravity fields ϕ and gauge-invariant operators $\mathcal{O}$ of the CFT were provided by Witten [67] and also by Gubser, Klebanov and Polyakov [68]. According to these works, the boundary values of supergravity fields are considered to be sources which couple to operators in the dual gauge theory, and the supergravity partition function is identified with the generating functional of gauge theory correlation functions,

$$\left\langle \mathrm{e}^{\int d^4x \phi_0 \mathcal{O}} \right\rangle_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{S}}[\phi_0] \,, \tag{2.29}$$

where the left hand side is the generating functional containing the coupling of a gauge theory operator to the boundary value ϕ_0 of a supergravity field ϕ. The right hand side is the supergravity partition function evaluated at the boundary which can be computed via the classical supergravity action $\mathcal{I}_S$,

$$\mathcal{Z}_S(\phi_0) = \mathrm{e}^{-\mathcal{I}_S(\phi)}\Big|_{\phi=\phi_0} . \tag{2.30}$$

With relation (2.29) at hand, we can compute correlation functions of $\mathcal{O}$ simply by taking functional derivatives of the classical supergravity action with respect to ϕ_0. The formula (2.29) applies in general such that each field propagating in the bulk is in a one-to-one correspondence with an operator in the gauge theory. However, the supergravity action diverges, because of the infinite volume of the AdS_5 spacetime and thus needs to be appropriately renormalized. A renormalization procedure was developed in [69, 70] which consists in adding covariant *counterterms* to the divergent action yielding a finite action. Furthermore, as in any background with boundary, the action has to be supplemented by a *Gibbons-Hawking term* [71] which is a boundary term rendering the variational problem well-defined. Finally, the resulting five-dimensional renormalized action S_{ren} consists of three terms,

$$S_{\text{ren}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{counter}} , \tag{2.31}$$

where S_{GH} denotes the Gibbons-Hawking term and S_{counter} includes the necessary counterterms. Evaluating this action on the solution then yields the renormalized on-shell action.

The general solution of the bulk field equations has to satisfy Dirichlet boundary conditions. An ansatz for the supergravity fields is given by the decomposition with respect to a basis of spherical harmonics Y_Δ on the S^5 [8],

$$\phi(r, x, y) = \sum_{\Delta=0}^{\infty} \phi_\Delta(r, x) Y_\Delta(y) , \tag{2.32}$$

where r, x and y denote the coordinates on AdS_5 and on S^5, respectively. Now, the fields $\phi_\Delta(r)$ are effectively defined on AdS_5[1]. Asymptotically, the supergravity fields $\phi_\Delta(r)$ are free and satisfy the free field equations of motion. The S^5 compactification contributes to the masses of the fields. Thus, for instance, for a scalar field this yields the mass relation [67]

$$m^2 = \Delta(\Delta - 4) , \tag{2.33}$$

[1]It is important to note, that the truncation of ten-dimensional supergravity on $AdS_5 \times S^5$ to supergravity on AdS_5 is not evident, since the usual Kaluza-Klein method is not applicable. Kaluza-Klein compactification consists in dimensional reduction on a space of very small radius which leads to large masses of the Kaluza-Klein modes. In a low-energy approximation, these Kaluza-Klein modes decouple from the low-energy states. However, in the case of $AdS_5 \times S^5$ both spaces have equal radius L which spoils the usual Kaluza-Klein argument. Nevertheless, there is a consistent truncation to $\mathcal{N} = 2$ supergravity on AdS_5. Examples are given in [72–74].

and the asymptotic solution for $r \to \infty$ takes the form

$$\phi_\Delta(r_\infty, x) = r^{\Delta-4}\phi_0(x) + r^{-\Delta}\langle\mathcal{O}(x)\rangle \ , \tag{2.34}$$

where x denotes coordinates along the boundary of AdS_5. Since the supergravity field ϕ_Δ is dimensionless, the operator $\mathcal{O}$ has dimension Δ. The boundary condition on the supergravity field becomes $\phi_0 = \lim_{r\to\infty} r^{\Delta-4}\phi_0(x)$ [67]. Similar relations exist for non-scalar fields such as fermions and tensor fields on AdS. Thus, the boundary condition is determined such that the mass of the supergravity and the conformal dimension of the gauge theory operator match in a certain way.

In this thesis, we are only concerned with one-point functions of operators $\mathcal{O}$ in the presence of sources. Thus we ask questions as given a supergravity field to which operator does it correspond on the gauge theory side. The answer is given by the computation rule [69, 70],

$$\langle\mathcal{O}(x)\rangle = \frac{1}{\sqrt{G_0(x)}}\frac{\delta S_{\text{ren}}}{\delta\phi_0(x)} \ , \tag{2.35}$$

where S_{ren} is now the renormalized on-shell action.

Examples of the field-operator mappings (2.35) are, for instance, the relation between the metric field G_{MN} on the supergravity side and the energy-momentum tensor $T_{\mu\nu}$ on the gauge theory side, as well as the relation between a gauge field A_M on the supergravity side and a current J_μ on the gauge theory side which are given by

$$A_M(r,x) \to \langle J_\mu(x)\rangle = \frac{1}{\sqrt{G_0(x)}}\frac{\delta S_{\text{ren}}}{\delta A_{0\mu}(x)} \ , \tag{2.36}$$

$$G_{MN}(r,x) \to \langle T_{\mu\nu}(x)\rangle = \frac{2}{\sqrt{G_0(x)}}\frac{\delta S_{\text{ren}}}{\delta G_{0\mu\nu}(x)} \ . \tag{2.37}$$

So far, we have seen how the correspondence is implemented. In the following section, we explain how the correspondence is extended to describe CFTs at finite temperature as well as in the presence of a finite charge density.

2.6 Extensions of the correspondence

As we have seen so far, the AdS/CFT correspondence provides a technique to describe strongly-coupled $\mathcal{N} = 4$ super Yang-Mills gauge theories. However, these field theories are far from being real field theories present in our world such as QCD. Therefore, a great desire of the AdS/CFT experts is to approximate real field theories as good as possible. An important step in this direction is to introduce temperature in the field theory. This locally breaks the conformal invariance of the field theory while globally it is still preserved, as well as supersymmetry, leaving a far more realistic field theory to investigate.

How finite temperature comes into play is explained in the following section. Moreover, we work out some special properties of thermal CFTs in finite volume in section 2.6.2. An additional extension is to turn on a chemical potential in the field theory which is demonstrated in section 2.6.3.

2.6.1 Finite temperature

In the context of the AdS/CFT correspondence, a thermal field theory on the boundary of AdS is obtained by embedding a black hole in AdS. Thus, the boundary theory is considered as a field theory in the background of an AdS black hole radiating with sufficiently high temperature. The temperature of the field theory is identified with the *Hawking temperature* [75] of the black hole [18]. This situation is obtained as follows. The role of the ten-dimensional extremal D3-brane in the zero temperature AdS/CFT correspondence is here played by a ten-dimensional black D3-brane which can be viewed as a higher dimensional analog of an ordinary black hole with flat horizon. These black D3-branes are non-extremal, since they have a mass which is larger than the BPS mass satisfying the BPS bound.

The line element of the black D3-brane reads [9]

$$ds^2 = -f_+(r)f_-(r)^{-1/2}dt^2 + f_-(r)^{1/2}dx^idx^i + f_+(r)^{-1}f_-(r)^{-1}dr^2 + r^2 d\Omega_5^2\,, \quad (2.38)$$

where x^i with $i = 1, 2, 3$ are the spatial Poincaré coordinates along the brane, r is the radial coordinate perpendicular to the D3-brane, $d\Omega_5$ denotes the volume element of the S^5 and $f_\pm$ is given by

$$f_\pm = 1 - \left(\frac{r_\pm}{r}\right)^4\,, \quad (2.39)$$

where $r_\pm$ are the outer and inner horizon, respectively. The extremal limit follows from setting $r_+ = r_-$. In this case, (2.38) reduces to (2.17).

Now, taking again the decoupling limit $\alpha' \to 0$, $r \to 0$ and $r/\alpha' = \text{const.}$ yields the near-horizon line element

$$ds^2 = \frac{r^2}{L^2}\left(-hdt^2 + dx^idx^i\right) + \frac{L^2 h}{r^2}dr^2 + L^2 d\Omega_5^2\,, \quad (2.40)$$

with

$$h = 1 - \frac{r_0^4}{r^4}\,. \quad (2.41)$$

Here r_0 denotes the horizon radius of the non-extremal black brane which is proportional to the mass. The resulting line element is that of the AdS_5-*Schwarzschild black brane* in Poincaré coordinates with flat horizon times an S^5. The corresponding spacetime is asymptotically AdS_5 times S^5, whereas in the bulk of AdS_5 the geometry is dominated by the black brane. The boundary at $r = \infty$ has the topology $\mathbb{R} \times \mathbb{R}^3$. This is identified with the field theory domain. Since the metric on the boundary is defined only up to conformal transformations, we can choose coordinates such that the CFT time equals the AdS time. In this case the field theory domain has the volume $V = L^3$.

A global version of (2.40) with boundary topology $\mathbb{R} \times S^3$ is given by

$$ds^2 = -\frac{r^2}{L^2} h dt^2 + \frac{L^2}{r^2 h} dr^2 + r^2 d\Omega_3^2 + L^2 d\Omega_5^2 \,, \tag{2.42}$$

where $d\Omega_3$ is the volume element of the S^3 and the function h now has the form

$$h = 1 - \frac{r_0^4}{r^4} + \frac{L^2}{r^2} \,, \tag{2.43}$$

where here r_0 is again proportional to the mass of the black hole, but does not coincide with the horizon radius. It can be expressed in terms of the horizon radius as

$$r_0^4 = r_h^4 + r_h^2 L^2 \,. \tag{2.44}$$

The line element (2.42) is that of the usual *AdS_5-Schwarzschild black hole* with spherical horizon times an S^5. The CFT time and the AdS time coincide, since the three-sphere has radius L.

The two line elements (2.40) of the black brane and (2.42) of the black hole are related by a specific limit. Namely, taking r_0^4, which is proportional to the mass of the black hole (2.42), to be very large yields the line element of the black brane (2.40). This can be seen by first rescaling the black hole line element as $r \to (r_0/L) r$ and $t \to (L/r_0) t$ as well as taking $r_0^4 \to \infty$. Second, we locally have to set $d\Omega^2 = \sum_i dx^i$, and finally rescaling $x^i \to (L/r_0) x^i$ yields (2.40).

The Hawking temperature of the AdS-Schwarzschild black hole (2.42) is derived by introducing a Euclidean time coordinate $\tau = -it$ in (2.40) and requiring the periodicity of τ to be such that there is no conical singularity at the horizon r_h. This period β is interpreted as the inverse of the Hawking temperature which is

$$T = \frac{r_h}{2\pi L^2} \left(1 + \frac{r_0^4}{r_h^4} \right) \,. \tag{2.45}$$

Thus, the temperature of a thermal field theory on the three-sphere dual to an AdS black hole is given by (2.45). In the limit of very large mass as described above, (2.45) reduces to $T = r_h/(2\pi L^2)$ which is the Hawking temperature of the black brane (2.40), and simultaneously the temperature of the dual field theory in flat space.

The thermodynamic properties of the AdS-Schwarzschild black hole can be described within the canonical ensemble. The partition function of the canonical ensemble is given by

$$Z_{\mathrm{bh}} = \mathrm{e}^{-I} \equiv \mathrm{e}^{-\beta F_{\mathrm{bh}}} \,, \tag{2.46}$$

where I denotes the five-dimensional Euclidean renormalized on-shell action, $\beta = 1/T$ is the inverse temperature and $F_{\rm bh} = E_{\rm bh} - TS_{\rm bh}$ is the free energy with $E_{\rm bh}$ being the total energy and $S_{\rm bh}$ the entropy of the black hole. Given the partition function (2.46), we can compute the total energy and the entropy of the AdS black hole (2.42),

$$E_{\rm bh} = \frac{\partial I}{\partial \beta} = \frac{3\pi r_0^4}{8G_5 L^2}\,, \tag{2.47}$$

$$S_{\rm bh} = \beta E_{\rm bh} - I = \frac{V_h}{4G_5}\,, \tag{2.48}$$

where the five-dimensional Newton constant is related to the ten-dimensional Newton constant by $G_5 = L^5 G_{10}$ and $V_h = r_h^3 vol$ denotes the horizon area with vol being the unit volume of the spatial three-dimensional space. The energy (2.47) derived from the thermodynamic partition function does not always coincide with the ADM energy which is determined by the asymptotic behavior of the geometry. In case of the spherical AdS black hole (2.42), the ADM energy exhibits an additional contribution, $E_0 = 3\pi L^2/32G_5$, which is interpreted as the ground-state energy of global AdS. However, we are only interested in energies relative to this ground-state energy and therefore ignore this term in the total energy. For the flat black brane, thus in the large mass limit, this term is absent. The entropy is the usual *Bekenstein-Hawking entropy* [76].

According to the dictionary of the AdS/CFT correspondence (cf. section 2.5) we set the canonical partition function (2.42) equal to the partition function of the thermal field theory. Therefore we can identify $F_{\rm bh}, E_{\rm bh}, S_{\rm bh}$ with the free energy F, the energy E and the entropy S of the field theory [67]. Thus, after conversion of bulk theory coupling constants to field theory coupling constants ($G_5 = \pi L^3/(2N^2)$) we obtain for the energy of a field theory on a three-sphere dual to the AdS-Schwarzschild black hole

$$E = \frac{3N^2 r_0^4}{4L^5}\,. \tag{2.49}$$

In case of the black brane, r_0 given by (2.44) reduces to the horizon radius r_h which can be expressed in terms of the Hawking temperature as $r_0 = T\pi L^2$. Rewriting (2.49) in terms of the temperature leads to $E = 3/8\pi^2 V N^2 T^4$ which exhibits the familiar energy-temperature relation following the *Stefan-Boltzmann law* for radiation. By the way, the ground-state energy of global AdS mentioned above can be interpreted as a vacuum energy [77] resulting from the *Casimir effect* [78] of quantum field theories in finite volume. This effect is absent for a CFT on the non-compact boundary of a flat black brane.

In case of the black hole, r_0 has two contributions scaling inhomogeneously with the horizon radius r_h.

The entropy (2.48) expressed in field theory quantities reads

$$S = \frac{\pi^2}{2} N^2 V T^3 \,, \tag{2.50}$$

where $V = L^3 vol$. This result was obtained in the large N and large λ limit and is believed to be the correct value for the entropy of the strongly-coupled thermal CFT. However, the result is hard to check, since a dual field theory computation is only possible for small coupling λ. The result of the perturbative small λ computation differs from (2.50) by a factor of 4/3 [79].

2.6.2 Thermal conformal field theory in finite volume

A field theory in finite volume has special thermodynamic properties.

The total energy of a quantum field theory in finite volume is not a purely extensive quantity. Thus regarded as a function of the entropy and the volume, $E(S,V)$ is *not* a homogeneous function of degree one, thus $E(\lambda S, \lambda V) \neq \lambda E(S,V)$. The energy contains an additional *subextensive* contribution scaling with a power smaller than one. This subextensive contribution is caused by a finite-temperature analog of the Casimir effect mentioned before. Note that the finite-temperature Casimir energy must not be confused with the zero-temperature Casimir energy addressed above, and from now on we exclusively mean the former when referring to the Casimir energy. The Casimir energy is well-known in $(1+1)$-dimensional CFT where it is proportional to the central charge c.

The Casimir energy can be defined as the violation of the thermodynamic *Euler relation* [36] which is essentially given by the integrated first law of thermodynamics. Thus, for a system satisfying the first law $dE = TdS + pdV$ the Casimir energy is given by

$$E_c = 3(E + pV - TS) \,, \tag{2.51}$$

where $p = (1/3)(E/V)$ is the pressure. The factor of 3 was inserted for later convenience. The total energy E of the boundary field theory can be written as a sum of extensive energy E_e and subextensive Casimir energy E_c

$$E = E_e + \frac{1}{2} E_c \,, \tag{2.52}$$

where again the factor of 1/2 was inserted for later convenience. This is exactly the form of the total energy (2.49) with (2.44) of the CFT dual to the AdS-Schwarzschild black hole (2.42). There, the extensive energy contribution is proportional to r_h^4 and can be written as

$$E_e = \frac{3}{vol} \left(\frac{\pi^4}{4N^2} \right)^{1/3} \frac{S^{4/3}}{V^{1/3}} \,, \tag{2.53}$$

with $vol = vol(S^3)$ being the unit volume of the S^3 and where we used that $r_h^4 = S^{4/3}(4G_5)^{4/3}/(vol)^{4/3}$. Observe, that E_e scales as $E_e \to \lambda E_e$ under the simultaneous rescaling $S \to \lambda S$ and $V \to \lambda V$. The Casimir energy contribution is proportional to r_h^2 and can be written as

$$E_c = 3\left(\frac{\pi^2 N^2}{16vol}\right)^{1/3} \frac{S^{2/3}}{V^{1/3}}\,. \tag{2.54}$$

Observe, that E_c scales as $E_c \to \lambda^{1/3} E_c$ under the simultaneous rescaling $S \to \lambda S$ and $V \to \lambda V$. Furthermore, it can be verified that (2.54) is actually the Casimir energy according to the definition (2.51) by inserting the Hawking temperature (2.45), the entropy (2.48) as well as the energy (2.49) into the Euler relation (2.51).

Due to the two different energy contributions, the entropy of a strongly-coupled CFT, which has a gravity dual, living on a three-sphere with radius L can be written in the very special form,

$$S = \frac{2\pi L}{n}\sqrt{E_c(2E - E_c)}\,, \tag{2.55}$$

where n denotes the number of spatial dimensions which here is $n = 3$. This is known as the *Cardy-Verlinde formula* [36] which is the higher-dimensional generalization of the *Cardy formula* [80] for the entropy of a $(1+1)$-dimensional CFT. Identifying $EL = L_0$, $E_c L = c/12$, where L_0 is the *Virasoro generator* and c the *central charge* of the CFT, and setting $n = 1$ leads to the Cardy formula

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}\,. \tag{2.56}$$

While the formula (2.55) is applicable in finite volume, it also has a well-defined flat limit. Expressing the Casimir energy E_c in terms of the extensive energy E_e and then taking the total energy E to be large reduces to (2.50).

The relation between the Casimir energy E_c and the central charge c, in the text above (2.56) suggests that the higher-dimensional generalization of the central charge is proportional to the *Casimir entropy* which is the *Bekenstein entropy* $S_B = (2\pi/3)EL$ of a CFT on a three-sphere with radius L at an energy $E = E_c$. This makes sense, since both quantities can be interpreted as counting degrees of freedom of the theory [36].

2.6.3 Finite density

The AdS/CFT correspondence can be extended to describe CFTs at a finite charge density. For instance, a $U(1)$ gauge symmetry in the AdS black hole background is dual to a global $U(1)$ symmetry in the field theory. $U(1)$ gauge fields in AdS_5 naturally arise in the S^5 compactification of the ten-dimensional *spinning* near-horizon brane. These branes rotate in three independent rotation planes of the S^5 breaking the $SO(6)$ Lorentz symmetry down to $U(1) \times U(1) \times U(1)$, which is the maximal Abelian subgroup of $SO(6)$. The three remaining $U(1)$ isometries on the AdS side become three independent global $U(1)$ symmetries on the CFT side. The massless gauge fields A_M^A in AdS correspond to currents J_μ^A of conformal dimension four in the dual CFT, according to relation (2.33) of the AdS/CFT dictionary. These currents are considered as the three remaining $U(1)$ R-currents stemming from the broken $SO(6)$ R-symmetry.

Black holes which are charged under $U(1) \times U(1) \times U(1)$ gauge symmetry are often referred to as STU black holes. In chapter 3, we give a detailed description of the five-dimensional AdS-STU black holes and how they arise from S^5 compactifications of spinning branes. In chapter 4 and 5, we make use of this kind of background.

Instead of $U(1)$ charge density, we can also add $SU(2)$ charge density to the CFT. In this case, the dual supergravity solution is a non-Abelian AdS black hole and the field theory currents of the global $SU(2)$ symmetry are dual to the gauge fields of the $SU(2)$ gauge symmetry in AdS. The non-Abelian AdS black hole is considered in chapter 6.

Furthermore, in all cases, we work with a fixed chemical potential. The chemical potential μ is defined as the difference in the electric potential A_t at the boundary of AdS $r = r_b$ and the horizon at $r = r_h$ [72],

$$\mu = A_t(r_b) - A_t(r_h)\,. \tag{2.57}$$

Under this condition the thermodynamic properties of the field theory can be extracted from the grand potential of the grand canonical ensemble,

$$\Omega = E - TS - \mu Q\,, \tag{2.58}$$

where Q is the total charge. The grand potential plays the same role as the free energy of the canonical ensemble at zero charge density in section 2.6.1. Thus, in the grand canonical ensemble we can identify

$$\beta\Omega = I\,. \tag{2.59}$$

2.7 Summary and outlook on subsequent chapters

In this chapter, we reviewed basic facts about the AdS/CFT correspondence which constitutes the common basic principle behind the research results presented in chapter 4, 5 and 6. First, we introduced the two theories participating in the correspondence, namely $\mathcal{N} = 4$ superconformal Yang-Mills theory and type IIB supergravity in Anti-de Sitter space. After that, we presented the conjecture as well as a heuristic argument in favor of the AdS/CFT correspondence. Moreover, the extension of the correspondence to include CFTs at finite temperature as well as at finite density was presented. The upshot of this was that temperature is turned on on the field theory side by replacing the extremal D3-brane by a non-extremal black D3-brane in the bulk which emits Hawking radiation heating up the field theory on the boundary. A charge density is induced on the field theory side by charging the black hole under some gauge symmetry. Moreover, we pointed out that strongly-coupled thermal CFTs which are dual to a spherical black hole, and thus live on a three-sphere possessing a finite volume, show special thermodynamic properties. In this case, the entropy can be written as a Cardy-Verlinde formula in terms of the total energy and the Casimir energy. According to that, the Casimir entropy seems to play the role of a higher-dimensional generalization of the central charge of CFTs in $(1 + 1)$ dimensions.

In the rest of this thesis, we mainly make use of the AdS/CFT correspondence at finite temperature and at finite charge density to describe strongly-coupled charged thermal systems. In chapter 4 and 5, we consider the field theory to live on a three-sphere, whose special thermodynamic properties lead to interesting effects. Additionally, we turn on finite $U(1)\times U(1)\times U(1)$ charge in the field theory. The associated dual charged black hole solutions are, for that purpose, studied in detail in chapter 3. Chapter 6 deals with thermal CFTs in the presence of $SU(2)$ charge density and the associated dual black hole solutions are numerically derived in the same chapter.

Chapter 3

Black holes in five-dimensional $\mathcal{N} = 2$ gauged supergravity

This chapter is partly intended to be a review on the static black hole solutions of $\mathcal{N} = 2$ *gauged supergravity* in five dimensions. In addition, it contains a calculation by the author in section 3.3, namely the derivation of *first-order differential (flow) equations* from the five-dimensional gauged $\mathcal{N} = 2$ supergravity action in the context of non-extremal electrically charged static black hole solutions.

The chapter is structured as follows. After a short overview of the content of the chapter, in section 3.2 we review how five-dimensional $\mathcal{N} = 2$ gauged supergravity is derived from eleven-dimensional supergravity. For this purpose, the compactification of eleven-dimensional supergravity on a *Calabi-Yau three-fold* as well as the *gauging* of the resulting five-dimensional theory are explained. Moreover, relevant elements of the *very special geometry* of the scalar fields of $\mathcal{N} = 2$ gauged supergravity are shortly reviewed. In section 3.3, we derive the first-order flow equations for five-dimensional non-extremal electrically charged black holes of $\mathcal{N} = 2$ gauged supergravity by combining the flow equations for ungauged extremal electrically charged static black holes and gauged flat domain walls. Then we solve the flow equations and recover the solutions of [21]. In section 3.4, the general $\mathcal{N} = 2$ gauged supergravity is truncated to the *STU model* and the corresponding action as well as the AdS-STU black hole solutions are presented. The *STU model* allows for four special simplifying cases which are worked out for later use. The last section 3.5 sketches the embedding of the AdS-STU black hole in type IIB supergravity by compactifying a *spinning D*3*-brane* on a five-sphere.

The derivation of the first-order flow equations is part of the work which was done by the author of this thesis in collaboration with Gabriel Lopes Cardoso and which was published in [1].

3.1 Overview

The $\mathcal{N} = 2$ gauged supergravity action in five dimensions was derived in [22] in the mid 1980'ies. Since the advent of gauge/gravity dualities there has been renewed interest in gauged supergravity theories in various dimensions. An explicit solution to five-dimensional $\mathcal{N} = 2$ supergravity was found in [21] in the context of the STU model, which arises as a certain truncation of the full theory, by solving the associated equations of motion. These solutions correspond to non-extremal static AdS black holes charged under three $U(1)$ gauge symmetries. Special cases of these black holes have been discussed in [31, 72, 73, 81–85].

The general five-dimensional supergravity action can be derived by compactifying eleven-dimensional supergravity on a Calabi-Yau three-fold [23–25] and gauging the resulting five-dimensional theory [22]. This can be further truncated to the STU model. Fortunately, the STU model can also be embedded into type IIB string theory by performing a Kaluza-Klein reduction on a five-sphere and truncating the resulting theory to $\mathcal{N} = 2$ supersymmetry [72–74]. This enables us to study different applications of the AdS/CFT correspondence in the AdS black hole background found in [21], what we exploit in chapter 4 and 5.

A convenient feature of the supergravity action is that when we specialize to backgrounds which describe electrically charged static black hole solutions, it can be written, up to total derivative terms, as a sum of squares of expressions which are of first-order in derivatives. When these expressions are put to zero we obtain first-order differential (flow) equations which are consistent with the second-order equations of motion and which are, in addition, easier to solve. First-order flow equations are well-known from the *attractor mechanism* [26–29] of extremal solutions, such as [30, 31]. Therefore, it is very surprising that there exist classes of non-extremal solutions that also allow for such a description, since non-extremal black holes do not exhibit attractor behavior [32, 33]. First-order flow equations for non-extremal AdS-Einstein-Maxwell black holes have been discussed in the past in [86] and more recently in [87, 88] in the context of Einstein-dilaton-p-form systems. In this chapter we find flow equations for general non-extremal electrically charged black holes in $\mathcal{N} = 2$ gauged supergravity and we rederive the solution of [21].

3.2 $\mathcal{N} = 2$ supergravity on AdS_5

In this section, the five-dimensional $\mathcal{N} = 2$ gauged supergravity action is derived in two steps. First, eleven-dimensional supergravity is compactified to five-dimensional ungauged supergravity on a Calabi-Yau three-fold. After that the theory is gauged yielding supergravity which has AdS space as a solution. In the following, we refer to this supergravity as the *AdS supergravity*.

3.2.1 Calabi-Yau compactification of eleven-dimensional supergravity

By definition, a Calabi-Yau n-fold is a Kähler manifold having n complex dimensions and vanishing first Chern class[1],

$$c_1 = \frac{1}{2\pi}[R] = 0\,. \tag{3.1}$$

The latter implies that a Calabi-Yau manifold may be endowed with a Ricci flat Kähler metric. A Calabi-Yau n-fold is characterized by its Hodge numbers $h_{p,q}$ which count the number of independent harmonic (p,q) forms on the manifold. However, not all of the $h_{p,q}$ are independent, but related by symmetries and dualities. Let us list the non-zero Hodge numbers for the special case of $n = 3$ which is of interest in this chapter. Any compact connected Kähler manifold, which applies to Calabi-Yau manifolds, has $h_{0,0} = 1$ corresponding to constant functions. Then Poincaré duality gives $h_{3,3} = 1$. Furthermore, the Calabi-Yau condition requires that there exists a nowhere vanishing holomorphic three-form, thus $h_{3,0} = 1$. Complex conjugation then gives $h_{0,3} = 1$. In addition, there are two more independent non-zero Hodge numbers, $h_{2,1} = h_{1,2}$ and $h_{1,1}$, which have different values for different Calabi-Yau three-folds and which specify their cohomology structure. The full set of Hodge numbers can be displayed in the Hodge diamond (figure 3.1). The characterization in terms of Hodge numbers is not unique. Some

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Figure 3.1: Hodge numbers of a Calabi-Yau three-fold. (figure taken from [6])

Calabi-Yau manifolds with the same $h_{p,q}$ can be related by deformations of their metrics. These deformations are proportional to a linear combination of the harmonic (p,q) forms whose coefficients are called *moduli*. These moduli determine the size and shape of the manifold. In this sense, the moduli space is a continuous family of Calabi-Yau spaces. In the case of a Calabi-Yau three-fold the moduli space is $(2h_{2,1} + h_{1,1})$-dimensional.

[1]More information on Calabi-Yau manifolds can be found in [89–91].

Let us now come back to the compactification of eleven-dimensional supergravity on a Calabi-Yau three-fold. The eleven-dimensional supergravity action was derived in [92]. Its bosonic part reads

$$16\,\pi\,G_{11}\,S = \int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4\,, \tag{3.2}$$

where R is the scalar curvature, $F_4 = dA_3$ is the four-form field strength associated with the three-form gauge field A_3. G_{11} denotes the eleven-dimensional Newton constant and G is the determinant of the eleven-dimensional metric $G_{\hat{\mu}\hat{\nu}}$. The quantity $|F_4|^2$ is defined by

$$|F_4|^2 = \frac{1}{4!} F_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} F^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\,. \tag{3.3}$$

The compactification is performed according to the usual *Kaluza-Klein reduction* which was invented in [4,5]. In doing so, we compactify all the eleven-dimensional fields in (3.2) to five dimensions and then truncate to the massless sector[2]. Keeping only the massless modes of the fields is reasonable, since assuming the size of the compactification space is very small (on the order of the Planck length) the massive modes are too heavy to participate in the low-energy dynamics of the fields. The field content of the resulting five-dimensional theory depends strongly on the structure of the Calabi-Yau, and thus on its Hodge numbers. It consists of the gravity mulitplet, whose bosonic part comprises the graviton and the graviphoton, $h_{1,1} - 1$ vector multiplets each with a bosonic part comprising a one-form gauge field and a real scalar field, and $h_{2,1} + 1$ hyper multiplets each with a bosonic part comprising two complex scalar fields [23–25]. The theory can be further truncated by setting the hyper-multiplet fields to zero.

So far, the five-dimensional $\mathcal{N} = 2$ supergravity action does not contain a cosmological constant which is, however, necessary to have AdS_5 as a vacuum solution. How this comes into play is explained in the following section.

3.2.2 Gauged supergravity

The supersymmetry algebra of $\mathcal{N} = 2$ supergravity in five-dimensional flat space contains an $SU(2)$ R-symmetry. The AdS supergravity is obtained by gauging the $U(1)$ subgroup of the $SU(2)$ group, which breaks $SU(2)$ down to $U(1)$. The gauging is achieved by introducing a linear combination of the Abelian vector fields already present in the ungauged theory, *i.e.* $A_\mu = h_A\, A^A_\mu$, with a coupling constant $\mathfrak{g}$. The h_A are constants. The coupling of the fermionic fields to the $U(1)$ vector field breaks supersymmetry. In order to preserve $\mathcal{N} = 2$ supersymmetry, gauge-invariant $\mathfrak{g}$-dependent terms have to be added. In a bosonic background, these

[2] For an introduction into Kaluza-Klein theory see [93].

additional terms give a scalar potential [22]. The bosonic part of the resulting effective gauged supersymmetric $\mathcal{N} = 2$ action describing the coupling of vector multiplets to supergravity reads

$$\begin{aligned}16\pi\, G_5\, S = &\int d^5x\sqrt{-G}\left(R - \mathcal{G}_{ij}\,\partial_M\varphi^i\partial^M\varphi^j - \frac{1}{2}\mathcal{G}_{AB}\,F^A_{MN}F^{B\,MN} - V_{\text{pot}}\right)\\ &+\frac{\kappa}{3}\int C_{ABC}\,F^A\wedge F^B\wedge A^C\,,\end{aligned}\tag{3.4}$$

where now, R is the five-dimensional Ricci scalar, G is the determinant of the five-dimensional metric G_{MN}, φ^i are the real vector-multiplet scalar fields with $i = \{1, ..., h_{1,1} - 1\}$, $\mathcal{G}_{ij}$ denotes the metric of the scalar-field target space and $F^A = dA^A$ is the two-form field strength associated with the one-form gauge field A^A with $A = \{1, ..., h_{1,1}\}$. The C_{ABC} are given by the triple intersection of the $(1, 1)$-forms on the Calabi-Yau space. The term V_{pot} is the scalar potential defined below. It contains a constant part which gives a cosmological constant. Furthermore, G_5 denotes the five-dimensional Newton constant and the coefficient in front of the Chern-Simons term $\kappa = -1/(2\sqrt{3})$ is the Chern-Simons coupling constant.

The above action can be rewritten in terms of $h_{1,1}$ scalar fields X^A which satisfy the constraint

$$\frac{1}{6}\,C_{ABC}\,X^A\,X^B\,X^C = 1\,. \tag{3.5}$$

Their target space metric $\mathcal{G}_{AB}$ is given by

$$\mathcal{G}_{AB} = -\frac{1}{2}\,C_{ABC}\,X^C + \frac{9}{2}\,X_A\,X_B\,, \tag{3.6}$$

where

$$X_A = \frac{1}{6}\,C_{ABC}\,X^B\,X^C\,. \tag{3.7}$$

Observe that $X^A\,X_A = 1$ in view of (3.5). In addition,

$$X_A\,\partial_i X^A = 0\,, \tag{3.8}$$

where $X^A = X^A(\varphi^i)$ and $\partial_i X^A(\varphi) = \partial X^A/\partial\varphi^i$. The scalar fields φ^i have the target space metric

$$\mathcal{G}_{ij} = \mathcal{G}_{AB}\,\partial_i X^A\,\partial_j X^B\,. \tag{3.9}$$

The indices A are raised and lowered using $\mathcal{G}_{AB}$ according to

$$\begin{aligned}\mathcal{G}_{AB}\,X^B &= \frac{3}{2}\,X_A\,,\\ \mathcal{G}_{AB}\,\partial_i X^B &= -\,\frac{3}{2}\,\partial_i X_A\,.\end{aligned}\tag{3.10}$$

The potential V_{pot} is expressed in terms of the *superpotential*

$$W = h_A \, X^A \tag{3.11}$$

and reads

$$V_{\text{pot}} = \mathfrak{g}^2 \left(\mathcal{G}^{ij} \, \partial_i W \, \partial_j W - \frac{4}{3} W^2 \right) = \mathfrak{g}^2 \left(h_A \, \mathcal{G}^{AB} \, h_B - 2 \, W^2 \right) \,, \tag{3.12}$$

where in the second step we used

$$\mathcal{G}^{ij} \, \partial_i X^A \, \partial_j X^B = \mathcal{G}^{AB} - \frac{2}{3} X^A \, X^B \,. \tag{3.13}$$

The gauge coupling constant $\mathfrak{g}$ is identified with the inverse of the curvature radius of AdS_5, $\mathfrak{g} = L^{-1}$.

From (3.4) we can derive the equations of motion for the metric

$$\begin{aligned} R_{MN} =& \mathcal{G}_{ij} \, \partial_M \varphi^i \, \partial_N \varphi^j \\ & + \mathcal{G}_{AB} \left(F^A_{MP} \, F^{BP}_N - \frac{1}{6} \, G_{MN} \, F^A_{PQ} F^{BPQ} \right) + \frac{1}{3} \, G_{MN} \, V_{\text{pot}} \,, \end{aligned} \tag{3.14}$$

for the gauge fields

$$\nabla^N \left(\mathcal{G}_{AB} \, F^B_{NM} \right) = \frac{1}{16} \, C_{ABC} \, \epsilon_M{}^{NPQR} \, F^B_{NP} \, F^C_{QR} \tag{3.15}$$

and for the physical scalar fields φ^i

$$\begin{aligned} 2 \, \nabla^M \left(\mathcal{G}_{ij} \, \nabla_M \varphi^j \right) - \left(\partial_i \mathcal{G}_{jk} \right) \partial_M \varphi^j \, \partial^M \varphi^k & \\ - \frac{1}{2} \left(\partial_i \mathcal{G}_{AB} \right) F^A_{MN} \, F^{BMN} - \partial_i V_{\text{pot}} &= 0 \,. \end{aligned} \tag{3.16}$$

In terms of the real scalar fields X^A, (3.16) reads

$$\begin{aligned} \partial_i X^A \Big[2 \, \nabla^M \left(\mathcal{G}_{AB} \, \partial_M X^B \right) - \left(\partial_A \mathcal{G}_{BC} \right) \partial_M X^B \, \partial^M X^C & \\ - \frac{1}{2} \left(\partial_A \mathcal{G}_{BC} \right) F^B_{MN} \, F^{CMN} - \partial_A V_{\text{pot}} \Big] &= 0 \,. \end{aligned} \tag{3.17}$$

The way to solve the scalar field equations of motion (3.17) is to construct X^A that solve the equations

$$\begin{aligned} 2 \, \nabla^M \left(\mathcal{G}_{AB} \, \partial_M X^B \right) - \left(\partial_A \mathcal{G}_{BC} \right) \partial_M X^B \, \partial^M X^C & \\ - \frac{1}{2} \left(\partial_A \mathcal{G}_{BC} \right) F^B_{MN} \, F^{CMN} - \partial_A V_{\text{pot}} &= 0 \end{aligned} \tag{3.18}$$

up to leftover terms proportional to X_A. These are then projected out by the constraint $X_A \, \partial_i X^A = 0$.

3.3 First-order flow equations

The equations of motion (3.14), (3.15) and (3.16) allow for various classes of solutions that have a description in terms of first-order flow equations. In the ungauged case ($\mathfrak{g} = 0$) one such class consists of electrically charged static extremal black hole solutions with line element [29, 94, 95]

$$ds_5^2 = -\mathrm{e}^{-4U}\, dt^2 + \mathrm{e}^{2U}\, dr^2 + \mathrm{e}^{2U}\, r^2\, d\Omega_3^2\ . \tag{3.19}$$

The metric factor e^{2U} and the scalar fields φ^i supporting the spherically symmetric black hole solution only depend on the radial coordinate r. They satisfy the first-order flow equations

$$\frac{d\mathrm{e}^{2U}}{d\xi} = \frac{1}{3}\, Z\ , \tag{3.20}$$

$$\frac{d\varphi^i}{d\xi} = -\frac{1}{2}\, \mathrm{e}^{-2U}\, \mathcal{G}^{ij}\, \partial_j Z\ , \tag{3.21}$$

where ξ denotes the variable $\xi = 1/r^2$ and where $Z = q_A\, X^A$. These flow equations can be combined into

$$X_A' + 2U'\, X_A = -\frac{2}{3}\, \mathrm{e}^{-2U}\, \frac{q_A}{r^3}\ , \tag{3.22}$$

where $' = d/dr$. Indeed, contracting (3.22) with X^A results in the flow equation for e^{2U}, while contracting with $\partial_j X^A$ yields the flow equation for φ^i in view of the very special geometry relations (3.8) and (3.10).

The flow equations (3.20) and (3.21) are solved in terms of harmonic functions H_A,

$$\mathrm{e}^{2U} = \frac{1}{3}\, H_A\, X^A\ , \tag{3.23}$$

$$\mathrm{e}^{2U}\, X_A = \frac{1}{3}\, H_A\ , \tag{3.24}$$

where $H_A = c_A + q_A/r^2$, and where the c_A denote arbitrary integration constants.

In the gauged case ($\mathfrak{g} \neq 0$) a well-known class of solutions admitting a description in terms of first-order flow equations are flat domain wall solutions with line element [96–105]

$$ds_5^2 = \mathrm{e}^{2A}\, \eta_{\mu\nu}\, dx^\mu dx^\nu + d\rho^2\ , \tag{3.25}$$

where $\eta_{\mu\nu}$ denotes the four-dimensional Minkowski metric. The metric factor e^{2A} and the scalar fields φ^i supporting the domain wall solution only depend on the radial coordinate ρ. They satisfy the first-order flow equations

$$\frac{dA}{d\rho} = \frac{1}{3}\, \mathfrak{g}\, W\ , \tag{3.26}$$

$$\frac{d\varphi^i}{d\rho} = -\, \mathfrak{g}\, \mathcal{G}^{ij}\, \partial_j W\ . \tag{3.27}$$

Changing the radial variable from ρ to r such that $d\rho/dr = (\mathfrak{g}\, r\, \mathrm{e}^{2U(r)})^{-1}$, with $\mathrm{e}^{A(\rho)} = \mathfrak{g}\, r\, \mathrm{e}^{U(r)}$, yields the line element in the form

$$ds_5^2 = \mathrm{e}^{-4U} f\, \eta_{\mu\nu}\, dx^\mu dx^\nu + \mathrm{e}^{2U} f^{-1}\, dr^2 \ , \tag{3.28}$$

where

$$f = \mathfrak{g}^2\, r^2\, \mathrm{e}^{6U} \ . \tag{3.29}$$

The flow equations (3.26) and (3.27) now take the form

$$U'\, r = \frac{1}{3}\, \mathrm{e}^{-2U}\, W - 1 \ , \tag{3.30}$$

$$X'^A = \frac{\mathrm{e}^{-2U}}{r} \left(\frac{2}{3}\, W\, X^A - \mathcal{G}^{AB} h_B \right) \ , \tag{3.31}$$

where $' = d/dr$. Here we have displayed the flow equation for the X^A. The flow equation for the φ^i,

$$\varphi^{i\prime} = -\frac{\mathrm{e}^{-2U}}{r}\, \mathcal{G}^{ij}\, \partial_j W \ , \tag{3.32}$$

follows from the flow equation for X^A by contracting it with $\mathcal{G}_{AB}\, \partial_j X^B$.

The electrically charged black hole solutions of ungauged supergravity described above satisfy first-order flow equations based on $Z = q_A\, X^A$, whereas the flat domain wall solutions of gauged supergravity just described satisfy first-order flow equations based on $W = h_A\, X^A$. We may ask whether there exist charged solutions to gauged supergravity satisfying both sets of first-order flow equations (3.22) and (3.30), (3.31)[3]. That charged extremal solutions exist in gauged supergravity based on first-order flow equations was demonstrated in [31], where various examples with one real scalar field and one Abelian gauge field were discussed. Non-extremal electrically charged static black hole solutions were constructed in [21] by solving the equations of motion. Here we show that these solutions have a first-order flow description based on the two sets (3.22) and (3.30), (3.31), by rewriting the five-dimensional action (3.4) in terms of these equations. Then, the compatibility of the flow equations (3.22) and (3.30), (3.31) requires identifying the integration constants c_A appearing in the solution (3.23), (3.24) with h_A, so that now

$$H_A = h_A + \frac{q_A}{r^2} \ . \tag{3.33}$$

Following [21] we consider the ansatz for a non-extremal electrically charged static black hole solution

$$ds_5^2 = -\,\mathrm{e}^{-4U} f\, dt^2 + \mathrm{e}^{2U} f^{-1}\, dr^2 + \mathrm{e}^{2U}\, r^2\, d\Sigma_k^2 \ , \tag{3.34}$$

$$f = k - \frac{\mu}{r^2} + \mathfrak{g}^2\, r^2\, \mathrm{e}^{6U} \ , \tag{3.35}$$

[3]Common features of black hole and domain wall solutions were recently discussed in [30].

where $U = U(r), f = f(r)$. Here $d\Sigma_k^2$ denotes the line element of a three-dimensional space of constant curvature with metric η_{ij}^k, either flat space ($k = 0$), hyperbolic space ($k = -1$) or a unit three-sphere S^3 ($k = 1$). The presence of a non-vanishing parameter μ is necessary in order for the solutions to have a horizon. Observe that the line elements (3.19) and (3.28) are special cases of (3.34). In the following, we will always consider the case $k = 1$, but we keep k in the formulae as a book-keeping device. The scalar fields and the gauge fields supporting the solutions are taken to be functions of r, only. Inserting the line element (3.34) into the action (3.4) yields

$$16\pi\, G_5\, S = S_0 + S_2 + S_{\rm td}\ , \tag{3.36}$$

where S_0 and S_2 comprise the contributions to order $\mathfrak{g}^0$ and $\mathfrak{g}^2$, respectively, and where $S_{\rm td}$ contains total derivative terms. S_0 and S_2 read

$$\begin{aligned}
S_0 =& \int d^5x \sqrt{\eta^k}\, \Bigg[3\,\mu\, \frac{\mathrm{e}^{-2U}}{r^3}\, q_A \mathcal{G}^{AB} \left(2X_B - \frac{1}{3}\,\mathrm{e}^{-2U}\, H_B \right) - \frac{9}{4} \left(k - \frac{\mu}{r^2} \right) r^3 \\
& \times \left(X'_A + 2U'\, X_A + \frac{2}{3}\,\mathrm{e}^{-2U}\, \frac{q_A}{r^3} \right) \mathcal{G}^{AB} \left(X'_B + 2U'\, X_B + \frac{2}{3}\,\mathrm{e}^{-2U}\, \frac{q_B}{r^3} \right) \\
& + r^3\,\mathrm{e}^{4U} \left(F^A_{tr} - \mathrm{e}^{-4U}\, \frac{\mathcal{G}^{AC} Q_C}{r^3} \right) \mathcal{G}_{AB} \left(F^B_{tr} - \mathrm{e}^{-4U}\, \frac{\mathcal{G}^{BD} Q_D}{r^3} \right) \Bigg]\ , \\
S_2 =& \mathfrak{g}^2 \int d^5x \sqrt{\eta^k}\, \Big[\, \frac{4}{3}\, r^3\,\mathrm{e}^{2U} \left(W - 3\,\mathrm{e}^{2U}\, (U' r + 1) \right)^2 \\
& - r^5\,\mathrm{e}^{6U} \left(X'^A - \frac{\mathrm{e}^{-2U}}{r} \left(\frac{2}{3}\, W\, X^A - \mathcal{G}^{AC} h_C \right) \right) \mathcal{G}_{AB} \\
& \times \left(X'^B - \frac{\mathrm{e}^{-2U}}{r} \left(\frac{2}{3}\, W\, X^B - \mathcal{G}^{BD} h_D \right) \right) \Big]\ ,
\end{aligned} \tag{3.37}$$

where $' = d/dr$. Observe that the Chern-Simons term proportional to the intersection numbers C_{ABC} vanishes, since in the static case there are no magnetic fields present. In the expression for S_0, the physical electric charges Q_A are related to the q_A by

$$Q_A \mathcal{G}^{AB} Q_B = k\, q_A \mathcal{G}^{AB} q_B + \mu\, q_A \mathcal{G}^{AB} h_B\ , \tag{3.38}$$

and H_A is given by (3.33). $S_{\rm td}$ reads

$$\begin{aligned}
S_{\rm td} =& \int d^5x \sqrt{\eta^k}\, \Bigg[2\, Q_A\, F^A_{tr} + \Big(6\,\mu\, U - 2(k - \frac{\mu}{r^2})\, (r^3\, U' + \mathrm{e}^{-2U}\, q_A X^A) \Big)' \\
& + \mathfrak{g}^2 \left(-8 r^5\,\mathrm{e}^{6U}\, U' - 8 r^4\,\mathrm{e}^{6U} + 2 r^4\,\mathrm{e}^{4U}\, W \right)' \Bigg]\ .
\end{aligned} \tag{3.39}$$

Observe that S_0 and S_2 are given in terms of squares of first-order differential equations, with the exception of the first term of S_0, which is proportional to the parameter μ. Under variation with respect to U or with respect to X^A, the contribution from the first term of S_0 vanishes provided that

$$X_A = \frac{1}{3}\,\mathrm{e}^{-2U}\,H_A\ . \tag{3.40}$$

Thus, extremizing S_0 and S_2 with respect to the fields U, X^A and F^A_{tr} yields the relation (3.40), the first-order flow equations (3.22) and (3.30), (3.31) as well as

$$F^A_{tr} = \mathrm{e}^{-4U}\,\frac{\mathcal{G}^{AC}Q_C}{r^3}\ . \tag{3.41}$$

The flow equations (3.22) and (3.30), (3.31) are solved by (3.23), (3.24) with $c_A = h_A$, as discussed above, and the solution agrees with (3.40).

We take h_A and q_A in (3.33) to be positive to ensure that $H_A > 0$. We also take $X^A > 0$, so that $\mathrm{e}^{2U} > 0$ along the flow. We impose the normalization $\mathrm{e}^{2U} = 1$ at $r = \infty$. The asymptotic value of X_A is then $\frac{1}{3}\,h_A$. Denoting the asymptotic value of the X^A by h^A, we have $\frac{1}{3}h^A\,h_A = 1$ in view of (3.5). We introduce the *dual superpotential* $\tilde{W}$ as

$$\tilde{W} = h^A\,X_A\ , \tag{3.42}$$

for later convenience.

Summarizing, the line element (3.34) together with (3.23), (3.24), (3.33), (3.41) and (3.38) describe non-extremal electrically charged static black hole solutions to first-order flow equations. It can be checked that they solve the equations of motion (3.14), (3.15) and (3.17).

On the solution, the total derivative terms (3.39) can be written as

$$S_{\rm td} = \int d^5x\sqrt{\eta^k}\,\Big[6\,\mu\,U' - 2\left(r^3\,f\,U' + r^2(f-k)\right)'\Big]\ , \tag{3.43}$$

while the first term of S_0 yields

$$-\int d^5x\sqrt{\eta^k}\,6\,\mu\,U'\ . \tag{3.44}$$

Thus, on the solution, the action (3.4) evaluates to

$$16\pi\,G_5\,S = -2\int d^5x\sqrt{\eta^k}\,\left(r^3\,f\,U' + r^2(f-k)\right)'\ , \tag{3.45}$$

in agreement with [106].

The electric field F^A_{tr} is determined in terms of the potential $\phi^A(r)$, *i.e.* $F^A_{tr} = -\partial_r\phi^A(r)$. In the following, we compute the contraction $Q_A\,\phi^A$. To this end, we differentiate (3.20) and obtain

$$U'' = -\frac{1}{3}\,\mathrm{e}^{-2U}\,\left(-2U'\,\frac{Z}{r^3} - 3\frac{Z}{r^4} + \frac{q_A X'^A}{r^3}\right)\ . \tag{3.46}$$

Using (3.22) we compute

$$X'^A = -\frac{3}{2}\,\mathcal{G}^{AB}\,X'_B = 2U'\,X^A + \mathrm{e}^{-2U}\,\frac{\mathcal{G}^{AB}q_B}{r^3}\ . \tag{3.47}$$

Then, using (3.46) and (3.47) gives

$$U'' + \frac{3}{r}\,U' = -\frac{1}{3}\,\mathrm{e}^{-4U}\,\frac{q_A\mathcal{G}^{AB}q_B}{r^6}\ . \tag{3.48}$$

With the help of (3.20), (3.38) and (3.48) we obtain

$$-3\left[r^3\left(k - \frac{\mu}{r^2}\right)U'\right]' = \mathrm{e}^{-4U}\,\frac{Q_A\mathcal{G}^{AB}Q_B}{r^3}\ , \tag{3.49}$$

which equals $Q_A\,F^A_{tr}$, as can be seen from (3.41). Hence we establish that

$$Q_A\,\phi^A = -\left(k - \frac{\mu}{r^2}\right)\,\mathrm{e}^{-2U}\,q_A X^A + k\,q_A h^A\ , \tag{3.50}$$

where we used (3.20) once more. We chose the integration constant in such a way that $Q_A\,\phi^A$ vanishes at spatial infinity, as in [107]. In the context of the AdS/CFT correspondence [17, 67, 68] this means that the potentials ϕ^A associated with the $U(1)$ charges Q_A approach the boundary at a vev rate [31, 108].

The quantity appearing in the first law of black hole mechanics, $dM = T_H\,d\mathcal{S} + \phi^A\,dQ_A$, is not (3.50) but a rescaled one given by [72]

$$Q_A\,\phi^A = \frac{2}{3w_5}\left(-\left(k - \frac{\mu}{r^2}\right)\,\mathrm{e}^{-2U}\,q_A X^A + k\,q_A h^A\right)\ , \tag{3.51}$$

where

$$w_5 = \frac{16\pi}{3}\frac{G_5}{vol}\ . \tag{3.52}$$

Here, $vol = \int d^3x\sqrt{\eta^k}$ denotes the volume of the three-dimensional space of constant curvature with line element $d\Sigma^2_k$. For $k = 1$ this space is a unit three-sphere S^3 with volume $vol = vol(S^3)$. As already stated, here and in the following, the focus lies on the case $k = 1$, but we nevertheless keep k in the formulae as a book-keeping device.

Next, we compute the coefficient of the $1/r^2$-term in the metric factor $-\mathrm{e}^{-4U}\,f$ of the line element (3.34). We denote this coefficient by $w_5\,M$. Expanding $\mathrm{e}^{2U} = 1 + \kappa/r^2 + \ldots$ as well as $X^A = h^A + \beta^A/r^2 + \ldots$ and using (3.23) results in $\kappa = \frac{1}{3}\left(h^A\,q_A + h_A\,\beta^A\right)$. On the other hand, inserting the expansion of X^A into (3.5) yields $h_A\,\beta^A = 0$. It follows that the coefficient $w_5\,M$ is given by

$$w_5\,M = \mu + 2k\,\kappa = \mu + \frac{2}{3}\,k\,h^A\,q_A\ . \tag{3.53}$$

M denotes the mass of the black hole relative to the ground state energy of the AdS space (cf. section 2.6.1).

3.4 STU black hole solutions

In many cases, we further truncate the theory (3.4) by restricting the index A to values in the set $A=\{1,2,3\}$. In this case, there are only three scalar fields, $S=X^1$, $T=X^2$, $U=X^3$, which are parametrized in terms of the physical scalar fields φ^1 and φ^2 as, for instance,

$$X^1=\mathrm{e}^{-\frac{1}{\sqrt{6}}\varphi^1-\frac{1}{\sqrt{2}}\varphi^2}\,,\quad X^2=\mathrm{e}^{-\frac{1}{\sqrt{6}}\varphi^1+\frac{1}{\sqrt{2}}\varphi^2}\,,\quad X^3=\mathrm{e}^{-\frac{2}{\sqrt{6}}\varphi^1}\,. \tag{3.54}$$

The intersection numbers C_{ABC} reduce to one independent intersection number $C_{123}=1$, such that the constraint (3.5) becomes

$$X^1\,X^2\,X^3=1 \tag{3.55}$$

and the metric $\mathcal{G}_{AB}$ is now given by

$$\mathcal{G}_{AB}=\frac{1}{2}\,\delta_{AB}\,\left(X^A\right)^{-2}\,, \tag{3.56}$$

where here there is no summation over A. The superpotential (3.11) and the dual superpotential (3.42) become

$$W=\sum_{A=1}^{3}X^A\,,\quad \tilde{W}=\sum_{A=1}^{3}X_A\,, \tag{3.57}$$

where we set $h_A=h^A=1$. Consequently, the bulk action (3.4) simplifies to

$$\begin{aligned}16\pi\,G_5\,S=\int d^5x\sqrt{-G}\,\Bigg(&R-\frac{1}{2}\left(\partial\varphi^1\right)^2-\frac{1}{2}\left(\partial\varphi^2\right)^2+4\mathfrak{g}^2\sum_{A=1}^{3}\left(X^A\right)^{-1}\\&-\frac{1}{4}\sum_{A=1}^{3}\left(X^A\right)^{-2}\left(F^A_{MN}\right)^2-\frac{\sqrt{3}}{2}\kappa\,C_{ABC}\,\epsilon^{MNOPQ}\,F^A_{MN}F^B_{OP}A^C_Q\Bigg)\,.\end{aligned} \tag{3.58}$$

The solutions to the equations for the metric function e^{6U} (3.23) and the scalar fields X^A [94] read

$$\mathrm{e}^{6U}=\mathcal{H}=H_1H_2H_3\,,\quad X^A=H_A^{-1}\mathcal{H}^{1/3}\,,\quad X_A=\frac{1}{3}H_A\mathcal{H}^{-1/3}\,. \tag{3.59}$$

Inserting the solution for the metric function e^{6U} into the ansatz (3.34) and (3.35) yields the line element of the AdS-STU black hole solution

$$\begin{aligned}ds_5^2=&-\mathcal{H}^{-2/3}f\,dt^2+\mathcal{H}^{1/3}f^{-1}dr^2+\mathcal{H}^{1/3}r^2d\Sigma_k^2\\f=&k-\frac{\mu}{r^2}+\mathfrak{g}^2r^2\mathcal{H}\,.\end{aligned} \tag{3.60}$$

The gauge field corresponding to the charged AdS-STU black hole solution (3.60) reads [106]

$$A^A = -\frac{Q_A}{r^2} H_A^{-1} dt\,, \tag{3.61}$$

where there is no summation over A, and the Q_A are given by the square root of (3.38).

The STU model allows for four special cases for which calculations become relatively simple. These are deduced in the following subsections.

3.4.1 The Schwarzschild case ($q_1 = q_2 = q_3 = 0$)

The Schwarzschild case is obtained by setting the two physical scalar fields to $\varphi^1 = \varphi^2 = 0$ as well as the charges to $q_1 = q_2 = q_3 = 0$. Consequently, the scalar fields as well as the harmonic functions reduce to $X^A = 1$ and $H_A = 1$, such that $\mathcal{H} = 1$. For the superpotential, we then have $W = 3$ and for the dual superpotential, we have $\tilde{W} = 1$. Applying these simplifications to the line element (3.60) with $k = 1$ and the gauge field (3.61) yields the known AdS-Schwarzschild black hole solution with the ADM mass

$$w_5 M = \mu\,. \tag{3.62}$$

An event horizon exists as long as $\mu > 0$.

3.4.2 The Maxwell case ($q_1 = q_2 = q_3 = q$)

The Maxwell case is obtained by setting the two physical scalar fields to $\varphi^1 = \varphi^2 = 0$, and all the charges equal, $q_1 = q_2 = q_3 = q$. In this case, the scalar fields again reduce to $X^A = 1$ and the harmonic functions become all equal, $H_A = H = 1 + q/r^2$, such that $\mathcal{H} = H^3$. The superpotential and the dual superpotential again become $W = 3$ and $\tilde{W} = 1$. Applying these simplifications to the line element (3.60) with $k = 1$ and the gauge field (3.61) yields the AdS-Reissner-Nordström solution with the ADM mass and the electric energy[4]

$$w_5\, M = \mu + 2q\,, \quad Q_A\, \phi^A = \frac{2}{w_5}\,\frac{q(\mu + q)}{q + r^2}\,. \tag{3.63}$$

The horizon radius is given by the largest positive root of $f(r)$ of (3.60).

[4]More precisely, the true electric energy of the black hole is $Q_A \phi^A$ evaluated at the black hole horizon.

3.4.3 Two equal charges ($q_1 = q_2 = q$, $q_3 = 0$)

The case of two equal charges corresponds to setting $\varphi^1 = \varphi$ and $\varphi^2 = 0$, and simultaneously $q_1 = q_2 = q$, $q_3 = 0$. We then have for the harmonic functions $H_1 = H_2 = H = 1 + q/r^2$, $H_3 = 1$ such that $\mathcal{H} = H^2$, and consequently for the scalar fields $X^1 = X^2 = H_1^{-1/3}$, $X^3 = H_1^{2/3}$. For the superpotential and the dual superpotential, we obtain $W = 2H^{-1/3} + H^{2/3}$ and $\tilde{W} = \frac{1}{3}\left(2H^{1/3} + H^{-2/3}\right)$. The corresponding two-charge AdS black hole solution resulting from (3.60), with $k = 1$, together with (3.61) has the ADM mass and the electric energy

$$w_5\, M = \mu + \frac{4}{3}\, q\,, \quad Q_A\, \phi^A = \frac{4}{3w_5}\, \frac{q(\mu+q)}{q+r^2}\,. \tag{3.64}$$

The existence of a horizon shielding the singularity at $r = 0$ requires taking $\mu > \mathfrak{g}^2\, q^2$. There is no inner horizon [31, 83].

3.4.4 One charge ($q_1 = q$, $q_2 = q_3 = 0$)

Finally, the case of one non-zero charge is obtained by setting $\varphi^1 = \varphi$ and $\varphi^2 = \sqrt{3}\varphi^1$, and simultaneously $q_1 = q$, $q_2 = q_3 = 0$. We then have for the harmonic functions $H_1 = H = 1 + q/r^2$, $H_2 = H_3 = 1$, such that $\mathcal{H} = H$, and for the scalar fields $X^1 = H^{-2/3}$, $X^2 = X^3 = H^{1/3}$. The superpotential and the dual superpotential become $W = H^{-2/3} + 2H^{1/3}$ and $\tilde{W} = \frac{1}{3}\left(H^{2/3} + 2H^{-1/3}\right)$. The corresponding one-charge AdS black hole solution resulting from (3.60), with $k = 1$, together with (3.61) has the ADM mass and the electric energy

$$w_5\, M = \mu + \frac{2}{3}\, q\,, \quad Q_A\, \phi^A = \frac{2}{3w_5}\, \frac{q(\mu+q)}{q+r^2}\,. \tag{3.65}$$

The solution has a single horizon shielding the singularity at $r = 0$ whenever $\mu > 0$ [31, 83].

3.5 Embedding of the AdS-STU black hole into type IIB supergravity

In section 3.2, it was shown how five-dimensional $\mathcal{N} = 2$ supergravity can be derived from eleven-dimensional supergravity by compactifying on a Calabi-Yau three-fold. However, in order to justify the use of the theory (3.58), which is a truncation of the full $\mathcal{N} = 2$ supergravity, within the AdS/CFT correspondence, it should be embeddable into type IIB string theory. This is actually possible. In [72–74], it was shown that the AdS-STU black hole (3.60) arises from the compactification of a spinning D3-brane of type IIB supergravity as is explained in the following. Starting from a static black D3-brane as (2.38), we can turn on three different angular momenta in the six-dimensional transverse space such that the rotation axes lie in three independent planes. This breaks the $SO(6)$ rotational symmetry of (2.38) to $U(1) \times U(1) \times U(1)$. Next we take a decoupling limit of this spinning D3-brane similar to the one in section 2.4, where the radial coordinate r and α' are sent to zero while r/α' and the rotation parameters l_i, $i = 1, 2, 3$, are kept fixed. The resulting line element of the near-horizon spinning D3-brane becomes a product space consisting of a five-dimensional asymptotically AdS_5 part and a deformed S^5 part,

$$ds_{10}^2 = \sqrt{\tilde{\Delta}} ds_5^2 + \frac{1}{\mathfrak{g}^2 \sqrt{\tilde{\Delta}}} \sum_{A=1}^{3} \left(X^A\right)^{-1} \left(d\mu_A^2 + \mu_A^2 \left(d\phi_A + \mathfrak{g} A^A\right)^2\right) . \qquad (3.66)$$

The five-dimensional line element ds_5^2 is now given by the line element (3.60) with $k = 0$ of the AdS-STU black brane by virtue of the identification of the rotation parameters l_A^2 with the charges q_A. Thus, from the five-dimensional point of view the three rotations of the S^5 correspond to three Abelian $U(1)$ gauge fields associated with the three charges q_A.

The quantity $\tilde{\Delta}$ is given by

$$\tilde{\Delta} = \sum_{A=1}^{3} X^A \mu_A^2 \, , \qquad (3.67)$$

and $\mathfrak{g}$ in (3.66) again denotes the gauge coupling constant which is equal to the inverse AdS radius L^{-1}.

The X^A are the scalar fields introduced in (3.54). Here the S^5 is parametrized by the three parameters μ_A which are expressed in terms of angles on a two-sphere,

$$\mu_1 = \sin\theta \; , \quad \mu_2 = \cos\theta \sin\psi \; , \quad \mu_3 = \cos\theta \cos\psi \; , \qquad (3.68)$$

and which satisfy $\sum_A \mu_A^2 = 1$, as well as three angles ϕ_A. In (3.66), the S^5 is deformed by three one-form gauge fields, associated with the three rotation parameters l_A, given by

$$A^A = \frac{\sqrt{\mu}\, l_A}{r^2} H_A^{-1} dt \, , \quad H_A = 1 + \frac{l_A^2}{r^2} \, , \qquad (3.69)$$

where there is no summation over A and $\mu = r_0^4$ denotes the non-extremality parameter[5] of the near-horizon black brane solution. Identifying $\sqrt{\mu}\, l_A = -Q_A$ yields the gauge fields (3.61) associated with the AdS-STU black hole with $k = 0$.

[5] The non-extremality parameter μ must not be confused with the chemical potential μ introduced in section 2.6.3.

Note, that the near-horizon spinning D3-brane (3.66) exhibits the form of the reduction ansatz for dimensional reduction on the S^5. As we know from chapter 2 the D3-brane solution is supported by a constant dilaton equal to the string coupling g_s as well as the self-dual five-form field strength $\tilde{F}_5$. All other fields of type IIB supergravity vanish. Inserting the ansatz (3.66) and an appropriate reduction ansatz for the five-form field strength into the ten-dimensional equations of motion yields five-dimensional equations of motion which can be derived from the action (3.58).

Thus, the embedding of the AdS-STU black hole solution (3.60) with $k = 0$ gives a ten-dimensional solution which is precisely the decoupling limit of the spinning D3-brane. The parameters of the rotation in the compact directions are identified with the charges of the five-dimensional black hole. Finally, the $k = 1$ AdS-STU black hole is obtained by transforming the AdS_5 part to global coordinates.

In chapter 4 and 5, we use the asymptotically AdS_5 background (3.60) and its special cases introduced in sections 3.4.1-3.4.4 to study different applications of the AdS/CFT correspondence.

3.6 Summary

In this chapter, we reviewed the non-extremal charged static black hole solutions of five-dimensional $\mathcal{N} = 2$ gauged supergravity. For that purpose, we derived the action of five-dimensional $\mathcal{N} = 2$ gauged supergravity from eleven-dimensional supergravity by compactifying on a Calabi-Yau three-fold. Then, the resulting action was gauged to obtain the gauged supergravity action which admits AdS space as a solution.

In addition, we presented a calculation carried out by the author of this thesis, namely the derivation of first-order flow equations from the five-dimensional $\mathcal{N} = 2$ gauged supergravity action in the context of non-extremal electrically charged static black holes. It turned out that these equations are given by the combination of the flow equations that were derived earlier for the electrically charged black hole solutions of ungauged supergravity, which are based on the central charge $Z = q_A X^A$, as well as for flat domain wall solutions of gauged supergravity, which are based on the superpotential $W = h_A X^A$. The compatibility of these two sets of flow equations for the non-extremal charged black hole solutions of gauged supergravity requires to identify the integration constants c_A appearing in the solution to the central-charge based flow equations with the coefficients h_A appearing in the superpotential W. Moreover, it can be shown that these flow equations are consistent with the equations of motion and that they admit the solutions found in [21]. A striking advantage of first-order flow equations is that they are easier to solve compared to the second-order equations of motion.

After that, we reviewed the special case of the AdS-STU black hole solution which constitutes a full solution to the five-dimensional gauged supergravity. This solution, in turn, can be further specialized to four different relatively simple solutions, the AdS-Schwarzschild, the AdS-Maxwell, the two-charge AdS and the one-charge AdS black hole.

Finally, we demonstrated that the five-dimensional AdS-STU black hole can be embedded into type IIB string theory for which reason this solution qualifies for being investigated in the context of the AdS/CFT correspondence. We exploit this fact in chapter 4 and 5.

Chapter 4

Holographic brane cosmology vs. entropy and Casimir energy

In this chapter, a four-dimensional *Randall-Sundrum-type brane universe* is studied in the context of the AdS/CFT correspondence. The bulk background is taken to be the static non-extremal charged AdS black hole solution of $\mathcal{N} = 2$ gauged supergravity in five dimensions introduced in chapter 3. The thermal field theory that is dual to the AdS black hole is assumed to live on the brane, representing the matter content of the brane universe. The equations of motion of the brane are shown to be equal to the *Friedmann equations*, which are in the following often referred to as *Friedmann-Robertson-Walker (FRW)* equations, of a closed universe expressed in terms of bulk $\mathcal{N} = 2$ supergravity quantities. The entropy of the brane field theory is shown to take the form of a Cardy-Verlinde-type formula modified by the presence of $\mathcal{N} = 2$ supergravity quantities. At the horizon of the black hole, the equations for the entropy and the Casimir energy of the field theory surprisingly merge with the two FRW equations. This might indicate that both sets of equations stem from a single more fundamental theory.

The outline of the chapter is as follows. Section 4.1 gives a brief introduction to *brane-world holography*. In section 4.2, the bulk-brane system is viewed from the bulk perspective. The brane action is presented and the equations of motion are derived. Then, a Cardy-Verlinde-type formula for the entropy is derived and the Casimir energy is computed in terms of bulk black hole quantities. In section 4.3, the system is viewed from the brane perspective and the merging of the equations for the entropy and the Casimir energy with the two FRW equations at the black hole horizon is demonstrated. In section 4.4, an outlook on how to extend the analysis to a five-dimensional theory including higher-derivative curvature terms is given. As an explicit example, a Gauss-Bonnet term is added to the five-dimensional action and the first FRW equation is derived. The last section 4.5 contains a summary and a discussion of the results.

The present chapter is based on work which was done by the author of this thesis in collaboration with Gabriel Lopes Cardoso and which was published in [1].

4.1 Introduction

The notion of a brane universe, which we also refer to as *brane world*, implies that the observable part of our world, as for instance all standard model fields, are restricted to a 3-brane with four-dimensional world-volume which is embedded in a higher-dimensional space. However, gravity is not bounded to the brane and may propagate in the extra dimensions. Such brane worlds were first investigated in [109–112].

In [34, 113], Randall and Sundrum proposed a model with one single infinite dimension transverse to the brane universe. The resulting five-dimensional space is AdS_5 whose radial direction represents the extra dimension. Only gravity is, in principle, allowed to act in this dimension. However, it was shown that there exists a normalizable zero-mode bound state which is localized on the brane. Thus, four-dimensional gravity is effectively also confined to the brane in accordance with Newton's law [34]. This way of dimensional reduction to obtain a four-dimensional spacetime from a higher-dimensional spacetime is different from the Kaluza-Klein idea where all extra dimensions are considered to be small and compact compared to the observed extended four dimensions. Therefore, it can be viewed as an *alternative to compactification* as it was first termed by its inventors.

The scenario of [34] can also be viewed in light of the AdS/CFT correspondence. This is sometimes referred to as *brane-world holography* [14], since the four-dimensional brane field theory is considered to be dual to the five-dimensional bulk theory. In spite of this, brane-world holography differs in some respects from the usual AdS/CFT correspondence. The five-dimensional bulk theory is dual to the four-dimensional theory on the brane which is, however, not considered to be at infinity of the radial direction, but at some finite IR cut-off. From the UV/IR relation we know that an IR cut-off in the bulk corresponds to a UV cut-off in the brane field theory. Thus, the brane theory is not conformally invariant due to the presence of a scale set by the cut-off. This, in turn, is cured, if the brane is sent to infinity where the true boundary of the AdS space is situated. In spite of the broken conformal symmetry, in the following we often stick to the term CFT for the brane field theory. Another departure is that the bulk theory couples to gravity on the brane. This is possible, since here the metric does not blow up at the position of the brane along the radial direction which is given by the IR cut-off rather than infinity as in the original AdS/CFT construction. Thus, from the brane point of view, the energy red-shift, which presents a potential well for the bulk theory modes in the direction to the brane, is finite allowing a bulk graviton to reach the brane and couple to the brane field theory.

In addition, such a holographic brane world can now be shown to behave like a standard FRW universe[1]. Putting the brane in the background of an AdS black hole turns on a temperature in the brane field theory. Then making the position of the brane time-dependent enables the brane to move in this background. The motion of the brane turns out to be exactly described by the FRW equations of a closed universe [35]. The size of the universe corresponds directly to the distance between the brane and the black hole singularity. The matter content of the universe is represented by the finite-temperature field theory on the brane dual to the supergravity theory in the bulk. From the five-dimensional perspective there are two special moments in the history of the universe, namely when the brane coincides with the black hole horizon during the expansion phase and again during the recontraction phase. At these moments, the entropy and the Casimir energy of the brane field theory surprisingly merge with the two FRW equations [36, 37]. The main goal of this chapter is to investigate this phenomenon for the case of the non-extremal charged black holes with spherical horizons ($k = 1$) in $\mathcal{N} = 2$ gauged supergravity supported by scalar fields which were introduced in chapter 3. Earlier investigations for an AdS-Schwarzschild and an AdS-Maxwell black hole can be found in [36–41].

A key ingredient to observe this merging is that the entropy can be written as a Cardy-Verlinde-type formula for conformal field theories presented in section 2.6.1. This expresses the square of the entropy S in terms of the product of the extensive part E_e of the energy and the Casimir energy E_c on the brane. As already shown in section 2.6.1, the energies E_e and E_c behave as

$$E_e\, a \propto S^{4/3} \quad , \quad E_c\, a \propto S^{2/3} \, , \tag{4.1}$$

where now a denotes the radius of the spherical brane. For brane theories dual to AdS-Schwarzschild or AdS-Maxwell black holes, the proportionality coefficients in (4.1) are independent of a and S. However, in case of the general AdS black hole solution of $\mathcal{N} = 2$ supergravity, non-trivial scalar fields are present which show up in the proportionality coefficients of (4.1) in terms of the superpotentials (3.11) and (3.42). These superpotentials do not have a simple dependence on extensive quantities. The Cardy-Verlinde formula in the context of the STU model (see section 3.4) was also discussed in [116, 117]. We show that the two FRW equations, describing the motion of the brane in the charged black hole background, take a form that is similar to the Cardy-Verlinde-type formula and to the equation for the Casimir energy on the brane, respectively. Then, as the brane crosses the event horizon of the black hole, these two sets of equations coincide. This merging of the two a priori unrelated sets of equations indicates that there might be a common origin relating the entropy and the energy of a CFT with the standard FRW equations [36, 37].

[1]This topic is not reviewed here. The reader interested in some more details is referred to [114, 115] for an introduction.

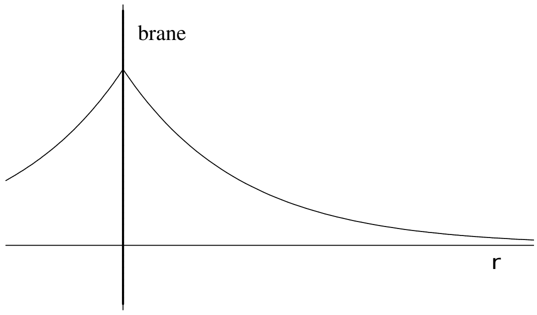


Figure 4.1: Randall-Sundrum construction. Two identical copies of AdS_5 are separated by a 3-brane at some position r along the common radial direction. The true boundaries of both AdS spaces are thus cut off by the 3-brane. (figure taken from [14])

In [36] it was argued that the entropy of a thermal CFT with a gravity dual is always less than the *Bekenstein entropy* [118]. The Bekenstein entropy is proportional to the energy and the linear size of the system. For a radiation dominated universe, it is constant throughout the entire evolution. Thus, as long as the entropy of the CFT does not change, the Bekenstein bound is satisfied at all times. We investigate the *Bekenstein bound* on the entropy [36, 118] for special cases of the AdS-STU black hole solution introduced in section 3.4.

4.2 The bulk perspective

In the first part of this section, we derive the FRW equations for a closed universe from the motion of a 3-brane in the background of the static black hole solution (3.34) of gauged $\mathcal{N} = 2$ supergravity. For this purpose, we take the bulk point of view and express all equations in terms of black hole quantities. In the second part, it is shown that for any model (3.5) the entropy of a charged AdS black hole (3.34) can be written as a Cardy-Verlinde-type formula [36]. This has already been discussed in [38, 40, 41, 116, 117] for various black holes in the context of the STU model. In contrast to the AdS-Schwarzschild black hole or the AdS-Maxwell black hole considered in the analysis of [37, 38], the black hole solutions (3.34) are supported by non-trivial scalar fields which complicate the situation significantly.

4.2.1 FRW cosmology on the brane

The *Randall-Sundrum model* [34] can be roughly described as consisting of two identical copies of AdS_5 separated by a 3-brane at some position near the true boundaries of the two AdS spaces, respectively (see figure 4.1). The brane has constant tension which is fine-tuned against the bulk cosmological constants such that the brane cosmological constant vanishes. In this chapter, we focus on one half of the spacetime depicted in figure 4.1. Thus, following [35, 37, 119], we consider AdS_5 which is cut off by the 3-brane near its true boundary and do not care about the region beyond the cut-off brane. Furthermore, similar to [120], we do not take the brane tension to be constant, but allow it to vary along the radial direction. This is necessary for the vanishing of the cosmological constant of a brane moving in the background of the AdS black hole (3.34).

The total action describing the bulk-brane system is given by the bulk-spacetime action (3.4) and the brane action consisting of a Gibbons-Hawking term [71], the coupling to the scalar fields through the superpotential as well as a four-dimensional Einstein term,

$$-\frac{1}{8\pi\, G_5}\int_\Sigma d^4x\sqrt{-\gamma}\left(K+\frac{W}{L}+\frac{L}{4}\,\mathcal{R}\right)\ , \tag{4.2}$$

where Σ denotes the brane world-volume, $\mathcal{R}$ denotes the Ricci scalar on the brane, K is the Gibbons-Hawking term and W is the superpotential defined in (3.11). The latter depends on the scalar fields $X^A(r)$ yielding a brane tension W/L that changes with r. In case of the AdS-Schwarzschild or the AdS-Maxwell black hole, the superpotential becomes a constant, $W=3$, yielding a constant brane tension. In the context of the original AdS/CFT correspondence where the 3-brane coincides with the true boundary of the AdS space at $r=\infty$, the second and the third term are counterterms [77, 106, 121, 122]. These ensure that the combined action of (3.4) and (4.2) is finite when the brane is moved to infinity as explained in section 2.5. Observe that since we focus on the case $k=1$ for which the boundary topology is $\mathbb{R}\times S^3$, the holographic trace anomaly [123] vanishes and no further counterterms are required [121, 124].

The extrinsic curvature K is given by

$$K=\gamma^{MN}\,K_{MN}\quad ,\quad K_{MN}=\gamma_M{}^P\,\gamma_N{}^Q\,\nabla_{(P}\,n_{Q)}\ . \tag{4.3}$$

Here the tensor $\gamma_{MN}=G_{MN}-n_M\,n_N=\mathrm{diag}(0,\gamma_{\mu\nu})$ denotes the projection of G_{MN} onto Σ, so that the induced metric $\gamma_{\mu\nu}$ on Σ is given by the tangential components of γ_{MN}. We note that $\gamma^{MN}=G^{MN}-n^M\,n^N$ and $K=\nabla_M\,n^M$, where $n^M=G^{MN}\,n_N$. The vector $n=n^M\,\partial_M$ is the unit normal to Σ, *i.e.* $n^M\,n_M=1$.

We view the brane as a dynamical entity [35, 37] moving in a background of the form

$$ds_5^2=G_{MN}\,dx^M\,dx^N=-A(a)\,dt^2+B(a)\,da^2+a^2\,d\Sigma_k^2\ , \tag{4.4}$$

where the spatial three-sphere has radius a. Note that the black hole metric (3.34) is of this type with

$$a=r\,e^U \tag{4.5}$$

and

$$A=\mathrm{e}^{-4U}\,f\quad ,\quad B=\frac{1}{(1+r\,U')^2\,f}\quad ,\quad U'=\frac{dU}{dr}\ , \tag{4.6}$$

where A and B are related by

$$A = \left(\frac{3}{W}\right)^2 \frac{1}{B} \tag{4.7}$$

by virtue of equation of (3.30).

We take the induced metric $\gamma_{\mu\nu}$ on the brane to have the form of a standard FRW metric with *cosmic scale factor* $a(\tau)$ [37],

$$ds_4^2 = -d\tau^2 + \gamma_{ij}\, dx^i dx^j = -d\tau^2 + a^2(\tau)\, d\Sigma_k^2 \,. \tag{4.8}$$

Comparing (4.8) with (4.4) shows that the induced metric is obtained from the bulk metric by requiring

$$-A\left(\frac{dt}{d\tau}\right)^2 + B\left(\frac{da}{d\tau}\right)^2 = -1\,. \tag{4.9}$$

This results in

$$\frac{dt}{d\tau} = A^{-1}\sqrt{A + A\,B\,\dot{a}^2} \quad , \quad \dot{a} = \frac{da}{d\tau}\,. \tag{4.10}$$

In the limit $r \to \infty$, where (4.4) becomes the AdS metric, the left equation simplifies to $dt/d\tau = L/a$ for the line element (3.34). It provides a conversion factor to translate between the AdS time t and the CFT time τ, and thus becomes important when translating bulk energy and temperature into brane energy and temperature. From this we see that both time-coordinates coincide, if the radius a of the spatial S^3 is equal to the AdS radius L.

To compute the associated vector n^M, we follow [125] and introduce the velocity vector v^M ($v^M\, v_M = -1$),

$$v^M = \frac{dx^M}{d\tau} = (\frac{dt}{d\tau}, \dot{a}, \vec{0})\,. \tag{4.11}$$

Using

$$v^M\, n_M = 0\,, \tag{4.12}$$

we find that the unit normal vector n^M is given by

$$n^M = \pm\frac{1}{\sqrt{A\,B}}(B\,\dot{a}, \sqrt{A + A\,B\,\dot{a}^2}, \vec{0})\,. \tag{4.13}$$

Below we see that we have to take the minus sign for consistency.

Next, proceeding as in [120], we vary the combined action (3.4) and (4.2) with respect to the metric G_{MN}. Setting the variation to zero results in the equations [77, 120, 126]

$$K_{\mu\nu} - \left(K + \frac{W}{L}\right)\gamma_{\mu\nu} = 0 \tag{4.14}$$

and

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\,\gamma_{\mu\nu} = 8\pi\, G_4\, T^{\text{matter}}_{\mu\nu} \,. \tag{4.15}$$

Here we have split the brane equation of motion into two equations, where the first one (4.14) is given in terms of the extrinsic curvature, while the second one (4.15) is given in terms of the Ricci tensor on the brane Σ. This splitting can be motivated by noting that the five- and four-dimensional Newton constants are related by [35]

$$G_5 = \frac{1}{2} G_4\, L \,, \tag{4.16}$$

as can be seen from (4.2). Then, the terms in (4.15) stem from those terms in the action (4.2) that are proportional to G_4^{-1} and hence are intrinsically four-dimensional, whereas the terms in (4.14) come from terms in (4.2) that are multiplied by powers of G_5 and L in such a way that these factors do not combine into powers of G_4 only. Thus, W only contributes to (4.14), and there is no induced cosmological constant on the brane (cf. equation (4.29)) [37]. Note that in order for the two equations (4.14) and (4.15) to be consistent with each other, we have to supplement the action (4.2) with terms describing the non-gravitational degrees of freedom on the brane. This results in the presence of an energy-momentum tensor $T^{\text{matter}}_{\mu\nu}$ in (4.15) that is homogeneous and isotropic, *i.e.* $T^{\text{matter}}_{\mu\nu} = \text{diag}(-\rho_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}}, p_{\text{eff}})$, and that is conserved,

$$\frac{d\rho_{\text{eff}}}{d\tau} = -3H\,\left(\rho_{\text{eff}} + p_{\text{eff}}\right) \,, \tag{4.17}$$

where

$$H = \frac{\dot{a}}{a} \tag{4.18}$$

denotes the Hubble parameter[2].

Tracing (4.14) yields

$$K_{\mu\nu} = -\frac{W}{3L}\,\gamma_{\mu\nu} \,, \tag{4.19}$$

which we now evaluate. We first compute the ij-component of $K_{\mu\nu}$. Using the definition (4.3) as well as (4.13) we obtain

$$K_{ij} = \frac{1}{2} n^a \partial_a \gamma_{ij} = a^{-1} n^a\, \gamma_{ij} = \pm\frac{\sqrt{A + A\,B\dot{a}^2}}{\sqrt{A\,B}\,a}\,\gamma_{ij} \,, \tag{4.20}$$

and hence we get from (4.19),

$$\pm\frac{\sqrt{A + A\,B\dot{a}^2}}{\sqrt{A\,B}\,a} = -\frac{W}{3L} \,. \tag{4.21}$$

[2] H should not be confused with the H_A defined in (3.33).

Since the right hand side is negative, we take the minus sign in (4.13). Then, squaring (4.21) results in

$$H^2 = \left(\frac{W}{3L}\right)^2 - \frac{1}{B\,a^2}\,. \tag{4.22}$$

This is the Friedmann equation describing the dynamics of the scale factor $a(\tau)$.

Next, we compute the $\tau\tau$-component of (4.19) using the method of [120, 127], which we now review. We express $K_{\tau\tau}$ as

$$K_{\tau\tau} = K_{MN}\, v^M\, v^N = -n_N\, \mathcal{A}^N\,, \tag{4.23}$$

where we used (4.12) and where $\mathcal{A}^N = v^M\, \nabla_M\, v^N$. Since $v^N\, v_N = -1$ we have $v_N\, \mathcal{A}^N = 0$, and hence $\mathcal{A}^N$ is proportional to the normal n^N, *i.e.* $\mathcal{A}^N = \hat{\mathcal{A}}\, n^N$. To compute $\hat{\mathcal{A}}$ we use the fact that the black hole metric (3.34) has a timelike Killing vector $l = l^M\, \partial_M = \partial_t$. We compute $\partial_\tau \left(l_M\, v^M\right) = v^N\, \nabla_N \left(l_M\, v^M\right) = l_N\, \mathcal{A}^N = l_N\, n^N\, \hat{\mathcal{A}}$, where we used the Killing equation $\nabla_M\, l_N = -\nabla_N\, l_M$ once. Hence [120, 127]

$$K_{\tau\tau} = -\frac{\partial_\tau \left(l^M\, v_M\right)}{l^N\, n_N} = -\frac{\partial_\tau v_t}{n_t}\,. \tag{4.24}$$

Thus, the $\tau\tau$-component of (4.19) yields

$$\partial_\tau v_t = \frac{W}{3L}\, n_t\,. \tag{4.25}$$

Using (4.7), (4.10) and (4.22), we find that (4.25) results in

$$\dot{H} + H^2 = \left(\frac{W}{3L}\right)^2 \left(1 + \frac{a}{W}\,\frac{\partial W}{da}\right) + \frac{1}{2B^2\, a}\,\frac{\partial B}{\partial a}\,, \tag{4.26}$$

where $\dot{H} = dH/d\tau$. This equation is precisely the τ-derivative of (4.22).

Finally, we compute $K_{\tau i} = K_{Mi}\, v^M$ using (4.3) and find $K_{\tau i} = 0$, which is consistent with (4.19). Thus, we conclude that the equations (4.19) consistently reduce to the Friedmann equation (4.22) and its τ-derivative.

The Friedmann equation (4.22) can be rewritten in terms of the black hole data M and $Q_A\, \phi^A$ as follows. Using (4.5) as well as (4.45) we obtain

$$\mathrm{e}^{2U} = \frac{W}{3\, a^2 - q_A\, X^A}\, a^2\,. \tag{4.27}$$

Using (4.27), $Q_A\, \phi^A$ given in (3.51) becomes

$$Q_A\, \phi^A = \frac{2}{3w_5}\left(-3k\,\frac{q_A X^A}{W} + k\,\frac{\left(q_A X^A\right)\,\left(q_B X^B\right)}{W a^2} + \mu\,\frac{q_A X^A}{a^2} + k\, q_A h^A\right)\,. \tag{4.28}$$

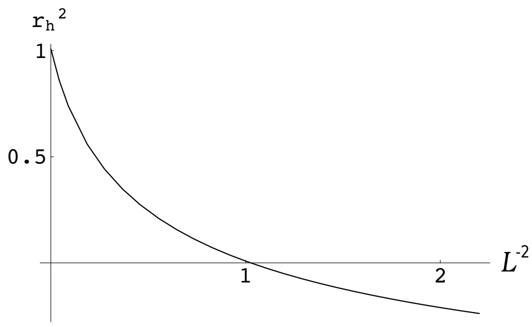


Figure 4.2: The horizon radius r_h is displayed as a function of the square of the inverse AdS radius L^{-2} for the solution of section 3.4.3 with $\mu = q = 1$. For $L^{-1} > 1$, r_h becomes negative. Thus, a horizon exists for $L^{-1} < 1$.

Inserting (4.6), (3.30) and (4.27) into (4.22) and using (4.28) yields

$$H^2 = -k\,\frac{W\,\tilde{W}}{3\,a^2} + \frac{w_5 W}{3\,a^4} M - \frac{w_5\,W}{6\,a^4}\,Q_A\,\phi^A\ . \tag{4.29}$$

It is instructive to check whether $H^2 \geq 0$ along the motion of the brane in the AdS black hole background. For this purpose, let us consider the black hole background of section 3.4.3 for concreteness. We set $\mu = q = 1$ (in appropriate units). The existence of a horizon then requires taking $L^{-1} < 1$, as shown in figure 4.2. Picking the value $L^{-1} = 0.30$ we find that the horizon is at $r_h = 0.64$ and that H^2 vanishes at $r = 1$. In figure 4.3, H^2 is displayed as a function of r using (4.29). We find that $H^2 \geq 0$ in the outside region between the horizon and the turning point at $r = 1$. The brane thus expands until it reaches its maximal radius at $r = 1$ after which it recontracts and falls through the horizon.

4.2.2 A Cardy-Verlinde-type formula for charged black holes

Now, we show that for any model (3.5) the entropy of a charged spherical black hole (3.34) can be written as a Cardy-Verlinde-type formula [36] which has already been discussed in [38, 40, 41, 116, 117] for various black holes in the context of the STU model.

In the coordinates (3.34), the exterior horizon r_h of the black hole is located at the largest real positive root of

$$f(r_h) = 0\ . \tag{4.30}$$

Remember that we assume $\mathrm{e}^{-4U(r_h)} \neq 0$ (cf. section 3.3). The associated Hawking

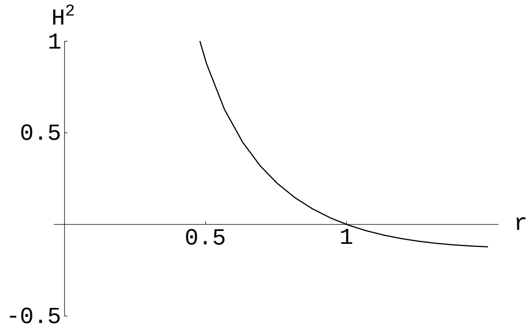


Figure 4.3: The square of the Hubble constant H^2 is displayed as a function of the radial coordinate r for the solution of section 3.4.3 with $\mu = q = 1$. At the point where H vanishes, the expansion of the brane universe breaks down. After that, the universe recontracts. Thus, the physical range is $r \leq 1$, and the horizon is at $r_h = 0.64$.

temperature T_H is given by [106]

$$T_H = \frac{1}{4\pi}\, f'(r_h)\, \mathrm{e}^{-3U(r_h)} \ . \tag{4.31}$$

The Bekenstein-Hawking entropy of the black hole is given by a quarter of the area $V_h = a_h^3 vol(S^3)$ of the event horizon which is a three-sphere with radius $a_h = r_h\, \mathrm{e}^{U(r_h)}$,

$$S = \frac{V_h}{4\, G_5} = \frac{4\pi}{3w_5}\, a_h^3 \ , \tag{4.32}$$

with w_5 given by (3.52). Hence

$$T_H\, S = \frac{1}{3w_5}\, f'(r_h)\, r_h^3 \ , \tag{4.33}$$

which we now compute. Using (3.20) we obtain

$$f'(r)\, r^3 = 2\mu + 2\,\frac{2}{L^2}\, r^4\, \mathrm{e}^{6U} \left(1 - \mathrm{e}^{-2U}\, \frac{q_A X^A}{r^2} \right) \ . \tag{4.34}$$

At the horizon, it follows from (4.30) that

$$\mu - k\, r_h^2 = \frac{1}{L^2}\, r_h^4\, \mathrm{e}^{6U(r_h)} \ , \tag{4.35}$$

and hence

$$T_H\, S = \frac{2}{3w_5} \left[2\mu - k\, r_h^2 + \left(k - \frac{\mu}{r_h^2} \right) \mathrm{e}^{-2U(r_h)}\, q_A X_h^A \right] \ . \tag{4.36}$$

Next, let us consider the Smarr-type combination

$$\frac{4}{3}M - T_H\, S - Q_A\, \phi_h^A \ , \tag{4.37}$$

with $Q_A\, \phi_h^A$ given by (3.51) and evaluated at the horizon. This combination is the gravitational counterpart of the Casimir energy $E_c/3$ on the brane. E_c is defined as the violation of the thermodynamic Euler relation [36], as we have learned in section 2.6.2. The combination (4.37) can also be motivated by exhibiting its relation to the Smarr formula, as follows. In the absence of charges, the area V_h of the event horizon is determined in terms of the mass parameter μ and the AdS radius L using (4.35). We can view this as a relation $\mu = \mu(V_h, L)$. Under the simultaneous rescaling $r_h \to \lambda\, r_h$ and $L \to \lambda\, L$ we have $V_h \to \lambda^3\, V_h$ as well as $\lambda^2\, \mu = \mu(\lambda^3\, V_h, \lambda\, L)$. Differentiating with respect to λ, setting $\lambda = 1$ and multiplying with w_5^{-1} results in

$$2M = 3T_H\, S + L\, \frac{\partial M}{\partial L} \ , \tag{4.38}$$

where we used the first law of black hole thermodynamics, $dM = T_H\, dS$. Using (4.35), we compute

$$L\, \frac{\partial M}{\partial L} = -2M + \frac{2k}{w_5}\, r_h^2 \ . \tag{4.39}$$

Inserting (4.39) into (4.38) we obtain

$$\frac{4}{3}M - T_H\, S = \frac{2k}{3w_5}\, r_h^2 \ . \tag{4.40}$$

This is the result for the Smarr-type combination (4.37) for uncharged black holes. In the ungauged case ($L^{-1} = \mathfrak{g} = 0$)[3], we have $k\, r_h^2 = \mu$ and (4.40) yields the Smarr formula $\frac{2}{3}M = T_H\, S$ [128].

In analogy to [36], we denote the Smarr-type combination (4.37) by $\tilde{E}_c/3$. Using (3.53), (4.36) and (3.51) we obtain

$$\tilde{E}_c = \frac{2\, k}{w_5} \left(r_h^2 + \frac{1}{3}\, h^A q_A \right) \ . \tag{4.41}$$

On the other hand, contracting (3.40) with h^A gives

$$r^2 \mathrm{e}^{2U}\, h^A X_A = r^2 + \frac{1}{3} h^A q_A \ , \tag{4.42}$$

[3] In chapter 3, we explained that the inverse AdS radius L^{-1} is identified with the gauge coupling constant $\mathfrak{g}$ of gauged supergravity. In the ungauged case $L^{-1} = \mathfrak{g} = 0$, the vacuum solution is Minkowski space, instead of AdS space in the gauged case, where $L^{-1} = \mathfrak{g}$ is finite.

and hence

$$\tilde{E}_c = \frac{2\,k}{w_5}\,\tilde{W}_h\,a_h^2\;, \tag{4.43}$$

where $\tilde{W}_h$ denotes the dual superpotential (3.42) evaluated at the horizon. The quantity $\tilde{E}_c$ is thus non-vanishing for a horizon of spherical topology. Observe that $a_h^2 \propto S^{2/3}$ (cf. equation (4.1)). In the ungauged case ($L^{-1} = \mathfrak{g} = 0$), we have $k\,r_h^2 = \mu$ as well as $\tilde{E}_c = 2M - Q_A\,\phi_h^A$, and (4.37) yields the Smarr formula $\frac{2}{3}M = T_H\,S + \frac{2}{3}Q_A\,\phi_h^A$ [107, 128].

In the gauged case ($L^{-1} = \mathfrak{g} \neq 0$), the combination $2M - Q_A\,\phi_h^A - \tilde{E}_c$ is no longer vanishing. We find

$$2M - Q_A\,\phi_h^A - \tilde{E}_c = \frac{2}{3w_5}\,(\mu - k\,r_h^2)\left(3 - \mathrm{e}^{-2U(r_h)}\,\frac{q_A X_h^A}{r_h^2}\right)\;, \tag{4.44}$$

where we used (3.53), (3.51) and (4.41). With the help of (3.23) and (3.11) we have

$$\frac{q_A X^A}{r^2} = 3\mathrm{e}^{2U} - W\;, \tag{4.45}$$

so that

$$2M - Q_A\,\phi_h^A - \tilde{E}_c = \frac{2}{3w_5}\,(\mu - k\,r_h^2)\,\mathrm{e}^{-2U(r_h)}W_h\;, \tag{4.46}$$

where W_h denotes (3.11) evaluated at the horizon. Using (4.35) this can be written as

$$2M - Q_A\,\phi_h^A - \tilde{E}_c = \frac{2}{3L^2 w_5}\,W_h\,a_h^4\;, \tag{4.47}$$

which is positive. Observe that $a_h^4 \propto S^{4/3}$ (cf. equation (4.1)). We denote this combination by $2\,\tilde{E}_e$, and we note that $2M = 2\,\tilde{E}_e + \tilde{E}_c + Q_A\,\phi_h^A$.

It follows that we can express the square of the entropy (4.32) as

$$k\,W_h\,\tilde{W}_h\,S^2 = \frac{8\pi^2 L^2}{3}\,\tilde{E}_e\,\tilde{E}_c = \frac{4\pi^2 L^2}{3}\,\tilde{E}_c\left(2M - Q_A\,\phi_h^A - \tilde{E}_c\right)\;. \tag{4.48}$$

This is a Cardy-Verlinde-type formula for charged AdS black holes, here expressed in terms of gravitational quantities. It makes use of both W and $\tilde{W}$ evaluated at the horizon. Using (3.13), we note the relation

$$W\,\tilde{W} = 3\left(1 + \frac{1}{2}\,\mathcal{G}^{ij}\,\partial_i W\,\partial_j \tilde{W}\right)\;, \tag{4.49}$$

which shows that in general $W_h\,\tilde{W}_h \neq 3$.

Summarizing, we find that three combinations are naturally expressed in terms of the superpotential quantities W and $\tilde{W}$, namely (4.43), (4.47) and (4.48).

AdS	CFT
M	$E = M\,L/a$
T_H	$T = T_H\,L/a$
$Q_A\,\phi_h^A$	$Q_A\,\hat{\Phi}^A = Q_A\,\phi_h^A\,L/a$
$Q_A\,\phi^A$	$Q_A\,\Phi^A = Q_A\,\phi^A\,L/a$
S	S

Table 4.1: Relation between black hole and field theory data.

4.3 The brane perspective

As we have learned in chapter 2, the AdS/CFT correspondence states that the bulk theory provides a dual description of the CFT on the brane, at least if it resides at the true boundary of the AdS space. According to this, the results obtained in the preceding section can be transferred to the CFT side which reveals new insights about the properties of the CFT on the brane. In the first part of this section, a Cardy-Verlinde-type formula for the brane field theory is presented, derived from the bulk counterpart (4.48). We also convert the FRW equation (4.29) and its time-derivative, which are expressed in bulk quantities, into the standard form in terms of four-dimensional brane quantities. The second and third part present the merging between the first FRW equation and the Cardy-Verlinde-type formula, and between the second FRW equation and the Casimir energy of the brane field theory.

4.3.1 Dual description on the brane

The mass M, the Hawking temperature T_H and the entropy S of the black hole are related to the energy E of the field theory on the brane, to its temperature T and to its entropy S [18]. The relation makes use of a conversion factor determined by the asymptotic behavior of $dt/d\tau = A^{-1/2}$ [37], which arises due to the different time-coordinates on the brane and in the bulk. For the line element (3.34) it is given by L/a. In the charged case, the black hole and field theory data are thus related as in table 4.1 [37,38,40], where $Q_A\,\phi_h^A$ denotes the horizon value of (3.51).

The spatial volume of the brane is given by

$$V = a^3\,vol\;, \tag{4.50}$$

with vol described in the text below (3.52). The energy density ρ of the radiation is

$$\rho = \frac{E}{V} \tag{4.51}$$

and its pressure satisfies $p = \frac{1}{3}\rho$. As already explained in section 2.6.1 the energy E of a CFT on a three-sphere is not a purely extensive quantity. It contains a subextensive part called the Casimir energy defined by [36]

$$E_c = 3\left(E + p\,V - T\,S - Q_A\,\hat{\Phi}^A\right) \,. \tag{4.52}$$

Here we have defined E_c in terms of $\hat{\Phi}^A$ rather than Φ^A. The Casimir energy (4.52) denotes the violation of the thermodynamic Euler relation. The Euler relation for a system based on the first law of thermodynamics $dE = T\,dS - p\,dV + \hat{\Phi}^A\,dQ_A$ states that if the energy $E(S, V, Q)$ is extensive, *i.e.* if it satisfies $E(\lambda\,S, \lambda\,V, \lambda\,Q) = \lambda\,E(S, V, Q)$, then the energy takes the form $E = T\,S - p\,V + Q_A\,\hat{\Phi}^A$. This relation is derived by differentiating once with respect to λ, then setting $\lambda = 1$ and subsequently using the first law of thermodynamics.

The gravitational counterpart of the quantity E_e and of the Casimir energy E_c is given in (4.47) and (4.43), respectively. Using table 4.1, we obtain

$$\begin{aligned} E_e =& \frac{L}{a}\tilde{E}_e = \frac{1}{4\pi}\left(\frac{2\,G_4}{vol}\right)^{1/3}\frac{W_h}{L^{2/3}}\frac{S^{4/3}}{a} \,, \\ E_c =& \frac{L}{a}\tilde{E}_c = \frac{3k}{2\pi}\left(\frac{vol}{2\,G_4}\right)^{1/3} L^{2/3}\,\tilde{W}_h\,\frac{S^{2/3}}{a} \,, \end{aligned} \tag{4.53}$$

where we employed (4.16) to rewrite (3.52) in terms of G_4. Observe that W_h and $\tilde{W}_h$ do not have a simple scaling behavior under $a \to \lambda^{1/3}\,a\,,\; S \to \lambda\,S, Q \to \lambda\,Q$. Observe that, if the charges are absent, the superpotentials reduce to $W_h = 3$ and $\tilde{W}_h = 1$. In this case, the energies (4.53) become identical to (2.53) and (2.54), if the radius of the three-sphere is set to $a = L$, that is, if the AdS time and the CFT time coincide.

Then, it follows that the entropy relation (4.48) can be written as

$$k\,W_h\,\tilde{W}_h\,S^2 = \frac{8\pi^2 a^2}{3}\,E_c\,E_e = \frac{4\pi^2 a^2}{3}\,E_c\left(2E - Q_A\,\hat{\Phi}^A - E_c\right) \tag{4.54}$$

in terms of field theory data on the brane. This is of the type of the Cardy-Verlinde formula [36] introduced in section 2.6.1. Observe that it reduces to (2.55), if the charges are absent and the radius of the three-sphere is set to $a = L$.

In [36], it was proposed that the entropy S of a CFT with an AdS-dual description satisfies the bound $S \leq S_B$, where $S_B = \frac{2}{3}\pi\,a\,E$ denotes the Bekenstein entropy [118]. To analyze whether this also holds for the expression (4.54) is not straightforward due to the presence of the factor $W_h\,\tilde{W}_h$, which satisfies the relation (4.49). For the specific STU models 3.4.3 and 3.4.4, however, it is possible to check that the bound holds. For these models, we have $3\,W\,\tilde{W} = 5 + 2H_1^{-1} + 2H_1 = 9 + 2q^2/(r^4\,H_1) > 9$, since $H_1 > 0$. It follows that $W_h\,\tilde{W}_h > 3$. From (3.64) and

(3.65) we see that the quantity $Q_A\,\hat{\Phi}^A$ is positive. Therefore, it follows that $S < \frac{2}{3}\pi\, a\sqrt{E_c\,(2E - E_c)}$, which has a maximum at $E = E_c$ for a given energy E [36], and hence the bound $S \leq S_B$ is satisfied for these two models.

Now, we write the Friedmann equation (4.29) in terms of field theory data (cf. table 4.1). Introducing the charge density

$$\rho_A = \frac{Q_A}{V} \tag{4.55}$$

we obtain

$$H^2 = -k\,\frac{W\,\tilde{W}}{3\,a^2} + \frac{8\pi\,G_4}{9}\,W\,\left(\rho - \frac{1}{2}\rho_A\,\Phi^A\right)\,, \tag{4.56}$$

where we used the relation (4.16) to express the Friedmann equation in terms of four-dimensional quantities. Observe that for any model (3.5), W, $\tilde{W}$ and Φ^A are generically complicated functions of a. Defining

$$\rho_{\text{eff}} = \frac{W}{3}\rho - \frac{W}{6}\,\rho_A\,\Phi^A + k\,\frac{(3 - W\,\tilde{W})}{8\pi\,G_4\,a^2}\,, \tag{4.57}$$

yields the Friedmann equation in the usual form,

$$H^2 = -\frac{k}{a^2} + \frac{8\pi\,G_4}{3}\rho_{\text{eff}}\,. \tag{4.58}$$

This is the form following from (4.15) with a suitably chosen $T^{\text{matter}}_{\mu\nu}$. Due to the presence of the charges, the effective energy density does not have a standard uniform scaling behavior with a constant *equation of state parameter* w, for which $\rho \propto a(\tau)^{-3(1+w)}$ [114]. This is different to, for instance, radiation scaling as $a(\tau)^{-4}$ or dust scaling as $a(\tau)^{-3}$. If we restrict ourselves to the STU model and set all the three remaining charges equal (Maxwell case of 3.4.2), the effective energy density on the brane reduces to that of *stiff matter* [38]. This term is used for matter whose equation of state parameter equals $w = 1$. In the uncharged case, we obtain $\rho_{\text{eff}} = \rho$ showing a scaling behavior as radiation, $\rho_{\text{eff}} \propto a(\tau)^{-4}$.

Next, we differentiate the Friedmann equation (4.56) with respect to proper time τ to obtain

$$\begin{aligned}\dot{H} = \frac{dH}{d\tau} =& k\,\frac{W\,\tilde{W}}{3\,a^2} - \frac{4\pi\,G_4 W}{3}\,(\rho + p) - \frac{k}{6\,a}\,\frac{d}{da}\left(W\tilde{W}\right)\\ &+ \frac{4\pi\,G_4\,a}{9}\left(\frac{dW}{da}\,\rho - \frac{1}{2}\,\frac{d}{da}\left(W\,\rho_A\,\Phi^A\right)\right)\,,\end{aligned} \tag{4.59}$$

where we used that

$$\frac{d\rho}{d\tau} = -3H\,\left(\rho + p\right)\,. \tag{4.60}$$

The equation for $\dot{H}$ can be written in the standard FRW form following from (4.15),

$$\dot{H} = \frac{k}{a^2} - 4\pi\, G_4\, (\rho_{\text{eff}} + p_{\text{eff}}) \ , \tag{4.61}$$

with ρ_{eff} given by (4.57) and with p_{eff} given by

$$\begin{aligned} p_{\text{eff}} =& \frac{W}{3} p + \frac{W}{6} \rho_A\, \Phi^A + \frac{k}{24\pi\, G_4\, a^2} \left(a\, \frac{d}{da} \left(W\tilde{W} \right) + W\tilde{W} - 3 \right) \\ & - \frac{a}{9} \left(\frac{dW}{da}\, \rho - \frac{1}{2} \frac{d}{da} \left(W\, \rho_A\, \Phi^A \right) \right) \ . \end{aligned} \tag{4.62}$$

We see that similar to the expression for the effective energy density, the expression for the effective pressure is very complicated due to the presence of different charges. In addition, p_{eff} and ρ_{eff} are related by an equation of state, $p_{\text{eff}} = w\rho_{\text{eff}}$, where w again turns out to be non-constant. This again shows that the matter content considered here is not of any standard form as, for instance, radiation with $w = 1/3$ or dust with $w = 0$. Restricting to the Maxwell case of the STU model with all three charges set equal, p_{eff} and ρ_{eff} satisfy the equation of state for stiff matter with $w = 1$. If all charges are set to zero, the effective pressure reduces to $p_{\text{eff}} = p$ which together with the corresponding effective energy density satisfies the equation of state for radiation with $w = 1/3$.

Observe that $T^{\text{matter}}_{\mu\nu}$ is, in general, not traceless,

$$\begin{aligned} T_\mu{}^{\mu\ \text{matter}} = -\rho_{\text{eff}} + 3p_{\text{eff}} =& \frac{2}{3}\, W\, \rho_A\, \Phi^A + k\, \frac{(W\tilde{W} - 3)}{4\pi\, G_4\, a^2} + \frac{k}{8\pi\, G_4\, a} \frac{d}{da} \left(W\tilde{W} \right) \\ & - \frac{a}{3} \left(\frac{dW}{da}\, \rho - \frac{1}{2} \frac{d}{da} \left(W\, \rho_A\, \Phi^A \right) \right) \ . \end{aligned} \tag{4.63}$$

Since ρ_{eff} is not one of the known standard matter forms, we now check that $\rho_{\text{eff}} > 0$ in the example considered before describing the motion of the brane in the black hole background of section 3.4.3 with the parameters μ, q set to the values $\mu = q = 1$. We find that $\rho_{\text{eff}} > 0$ for $r > 0$, as shown in figure 4.4. The behavior of the pressure p_{eff} for the same values of the parameters is displayed in figure 4.5. It is positive throughout. We also find that outside of the horizon, the trace of the energy-momentum tensor $a^4\, T_\mu{}^{\mu\ \text{matter}}$ only vanishes asymptotically. This is displayed in figure 4.6.

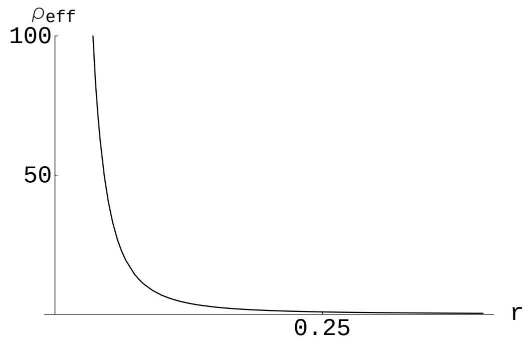


Figure 4.4: The effective energy density ρ_{eff} is displayed as a function of the radial coordinate r for the solution of section 3.4.3 with $\mu = q = 1$. ρ_{eff} is positive throughout the r interval.

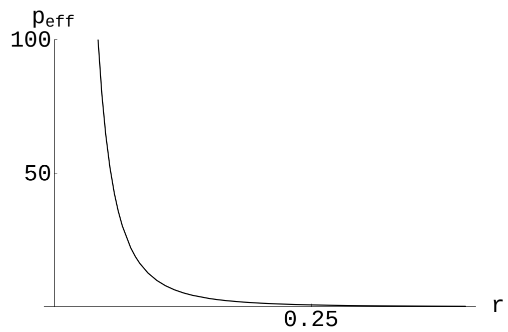


Figure 4.5: The effective pressure p_{eff} is displayed as a function of the radial coordinate r for the solution of section 3.4.3 with $\mu = q = 1$. p_{eff} is positive throughout the r interval.

4.3.2 Correspondence between the first FRW equation and the entropy

The Friedmann equation (4.56), when written as

$$k\,W\,\tilde{W}\,\left(\frac{9\,V^2}{4\,G_4^2\,W^2}\,H^2\right) = \frac{4\pi^2}{3}\,a^2\,\left(k\,\frac{3\,V\,\tilde{W}}{4\pi\,G_4\,a^2}\right)\,\left(2E - Q_A\,\Phi^A - k\frac{3\,V\,\tilde{W}}{4\pi\,G_4\,a^2}\right)\,, \tag{4.64}$$

has a structure that is similar to that of the Cardy-Verlinde-type formula (4.54). Moreover, when the brane crosses the event horizon, both equations coincide. Namely, at the horizon where $f(r_h) = 0$, (4.22) yields

$$H_h^2 = \left(\frac{W_h}{3L}\right)^2\,, \tag{4.65}$$

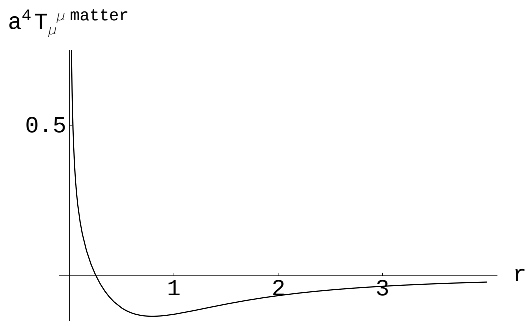


Figure 4.6: The rescaled trace of the energy-momentum tensor $a^4\, T_\mu^{\mu\ \text{matter}}$ is displayed as a function of the radial coordinate r for the solution of section 3.4.3 with $\mu = q = 1$. It vanishes only asymptotically.

which can be used to express the AdS radius L in terms of H_h and W_h [37]. Inserting this relation into the expression for the entropy (4.32) gives

$$S^2 = \frac{9\, V_h^2}{4\, G_4^2\, W_h^2}\, H_h^2\ , \tag{4.66}$$

where V_h denotes the volume (4.50) evaluated at the horizon, and where we used the relation (4.16). On the other hand, at the horizon the Casimir energy E_c can be written as

$$E_{c,h} = k\, \frac{3\, V_h}{4\pi\, G_4}\, \frac{\tilde{W}_h}{a_h^2} \tag{4.67}$$

by virtue of (4.66). Inserting the expressions (4.66) and (4.67) into (4.64) then yields the Cardy-Verlinde-type formula (4.54).

4.3.3 Correspondence between the second FRW equation and the Casimir energy

Now, we rewrite the second FRW equation (4.59) in order to exhibit its similarity with equation (4.52) for the Casimir energy E_c. Using (3.24) and (4.27) we obtain

$$\begin{aligned} \tilde{W} =& \mathrm{e}^{-2U} + \frac{q_A\, h^A}{3\, a^2}\ , \\ W\, \tilde{W} =& 3 - \frac{q_A\, X^A}{a^2} + \frac{W}{3}\, \frac{q_A\, h^A}{a^2}\ , \end{aligned} \tag{4.68}$$

while using (3.51) and (4.27) we get

$$\begin{aligned} \frac{d\left(\rho_A\, \Phi^A\right)}{da} = -\frac{6}{a}\, \rho_A\, \Phi^A + & \frac{1}{4\pi\, G_4\, a^5} \left(2k\, q_A\, h^A - 6k\, \frac{q_A\, X^A}{W} \right. \\ & \left. - \left(k - \frac{\mu}{a^2}\, \mathrm{e}^{2U}\right) \mathrm{e}^{-2U}\, a\, q_A \frac{dX^A}{da} \right) . \end{aligned} \tag{4.69}$$

Inserting (4.68) and (4.69) into (4.59) and rearranging the terms we find that the second FRW equation (4.59) can be written as

$$\begin{aligned} k\,\frac{3V\,\tilde{W}}{4\pi\,G_4\,a^2} =& 3\Bigg(E + p\,V - Q_A\,\Phi^A \\ & + \frac{3V}{4\pi\,G_4 W}\left(\frac{1}{9\,a}\left(k - \frac{\mu}{a^2}\,\mathrm{e}^{2U}\right)\mathrm{e}^{-4U}\,W\frac{dW}{da} + \dot{H}\right)\Bigg) \end{aligned} \tag{4.70}$$

by virtue of the relation $H_A\,dX^A/da = 0$, which holds due to (3.24) and the very special geometry relation (3.8).

Now, let us consider the Casimir energy E_c. Following [37], we first relate the brane temperature T to $\dot{H}$. Using equation (3.30), the Hawking temperature (4.31) can be written as

$$T_H = \frac{1}{12\pi}\left[\frac{df}{da}\,\mathrm{e}^{-4U}\,W\right]_h . \tag{4.71}$$

Then, taking the τ-derivative of (4.22) and using that at the horizon $f(r_h) = 0$ as well as (4.65), we obtain for the temperature on the brane,

$$T = T_H\,\frac{L}{a} = \frac{a_h}{2\pi\,a}\left[\frac{|H|}{W}\,a\,\frac{dW}{da} - \frac{\dot{H}}{|H|}\right]_h . \tag{4.72}$$

Inserting the expressions (4.66), (4.67) and (4.72) into the defining relation (4.52) yields the equation for the Casimir energy in the form

$$k\,\frac{3V_h\,\tilde{W}_h}{4\pi\,G_4\,a_h\,a} = 3\left(E + p\,V - Q_A\,\hat{\Phi}^A + \frac{3V_h\,a_h}{4\pi\,G_4 W_h\,a}\left[-\frac{H^2}{W}\,a\,\frac{dW}{da} + \dot{H}\right]_h\right) . \tag{4.73}$$

Comparing (4.70) with (4.73) shows that both equations have a similar structure. Moreover, they coincide at the event horizon of the black hole, since

$$\left[-\frac{H^2}{W}\,a\,\frac{dW}{da}\right]_h = \left[\frac{1}{9\,a}\left(k - \frac{\mu}{a^2}\,\mathrm{e}^{2U}\right)\mathrm{e}^{-4U}\,W\frac{dW}{da}\right]_h , \tag{4.74}$$

where we used (4.65) as well as $f(r_h) = 0$.

4.4 Comments about the inclusion of higher-curvature terms

It would be interesting to extend the analysis above to a modified five-dimensional supergravity action including higher derivative curvature terms[4]. These terms are

[4] See also [129–133] for related discussions.

naturally present in, for instance, the effective low-energy actions of superstring theories. In the context of the AdS/CFT correspondence, these are viewed as corrections in the large N expansion of the dual field theory. In general, it is very difficult to find non-trivial exact analytical solutions of the Einstein equations with higher derivative terms. An exact analytical solution is known for Einstein gravity with a negative cosmological constant supplemented by a *Gauss-Bonnet term* [134–136]. This combination consists of squares of the curvature scalar, the Ricci tensor and the Riemann tensor. The associated coupling is not fixed, but depends on a scalar field. The presence of higher curvature terms leads to further subextensive contributions that modify the form of the Cardy-Verlinde-type formula, the Casimir energy and the FRW equations. The Friedmann equation for a spherical brane moving in this black hole background has been derived in [137–139]. The bulk action is given by

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - 2\Lambda + \alpha \left(R^2 - 4R^{MN} R_{MN} + R^{MNPQ} R_{MNPQ} \right) \right] \ , \tag{4.75}$$

where $\Lambda = -6/L^2$, which corresponds to setting $W = 3$ in (3.12). In the context of string theory, α is proportional to the Regge slope parameter α'. The associated black hole solution with a spherical horizon has the line element [134–136][5]

$$ds_5^2 = -f dt^2 + f^{-1} dr^2 + r^2 \, d\Omega_3^2 \quad , \quad f = 1 + \frac{r^2}{4\alpha} \left(1 - \sqrt{1 + 8\alpha \left(\frac{\mu}{r^4} - \frac{1}{L^2} \right)} \right) \ , \tag{4.76}$$

where $d\Omega_3^2$ denotes the line element of a unit three-sphere. The horizon of the black hole is located at $f(r_h) = 0$, which yields

$$\mu = \frac{r_h^4}{L^2} + r_h^2 + 2\alpha \ . \tag{4.77}$$

The mass of the black hole is $M = \mu/w_5$. The Gauss-Bonnet corrected entropy reads [135, 140]

$$S = \frac{4\pi}{3w_5} \left(r_h^3 + 12\alpha \, r_h \right) \ , \tag{4.78}$$

and the Hawking temperature T_H is given by

$$T_H = \frac{1}{4\pi} f'(r_h) = \frac{1}{2\pi \, r_h} \left(1 + \frac{2}{L^2} \frac{r_h^4 - 2L^2\alpha}{r_h^2 + 4\alpha} \right) \ . \tag{4.79}$$

The relation (4.78) can be inverted to express the horizon radius r_h in terms of the entropy,

$$r_h(s) = \frac{-8\,\alpha + 2^{1/3} \left(s + \sqrt{s^2 + 256\,\alpha^3} \right)^{2/3}}{2^{2/3} \left(s + \sqrt{s^2 + 256\,\alpha^3} \right)^{1/3}} \ , \tag{4.80}$$

[5] We use the notation of [139].

where

$$s = \frac{3w_5}{4\pi}\, S\ . \tag{4.81}$$

Using (4.77) and (4.80) the mass M can be expressed as a power series in s. At quadratic order in α, this gives

$$M = \frac{1}{w_5}\left(\frac{1}{L^2}\, s^{4/3} + \left(1 - 16\frac{\alpha}{L^2}\right) s^{2/3} - 6\alpha \left(1 - 16\frac{\alpha}{L^2}\right) + 16\alpha^2\, s^{-2/3}\right)\ , \tag{4.82}$$

where the first term proportional to $s^{4/3}$ is the only extensive contribution. In addition, the Gauss-Bonnet term induces an infinite series of subextensive corrections expressed in powers of $\alpha\, s^{-2/3}$. Thus, we see that in contrast to the two-derivative case, it is not straightforward to read off the contribution of the Casimir energy to the total energy, because of the infinite series of subextensive contributions. The total energy seems to exhibit further subextensive corrections which might or might not be caused by the Casimir effect. Therefore, it is not clear whether the entropy of the field theory on the brane can still be written as a Cardy-Verlinde-type formula, and thus as a product of the Casimir energy and the extensive energy, in the presence of higher curvature terms. For this reason, we leave the problem of finding a generalized Cardy-Verlinde-type formula for a future project and restrict ourselves here to present the Gauss-Bonnet corrected Friedmann equation.

As it was the case at the two-derivative level, the bulk action (4.75) needs to be supplemented by both boundary terms [141] and counterterms. The latter are given by [139]

$$-\frac{1}{8\pi\, G_5}\int_\Sigma d^4x\sqrt{-\gamma}\left(c_1 + \frac{c_2}{2}\,\mathcal{R}\right)\ , \tag{4.83}$$

where

$$c_1 = \frac{1 + 8\frac{\alpha}{L^2} - \sqrt{1 - 8\frac{\alpha}{L^2}}}{2\sqrt{\alpha}\,\sqrt{1 - \sqrt{1 - 8\frac{\alpha}{L^2}}}}\ , \quad c_2 = \frac{\sqrt{\alpha}\left(3 - 8\frac{\alpha}{L^2} - 3\sqrt{1 - 8\frac{\alpha}{L^2}}\right)}{\left(1 - \sqrt{1 - 8\frac{\alpha}{L^2}}\right)^{3/2}}\ . \tag{4.84}$$

Inspection of (4.83) shows that the five- and four-dimensional Newton constants are related by

$$G_5 = c_2\, G_4\ . \tag{4.85}$$

In the limit $\alpha \to 0$, we recover both the counterterms in (4.2) and (4.16).

The equation of motion for the brane moving in the black hole background (4.76) is expressed in terms of both the extrinsic curvature tensor K_{MN} and the Riemann tensor on the brane [139]. As in the two-derivative case (see equation (4.14) and (4.15)), we separate the term proportional to the Einstein tensor on the brane from the other terms. We take the induced metric on the brane to have the form of a standard FRW metric (4.8) with scale factor $a(\tau) = r(\tau)$ and

Hubble parameter $H = \dot{r}/r$, where $\dot{r} = dr/d\tau$. The resulting Friedmann equation takes the form [137, 138]

$$\left(H^2 + \frac{f}{r^2}\right)\left(3 + 8\alpha\left(H^2 + \frac{1}{r^2}\right) + 4\alpha\frac{(1-f)}{r^2}\right)^2 = c_1^2 \,. \tag{4.86}$$

It can be checked that there is no induced cosmological constant on the brane. Indeed, dropping all the terms that involve powers of μ/r^4 and of $H^2 + 1/r^2$ in (4.86) yields a perfect cancellation of all the remaining terms.

At quadratic order in α, (4.86) yields

$$H^2 = -\frac{1}{r^2} + \left(1 - 4\frac{\alpha}{L^2} + 8\frac{\alpha^2}{L^4}\right)\frac{\mu}{r^4} - 2\alpha\left(1 - 20\frac{\alpha}{L^2}\right)\frac{\mu^2}{r^8} + 8\alpha^2\,\frac{\mu^3}{r^{12}} \,. \tag{4.87}$$

The energy E on the brane is related to the mass M by a conversion factor which is determined by the asymptotic behavior of $dt/d\tau = f^{-1/2}$ [37]. For the line element (4.76) this yields the conversion factor $L/r(1-\alpha/L^2 - \frac{5}{2}\alpha^2/L^4)$ at quadratic order in α. Using this as well as (4.85), the Friedmann equation (4.87) can be expressed in terms of the energy density ρ (4.51) on the brane,

$$H^2 = -\frac{1}{r^2} + \frac{8\pi\, G_4}{3}\,\rho - 2\alpha\left(1 - 16\frac{\alpha}{L^2}\right)\left(\frac{8\pi\, G_4}{3}\,\rho\right)^2 + 8\alpha^2\left(\frac{8\pi\, G_4}{3}\,\rho\right)^3 \,. \tag{4.88}$$

Observe that the coefficient of the term linear in ρ does not receive α-corrections. It would be interesting to show that the equation for the entropy on the brane, expressed as a Cardy-Verlinde formula, and the equation for the Casimir energy have a structure that is similar to (4.88) and its τ-derivative.

4.5 Summary and discussion

In this chapter, a four-dimensional brane universe moving in the background of a five-dimensional AdS black hole of $\mathcal{N} = 2$ supergravity was investigated in the light of the AdS/CFT correspondence. In contrast to the original AdS/CFT prescription, the AdS space is cut off by a 3-brane which here accommodates the dual field theory. Since this introduces a UV cut-off in the field theory, the conformal symmetry is broken yielding an exotic type of matter on the brane. It was shown that the induced metric on the brane can take the standard FRW form, where the time-coordinate differs from the AdS time by a factor of $\lim_{a\to\infty} dt/d\tau = L/a$, and the radius of the spatial S^3 is given by the cosmic scale factor $a(\tau)$. The motion in the five-dimensional background turned out to be governed by the FRW equations for a closed universe whose matter content is represented by the exotic field theory on the brane. The evolution of the brane universe can be imagined as follows. Starting out in the center of the black hole, the brane first expands until it reaches a maximal radius after which it recontracts and falls through the horizon. The effective energy density took a complicated form, expressed in $\mathcal{N} = 2$ supergravity quantities, which is therefore not of the standard form. After reducing the number of electric charges to three and setting all equal (Maxwell case), the effective energy density became that of stiff matter, whereas in the case where all charges vanish (Schwarzschild case) the effective energy density reduced to that of radiation.

Due to the differing time-coordinates, the energy and temperature in the bulk and on the brane are related by a conversion factor L/a which makes these quantities dependent on the size of the universe. The entropy, however, is independent of such a conversion factor and thus remains constant during the cosmological evolution. In addition, it respects the Bekenstein bound which was conjectured to hold for CFTs with a gravity dual. The energy of the brane field theory exhibits two different contributions. The one proportional to the superpotential W is called the extensive energy in analogy to the uncharged case. Similarly, the contribution proportional to $\tilde{W}$ is called the Casimir energy. Making use of this splitting, the entropy was shown to take a form that is similar to the Cardy-Verlinde formula for CFTs. The departure compared to the original Cardy-Verlinde formula [36] consists in the presence of the non-constant superpotentials W and $\tilde{W}$.

However, the main result of this chapter is the miraculous merging of the first FRW equation and the Cardy-Verlinde-type formula as well as that of the second FRW equation and the equation for the Casimir energy of the field theory at the horizon of the black hole. How this arises can be seen as follows. First, the right hand side of the first FRW equation written in the form (4.64) is already similar to the right hand side of the Cardy-Verlinde formula. The left hand side is actually proportional to the Hubble entropy $S_H = (VH)/(2G_4)$ [142] for a closed universe which was defined in the context of the pre-Big-Bang scenario [143]. Here $V = V_H n_H$ and V_H denotes the volume of a causally connected Hubble region and n_H is the number of Hubble regions in the universe. The Hubble entropy thus establishes the connection between entropy and Hubble parameter. At the horizon of the five-dimensional black hole, where $V = V_h$, with V_h being the area of the black hole horizon, the Hubble entropy of the brane universe becomes equal to the Bekenstein-Hawking entropy of the five-dimensional black hole which is, according to the AdS/CFT correspondence, simultaneously the entropy of the brane field theory. Second, the connection between the equation for the Casimir energy (4.52) and the second FRW equation is essentially established through the Hawking temperature which can be expressed in terms of the Hubble parameter and its time derivative. It was suggested in [36] that this merging exists independently of the kind of matter contained in the brane universe. Our analysis further supports this idea.

In the last section, an outlook was given on how to possibly extend the analysis of this chapter to a higher-derivative curvature bulk gravity. For this purpose, a Gauss-Bonnet term consisting of curvature squares was added to the action. For simplicity, all charges were set to zero and the first FRW equation expressed in brane quantities was computed. However, a higher-derivative curvature generalization of the Cardy-Verlinde-type formula could not be found, since the total energy this time contained an infinite series of subextensive corrections which could not be simply grouped in an extensive and a Casimir part. Therefore, it is not clear whether the entropy can still be written at all as being proportional to the product of a Casimir energy part and an extensive energy part. For this reason, we leave the problem of finding a generalized Cardy-Verlinde-type formula for a field theory dual to Gauss-Bonnet gravity for future investigations.

Chapter 5

Fluid dynamics on the three-sphere from gravity

In this chapter, we deal with another application of the AdS/CFT correspondence which is the *fluid/gravity correspondence*. This correspondence is used to compute subextensive corrections, which are proportional to the shear tensor, to the energy-momentum tensor of fluids on three-spheres. The dual configurations we consider are charged black hole solutions of $\mathcal{N} = 2$ gauged supergravity theories in five dimensions.

The chapter is organized as follows. In section 5.2, we explain the fluid/gravity correspondence and how it emerges from the AdS/CFT correspondence. Section 5.3 reviews relevant background material about relativistic fluid dynamics, and an a-priori argument is presented how the ratio η/s might be corrected due to the finite size and the curvature of the three-sphere accommodating the fluid. In section 5.4, the fluid/gravity correspondence is used to construct different five-dimensional deformed AdS black hole solutions which are dual to incompressible viscous conformal fluids on the three-sphere. These are the deformed AdS-Schwarzschild black hole and three special cases of the deformed AdS-STU black hole of $\mathcal{N} = 2$ supergravity, namely with three equal charges (Maxwell), with two equal charges and with one non-zero charge. For all these black hole solutions, the energy-momentum tensor of the dual fluid is computed and corrections to η/s are obtained. Section 5.6 contains a summary and conclusions. For the sake of comparison with the deformed solutions in section 5.4, various known rotating solutions of the STU model are summarized in appendix B.1, B.2 and B.3. Appendix C contains an example calculation of the boundary energy-momentum tensor for one of the black hole solutions of the STU model.

This chapter is based on work which was done by the author of this thesis in collaboration with Gabriel Lopes Cardoso and Gianguido Dall'Agata, and which was published in [2].

5.1 Introduction

The fluid/gravity correspondence [144–146][1] arises by taking a specific limit of the AdS/CFT correspondence in which the dynamics of the boundary conformal field theory simplifies to effective classical conformal fluid dynamics[2]. This *hydrodynamic limit* consists in focussing on near-equilibrium dynamics and restricting to long-wavelength fluctuations. On the bulk side, this requires to consider dynamical black hole solutions which must be regular in order to meet the requirements for the fluid dynamical stress tensor. Thus, the fluid/gravity correspondence constitutes a concrete relation between the physics of fluids and that of gravity.

Exploring the fluid/gravity correspondence is of important significance for theoretical as well as experimental physics. First, fluid dynamics provides many interesting and yet unmanageable long-term challenges as for instance the search for globally regular solutions to the Navier-Stokes equations for non-relativistic incompressible viscous fluids, or a detailed understanding of turbulence in fluid dynamical evolution. A holographic mapping of the fluid dynamical system to classical gravitational dynamics opens a new perspective on these issues [44–46].

Second, the fluid/gravity correspondence provides a new way to understand the physics of black holes, namely in terms of fluid dynamics. The idea of modeling black holes by fluids living on a lower-dimensional brane was developed even before in the works on the *membrane paradigm* [147]. In this description any black hole has a fictitious fluid living on its horizon, whose dynamics constitutes an analogy to qualitatively understand the physics of the black hole. In contrast to this, the fluid/gravity correspondence is a true duality between fluid dynamics and gravitational dynamics. It provides an algorithm to systematically construct regular black hole solutions whenever a solution to the fluid equations of motion is given. Thus every fluid flow in the boundary field theory is mapped to a black hole solution in the bulk with a regular event horizon. This enables us to understand the phase structure and the stability of the black hole solutions in terms of the fluid model.

Third, the fluid/gravity correspondence makes contact with real world physics. In particular, it proves to be useful in describing the dynamics of the QGP which is a state of matter consisting of quarks and gluons and which is believed to have existed shortly ($\approx 10^{-33}$ s) after the Big Bang. Nowadays, it can be produced at heavy-ion colliders and has been studied, for instance, at the Relativistic Heavy Ion Collider (RHIC) in Brookhaven. There, QGP was created at a temperature between 200 and 300 MeV which is just above the confinement-deconfinement temperature $T_c = 170$ MeV. However, although the quarks and gluons are deconfined, experimental data reveals that the QGP is still strongly coupled. Furthermore,

[1] For an introduction on this topic see [42, 43].

[2] The hydrodynamic limit is a feature of quantum field theories in general, not only of conformal field theories.

the data suggests that it behaves like an almost perfect fluid. Soon, the QGP will be more intensely studied at the Large Hadron Collider (LHC) at CERN at a temperature of about $5T_c$ [47].

The hydrodynamic regime is characterized by a set of transport coefficients which can be determined either by measurements or by computations on the basis of a microscopic theory. For a strongly-coupled fluid these coefficients cannot be computed using conventional techniques such as perturbation theory or Lattice QCD. However, the AdS/CFT correspondence provides a theoretical framework to understand the qualitative features of the hydrodynamics seen in the QGP by providing information about the hydrodynamics of strongly-coupled superconformal theories. In fact, QCD and large N superconformal field theories are qualitatively different, since QCD is an $SU(3)$ gauge theory, not supersymmetric, not conformal and has confinement. However, it is expected that these differences become less crucial for high enough temperatures $T \gtrsim T_c$, such that large N superconformal fluids can serve as toy-models for the QGP. An example for a qualitative feature which seems to be universal for all strongly-coupled fluids is the behavior of the shear viscosity. The shear viscosity is defined as a measure for the dissipation of momentum in a fluid transverse to the velocity flow. In [48], it was conjectured that for conformal fluids in the limit of an infinite number of colors N and infinite 't Hooft coupling λ, and which are dual to black branes, the ratio of shear viscosity to entropy density is always given by [48]

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}\,. \tag{5.1}$$

where $\hbar/(4\pi k_B) \approx 0.06 \times 10^{-11}\,\mathrm{K\ s}$. Furthermore, it was argued that (5.1) constitutes a lower bound on the viscosity of fluids in general implying that a fluid cannot be arbitrarily close to being a perfect fluid. Thus, for any fluid holds

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}\,. \tag{5.2}$$

This is far below the value for any laboratory liquid. In units where $\hbar = k_B = 1$ (5.1) becomes $\eta/s = 1/(4\pi) \approx 0.08$, whereas at a temperature $T = 20°\mathrm{C}$ water has $\eta/s \approx 30.01$ [148]. In figure 5.1 [48], the ratio η/s is depicted in units of $\hbar/(4\pi)$, where $k_B = 1$, for three different real liquids at different pressures, helium at 0.1 MPa, nitrogen at 10 MPa and water at 100 MPa. The ratio η/s is always substantially larger than the value (5.1) which is represented by the red horizontal line. First calculations based on RHIC data show that the lowest value for η/s for the QGP also respects (5.2) and is with $\eta/s \approx 0.1$, in units where $\hbar = k_B = 1$, quite close to $1/(4\pi) \approx 0.08$ [10,49]. The bound (5.2) was probed and proved true for different charged and uncharged conformal fluids [144–146, 149–159].

The bound (5.2) was derived on the basis of certain conditions, namely that λ and N are infinite, the fluctuations of the hydrodynamic variables as well as

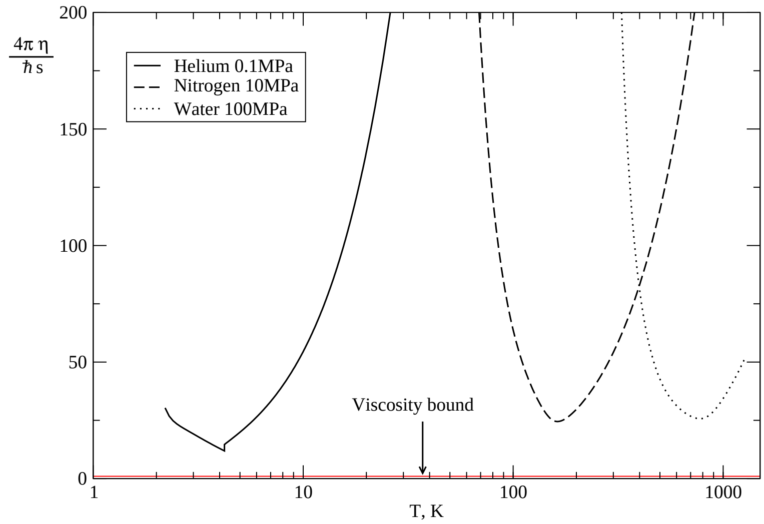


Figure 5.1: The ratio $4\pi\eta/(\hbar s)$ versus the temperature for helium, nitrogen and water, each at a different pressure, is shown. The ratio is always larger than its value in theories with gravity duals represented by the red horizontal line. (figure taken from [48])

the background curvature are of long wavelength and the volume accommodating the fluid is of infinite size. Relaxing these conditions might lead to deviations from the bound (5.2). For instance, the first correction in the case of finite λ was computed in [50,51]. This correction is positive and thus in accordance with (5.2). Finite N corrections at infinite λ were first considered in [160]. Those came out negative indicating that away from the $N \to \infty$ limit, (5.2) can be violated. In addition to that, including background curvature and finite-size effects might also lead to deviations from (5.2). For instance, curvature effects on η/s for fluids on hyperbolic spaces were discussed in [52].

The intention of the research presented in this chapter is to investigate the ratio η/s for the $\mathcal{N} = 4$ super Yang-Mills plasma[3] at infinite N and infinite λ, and including background curvature as well as finite-size effects by assuming the fluid to propagate on a three-sphere. We consider both effects, finite size and curvature, simultaneously, since our method is not able to distinct between the two effects. The three-sphere constitutes the asymptotic boundary of a spherical AdS black hole. For that purpose, we use the fluid/gravity correspondence as a tool to find new solutions to the fluid-dynamical equations on the three-sphere by constructing dual black hole solutions with spherical horizons.

[3]With $\mathcal{N} = 4$ super Yang-Mills plasma, we mean the hydrodynamic effective description of $\mathcal{N} = 4$ super Yang-Mills theory.

5.2 The nature of the fluid/gravity correspondence

The conformal fluid/gravity correspondence relates the hydrodynamic regime of strongly-coupled four-dimensional conformal field theories to regular black brane solutions in asymptotically AdS_5 backgrounds. In the hydrodynamic regime, any quantum field theory can be effectively described by fluid dynamics. The hydrodynamic regime is characterized by certain conditions. The system under study is assumed to be in local thermal equilibrium at each point in space, even though the energy and the charge density may vary over large distances. This means that the length scales on which the thermodynamic variables and the curvature of the manifold vary are large compared to the equilibration length scale, or in other words the mean free path l_{mfp}, of the fluid.

On the gravity side, these conditions are met whenever the horizon r_h of the dual black hole is large compared to the AdS radius L [161]. This includes for instance all non-extremal black holes whose temperature is large compared to unity. It is then possible to expand the formulae of black hole mechanics in a power series in $L/r_h \sim l_{\text{mfp}}$. The leading order term of this expansion is mapped under the AdS/CFT correspondence to the results of fluid dynamics. At subleading orders, deviations from the predictions of classical fluid dynamics can appear. Thus, black hole solutions in AdS provide exact near-equilibrium solutions to the equations of fluid dynamics to all orders in l_{mfp}. In addition, the study of higher-order corrections of these solutions away from the limit $l_{\text{mfp}} \ll 1$ might yield useful information about the nature of the fluid dynamical approximation of quantum field theories.

In practice, the expansion in the parameter L/r_h is implemented as an expansion in field theory derivatives of the fluid velocity and thermodynamic variables. The energy-momentum tensor of the fluid is then likewise expanded in powers of derivatives. In [144, 161–163] a formalism was developed to systematically construct AdS black hole solutions in a derivative expansion. According to the dictionary of the AdS/CFT correspondence, those black hole solutions are then mapped to the derivative expansion of the fluid's energy momentum tensor. This simultaneously determines all possible transport coefficients at each order in the gradient expansion which can be simply read off from the expansion of the energy-momentum tensor. This *black hole approach* to the hydrodynamics of strongly-coupled field theories was also generalized to charged black holes [156, 157] and to black holes in arbitrary dimensions [164, 165]. In all these cases, the energy-momentum tensor for an arbitrary fluid flow was computed up to second order in derivatives. At this subleading order, novel terms appear which are linearly independent of the first-order terms. These terms are accompanied by novel transport coefficients which are not necessarily known in classical hydrodynamics. In principle, the whole construction can be extended to arbitrary orders in the derivative expansion albeit, in general, with increasing computational complexity. Besides obtaining new transport coefficients, the coefficients of lower orders might receive corrections at higher orders. At second order, this is not the case and thus, in particular, the ratio η/s is still given by (5.1). However, in general it is conceivable that the shear viscosity and consequently η/s receives corrections.

In section 5.4, we use the black hole approach to construct four different black hole solutions, namely the deformed AdS-Schwarzschild black hole and three deformed differently charged AdS-STU black holes. By mapping each of these black hole solutions to the energy-momentum tensor of the corresponding fluid higher-order terms appear in the gradient expansion. Actually, the shear viscosity turns out to receive contributions from a third-order term which leads to a correction of the ratio η/s.

This black hole approach is not the first technique for the computation of hydrodynamic transport coefficients by means of the fluid/gravity correspondence. The first calculation of this kind was the computation of the shear viscosity of a super Yang-Mills fluid dual to a near-extremal black three-brane solution [145]. There, the shear viscosity was related to the absorption cross section of low-energy gravitons by the near-extremal black three-brane. In general, hydrodynamic transport coefficients can be expressed in terms of correlation functions of the corresponding currents through *Kubo relations*. In [146, 166], a recipe was formulated to compute Minkowski-space correlation functions from gravity which are then used in the Kubo relations. This technique also works beyond the first order in derivatives, but only for linear fluctuations, and is therefore less general compared to the black hole approach. In [167], non-linear hydrodynamics was considered in the context of the *Bjorken boost-invariant flow* [168]. The results of [167] are in agreement with those found in [144], wherein the authors made use of the black hole approach. However, the black hole approach again is more general in this respect since the computations can be done for an arbitrary fluid flow.

5.3 Elements of relativistic fluid dynamics

In this section, a few selected facts about fluid dynamics are reviewed which are relevant for the considerations here. For a comprehensive introduction to the subject, see [169, 170].

In contrast to the dynamics of point particles which is described by field theories, fluid dynamics as an effective theory describes the motion of fluid elements. A fluid element is an infinitesimal volume element in the fluid which still contains sufficiently many fluid particles such as molecules or atoms. In this sense, fluid dynamics governs the collective motion of several fluid particles. The state of a fluid is completely characterized by the velocity field $u^\mu = \gamma\,(1, \vec{v}(t,x,y,z))$ as well as two thermodynamic quantities, for instance the pressure $p(t,x,y,z)$ and the energy density $\rho(t,x,y,z)$. The factor γ normalizes the velocity four-vector such that $g^{\mu\nu}u_\mu u_\nu = -1$, where $g_{\mu\nu}$ denotes the metric of the space in which the fluid propagates. In classical fluid dynamics, $g_{\mu\nu} = \eta_{\mu\nu}$ is the Minkowski metric, whereas in general $g_{\mu\nu}$ can be arbitrary.

The equations of fluid dynamics are the equations of local conservation of the energy-momentum tensor $T^{\mu\nu}$, the charge currents J_i^μ as well as the entropy current J_s^μ

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{5.3}$$

$$\nabla_\mu J^\mu_{i,s} = 0\,, \tag{5.4}$$

supplemented by thermodynamic constitutive relations that express these currents in terms of the fluid dynamical variables. For an arbitrary metric, ∇ denotes the spacetime covariant derivative. In the following, we focus on the energy-momentum tensor and its conservation equation (5.3).

5.3.1 The energy-momentum tensor

As was explained in section 5.2, the equations of fluid dynamics can be expanded in a derivative expansion. The energy-momentum tensor to zeroth order in derivatives corresponds to the ideal fluid part

$$T^{\mu\nu}_{\text{ideal}} = \rho\, u^\mu u^\nu + pP^{\mu\nu}\,, \tag{5.5}$$

where ρ is the total energy density, p is the pressure and $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor which projects four-vectors onto the three-dimensional submanifold orthogonal to u^μ.

The ideal fluid approximation neglects energy dissipative effects such as friction among fluid elements, heat conduction and charge diffusion. Including inner friction leads to viscous fluids whose energy-momentum tensor is (5.5) plus the viscous part

$$T^{\mu\nu}_{\text{visc}} = -\zeta\,\vartheta\,P^{\mu\nu} - 2\eta\,\sigma^{\mu\nu}\,. \tag{5.6}$$

The viscous part is first order in the derivative expansion, since the shear tensor $\sigma^{\mu\nu}$ and the quantity ϑ are both proportional to first derivatives of the velocity [144, 162, 165],

$$\sigma^{\mu\nu} = \frac{1}{2}\left(P^{\mu\lambda}\nabla_\lambda u^\nu + P^{\nu\lambda}\nabla_\lambda u^\mu\right) - \frac{1}{3}\vartheta\,P^{\mu\nu}\,, \tag{5.7}$$

$$\vartheta = \nabla_\mu u^\mu\,. \tag{5.8}$$

The transport coefficients ζ and η are called bulk and shear viscosity, respectively, and are generally considered to be constants. The bulk viscosity of a fluid measures the inner friction during volume changes, whereas the shear viscosity is a measure for momentum dissipation transverse to the fluid flow.

For an incompressible fluid, ϑ is zero and the first term in (5.6) as well as the last term in (5.7) vanishes.

Transport coefficients of strongly-coupled fluids can be determined either by measurements or by microscopic computations by means of the fluid/gravity correspondence as was explained in section 5.2.

5.3.2 Conformal fluids

A conformal field theory in the hydrodynamic regime is often referred to as conformal fluid. Conformal invariance further restricts the form of the energy-momentum tensor. Since, for a conformal fluid, the trace of the energy-momentum tensor vanishes[4], the pressure and the energy density are related as $\rho = 3p$ and the bulk viscosity ζ is zero.

5.3.3 A first glance on finite-size effects

Before constructing the black hole solution dual to a fluid on a three-sphere and computing its energy-momentum tensor, we present an a-priori argument whether and how the finite volume of the three-sphere might effect the ratio η/s.

The ratio η/s can be computed using the definition of the momentum diffusion constant

$$\mathcal{D} = \frac{\eta}{\rho + p}\,, \tag{5.10}$$

which can be read off from the linearized hydrodynamic equations as follows. Consider an incompressible conformal fluid in thermal equilibrium such that the energy-momentum tensor is constant. Now, let the velocity vector fluctuate slightly transversally to the flow direction and in time. For a small fluctuation we can expand the hydrodynamic equations to linear order,

$$\nabla_t T^{tj} + \nabla_i T^{ij} = 0\,. \tag{5.11}$$

The constitutive equation

$$T^{ij} = -\eta \left(\nabla^i u^j + \nabla^j u^i\right) = -\frac{\eta}{\rho + p}\left(\nabla^i T^{tj} + \nabla^j T^{ti}\right)\,, \quad i \neq j \tag{5.12}$$

[4] In general, in curved space the scaling symmetry of $\mathcal{N} = 4$ super Yang-Mills theory is broken due to the *Weyl anomaly* of the energy-momentum tensor. In four dimensions, the trace of the energy-momentum tensor is given by [77, 123]

$$T^\mu_\mu = -\frac{L^3}{8\pi G_4}\left(-\frac{1}{8}R^{\mu\nu}R_{\mu\nu} + \frac{1}{24}R^2\right)\,. \tag{5.9}$$

On the three-sphere, the right hand side is zero such that $T_{\mu\nu}$ is traceless.

expresses the spatial components T^{ij} of the energy-momentum tensor through the momentum density T^{tj}. Inserting (5.12) in (5.11) yields a diffusion equation for the momentum density T^{tj},

$$\partial_t T^{tj} - \mathcal{D}\vec{\nabla}^2 T^{tj} = 0\,, \tag{5.13}$$

with the momentum diffusion constant $\mathcal{D} = \eta/(\rho + p)$.

In [146], the momentum diffusion constant was computed for the $\mathcal{N} = 4$ super Yang-Mills plasma in the limit of an infinite number of colors N and infinite 't Hooft coupling λ dual to an uncharged black brane solution. Using two-point correlation functions of the energy-momentum tensor, the diffusion constant was determined in terms of the temperature as

$$\mathcal{D} = \frac{1}{4\pi T_0}\,, \tag{5.14}$$

where $T_0 = \frac{r_h}{\pi L^2}$. The computation of (5.14) is only valid up to first order in the gradient expansion and (5.14) might receive corrections at higher orders. Such corrections have not yet been considered. For this reason, we assume for now the first-order value (5.14) to be valid at all orders and keep this issue in mind when considering the sphere case.

Expressing (5.14) in terms of the entropy density $s = 4\pi\, r_h^3$ as

$$\mathcal{D} = \pi^{1/3}/(4^{2/3} s^{1/3})\,, \tag{5.15}$$

it follows from (5.10) that

$$\frac{\eta}{s} = \frac{\pi^{1/3}}{4^{2/3}}\frac{\rho + p}{s^{4/3}}\,. \tag{5.16}$$

Note that we use units where $L = 16\pi G_5 = 1$. For a conformal fluid in flat space, (5.16) equals $\eta/s = 1/(4\pi)$, since for the total energy density we have $\rho = 3\,p = 3\,s^{4/3}/(4\pi)^{4/3}$. Thus, the fluid in question satisfies the bound (5.2).

In contrast to fluids in flat space, the energy of a fluid on a three-sphere is not a purely extensive quantity [36,37]. It contains additional subextensive parts as for instance the Casimir energy E_c introduced in section 2.6.2 which is proportional to the curvature of the three-sphere. Remember that this energy was defined as the violation of the thermodynamic Euler relation such that

$$\frac{1}{3}E_c = \frac{4}{3}M - T_H\,\mathcal{S} - Q\,\phi\,. \tag{5.17}$$

We may ask whether this non-extensivity results in a correction of the coefficient η in front of the shear tensor and hence in a deviation from the value $\eta/s = 1/(4\pi)$. For instance, consider the conformal fluid dual to an AdS-Schwarzschild black hole. Its total energy density reads [1]

$$\rho = \rho_{\rm e} + \frac{1}{2}\rho_{\rm c} = 3\left(\frac{s}{4\pi}\right)^{4/3} + 3k\left(\frac{s}{4\pi}\right)^{2/3}\,, \tag{5.18}$$

where k is the inverse radius of the three-sphere. Here $\rho_{\rm e} \propto s^{4/3}$ denotes the extensive part and $\rho_{\rm c} \propto s^{2/3}$ denotes the subextensive part proportional to k of the total energy density. Taking the flat-space relation (5.16) at face value then suggests that

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{1}{2}\frac{\rho_{\rm c}}{\rho_{\rm e}}\right) \tag{5.19}$$

$$= \frac{1}{4\pi}\left(1 + \frac{k}{a_h^2}\right), \tag{5.20}$$

where we used that $a_h^2 = (s/4\pi)^{2/3}$. Thus the ratio η/s receives a correction proportional to the normalized curvature k of the three-sphere. This reflects the indistinguishability between finite size and curvature effects. While we can argue that the correction in (5.20) appears due to a Casimir contribution E_c in the energy of the fluid, which is a true finite-size effect, in (5.20) it appears to be a curvature effect, since k is proportional to the curvature scalar of the three-sphere.

In section 5.5, we use the black hole approach described in section 5.2 to compute corrections to η/s and to probe (5.20) as well as its universality. It is shown that (5.20) indeed holds for conformal fluids dual to AdS-Schwarzschild black holes as well as for charged fluids dual to AdS-STU black holes with three equal charges. Furthermore, we look at charged fluids dual to the AdS-STU black hole with two equal charges and with one non-zero charge.

5.4 The boosted black hole solution in $\mathcal{N} = 2$ supergravity

In this section, we construct static black hole solutions with spherical horizons of $\mathcal{N} = 2$ supergravity in AdS_5. These black holes are dual to fluids near equilibrium living on the four-dimensional boundary of global AdS_5 which is a three-sphere.

We search for black hole solutions to the Einstein equations of motion (3.14). As explained in chapter 3, one class of solutions are the electrically charged static black hole solutions (3.34) with $d\Sigma_k^2 = d\Omega_3^2$,

$$\begin{aligned} ds^2 &= G_{MN}\, dx^M dx^N \\ &= -\mathrm{e}^{-4U(r)}\, p(r)\, dt^2 + \mathrm{e}^{2U(r)}\, p^{-1}(r)\, dr^2 + \mathrm{e}^{2U(r)}\, r^2\, d\Omega_3^2 \,, \end{aligned} \tag{5.21}$$

where

$$p(r) = k - \frac{\mu}{r^2} + \frac{\mathrm{e}^{6U}\, r^2}{L^2} \quad , \quad k > 0 \,. \tag{5.22}$$

The line element $d\Omega_3^2$ of the three-sphere can be written as

$$d\Omega_3^2 = g_{ij}\, dx^i dx^j = k^{-1}\left(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2\right) \,, \tag{5.23}$$

with $0 \leq \theta \leq \pi/2\,,\, 0 \leq \phi < 2\pi\,,\, 0 \leq \psi < 2\pi$. The curvature tensor of the three-sphere is $R_{ij} = 2\,k\,g_{ij}$, and the associated curvature scalar is $R = 6\,k$. These black hole solutions are supported by scalar fields $X^A(r)$ which satisfy the relation (3.24). A definition of the various quantities of $\mathcal{N} = 2$ supergravity can be found in chapter 3.

The solution (5.21) is written in *Schwarzschild-type* coordinates which is not a proper coordinate system for a dual description of fluid dynamics. The reason for that is that Schwarzschild coordinates are not regular everywhere away from the curvature singularity at $r = 0$, since there is a coordinate singularity at the horizon at $r = r_h$ where $p^{-1}(r_h) = \infty$. Regularity of the black hole solution is necessary to obtain a fluid dynamical energy-momentum tensor [42] which belongs to a special class of conserved energy-momentum tensors. For instance, the energy-momentum tensor of an uncharged fluid in four dimensions has four degrees of freedom, the velocity and the temperature, and hence is completely determined by the four fluid dynamical equations of motion (5.3). In contrast, the singular solution (5.21) would lead to a general traceless, symmetric energy-momentum tensor with nine degrees of freedom resulting in an underdetermined system of fluid dynamical equations. This is successfully circumvented by the regular boosted black hole solution whose field theory degrees of freedom match those of the fluid energy-momentum tensor. In order to construct the boosted black hole solution we have to transform (5.21) to *Eddington-Finkelstein* coordinates.

Inspection of the line element (5.21) shows that the radius of the three-sphere is $\mathrm{e}^U\, r$ in units of $1/\sqrt{k}$. It is thus convenient to introduce a new radial coordinate $a = \mathrm{e}^U\, r$. We also introduce the function

$$f = \mathrm{e}^{-4U}\,\frac{p}{a^2} = \frac{1}{L^2} + \mathrm{e}^{-4U}\,\frac{k}{a^2} - \mathrm{e}^{-2U}\,\frac{\mu}{a^4}\,. \tag{5.24}$$

Then, using the flow equation (3.30) involving the superpotential W defined in (3.11), the line element takes the form

$$ds^2 = -a^2\, f(a)\, dt^2 + 9\left(a^2\, f(a)\, W^2(a)\right)^{-1} da^2 + a^2\, d\Omega_3^2\,. \tag{5.25}$$

Next, we introduce Eddington-Finkelstein type coordinates by

$$v = t + g(a) \quad , \quad \frac{dg}{da} = \frac{3}{W(a)\, a^2\, f(a)}\,, \tag{5.26}$$

so that the line element (5.25) becomes

$$ds^2 = -a^2\, f(a)\, dv^2 + \frac{6}{W(a)}\, dv\, da + a^2\, d\Omega_3^2\,. \tag{5.27}$$

Following [144, 161], we define boundary coordinates $x^\mu = (v, \theta, \phi, \psi)$ and we introduce the associated four-dimensional metric $g_{\mu\nu} = (g_{vv}, g_{ij}) = (-1, g_{ij})$, which is kept fixed throughout. Then, the static black hole metric (5.27) can be written as

$$ds^2 = -a^2 f(a)\, u_\mu\, u_\nu\, dx^\mu\, dx^\nu - \frac{6}{W(a)}\, u_\mu\, dx^\mu\, da + a^2\, P_{\mu\nu}\, dx^\mu\, dx^\nu \ , \tag{5.28}$$

where here u_μ denotes the four-vector $u_\mu = (-1, 0, 0, 0)$ and where

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu\, u_\nu \ . \tag{5.29}$$

The line element (5.28) is the boosted black hole solution of $\mathcal{N} = 2$ supergravity which is regular throughout the r interval except for the black hole singularity at $r = 0$ and can thus be mapped to solutions of fluid dynamics. The four-vector u_μ denotes the velocity vector of the dual fluid. Indices of boundary tensor quantities are lowered or raised using the boundary metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, such as, for instance, $u^\mu = g^{\mu\nu}\, u_\nu$.

In the following, we set $L = 1$ for convenience. Following [162, 165], we introduce the Schouten tensor $S_{\mu\nu} = \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}\, R\, g_{\mu\nu}\right)$. Here $R_{\mu\nu}$ and R are the four-dimensional Ricci tensor and Ricci scalar computed from the metric $g_{\mu\nu}$. Then, the line element (5.28) can also be expressed as

$$ds^2 = -\frac{6}{W(a)}\, u_\mu\, dx^\mu\, da + \left[a^2\, g_{\mu\nu} + \mathrm{e}^{-4U}\, u_{(\mu}\, S_{\nu)\lambda}\, u^\lambda + \mathrm{e}^{-2U}\, \frac{\mu}{a^2}\, u_\mu\, u_\nu\right] dx^\mu\, dx^\nu \ , \tag{5.30}$$

where $a_{(\mu}\, b_{\nu)} = a_\mu\, b_\nu + a_\nu\, b_\mu$. Observe that (5.30) is invariant under the global rescaling [162, 165]

$$\begin{aligned} a \to \mathrm{e}^{-\chi}\, a \ , \quad g_{\mu\nu} \to \mathrm{e}^{2\chi}\, g_{\mu\nu} \ , \quad u_\mu \to \mathrm{e}^{\chi}\, u_\mu \ , \\ \mathrm{e}^U \to \mathrm{e}^U \ , \quad \mu \to \mathrm{e}^{-4\chi}\, \mu \ , \end{aligned} \tag{5.31}$$

which also implies the rescaling

$$\begin{aligned} W \to W \ , \quad q_A \to \mathrm{e}^{-2\chi}\, q_A \ , \quad k \to \mathrm{e}^{-2\chi}\, k \ , \\ w_5 M \to \mathrm{e}^{-4\chi}\, w_5 M \ , \quad Q_A \to \mathrm{e}^{-3\chi}\, Q_A \ . \end{aligned} \tag{5.32}$$

For later convenience, we note that the scaling symmetry can be used to rescale the radial coordinate as $a/a_h = \rho$.

5.5 Deformed black hole solution

In the following, the static solutions described in the previous section is deformed by a slowly varying velocity field $u^\mu(x)$ of the form

$$u^\mu = (1, \epsilon\, \beta^\theta(x), \epsilon\, \beta^\phi(x), \epsilon\, \beta^\psi(x)) \ . \tag{5.33}$$

Here, the deformation β is multiplied with a small parameter ϵ. Thus, the deformation β^i is taken to be small in amplitude. We work at linear order in ϵ. At this order, u^μ satisfies the normalization condition $u^\mu\, u_\mu = -1$.

In addition, and following [144], a counting parameter δ is introduced by performing the rescaling $x^\mu \to \delta\, x^\mu$, so that an expansion in powers of δ counts covariant derivatives. For instance, the curvature tensor R_{ij} of the three-sphere, which is referred to as the background curvature tensor in the following, then comes multiplied by a factor δ^2.

The boundary energy-momentum tensor $T_{\mu\nu}$ of the deformed solutions contains a term proportional to the shear tensor $\sigma_{\mu\nu}$, with a coefficient denoted by η. We are interested in computing corrections to the ratio η/s due to the background curvature scalar $R = 6k$. These corrections, if present, give rise to deviations from the value $4\pi\,\eta/s = 1$, which are written as $4\pi\,\eta/s - 1 = \sum_{p\geq 1} \alpha_{2p}\,\delta^{2p}$. To compute these corrections, the perturbations of the black hole metric are organized in powers of ϵ and δ. In this work, we are only interested in the first subleading correction $\alpha_2\,\delta^2$. It corresponds to a term of the type $k\,\sigma_{\mu\nu}$, and hence of order $\epsilon\,\delta^3$, in the boundary energy-momentum tensor $T_{\mu\nu}$. Thus, only terms in the perturbed line element are kept that are at most of order $\epsilon\,\delta^3$.

First, the Schwarzschild case is considered which corresponds to setting $W = 3$ and $\mathrm{e}^U = 1$ in (5.30). The static Schwarzschild line element contains a term proportional to the background curvature scalar $R = 6k$. Thus, it contains a term of order $\epsilon^0\,\delta^2$. The deformed Schwarzschild solution, on the other hand, contains terms that are of order ϵ and higher. Its line element has been worked out in [164, 165] at order δ^2, and there are only two perturbations that are also of order ϵ, namely the shear tensor $\sigma_{\mu\nu}$ and the perturbation proportional to $u_\mu\, R_{\nu\lambda}\, u^\lambda$. The latter contains the term $u_t\, R_{ij}\, u^j$, which is of order $\epsilon\,\delta^2$. At order δ^3, new perturbations have to be added to the line element. Out of these, only perturbations that are proportional to the shear tensor $\sigma_{\mu\nu}$ can contribute to η. At order $\epsilon\,\delta^3$ there is only one such term, namely $R\,\sigma_{\mu\nu}$, which for constant R can be absorbed into the term proportional to $\sigma_{\mu\nu}$ at order δ. Thus, up to order $\epsilon\,\delta^3$, the metric perturbations may be restricted to those involving $\sigma_{\mu\nu}$ and to one particular perturbation of order δ^2 associated with the background curvature, namely $u_\mu\, R_{\nu\lambda}\, u^\lambda$.

Next, deformed charged black hole solutions are discussed. In this case there are new perturbations present at each order in δ. For instance, the case of the electrically charged Maxwell black hole corresponds to setting $W = 3$ and $\mathrm{e}^{6U} = H_1 H_2 H_3 = H$ in (5.30) where H_A is the harmonic function defined in (3.33). The new perturbations in the Maxwell case were computed up to order δ^2 in [156, 157]. Rather than taking all of these new terms into account, we follow the same strategy as in the Schwarzschild case. Namely, we start with the deformed solution at order δ and add one particular perturbation of order δ^2 to its line element, namely the one proportional to $u_\mu\, R_{\nu\lambda}\, u^\lambda$.

Now that we have clarified the ingredients we need, we make a solution ansatz using these and solve the associated equations of motion up to first order in ϵ. However, the equations of motion are not expanded in field theory derivatives. In other words, the solution we thus construct at order ϵ is an exact solution containing all field theory derivatives. It is determined in terms of a specific velocity field that is slowly varying in a certain coordinate range. Computing the associated boundary energy-momentum tensor, we find a correction to η/s proportional to the background curvature k. The addition of further deformations to the line element presumably results in a modified solution that contributes additional terms to η/s. If present, these new contributions should be qualitatively different from the one we compute here.

The ratio η/s should not receive corrections in ϵ, since that would make it depend on the amplitude ϵ of the velocity field. Indeed, using the results of [165], we have checked that for the Schwarzschild black hole, the second order metric perturbations that are of order $\epsilon^2\,\delta^2$ do not contribute to η.

The solutions constructed at order ϵ are based on the specific velocity field

$$u^\mu = (1, 0, \epsilon\, \beta^\phi(\theta), \epsilon\, \beta^\psi(\theta))\ , \tag{5.34}$$

which is a justified ansatz for a viscous $(u^i(x^j)) \neq 0)$, incompressible $(\theta = 0)$ and slightly fluctuating (order ϵ) fluid. As a consequence of linearity in velocities, the Weyl connection[5]

$$\mathcal{A}_\mu = u^\nu\, \nabla_\nu u_\mu - \frac{1}{3}(\nabla_\nu u^\nu)\, u_\mu \tag{5.35}$$

introduced in [162] vanishes. The second term in (5.35), which is proportional to ϑ introduced in (5.8), is absent for an incompressible fluid. In addition, we demand that the mass and the charges of the black hole solution are kept constant at order $\epsilon\,\delta^2$.

In the following four subsections, the case of the deformed Schwarzschild black hole is discussed, followed by three deformed special STU black holes of $\mathcal{N} = 2$ gauged supergravity which were defined in 3.4.2, 3.4.3 and 3.4.4.

5.5.1 Deformed Schwarzschild black hole

The construction of a black hole solution dual to a conformal fluid starts from a stationary black hole solution in Eddington-Finkelstein coordinates, which then gets deformed by a slowly varying velocity field [144]. Let us consider the static Schwarzschild solution in Eddington-Finkelstein coordinates which, according to (5.30), is given by

$$ds^2 = -2\, u_\mu\, dx^\mu\, da + \Big[a^2\, g_{\mu\nu} + u_{(\mu}\, S_{\nu)\lambda}\, u^\lambda + \frac{\mu}{a^2}\, u_\mu\, u_\nu\Big]\, dx^\mu\, dx^\nu\ , \tag{5.36}$$

[5] Here the covariant derivative ∇_μ is computed using the boundary metric $g_{\mu\nu}$.

where $u_\mu = (-1,0,0,0)$. Observe that the term proportional to the Schouten tensor is of order $\epsilon^0\,\delta^2$. The associated function f reads $f = 1 + k/a^2 - \mu/a^4$. The event horizon is at $f(a_h) = 0$. It is useful to introduce rescaled variables $\rho = a/a_h$ and $m = \mu/a_h^4$, in terms of which f is given by

$$f(\rho) = 1 + \frac{k}{a_h^2\,\rho^2} - \frac{m}{\rho^4}\;. \tag{5.37}$$

The event horizon is at $\rho = 1$ and m satisfies $m = 1 + k/a_h^2$.

Now, (5.36) is deformed by taking the velocity field to be non-trivial. The perturbed line element is then written in terms of Weyl covariant combinations [162, 165]. We work at first order in ϵ, and we take the velocity field to be of the form (5.34), for which the Weyl connection vanishes at first order in ϵ. The vanishing of the latter implies that the Weyl-covariantized Schouten tensor $\mathcal{S}_{\mu\nu}$ coincides with the ordinary Schouten tensor $S_{\mu\nu}$.

In general, when deforming the static black hole solution, not only the velocity field u^μ but also the mass μ becomes a slowly varying function of x^μ [144]. For the velocity field (5.34), inspection of equation (C.1) in [165] shows that μ remains constant at order $\epsilon\,\delta^2$ provided that $\mathcal{D}^\nu\sigma_{\nu\mu} = 0$. Here $\mathcal{D}$ denotes the Weyl covariant derivative introduced in [162], and the shear tensor $\sigma_{\mu\nu}$ is defined below. Using this information, we make an ansatz for the line element that captures effects of order $\epsilon\,\delta^2$, and we take μ to be constant.

(5.36) is deformed by adding a term proportional to the shear tensor $\sigma_{\mu\nu}$ defined in (5.7). For the velocity field (5.34) this yields $\sigma_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu\right)$ to first order in ϵ. Thus we make the following ansatz for the perturbed line element at order ϵ,

$$\begin{aligned} ds^2 = &- 2\,u_\mu\,dx^\mu\,da + \left[a^2\,g_{\mu\nu} + u_{(\mu}\,S_{\nu)\lambda}\,u^\lambda + \frac{\mu}{a^2}\,u_\mu\,u_\nu\right]dx^\mu\,dx^\nu \\ &+ 2\,\frac{a^2}{a_h}\,F(a)\,\sigma_{\mu\nu}\,dx^\mu\,dx^\nu\;. \end{aligned} \tag{5.38}$$

Here, F has Weyl weight zero, so that (5.38) is invariant under the rescaling (5.31). Observe that according to the counting described above, $\sigma_{\mu\nu}$ is of order $\epsilon\,\delta$, while $u_{(\mu}\,S_{\nu)\lambda}\,u^\lambda$ contains the deformation term $u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda$ which is of order $\epsilon\,\delta^2$.

Imposing the condition $\mathcal{D}^\nu\sigma_{\nu\mu} = 0$ we find the following expression for the velocity field,

$$\begin{aligned} \beta^\phi(\theta) =&\omega_1 + c_1\left(-\frac{1}{4}\log[\cos\,\theta] + \frac{1}{4}\log[\sin\,\theta] + \frac{1}{8\,\cos^2\,\theta}\right)\;, \\ \beta^\psi(\theta) =&\omega_2 + c_2\left(-\frac{1}{4}\log[\cos\,\theta] + \frac{1}{4}\log[\sin\,\theta] - \frac{1}{8\,\sin^2\,\theta}\right)\;, \end{aligned} \tag{5.39}$$

with constants $\omega_1, \omega_2, c_1, c_2$. Observe that in obtaining (5.39) we have not resorted to any approximation, *i.e.* at order ϵ (5.39) solves $\mathcal{D}^\nu\sigma_{\nu\mu} = 0$ exactly. The small amplitude approximation, however, breaks down at $\theta = 0, \pi/2$, where the norm of the velocity field diverges. Therefore, the range of θ has to be restricted to be consistent with the small amplitude expansion. This may be achieved by restricting θ to be in the range $\lambda < \theta < \pi/2 - \lambda$ with $\epsilon << \lambda^2$.

In case that both the c_i $(i = 1, 2)$ vanish, (5.38) describes an uncharged stationary black hole solution (at order ϵ) with $\sigma_{\mu\nu} = 0$. In the following, we are interested in non-stationary solutions, and hence we take at least one of the c_i to be non-vanishing. Using (5.39), and inserting the ansatz (5.38) into the Einstein equations of motion, we find that they are satisfied to first order in ϵ provided that F satisfies the differential equation

$$\frac{d}{d\rho}\left(\rho^5 f(\rho)\,\frac{d}{d\rho}F(\rho)\right) = -\left(3\rho^2 + \frac{k}{a_h^2}\right) . \tag{5.40}$$

When solving the Einstein equations, we do not resort to any truncation. Thus, (5.39) and (5.40) yield an exact solution to the Einstein equations at first order in ϵ.

Integrating (5.40) once gives

$$\rho^5 f(\rho)\,\frac{d}{d\rho}F = -\left(\rho^3 + \frac{k}{a_h^2}\,\rho - \zeta\right) , \tag{5.41}$$

where the integration constant ζ is set to the value $\zeta = 1 + k/a_h^2$ so as to account for the vanishing of $f(\rho)$ at the horizon $\rho = 1$. Note that (5.41) can be written as

$$\frac{d}{d\rho}F = -\frac{(\rho^2 + \rho + \zeta)}{\rho(\rho + 1)(\rho^2 + \zeta)} . \tag{5.42}$$

Integrating (5.42) once results in

$$F(\rho) = \int_\rho^\infty du\,\frac{(u^2 + u + \zeta)}{u(u + 1)(u^2 + \zeta)} , \tag{5.43}$$

which is well-behaved as long as $\rho > 0$. In the limit of large ρ this yields

$$\frac{F(\rho)}{a_h} = \frac{1}{a} - \frac{\eta}{4\,a^4} , \tag{5.44}$$

where

$$\eta = \zeta\,a_h^3 = a_h^3 + k\,a_h \tag{5.45}$$

is in fact the shear viscosity as is shown below.

Next we consider the fluid on a three-sphere dual to (5.38). Its energy-momentum tensor $T_{\mu\nu}$ can be computed using standard techniques [77, 106, 121], see appendix C. We obtain

$$\begin{aligned}16\pi G_5\,\langle T_{\mu\nu}\rangle =& \frac{1}{4}\left(R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} - \frac{R^2}{12}g_{\mu\nu}\right)\\ &+ \mu\,(g_{\mu\nu} + 4\,u_\mu\,u_\nu) - 2\,\eta\,\sigma_{\mu\nu} .\end{aligned} \tag{5.46}$$

The terms in the first line of this expression denote the contribution to the energy-momentum tensor of global AdS_5 [77, 163], while the terms proportional to μ denote the perfect fluid contribution[6]. The last term is the shear term with the shear viscosity η determined by (5.45). Thus, at order δ^3 in the derivative expansion of the energy-momentum tensor, η receives a positive correction of order δ^2 proportional to the curvature of the three-sphere. Consequently, this leads to a positive correction of order δ^2 in the ratio η/s. In units where $L = 16\pi G_5 = 1$ the entropy density s of the fluid on a unit three-sphere is $s = \mathcal{S}/vol(S^3) = 4\pi\, a_h^3$, so that the ratio η/s reads

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{k}{a_h^2}\right) . \tag{5.47}$$

This result is in accordance with the prediction in section 5.3.3. Thus, η/s for a fluid dual to a Schwarzschild black hole receives a positive correction due to the finite size of the three-sphere.

5.5.2 Deformed Maxwell black hole

Next, we consider the Maxwell black hole in the context of the STU model introduced in section 3.4.2. Remember that the scalar fields are set to $X^1 = X^2 = X^3 = 1$ and the gauge fields are set to $A^1 = A^2 = A^3 = 2A/\sqrt{3}$. Then, from (5.30), we obtain the following line element for the static Maxwell black hole,

$$ds^2 = -2\, u_\mu\, dx^\mu\, da + \left[a^2\, g_{\mu\nu} + u_{(\mu}\, S_{\nu)\lambda}\, u^\lambda + \left(\frac{w_5\, M}{a^2} - \frac{Q^2}{a^4}\right) u_\mu\, u_\nu\right] dx^\mu\, dx^\nu\ . \tag{5.48}$$

The Maxwell gauge potential reads

$$A_\mu = -\frac{\sqrt{3}}{2}\,\frac{Q}{a^2}\, u_\mu \quad , \quad A_a = 0\ , \tag{5.49}$$

where $u_\mu = (-1, 0, 0, 0)$. The function f in (5.24) reads $f(a) = 1 + k/a^2 - w_5\, M/a^4 + Q^2/a^6$. The location a_h of the outer event horizon is given by the largest positive root of $f(a)$. In terms of the rescaled variables $\rho = a/a_h$, $m = w_5\, M/a_h^4$ and $\mathcal{Q} = Q/a_h^3$, the function f is given by

$$f(\rho) = 1 + \frac{k}{a_h^2\, \rho^2} - \frac{m}{\rho^4} + \frac{\mathcal{Q}^2}{\rho^6}\ . \tag{5.50}$$

The outer event horizon is at $\rho = 1$ and m satisfies $m = 1 + k/a_h^2 + \mathcal{Q}^2$.

Now, the static Maxwell solution is deformed by taking the velocity field to be of the form (5.34) with β^ϕ and β^ψ given by (5.39). We work at first order in ϵ,

[6] μ is related to the pressure $p = M/(3\, vol(S^3))$ by $\mu = 16\pi\, G_5\, p$.

as before. The results of [156, 157] show that at order $\epsilon\,\delta^2$, the electric charge Q can be kept constant when M is constant. In the following, we take both M and Q to be constant.

We construct a solution to the combined Einstein-Maxwell equations of motion as follows. We take the gauge potential to be of the form (5.49) with the velocity field given by (5.39). Inserting this ansatz into the equations of motion, we find that we can solve the combined system at first order in ϵ and to all orders in δ with the following line element,

$$\begin{aligned} ds^2 = & -2\,u_\mu\,dx^\mu\,da + \left[a^2\,g_{\mu\nu} + u_{(\mu}\,S_{\nu)\lambda}\,u^\lambda + \left(\frac{w_5\,M}{a^2} - \frac{Q^2}{a^4}\right)u_\mu\,u_\nu\right]dx^\mu\,dx^\nu \\ & + \left[2\sqrt{3}\,\kappa\,\frac{Q}{a^2}\,u_{(\mu}\,l_{\nu)} + 2\,\frac{a^2}{a_h}\,F(a)\,\sigma_{\mu\nu}\right]dx^\mu\,dx^\nu \\ & + 4\sqrt{3}\,\kappa\,\frac{Q}{a^4\,f(a)}\,l_\mu\,dx^\mu\,da\;, \end{aligned} \tag{5.51}$$

where we recall that $u_{(\mu}\,l_{\nu)} = u_\mu\,l_\nu + u_\nu\,l_\mu$, and where [156, 157, 163]

$$l_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma}\,u^\nu\,\mathcal{D}^\lambda u^\sigma = \frac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma}\,u^\nu\,\nabla^\lambda u^\sigma\;, \tag{5.52}$$

with $\epsilon_{\mu\nu\lambda\sigma} = e_\mu{}^a\,e_\nu{}^b\,e_\lambda{}^c\,e_\sigma{}^d\,\epsilon_{abcd}$. Observe that l_μ and $F(a)$ have Weyl-weight zero, and that the associated terms in (5.51) are of order $\epsilon\,\delta$, while $u_{(\mu}\,S_{\nu)\lambda}\,u^\lambda$ contains the deformation term $u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda$ which is of order $\epsilon\,\delta^2$. The line element (5.51) is invariant under the rescaling (5.31) and (5.32).

The quantity F now satisfies the differential equation

$$\frac{d}{d\rho}\left(\rho^5\,f(\rho)\,\frac{d}{d\rho}F(\rho)\right) = -\left(3\rho^2 + \frac{k}{a_h^2}\right)\;, \tag{5.53}$$

with $f(\rho)$ given by (5.50). Integrating (5.53) once gives

$$\rho^5\,f(\rho)\,\frac{d}{d\rho}F = -\left(\rho^3 + \frac{k}{a_h^2}\,\rho - \zeta\right)\;, \tag{5.54}$$

where the integration constant ζ is set to the value $\zeta = 1 + k/a_h^2$ so as to account for the vanishing of $f(\rho)$ at the outer horizon $\rho = 1$. Note that (5.54) can be written as

$$\frac{d}{d\rho}F = -\frac{\rho\,(\rho^2 + \rho + \zeta)}{(\rho+1)(\rho^4 + \zeta\,\rho^2 - \mathcal{Q}^2)}\;. \tag{5.55}$$

Integrating (5.55) once results in

$$F(\rho) = \int_\rho^\infty du\,\frac{u\,(u^2 + u + \zeta)}{(u+1)(u^4 + \zeta\,u^2 - \mathcal{Q}^2)}\;. \tag{5.56}$$

Here ρ should be taken to be larger than the largest positive root of $u^4 + \zeta\, u^2 - \mathcal{Q}^2$ to avoid a singularity in $F(\rho)$. In the limit of large ρ this yields

$$\frac{F(\rho)}{a_h} = \frac{1}{a} - \frac{\eta}{4\, a^4}\ , \tag{5.57}$$

where

$$\eta = \zeta\, a_h^3 = a_h^3 + k\, a_h \tag{5.58}$$

is in fact the shear viscosity as is shown below.

The line element (5.51) is not in the customary gauge $g_{a\mu} = -u_\mu$ [165]. It can be brought into this gauge by the following coordinate transformation at order ϵ,

$$dx^\mu \to dx^\mu - h(a)\, l^\mu\, da - \left(\int^a h(b)\, db\right) dl^\mu\ , \tag{5.59}$$

where $h(a) = 2\sqrt{3}\,\kappa\, Q/(a^6\, f(a))$. Here the term proportional to l^μ is of order $\epsilon\,\delta$, while the term proportional to dl^μ is of order $\epsilon\,\delta^2$. The resulting line element is then regular at the outer horizon $f(a_h) = 0$ of the undeformed static black hole solution.

In the stationary case, the velocity field has the form (5.39) with $c_i = 0$. Due to the curvature k of the background, l^μ is non-vanishing but constant and given by $l^\mu = \sqrt{k}\,(0, 0, -\omega_2, -\omega_1)$. Then the second term in (5.59) vanishes, and the line element takes the form

$$\begin{aligned} ds^2 = &- 2\, u_\mu\, dx^\mu\, da + \left[a^2\, g_{\mu\nu} + u_{(\mu}\, S_{\nu)\lambda}\, u^\lambda + \left(\frac{w_5\, M}{a^2} - \frac{Q^2}{a^4}\right) u_\mu\, u_\nu \right. \\ &\left. + 2\sqrt{3}\,\kappa\, \frac{Q}{a^2}\, u_{(\mu}\, l_{\nu)}\right] dx^\mu\, dx^\nu \end{aligned} \tag{5.60}$$

in the gauge $g_{a\mu} = -u_\mu$. It is straightforward to relate this line element to the usual one [171] written in *Boyer-Lindquist-type* coordinates, to linear order in ω_1 and ω_2, see appendix B.1.

Next we compute the associated boundary energy-momentum tensor $T_{\mu\nu}$ of the fluid dual to (5.51), see appendix C. We obtain

$$\begin{aligned} 16\pi G_5\, \langle T_{\mu\nu}\rangle = &\frac{1}{4}\left(R_{\alpha\beta} R^{\alpha\ \ \beta}_{\ \ \mu\ \ \nu} - \frac{R^2}{12} g_{\mu\nu}\right) \\ &+ w_5\, M\,(g_{\mu\nu} + 4\, u_\mu\, u_\nu) + 8\sqrt{3}\,\kappa\, Q\, u_{(\mu}\, l_{\nu)} - 2\,\eta\,\sigma_{\mu\nu}\ . \end{aligned} \tag{5.61}$$

Again we see that the quantity determined in (5.58) is indeed the shear viscosity which receives a correction of order δ^2 proportional to the curvature of the three-sphere. In units where $L = 16\pi G_5 = 1$, and using that the entropy density s of the fluid on a unit three-sphere is $s = \mathcal{S}/vol(S^3) = 4\pi\, a_h^3$, the ratio η/s reads

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{k}{a_h^2}\right)\ , \tag{5.62}$$

as in the Schwarzschild case. We note that the correction to $\eta/s = 1/(4\pi)$ proportional to the curvature is determined by the coefficient of the $u\,R\,u$-term in the line element (5.51).

In the stationary case, where $\sigma_{\mu\nu} = 0$, $T_{\mu\nu}$ takes the form given in [172]. It contains additional non-dissipative terms proportional to l_μ associated with the rotation of the fluid in a background of constant curvature k.

In [156,157], the authors constructed charged black brane solutions up to order δ^2. At order δ, their solution is based on the gauge field

$$A_\mu = -\frac{\sqrt{3}\,Q}{2\,a^2}\left(u_\mu - 2\sqrt{3}\,\kappa\,\frac{Q}{w_5\,M}\,l_\mu\right) \quad , \quad A_a = 0\;. \tag{5.63}$$

For the sake of comparison, let us construct a black hole solution based on (5.63) with the velocity field given by (5.39). Inserting this ansatz into the equations of motion, we find that they can be solved exactly at first order in ϵ with the following line element,

$$\begin{aligned} ds^2 =&-2\,u_\mu\,dx^\mu\,da + \left[a^2\,g_{\mu\nu} + u_{(\mu}\,S_{\nu)\lambda}\,u^\lambda + \left(\frac{w_5\,M}{a^2} - \frac{Q^2}{a^4}\right)u_\mu\,u_\nu\right]dx^\mu\,dx^\nu \\ &+\left[-\frac{6\,\kappa^2 Q^2}{w_5\,M\,a^2}\,u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda + \frac{2\sqrt{3}\,\kappa\,Q^3}{w_5\,M\,a^4}\,u_{(\mu}\,l_{\nu)} + 2\,\frac{a^2}{a_h}\,F(a)\,\sigma_{\mu\nu}\right]dx^\mu\,dx^\nu \\ &+\left[\frac{4\sqrt{3}\,\kappa\,Q^3}{w_5\,M\,a^6\,f}\,l_\mu - \frac{12\,\kappa^2\,Q^2}{w_5\,M\,a^4 f}\,R_{\mu\lambda}\,u^\lambda\right]dx^\mu\,da\;, \end{aligned} \tag{5.64}$$

with l_μ defined as in (5.52). The quantity F satisfies the differential equation (5.53). The line element (5.64) is invariant under the rescaling (5.31) and (5.32). It is again not in the customary gauge $g_{a\mu} = -u_\mu$ [165]. It can be brought into this gauge by the coordinate transformation (5.59) at order ϵ. The resulting line element is then regular at the outer horizon $f(a_h) = 0$ of the undeformed static black hole solution.

We may ask whether the two line elements (5.51) and (5.64) can be transformed into each other. The associated gauge fields are related by the shift $u^\mu \to u^\mu - 2\sqrt{3}\,\kappa\,\frac{Q}{w_5\,M}\,l^\mu$. Applying this shift to the line element (5.51) induces terms that are of order $\epsilon\,\delta^3$. The resulting line element thus has terms of different order in δ than the line element (5.64). A matching of these two line elements is thus only expected to occur when the full set of $\epsilon\,\delta^3$-terms is taken into account. However, in the stationary case ($c_i = 0$), the solution (5.51) is mapped into (5.64) at order ϵ by the shift of u^μ described above, under which $l_i \to l_i - \sqrt{3}\,\kappa\,\frac{Q}{w_5\,M}\,R_{ij}\,u^j$. The two line elements are then identical in the gauge $g_{a\mu} = -u_\mu$, as expected.

Let us now compare the line element (5.64) with the one obtained in [156, 157]. Since the gauge field (5.63) is at most of order $\epsilon\,\delta$, the comparison is only meaningful up to this order. Since the terms in (5.64) proportional to $R_{\mu\nu}$ are of order $\epsilon\,\delta^2$ they should be dropped in the comparison. Then, by going into the gauge $g_{a\mu} = -u_\mu$ via the coordinate transformation (5.59) (and dropping the term proportional to dl^μ which is also of order $\epsilon\,\delta^2$) we find that the line element (5.64) goes over into the one obtained in [156, 157].

Computing the associated boundary energy-momentum tensor $T_{\mu\nu}$ we obtain

$$\begin{aligned}16\pi G_5\,\langle T_{\mu\nu}\rangle =& \frac{1}{4}\left(R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} - \frac{R^2}{12}g_{\mu\nu}\right)\\ &+ w_5\,M\,\left(g_{\mu\nu} + 4\,u_\mu\,u_\nu\right) - \frac{24\,\kappa^2\,Q^2}{w_5\,M}\,u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda - 2\,\eta\,\sigma_{\mu\nu}\ ,\end{aligned} \tag{5.65}$$

with η/s given by (5.62). It contains non-dissipative terms proportional to the background curvature tensor $R_{\mu\nu}$. In the stationary case, the boundary energy-momentum tensor (5.61) matches (5.65) under the constant shift $u^\mu \to u^\mu - 2\sqrt{3}\,\kappa\,\frac{Q}{w_5\,M}\,l^\mu$ discussed above.

5.5.3 Deformed black hole solutions supported by scalar fields

Next, we consider black hole solutions in the STU model that are supported by non-trivial scalar fields, and that carry either one or two non-vanishing charges. In the two-charge case, the charges are taken to be equal, for simplicity. We deform the static solutions in the manner described above. We find that the scalar fields do not need to be deformed at order ϵ.

Two equal charges

We begin by first considering the case of two equal charges defined in section 3.4.3. The line element of the static solution is given by (5.28) and the gauge potentials and scalar fields are

$$\begin{aligned}A^1_\mu =& A^2_\mu = -\frac{Q}{a^2}\,H^{-\frac{1}{3}}\,u_\mu\ , \quad A^3_\mu = 0\ , \quad A^i_a = 0\ , \quad i = 1,2,3\ ,\\ X^1 =& X^2 = H^{-\frac{1}{3}}\ , \quad X^3 = H^{\frac{2}{3}}\ ,\end{aligned} \tag{5.66}$$

where $u_\mu = (-1,\,0,\,0,\,0)$. A definition of the various quantities involved can be found in chapter 3. The function $f(a)$ appearing in (5.28), when expressed in terms of the rescaled coordinates $\rho = a/a_h$, reads

$$f(\rho) = 1 + \mathrm{e}^{-4U}\,\frac{k}{a_h^2\,\rho^2} - \mathrm{e}^{-2U}\,\frac{m}{\rho^4}\ , \quad m = \frac{\mu}{a_h^4} = \left(1 + \frac{k}{a_h^2}\,\mathrm{e}^{-4U(a_h)}\right)\mathrm{e}^{2U(a_h)}\ . \tag{5.67}$$

The outer horizon is at $\rho = 1$.

This static solution is perturbed by again taking the velocity field to have the form (5.34) and (5.39). This results in a modification of the line element, and it also induces a non-vanishing A^3. We find that at first order in ϵ, but no approximation otherwise, the combined system of equations of motion is solved by

$$\begin{aligned} ds^2 =& -a^2 f(a)\, u_\mu\, u_\nu\, dx^\mu\, dx^\nu - \frac{6}{W(a)}\, u_\mu\, dx^\mu\, da + a^2\, P_{\mu\nu}\, dx^\mu\, dx^\nu \\ &+ \frac{1}{2}\, H^{-\frac{1}{3}}\, u_{(\mu}\, R_{\nu)\lambda}\, u^\lambda\, dx^\mu\, dx^\nu + 2\,\frac{a^2}{a_h}\, F(a)\, \sigma_{\mu\nu}\, dx^\mu\, dx^\nu \;, \\ A^1_\mu =& A^2_\mu = -\frac{Q}{a^2}\, H^{-\frac{1}{3}}\, u_\mu \;, \quad A^3_\mu = -\frac{q}{a^2}\, H^{\frac{2}{3}}\, l_\mu \;, \quad A^i_a = 0 \;, \quad i = 1,2,3 \end{aligned} \tag{5.68}$$

with the scalar fields given as in (5.66). Here l_μ and the velocity field are again given by (5.52) and (5.39), respectively. The stationary limit of this solution can be easily related to the solution found in [173] written in Boyer-Lindquist type coordinates, to linear order in rotation parameters (see appendix B.2).

The quantity F now satisfies the differential equation

$$\frac{1}{3}\,\frac{d}{d\rho}\left(\rho^5\, W(\rho)\, f(\rho)\, \frac{d}{d\rho} F(\rho)\right) = -\left(3\rho^2 + \frac{k}{a_h^2}\, \mathrm{e}^{-U}\,(1 - U'\rho)\right) , \tag{5.69}$$

where $U' = dU/d\rho$, with $\mathrm{e}^{3U} = H$. We note the appearance of the superpotential $W(a)$ on the left hand side which takes the constant value $W(a) = 3$ in both the Schwarzschild and the Maxwell case. The right hand side of (5.69) can be easily integrated by noting that the second term is a total derivative,

$$\mathrm{e}^{-U}\,(1 - U'\rho)\, d\rho = d\left(\rho\, \mathrm{e}^{-U}\right) . \tag{5.70}$$

Thus, integrating (5.69) once gives

$$\frac{1}{3}\,\rho^5\, W(\rho)\, f(\rho)\, \frac{d}{d\rho} F(\rho) = -\left(\rho^3 + \frac{k}{a_h^2}\, \mathrm{e}^{-U}\rho - \zeta\right) . \tag{5.71}$$

The integration constant ζ is set to the value $\zeta = 1 + \left(k\, \mathrm{e}^{-U(a_h)}\right)/a_h^2$ to allow for the vanishing of (5.69) at the outer horizon $\rho = 1$, where $f = 0$. Then, integrating (5.71) once results in

$$F(\rho) = \int_\rho^\infty \frac{du}{W(u)}\, \frac{3\left(u^3 + u\,(\zeta - 1)\, \mathrm{e}^{U(a_h)}\, \mathrm{e}^{-U} - \zeta\right)}{u^5 + u^3\,(\zeta - 1)\, \mathrm{e}^{U(a_h)}\, \mathrm{e}^{-4U} - u\,\left(\mathrm{e}^{2U(a_h)} + (\zeta - 1)\, \mathrm{e}^{-U(a_h)}\right)\, \mathrm{e}^{-2U}} . \tag{5.72}$$

For large ρ we have $\mathrm{e}^{3U} = H \approx 1 + q\,/(a_h^2\,\rho^2)$, and hence we obtain

$$\frac{F(\rho)}{a_h} = \frac{1}{a} - \frac{\eta}{4\,a^4}\,, \tag{5.73}$$

where

$$\eta = \zeta\, a_h^3 = a_h^3 + k\,\mathrm{e}^{-U(a_h)}\,a_h \tag{5.74}$$

is in fact the shear viscosity η as is shown below.

Computing the associated boundary energy-momentum tensor we obtain (see appendix C)

$$\begin{aligned} 16\pi G_5\,\langle T_{\mu\nu}\rangle =& \frac{1}{4}\left(R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} - \frac{R^2}{12}g_{\mu\nu}\right) \\ &+ w_5\,M\,\left(g_{\mu\nu} + 4\,u_\mu\,u_\nu\right) - \frac{2\,q}{3}\,u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda - 2\,\eta\,\sigma_{\mu\nu}\,. \end{aligned} \tag{5.75}$$

Again we see that the quantity determined in (5.74) is indeed the shear viscosity which receives a correction of order δ^2 proportional to the curvature of the three-sphere. Furthermore, (5.75) contains a non-dissipative term proportional to the background curvature tensor $R_{\mu\nu}$. In units where $L = 16\pi G_5 = 1$, the ratio η/s reads

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1 + \frac{k\,\mathrm{e}^{-U(a_h)}}{a_h^2}\right)\,. \tag{5.76}$$

We note that the correction to $\eta/s = 1/(4\pi)$ is determined by the coefficient of the $u\,R\,u$-term in the line element (5.68). Observe that compared to the Schwarzschild and the Maxwell case the correction to η/s comes multiplied with the factor $\mathrm{e}^{-U(a_h)}$ and thus differs from the prediction in section 5.3.3. A reason for that might be that now in the presence of non-trivial scalar fields, higher-order corrections to the diffusion constant are involved which were neglected in section 5.3.3. This together with the subextensive contribution in the energy then leads to the correction (5.76).

One charge

Next, we consider the case of one non-vanishing charge defined in section 3.4.4. Proceeding as before, *i.e.* taking the velocity field to be given by (5.39), we find that at first order in ϵ, but no approximation otherwise, the perturbed solution to the combined system of equations of motion is given by

$$\begin{aligned} ds^2 =& - a^2 f(a) \, u_\mu \, u_\nu \, dx^\mu \, dx^\nu - \frac{6}{W(a)} u_\mu \, dx^\mu \, da + a^2 \, P_{\mu\nu} \, dx^\mu \, dx^\nu \\ & + \frac{1}{2} H^{\frac{1}{3}} \, u_{(\mu} \, R_{\nu)\lambda} \, u^\lambda \, dx^\mu \, dx^\nu + 2 \frac{a^2}{a_h} F(a) \, \sigma_{\mu\nu} \, dx^\mu \, dx^\nu \, , \\ A^1_\mu =& - \frac{Q}{a^2} H^{-\frac{2}{3}} \, u_\mu \, , \quad A^2_\mu = A^3_\mu = 0 \, , \quad A^i_a = 0 \, , \quad i = 1,2,3 \, , \\ X^1 =& H^{-\frac{2}{3}} \, , \quad X^2 = X^3 = H^{\frac{1}{3}} \, . \end{aligned} \tag{5.77}$$

The stationary limit of this solution can be related to the solution found in [174, 175], to linear order in rotation parameters (see appendix B.3). The quantity F satisfies the differential equation

$$\frac{1}{3} \frac{d}{d\rho} \left(\rho^5 \, W(\rho) \, f(\rho) \, \frac{d}{d\rho} F(\rho) \right) = - \left(3\rho^2 + \frac{k}{a_h^2} \, e^{2U} \, (1 + 2 \, U' \rho) \right) \, , \tag{5.78}$$

where $U' = dU/d\rho$, with $e^{6U} = H$. The right hand side of (5.78) can be easily integrated by noting that the second term is a total derivative,

$$e^{2U} \, (1 + 2 \, U' \rho) \, d\rho = d \left(\rho \, e^{2U} \right) \, . \tag{5.79}$$

Integrating (5.78) once gives

$$\frac{1}{3} \rho^5 \, W(\rho) \, f(\rho) \, \frac{d}{d\rho} F(\rho) = - \left(\rho^3 + \frac{k}{a_h^2} \, e^{2U} \rho - \zeta \right) \, . \tag{5.80}$$

The integration constant ζ is set to the value $\zeta = 1 + \left(k \, e^{2U(a_h)} \right) / a_h^2$ to allow for the vanishing of (5.78) at the outer horizon $\rho = 1$, since $f = 0$ there. Then, integrating (5.80) once results in

$$F(\rho) = \int_\rho^\infty \frac{du}{W(u)} \frac{3 \left(u^3 + u \, (\zeta - 1) e^{-2U(a_h)} e^{2U} - \zeta \right)}{u^5 + u^3 (\zeta - 1) e^{-2U(a_h)} e^{-4U} - u \left(e^{2U(a_h)} + (\zeta - 1) e^{-4U(a_h)} \right) e^{-2U}} \, . \tag{5.81}$$

For large ρ, we have $e^{6U} = H \approx 1 + q/(a_h^2 \, \rho^2)$, and hence we obtain

$$\frac{F(\rho)}{a_h} = \frac{1}{a} - \frac{\eta}{4 \, a^4} \, , \tag{5.82}$$

where

$$\eta = \zeta \, a_h^3 = a_h^3 + k \, e^{2U(a_h)} \, a_h \tag{5.83}$$

is in fact the shear viscosity as is shown below.

Computing the associated boundary energy-momentum tensor yields

$$\begin{aligned}16\pi G_5\,\langle T_{\mu\nu}\rangle =&\frac{1}{4}\left(R_{\alpha\beta}R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}-\frac{R^2}{12}g_{\mu\nu}\right)\\&+w_5\,M\,\left(g_{\mu\nu}+4\,u_\mu\,u_\nu\right)+\frac{2\,q}{3}\,u_{(\mu}\,R_{\nu)\lambda}\,u^\lambda-2\,\eta\,\sigma_{\mu\nu}\,.\end{aligned} \tag{5.84}$$

Again we see that the quantity determined in (5.83) is indeed the shear viscosity which receives a correction of order δ^2 proportional to the curvature of the three-sphere. In units where $L=16\pi G_5=1$, the ratio η/s reads

$$\frac{\eta}{s}=\frac{1}{4\pi}\left(1+\frac{k\,\mathrm{e}^{2U(a_h)}}{a_h^2}\right)\,. \tag{5.85}$$

Note that the correction to $\eta/s=1/(4\pi)$ proportional to the curvature is determined by the coefficient of the $u\,R\,u$-term in the line element (5.77). Similar as in the two-charge case, the correction to η/s is multiplied by a factor of $\mathrm{e}^{2U(a_h)}$ which deviates from the prediction in section 5.3.3. This seems to imply that corrections to the diffusion constant are involved.

5.6 Summary and discussion

In this chapter, the black hole approach was used to construct deformed spherical AdS-STU black hole solutions of $\mathcal{N}=2$ supergravity which are dual to incompressible viscous conformal charged fluids, in the infinite N and infinite λ limit, propagating on a three-sphere. In particular, the deformed AdS-Schwarzschild black hole, the deformed AdS-STU black hole with three equal charges, with two equal charges and with one non-zero charge were constructed, and the ratio η/s for the corresponding fluids was computed. In all these cases, we found a positive deviation from the value $\eta/s=1/4\pi$ proportional to the curvature of the three-sphere at third order in the derivative expansion of the energy-momentum tensor.

As mentioned in the introduction, the energy of a perfect fluid on a three-sphere dual to a static black hole is not a purely extensive quantity [36, 37]. It contains a subextensive piece, the Casimir energy E_c, which is defined as the violation of the thermodynamic Euler relation. In the context of $\mathcal{N}=2$ gauged supergravity theories, the ratio of ρ_c and the extensive part ρ_e of the energy density, when expressed in terms of black hole data, reads (in units where $L=16\pi G_5=1$) [1]

$$\frac{\rho_c}{\rho_e}=\frac{6\,k}{a_h^2}\frac{\tilde{W}_h}{W_h}\,, \tag{5.86}$$

where a_h denotes the horizon radius, and $\tilde{W}_h$ as well as W_h denote the superpotentials evaluated at the horizon. The Schwarzschild and the Maxwell black hole both satisfy $W_h = 3, \tilde{W}_h = 1$. For these two black holes, the ratio η/s can be written as

$$\frac{\eta}{s} = \frac{1}{4\pi} \frac{\rho_e + \frac{1}{2}\rho_c}{\rho_e} = \frac{1}{4\pi} \left(1 + \frac{3\,k}{a_h^2} \frac{\tilde{W}_h}{W_h} \right) . \tag{5.87}$$

The ratio displayed in (5.87) takes a form that is written in manifest $\mathcal{N} = 2$ language and that could, a priori, be applicable to any black hole in an $\mathcal{N} = 2$ model. However, inspection of the two-charge result (5.76) and of the one-charge result (5.85) shows that they are not simply captured by (5.87). These two cases involve non-trivial scalar fields, and it is conceivable that additional terms involving these have to be added to (5.87) in order to obtain an expression that is valid for a general $\mathcal{N} = 2$ model.

Let us now discuss the diffusion coefficient D, defined as in (5.10). Let us first consider the Schwarzschild case, for which (5.87) implies that the ratio $D = \eta/(\rho + p) = 3\eta/(4\rho)$ equals $D = \pi^{1/3}/(4^{2/3}\, s^{1/3})$, as in the black brane case (5.16). Thus, when viewed as a function of s, D does not change its functional form. On the other hand, if D is viewed as a function of the temperature (the energy), then D changes its functional form due to the subextensive contribution $\rho_c \propto k$ to the total energy, *i.e.* D is not any longer simply given in terms of the inverse of the temperature. Either way, $\eta = D\,(\rho + p)$ receives a correction proportional to $\rho_c \propto k$ (see (5.86)).

Next, let us consider the Maxwell case. Viewing D as a function of s, we find that D is not any longer given by $D = \pi^{1/3}/(4^{2/3}\, s^{1/3})$. This can be understood as follows. The total energy of the system is not simply $\rho_e + \frac{1}{2}\rho_c$, but rather $\rho_e + \frac{1}{2}\rho_c + \frac{1}{2}\tilde{\rho}_A\, \phi_h^A$, where ϕ_h^A denote the electrostatic potentials at the horizon [1] and $\tilde{\rho}_A$ the charge densities. The contribution $\tilde{\rho}_A\, \phi_h^A$ is a subextensive contribution that is distinct from the contribution ρ_c, since the former is more subleading in the hydrodynamic expansion than the latter. Furthermore, $\tilde{\rho}_A\, \phi_h^A$ is proportional to the square of the charge, while ρ_c is proportional to k. Using (5.87), we find that the diffusion coefficient D is proportional to the ratio $(\rho_e + \frac{1}{2}\rho_c)/(\rho_e + \frac{1}{2}\rho_c + \frac{1}{2}\rho_A^Q\, \phi_h^A)$. At order $(L/a_h)^2$ in the expansion of the diffusion coefficient, the correction proportional to k cancels out, while the term proportional to $\tilde{\rho}_A\, \phi_h^A$ can be neglected. Thus, when D is viewed as a function of s, it does not receive a correction of order k. However, if D is viewed as a function of the temperature (the energy), then D changes its functional form (at first order in k) due to the subextensive contribution ρ_c to the total energy. Either way, $\eta = D\,(\rho + p)$ receives a correction proportional to $\rho_c \propto k$.

And finally, in the case of charged black holes with scalar fields, we find that D, when viewed as a function of s, receives a correction of order k, since in these cases the term proportional to k in η does not equal ρ_c, and hence it differs from the contribution ρ_c contained in the total energy.

The corrected shear viscosity at third order in the hydrodynamic expansion of the energy-momentum tensor can be viewed as an effective shear viscosity in curved space. In all the cases we considered, the correction proportional to k stems from a term in the black hole line element of the form $u\,R\,u$ which on the fluid side transforms to a third-order term in the energy-momentum tensor $T_{\mu\nu}$ of the form $R\,\sigma_{\mu\nu}$, where R denotes the curvature scalar which for the three-sphere is proportional to k. Thus our result, that at third order the shear viscosity receives corrections proportional to the curvature of the space on which the fluid propagates, might be true not only on the three-sphere, but also on generally curved backgrounds.

The third-order corrections in all the cases we considered are positive and therefore respect the bound (5.2).

In deriving the expressions for η/s we restricted ourselves to corrections of order k. Higher corrections in k are in principle also possible. For simplicity, we took the velocity field u^μ of the fluid to be of the specific form (5.39). Our expressions for η/s should, however, be independent of this particular choice of the velocity field.

Chapter 6

A holographic p-wave superfluid with back-reaction

In this chapter, the AdS/CFT correspondence is applied to study the strongly-coupled CFT dual to a *non-Abelian AdS black hole* in $SU(2)$ Einstein-Yang Mills theory. For sufficiently low temperature, these black hole solutions develop *vector hair* which in the dual field theory corresponds to a phase transition to a superfluid state with spontaneously broken rotational symmetry. Such a state is called a *p-wave superfluid* state. While numerically constructing the non-Abelian AdS black hole solution with a flat horizon we also take the back-reaction of the gauge fields into account. The bulk theory has a single free parameter, the ratio of the five-dimensional gravitational constant to the Yang-Mills coupling, which we denote as α. Previous analyses have shown that in the *probe limit*, where α goes to zero, and hence the gauge fields are ignored in the Einstein equations, the transition to the superfluid state is second order. We construct fully back-reacted solutions, where α is finite and the gauge fields are included in the Einstein equations, and find that for values of α above a critical value $\alpha_c = 0.365 \pm 0.001$, the transition becomes first order.

The outline of the chapter is as follows. After a brief introduction, in section 6.2 the action of the model is presented and the ansatz for the bulk fields is discussed. In section 6.3 we describe how to extract thermodynamic information from the solutions. In section 6.4 the numerical results are presented demonstrating that increasing α changes the order of the phase transition. Section 6.5 contains a conclusion.

This chapter is based on work which was done by the author of this thesis in collaboration with Martin Ammon, Johanna Erdmenger, Patrick Kerner and Andy O'Bannon and which was published in [3].

6.1 Introduction

The AdS/CFT correspondence [17] as a novel method for studying strongly-coupled systems at finite density may have useful applications in condensed matter physics. Especially suitable are quantum critical low-temperature systems[1]. Such systems undergo a *quantum phase transition* at zero temperature. These transitions are driven by quantum fluctuations rather than thermal fluctuations and can be accessed by varying a physical parameter. The position in the phase diagram where the transition occurs is referred to as *quantum critical point*. Near these points the system becomes scale-invariant. Moreover, quantum critical points can dominate regions away from the zero temperature limit, and it is believed that the scale invariant field theory at zero temperature can be generalized to describe the behavior of the system in the quantum critical region at finite temperature. Quantum critical systems are not purely theoretical: the thermodynamics of some high-T_c superconductors may be controlled by a quantum critical point. By now AdS/CFT can model many basic phenomena in condensed matter physics, such as the quantum Hall effect [176], non-relativistic scale-invariance [177, 178], and Fermi surfaces [179, 180].

The AdS/CFT correspondence can also describe phase transitions to *superfluid* states. These are phase transitions in which a sufficiently large $U(1)$ charge density triggers spontaneous breaking of the $U(1)$ symmetry: an operator charged under the $U(1)$ acquires a nonzero expectation value [53–55]. We refer to this as the *operator condensing*. The simplest bulk action that can describe such a transition is Einstein-Maxwell theory coupled to a charged scalar. In the bulk, a charged black hole is said to develop *scalar hair* which means that the black hole solution is supported by a scalar field. In the CFT, this charged scalar field operator condenses.

A simple bulk action has one great virtue, namely a kind of universality: the results may be true for many different dual CFTs, independent of the details of their dynamics. For the Einstein-Maxwell-scalar case, a fruitful exercise is to study various functional forms for the scalar potential and to scan through values of couplings in that potential [181–183]. Generally speaking, scanning through values of these parameters corresponds to scanning through many different dual CFTs. As shown in [181–183], such changes can have a dramatic effect, for example, the phase transition can change from second to first order.

AdS/CFT can also describe superfluid states in which the condensing operator is a vector and hence rotational symmetry is broken, that is, p-wave superfluid states [56, 57]. Here the CFT has a global $SU(2)$ symmetry and hence three conserved currents J^μ_a, where $a = 1, 2, 3$ label the generators of $SU(2)$. For a

[1] For an introduction on the application of AdS/CFT methods to quantum critical condensed matter systems see for example [11–13] and references therein.

sufficiently large charge density for some $U(1)$ subgroup of $SU(2)$, say a sufficiently large $\langle J_3^t \rangle$, holographic calculations reveal that, of the known available states, those with lowest free energy have a nonzero $\langle J_1^x \rangle$. Not only is the $U(1)$ broken, but spatial rotational symmetry is also broken to some subgroup.

On the AdS side, a simple bulk action that can describe such a transition is Einstein-Yang-Mills theory with gauge group $SU(2)$. CFT states with nonzero $\langle J_3^t \rangle$ are dual to black hole solutions with nonzero vector field $A_t^3(r)$ in the bulk, where r is the radial coordinate of AdS space. States with nonzero $\langle J_1^x \rangle$ are dual to black hole solutions with a nontrivial $A_x^1(r)$. The superfluid phase transition is thus dual to charged AdS black holes developing vector hair. A string theory realization for this model is given in [184–187].

Unlike the Einstein-Maxwell-scalar case, $SU(2)$ Einstein-Yang-Mills theory has only a *single* free parameter, $\alpha \equiv \kappa_5/\hat{g}$, where κ_5 is the five-dimensional gravitational constant and $\hat{g}$ is the Yang-Mills coupling. The Yang-Mills source terms on the right hand side of the Einstein equations are proportional to α^2. To date, most analyses of the holographic p-wave superfluid transition have employed the *probe limit*, which consists in taking $\alpha \to 0$ so that the gauge fields have no effect on the geometry, which becomes simply AdS-Schwarzschild. The probe limit was sufficient to show that a superfluid phase transition occurs and is second order.

Our goal is to study the effect of finite α, that is, to study the back-reaction of the gauge fields on the metric. We work with five-dimensional $SU(2)$ Einstein-Yang-Mills theory, with finite α. We numerically construct asymptotically AdS charged black hole solutions with vector hair[2]. Our principal result is that for a sufficiently large value of α the phase transition becomes first order. More specifically, we find a critical value $\alpha_c = 0.365 \pm 0.001$, such that the transition is second order when $\alpha < \alpha_c$ and first order when $\alpha > \alpha_c$.

We can provide some intuition for what increasing α means, in CFT terms, as follows. Generically, in AdS/CFT $1/\kappa_5^2 \propto c$, where c is the central charge of the CFT [77, 123], which, roughly speaking, counts the total number of degrees of freedom in the CFT. Correlation functions involving the $SU(2)$ current are generically proportional to $1/\hat{g}^2$ [11–13]. We may, again roughly, think of $1/\hat{g}^2$ as counting the number of degrees of freedom in the CFT that carry $SU(2)$ charge. Intuitively, then, in the CFT increasing α means increasing the ratio of charged degrees of freedom to total degrees of freedom.

[2] For similar studies see [188–190].

6.2 Holographic setup

We consider $SU(2)$ Einstein-Yang-Mills theory in five-dimensional asymptotically AdS space. The action is

$$S = \int \mathrm{d}^5x \sqrt{-G} \left[\frac{1}{2\kappa_5^2} (R - \Lambda) - \frac{1}{4\hat{g}^2} F^a_{MN} F^{aMN} \right] + S_{bdy} , \qquad (6.1)$$

where κ_5 is the five-dimensional gravitational constant, $\Lambda = -\frac{12}{L^2}$ is the cosmological constant, with L being the AdS radius, and $\hat{g}$ is the Yang-Mills coupling constant. The $SU(2)$ field strength F^a_{MN} is

$$F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + \epsilon^{abc} A^b_M A^c_N , \qquad (6.2)$$

where $M, N = \{t, r, x, y, z\}$, with r being the AdS radial coordinate, and ϵ^{abc} is the totally antisymmetric tensor with $\epsilon^{123} = +1$. The A^a_M are the components of the matrix-valued gauge field, $A = A^a_M \tau^a dx^M$, where the τ^a are the $SU(2)$ generators, which are related to the Pauli matrices by $\tau^a = \sigma^a/2i$. S_{bdy} includes boundary terms that do not affect the equations of motion, namely the Gibbons-Hawking boundary term as well as counterterms required for the on-shell action to be finite. We write S_{bdy} explicitly in section 6.3.

The Einstein and Yang-Mills equations derived from the above action are

$$R_{MN} + \frac{4}{L^2} G_{MN} = \kappa_5^2 \left(T_{MN} - \frac{1}{3} T_P{}^P G_{MN} \right) , \qquad (6.3)$$

$$\nabla_M F^{aMN} = -\epsilon^{abc} A^b_M F^{cMN} , \qquad (6.4)$$

where the Yang-Mills energy-momentum tensor T_{MN} is

$$T_{MN} = \frac{1}{\hat{g}^2} \mathrm{tr} \left(F_{PM} F^P{}_N - \frac{1}{4} G_{MN} F_{PO} F^{PO} \right) . \qquad (6.5)$$

Following [56], to construct charged black hole solutions with vector hair we choose a gauge field ansatz

$$A = \phi(r) \tau^3 \mathrm{d}t + w(r) \tau^1 \mathrm{d}x . \qquad (6.6)$$

The motivation for this ansatz is as follows. In the field theory we introduce a chemical potential for the $U(1)$ symmetry generated by τ^3. We denote this $U(1)$ as $U(1)_3$. The bulk operator dual to the $U(1)_3$ density is A^3_t, hence we include $A^3_t(r) \equiv \phi(r)$ in our ansatz. We want to allow for states with a nonzero $\langle J^x_1 \rangle$, so in addition we introduce $A^1_x(r) \equiv w(r)$. Solutions with nonzero $w(r)$ preserve only an $SO(2)$ subgroup of the $SO(3)$ rotational symmetry, so our metric ansatz respects only $SO(2)$. We also pattern our metric ansatz after the ones used in [190] since these tame singular points in the equations of motion. Our metric ansatz is

$$\mathrm{d}s^2 = -N(r)\sigma(r)^2\mathrm{d}t^2 + \frac{1}{N(r)}\mathrm{d}r^2 + r^2 f(r)^{-4}\mathrm{d}x^2 + r^2 f(r)^2\left(\mathrm{d}y^2 + \mathrm{d}z^2\right)\,, \qquad (6.7)$$

with $N(r) = -\frac{2m(r)}{r^2} + \frac{r^2}{L^2}$. For our black hole solutions we denote the position of the horizon as r_h. The AdS boundary is at $r \to \infty$.

Inserting our ansatz into the Einstein and Yang-Mills equations yields five equations of motion for $m(r)$, $\sigma(r)$, $f(r)$, $\phi(r)$, $w(r)$ and one constraint equation from the rr component of the Einstein equations. The dynamical equations can be recast as (prime denotes $\frac{\partial}{\partial r}$)

$$\begin{aligned}
m' &= \frac{\alpha^2 r f^4 w^2 \phi^2}{6N\sigma^2} + \frac{\alpha^2 r^3 \phi'^2}{6\sigma^2} + N\left(\frac{r^3 f'^2}{f^2} + \frac{\alpha^2}{6} r f^4 w'^2\right)\,,\\
\sigma' &= \frac{\alpha^2 f^4 w^2 \phi^2}{3rN^2\sigma} + \sigma\left(\frac{2rf'^2}{f^2} + \frac{\alpha^2 f^4 w'^2}{3r}\right)\,,\\
f'' &= -\frac{\alpha^2 f^5 w^2 \phi^2}{3r^2N^2\sigma^2} + \frac{\alpha^2 f^5 w'^2}{3r^2} - f'\left(\frac{3}{r} - \frac{f'}{f} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right)\,,\\
\phi'' &= \frac{f^4 w^2 \phi}{r^2 N} - \phi'\left(\frac{3}{r} - \frac{\sigma'}{\sigma}\right)\,,\\
w'' &= -\frac{w\phi^2}{N^2\sigma^2} - w'\left(\frac{1}{r} + \frac{4f'}{f} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right)\,.
\end{aligned} \qquad (6.8)$$

The equations of motion are invariant under four scaling transformations,

$$\begin{aligned}
&(I) && \sigma \to \lambda\sigma, \quad \phi \to \lambda\phi,\\
&(II) && f \to \lambda f, \quad w \to \lambda^{-2} w,\\
&(III) && r \to \lambda r\,, \quad m \to \lambda^4 m\,, \quad w \to \lambda w\,, \quad \phi \to \lambda\phi,\\
&(IV) && r \to \lambda r\,, \quad m \to \lambda^2 m\,, \quad L \to \lambda L\,, \quad \phi \to \tfrac{\phi}{\lambda}\,, \quad \alpha \to \lambda\alpha,
\end{aligned}$$

where in each case λ is some real positive number. Using (I) and (II) we can set the boundary values of both $\sigma(r)$ and $f(r)$ to one, so that the metric is asymptotically AdS. We are free to use (III) to set r_h to be one, but we retain r_h as a book-keeping device. We use (IV) to set the AdS radius L to one.

A known analytic solution of the equations of motion is an asymptotically AdS Reissner-Nordström black hole, which has $\phi(r) = \mu - q/r^2$, $w(r) = 0$, $\sigma(r) = f(r) = 1$, and $N(r) = \left(r^2 - \frac{2m_0}{r^2} + \frac{2\alpha^2 q^2}{3r^4}\right)$, where $m_0 = \frac{r_h^4}{2} + \frac{\alpha^2 q^2}{3r_h^2}$ and $q = \mu r_h^2$. Here μ is the value of $\phi(r)$ at the boundary[3], which in CFT terms is the $U(1)_3$ chemical potential.

[3]Note that in this chapter μ has a different meaning than in chapter 3, 4 and 5. In particular, it is not related to the non-extremality of a black hole solution.

To find solutions with nonzero $w(r)$ we resort to numerics. We solve the equations of motion using a shooting method. We vary the values of functions at the horizon until we find solutions with suitable values at the AdS boundary. We thus need the asymptotic form of solutions both near the horizon $r = r_h$ and near the boundary $r = \infty$.

Near the horizon, we define $\epsilon_h \equiv \frac{r}{r_h} - 1 \ll 1$ and then expand every function in powers of ϵ_h with some constant coefficients. Two of these we can fix as follows. We determine r_h by the condition $N(r_h) = 0$, which gives that $m(r_h) = r_h^4/2$. Additionally, we must impose $A_t^3(r_h) = \phi(r_h) = 0$ for A to be well-defined as a one-form (see for example [191]). The equations of motion then impose relations among all the coefficients. A straightforward exercise shows that only four coefficients are independent,

$$\left\{\phi_1^h, \sigma_0^h, f_0^h, w_0^h\right\} , \tag{6.9}$$

where the subscript denotes the order of ϵ_h, so, for instance, σ_0^h is the value of $\sigma(r)$ at the horizon. All other near-horizon coefficients are determined in terms of these four.

Near the boundary $r = \infty$ we define $\epsilon_b \equiv \left(\frac{r_h}{r}\right)^2 \ll 1$ and then expand every function in powers of ϵ_b with some constant coefficients. The equations of motion again impose relations among the coefficients. The independent coefficients are

$$\left\{m_0^b, \mu, \phi_1^b, w_1^b, f_2^b\right\} , \tag{6.10}$$

where here the subscript denotes the power of ϵ_b. All other near-boundary coefficients are determined in terms of these.

We used scaling symmetries to set $\sigma_0^b = f_0^b = 1$. Our solutions also have $w_0^b = 0$ since we do not want to source the operator J_1^x in the CFT, the $U(1)_3$ gets *spontaneously* broken). In our shooting method we choose a value of μ and then vary the four independent near-horizon coefficients until we find a solution which produces the desired value of μ and has $\sigma_0^b = f_0^b = 1$ and $w_0^b = 0$.

In what follows, we often work with dimensionless coefficients by scaling out factors of r_h. We thus define the dimensionless functions $\tilde{m}(r) \equiv m(r)/r_h^4$, $\tilde{\phi}(r) \equiv \phi(r)/r_h$ and $\tilde{w}(r) \equiv w(r)/r_h$, while $f(r)$ and $\sigma(r)$ are already dimensionless.

6.3 Thermodynamics

Next, we describe how to extract thermodynamic information from our solutions. Our solutions describe thermal equilibrium states in the dual CFT. We work in the grand canonical ensemble, with fixed chemical potential μ.

We can obtain the temperature and entropy from horizon data. The temperature T is given by the Hawking temperature of the black hole,

$$T = \frac{\kappa}{2\pi} = \frac{\sigma_0^h}{12\pi}\left(12 - \alpha^2 \frac{\left(\tilde{\phi}_1^h\right)^2}{{\sigma_0^h}^2}\right) r_h \,. \tag{6.11}$$

Here $\kappa = \sqrt{\partial_\mu \xi \partial^\mu \xi}\big|_{r_h}$ is the surface gravity of the black hole, with ξ being the norm of the timelike Killing vector. In the second equality we write T in terms of near-horizon coefficients. In what follows we often convert from r_h to T simply by inverting the above equation. The entropy S is given by the Bekenstein-Hawking entropy of the black hole,

$$S = \frac{2\pi}{\kappa_5^2} V_h = \frac{2\pi V}{\kappa_5^2} r_h^3 = \frac{2\pi^4}{\kappa_5^2} V T^3 \frac{12^3 {\sigma_0^h}^3}{\left(12 {\sigma_0^h}^2 - \left(\tilde{\phi}_1^h\right)^2 \alpha^2\right)^3} \,, \tag{6.12}$$

where V_h denotes the area of the horizon and $V = \int \mathrm{d}^3 x$.

The central quantity in the grand canonical ensemble is the grand potential Ω. In AdS/CFT we identify Ω with T times the on-shell bulk action in Euclidean signature. We thus analytically continue to Euclidean signature and compactify the time direction with period $1/T$. We denote the Euclidean bulk action as I and $I_{\text{on-shell}}$ as its on-shell value, and similarly for other on-shell quantities. Our solutions are always static, hence $I_{\text{on-shell}}$ always includes an integration over the time direction, producing a factor of $1/T$. To simplify expressions, we define $I \equiv \tilde{I}/T$. Starting now, we refer to $\tilde{I}$ as the action. $\tilde{I}$ includes a bulk term, a Gibbons-Hawking boundary term, and counterterms,

$$\tilde{I} = \tilde{I}_{\text{bulk}} + \tilde{I}_{\text{GH}} + \tilde{I}_{\text{CT}} \,. \tag{6.13}$$

$\tilde{I}_{\text{bulk}}^{\text{on-shell}}$ and $\tilde{I}_{GH}^{\text{on-shell}}$ exhibit divergences, which are canceled by the counterterms in $\tilde{I}_{\text{CT}}$. To regulate these divergences we introduce a hypersurface $r = r_{bdy}$ with some large but finite r_{bdy}. We always ultimately remove the regulator by taking $r_{bdy} \to \infty$. Using the equations of motion, for our ansatz $\tilde{I}_{\text{bulk}}^{\text{on-shell}}$ is

$$\tilde{I}_{\text{bulk}}^{\text{on-shell}} = \frac{V}{\kappa_5^2} \frac{1}{2f^2} r N \sigma (r^2 f^2)' \Big|_{r=r_{\text{bdy}}} \,. \tag{6.14}$$

For our ansatz, the Euclidean Gibbons-Hawking term is

$$\tilde{I}_{\text{GH}}^{\text{on-shell}} = -\frac{1}{\kappa_5^2} \int \mathrm{d}^3 x \sqrt{\gamma}\, \nabla_M n^M = -\frac{V}{\kappa_5^2} N \sigma r^3 \left(\frac{N'}{2N} + \frac{\sigma'}{\sigma} + \frac{3}{r}\right) \Big|_{r=r_{\text{bdy}}} \,, \tag{6.15}$$

where γ is the induced metric on the $r = r_{bdy}$ hypersurface and $n_M \mathrm{d}x^M = 1/\sqrt{N(r)}\, \mathrm{d}r$ is the outward-pointing normal vector. The only divergence in $\tilde{I}_{\text{bulk}}^{\text{on-shell}} +$

$\tilde{I}^{\text{on-shell}}_{\text{GH}}$ comes from the infinite volume of the asymptotically AdS space, hence, for our ansatz, the only nontrivial counterterm is

$$\tilde{I}^{\text{on-shell}}_{\text{CT}} = \frac{3}{\kappa_5^2}\int \mathrm{d}^3x\sqrt{\gamma} = \frac{3V}{\kappa_5^2} r^3\sqrt{N}\sigma\Big|_{r=r_{\text{bdy}}} . \tag{6.16}$$

Finally, Ω is related to the on-shell action, $\tilde{I}_{\text{on-shell}}$, as

$$\Omega = \lim_{r_{bdy}\to\infty} \tilde{I}_{\text{on-shell}}. \tag{6.17}$$

The chemical potential μ is simply the boundary value of $A^3_t(r) = \phi(r)$. The charge density $\langle J^t_3\rangle$ of the dual field theory can be extracted from $\tilde{I}_{\text{on-shell}}$ by

$$\langle J^t_3\rangle = \frac{1}{V}\lim_{r_{bdy}\to\infty}\frac{\delta\tilde{I}_{\text{on-shell}}}{\delta A^3_t(r_{bdy})} = -\frac{2\pi^3\alpha^2}{\kappa_5^2}T^3\frac{12^3{\sigma^h_0}^3}{\left(12{\sigma^h_0}^2 - (\tilde{\phi}^h_1)^2\alpha^2\right)^3}\,\tilde{\phi}^b_1 . \tag{6.18}$$

Similarly, the current density $\langle J^x_1\rangle$ is

$$\langle J^x_1\rangle = \frac{1}{V}\lim_{r_{bdy}\to\infty}\frac{\delta\tilde{I}_{\text{on-shell}}}{\delta A^1_x(r_{bdy})} = +\frac{2\pi^3\alpha^2}{\kappa_5^2}T^3\frac{12^3{\sigma^h_0}^3}{\left(12{\sigma^h_0}^2 - (\tilde{\phi}^h_1)^2\alpha^2\right)^3}\,\tilde{w}^b_1 . \tag{6.19}$$

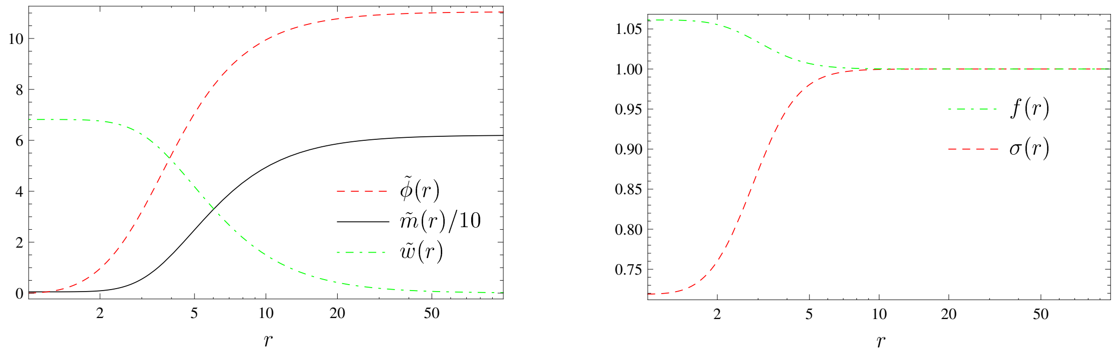
The expectation value of the energy-momentum tensor of the CFT is [69, 77]

$$\langle T_{\mu\nu}\rangle = \lim_{r_{bdy}\to\infty}\frac{2}{\sqrt{\gamma}}\frac{\delta\tilde{I}_{\text{on-shell}}}{\delta\gamma^{\mu\nu}} = \lim_{r_{bdy}\to\infty}\left[\frac{r^2}{\kappa_5^2}\left(-K_{\mu\nu} + K^\lambda{}_\lambda\gamma_{\mu\nu} - 3\,\gamma_{\mu\nu}\right)\right]_{r=r_{\text{bdy}}} , \tag{6.20}$$

where $\mu,\nu,\lambda = \{t,x,y,z\}$ and $K_{\mu\nu} = \frac{1}{2}\sqrt{N(r)}\,\partial_r\gamma_{\mu\nu}$ is the extrinsic curvature. We find

$$\begin{aligned}
\langle T_{tt}\rangle &= 3\frac{\pi^4}{\kappa_5^2}VT^4\frac{12^4{\sigma^h_0}^4}{\left(12{\sigma^h_0}^2 - (\tilde{\phi}^h_1)^2\alpha^2\right)^4}\,\tilde{m}^b_0 ,\\
\langle T_{xx}\rangle &= \frac{\pi^4}{\kappa_5^2}VT^4\frac{12^4{\sigma^h_0}^4}{\left(12{\sigma^h_0}^2 - (\tilde{\phi}^h_1)^2\alpha^2\right)^4}\left(\tilde{m}^b_0 - 8f^b_2\right) ,\\
\langle T_{yy}\rangle = \langle T_{zz}\rangle &= \frac{\pi^4}{\kappa_5^2}VT^4\frac{12^4{\sigma^h_0}^4}{\left(12{\sigma^h_0}^2 - (\tilde{\phi}^h_1)^2\alpha^2\right)^4}\left(\tilde{m}^b_0 + 4f^b_2\right) .
\end{aligned} \tag{6.21}$$

Notice that $\langle T_{tx}\rangle = \langle T_{ty}\rangle = \langle T_{tz}\rangle = 0$. Even in phases where the current $\langle J^x_1\rangle$ is nonzero, the fluid has zero net momentum. Indeed, this result is guaranteed by our ansatz for the metric, which is diagonal.



(a) (b)

Figure 6.1: (a) The dimensionless gauge field components $\tilde{\phi}(r)$ (red dashed) and $\tilde{w}(r)$ (green dot-dashed) and the dimensionless metric function $\tilde{m}(r)$, scaled down by a factor of 10, (black solid) versus the AdS radial coordinate r for $\alpha = 0.316$ at $T \approx 0.45T_c$. (b) The dimensionless metric functions $\sigma(r)$ (red dashed) and $f(r)$ (green dot-dashed) versus the AdS radial coordinate r for $\alpha = 0.316$ at $T \approx 0.45T_c$.

For $\tilde{m}_0^b = \frac{1}{2} + \frac{\alpha^2\tilde{\mu}^2}{3}$, $\sigma_0^h = 1$, $\tilde{\phi}_1^h = 2\tilde{\mu}$, $f_2^b = 0$, and $\tilde{\phi}_1^b = -\tilde{\mu}$ we recover the correct thermodynamic properties of the Reissner-Nordström black hole, which preserves the $SO(3)$ rotational symmetry. For example, we find that $\langle T_{xx}\rangle = \langle T_{yy}\rangle = \langle T_{zz}\rangle$ and $\Omega = -\langle T_{yy}\rangle$. For solutions with nonzero $\langle J_1^x\rangle$, the $SO(3)$ is broken to $SO(2)$. In these cases, we find that $\langle T_{xx}\rangle \neq \langle T_{yy}\rangle = \langle T_{zz}\rangle$. Just using the equations above, we also find $\Omega = -\langle T_{yy}\rangle$. In the superfluid phase, both the nonzero $\langle J_1^x\rangle$ and the energy-momentum tensor indicate breaking of $SO(3)$.

Tracelessness of the energy-momentum tensor implies $\langle T_{tt}\rangle = \langle T_{xx}\rangle + \langle T_{yy}\rangle + \langle T_{zz}\rangle$, which is indeed true for eq. (6.21), so in the dual CFT we always have a conformal fluid. The only physical parameter in the CFT is thus the ratio μ/T.

6.4 Phase transitions

In this section, we present our numerical results. We scanned through values of α from $\alpha = 0.032$ to $\alpha = 0.548$. Typical solutions for the metric and gauge field functions appear in figure 6.1. The solutions for other values of α are qualitatively similar. Notice that all boundary conditions are met: at the horizon $\tilde{\phi}(r)$ vanishes, and at the boundary $f_0^b = \sigma_0^b = 1$ and $\tilde{w}_0^b = 0$.

For every value of α that we use, we find Reissner-Nordström solutions for all temperatures, and for sufficiently low temperatures we always find additional solutions, with nonzero $w(r)$, that are thermodynamically preferred to the Reissner-Nordström solution. In other words, for every value of α that we use, we find a phase transition, at some temperature T_c, in which a charged black hole grows vector hair, which in the CFT is a p-wave superfluid phase transition. Our nu-

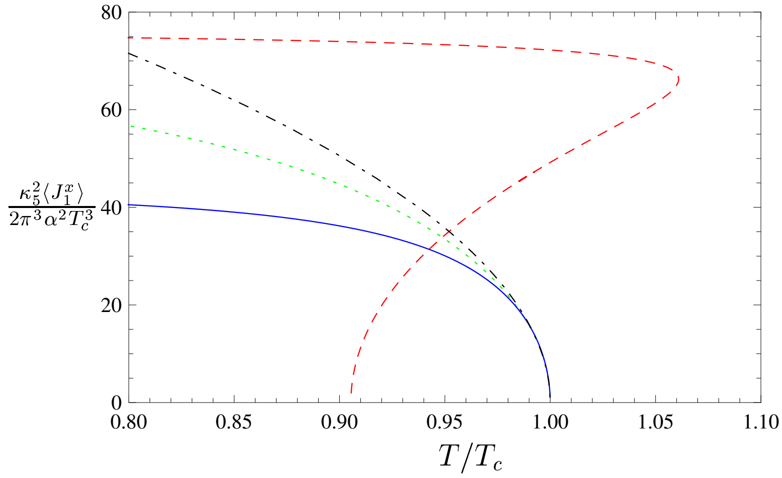
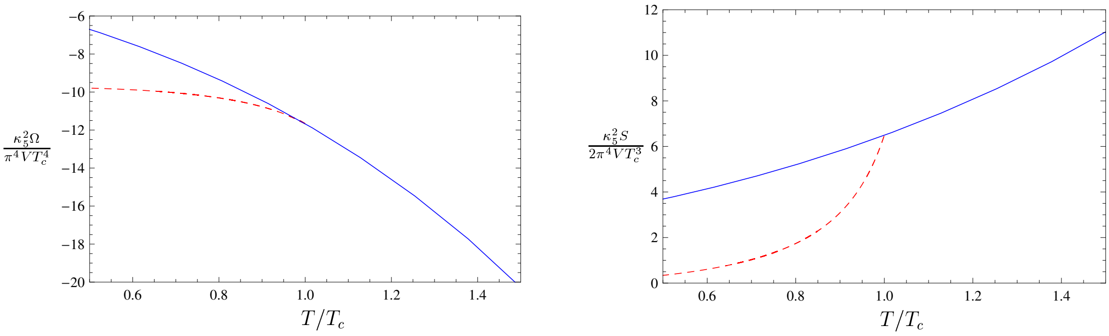


Figure 6.2: The order parameter $\langle J_1^x\rangle$, multiplied by $\kappa_5^2/(2\pi^3\alpha^2T_c^3)$, versus the rescaled temperature T/T_c for different α: $\alpha = 0.032 < \alpha_c$ (green dotted), $\alpha = 0.316 < \alpha_c$ (blue solid) and $\alpha = 0.447 > \alpha_c$ (red dashed). The black dot-dashed curve is the function $a(1-T/T_c)^{1/2}$ with $a = 160$. The green dotted curve is scaled up by a factor of 8 while the red dashed curve is scaled down by a factor of 5 such that a, which depends on α, coincides for the green dotted and blue solid curves. If we decrease T toward T_c, entering the figure from the right, we see that the blue solid and the green dotted curves rise continuously and monotonically from zero at $T = T_c$, signaling a second-order phase transition. The close agreement with the black dot-dashed curve suggests that these grow from zero as $(1-T/T_c)^{1/2}$. In the $\alpha = 0.447$ case, the red dashed curve becomes multi-valued at $T = 1.061\,T_c$. In this case, at $T = T_c$, the value of $\kappa_5^2\langle J_1^x\rangle/(2\pi^3\alpha^2T_c^3)$ jumps from zero to the upper part of the red dashed curve, signalizing a first-order transition.

merical results show that the phase transition is second order for $\alpha < \alpha_c$ and first order for $\alpha > \alpha_c$ where $\alpha_c \approx 0.365 \pm 0.001$.

For example, for $\alpha = 0.316 < \alpha_c$, we only find solutions with $\langle J_1^x\rangle = 0$ until a temperature T_c where a second set of solutions, with nonzero $\langle J_1^x\rangle$, appears. Figure 6.2 shows that $\langle J_1^x\rangle$ rises continuously from zero as we decrease T below T_c. Figure 6.4 (a) shows the grand potential Ω, divided by $\pi^4VT_c^4/\kappa_5^2$, versus the rescaled temperature T/T_c for $\alpha = 0.316$. The blue solid curve in figure 6.4 (a) comes from solutions with $\langle J_1^x\rangle = 0$ and the red dashed curve comes from solutions with $\langle J_1^x\rangle \neq 0$. We see clearly that for $T < T_c$ the states with $\langle J_1^x\rangle \neq 0$ have the lower $\kappa_5^2\Omega/(\pi^4VT_c^4)$ and hence are thermodynamically preferred. We thus conclude that a phase transition occurs at $T = T_c$. The nonzero $\langle J_1^x\rangle$ indicates spontaneous breaking of $U(1)_3$ and of $SO(3)$ rotational symmetry down to $SO(2)$, and hence is an order parameter for the transition. Figure 6.4 (b) shows the entropy S, divided by $2\pi^4VT_c^3/\kappa_5^2$, versus the rescaled temperature T/T_c for $\alpha = 0.316$. The blue solid curve and the red dashed curve have the same meaning as in figure



(a) (b)

Figure 6.3: (a) $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ versus the rescaled temperature T/T_c for $\alpha = 0.316$. The blue solid curve comes from solutions with $\langle J_1^x \rangle = 0$ while the red dashed curve comes from solutions with nonzero $\langle J_1^x \rangle$. For $T > T_c$, we have only the blue curve, but when $T \leq T_c$ the red dashed curve appears and has the lower $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$, indicating a phase transition at $T = T_c$. $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ is continuous and differentiable at $T = T_c$. (b) $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ versus T/T_c for $\alpha = 0.316$. The blue solid and red dashed curves have the same meanings as in (a). $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ is continuous but not differentiable at $T = T_c$, indicating a second-order transition.

6.4 (a). Here we see that $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ is continuous but has a kink, *i.e.* a discontinuous first derivative, clearly indicating a second-order transition. For other values of $\alpha < \alpha_c$, the figures are qualitatively similar.

A good question concerning these second-order transitions is: what are the critical exponents? In the probe limit, $\alpha = 0$, an analytic solution for the gauge fields exists for T near T_c [185], which was used in [57] to show that for $T \lesssim T_c$, $\langle J_1^x \rangle \propto (1 - T/T_c)^{1/2}$. In other words, in the probe limit the critical exponent for $\langle J_1^x \rangle$ takes the mean-field value 1/2. Does increasing α change the critical exponent? Our numerical evidence suggests that the answer is no: for all $\alpha < \alpha_c$, we appear to find $\langle J_1^x \rangle \propto (1 - T/T_c)^{1/2}$ (see figure 6.2).

As α increases past $\alpha_c = 0.365 \pm 0.001$, we see a qualitative change in the thermodynamics. Consider for example $\alpha = 0.447$. Here again we only find solutions with $\langle J_1^x \rangle = 0$ down to some temperature where *two* new sets of solutions appear, both with nonzero $\langle J_1^x \rangle$. In other words, three states are available to the system: one with $\langle J_1^x \rangle = 0$ and two with nonzero $\langle J_1^x \rangle$. Figure 6.2 shows that as we cool the system, $\langle J_1^x \rangle$ becomes multi-valued at $T = 1.061\, T_c$. To determine which state is thermodynamically preferred, we compute the grand potential Ω. Figure 6.4 (a) shows $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ versus T/T_c. The blue solid curve and the red dashed curve have the same meanings as in figure 6.4. We immediately see the characteristic “swallowtail” shape of a first-order phase transition. If we decrease T, entering the figure along the blue solid curve from the right, we reach the

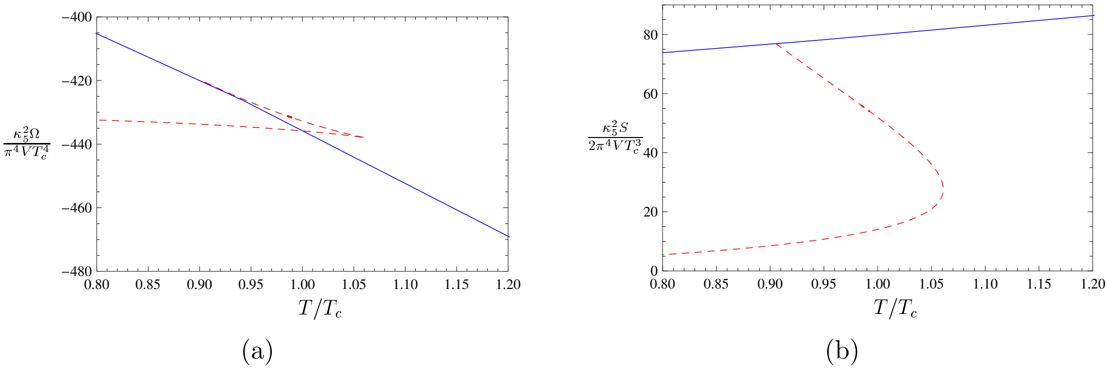


Figure 6.4: (a) $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ versus the rescaled temperature T/T_c for $\alpha = 0.447$. The blue solid and red dashed curves have the same meanings as in figure 6.3. For $T > T_c$ we have only the blue solid curve. At $T = 1.061\,T_c$, the red dashed curve appears and $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ becomes multi-valued. When $T \leq T_c$ the red dashed curve has the lowest $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$, indicating a phase transition at $T = T_c$. $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ is continuous but not differentiable at $T = T_c$, signalizing a first-order transition. (b) $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ versus T/T_c for $\alpha = 0.447$. $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ is not continuous at $T = T_c$, but rather jumps from the blue solid curve to the lowest branch of the red dashed curve, indicating a first-order transition.

temperature $T = 1.061\,T_c$ where the new solutions appear (as the red dashed curve). The blue solid curve still has the lowest $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ until $T = T_c$. If we continue reducing T below T_c, then the red curve has the lowest $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$. The transition is clearly first order: $\kappa_5^2\Omega/\left(\pi^4 V T_c^4\right)$ has a kink at $T = T_c$. We can also see from the entropy that the transition is first order. Figure 6.4 (b) shows $\kappa_5^2 S/\left(2\pi^4 V T_c^3\right)$ versus T/T_c. The entropy, like the grand potential, is multi-valued, and jumps discontinuously from the blue solid curve to the lowest part of the red dashed curve at $T = T_c$, indicating a first-order transition.

Notice that a crucial difference between $\alpha < \alpha_c$ (second order) and $\alpha > \alpha_c$ (first order) is that for $\alpha > \alpha_c$ the critical temperature T_c is not simply the temperature at which $\langle J_1^x \rangle$ becomes nonzero. We need more information to determine T_c when $\alpha > \alpha_c$, for example we can study Ω.

A good question is: how does increasing α change T_c? Figure 6.5 depicts the phase diagram[4] of the p-wave superfluid showing α^2 as a function of the temperature T/μ. The blue curve divides the diagram into a lower left (blue-shaded area) and an upper right (uncoloured and red-shaded area) part. In the lower left part, the Reissner-Nordström solution is unstable. In the upper right part, the Reissner-Nordström solution is metastable in the red-shaded area and stable in

[4]This phase diagram is part of the work subsequent to [3] which was done in collaboration with Martin Ammon, Johanna Erdmenger, Patrick Kerner and Andy O'Bannon. The phase diagram of the p-wave superfluid was first presented in [192].

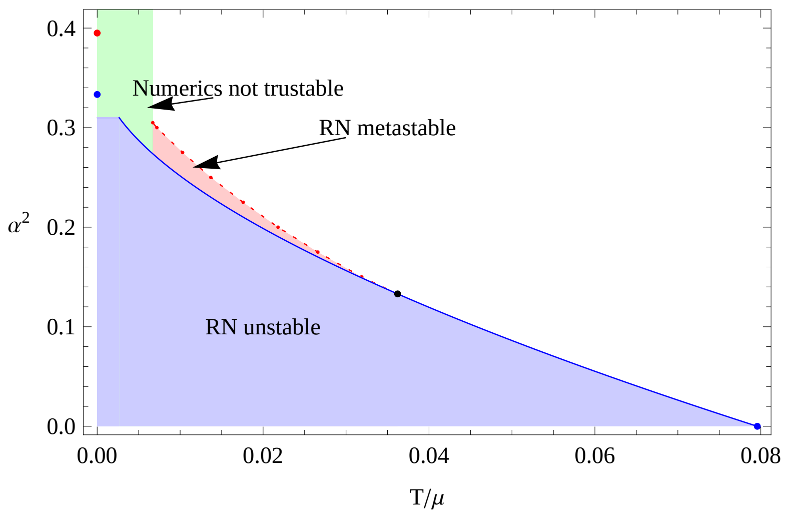


Figure 6.5: The phase diagram of the p-wave superfluid showing α^2 as a function of T/μ. The superfluid transition occurs at $T_c/\mu = 1/4\pi \approx 0.796$ [57]. Increasing α decreases T_c/μ. For $\alpha^2 < \alpha_c^2 \approx 0.133$ the transition is second order and coincides with the occurrence of the instability (blue curve). For $\alpha^2 < \alpha_c^2 \approx 0.133$ the transition is first order (red dotted curve) and occurs at a higher temperature than the instability (blue curve). At zero temperature, the transition occurs at $\alpha^2 \approx 0.394$ (red dot) and is still first order as was shown in [192], and the instability occurs at $\alpha^2 \approx 1/3$ (blue dot) as was shown in [189].

the uncoloured area. In the probe limit, $\alpha = 0$, the superfluid phase transition sets in at $T_c/\mu = 1/4\pi \approx 0.796$ [57]. Then, increasing α further decreases the temperature T_c/μ. As long as $\alpha^2 < \alpha_c^2 \approx 0.133$, the transition is second order and coincides with the occurrence of the instability, whereas for $\alpha^2 > \alpha_c^2 \approx 0.133$ the transition is first order (red dotted curve) and sets in at a higher temperature than the instability. For $\alpha^2 \approx 0.394$, the transition occurs at zero temperature and is still first order which is in accordance with [192]. The instability at zero temperature occurs for $\alpha^2 \approx 1/3$ which is in accordance with [189]. In the green shaded area our numerics is not trustable.

6.5 Discussion

We studied asymptotically AdS charged black holes in five-dimensional $SU(2)$ Einstein-Yang-Mills theory with finite $\alpha = \kappa_5/\hat{g}$, that is, with back-reaction of the gauge fields. Our numerical solutions show that, for a given value of α, as the temperature decreases the black holes grow vector hair. Via AdS/CFT, this process appears as a phase transition to a p-wave superfluid state in a strongly-coupled CFT. We have shown that the order of the phase transition depends on the value of α: for values below $\alpha_c = 0.365 \pm 0.001$, the transition is second order, while for larger values the transition is first order. Moreover, the temperature T_c/μ for which a phase transition occurs seems to decrease down to $T = 0$ as α is increased. This is in accordance with the results of [192]. The second-order phase transition occurs due to an instability of the gauge field in the Reissner-Nordström background [56]. This instability only exists for $\alpha < 1/\sqrt{3}$ as found in [189]. For $\alpha > 1/\sqrt{3}$ and $T = 0$ the Reissner-Nordström solution becomes metastable, and hence a phase transition occurs at temperatures above those where the instability appears. Above $\alpha > 0.628$ there is no phase transition and the Reissner-Nordström solution is stable.

The superfluid condensate, which is given by the $SU(2)$ current $\langle J_x^1 \rangle$, should not be interpreted as a current of massive charge carriers, since the superfluid has zero spatial momentum. We might speculate and possibly think of $\langle J_x^1 \rangle$ as the current of constant spin changes along the x-direction similar as the varying alignment of magnetic moments in spiral magnets.

As we mentioned in section 6.1, intuitively we may think of increasing α as increasing the ratio of charged degrees of freedom to total degrees of freedom in the CFT. To make that intuition precise, we can consider a specific system. One string theory realization of $SU(2)$ gauge fields in AdS space is type IIB supergravity in five-dimensional AdS space times a five-sphere plus two coincident D7-branes that provide the $SU(2)$ gauge fields [184–187]. The dual field theory is $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory[5], in the limits of large N_c and large 't Hooft coupling, coupled to a number $N_f = 2$ of massless $\mathcal{N} = 2$ supersymmetric hypermultiplets in the N_c representation of $SU(N_c)$, *i.e.* flavor fields. The global $SU(N_f) = SU(2)$ is an isospin symmetry. Translating from gravity to field theory quantities, we have $1/\kappa_5^2 \propto N_c^2$ and $1/\hat{g}^2 \propto N_f N_c$, hence $\alpha \propto \sqrt{N_f/N_c}$, which supports our intuition. We must be cautious, however. In the field theory, the probe limit consists in neglecting quantum effects due to the flavor fields because these are suppressed by powers of N_f/N_c. If N_f/N_c becomes finite, then, for example, in the field theory the coupling would run, the dual statement being that in type IIB supergravity the dilaton would run, which is an effect absent in our model. We should not draw too close an analogy between our simple model and this particular string theory system.

[5]Here the index c in $SU(N_c)$ stands for color.

Chapter 7

Summary of results and conclusion

The aim of the research presented in this thesis was to be suggestive of the broad range of applications of the AdS/CFT correspondence. For that purpose the AdS/CFT correspondence was applied to three different strongly-coupled systems at finite temperature and finite charge density: a brane world, a fluid on a three-sphere and a p-wave superfluid. These three applications were investigated in detail in view of specific questions. This uncovered interesting new insights into these systems as well as into the physics of the dual black holes. In addition, these applications simultaneously functioned as tests of the usability of the AdS/CFT correspondence.

To investigate strongly-coupled systems on the field theory side we always started on the gravity side and used the supergravity set up to substitutionally carry out computations. The results of these computations were then translated to the field theory side. This reveals that a broad knowledge of black hole physics, in particular the knowledge of various black hole solutions and their thermodynamic properties, is essential for the work with holographic techniques. Since we strongly made use of the non-extremal charged static black hole solutions (3.34) of $\mathcal{N} = 2$ gauged supergravity in chapter 4 and 5, in chapter 3 we first presented some background material and then studied these solutions in more detail. Surprisingly, we found out that the gauged $\mathcal{N} = 2$ supergravity action admits a rewriting in terms of squares of first-order differential (flow) equations in the context of the non-extremal solutions (3.34). Such a rewriting is well-known from the attractor mechanism of extremal black holes.

In chapter 4, the non-extremal charged static black hole background (3.34) was employed to describe the cosmological evolution of a spherical brane world. The total energy on the brane receives a subextensive Casimir contribution due to finite-size effects. This additional energy contribution enables to express the entropy of the brane as a Cardy-Verlinde-type formula. We then investigated a possible merging of the two Friedmann equations, governing the brane evolution, with the Cardy-Verlinde-type formula and the Casimir energy on the brane. In the last section, we investigated whether this merging can also be observed when the bulk theory is modified by a Gauss-Bonnet curvature term.

In chapter 5, a conformal fluid dual to the non-extremal charged static black hole solution (3.34) was investigated assuming that it propagates on a three-sphere. The total energy of the fluid again receives a Casimir contribution. We investigated the question whether this Casimir energy could lead to a correction of the ratio η/s. For that purpose, we constructed different new deformed black hole solutions on the basis of special cases of the general static black hole solution (3.34). Then, we computed the boundary energy-momentum tensors corresponding to these solutions and extracted the shear viscosity from their derivative expansions.

In chapter 6, we modeled a holographic p-wave superfluid and investigated its phase structure. For that purpose, we considered an $SU(2)$ Einstein-Yang-Mills system and we numerically constructed new non-Abelian AdS black hole solutions including the back-reaction of the gauge fields. The strength of the back-reaction depends on the parameter $\alpha = \kappa_5/\hat{g}$. We investigated the superfluid phase transition by scanning through different values of α.

For the three applications, the AdS/CFT correspondence seems to provide reasonable results. The main new results of the chapters 3 to 6 are listed below.

Chapter 3: First-order flow equations for non-extremal black holes

- In the context of the non-supersymmetric non-extremal electrically charged static black hole solutions (3.34), the $\mathcal{N} = 2$ gauged supergravity action admits a rewriting in terms of first-order differential (flow) equations. These are consistent with the second-order equations of motion. The black hole solutions (3.34) can thus be alternatively derived by solving the first-order equations.

Chapter 4: Holographic brane cosmology and thermodynamics

- The equations of motion of the spherical brane in the background of the charged static black hole with the line element (3.34) of $\mathcal{N} = 2$ gauged supergravity are the Friedmann (FRW) equations for a closed universe. The effective energy density and the effective pressure entering the Friedmann equations show a very complicated behavior under the rescaling of the cosmic scale factor. This indicates that the matter content represented by the brane field theory is not of the standard form, but of some exotic type.

- The entropy on the brane can be written as a Cardy-Verlinde-type formula which differs from the original Cardy-Verlinde formula by the appearance of two functions in the prefactor, namely the superpotentials which depend on the scalar fields supporting the black hole solution. The generalized Casimir energy as well as the generalized extensive energy also contain these functions.

- The first and the second FRW equation take forms that are similar to the Cardy-Verlinde-type formula for the entropy and the equation for the Casimir energy, respectively. At the horizon, these two sets of equations even coincide. This phenomenon was first discovered by Verlinde in [36] for a brane in a Schwarzschild black hole background. Here we have shown that this phenomenon still exists for a brane in the background of a general charged black hole of $\mathcal{N} = 2$ gauged supergravity.

- Including a Gauss-Bonnet curvature term in the five-dimensional bulk action produces an infinite series of subextensive contributions in the total energy on the brane. In this case, it is unclear whether the entropy can be written at all as a Cardy-Verlinde formula.

Chapter 5: Fluid dynamics on the three-sphere from gravity

- New deformed regular black hole solutions with spherical horizons were constructed on the basis of special cases of the charged static black hole solutions (3.34), namely AdS-Schwarzschild, AdS-Maxwell, the two-charge AdS black hole and the one-charge AdS black hole. All these special solutions were found to be dual to special incompressible viscous fluids, in the large N, large λ limit, propagating on the three-sphere. Moreover, we computed the energy-momentum tensors of these fluids.

- For all these fluids we found that the ratio η/s receives a positive correction Δ proportional to the curvature of the three-sphere. The correction corresponds to a third-order term in the derivative expansion of the fluid's energy-momentum tensor. In case of constant scalar fields, it amounts to $\Delta = (1/4\pi)(k/a_h^2)$, in which $\hbar = k_B = 1$. If the scalars are non-trivial, the correction becomes $\Delta = (1/4\pi)(k/a_h^2)\mathrm{e}^{-U(a_h)}$ in the two-charge case and $\Delta = (1/4\pi)(k/a_h^2)\mathrm{e}^{2U(a_h)}$ in the one-charge case where $\mathrm{e}^{U(a)}$ is positive definite and a_h denotes the horizon radius.

Chapter 6: A holographic p-wave superfluid with back-reaction

- New non-Abelian AdS black hole solutions were numerically constructed including the full back-reaction of the gauge fields. We showed that for a given value of α and sufficiently low temperature these black hole solutions develop vector hair. On the dual field theory side, this process corresponds to a phase transition to a p-wave superfluid state.

- The superfluid condensate is represented by the vev of the $SU(2)$ current $\langle J_x^1 \rangle$ which breaks the rotational symmetry in the superfluid phase. Thus, the x-direction is singled out as the direction in which superfluidity occurs.

- The order of the superfluid transition was found to be second order for $\alpha < \alpha_c = 0.365 \pm 0.001$ and first order for $\alpha > \alpha_c$. Moreover, the critical temperature T_c/μ, at which a phase transition occurs, seems to decrease down to $T = 0$ as α is increased which is in accordance with the results of [192].

Let us comment on these results in the following paragraphs. The fact that *first-order flow equations* exist for non-extremal black holes is rather surprising, since such flow equations were first thought to be strictly connected to the attractor mechanism of extremal black holes. However, non-extremal black holes cannot be attractive [32,33], since the distance to the horizon, which is covered by the scalar fields, is finite leaving not enough "time" to forget about the "initial" conditions at infinity. In contrast, for extremal black holes, the distance to the horizon is infinite, because of the infinite throat geometry which guarantees that the information about the values of the scalar fields at infinity gets lost. Thus, the existence of first-order flow equations is not necessarily tied to attractor behavior. A striking advantage of the first-order equations is that they are easier to solve than the second-order equations of motion.

The merging of the FRW equations with the Cardy-Verlinde formula and the equation for the Casimir energy of the *brane world* can be understood as follows. First, the right hand side of the first FRW equation written in the form (4.64) is already similar to the right hand side of the Cardy-Verlinde formula. The left hand side is actually proportional to the Hubble entropy $S_H = (VH)/(2G_4)$ [142] for a closed universe which was defined in the context of the pre-Big-Bang scenario [143]. Here $V = V_H n_H$ and V_H denotes the volume of a causally connected Hubble region and n_H is the number of Hubble regions in the universe. The Hubble entropy thus establishes the connection between entropy and Hubble parameter. At the horizon of the five-dimensional black hole, where $V = V_h$ with V_h being the area of the black hole horizon, the Hubble entropy of the brane universe then becomes equal to the Bekenstein-Hawking entropy of the five-dimensional black hole which is, according to the AdS/CFT correspondence, simultaneously the entropy of the brane field theory. Second, the connection between the equation for the Casimir energy (4.52) and the second FRW equation is essentially established through the Hawking temperature which can be expressed in terms of the Hubble parameter and its time derivative. It was suggested in [36] that this merging exists independently of the kind of matter contained in the brane universe. Our analysis further supports this idea. In case of higher-derivative gravity in the five-dimensional bulk theory, it turned out to be difficult to write the entropy of the brane field theory as a Cardy-Verlinde-type formula, since an infinite series of subextensive energy contributions appears in the total energy. Thus, for the future, it remains to clarify whether the entropy can be written at all as a Cardy-Verlinde formula and whether, if this is the case, this finally still merges with the first FRW equation at the black hole horizon.

The corrections to the shear viscosity to entropy density ratio η/s, that were obtained for different *fluids on a three-sphere* dual to AdS-STU black hole solutions, all turned out to be positive. This is in accordance with the lower bound on η/s which was conjectured in [48] to be universal for all fluids. These corrections stem from a third-order term in the derivative expansion of the energy-momentum tensors of the fluids and are all proportional to the curvature of the three-sphere. In the Schwarzschild and Maxwell case these seem to be caused by a subextensive Casimir contribution in the total energy of the fluid, which is a finite-size effect. However, in the presence of scalar fields additional effects, are involved, since the corrections take forms that are not purely related to the Casimir energy. An additional effect might be possible higher-order corrections to the shear diffusion constant, whose form is not known yet and which we therefore neglected in our considerations in section 5.3.3. Thus, maybe a combination of this effect together with the Casimir energy leads to the correction of η/s when scalar fields are involved.

The fluid flow (5.39) on the three-sphere that we constructed by means of the AdS/CFT correspondence arose as a very special solution of some components of the Einstein equations and depends only on the angular coordinate θ. It would be interesting to solve the Einstein equations for a more general fluid flow on the three-sphere which might depend on time and all spatial coordinates and which might not be restricted to incompressibility and small amplitudes. Nevertheless, the correction to η/s should be independent of the specific form of the fluid flow.

The description of the *p-wave superfluid* constructed here by means of the AdS/CFT correspondence can be viewed as an effective description comparable with Landau-Ginzburg theory which describes the macroscopic phenomena of superconductivity near the superconducting phase transition without explaining the microscopic degrees of freedom. As in this theory, we can deal with currents and order parameters, but we cannot reveal the pairing mechanism or the Lagrangian for the degrees of freedom that form the Cooper pairs.

The superfluid condensate, which is given by the $SU(2)$ current $\langle J_x^1 \rangle$, should not be interpreted as a current of massive charge carriers, since the superfluid has zero spatial momentum. We might speculate and possibly think of $\langle J_x^1 \rangle$ as the current of constant spin changes along the superfluid x-direction similar as the periodically varying alignment of magnetic moments in spiral magnets.

With our fully back-reacted solution at hand, many questions still remain to investigate. For instance, it would be interesting to study the transport properties of the dual conformal fluid, for example the electrical conductivity, which at zero temperature should exhibit a “hard gap”, as explained in [189]. Moreover, we could turn on superfluid velocities and study the fluid's response. In similar systems, sufficiently large superfluid velocities also changed the transition from second to first order [193, 194]. A further question is, what is the speed of sound. In a p-wave superfluid this question is not so easy to answer, since the speed of sound need not be the same in all directions due the broken rotational symmetry. In addition, we could study the speeds of second and fourth sounds [57, 195, 196].

In conclusion, this thesis has shown that the AdS/CFT correspondence usefully applies to various systems at strong coupling. While the first application dealing with a brane universe in the context of the AdS/CFT correspondence is more of theoretical nature, the other two applications, in some sense are relevant for experiments. Although the systems which can be described by the AdS/CFT correspondence are all more or less toy-models of realistic systems, there seem to exist properties, as for instance the ratio η/s, which are universal for a larger class of strongly-coupled systems and which therefore can be investigated by holographic techniques. For the future, it would be desirable to discover further universal quantities, for instance a quantity related to condensed matter physics, which can be explored by the AdS/CFT correspondence. Nevertheless, already at the present stage, the AdS/CFT correspondence has achieved what could not be done before. It provides a tool to calculate non-equilibrium observables in strongly-coupled systems.

Appendix A

Notation and conventions

In this thesis the following notation and conventions are employed. The four-dimensional Minkowski metric is taken to be

$$\eta_{\mu\nu} = \mathrm{diag}(-1, \eta_{ij}) = \mathrm{diag}(-1, 1, 1, 1)\,, \tag{A.1}$$

where $\mu, \nu = \{0, ..., 3\}$ are four-dimensional Lorentz indices, and $i, j = \{1, 2, 3\}$ denote spatial directions.

The metric of a three-dimensional space of constant curvature is generally denoted by η^k_{ij}. This includes three-dimensional flat space ($k = 0$), three-dimensional hyperbolic space ($k = -1$) and the unit three-sphere ($k = 1$).

The four-dimensional AdS boundary metric is denoted by

$$g_{\mu\nu} = \mathrm{diag}(-1, g_{ij})\,, \tag{A.2}$$

where μ, ν are spacetime indices tangential to the boundary and i, j denote the corresponding spatial directions. The induced metric on a four-dimensional slice tangential to the AdS boundary is denoted by $\gamma_{\mu\nu}$.

The five-dimensional spacetime metric is denoted by G_{MN} where $M, N = \{0, ..., 4\}$ are five-dimensional Lorentz indices.

The ten-dimensional spacetime metric occurring only in chapter 2 and the eleven-dimensional spacetime metric occurring only in chapter 3 are both denoted by $G_{\hat{\mu}\hat{\nu}}$ where $\hat{\mu}, \hat{\nu}$ run from 0 to 9 and 0 to 10, respectively.

The target space metric of the moduli fields φ^i occurring in chapter 3 is denoted by $\mathcal{G}_{ij}$ with $i, j = \{1, ..., h_{1,1} - 1\}$. The target space metric of the scalar fields X^A is denoted by $\mathcal{G}_{AB}$ with $A, B = \{1, ..., h_{1,1}\}$.

The totally antisymmetric Levi-Civita tensor, whose components are $\pm\sqrt{|G|}$ or 0, is given by

$$\epsilon_{\mu_1 \dots \mu_d} = \sqrt{|G|}\varepsilon_{\mu_1 \dots \mu_d} \,, \tag{A.3}$$

where $\varepsilon_{\mu_1 \dots \mu_d}$ is the totally antisymmetric Levi-Civita tensor density with

$$\varepsilon_{\mu_1 \dots \mu_d} \equiv (+1, \, -1, \, 0) \,, \tag{A.4}$$

depending on whether $\mu_1 ... \mu_d$ is an even permutation of the canonically-ordered set of index values, an odd permutation or no permutation at all.

A p-form A is defined as

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \,. \tag{A.5}$$

The Hodge $\star$ operator transforms a p-form A into a $q = (d-p)$-form B according to

$$B = \star A = \frac{1}{p!q!} \epsilon_{\nu_1 \dots \nu_q}{}^{\mu_1 \dots \mu_p} A_{\mu_1 \dots \mu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_q} \,. \tag{A.6}$$

Appendix B

Rotating AdS-STU black hole solutions

B.1 Rotating Maxwell black hole in Eddington-Finkelstein-type coordinates

The general non-extremal rotating black hole solution in minimal five-dimensional gauged supergravity was derived in [171] in Boyer–Lindquist type coordinates. At linear order in angular velocities $\epsilon\,\omega_1$ and $\epsilon\,\omega_2$ and with $w_5 = L = k = 1$

$$
\begin{aligned}
ds^2 =& \left(-(1+a^2) + \frac{\Sigma}{a^4}\right) dt^2 + \frac{a^2}{\Delta_a} da^2 + a^2 d\theta^2 + a^2 \left(\sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2\right) \\
& - \frac{2}{a^4}\left(\epsilon\,\omega_2\,\Sigma + \epsilon\,\omega_1\,Q\,a^2\right)\cos^2\theta\, d\psi\, dt - \frac{2}{a^4}\left(\epsilon\,\omega_1\,\Sigma + \epsilon\,\omega_2\,Q\,a^2\right)\sin^2\theta\, d\phi\, dt\ , \\
A =& \frac{\sqrt{3}\,Q}{a^2}\left(dt - \epsilon\,\omega_1\,\sin^2\theta\, d\phi - \epsilon\,\omega_2\,\cos^2\theta\, d\psi\right)\ ,
\end{aligned}
\tag{B.1}
$$

where

$$
\Delta_a = a^2(1+a^2) + \frac{Q^2}{a^2} - M \quad , \quad \Sigma = M\,a^2 - Q^2\ . \tag{B.2}
$$

The line element in (B.1) can be rewritten in terms of Eddington-Finkelstein-type coordinates by applying the following transformations,

$$
dt \to dt - \frac{a^2}{\Delta_a}\, da\ , \quad d\phi \to d\phi - \frac{\epsilon\,\omega_1}{\Delta_a}\left(1+a^2\right) da\ , \quad d\psi \to d\psi - \frac{\epsilon\,\omega_2}{\Delta_a}\left(1+a^2\right) da\ . \tag{B.3}
$$

Then, at first order in ϵ, the line element becomes

$$\begin{aligned} ds^2 = & -\frac{\Delta_a}{a^2}\,dt^2 + a^2\,d\Omega_3^2 + 2\,dt\,da \\ & + \frac{2\,\epsilon}{a^4}\left(\omega_1\,Q^2 - \omega_1\,M\,a^2 - \omega_2\,Q\,a^2\right)\sin^2\theta\,dt\,d\phi \\ & + \frac{2\,\epsilon}{a^4}\left(\omega_2\,Q^2 - \omega_2\,M\,a^2 - \omega_1\,Q\,a^2\right)\cos^2\theta\,dt\,d\psi \\ & + 2\,\epsilon\,\sin^2\theta\left(\frac{\omega_2\,Q}{\Delta_a} - \omega_1\right)da\,d\phi + 2\,\epsilon\,\cos^2\theta\left(\frac{\omega_1\,Q}{\Delta_a} - \omega_2\right)da\,d\psi\,, \end{aligned} \tag{B.4}$$

while the gauge field is still given by (B.1). Rewriting the five-dimensional line element (B.4) in terms of the four-dimensional quantities $u^\mu = (\,1,\,0,\,\epsilon\,\omega_1, \epsilon\,\omega_2)$, $l^\mu = (\,0,\,0,\,-\epsilon\,\omega_2,\,-\epsilon\,\omega_1)$ and $g_{\mu\nu} = \mathrm{diag}(\,-1,\,1,\,\sin^2\theta,\,\cos^2\theta)$ yields the line element (5.51) with $\sigma_{\mu\nu} = 0$ and $\kappa = -1/(2\sqrt{3})$.

B.2 A two-charge rotating STU black hole in Eddington-Finkelstein-type coordinates

A rotating version of the static two-charge STU black hole solution (5.66) has been constructed in [173]. At linear order in rotation parameters $\epsilon\,\omega_1$ and $\epsilon\,\omega_2$, it reads

$$\begin{aligned} ds^2 =& H^{-\frac{4}{3}}\left[-\frac{X}{r^2}\,dt^2 + \frac{2\,\epsilon}{r^2}\left(X - \frac{f_3}{r^2}\right)\left(\omega_1\,\sin^2\theta\,dt\,d\phi + \omega_2\,\cos^2\theta\,dt\,d\psi\right)\right. \\ & \left. + \,\frac{f_3^2}{r^6}\left(\sin^2\theta\,d\phi^2 + \cos^2\theta\,d\psi^2\right)\right] + H^{\frac{2}{3}}\left[\frac{r^2}{X}\,dr^2 + r^2\,d\theta^2\right]\,, \\ H =& 1 + \frac{\mu\,s^2}{r^2}\,, \\ X =& r^2 - \mu + \mathfrak{g}^2\,(r^2 + \mu\,s^2)^2\,, \\ f_3 =& r^4 + \mu\,s^2\,r^2\,, \end{aligned} \tag{B.5}$$

where[1]

$$s = \sinh\,\delta\,, \qquad c = \cosh\,\delta\,. \tag{B.6}$$

The associated gauge potentials are

$$\begin{aligned} A^1 =& A^2 = \frac{\mu\,s\,c}{r^2 H}\left(dt - \epsilon\,(\,\omega_1\,\sin^2\theta\,d\phi + \omega_2\,\cos^2\theta\,d\psi)\right)\,, \\ A^3 =& \frac{\mu\,s^2}{r^2}\,\epsilon\,\left(\omega_2\,\sin^2\theta\,d\phi + \omega_1\,\cos^2\theta\,d\psi\right)\,. \end{aligned} \tag{B.7}$$

[1] Here, s should not be confused with the entropy density in the main text.

Setting $\mathfrak{g}^2 = 1$, $\mu\, s^2 = q$, $\mu\, s\, c = Q$ as well as changing the radial coordinate to $a = r\, H^{\frac{1}{3}}$ and performing the transformations

$$\begin{aligned} dt \to & dt - \frac{3\, a^2\, H^{\frac{2}{3}}}{X\, W(a)}\, da\,, \\ d\phi \to & d\phi + \epsilon\, \omega_1 \left(dt - \frac{3\, H^{-\frac{1}{3}}}{a^4 f(a)\, W(a)}\, da \right)\,, \\ d\psi \to & d\psi + \epsilon\, \omega_2 \left(dt - \frac{3\, H^{-\frac{1}{3}}}{a^4 f(a)\, W(a)}\, da \right)\,, \end{aligned} \tag{B.8}$$

where $f(a)$ is given in (5.24) with k set to $k = 1$ and with $\mathrm{e}^{3U} = H$, yields (B.5) and (B.7) in Eddington-Finkelstein-type coordinates at first order in ϵ,

$$\begin{aligned} ds^2 = & - a^2\, f(a)\, dt^2 + \frac{6}{W(a)}\, da\, dt - \frac{6}{W(a)}\, \epsilon\, \left(\omega_1\, \sin^2\theta\, d\phi + \omega_2 \cos^2\theta\, d\psi\right)\, da \\ & + a^2\, d\Omega_3^2 + 2\, \epsilon\, \left(a^2\, f(a) + a^2 - H^{-\frac{1}{3}}\right) \left(\omega_1 \sin^2\theta\, d\phi + \omega_2 \cos^2\theta\, d\psi\right)\, dt\,, \\ A^1 = & A^2 = -\frac{Q}{a^2}\, H^{-\frac{1}{3}}\, u_\mu\, dx^\mu\,, \quad A^3 = -\frac{q}{a^2}\, H^{\frac{2}{3}}\, l_\mu\, dx^\mu\,, \end{aligned} \tag{B.9}$$

where $W(a) = 2\, H^{-\frac{1}{3}} + H^{\frac{2}{3}}$ is the superpotential. Then, rewriting the five-dimensional line element (B.9) and the gauge potentials (B.7) in terms of the four-dimensional quantities $u^\mu = (\,1,\, 0,\, \epsilon\, \omega_1,\, \epsilon\, \omega_2)$, $l^\mu = (\,0,\, 0,\, -\epsilon\, \omega_2,\, -\epsilon\, \omega_1)$ and $g_{\mu\nu} = \mathrm{diag}(\,-1,\, 1,\, \sin^2\theta,\, \cos^2\theta)$ yields (5.68) with $\sigma_{\mu\nu} = 0$.

B.3 Three-charge rotating STU black hole with equal rotation parameters in Eddington-Finkelstein coordinates

A rotating three-charge STU black hole with equal rotation parameters $\omega_1 = \omega_2 = \tilde{\omega}$ has been constructed in [174]. At first oder in the rotation parameter $\epsilon\, \tilde{\omega}$, it reads

$$\begin{aligned} ds^2 = & - \frac{Y}{R^2}\, dt^2 + \frac{r^2\, R}{Y}\, dr^2 + R\, d\Omega_3^2 - \frac{2\, f_2}{R^2}\, dt\, \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\,, \\ A^i = & \frac{\mu}{r^2\, H_i}\, \left(s_i\, c_i\, dt + \epsilon\, \tilde{\omega}\, \left(c_i\, s_j\, s_k - s_i\, c_j\, c_k\right) \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\right)\,, \\ X^i = & \frac{R}{r^2\, H_i}\,, \quad i = 1,\, 2,\, 3\,, \quad i \neq j \neq k \neq i\,, \end{aligned} \tag{B.10}$$

where

$$
\begin{aligned}
Y =& R^3 + r^4 - \mu\, r^2\,, \\
R =& r^2 \left(\prod_{i=1}^{3} H_i\right)^{\frac{1}{3}}\,, \qquad H_i = 1 + \frac{\mu\, s_i^2}{r^2}\,, \\
f_2 =& \epsilon\, \tilde{\omega} \left(-\gamma\, R^3 + \mu \left(\prod_i c_i - \prod_i s_i\right) r^2 + \mu^2 \prod_i s_i\right)\,, \\
s_i =& \sinh \delta_i\,, \qquad c_i = \cosh \delta_i\,.
\end{aligned}
\tag{B.11}
$$

Changing the radial coordinate to $a = r\,\mathrm{e}^U = r\,(H_1\,H_2\,H_3)^{\frac{1}{6}}$ and applying the transformation

$$
dt \to dt - \frac{3}{W(a)\, a^2\, f(a)}\, da\,, \tag{B.12}
$$

yields the line element (B.10) in the form

$$
\begin{aligned}
ds^2 =& -a^2\, f(a)\, dt^2 + \frac{6}{W(a)} \left(dt + \frac{\epsilon\, \tilde{\omega}\, h(a)}{a^2\, f} \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\right) da \\
&+ a^2\, d\Omega_3^2 - 2\,\epsilon\, \tilde{\omega}\, h(a) \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right) dt\,,
\end{aligned}
\tag{B.13}
$$

where

$$
h(a) = -\gamma\, a^2 + \frac{\mu}{a^2}\, \mathrm{e}^{-2U} \left(\prod_i c_i - \prod_i s_i\right) + \frac{\mu^2}{a^4} \prod_i s_i\,. \tag{B.14}
$$

For later convenience, we define $\omega = \gamma\, \tilde{\omega}$ and $\tilde{h} = \gamma^{-1}\, h$ such that $\omega\, \tilde{h} = \tilde{\omega}\, h$. Then carrying out the transformations

$$
\begin{aligned}
d\phi \to& d\phi + \epsilon\, \omega \left(dt - \frac{3\left(a^2 f(a) + \tilde{h}(a)\right)}{a^4\, f(a)\, W(a)}\, da\right)\,, \\
d\psi \to& d\psi + \epsilon\, \omega \left(dt - \frac{3\left(a^2 f(a) + \tilde{h}(a)\right)}{a^4\, f(a)\, W(a)}\, da\right)
\end{aligned}
\tag{B.15}
$$

yields (B.10) in Eddington-Finkelstein-type coordinates at first order in ϵ,

$$\begin{aligned}
ds^2 =& - a^2\, f(a)\, dt^2 + \frac{6}{W(a)} \left(dt - \epsilon\, \omega \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\right)\, da \\
&+ a^2\, d\Omega_3^2 + 2\,\epsilon\,\omega \left(a^2\, f(a) + a^2 - \left(a^2\, f(a) + \tilde{h}(a)\right)\right) \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\, dt\,, \\
A^i =& \frac{\mu}{a^2\, H_i}\, e^{2U} \left(s_i\, c_i\, dt + \epsilon\, \frac{\omega}{\gamma}\, \left(c_i\, s_j\, s_k - s_i\, c_j\, c_k\right) \left(\sin^2\theta\, d\phi + \cos^2\theta\, d\psi\right)\right)\,, \\
X^i =& \frac{1}{H_i} \left(\prod_{i=1}^{3} H_i\right)^{\frac{1}{3}}\,, \quad i = 1,\, 2,\, 3\,, \quad i \neq j \neq k \neq i\,.
\end{aligned} \tag{B.16}$$

The line element in (B.16) is related to the various line elements used in the main text, as follows. Let us first consider the stationary limit of the Maxwell solution (5.51) with $\omega_1 = \omega_2 = \omega$. It is obtained from (B.16) by setting $\delta_1 = \delta_2 = \delta_3 = \delta$, $W(a) = 3$ and $\Delta_a = f a^4$ with f given by $f(a) = 1 + k/a^2 - M/a^4 + Q^2/a^6$. Then, the function h becomes (with $s_i = s, c_i = c$)

$$h(a) = -\gamma\, a^2 + \frac{\mu}{a^2}\, \mathrm{e}^{-2U} \left(c^3 - s^3\right) + \frac{\mu^2}{a^4}\, s^3\,, \tag{B.17}$$

which can also be written as

$$h(a) = \gamma\, \tilde{h}(a) = (c - s) \left(-a^2 + \frac{\mu}{a^2}\, \mathrm{e}^{-2U}\, (c^2 + s^2 + c\, s) + \frac{\mu^2}{a^4}\, s^3\, (c + s)\right)\,. \tag{B.18}$$

Setting $M = \mu + 2\,\mu\, s^2$, $Q = \mu\, s\, c$ and $\mathrm{e}^{-2U} = (a^2 - \mu\, s^2)/a^2$ gives

$$\tilde{h}(a) = -a^2\, f(a) + 1 + \frac{Q}{a^2}\,. \tag{B.19}$$

The terms in this expression are related as follows to the ones in (5.51): the second term is the coefficient of the $u\, R\, u$-term, while the third term is the coefficient of the $u\, l$-term.

Next, let us consider the stationary limit of the two-charge solution (5.68). It is obtained from (B.16) by setting $\delta_1 = \delta_2 = \delta$, $\delta_3 = 0$, $\gamma = 1$ and $H = \mathrm{e}^{3U}$. Then the function h becomes

$$h(a) = \tilde{h}(a) = -a^2\, f(a) + \mathrm{e}^{-U}\,, \tag{B.20}$$

with f given by (5.24). In this expression, the second term is the coefficient of the $u\, R\, u$-term in (5.68).

And finally, the stationary limit of the one-charge solution (5.77) is obtained from (B.16) by setting $\delta_1 = \delta$ and $\delta_2 = \delta_3 = 0$. Now the function h reads (with $c_1 = c$)

$$h(a) = -\gamma\, a^2 + \frac{\mu}{a^2}\, \mathrm{e}^{-2U}\, c\,. \tag{B.21}$$

This can be written as

$$h(a) = \gamma\, \tilde{h}(a) = \frac{1}{c}\left(-a^2 + \frac{\mu}{a^2}\, \mathrm{e}^{-2U}\, c^2\right) . \tag{B.22}$$

Setting $\gamma = c^{-1}$, and with $H = \mathrm{e}^{6U}$, we obtain

$$\tilde{h}(a) = -a^2\, f(a) + \mathrm{e}^{2U} , \tag{B.23}$$

with f given by (5.24). In this expression, the second term is the coefficient of the $u\, R\, u$-term in (5.77).

Appendix C

Boundary energy-momentum tensor for the STU black hole

Here we compute the boundary energy-momentum for the STU black hole carrying two equal charges. A similar calculation applies to the other cases discussed in the main text, namely no charge (the Schwarzschild case), one non-vanishing charge and three equal charges (the Maxwell case).

The boundary energy-momentum tensor is given by [77, 106, 121]

$$8\pi G_5 \langle T_{\mu\nu} \rangle = \lim_{a\to\infty} \left[a^2 \left(K_{\mu\nu} - K\,\gamma_{\mu\nu} - \frac{W(a)}{L}\,\gamma_{\mu\nu} + \frac{L}{2}\,G_{\mu\nu} \right) \right] , \tag{C.1}$$

where the boundary metric $\gamma_{\mu\nu}$ is read off from the bulk metric written in the form

$$ds^2 = N^2 da^2 + \gamma_{\mu\nu}\,(dx^\mu + n^\mu da)\,(dx^\nu + n^\nu da) \ , \tag{C.2}$$

$G_{\mu\nu} = R_{\mu\nu}[\gamma] - \frac{1}{2}\gamma_{\mu\nu}\,R[\gamma]$ is the four-dimensional Einstein tensor of $\gamma_{\mu\nu}$, and the extrinsic curvature tensor is given by [164]

$$K_{\mu\nu} = -\frac{1}{2N}\,(\partial_a \gamma_{\mu\nu} - \nabla_\mu[\gamma]\,n_\nu - \nabla_\nu[\gamma]\,n_\mu) \ , \tag{C.3}$$

with $K = \gamma^{\mu\nu}\,K_{\mu\nu}$. Here $n_\mu = \gamma_{\mu\nu}\,n^\nu$, and $W(a)$ is the superpotential.

Imposing the tracelessness of $T_{\mu\nu}$ results in $K = -4\,W(a)/(3\,L) - L\,R[\gamma]/6$, and reinserting this into (C.1) yields

$$8\pi G_5 \langle T_{\mu\nu} \rangle = \lim_{a\to\infty} \left[a^2 \left(K_{\mu\nu} + \frac{W(a)}{3\,L}\,\gamma_{\mu\nu} + \frac{L}{2} \left(R_{\mu\nu}[\gamma] - \frac{1}{6}\,\gamma_{\mu\nu}\,R[\gamma] \right) \right) \right] . \tag{C.4}$$

In the following we set $L = 1$.

Comparing (C.2) with the line element (5.68) for the deformed STU black hole, and using (5.73), we infer that for large a,

$$\begin{aligned} n_\mu &= -\frac{3}{W(a)}\,u_\mu\,, \qquad N^2 = -\frac{9}{W(a)^2}\,\gamma^{\mu\nu}\,u_\mu\,u_\nu\,,\\ \gamma_{\mu\nu} &= a^2 g_{\mu\nu} - \left(\mathrm{e}^{-4U}\,k - \mathrm{e}^{-2U}\,\frac{\mu}{a^2}\right)u_\mu\,u_\nu + \frac{1}{2}\,\mathrm{e}^{-U}\left(u_\mu\,R_{\nu\lambda}\,u^\lambda + u_\nu\,R_{\mu\lambda}\,u^\lambda\right)\\ &\quad + \left(2a - \frac{\eta}{2a^2}\right)\sigma_{\mu\nu}\,. \end{aligned} \tag{C.5}$$

Here $W(a) \approx 3 + q^2/(3\,a^4)$ and the exponential functions $\mathrm{e}^{-\chi U}$ in $\gamma_{\mu\nu}$ behave as $\mathrm{e}^{-\chi U} \approx 1 - \chi\,q/(3\,a^2)$ so that

$$\begin{aligned} \gamma_{\mu\nu} &= a^2 g_{\mu\nu} - \left(k - \frac{w_5\,M}{a^2}\right)u_\mu\,u_\nu + \frac{1}{2}\left(1 - \frac{q}{3\,a^2}\right)\left(u_\mu\,R_{\nu\lambda}\,u^\lambda + u_\nu\,R_{\mu\lambda}\,u^\lambda\right)\\ &\quad + \left(2a - \frac{\eta}{2a^2}\right)\sigma_{\mu\nu}\,, \end{aligned} \tag{C.6}$$

where $w_5\,M = \mu + \frac{4}{3}\,k\,q$ is the physical mass. At first order in ϵ and at large a, the inverse metric $\gamma^{\mu\nu}$ is then given by

$$\gamma^{\mu\nu} = \frac{1}{a^2}\,g^{\mu\nu} + \frac{k}{a^4}\,u^\mu\,u^\nu - \frac{1}{2a^4}\left(u^\mu\,R^\nu_{\ \lambda}\,u^\lambda + u^\nu\,R^\mu_{\ \lambda}\,u^\lambda\right) - \frac{2}{a^3}\,\sigma^{\mu\nu}\,, \tag{C.7}$$

where the indices on the right hand side are raised with the metric $g^{\mu\nu}$.

Computing the terms in (C.4) for large a and at first order in ϵ, we obtain

$$
\begin{aligned}
W(a) =& 3 + \frac{q^2}{3\,a^4} \ , \\
N^{-1} =& a + \frac{k}{2a} - \frac{k^2}{8a^3} - \frac{w_5\,M}{2a^3} + \frac{q^2}{9a^3} \ , -\frac{1}{2N}\partial_a\gamma_{\mu\nu} + \frac{W(a)}{3}\,\gamma_{\mu\nu} \\
=& \frac{1}{2a^2}\left(\frac{k^2}{4}g_{\mu\nu} + w_5\,M\,(g_{\mu\nu} + 4\,u_\mu\,u_\nu) - 2\,\eta\,\sigma_{\mu\nu}\right) \\
& - \frac{k}{2}\,(g_{\mu\nu} + 2\,u_\mu\,u_\nu) + \left(a - \frac{k}{2a}\right)\sigma_{\mu\nu} \\
& + \left(\frac{1}{2} - \frac{q}{3\,a^2}\right)\left(u_\mu\,R_{\nu\lambda}\,u^\lambda + u_\nu\,R_{\mu\lambda}\,u^\lambda\right) \ , \\
\Gamma^\gamma_{\alpha\beta}[\gamma] =& \Gamma^\gamma_{\alpha\beta} + \frac{1}{a}g^{\gamma\lambda}\left(\nabla_\alpha\sigma_{\lambda\beta} + \nabla_\beta\sigma_{\alpha\lambda} - \nabla_\lambda\sigma_{\alpha\beta}\right) \ , \\
\nabla_\mu[\gamma]\,n_\nu =& -\nabla_\mu\,u_\nu \ , \\
R_{\mu\nu}[\gamma] =& R_{\mu\nu} + \frac{4k}{a}\,\sigma_{\mu\nu} \ , \\
R[\gamma] =& \frac{R}{a^2} \ , \\
R_{\mu\nu}[\gamma] - \frac{1}{6}\gamma_{\mu\nu}R[\gamma] =& R_{\mu\nu} - kg_{\mu\nu} + \frac{k^2}{a^2}\,u_\mu\,u_\nu \\
& - \frac{k}{2a^2}\left(u_\mu\,R_{\nu\lambda}\,u^\lambda + u_\nu\,R_{\mu\lambda}\,u^\lambda\right) + \frac{2k}{a}\sigma_{\mu\nu} \ .
\end{aligned}
\tag{C.8}
$$

Inserting these expressions into (C.4) yields the energy-momentum tensor (5.75).

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