Holographic Renormalization of Fake Supergravities

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Abstract

The string/gauge theory correspondence allows to calculate correlators in certain strongly coupled gauge theories via solving the equations of motion of supergravity (SUGRA). In this work we propose a method to calculate correlators for theories with logarithmically running gauge couplings, which corresponds to a logarithmically wrapped bulk-metric on the gravity side. One of the most prominent examples is the Klebanov-Strassler background, for which calculations were carried out. However, the proposed method is more general in nature and is applicable for all theories that are known as "fake" SUGRA theories. Such "fake" SUGRA theories allow for BPS domain-wall solutions, which are the holographic duals of renormalization group flows. It would be a daunting task to find the full counterterms for such general theories. Thus, at present we content ourselves with calculating the 2-point function and to some extent the 1-point function. Furthermore, we only consider gauge theories living on the flat space-time, which allows us to neglect all the counterterms involving the space-time curvature.

We start with the string/gauge correspondence formula

$$\mathrm{e}^{-S_{on-shell}[\mathfrak{s}]} = \int \mathcal{D}\Phi \ \mathrm{e}^{-S_{\mathrm{QFT}}[\Phi]+\int \mathcal{O}_i \mathfrak{s}_i \, \mathrm{d}^d x} \quad , \tag{1}$$

where $S_{on-shell}[\mathfrak{s}]$ is the renormalized on-shell bulk action evaluated as a functional of suitably defined boundary values $\mathfrak{s}_i$ of the various bulk fields, which are identified with the *sources* coupling to certain QFT operators $\mathcal{O}_i$. $S_{on-shell}[\mathfrak{s}]$ is then identified with the generating functional of the connected correlation functions of various QFT operators. In particular, the exact 1-point functions of the QFT operators are given by

$$\langle \mathcal{O}_i(x) \rangle = -\frac{\delta S_{on-shell}}{\delta \mathfrak{s}_i(x)} \ . \tag{2}$$

In order to evaluate the 2-point functions, one has to know the dependence of the right hand side of (1) on the sources up to linear order, which requires to solve the linearized equations of motion. Hence, in order to calculate 1- and 2-point functions, it suffices to know the action up to quadratic order in fluctuations.

In this work we give a recipe to determine the *quadratic* terms. In doing so we use the so-called gauge invariant mechanism. In order to calculate the vacuum expectation values we would need to know the boundary terms linear in fluctuations. We do not know yet how to generalize our prescription for those terms. Thus, we calculate only the linear term of the 1-point function, i.e excluding the VEV. However, we make an observation that the response function of the zero mode seems to encode some information about the VEV. One advantage of our approach is that it allows to discuss the scheme dependence of 1- and 2-point functions in a rather general way.

Zusammenfassung

Die String/Eichtheorie Korrespondenz erlaubt die Berechnung von Korrelatoren in bestimmten stark gekoppelten Eichtheorien via Lösung von Bewegungsgleichungen in der Supergravitation (SUGRA). In der Arbeit wird eine Methode vorgeschlagen die holographisch renormierten Korrelatoren für die Eichtheorien auszurechnen, in denen die Eichkopplungen im gesamten Energiebereich logarithmisch rennen, was auf der Gravitationsseite einer logarithmisch gewarpten Bulkmetrik entspricht. Eines der prominentesten Beispiele is der Klebanov-Strassler Hintergrund, auf welchen wir unsere Methode anwenden. Die entwickelte Methode ist allerdings allgemeiner und ist für alle Theorien anwendbar, die als "fake" SUGRA bezeichnet werden. Solche "fake" SUGRA Theorien haben BPS Domänenwandlösungen, die dual sind zu Renormierungsgruppenflüssen. Es wäre sehr kompliziert, die kompletten Gegenterme für solche allgemeinen Theorien auszurechnen. Deswegen konzentrieren wir uns erstmal auf die Bestimmung von renormierten 2-Punkt Funktion und in gewissem Maße von 1-Punk Funktionen. Außerdem betrachten wir flache Räume entlang des Randes, um die durch die räumliche Krümmung entstehenden Gegenterme vernachlässigen zu können.

Wir beginnen mit der String/Eich Korrespondenzformel

$$
\mathrm{e}^{-S_{on-shell}[\mathfrak{s}]} = \int \mathcal{D}\Phi \; \mathrm{e}^{-S_{\mathrm{QFT}}[\Phi] + \int \mathcal{O}_i \mathfrak{s}_i \, \mathrm{d}^d x} \quad , \tag{1}
$$

wo $S_{on-shell}[\mathfrak{s}]$ die renormierte on-shell Bulkwirkung ist, die als Funktional von geeignet definierten Grenzwerten $\mathfrak{s}_i$ von verschiedenen Bulkfeldern ausgewertet wird. Die $\mathfrak{s}_i$ werden identifiziert als *Quellen* für bestimmte QFT Operatoren $\mathcal{O}_i$ und $S_{on-shell}[\mathfrak{s}]$ ist identifiziert mit dem generierenden Funktional für die zusammenhängenden Korrelatoren von QFT Operatoren. Die exakte 1-Punkt Funktion ist dann gegeben durch

$$
\langle \mathcal{O}_i(x) \rangle = -\frac{\delta S_{on-shell}}{\delta \mathfrak{s}_i(x)} \quad . \tag{2}
$$

Um die 2-Punkt Funktionen auszurechnen, muss man die Abhängigkeit von der rechten Seite von (1) von den Quellen bis zur ersten Ordnung kennen, was der Lösung der linearisierten Bulkbewegungsgleichungen bedarf. Deswegen reicht es für die Berechnung von 1- und 2-Punkt Funktionen die Wirkung bis zur quadratischen Ordnung in Fluktuationen zu kennen. In der Arbeit zeigen wir, wie man die *quadratischen* Terme bestimmt und benutzen dabei den so genannten eichinvarianten Formalismus. Um die Vakuumerwartungswerte zu berechnen, müsste man auch die Terme, die linear in Fluktuationen sind, kennen. Wir wissen noch nicht, wie man unsere Methode auf diese Terme erweitert. Deswegen berechnen wir nur den linearen Term von der 1-Punkt Funktion, also ohne den VEW. Wir bemerken allerdings, dass die Antwortfunktion des zero-modes die Information über den VEW zu beinhalten scheint. Ein Vorteil unserer Methode ist, dass man die Schemenabhängigkeit von 1- und 2-Punkt Funktionen in ziemlich allgemeiner Weise diskutieren kann.

Contents

1 Introduction

String theory is a well known candidate for quantizing gravity and unifying it with other interactions. In the recent research [1, 2, 3], however, a somewhat different approach has been taken. Rather than unifying the other interactions with gravity, it has been shown, that at least some gauge theories have a dual gravity (string theory) description. Besides of being an interesting theoretical problem, these dualities have become a useful tool for studying strongly coupled gauge theories, where the perturbative approach fails.

As is well known string theory first emerged in 60's as a possible description for strong interactions. Empirical evidence for the string-like structure of the hadrons came from arranging mesons and baryons into approximately linear Regge-trajectories. Studies of the πN scattering led Dolen, Horn and Schmidt [4] to a duality conjecture which stated that the sum over s-channels exchanges is equal to the sum over the t-channels. This fact, however, posed a problem on finding analytical form of such dual amplitude. The first and rather simple expression for a manifestly dual 4-point amplitude was found by Veneziano [5]:

$$A(s,t) \sim \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{1.1}$$

and it has an exactly linear Regge-trajectory $\alpha(s) = \alpha(0) + \alpha' s$. An open string interpretation of this amplitude was proposed [6, 7] and in 70's string theory became a very popular candidate for the theory of strong interactions. The basic idea was to think of a meson as a string with a quark attached to one end and antiquark to another. Rotational and vibrational excitations of such open string then give rise to the various meson states. Decay of a meson is described by a splitting of the string.

The dynamics of the string world-sheet is described by the Nambu-Goto area action

$$S_{NG} = -T \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{-\det \partial_a X^\mu \partial_b X_\mu}\,, \tag{1.2}$$

where a, b take two values ranging over the σ and τ directions of the string world-sheet. T is the string tension and is related to the Regge-slope via $T^{-1} = 2\pi\alpha'$. Furthermore, the quantum consistency of the Veneziano model requires that the Regge intercept is $\alpha(0) = 1$, implying that spin 1 state is massless and spin 0 is a tachyon. And this poses a big problem, since a ρ-meson is not massless and the presence of a tachyonic state in the theory is an indicator for an instability. Further problems became apparent, when the string theory zero-point energy was calculated. It leads to

$$\alpha(0) = \frac{d-2}{24} \tag{1.3}$$

where d is the space-time dimension, so that (1.3) implies that the model has to be defined in 26 space-time dimensions. It was possible to construct consistent supersymmetric string theories in 10 dimensions, but it was still unclear how these are related to the 4-dimensional world. Finally, the Asymptotic Freedom of the strong interactions has been discovered [8, 9] and singled out the Quantum Chromodynamics as the exact theory of the strong interactions, delivering the final blow to the string theory. Instead it was observed that the graviton emerges naturally in the framework of the string theory and it became a promising candidate for quantization of the gravity and the unification of quantum gravity with other forces [10, 11].

In the mid 90's studies of the Dirichlet branes (D-branes) brought the string theory and gauge theory back together. D-branes are soliton-like objects (”membranes”) of various internal dimensionalities, present in theories of closed superstrings [12]. A Dp-brane is a p-dimensional hyperplane in the $9+1$ dimensional space-time where the strings are allowed to end. Dp-brane acts like a topological defect, in a sense that upon touching it a closed string may become an open string, whose ends are free to move on the brane. For the endpoints of such strings the $p+1$ longitudinal coordinates satisfy the free Neumann boundary conditions and the $9-p$ coordinates transversal to the Dp-brane satisfy the fixed Dirichlet boundary conditions (this gave rise to the name ”Dirichlet brane”). Polchinski has shown [12] that a Dp-brane preserve 1/2 of the bulk supersymmetries and carries an elementary unit of charge with respect to the $p+1$ form gauge potential from the Ramond-Ramond sector of the type II superstring.

The most important property of the D-branes for the purposes of this work, is the fact that they realize gauge theories on their world-volume; a special role is then played by the D3-branes, which realize the $3+1$ gauge theory. The massless spectrum of the open strings living on the Dp-brane is the one of the maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. The $9-p$ massless fields of this supermultiplet are the Goldstone modes associate with the transverse excitations of the Dp-branes whereas the photons and fermions provide unique supersymmetric completion. Considering N parallel branes raises the number of species of the strings to N^2, since they can end on any of the branes, and the gauge theory becomes $U(N)$. For our purposes we will be interested in the cases where N branes are stacked on top of each other, so that the relative separations between the branes and, consequently, the expectation values of the scalar fields vanish. If N is large, then the stack of branes becomes a massive object embedded into a theory of closed strings with gravity. This massive object will curve the space-time around it and it is possible to describe it in terms of some classical metric and other background fields. Hence we obtain two different descriptions of a stack of N Dp-branes: one in terms of the gauge theory ”living” on their world-volume and another in terms of the classical charged p-brane background of the type II closed superstring theory.

Now, as we mentioned above, D3-branes play a special role. A stack of N parallel D3-branes realize a $3+1$ dimensional $U(N)$ theory. Studies of the metric of such a stack have revealed, that close to the branes the space-time metric factorizes into a direct product of two smooth spaces, AdS_5 and S^5 with equal radii L, whereas for large distances the metric becomes the metric of the flat Minkowski space. Hence, the brane geometry can be viewed as semi-infinite throat of radius L, which opens up into flat 9+1 dimensional space-time for distances $r \gg L$. For L much larger than the string length scale, $\sqrt{\alpha'}$, the entire 3-brane geometry has small curvatures everywhere and is well described by the supergravity approximation to the Type IIB string theory. The exact relation between L and α' can be found by equating the gravitational tension of the extremal 3-brane classical solution to N times the tension of the single brane

$$L^4 = g_{YM} N \alpha'^2. \tag{1.4}$$

Studies of massless particle absorption by the 3-branes [13, 14, 15] have shown, that in the low-energy limit, the throat region ($r \ll L$) decouples from the asymptotically flat region ($r \gg L$). In the similar way, the $\mathcal{N} = 4$ supersymmetric $SU(N)$ theory on the stack of N D3-branes decouples in the low-energy limit from the bulk closed string theory. These considerations led Maldacena to make his famous conjecture [1], that type IIB string theory on $AdS_5 \times S^5$, of radius given by (1.4) is dual to the $\mathcal{N} = 4$ Super Yang-Mills theory. The number of colors in the gauge theory, N, is dual to the number of the flux units of the 5-form Ramond-Ramond field strength. In the following it was conjectured [2, 3] that there exists a one-to-one map between gauge invariant operators in the Conformal Field Theory and fields in AdS_5. The dimension of an operator Δ is then determined by the mass of the dual field in AdS_5.

Now, the discovery of this duality is a truly remarkable result. The direct mapping between the quantities of gauge theory and type IIB string theory on AdS_5 implies that we can calculate correlation functions of various operators in CFT using its dual formulation (precise methods were developed in [13, 14, 15]). Moreover, (1.4) implies that the size of the throat in string units is $\lambda^{1/4}$ where λ is the 't Hooft coupling $\lambda \equiv g_{YM}^2 N = g_s N$. The requirement $L \gg \sqrt{\alpha'}$ translates into $\lambda \gg 1$, hence the supergravity approximation is valid when the 't Hooft coupling is very large and perturbative field theoretic methods are not applicable. Thus, the Maldacena conjecture gave us a nice way to circumvent the problem arising in strongly coupled gauge theories by computing the correlation functions in the dual supergravity background.

The methods developed in [13, 14, 15] can be extended to describe non-conformal theories, which are obtained by deforming CFT's either by addition of relevant operators to the Lagrangian or by turning on VEV's for these operators. The gravity duals of such theories are domain wall solutions of $(d+1)$-dimensional bulk theories with isometry group being the d-dimensional Poincaré group. There is a number of known classical solutions, e.g. [16, 17, 18, 19, 20].

Now a well known fact is that in quantum field theory, the correlation functions suffer from UV divergences and a renormalization of the theory is needed. These divergences are related to the IR divergences on the gravitational side. On the gravitational side IR - long distance - corresponds to "near the boundary". To deal with these divergences a method of holographic renormalization was developed [21, 22] and is described in detail in [36]. The first step of this method is to write bulk fields as series expansions in the radial coordinate r which is transverse to the boundary. This allows us to determine an asymptotic solution of the field equations given arbitrary Dirichlet boundary conditions. The solutions are obtained by substituting the series expansions into the nonlinear bulk field equations and solving term-by-term. This process is referred to as near-boundary analysis and it allows to determine the first $\Delta - d/2$ terms in the expansion. To determine further terms, additional information is required in order to obtain a unique solution. This is due to the fact that we have second order field equations and only the Dirichlet conditions on the boundary have been specified so far. To specify a unique solution we demand that the fluctuations about the domain wall background vanish in the deep interior. The coefficients determined by near-boundary analysis are local functions of the boundary data, whereas higher order terms may contain non-local contributions.

The next step is the construction of the renormalized action S_{ren} by a process of regularization and renormalization. The bulk theory is regulated by introducing a cut-off at some large but finite value of the radial coordinate. The series solution is then inserted in the regulated classical action. One can then observe that the on-shell action contains a finite number of terms which diverge if the cut-off were removed. These divergent terms contain only coefficients from the solution that are fixed by the near-boundary analysis and can be removed by adding counterterms to the action expressed as invariant local functionals of the induced metric and other fields at the cut-off. These fields depend locally on the Dirichlet conditions on the true boundary, which is approached, as the cut-off is removed. Similar to the usual procedure of regularization in the field theory, we still have the freedom of adding some finite local counterterms, which corresponds to a choice of a particular scheme (e.g. supersymmetric scheme, etc.). The sum of the regulated action and the counterterms is finite as the cut-off is removed and S_{ren} is then defined by this limit. By construction S_{ren} is invariant under 5D diffeomorphisms except the ones generating Weyl transformations of the boundary metric. The violation of the Weyl invariance results in the emergence of the logarithmically divergent counterterms and one can read off conformal anomalies directly from these terms.

After having obtained the renormalized action, one can compute the finite correlation functions by functional differentiating S_{ren} with respect to the sources. The correlation functions then involve the lowest order series coefficients that are not determined by near-boundary analysis, which is an expected result, since correlation functions are non-local. In contrary, field theory UV divergences and anomalies are local and can be fully determined by the near-boundary analysis, as well as the Ward identities.

Now, gauge theories that are dual to anti de-Sitter or asymptotically anti de-Sitter spacetimes are either maximally supersymmetric or at least flow to the maximally supersymmetric and conformal theory in the UV. However, we would like to approach the observed physical world, i.e. QCD, and hence the gauge theory in question should not be conformal, supersymmetric and should exhibit confinement. It is a daunting task to fulfill all of these requirements, but several attempts have been made to achieve at least some of them [23, 24, 25]. One of the ways to reduce the amount of SUSY's is to place the stack of D3-branes on the tip of a 6-dimensional cone (conifold) instead of the flat Minkowski space [23]. The corresponding geometry, as we approach the N D3-branes is then changed to $AdS_5 \times T^{1,1}$, where $T^{1,1}$ is the base of the cone with topology $S^2 \times S^3$(see Section 4.1 for the details). Same arguments as in previous paragraphs lead to the conclusion that the supergravity on $AdS_5 \times T^{1,1}$ should be dual to the gauge theory on the branes, which is in this case an $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N)$ gauge theory [23]. Again, we can match the geometrical symmetries of the conifold to the continuous symmetries of the gauge theory and certain coordinates on the conifold can be matched with the bifundamental fields of the dual theory. However, while breaking the number of supersymmetries to 1, this theory still preserves conformality [23, 26] and hence does not provide us with confinement.

Studies of the duality between the type IIB string theory on the $AdS_5 \times T^{1,1}$ and the corresponding field theory also led to studies of branes wrapped around the cycles of the conifold and attempts to identify these states in the field theory [27, 28]. In particular one could add M units of so-called fractional D3-branes at the tip of the conifold, which corresponds to wrapping D5-branes around the S^2 cycle. This results in appearance of M units of the 3-form flux through the S^3 in addition to the N units of the 5-form flux through $T^{1,1}$ coming from the regular D3-branes. On the gauge theory side we have an $SU(N + M) \times SU(N)$ gauge theory, where M is the number of the fractional D3-branes. It still has the same bifundamental field content as the theory dual to the singular conifold, however, the addition of M wrapped branes renders the theory non-conformal. Instead, one can show [28] that the relative coupling of $SU(N + M) \times S(N)$ runs logarithmically. Furthermore, this theory exhibits a so-called duality cascade. One can show that the couplings g_1 and g_2 of the $SU(N + M) \times SU(N)$ flow in different directions, and at some point the coupling of the $SU(N + M)$ becomes infinite. In order to make sense of the theory past this infinite coupling we must perform the so-called Seiberg duality [29], which results in transforming the gauge group of the theory into $SU(N) \times SU(N - M)$ (details are given in Sections 4.2, 4.3). The theory is self-similar under this transformation, so that we obtain a cascade. In the dual supergravity solution, conifold with fractional D3-branes [30], this corresponds to decreasing the 5-form flux by M-units. However, as the cascade flows to the IR, the cascade must stop, since eventually we reach negative N and that would be unphysical. On the SUGRA side this corresponds to the D3-branes charge becoming negative, rendering the metric singular. It has been therefore suggested [24], that we may choose N and M in such a way, that we will finally arrive at the gauge group $SU(2M) \times SU(M)$ and the strong dynamics of this theory will resolve the naked singularity in the metric. (In the

6-dimensional internal space wrapping the branes around the S^2 of the conifold results in blowing up of the S^3 by the emerging 3-form flux, so that it does not shrink to zero size at the tip). The flow then becomes an infinite series of Seiberg duality transformations. The supergravity dual of this cascade is referred to as *warped deformed conifold* or Klebanov-Strassler background [24] and it will be the main focus of our work.

The method of holographic renormalization, briefly described in one of the previous paragraphs, does not cover the cases in which the field theory has logarithmically running coupling even in ultra violet (holographically that means that the metric has logarithmic warping). Klebanov-Strassler background [24] is a prime example of such a background and is well approximated by the Klebanov-Tseytlin (KT) solution [30] in the UV. Calculating correlation functions for such cases is far more complicated, partly also due to the fact that holographic renormalization has not been worked out yet in a systematic way as in aAdS cases. Recently, some progress was made on the holographic renormalization in bulk backgrounds conformal to $AdS_{p+2} \times S^{8-p}$ with a non-vanishing dilaton [31]; however, these cases imply only couplings that run with a power law in the UV. Hence only few attempts to calculate correlators using the KT background have been made [32, 33, 34, 35], and only in last two the program of holographic renormalization [36] was applied. In addition, calculations of mass spectra in the KS background [32, 37, 38, 39, 40, 41] have been done with the assumption that a consistent method of HR in non-aAdS backgrounds exists.

In our work we would like to propose a method for calculation of renormalized correlators holographically from backgrounds which are not aAdS, given that this is a feature one would expect for the dual description of any gauge theory with a running coupling in UV, like QCD. We will concentrate on the KS background, but our approach is more general in nature and it would be interesting to test it also on other cases like for example Maldacena-Nunez background [25].
We consider a general bulk theory of gravity coupled to an arbitrary number of scalars, whose potential can be expressed via a "superpotential". Such theories are known as “fake SUGRA” theories [42], where “fake” does not mean that the theory is necessarily non-supersymmetric, just that the formalism is applicable more generally. The relation between supergravity and fake supergravity was analyzed in [43, 44]. Furthermore, they allow for BPS domain wall background solutions, which are the holographic duals of renormalization group flows. The fake SUGRA systems include the case of KS (and also KT), when viewed as a consistent truncation of type-IIB SUGRA [45, 46]. For such a general theory, it would be a very difficult task to find the complete counterterms. Thus, at present we content ourselves with giving a recipe how to calculate renormalized two- (and to some extent one-) point functions of the operators dual to the scalars of the theory. Furthermore, we only consider field theories living on a flat space-time, which allows us to ignore all counterterms involving the space-time curvature. In a sense, our approach is inspired by [47, 48], where the philosophy was put forward to concentrate on the part of the counterterm action which is really necessary to calculate n-point functions for a given n, i.e. the terms of n-th order in the fluctuations. In

this spirit, we consider the case $n = 2$. The counterterms we propose involve the fluctuations in a covariant way, but, otherwise, do depend on the background. It might be possible to derive them from a fully covariant expression, but we have not attempted to do so.

The starting point of the holographic calculation of correlation functions in AdS/CFT is the correspondence formula [3]

$$\mathrm{e}^{-S_{on-shell}[\mathfrak{s}]} = \int \mathcal{D}\Phi \; \mathrm{e}^{-S_{\mathrm{QFT}}[\Phi]+\int \mathcal{O}_i \mathfrak{s}_i \, \mathrm{d}^d x} \quad , \tag{1.5}$$

where $S_{\text{on-sh}}[\mathfrak{s}]$ denotes the *renormalized* bulk on-shell action evaluated as a functional of suitably defined boundary values $\mathfrak{s}_i$ of the various bulk fields, which are identified with the *sources* coupling to certain QFT operators $\mathcal{O}_i$. Hence, the bulk quantity $S_{on-shell}[\mathfrak{s}]$ is identified with the generating functional of the connected correlation functions of various QFT operators. In particular, the *exact* one-point functions of the QFT operators are given by

$$\langle \mathcal{O}_i(x) \rangle = -\frac{\delta S_{on-shell}}{\delta \mathfrak{s}_i(x)} \ . \tag{1.6}$$

In order to calculate the two-point functions, one has to know the dependence of the right hand side of (1.6) on the sources $\mathfrak{s}_j$ up to linear order. Determining this dependence requires to solve the linearized bulk equations of motion. Thus, if we are interested in one- and two-point functions, a knowledge of the action, which consists of bulk and boundary terms, up to quadratic order in fluctuations is sufficient. We will give a recipe how to calculate the *quadratic* terms. In doing so, we make use of the gauge-invariant formalism for the fluctuations developed in [49, 50, 46], in which the scalar fluctuations explicitly decouple from those of the metric at the linearized level. Thereby, the gauge-invariant fields are identified with the relevant bulk degrees of freedom that encode the information on the boundary correlation functions. We restrict our attention to the scalar sector, but the recipe can be extended easily to the traceless transversal fluctuations of the metric.

In order to calculate the vacuum expectation values (VEV s), one would also have to know the boundary terms *linear* in the fluctuations (of course, also a term independent of the fluctuations has to be added in order to obtain a finite action; in our calculations we assume that such a term has been added). We do not know yet how to generalize our prescription to those terms. Thus, strictly speaking, we can only calculate the contributions to the one-point functions, which are linear in fluctuations (i.e. excluding the VEV s). However, we shall observe that the linearized equations of motion have a zero mode solution (depending only on the radial coordinate) which seems to encode some information about the VEV s. We will make this more explicit in the examples that we discuss in later sections.

One advantage of our approach is that it allows to discuss the scheme dependence of one- and two-point functions in a rather general way, as we will do in section 5.2.3 and then more concretely in the examples.

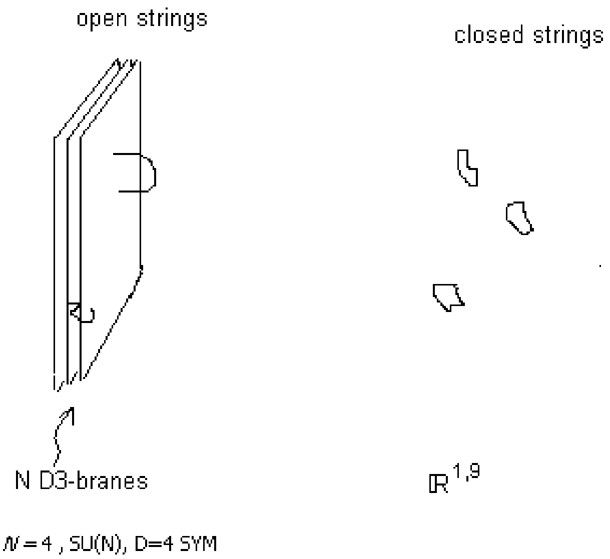
The work is built up as follows. First we will give an overview over relevant aspects of $\mathcal{N}=4$ Super Yang Mills gauge theory and type IIB Supergravity and introduce the notion of the string/gauge duality. In the following section, we will give an introduction to the method of holographic renormalization and consider the examples of its applications. Furthermore we introduce the Klebanov Strassler theory, starting with a more general overview about backgrounds consisting of D3-brane stacks placed on the conical singularity. We will show how the KS background exhibits the interesting features, like confinement. Finally, in the next section we will introduce the method of Perturbative Holographic Renormalization, test this method on the simpler examples of the Coulomb branch flow and GPPZ flow, introduced in Section 3 and apply it to the Klebanov-Strassler theory. We will discuss the results and related issues.

2 Anti de-Sitter/Conformal Field Theory correspondence

In this section we will present a more detailed motivation for the AdS/CFT correspondence.

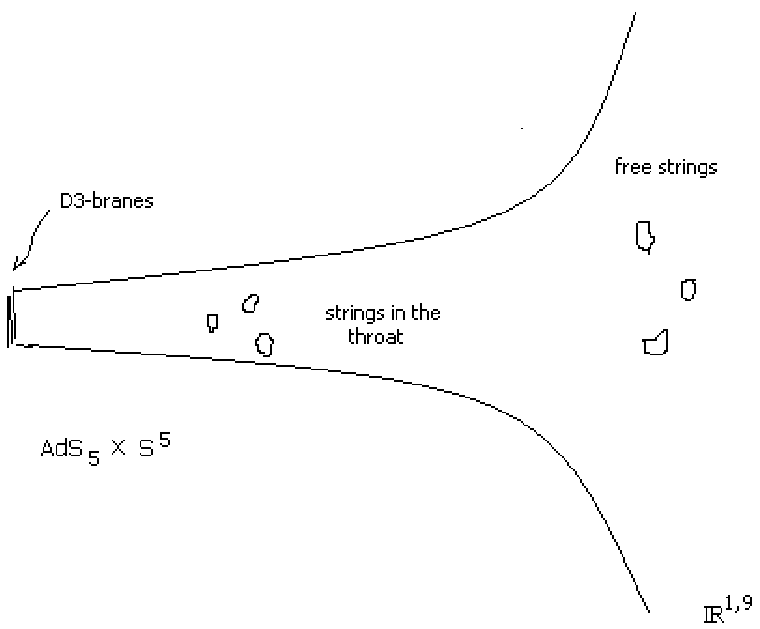
Let us consider a system in Type IIB string theory consisting of a stack of D3 branes placed in a flat space (see Figure 1). Such a system can be described in two different ways. One way is to describe it in terms of the gauge theory of their world volume. In type IIB string theory we have both open and closed strings interacting with each other. Open strings can end on D3 branes, closed strings can split upon meeting the brane and become open an vice versa. For the matter at hand we want to take the low energy limit where these two sets of degrees of freedom (open strings and closed strings) decouple from each other. This is indeed possible due to the fact that the coupling constant between two strings is governed by the *dimensionful* Newton constant G_N so that in 10-dim the effective constant becomes $G_N E^8$ (where E is energy) and in the limit $E \to 0$ this coupling constant vanishes and so does the interaction between closed strings and closed strings with open strings. This essentially means that gravity is free in IR. As a result we obtain two separate kinds of degrees of freedom: 1) low energy closed strings (gravitons) propagating in 10-dim flat space. 2) low energy excitations of the open strings. Interactions between open strings remain since their interactions are governed by $g_{YM} = 2\pi g_s$ which is small but finite. D3 branes preserve half of the supersymmetries of the type IIB superstring theory, so that we have at the end $\mathcal{N}=4$ $SU(N_c)$ D=4 SYM effective low energy description.

Figure 1: D3-branes in 10-dim Minkowski space-time



On the other hand Dp-branes are massive objects which are also charged under a $p+1$-form potential so that they will naturally curve the space around them (Figure 2). If we take a large stack of branes (N_c large) we will get a smooth solution which describes the curvature of the space-time around them: we will still have flat space far away from the stack of the branes; as we approach the stack of the branes, however, we find the space-time deformed in such way, that we essentially have a coset space consisting of 5 compact dimensions (S^5 in the simplest case) and 5 non-compact dimensions are deformed in an AdS_5 throat. The gravitational radius of this throat is given by $\frac{L}{l_s} \sim (g_s N_c)^{1/4}$, where l_s is the string length and g_s string coupling. So we see that this is only a good description as long as $g_s N_c \gg 1$. In the other limit, $g_s N_c \ll 1$ the gravitational radius becomes vanishing small and one can replace the geometry by placing a boundary condition at the location of the branes, so that the description in terms of $N = 4$ SYM becomes appropriate.

Figure 2: Strings in $AdS_5 \times S^5$ background



Now let us take the low energy limit in both descriptions. In the gauge theory description we will have $N = 4$ SYM (interacting open strings) + free gravity (decoupled closed strings). In the geometrical description we have closed strings both in flat space as also in the throat. At the low energy limit closed strings in the flat space do not see the throat anymore due to the rising wavelength, whereas the strings in the throat find it increasingly difficult to "climb up" the throat due to the rising gravitational potential. So we again get two decoupled sets of degrees of freedom: strings in $AdS_5 \times S^5$ + free gravity. The AdS/CFT conjecture, as proposed by Maldacena [1], states that the following two theories are equivalent to one another:

- 10-dim Type II B string theory on the product space $AdS_5 \times S^5$, where the type IIB 5-form flux through S^5 is an integer N_c and the equal radii of AdS_5 and S^5 are given by $L^4 = 4\pi g_s N \alpha'^2$, with g_s being the string coupling

- a 4-dimensional Super-Yang-Mills theory with maximal $\mathcal{N} = 4$ supersymmetry, gauge group $SU(N)$, Yang-Mills coupling $g_{YM}^2 = g_s$ in the conformal phase.

In the strongest form of the conjecture, the correspondence is to hold for all values of N and all regimes of coupling $g_s = g_{YM}^2$. However there are also some interesting and highly non-trivial limits. The 't Hooft limit on the SYM side, where $\lambda \equiv g_{YM}^2 N$ is fixed while $N \to \infty$, corresponds to classical string theory on $AdS_5 \times S^5$ on the AdS side. In this sense, classical string theory on

$AdS_5 \times S^5$ provides a classical Lagrangian formulation of the large N dynamics of $\mathcal{N} = 4SYM$ theory.
A further limit, $\lambda \to \infty$ reduces classical string theory to classical Type IIB supergravity on $AdS_5 \times S^5$. Hence, strong coupling dynamics in SYM theory (at large N limit) is mapped onto classical low energy dynamics in supergravity.

In the following, we will introduce the components of the duality, $\mathcal{N} = 4$ SYM and SUGRA on $AdS_5 \times S^5$ in some more detail and also formulate the conjecture more precisely. For more details we refer to [51, 52] and references therein.

2.1 $\mathcal{N} = 4$ Super Yang Mills

The Lagrangian for the $\mathcal{N} = 4$ super-Yang-Mills theory is unique and given by [53, 54]

$$\begin{aligned}\mathcal{L} = \mathrm{tr}\Bigg\{ & - \frac{1}{2g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \Sigma_a i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \Sigma_i D_\mu X^i D^\mu X^i \\ & + \Sigma_{a,b,i} g C_i^{ab} \lambda_a \left[X^i, \lambda_b\right] + \Sigma_{a,b,i} g \bar{C}_{iab} \bar{\lambda}_a \left[X^i, \bar{\lambda}_b\right] + \frac{g}{2}\Sigma_{i,j}\left[X^i, X^j\right]^2 \Bigg\} \end{aligned} \tag{2.1}$$

The constants C_i^{ab} and C_{iab} are related to the Clifford Dirac matrices for $SO(6)_R \sim SU(4)_R$. By construction, this Lagrangian is invariant under $\mathcal{N} = 4$ Poincaré symmetry with following transformation laws [51]

$$\begin{aligned} \delta X^i &= [Q^a_\alpha, X^i] = C^{iab}\lambda_{\alpha b} \\ \delta\lambda_b &= \{Q^a_\alpha, \lambda_{\beta b}\} = F^+_{\mu\nu}(\sigma^{\mu\nu})^\alpha{}_\beta \delta^a_b + [X^i, X^j]\epsilon_{\alpha\beta}(C_{ij})^a{}_b \\ \delta\bar{\lambda}^b_{\dot\beta} &= \{Q^a_\alpha, \bar{\lambda}^b_{\dot\beta}\} = C^{ab}_i \bar{\sigma}^\mu_{\alpha\dot\beta} D_\mu X^i \\ \delta A_\mu &= [Q^a_\alpha, A_\mu] = (\sigma_\mu)_\alpha{}^{\dot\beta}\bar{\lambda}^a_{\dot\beta}\,. \end{aligned} \tag{2.2}$$

The constants $(C_{ij})^a{}_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_R$.
Classically $\mathcal{L}$ is scale invariant. One can see this by assigning the standard mass-dimensions to the fields and couplings

$$[A_\mu] = [X^i] = 1 \qquad [\lambda_a] = \frac{3}{2} \qquad [g] = [\theta] = 0\,. \tag{2.3}$$

All terms in the Lagrangian are of dimension 4, which implies the scale invariance. In relativistic field theory, scale invariance and Poincaré invariance combine into conformal symmetry with the group $SO(2,4) \sim SU(2,2)$. Furthermore, $\mathcal{N} = 4$ Poincaré symmetry and conformal invariance combined give rise to the superconformal symmetry described by the supergroup $SU(2,2|4)$.

It is believed that the perturbatively quantized $\mathcal{N} = 4$ SYM theory is UV finite and hence the renormalization group β-function vanishes identically. The theory is exactly scale invariant at the quantum level, so that $SU(2,2|4)$ is a fully quantum mechanical symmetry.

There is also a discrete global symmetry of the theory stemming from S-duality conjecture [53]. To state this invariance, one combines the real coupling g and the real instanton angle θ into one complex coupling

$$\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \,. \tag{2.4}$$

The quantum theory is invariant under $\theta \to \theta + 2\pi$ and $\tau \to \tau + 1$. The S-duality conjecture states that the theory is also invariant under the $\tau \to -\frac{1}{\tau}$. Both symmetries taken together give us the S-duality group $SL(2, \mathbb{Z})$, generated by

$$\tau \to \frac{a\tau + b}{c\tau + d}, \; ad - bc = 1, \; a, b, c, d, \in \mathbb{Z}. \tag{2.5}$$

When $\theta = 0$, the S-duality transformation reduces to $g \to 1/g$, thereby exchanging strong and weak coupling.

2.1.1 Super Conformal $\mathcal{N} = 4$ Super Yang- Mills

The global continuous symmetry group of the $\mathcal{N} = 4$ SYM is given by the supergroup $SU(2, 2|4)$ [55, 56]. It includes following constituents.

- *Conformal symmetry*, forming the group $SO(2, 4) \sim SU(2, 2)$ is generated by translations P^μ, Lorentz transformations $L_{\mu\nu}$, dilations D and special conformal transformations K^μ.
- *R-symmetry*, forming the group $SO(6)_R \sim SU(4)_R$, generated by T^A, $A = 1, \cdots, 15$;
- *Poincaré supersymmetries* generated by the supercharges Q^a_α and their complex conjugates $\bar{Q}_{\dot{\alpha}a}$, $a = 1, \cdots, 4$. The presence of these charges results immediately from $\mathcal{N} = 4$ Poincaré supersymmetry;
- *Conformal supersymmetries* generated by the supercharges $S_{\alpha a}$ and their complex conjugates $\bar{S}^a_{\dot{\alpha}}$. These symmetries arise from commutators of the Poincaré supersymmetries and the special conformal transformations K_μ. Since both are symmetries, their commutator must also be a symmetry.

The two bosonic subalgebras $SO(2, 4)$ and $SU(4)_R$ commute. The supercharges Q^a_α and $\bar{S}^a_{\dot{\alpha}}$ transform under the $\mathbf{4}$ of $SU(4)_R$, while $\bar{Q}_{\dot{\alpha}a}$ and $S_{\alpha a}$ transform under the $\mathbf{4}^*$. Hence we see that the generators fit into a super algebra in following way

$$\begin{pmatrix} P_\mu \, K_\mu \, L_{\mu\nu} \, D & Q^a_\alpha, \bar{S}^a_{\dot{\alpha}} \\ \bar{Q}_{\dot{\alpha}a} \; S_{\alpha a} & T^A \end{pmatrix}$$

Most structure relations are straightforward, except the relations between the supercharges. To organize the structure relations, we will use the natural grading of the algebra given by the dimension of the generators,

$$
\begin{aligned}
[D] = [L_{\mu\nu}] = [T^A] = 0 \qquad\qquad & [P^\mu] = +1 \qquad [K_\mu] = -1 \\
& [Q] = +1/2 \qquad [S] = -1/2
\end{aligned}
\tag{2.6}
$$

Thus, we have

$$
\begin{aligned}
\{Q^a_\alpha, Q^b_\beta\} &= \{S_{\alpha a}, S_{\beta b}\} = \{Q^a_\alpha, \bar{S}^b_{\dot\beta}\} = 0 \\
\{Q^a_\alpha, \bar{Q}_{\dot\beta b}\} &= 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta_b{}^a \\
\{S_{\alpha a}, \bar{S}^b_{\dot\beta}\} &= 2\sigma^\mu_{\alpha\dot\beta} K_\mu \delta_a{}^b \\
\{Q^a_\alpha, S_{\beta b}\} &= \epsilon_{\alpha\beta}\big(\delta_b{}^a D + T^a{}_b\big) + \frac{1}{2}\delta_b{}^a L_{\mu\nu}\sigma^{\mu\nu}_{\alpha\beta}
\end{aligned}
\tag{2.7}
$$

2.2 Supergravity and Superstrings

2.2.1 D=10 Supergravity action, particles and fields

For the action of the Type IIB theory, there is no completely satisfactory action as it involves an antisymmetric field A_4^+ with selfdual field strength. However, one can write an action involving both dualities of A_4 and then impose the self-duality in an extra equation. We will get therefore [52]:

$$
\begin{aligned}
S_{IIB} = & +\frac{1}{4\kappa_B^2}\int \sqrt{G} e^{-2\Phi}(2R_G + 8\partial_\mu\Phi\partial^\mu\Phi - |H_3|^2) \qquad (2.8) \\
& -\frac{1}{4\kappa_B^2}\int \left[\sqrt{G}(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2) + A_4^+ \wedge H_3 \wedge F_3\right] + \text{fermions}
\end{aligned}
$$

with field strengths defined as follows

$$
\begin{cases}
F_1 = dC \\
H_3 = dB \\
F_3 = dA_2 \\
F_5 = dA_4^+
\end{cases}
\qquad
\begin{cases}
\tilde{F}_3 = F_3 - CH_3 \\
\tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3
\end{cases}
\tag{2.9}
$$

and there is a supplementary self-duality condition $*\tilde{F}_5 = \tilde{F}_5$. Type IIB supergravity is invariant under the non-compact symmetry group $SU(1,1) \sim SL(2,\mathbb{R})$. While the above form of the metric arises naturally from the string low energy approximation, the symmetry in question is not manifest in it. In order to render it manifest, one can redefine fields from the string metric $G_{\mu\nu}$ used in (2.8) to the Einstein metric $G_{E\mu\nu}$ and express the tensor fields in terms of complex fields:

$$
G_{E\mu\nu} \equiv e^{-\Phi/2} G_{\mu\nu} \qquad\qquad \tau \equiv C + ie^{-\Phi}
$$
$$
G_3 \equiv (F_3 - \tau H_3)/\sqrt{\operatorname{Im}\tau} \tag{2.10}
$$

Then the action becomes

$$
\begin{aligned}
S_{\text{IIB}} &= \frac{1}{4\kappa_B^2}\int\sqrt{G_E}\left(2R_{G_E} - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{(\operatorname{Im}\tau)^2} - \frac{1}{2}|F_1|^2 - |G_3|^2 - \frac{1}{2}|\tilde{F}_5|^2\right) \\
&\quad -\frac{1}{4i\kappa_B^2}\int A_4\wedge\bar{G}_3\wedge G_3
\end{aligned} \tag{2.11}
$$

Under the $SU(1,1)$ symmetry of Type IIB supergravity, the metric and A_4^+ fields are invariant. The dilaton-axion field τ changes under s Möbius transformation,

$$
\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d} \qquad\qquad ad - bc = 1,\ a, b, c, d \in \mathbb{R} \tag{2.12}
$$

$B_{\mu\nu}$ and $A\mu\nu$ fields rotate into one another under the linear transformation associated with above Möbius transformation.

The $\mathcal{N}=2$, $D=10$ Type IIB theory has the following field and particle contents [51],

$$
\text{Type IIB}\begin{cases}
G_{\mu\nu} & SO(8) & 35_B & \text{metric} - \text{graviton} \\
C + i\Phi & & 2_B & \text{axion} - \text{dilaton} \\
B_{\mu\nu} + iA_{2\mu\nu} & & 56_B & \text{rank 2 antisymmetric} \\
A^+_{4\mu\nu\rho\sigma} & & 35_B & \text{antisymmetric rank 4} \\
\psi^I_{\mu\alpha}\ {}_{I=1,2} & & 112_F & \text{Majorana} - \text{Weyl gravitinos} \\
\lambda^I_\alpha\ {}_{I=1,2} & & 16_F & \text{Majorana} - \text{Weyl dilatinos}
\end{cases} \tag{2.13}
$$

The rank 4 antisymmetric tensor $A^+_{\mu\nu\rho\sigma}$ has self-dual field strength, which is indicated with the + superscript. The gravitinos are Γ-traceless. The two gravitinos $\psi^I_{\mu\alpha}$ have the same chirality, while the two dilatinos λ^I_α also have the same chirality but opposite to that of the gravitinos. The theory is chiral (parity violating).

2.3 Branes in Supergravity

A rank $p+1$ antisymmetric tensor field $A_{\mu_1\ldots\mu_{p+1}}$ may be identified with a (p+1)-form,

$$
A_{p+1} \equiv \frac{1}{(p+1)!} A_{\mu_1\cdots\mu_{p+1}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p+1}} \tag{2.14}
$$

$(p+1)$-form couples to geometrical objects Σ_{p+1} of space-time dimension $p+1$, since a diffeomorphism invariant action may be constructed as follows

$$S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A_{p+1} \tag{2.15}$$

It is invariant under Abelian gauge transformations $\rho_p(x)$ of rank p

$$A_{p+1} \to A_{p+1} + d\rho_p\,. \tag{2.16}$$

Furthermore the field A_{p+1} has a gauge invariant field strength F_{p+2}, which is a $p+2$-form with conserved flux. p-branes are 1/2 BPS solutions to supergravity with non-trivial A_{p+1} charge.
A p-brane has a $(p+1)$-dimensional flat hypersurface, with Poincaré invariance group $\mathbb{R}^{p+1} \times SO(1,p)$.The dimension of the transverse space is $D-p-1$ and it is always possible to find solutions with maximal rotational symmetry $SO(D-p-1)$ in this transverse space. Hence, one can think of p-branes in 10-dimensional supergravity as solutions with symmetry group

$$\mathbb{R}^{p+1} \times SO(1,p) \times SO(10-p) \tag{2.17}$$

Let as denote the space-time coordinates as follows

$$\begin{array}{lll} \text{Coordinates } || \text{ to brane} & x^\mu & \mu = 0,1,\cdots,p \\ \text{Coordinates } \perp \text{ to brane} & y^u = x^{p+u} & u = 1,2,\cdots,D-p-1 \end{array}$$

Poincaré invariance in $p+1$ directions means that metric in those directions is a rescaling of the Minkowski flat metric, while rotation invariance in the transverse dimensions forces the metric in those directions to be a rescaling of Euclidean metric. Furthermore, rescaling functions have to be independent on $x^\mu, \mu = 0,1,...,p$. With above restrictions we find that the solution to field equations may be expressed in terms of a single function h [57]

$$\text{Dp} \qquad ds^2 = h(\vec{y})^{-1/2} dx^\mu dx_\mu + h(\vec{y})^{1/2} d\vec{y}^2 \qquad\qquad e^\Phi = h(\vec{y})^{(3-p)/4} \tag{2.18}$$

with metric being expressed in the string frame. Function h must be harmonic with respect to $\vec{y}$. Assuming maximal rotational symmetry by $S(10-p-1)$ in the transversal dimensions and remembering that the metric should tend to flat space-time at $y \to \infty$, the most general solution is parametrized by a single scale factor L and is given by

$$h(y) = 1 + \frac{L^{10-p-3}}{y^{10-p-3}} \tag{2.19}$$

Since α' is the only dimensionfull parameter of the theory, L must be a constant times the α' dependence. For our purposes we a interested the solution of N coincident Dp-branes, for which

$L^{10-p-1} = N\rho_P$ with $\rho_p = g_s\ (4\pi)^{(5-p)/2}\Gamma((7-p)/2)(\alpha')^{(D-p-3)/2}$.

While originally found as solutions to supergravity field equations, it is expected that the p-branes of Type IIA/B supergravity extend to solutions of the full Type IIA/B string theory. These solutions will then break half of the symmetries of the string theory. They may also be subject to α' corrections as compared to supergravity solutions.

2.3.1 D3-branes

For the AdS/CFT purposes D3-branes are especially interesting, due to several reasons: (1) its worldbrane has 4-dimensional Poincaré invariance; (2) it has constant axion and dilaton fields; (3) it is regular at $y = 0$; (4) it is self-dual.
The D3-brane solution is characterized by [51]

$$\begin{cases} g_s = e^{\phi},\ C \text{ constant} \\ B_{\mu\nu} = A_{2\mu\nu} = 0 \\ ds^2 = h(y)^{-1/2}dx^{\mu}dx_{\mu} + h(y)^{1/2}(dy^2 + y^2 d\Omega_5^2) \\ F^{+}_{5\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau\upsilon}\partial^{\upsilon}h \end{cases} \tag{2.20}$$

$\epsilon_{\mu\nu\rho\sigma\tau\upsilon}$ is the volume element transverse to the 4-dimensional Minkowski D3-brane in 10-dimensions. The N-brane solution with general locations of N_I parallel D3-branes located at transverse position $\vec{y}_i$ is described by

$$h(\vec{y}) = 1 + \sum_{I=1}^{N} \frac{4\pi g_s N_I (\alpha')^2}{|\vec{y} - \vec{y}_I|^4} \tag{2.21}$$

where $N = \sum I N_I$ is the total number of branes.

The radius L of the D3-brane solution is given by $L^4 = 4\pi g_s N \ell_p^4$ with ℓ_p being the Planck length $\ell_p^2 = \alpha'$. Thus, for $g_s N \ll 1$, the radius L is smaller than the string length and the supergravity approximation is not expected to be a reliable approximation to the full string solution. In this regime $g_s \ll 1$, so the string perturbation theory is expected to be reliable. For $g_s N \gg 1$, however, $L \gg \ell_p$, and the supergravity approximation is expected to be a good approximations to the full string solution.

The D3-brane solution is more properly a two-parameter family of solutions, characterized by the string coupling g_s and the instanton angle $\theta = 2\pi C$, which may be combined into the single complex parameter $\tau = C + ie^{-\phi}$. The $SU(1,1) \sim SL(2,\mathbf{R})$ symmetry of Type IIB supergravity acts transitively on τ, so all solutions lie in a single orbit of this group. In full superstring theory, however, the range of θ is quantized so that the identification $\theta \sim \theta + 2\pi$ may be made, and as a result also $\tau \sim \tau + 1$. Therefore, the allowed Möbius transformations must be elements of the $SL(2,\mathbf{Z})$ subgroup of $SL(2,\mathbf{R})$, for which $a,b,c,d \in \mathbf{Z}$. These transformations map between equivalent solutions in string theory. Thus, the string theories defined on D3 backgrounds which are related by an $SL(2,\mathbf{Z})$ duality will be equivalent to one another. This property will be of crucial importance in the AdS/CFT correspondence where it will emerge as the reflection of S-duality in $\mathcal{N}=4$ SYM theory.

2.4 AdS/CFT Correspondence

Now we are almost ready to formulate the Maldacena conjecture a bit more precise, but before we do that, let us consider some additional points and introduce the AdS geometry.

2.4.1 Non-abelian Gauge symmetry on D3 branes

Open strings whose both end points are attached to a single brane can have arbitrary short length and must therefore be massless. This excitation mode induces a massless $U(1)$ gauge theory on the worldbrane which is effectively 4-dimensional flat space time [58]. The brane is a 1/2 BPS object so that it breaks half of the total number of the symmetries, thus the $U(1)$ gauge theory must have $\mathcal{N}=4$ Poincaré supersymmetry. In the low energy approximation the $\mathcal{N}=4$ supersymmetric $U(1)$ gauge theory is free.

Not lets us consider a system with $N>1$ parallel separated D3-branes. The end points of the string can be attached to the same brane, giving rise to massless excitation modes. These modes induce a massless $U(1)^N$ gauge theory with $\mathcal{N}=4$ supersymmetry in the low energy limit. Alternatively, the open string can also have its ends attached to different branes. The mass of such string cannot be arbitrarily small, since it is bounded from below by the separation distance between the two branes. There are N^2-N such possible strings. In the limit where the branes coincide, all string states would be massless and the $U(1)^4$ theory is enhanced to a full $U(N)$ gauge symmetry. Separating the branes should be interpreted as Higgsing the gauge theory to the Coulomb branch where the gauge symmetry is spontaneously broken. The overall $U(1)=U(N)/SU(N)$ factor corresponds to the overall position of the branes and may be ignored when considering dynamics on the branes, therefore leaving only a $SU(N)$ gauge theory [59].

In the low energy limit, N coincident branes support an $\mathcal{N}=4$ SYM theory in 4 dimensions with gauge group $SU(N)$.

2.4.2 The Maldacena limit

The space-time metric of N coincident D3-branes may be written in the following form:

$$ds^2 = \left(1+\frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1+\frac{L^4}{r^4}\right)^{\frac{1}{2}} \left(dr^2 + y^2 d\Omega_5^2\right) \tag{2.22}$$

with the radius L of the brane given by

$$L^4 = 4\pi g_s N \alpha'^2 \,. \tag{2.23}$$

Now, if we take $r \gg L$ then we recover flat space-time $\mathbb{R}^{10}$. As we approach the stack of the branes ($r < L$) then a so called *throat* is building and geometry appears to be singular as $r \ll L$. However we can redefine the coordinate

$$\rho \equiv \frac{L^2}{r} \tag{2.24}$$

and take large ρ limit. Then the metric transforms into following asymptotic form

$$ds^2 = L^2 \left(\frac{1}{\rho^2} \eta_{ij} dx^i dx^j + \frac{d\rho^2}{\rho^2} + d\Omega_5^2 \right) \tag{2.25}$$

which is a metric of a product space. One component is the five-sphere S^5 with metric $L^2 d\Omega_5^2$ and the other is the hyperbolic space AdS_5 with constant negative curvature $\frac{L^2}{\rho^2}\left(\eta_{ij} dx^i dx^j + d\rho^2\right)$. Thus we get a regular and highly symmetrical geometry close to the stack of the branes which can be summarized as $AdS_5 \times S^5$ with both components having equal radii.

The Maldacena limit [1] corresponds to keeping fixed g_s and N as well as all physical length scales, while letting $\alpha' \to 0$. In this limit, only $AdS_5 \times S^5$ region of the geometry survives and contributes to the string dynamics of physical processes, while the dynamics of the asymptotic flat region decouples from the theory.

An easy way to see this decoupling is to consider the effective action $\mathcal{L}$ and to carry out the α' expansion in an arbitrary background with Riemann tensor R (with indices omitted for simplicity). Schematically the expansion has following form

$$\mathcal{L} = a_1 \alpha' R + a_2 \alpha'^2 R^2 + a_3 \alpha'^3 R^3 + ... \tag{2.26}$$

Physical objects and length scales in the asymptotically flat region are characterized by the scale $r \gg L$, so that scaling arguments give us $R \sim \frac{1}{r^2}$. Substitution in the equation above yields

$$\mathcal{L} = a_1 \alpha' \frac{1}{r^2} + a_2 \alpha'^2 \frac{1}{r^4} + a_3 \alpha'^3 \frac{1}{r^6} + ... \tag{2.27}$$

Now if we keep the physical size fixed, we see that entire contribution to the effective action from the flat space vanishes as $\alpha' \to 0$.

2.4.3 Geometry of the AdS spaces

Let us consider both Minkowskian and Euclidean AdS spaces.

Minkowskian AdS_{d+1} of the unit radius may be defined in $\mathbb{R}^{d+1}$ with coordinates $(Y_{-1}, Y_0, Y_1, ..., Y_d)$ as the $d+1$ hyperboloid with isometry $SO(2,d)$ given by the equation:

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_d^2 = -1 \tag{2.28}$$

with induce metric $ds^2 = -Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_d^2$. The topology of the manifold is that of the cylinder $S^1 \times \mathbb{R}$ times the sphere S^{d-1}, thus not simply connected. The topology of the boundary is given by $\partial AdS_{d+1} = S^1 \times S^{d-1}$. The manifold may be represented by the coset $SO(2,d)/SO(1,d)$. Euclidean AdS_{d+1} of unit radius may be defined in Minkowski flat space $\mathbb{R}^{d+1}$ with coordinates $(Y_{-1}, Y_0, Y_1, ..., Y_d)$ as the $d+1$ dimensional disconnected hyperboloid with isometry $SO(1,d)$ given by the equation

$$-Y_{-1}^2 + Y_0^2 + Y_1^2 + ... + Y_d^2 = -1 \tag{2.29}$$

with induced metric $ds^2 = -Y_{-1}^2 + Y_0^2 + Y_1^2 + ... + Y_d^2$. The topology of the manifold is that of the $\mathbb{R}^{d+1}$. The topology of the boundary is that of the d-sphere $\partial AdS_{d+1} = S^d$. The manifold can be represented by the coset $SO(1,d+1)/SO(d+1)$.

2.4.4 The AdS/CFT Conjecture

The AdS/CFT or Maldacena conjecture [1] states the equivalence (duality) between the following theories

- Type IIB superstring theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius L, the 5-form F_5^+ has integer flux $N = \int_{S^5} F_5^+$ and string coupling is g_s;
- $\mathcal{N} = 4$ super-Yang-Mills theory in 4-dimensions, with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its superconformal phase;

with following identifications between the parameters of both theories

$$g_s = g_{YM}^2\,, \qquad L^4 = 4\pi g_s N \alpha'^2 \tag{2.30}$$

and the axion expectation value equals the SYM instanton angle $\langle C \rangle = \theta_I$. The *equivalence* means a precise map between the states and fields on the superstring side and the local gauge invariant operators on the $\mathcal{N} = 4$ SYM side, as well as a correspondence between the correlators in both theories.

The above formulation of the conjecture is referred to as the *strong form*, as it is to hold for all values of N and of $g_s = g_{YM}^2$. However, string theory quantization on a general curved manifold is very difficult and at present out of the reach. Thus some limits were studied where Maldacena conjecture becomes more tractable but still remains nontrivial.

't Hooft limit The 't Hooft limit [60] corresponds to keeping the 't Hooft coupling $\lambda \equiv g^2_{YM} N = g_s N$ fixed and letting $N \to \infty$. In Yang-Mills theory, this limit is well defined in perturbation theory and corresponds to a topological expansion of the field theory's Feynman diagrams. On the AdS side this limit corresponds to *weak coupling string perturbation theory.* One can see it by rewriting the string coupling in terms of the 't Hooft coupling $g_s = \lambda/N$.

The large λ limit After the 't Hooft limit has been taken the only parameter left is λ. Quantum field theory perturbation theory corresponds to $\lambda \ll 1$. On the AdS side of the correspondence it is naturally to rather take $\lambda \gg 1$. Lets look again the α' expansion of the effective action

$$\mathcal{L} = a_1 \alpha' R + a_2 \alpha'^2 R^2 + a_3 \alpha'^3 R^3 + ... \tag{2.31}$$

The distance scales of interest now are those typical of the throat and hence are governed by the AdS radius L. This means that the scale of the Riemann tensor is set by

$$R \sim 1/L^2 = (g_s N)^{-1/2} \alpha; = \sqrt{\lambda}/\alpha' \tag{2.32}$$

hence, the expansion in α' becomes an expansion in powers of $\lambda^{-\frac{1}{2}}$,

$$\mathcal{L} = a_1 \lambda^{-\frac{1}{2}} + a_2 \lambda^{-1} + a_3 \lambda^{-\frac{3}{2}} + ... \tag{2.33}$$

So now any α' dependence has disappeared from the string theory and the role of α' as a scale has been taken over by the parameter $\lambda^{-\frac{1}{2}}$.

2.4.5 Mapping Global Symmetries

In order for the AdS/CFT correspondence to hold it is crucial to ensure that the global unbroken symmetries on both sides match exactly. It has been shown previously that the continuous global symmetry of $\mathcal{N} = 4$ super-Yang-Mills theory in its conformal phase is the superconformal group $SU(2,2|4)$ with maximal bosonic subgroup $SU(2,2) \times SU(4)_R \sim SO(2,4) \times SO(6)_R$. The bosonic subgroup arises as the product of the conformal group $SO(2,4)$ in 4-dimensions by the $SU(4)_R$ automorphism group of the Poincaré supersymmetry algebra. On the AdS side this bosonic group is recognized as the isometry group of the $AdS_5 \times S^5$ background. The completion in the full supergroup $SU(2,2|4)$ arises on the AdS side because 16 of the 32 Poincaré supersymmetries are preserved by the stack of N parallel D3-branes [12], and in the AdS limit, are supplemented by another 16 conformal supersymmetries (which are broken in the full D3-brane geometry).

$\mathcal{N} = 4$ SYM theory also has S-duality symmetry, realized on the complex coupling constant τ by Möbius transformations in $SL(2,\mathbb{Z})$. On the AdS side this is a global discrete symmetry of Type IIB string theory, which is unbroken by the D3-brane solution, in the sense that it maps non-trivially only the dilaton and axion expectation values. However, S-duality is a useful symmetry only in the strongest form of the conjecture. In 't Hooft limit, S-duality no longer has a consistent action.

2.4.6 Mapping Type IIB fields and CFT operators

After we saw that the global symmetry groups on the both sides of AdS/CFT correspondence coincide, we have to show that the actual representations of the supergroup $SU(2,2|4)$ also coincide on both sides. As we recall, especially important are the short multiplet (BPS) representations. *Single color trace operators* play a special role, since one may construct all higher trace operators using OPE. Hence we can expect, that single trace operators on the SYM side, correspond to single particle states on the AdS side [1, 61]. In turn, multiple trace BPS operators, should be then interpreted as bound state of these one particle states.

In order to identify the contents of irreducible representations of $SU(2,2|4)$ on the AdS side, let us describe all Type IIB massless supergravity and massive string degrees of freedom by fields φ living on $AdS_5 \times S^5$. One can decompose the metric as following:

$$ds^2 = g^{AdS}_{\mu\nu} dz^\mu dz^\nu + g^{S}_{uv} dy^u dy^v \,, \tag{2.34}$$

where $z^\mu,\ \mu = 0, 1, .., 4$ are the coordinates on AdS_5, and $y^u,\ u = 1, ..., 5$ the coordinates on S^5. The fields become functions $\varphi(z, y)$ associated with 10-dimensional degrees of freedom. It is useful to decompose $\varphi(z, y)$ in a series on S^5

$$\varphi(z, y) = \sum_{\Delta=0}^{\infty} \varphi_\Delta(z) Y_\Delta(y) \,, \tag{2.35}$$

where Y_Δ is the basis of spherical harmonics on S^5. Compactification of the fields on the S^5 leads to a contribution to their mass. One finds following relations between mass and scaling dimensions for different spins:

$$\begin{aligned}
\text{scalars} \qquad & m^2 = \Delta(\Delta - 4) \\
\text{spin } 1/2,\ 3/2 \qquad & |m| = \Delta - 2 \\
p - \text{form} \qquad & m^2 = (\Delta - p)(\Delta + p - 4) \\
\text{spin2} \qquad & m^2 = \Delta(\Delta - 4)
\end{aligned} \tag{2.36}$$

The complete correspondence between the representations of $SU(2,2|4)$ is presented in the table below. For our purposes only the first two lines are important.

Type IIB string theory	$\mathcal{N}=4$ conformal super-Yang-Mills
Supergravity Excitations 1/2 BPS, spin ≤ 2	Chiral primary + descendants $\mathcal{O}_2 = \mathrm{tr}\, X^{\{i}X^{j\}}$ + desc.
Supergravity Kaluza-Klein 1/2 BPS, spin ≤ 2	Chiral primary + Descendants $\mathcal{O}_\Delta = \mathrm{tr}\, X^{\{i_1} \cdots X^{i_\Delta\}}$ + desc.
Type IIB massive string modes non-chiral, long multiplets	Non-Chiral operators, dimensions $\sim \lambda^{1/4}$ e.g. Konishi $\mathrm{tr}\, X^i X^i$
Multiparticle states	products of operators at distinct points $\mathcal{O}_{\Delta_1}(x_1) \cdots \mathcal{O}_{\Delta_n}(x_n)$
Bound states	product of operators at same point $\mathcal{O}_{\Delta_1}(x) \cdots \mathcal{O}_{\Delta_n}(x)$

Table 1: Mapping of String and SUGRA states onto SYM Operators [51]

3 Holographic renormalization

3.1 Asymptotically anti-de Sitter spacetimes

First of all let us introduce the asymptotically anti-de Sitter (aAdS) space-times in some more detail.
AdS spacetime is a maximally symmetric solution of Einstein's equations with negative cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = \Lambda G_{\mu\nu} \, . \tag{3.1}$$

The AdS space is conformally flat which implies that the Weyl tensor vanishes:

$$W_{\mu\nu\kappa\lambda} = 0 \, . \tag{3.2}$$

Hence, the curvature tensor of the AdS_{d+1} spacetime is given by:

$$R_{\mu\nu\kappa\lambda} = \frac{1}{l^2} \left(G_{\kappa\mu} G_{\nu\lambda} - G_{\mu\lambda} G_{\nu\kappa} \right) \, , \tag{3.3}$$

with l being the radius of the AdS space: $\Lambda = -d(d-1)/2l^2$.
The metric for the AdS space is given by

$$ds^2 = \frac{l^2}{cos^2\theta} \left(-dt^2 + d\theta^2 + sin^2\theta d\Omega_{d-1}^2 \right) \tag{3.4}$$

with $0 \leq \theta \leq \frac{\phi}{2}$. The metric has a second order pole at $\theta = \frac{\pi}{2}$ which is where the boundary of the AdS space is located. This is a general feature of the metrics that satisfy (3.3) and therefore those metrics do not induce a metric at infinity, rather a conformal structure, i.e. metric up to a conformal transformation. This is achieved by introducing a so-called *defining function* U, a positive function in the interior of the manifold M which has a single zero and non-vanishing derivative at the boundary. Then the metric on the boundary is given by $g_{(0)} = U^2 G|_{\partial M}$. Since any other defining function given by $U' = U\exp(w)$ is good as well, metric $g_{(0)}$ is defined only up to a conformal transformation.

Now we are interested in solving to the Einstein equations with a given conformal structure at the boundary. In the framework of holographic renormalization this is usually done by working in the coordinate system introduced by Fefferman and Graham [62]:

$$\begin{aligned} ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)dx^i dx^j \,, && (3.5)\\ g_{ij}(x,\rho) &= g_{(0)} + ... + \rho^{d/2} g_{(d)} + h_{(d)}\rho^{d/2}\log\rho + ... && (3.6)\end{aligned}$$

Logarithmic part appears only if d is even. ρ is the radial coordinate emanating from the boundary. The most general expansion in Fefferman-Graham framework may contain also half-integral powers of ρ. However, in all cases that we consider one can show that coefficients of those vanish. Note, that all asymptotically AdS metrics can be recast in the form 3.5 near the boundary [62]. The radial coordinate ρ is related to the usual radial coordinate r via $\rho = \mathrm{e}^{-2r}$; the boundary is located at $r = \infty$, i.e. $\rho = 0$.

The curvature of the bulk metric G is in this case

$$R_{\kappa\lambda\mu\nu}[G] = (G_{\kappa\mu}G_{\nu\lambda} - G_{\mu\lambda}G_{\nu\kappa}) + \mathcal{O}(\rho)\,. \tag{3.7}$$

So that asymptotically we will have anti-de Sitter metric. In the coordinate system (3.5) Einstein equations are given by

$$\begin{aligned} \rho[2g'' - 2g'g^{-1}g' + Tr\,(g^{-1}g')\,g'] + \mathrm{Ric}(g) - (d-2)\,g' - Tr\,(g^{-1}g')\,g &= 0\\ \nabla_i Tr\,(g^{-1}g') - \nabla^j g'_{ij} &= 0\\ Tr\,(g^{-1}g'') - \frac{1}{2}Tr\,(g^{-1}g'g^{-1}g') &= 0, \qquad (3.8)\end{aligned}$$

prime denotes differentiation with respect to ρ and ∇_i is the covariant derivative constructed from the metric g. These equations can be solved order by order in the ρ variable. The resulting equations are algebraic, so the solution is insensitive to the sign of the cosmological constant and the signature of the spacetime. However, if the cosmological constant vanishes, then the corresponding equations are differential and impose conditions on $g_{(0)}$ as well. This means, that in general, the various coefficients in the asymptotic expansion of the metric that can contribute to divergences in the on-shell actions are non-local with respect to each other. This means, that in the case of asymptotically flat spaces there is no universal set of local counterterms that can remove the divergences from the

on-shell action for any solution.
For the AAdS backgrounds, the equations uniquely determine the coefficients $g_{(2)}, ..., g_{(d-2)}, h_{(d)}$ and the trace and covariant divergence of $g_{(d)}$. The coefficient $h_{(d)}$ is present only when d is even, and is equal to the metric variation of the holographic conformal anomaly. $g_{(d)}$ is directly related to the 1-point function of the dual stress energy tensor. In general, the solution obtained by this procedure is only valid near the boundary. In order to obtain solutions that extend to the deep interior, one requires more powerful techniques.

The results above can be extended to the cases where matter couples to gravity. In this case the bulk equation reads

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = T_{\mu\nu} \tag{3.9}$$

where $T_{\mu\nu} = \Lambda G_{\mu\nu}$ + matter contribution. The equation in this case has a near boundary solution provided the matter contribution to $T_{\mu\nu}$ is softer than the cosmological constant contribution.

3.2 Holographic Renormalization Method

In this section we will outline the method of holographic renormalization, which was worked out in [21] and [22], and described in detail in [63] or [36]. First of all we need a consistent truncation of the full bulk theory, in our case to D=5, $\mathcal{N} = 8$ gauge supergravity, to a number of scalar fields interacting with gravity. For simplicity we will consider the case of a single scalar, which is sufficient for the examples discussed at the end of this section. For this case the truncated supergravity action is

$$S = \frac{N^2}{2\pi^2}\int_M d^5x\sqrt{G}\left[\frac{1}{4}R + \frac{1}{2}G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right] - \frac{1}{2}\int_{\partial M}\sqrt{\gamma}\mathcal{K} \tag{3.10}$$

where $\mathcal{K}$ is the trace of the second fundamental form and γ is the induced metric on the boundary.

3.2.1 Asymptotic solution

We are interested in the most general solution of the bulk field equations with prescribed, but arbitrary, Dirichlet boundary conditions. We will suppress all space-time and internal indizes for simplicity and denote bulk fields as $\mathcal{F}(x,\rho)$. Near the boundary, each field has an asymptotic expansion of the form

$$\mathcal{F}(x,\rho) = \rho^m\left(f_{(0)}(x) + f_{(2)}(x)\rho + ... + \rho^n\left(f_{(2n)}(x) + \log\rho\tilde{f}_{(2n)}(x)\right) + ...\right) \tag{3.11}$$

where ρ is the radial coordinate of aAdS and the aAdS metric is given by (3.5).

The field equations are second order differential equations in ρ, so there are two independent solutions, with asymptotic behavior ρ^m and ρ^{n+m}, respectively. The form of the subleading terms in the asymptotic expansion is determined by the bulk field equations. For the cases we consider here, n and $2m$ are non-negative integers.

The boundary field $f_{(0)}$ that multiplies the leading behavior, ρ^m, is interpreted as the source for the dual operator. In the near boundary analysis one solves the field equations iteratively by treating the ρ-variable as a small parameter. This gives algebraic equations for $f_{(2k)}, k < n$, that uniquely determine $f_{(2k)}$ in terms of $f_{(0)}$ and derivatives up to order 2k. These equations leave $f_{(2n)}$ undetermined. This is due to the fact, that the coefficient $f_{(2n)}(x)$ is the Dirichlet boundary condition for a solution which is linearly independent from the one that starts as ρ^m. The undetermined coefficient is related to the exact 1-point function of the corresponding operator. The logarithmic term in (3.11) is necessary in order to obtain a solution. It is related to conformal anomalies of the dual theory and is also fixed in terms of $f_{(0)}$.

3.2.2 Regularization and Counterterms

As was discussed earlier, correlation functions in QFT suffer divergences in UV, which corresponds to divergences in IR on the gravitational side. Hence, in order to regularize the on-shell action, we restrict the range of the ρ-integration to $\rho \geq \epsilon$ and we evaluate the boundary terms at $\rho = \epsilon$, where $\epsilon > 0$ is a small parameter. A finite number of terms which diverge as $\epsilon \to 0$ can be isolated and the action takes form:

$$S_{reg}[f_{(0)}; \epsilon] = \int_{\rho=\epsilon} d^4x \sqrt{g(0)} \left[\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu+1)} a_{(2)} + ... - \log \epsilon a_{(2\nu)} + \mathcal{O}(\epsilon^0) \right] \tag{3.12}$$

where ν is a positive number that only depends on the scale dimension of the dual operator and $a_{(2k)}$ are local functions of the source(s) $f_{(0)}$. The logarithmic divergence directly gives the conformal anomaly [21]. The divergences do not depend on $\tilde{f}_{(2k)}$, i.e the coefficient that is not determined by the near boundary analysis.

The counterterm action is defined as

$$S_{ct}[\mathcal{F}(x, \epsilon); \epsilon] = -divergent\ terms\ of\ S_{reg}[f_{(0)}; \epsilon] \tag{3.13}$$

where the divergent term are expressed in terms of the fields $\mathcal{F}(x, \epsilon)$ 'living' at the regulated surface $\rho = \epsilon$ and the induced metric there is given by $\gamma_{ij} = g_{ij}(x, \epsilon)/\epsilon$. This is required for covariance and includes an "inversion" of the expansion (3.11).

3.2.3 Renormalized on-shell action

To obtain the renormalized action, we first define a subtracted action at the cutoff

$$S_{sub}[\mathcal{F}(x,\epsilon);\epsilon] = S_{reg}[f_{(0)};\epsilon] + S_{ct}[\mathcal{F}(x,\epsilon);\epsilon]\,. \tag{3.14}$$

The subtracted action has a finite limit as $\epsilon \to 0$ and the renormalized action is a functional of the sources defined by this limit:

$$S_{ren}[f_{(0)}] = \lim_{\epsilon\to 0} S_{sub}[\mathcal{F};\epsilon]\,. \tag{3.15}$$

The distinction between S_{sub} and S_{ren} is needed because the variations required to obtain the correlation functions are performed before the limit $\epsilon \to 0$ is taken.
The procedure described above is referred to as "minimal" scheme where we only subtract the divergences from the S_{reg}. However, we still have freedom to add some finite counterterms, similar to the standard quantum field theory. Adding finite counterterms corresponds to a change of the scheme, they may be used to restore some symmetries of a theory, e.g. supersymmetry [64].

3.2.4 Exact 1-point functions

The one-point functions can be obtained by functionally differentiating S_{ren} with respect to the sources. With the renormalization procedure applied, the variation of (3.10) with respect to sources reads

$$\delta S_{ren}[g_{(0)ij},\phi_{(0)}] = \int d^4x\sqrt{g_{(0)}}\left[\frac{1}{2}\langle T_{ij}\rangle\,\delta g^{ij}_{(0)} + \langle O_\Phi\rangle\,\delta\phi_{(0)}\right] \tag{3.16}$$

where $g_{(0)ij},\phi_{(0)}$ are the sources for the dual operators, as discussed above.

The 1-point function of the scalar operator O_Φ in the presence of sources is defined as

$$\langle O_\Phi\rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{ren}}{\delta\phi_{(0)}} \tag{3.17}$$

It can be computed by rewriting it in terms of the fields living at the regulated boundary

$$\langle O_\Phi\rangle = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{d/2-m}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{sub}}{\delta\Phi(x,\epsilon)}\right) \tag{3.18}$$

$\gamma_{ij}(x) = g_{ij}(x)/\epsilon$ is the induced metric on the boundary and $\gamma = \det\gamma_{ij}$. (3.18) has a limit as $\epsilon \to 0$ by construction. Explicit evaluation of this limit gives us

$$\langle\mathcal{O}_\Phi\rangle \sim \phi_{(2n)} + C(\phi_{(0)}) \tag{3.19}$$

Here, $C(\phi_{(0)})$ is a function that depends locally on the sources and hence yields contact terms to higher point functions. Its exact form depends on the theory in question and is also in general scheme dependent. The coefficient of $f_{(2n)}$ also depends on the theory in question, is however scheme independent.
The expectation value of the stress-energy tensor is given by

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{ren}}{\delta g^{ij}_{(0)}} \tag{3.20}$$

which again can be rewritten in terms of boundary fields and amounts to

$$\langle T_{ij} \rangle = \lim_{\epsilon \to 0} \frac{2}{\sqrt{g(x,\epsilon)}} \frac{\delta S_{ren}}{\delta g^{ij}(x,\epsilon)} = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon} T_{ij}[\gamma] \right) \tag{3.21}$$

where $T_{ij}[\gamma]$ is the stress-energy tensor of the theory at the cutoff $\rho = \epsilon$

3.2.5 Ward identities

We have two general Ward identities [22]. Using (3.16), the invariance of (3.10) under diffeomorphisms

$$\delta g^{ij}_{(0)} = -\left(\nabla^i \xi^j + \nabla^j \xi^i \right), \qquad \delta \phi_{(0)} = \xi^i \nabla_i \phi_{(0)} \tag{3.22}$$

yield the Ward identity for the conservation of the stress-energy tensor

$$\nabla^i \langle T_{ij} \rangle = - \langle O_\Phi \rangle \nabla_j \phi_{(0)} \,. \tag{3.23}$$

Furthermore, the invariance under Weyl transformations

$$\delta g^{ij} = -2\sigma g^{ij}, \delta \phi_{(0)} = -(4-\Delta)\sigma \phi_{(0)} \tag{3.24}$$

results in the conformal Ward identity

$$\langle T^i_i \rangle = -(4-\Delta)\phi_{(0)} \langle O_\Phi \rangle + \mathcal{A} \tag{3.25}$$

where $\mathcal{A}$ stands for the conformal anomaly, which is obtained from the logarithmic counterterm of the bulk action, as mentioned earlier.

3.2.6 RG flows

The energy scale on the boundary theory is associated with the radial coordinate of the bulk space-time. One can study RG transformations by using bulk diffeomorphisms that induce a Weyl transformation on the boundary metric. The simplest of such transformations has the following form

$$\rho = \rho' \mu^2, \qquad x^i = x'^i \mu \tag{3.26}$$

This i s an isometry of AdS. We know how bulk fields transform under bulk diffeomorphisms, hence we can compute how the $f_{(2n)}$ transforms under (3.26) and therefore find the RG transformation on n-point functions.

3.2.7 n-point functions

For computing n-point functions we need exact solutions of the bulk field equations with prescribed but arbitrary boundary conditions. Given such solution we can read-off $f_{(2n)}$ as a function of $f_{(0)}$ by considering the asymptotics of the solution and hence compute the n-point functions.

General Dirichlet problem is not necessarily tractable since the bulk equations are coupled non-linear equations. However we can linearize the bulk equations and solve them for linearized fluctuations. This way we determine the linear in $f_{(0)}$ term of $f_{(2n)}$, which is sufficient for 2-point functions. Higher point functions can be then found even if we do not have an exact solution. We solve the bulk field equations perturbatively and hence determine the terms of $f_{(2n)}$ of higher order in $f_{(0)}$.

3.3 Massive scalar

Let us illustrate the method on a simple example of a free massive scalar field in AdS space-time. The action of the system is given by

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{G} \left(G^{\mu\nu} \partial_{mu} \Phi \partial_\nu \Phi + m^2 \Phi^2 \right) \tag{3.27}$$

Space-time metric is given by

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} dx^i dx^i \tag{3.28}$$

And the bulk field equation is

$$\left(-\Box_G + m^2 \right) \Phi = -\frac{1}{\sqrt{G}} \partial_\mu \left(\sqrt{G} G^{\mu\nu} \partial_\nu \Phi \right) + m^2 \Phi = 0 \tag{3.29}$$

3.3.1 Asymptotic solution

First of all we want to obtain asymptotic solutions of (3.29). However, in general, scalar field couples to the Einstein equation through its stress energy tensor and we need to solve the coupled system of gravity-scalar equations. In the case of the example at hand the equations decouple near the boundary and we can study the boundary field equations in a fixed gravitational background given by (3.28).

We are interested in the solutions of the form

$$\begin{aligned}\Phi(x,\rho) &= \rho^{(d-\Delta)/2}\phi(x,\rho)\,,\\ \phi(x,\rho) &= \phi_{(0)}+\rho\phi_{(2)}+\rho^2\phi_{(4)}+\ldots\end{aligned} \tag{3.30}$$

Now inserting this into (3.29) gives us following equation

$$\begin{aligned}0=\Big[& \left(m^2-\Delta\left(\Delta-d\right)\right)\phi(x,\rho)\\ & -\rho\left(\Box_0\phi(x,\rho)+2\left(d-2\Delta+2\right)\partial_\rho\phi(x,\rho)+4\rho\partial_\rho^2\phi(x,\rho)\right)\Big]\end{aligned} \tag{3.31}$$

with $\Box_0 = \delta^{ij}\partial_i\partial_j$. The easiest way to solve this equation is to successively differentiate with respect to ρ and then take $\rho = 0$. The first step (without differentiating) gives us

$$\left(m^2-\Delta\left(\Delta-d\right)\right)=0\,, \tag{3.32}$$

which is the known relation between the mass and the conformal weight Δ of the dual operator. Having (3.32) satisfied and setting $\rho = 0$ again we get a relation for $\phi_{(2)}$

$$\phi_{(2)}(x)=\frac{1}{2(2\Delta-d-2)}\Box_0\phi_{(0)} \tag{3.33}$$

Successive differentiation and setting $\rho = 0$ gives us all coefficients of the expansion (3.30)

$$\phi_{(2n)}=\frac{1}{2n(2\Delta-d-2n)}\Box_0\phi_{(2n-2)} \tag{3.34}$$

However, this procedure stops when $2\Delta - d - 2n = 0$. In this case it is necessary to introduce a logarithmic term at order $\rho^{\Delta/2}$ in the expansion to obtain a solution. For example, for $\Delta = d/2+1$, the asymptotic expansion gets modified to

$$\phi(x,\rho)=\phi_{(0)}+\rho\left(\phi_{(2)}+\log\rho\psi_{(2)}\right)+\ldots \tag{3.35}$$

solving for $\psi_{(2)}$ yields

$$\psi_{(2)}=-\frac{1}{4}\Box_0\phi_{(0)} \tag{3.36}$$

and we find that $\phi_{(2)}$ is not determined by the field equations in this case. This simple case is easily generalized for $\Delta = d/2 + k$ with and integer k, then we get

$$\psi_{(2\Delta-d)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\Box_0)^k\phi_{(0)} \tag{3.37}$$

and again $\phi_{(2\Delta-d)}$ remains undetermined by the bulk field equations.

3.3.2 Regularization, counterterms and renormalization

Now let us consider the regularized action of the asymptotic solution at hand

$$\begin{aligned} S_{\text{reg}} &= \frac{1}{2}\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{G}(G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + m^2\phi^2) \\ &= \frac{1}{2}\int_{\rho\geq\epsilon} d^{d+1}x\sqrt{G}\Phi(-\Box_G + m^2)\Phi - \frac{1}{2}\int_{\rho=\epsilon} d^dxG^{\rho\rho}\Phi\partial_\rho\Phi \end{aligned} \tag{3.38}$$

where $\rho = \epsilon$ is taken to be the lower bound of integration. Bulk field equations are satisfied, so that the bulk term vanishes and inserting the explicit asymptotic solution yields

$$\begin{aligned} S_{\text{reg}} &= -\int_{\rho=\epsilon} d^dx\epsilon^{-\Delta+\frac{d}{2}}\left(\frac{1}{2}(d-\Delta)\phi(x,\epsilon)^2 + \epsilon\phi(x,\epsilon)\partial_\epsilon\phi(x,\epsilon)\right) \\ &= \int_{\rho=\epsilon} d^dx\left(\epsilon^{-\Delta+\frac{d}{2}}a_{(0)} + \epsilon^{-\Delta+\frac{d}{2}+1}a_{(2)} + \cdots - \log\epsilon a_{(2\Delta-d)}\right) \end{aligned} \tag{3.39}$$

with

$$\begin{aligned} a_{(0)} &= -\frac{1}{2}(d-\Delta)\phi_{(0)}^2, \qquad a_{(2)} = -(d-\Delta+1)\phi_{(0)}\phi_{(2)} = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_{(0)}\Box_0\phi_{(0)}, \\ a_{(2\Delta-d)} &= -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_{(0)}(\Box_0)^k\phi_{(0)} \end{aligned} \tag{3.40}$$

So we see, that as discussed in section 2.2.2, the coefficients $a_{(2\nu)}$ of the divergent terms are local functions of the source $\phi_{(0)}$.

In order to obtain the counterterms we have to invert the series (3.30). This is required because it is $\Phi(x,\epsilon)$ rather than $\phi_{(0)}$ that transforms as a scalar under bulk diffeomorphisms at $\rho = \epsilon$. To second order we then have

$$\begin{aligned} \phi_{(0)} &= \epsilon^{-(d-\Delta)/2}\left(\Phi(x,\epsilon) - \frac{1}{2(2\Delta-d-2)}\Box_\gamma\Phi(x,\epsilon)\right) \\ \phi_{(2)} &= \epsilon^{-(d-\Delta)/2-1}\frac{1}{2(2\Delta-d-2)}\Box_\gamma\Phi(x,\epsilon) \end{aligned} \tag{3.41}$$

where $\Box_\gamma$ is the Laplacian of the induced metric $\gamma_{ij} = \frac{1}{\epsilon}\delta_{ij}$ at $\rho = \epsilon$. These results suffice to rewrite $a_{(0)}$ and $a_{(2)}$ in terms of $\Phi(x,\epsilon)$. We then obtain for the counterterm action

$$S_{\text{ct}} = \int\sqrt{\gamma}\left(\frac{d-\Delta}{2}\Phi^2 + \frac{1}{2(2\Delta-d-2)}\Phi\Box_\gamma\Phi\right) + \cdots \tag{3.42}$$

where the dots stand for higher derivative terms. If $\Delta = d/2+1$ the coefficient of the $\Phi\Box_\gamma\Phi$ is replaced by $-\frac{1}{4}\log\epsilon$; and similarly, when $\Delta = d/2+k$ we obtain a k-derivative logarithmic counterterm.

The renormalized action in the minimal subtraction scheme is then given by (3.15), where we still have freedom to add finite counterterms. This corresponds to the scheme dependence in the field theory.

3.3.3 Exact 1-pt function

Equation (3.18) yields

$$\langle\mathcal{O}_\Phi\rangle_s = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon^{\Delta/2}}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\rm sub}}{\delta\Phi(x,\epsilon)}\right) \tag{3.43}$$

Let us discuss the case of $\Delta = d/2+1$ for simplicity. In this case we have

$$\begin{aligned}
\mathrm{d}S_{\rm sub} &= \mathrm{d}S_{\rm reg} + \mathrm{d}S_{\rm ct} \\
&= \int_{\rho\geq\epsilon} \mathrm{d}^{d+1}x\delta\Phi(-\Box_G + m^2)\Phi \\
&\quad + \int_{\rho=\epsilon}\mathrm{d}^dx\delta\Phi\left(-2\epsilon\partial_\epsilon\Phi + (d-\Delta)\Phi - \frac{\log}{\epsilon}\Box_\gamma\Phi\right)
\end{aligned} \tag{3.44}$$

The bulk field equations hold, hence the first integral vanishes and we obtain

$$\frac{\mathrm{d}S_{\rm sub}}{\delta\Phi} = -2\epsilon\partial_\epsilon\Phi + (d-\Delta)\Phi - \frac{1}{2}\log\epsilon\Box_\gamma\Phi \tag{3.45}$$

Inserting this in (3.43) and substituting for Φ the explicit asymptotic solution, shows that the divergent terms cancel, as expected, and the final part is

$$\langle\mathcal{O}_\Phi\rangle_s = -2\left(\phi_{(2)} + \psi_{(2)}\right) \tag{3.46}$$

As discussed earlier, the 1-point function depends on the part of the asymptotic solution which is not determined by the near boundary analysis. $\psi_{(2)}$ is a local functions of the sources and is scheme dependent. In fact if we add completely by the following finite counterterm to the action

$$S_{ct,fin} = -\frac{1}{4}\int d^dx\phi_{(0)}\Box_0 = -\frac{1}{2}\int d^dx\sqrt{\gamma}\mathcal{A}\,, \tag{3.47}$$

then we can remove $\psi_{(2)}$ completely from the 1-point function. $\mathcal{A}$ is the matter conformal anomaly For general Δ the result is [22]

$$\langle\mathcal{O}_\Phi\rangle_s = -(2\Delta-d)\phi_{(2\Delta-d)} + C(\phi_{(0)}) \tag{3.48}$$

3.3.4 RG transformations

In order to determine the RG transformations of the correlation functions we need to determine how the coefficients in the asymptotic solution transform under (3.26). $\Phi(x,\rho)$ is scalar, so

$$\Phi'(x',\rho') = \Phi(x,\rho) \tag{3.49}$$

which leads to

$$\begin{aligned}
\phi'_{(0)}(x') &= \mu^{d-\Delta}\phi_{(0)}(x'\mu) && (3.50)\\
\phi'_{(2)}(x') &= \mu^{d-\Delta+2}\phi_{(2)}(x'\mu) && (3.51)\\
&\;\;\cdot\,\cdot \\
\psi'_{(2\Delta-d)}(x') &= \mu^{\Delta}\psi_{(2\Delta-d)}(x'\mu) && (3.52)\\
\phi'_{(2\Delta-d)}(x') &= \mu^{\Delta}(\phi_{(2\Delta-d))}(x'\mu) + \log\mu^2\psi_{(2\Delta-d)}(x'\mu)) && (3.53)
\end{aligned}$$

and from this we obtain the transformed 1-pt function

$$\left\langle \mathcal{O}(x')\right\rangle'_s = \mu^{\Delta}\left(\left\langle \mathcal{O}(x'\mu)\right\rangle_s - (2\Delta-d)\log\mu^2\psi_{(2\Delta-d)}(x'\mu)\right) \tag{3.54}$$

The new term can be obtained by addition of the following counterterm

$$S_{\text{ct,fin}}(\mu) = \int d^dx\sqrt{\gamma}\frac{1}{2}\log\mu^2\mathcal{A} \tag{3.55}$$

with $\mathcal{A}$ being the matter conformal anomaly. This is an expected result since we are computing conformal field theory correlation functions. Hence, the correlation functions should have a trivial scale dependence, up to the effects of conformal anomalies.

3.3.5 Correlation functions and n-point functions

Up to now all considerations involved only the near boundary analysis. Holographic 1-point functions were derived, but they involve coefficients that are not determined by the near boundary analysis. In order to obtain those we need an exact solution of the bulk field equations. For the case at hand, the field equation is linear in Φ and can be solved exactly, in general however, the field equations are non-linear and cannot be solved in full generality. In such cases, one can linearize around the background and solve the linearized fluctuation equations, which is sufficient to obtain 2-point functions since we only need to know $\phi_{(2\Delta-d)}$ up to linear order in the source in order to determine them.

3.4 RG flows

The discussed method to compute holographically renormalized correlation functions can be used to obtain correlation functions for all quantum field theories that can be obtained via a deformation or a vev from a CFT that has a holographic dual [64],[63]. We will discuss two examples, Coulomb branch flow and GPPZ flow, and present the results for these two cases which we will need later in order to validate the method used for Klebanov-Strassler background.

As we have discussed, the asymptotic expansion of a scalar field that is dual to a dimension Δ operator has following form

$$\Phi(x,\rho) = \rho^{(d-\Delta)/2}\phi_{(0)} + \cdots + \rho^{\Delta/2}\phi_{(2\Delta-d)} + \cdots \tag{3.56}$$

$\phi_{(0)}$ is interpreted as a source and $\phi_{(2\Delta-d)}$ as a 1-point function.

The most general solution preserving Poincaré invariance in 4-dimensions is

$$\begin{aligned} ds^2 &= e^{2A(r)}\delta_{ij}dx^i dx^j + dr^2\,, \\ \Phi &= \Phi(r)\,, \end{aligned} \tag{3.57}$$

which can be recast in the form (3.5) Section 3.1. The action that describes the dynamics of this system is given in (3.10). The asymptotic behavior of the scalar field distinguishes between two different kinds of solutions:

- *Operator deformation.* In this case the near-boundary expansion of Φ is $\Phi \sim \rho^{(d-\Delta)/2}\varphi_0$, and this corresponds to the addition of the term $\varphi_0 O$ in the Lagrangian of the boundary theory.
- *VEV deformation.* Here the near-boundary expansion of Φ is $\Phi \sim \rho^{\Delta/2}\varphi_0$, and the boundary Lagrangian is still the same, but the vev of the dual operator is non-zero, $\langle O\rangle \sim \varphi_0$, and the vacuum spontaneously breaks conformal invariance.

In general it is difficult to solve the second order field equations of (3.10), however one can simplify matters further if the potential $V(\Phi)$ is derivable from the superpotential $W(\Phi)$ and is of the form

$$V(\Phi) = \frac{1}{2}(\partial_\Phi W)^2 - \frac{4}{3}W^2. \tag{3.58}$$

The BPS analysis of the domain wall action then yields the flow equations [65, 66]

$$\frac{dA(r)}{dr} = -\frac{2}{3}W(\Phi), \qquad \frac{d\Phi(r)}{dr} = \partial_\Phi W(\Phi)\,. \tag{3.59}$$

3.4.1 Coulomb branch flow

This solution corresponds to turning on a VEV of scalar operator of dimension 2. The flow describes the theory at a point on the Coulomb branch of $\mathcal{N} = 4$ SYM [16],[17].

The superpotential is given by

$$W(\Phi) = -e^{-\frac{2\Phi}{\sqrt{6}}} - \frac{1}{2}e^{\frac{4\Phi}{\sqrt{6}}}\,. \tag{3.60}$$

From this we can compute the potential $V(\Phi)$ to be

$$V(\Phi) = -e^{-\frac{4\Phi}{\sqrt{6}}} + 2e^{\frac{2\Phi}{\sqrt{6}}}\,. \tag{3.61}$$

This has following expansion near $\Phi = 0$

$$V(\Phi) = -3 - 2\Phi^2 + \frac{4}{3\sqrt{6}}\Phi^3 + \mathcal{O}(\Phi^4)\,, \tag{3.62}$$

which has a tachyonic mass $m^2 = -4$, so that Φ is dual to an operator of the scale dimension $\Delta = 2$.

One can express the domain-wall solution in terms of one variable v

$$v = e^{\sqrt{6}\Phi}\,, \qquad \mathrm{e}^{2A} = l^2\frac{v^{2/3}}{1-v}\,, \qquad \frac{dv}{dr} = 2v^{2/3}(1-v)\,. \tag{3.63}$$

The boundary lies at $v = 1$ and there is a curvature singularity at $v = 0$. l is the radius of the disc of branes in the 10-dimensional full theory and can be set to 1 for calculational purposes.

Asymptotic behavior of the scalar field is given by

$$\Phi \approx -\frac{1}{\sqrt{6}}(1-v) = -\frac{1}{\sqrt{6}}e^{-2r}\,. \tag{3.64}$$

For further calculations we can make a coordinate transformation from v to $\rho = \mathrm{e}^{-2r}$ so that (3.57) is recast in the form (3.5). Then we have a near-boundary expansion

$$1 - v = l^2\rho - \frac{2}{3}l^4\rho^2 + \mathcal{O}(\rho^3)\,. \tag{3.65}$$

In these coordinates we have following for Φ and A

$$\Phi(\rho) = \frac{1}{\sqrt{6}}\left(-l^2\rho + \frac{1}{6}l^4\rho^2 + \mathcal{O}(\rho^3)\right) \qquad e^{2A} = \frac{1}{\rho}\,. \tag{3.66}$$

and the decomposition of the active scalar field is

$$\Phi(x,\rho) = \rho\log\rho\phi(x,\rho) + \rho\tilde{\phi}(x,\rho) \tag{3.67}$$

Next we will need to solve the scalar field equations

$$\Box_G \Phi = \frac{\partial V}{\partial \Phi} \tag{3.68}$$

with potential given by (3.62) and we look for the solutions of the form (3.66) and with

$$\begin{aligned} \phi(x,\rho) &= \phi_{(0)} + \phi_{(2)} + \rho \log \rho \psi_{(2)} + ... \\ \tilde{\phi}(x,\rho) &= \tilde{\phi}_{(0)} + \tilde{\phi}_{(2)}\rho + ... \end{aligned} \tag{3.69}$$

where again $\phi_{(0)}$ is the source of the operator and $\tilde{\phi}_{(0)}$ is proportional to its VEV.
Solving those field equations order by order in ρ yields [63]

$$\begin{aligned} \phi_{(2)} &= -\frac{1}{4}\left(\Box_{(0)}\phi_{(0)} + \frac{2}{3}\phi_{(0)}R[g_{(0)}]\right) - \frac{4}{\sqrt{6}}(\phi_{(0)}^2 - \frac{1}{2}\phi_{(0)}\tilde{\phi}_{(0)}) \\ \tilde{\phi}_{(2)} &= -\frac{1}{4}\left(\Box_{(0)}\tilde{\phi}_{(0)} + \frac{1}{3}R[g_{(0)}](\tilde{\phi}_{(0)} + \phi_{(0)}) + 8(\phi_{(2)} + \psi_{(2)})\right) + \frac{1}{\sqrt{6}}\tilde{\phi}_{(0)}^2 \\ \psi_{(2)} &= \frac{1}{\sqrt{6}}\phi_{(0)}^2 \end{aligned} \tag{3.70}$$

And as discussed in previous sections, $\tilde{\phi}_{(0)}$ is not determined by these equations.
The regularized action is given by

$$\begin{aligned} S_{\text{reg}} &= \int_{\rho\geq\epsilon} d^5x\,\sqrt{G}\left(\frac{1}{2}G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2\Phi^2 + \frac{4}{3\sqrt{6}}\Phi^3\right) + ... \\ &= -\int d^dx\sqrt{g_{(0)}}\left[\log^2\epsilon\phi_{(0)}^2 + \log\epsilon(\phi_{(0)}^2 + 2\phi_{(0)}\tilde{\phi}_{(0)}) + \mathcal{O}(\epsilon^0)\right] \\ &= -\int d^4x\sqrt{\gamma}\left[\Phi^2(x,\epsilon) + \frac{\Phi^2(x,\epsilon)}{\log\epsilon}\right] + \mathcal{O}(\epsilon^0) \end{aligned} \tag{3.71}$$

Then correspondingly the renormalized action is given by

$$S_{\text{ren}} = \lim_{\epsilon\to 0} S_{\text{sub}} \equiv \lim_{\epsilon\to 0}\left[S_{\text{reg}} + \int d^4x\sqrt{\gamma}\left(\frac{\Delta}{2}\Phi^2(x,\epsilon) + \frac{\Phi^2(x,\epsilon)}{\log\epsilon}\right)\right] \tag{3.72}$$

Now we are ready to compute the 1-point function for the Coulomb branch case

$$\langle O_\Phi \rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta S_{\text{ren}}}{\delta\phi_{(0)}} = \lim_{\epsilon\to 0}\left(\frac{\log\epsilon}{\epsilon}\frac{1}{\sqrt{\gamma}}\frac{\delta S_{\text{sub}}}{\delta\Phi(x,\epsilon)}\right) \tag{3.73}$$

(3.72) yields

$$\frac{\delta S_{\text{sub}}}{\delta\Phi(x,\epsilon)} = 2\left(-\epsilon\partial_\epsilon\Phi(x,\epsilon) + \frac{\Delta}{2}\Phi(x,\epsilon) + \frac{\Phi(x,\epsilon)}{\log\epsilon}\right). \tag{3.74}$$

The two last terms in the above equation are coming from the counterterms. These are counterterms that are needed in order to cancel divergences, however we still have the freedom to add a scheme dependent finite counterterm of the form $u_2 \frac{\Phi^2(x,\epsilon)}{\log^2 \epsilon}$ (where u_2 is a proportionality factor), which yields an additional term in (3.74), namely $\frac{2u_2\Phi(x,\epsilon)}{\log^2 \epsilon}$ and thus we get

$$\langle O_\Phi \rangle = 2\tilde{\phi}_{(0)} + 2u_2\phi_{(0)} \,, \tag{3.75}$$

where the second terms comes from the finite counterterm. This result we will use later in order to justify the methods we use to compute correlation functions for the Klebanov-Strassler background.

3.4.2 GPPZ flow

The second example we would like to consider with the outlook for later is the GPPZ flow. This is the supergravity dual of a $\mathcal{N} = 1$ supersymmetry preserving mass deformation of $\mathcal{N} = 4$ SYM theory [67]. In general GPPZ flow has two active scalars, however for our purposes here we will consider only the case when only one of the scalars with scale dimension $\Delta = 3$ is turned on and will denote it Φ.

The superpotential reads

$$W(\Phi) = -\frac{3}{4}\left[1 + \cosh\left(\frac{2\Phi}{\sqrt{3}}\right)\right] \tag{3.76}$$

Thus we have for the potential

$$V(\Phi) = -\frac{9}{8} - \frac{3}{2}\cosh\left(\frac{2\Phi}{\sqrt{3}}\right) - \frac{3}{8}\cosh^2\left(\frac{2\Phi}{\sqrt{3}}\right) \,, \tag{3.77}$$

and it has following expansion near $\Phi = 0$

$$V(\Phi) = -3 - \frac{3}{2}\Phi^2 - \frac{1}{3}\Phi^4 + \mathcal{O}(\Phi^6) \,, \tag{3.78}$$

and again from the mass $m^2 = -3$ we read off the scale dimension of the dual scalar $\Delta = 3$. The domain-wall solution can be written in terms of variable $u = 1 - e^{(-2r)}$

$$\varphi_B = \frac{\sqrt{3}}{2}\log\frac{1+\sqrt{1-u}}{1-\sqrt{1-u}} \,, \qquad e^{2A} = \frac{u}{1-u} \,. \tag{3.79}$$

The boundary is at $u = 1$ and the solution is singular at $u = 0$. Asymptotic behavior of the Φ is given by $\Phi \approx \sqrt{3}e^{-r}$, so we have here an operator deformation by a dimension 3 operator.

Again we can transform into the coordinate system of (3.5) and then we have following near boundary expansion

$$\varphi_B = \rho^{1/2}\left[\sqrt{3} + \rho\frac{1}{\sqrt{3}} + \mathcal{O}(\rho^2)\right] \qquad e^{2A} = \frac{1}{\rho}(1-\rho) \tag{3.80}$$

The asymptotic expansion of the scalar field reads

$$\Phi(x,\rho) = \rho^{1/2}\left(\phi_{(0)}(x) + \rho\phi_{(2)}(x) + \rho\log\rho\psi_{(2)}(x) + ...\right) \tag{3.81}$$

Solving the scalar field equations order by order in ρ yields [63]

$$\psi_{(2)} = -\frac{1}{4}\Box\phi_{(0)}(x) + \frac{1}{6}R[g_{(0)}]\phi_{(0)}(x) \tag{3.82}$$

and dependence of $\phi_{(2)}$ on the sources $\phi_{(0)}$ and $g_{(0)}$ remains undetermined.
Solving the Einstein equations to the lowest order in ρ yields

$$g_{(2)ij} = \frac{1}{2}\left(R_{(0)ij} - \frac{1}{6}R_{(0)}g_{(0)ij}\right) = \frac{1}{3}\phi^2_{(0)ij}g_{(0)ij} \tag{3.83}$$

We see that the case of the GPPZ flow is different from CB flow case in that way, that we have here a backreaction of the scalar to $g_{(2)}$. Hence we cannot do fixed background calculations here, rather we need to solve coupled system of equations.

Regularized action in this case is

$$\begin{aligned} S_{\rm reg} = \int \rho d^4x\sqrt{g_{(0)}}\Bigg[-\frac{3}{2\epsilon^2} + \frac{1}{2\epsilon}\phi^2_{(0)} + \log\epsilon\Bigg(\frac{1}{32}(R_{ij}[g_{(0)}]R^{ij}[g_{(0)}] - \frac{1}{3}R^2[g_{(0)}]) \\ +\frac{1}{8}(\phi_{(0)}\Box_0\phi_{(0)} + \frac{1}{6}R[g_{(0)}]\phi^2_{(0)})\Bigg) + \mathcal{O}(\epsilon^0)\Bigg] \end{aligned} \tag{3.84}$$

Rewriting divergences in terms of induced fields γ_{ij} and $\Phi(x,\epsilon)$ yields the counterterm action

$$\begin{aligned} S_{\rm ct} = \int_{\rho=\epsilon} d^4x\sqrt{\gamma}\Bigg(\frac{3}{2} - \frac{1}{8}R[\gamma] + \frac{1}{2}\Phi^2(x,\epsilon) + \frac{1}{18}\Phi^4(x,\epsilon) \\ -\log\epsilon\left[\frac{1}{32}(R_{ij}[\gamma]R^{ij}[\gamma] - \frac{1}{3}R[\gamma]^2) + \frac{1}{4}[\Phi(x,\epsilon)\Box_\gamma\Phi(x,\epsilon) + \frac{1}{6}R\Phi^2(x,\epsilon)]\right]\Bigg). \end{aligned} \tag{3.85}$$

Note, that we again used the freedom of adding a finite counterterm which is the quartic term $\int_{\rho=\epsilon} d^4x\sqrt{\gamma}\frac{1}{18}\Phi^4$. As discussed in previous sections, addition of the finite counterterms corresponds to a choice of a scheme. In this case the quartic counterterm is added in order to preserve supersymmetry. (Note, that in this section we choose the coefficient in front of the quartic counterterm in the way that it agrees with SUSY scheme. In Section 5.3.2 we will discuss the notion of scheme dependence in more detail).

With above results we are ready to compute the 1-point function

$$\langle O_\Phi \rangle = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\rm ren}}{\delta \phi_{(0)}} = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^{3/2}\sqrt{\gamma}} \frac{\delta S_{\rm sub}}{\delta \Phi(x,\epsilon)} \right) = -2(\phi_{(2)} + \psi_{(2)}) + \frac{2}{9}\phi_{(0)}^3 \,, \tag{3.86}$$

with $\psi_{(2)} = -\frac{1}{4}\Box\phi_{(0)}$. Again, as discussed in Section 3.3 we can remove the $\psi_{(2)}$ term by adding an appropriate counterterm to the action. This result we will also need later for verification purposes.

4 Klebanov-Strassler Background

Until now we were considering only aAdS backgrounds which are dual to conformal gauge theories. However, the in the far reach the goal of gauge/string duality theories is to find a string dual description for the QCD. For that we need get rid of conformality, supersymmetry (or at least reduce the amount of the supersymmetrys) and, also very important, we need to have a confining theory. There are several approaches to that, such as Witten backgrounds, Klebanov-Strassler backgrounds and Maldacena-Nunez backgrounds [23, 24, 25]. In our work we are considering Klebanov-Strassler background and in this chapter we will give an introduction to it.

4.1 Conifold

One of the ways to formulate AdS/CFT duality at zero temperature and with reduced amount of SUSY is to place a stack of D3-branes at the tip of a 6-dimensional Ricci-flat cone, whose base is a 5-dim compact Einstein space, known as *conifold.*

Conifold is a singular non-compact Calabi-Yau 3-fold, described by following quadratic equations in $\mathbb{C}^4$

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \,. \tag{4.1}$$

This equation defines a real cone over a 5-dimensional manifold. Topology of the base can be shown to be $S^2 \times S^3$ and is called $T^{1,1}$. The metric of the base is given by:

$$d\Omega^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \tag{4.2}$$

$$+\frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + \frac{1}{6}\left(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) \tag{4.3}$$

So we see upon close examination of (4.2) that we can understand this space as $S^2 \times S^2$ with S^1 fibered over them. The 10-dim metric of the full space is then:

$$ds^2 = dr^2 + r^2 d\Omega^2_{T^{1,1}} \tag{4.4}$$

We can also introduce a new set of coordinates a_i, b_j by a following basis transformation

$$
\begin{aligned}
Z &= \begin{pmatrix} z_3 + iz_4 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 + iz_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} \\
&= \begin{pmatrix} -c_1 s_2 \, \mathrm{e}^{\frac{i}{2}(\psi+\phi_1-\phi_2)} & c_1 c_2 \, \mathrm{e}^{\frac{i}{2}(\psi+\phi_1+\phi_2)} \\ -s_1 s_2 \, \mathrm{e}^{\frac{i}{2}(\psi-\phi_1-\phi_2)} & s_1 c_2 \, \mathrm{e}^{\frac{i}{2}(\psi-\phi_1+\phi_2)} \end{pmatrix} ,
\end{aligned} \tag{4.5}
$$

with $c_i = cos\frac{\theta_i}{2}$ and $s_i = sin\frac{\theta_i}{2}$. This basis is the most suitable for studying the symmetries of the conifold.

Conifold exhibits $SU(2)\times SU(2)\times U(1)$ symmetry, where $SU(2)$'s are the rotational symmetries in respective S^2's and $U(1)$ acts by shifting ψ. The latter symmetry can be identified with the R-symmetry, $U(1)_R$, of the dual gauge theory. Coordinates a_i, b_j just defined transform in following way under the full symmetry group

$$
SU(2) \times SU(2) \;\; \text{symmetry :} \qquad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to L \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \tag{4.6}
$$

$$
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \to R \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} , \tag{4.7}
$$

$$
R - \text{symmetry :} \qquad (a_i, b_j) \to \mathrm{e}^{i\frac{\alpha}{2}} (a_i, b_j) \, , \tag{4.8}
$$

i.e. a and b transform as $(1/2, 0)$ and $(0, 1/2)$, respectively, under the $SU(2) \times SU(2)$ group with R-charge 1/2. From (4.6) we notice a certain ambiguity in the definition of a and b. Following redefinition

$$
a_i \to \lambda a_i \, , \qquad b_j \to \frac{1}{\lambda} b_j , \qquad \lambda \in \mathbb{C} \tag{4.9}
$$

leads to the same matrix Z. One can fix the magnitude of the transformation (4.9) by imposing the constraint $|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = 0$. In order to account for the remaining phase one can describe the conifold as the quotient of the a, b space with the above constraint by the relation $a \sim \mathrm{e}^{i\alpha} \, a, b \sim \mathrm{e}^{i\alpha} \, b$.

Apart from symmetry considerations the coordinates a_i and b_j have following importance. In the gauge theory on D3-branes at the tip of the conifold they are promoted to chiral superfields. The corresponding low energy gauge theory on N D3-branes is a $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N)$ gauge theory with bifundamental chiral superfields A_i, B_j $(i, j = 1, 2)$ in $(N, \bar{N})$ and $(\bar{N}, N)$ representations of the respective gauge groups and was constructed in [23] (hence is often referred to as Klebanov-Witten theory). The geometrical symmetries of the conifold correspond to the *continuous global symmetries* of the gauge theory, so that we have following symmetry: $SU(2) \times SU(2) \times U(1)_R \times U(1)_B$ with an additional baryon symmetry. $SU(2)$'s act on A_i and B_j, $U(1)_R$ is the

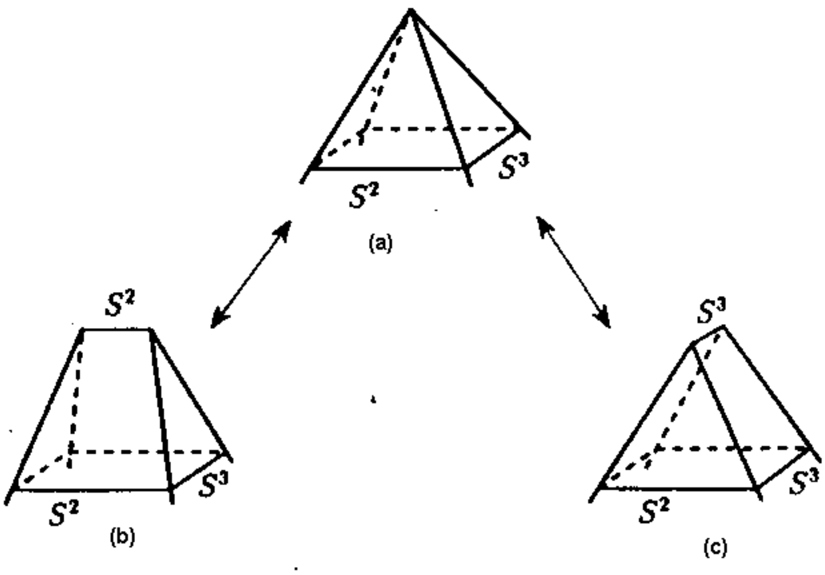
R-symmetry with the same charge for both fields $R_A = R_B = \frac{1}{2}$ and corresponds to the $U(1)$ acting on ψ at the string side. $U(1)_B$ corresponds on the SUGRA side to a gauged $U(1)$ symmetry of the vector field resulting form the Kaluza-Klein reduction of the RR 4-form $C^{(4)} \sim \omega^3 \wedge A$. On the gauge theory side it acts like $A_k \to e^{i\alpha} A_k$, $B_j \to e^{-i\alpha} B_j$, resulting in the opposite charges for the fields A_i and B_j. For consistency of the duality it is necessary to add an exact marginal superpotential, preserving the $SU(2) \times SU(2) \times U(1)_R$ symmetry of the theory. A marginal superpotential has R-charge 2, hence it has to be quartic and the symmetries fix it up to the overall renormalization. The superpotential for this gauge theory is given by:

$$W \sim Tr \det A_i B_j = Tr(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)\,. \tag{4.10}$$

One can also show that the gauge couplings in this theory do not run, so that the theory is superconformal for all values of gauge couplings and the superpotential coupling [23],[26]. Hence it exhibits no confinement, so that we need something different.

In the 6-dimensional space we have a singularity at the tip of the cone (note that 10-dimensional metric show no singularities as $r \to 0$). There are two ways to deal with it: *resolution* and *deformation*. *Resolving* corresponds to blowing up the S^2 at the tip of the conifold and *deforming* to the blowing up of the S^3 (see Figure 3).

Figure 3: Resolving and deformation of the conifold: (a) singular conifold; (b) resolved conifold; (c) deformed conifold [68]



4.2 Duality cascade

Before we proceed with resolving and deforming of conifolds, we will introduce some things, which will be useful later.
First of all let us introduce the notion of *Seiberg duality* [69]. In order to do this, let us consider two different theories.

- First, let us consider a theory, which is usually referred to as "SQCD": a supersymmetric version of QCD with N_f flavors. This model has an $\mathcal{N} = 1$ supersymmetric $SU(N)$ gauge theory, with matter consisting of N_f flavors of quarks and squarks. The left-handed quarks ψ^r and their supersymmetric partners squarks Q^r transform in the $\mathbf{N}$ representation of $SU(N)$ and are organized into N_f chiral multiplets, which are also denoted by Q^r with $r = 1, ..., N_f$. The gauge indices are omitted here for simplicity. Flavor indices imply that Q^r transforms as an $\mathbf{N_f}$ of an $SU(N_f)$ flavor group , which we will label $SU(N_f)_L$ in order to distinguish it from the $SU(N_f)_R$, under which the left-handed anti-quarks $\tilde{\psi}$ and their superpartner $\tilde{Q}$ transform. The latter are organized into N_f chiral multiplets $\tilde{Q}_u$, $u = 1, ..., N_f$ and transform under $\bar{\mathbf{N}}_\mathbf{f}$ of the $SU(N)$ group. The theory has a baryon number symmetry $U(1)_B$ under which Q^r has charge $1/N$ and $\tilde{Q}_u$ the opposite one. There is also an anomalous axial $U(1)$, analogous to QCD. Gluinos provide and additional non-anomalous axial symmetry referred to as $U(1)_\mathcal{R}$, under which gluinos λ have charge 1, the squarks Q and $\tilde{Q}$ the charge $1 - \frac{N}{N_f}$ and quarks ψ and $\tilde{\psi}$ the charge $-\frac{N}{N_f}$. The action of this theory consists of the kinetic terms for the fields, including the minimal couplings to the gauge fields and of the minimal number of additional terms required to preserve supersymmetry. In particular, the superpotential $W(Q, \tilde{Q})$ vanishes.

- Now let us consider another theory, denoted by "SQCD+M". The gauge group of this theory is $SU(\tilde{N})$ with

$$\tilde{N} = N_f - N. \tag{4.11}$$

  It has N_f flavors in the fundamental representation of the gauge group, labeled q_r and $\tilde{q}^u$, with q_r transforming as an $\bar{\mathbf{N}}_\mathbf{f}$ of $SU(N_f)_L$ and $\tilde{q}^u$ as an $\mathbf{N_f}$ of $SU(N_f)_R$. This theory is not quite SQCD since it has another set of gauge-singlet chiral superfields, which we denote by M^r_u, that couple to the matter fields by the superpotential

$$W = yM^r_u q_r \tilde{q}^u \tag{4.12}$$

  with y being the coupling constant. M^r_u transforms as $(\mathbf{N_f}, \bar{\mathbf{N}}_\mathbf{f})$ of the $SU(N_f)_L \times SU(N_f)_R$. Baryon number of the quarks is $1/\tilde{N}$ and the field M is uncharged. This theory also exhibits an anomaly free R symmetry under which both quark fields have charge $1 - \frac{\tilde{N}}{N_f}$, and the field M has R-charge $2\frac{\tilde{N}}{N_f}$, which ensures that the superpotential W has R-charge 2, as required by the supersymmetry.

Seiberg argued [29], that these two theories are dual to each other. More exactly this means, that this theory while being entirely different when they are both weakly coupled, nevertheless have the same physics at low momentum, i.e. in far infrared, where at least one of them is strongly coupled. In particular, the Green's functions of the two theories match identically in the limit where all external momenta are taken to zero, as long as one matches the gauge-invariant operators of one theory to those of the other.

In order to make contact to the Klebanov-Strassler theory, we consider "SQCD" theory with an additional quartic superpotential [69]

$$W = h(Q^r \tilde{Q}_u)(Q^u \tilde{Q}_r)\,, \tag{4.13}$$

where gauge indices are contracted inside the parentheses. The fields Q and $\tilde{Q}$ correspond to A's and B's in Section 4.1. This potential breaks explicitly part of the global flavor symmetry, however, still preserves the diagonal $SU(N_f)$ symmetry and charge conjugation. Now, in the dual "SQCD+M" theory, the operator $(Q^r \tilde{Q}_u)(Q^u \tilde{Q}_r)$ is mapped into the operator $M^r_u M^u_r$ and hence the superpotential reads

$$W = y M^r_u q_r \tilde{q}^u + \hat{h} M^r_u M^u_r\,. \tag{4.14}$$

The coupling $\hat{h}$ is proportional to the h from (4.13). Now, we can use the equation of motion of M in order to integrate it out

$$\bar{D}^2 (M^u_r)^\dagger = y q_r \tilde{q}^u + 2\hat{h} M^u_r\,. \tag{4.15}$$

As the fields M are massive, the left-hand side will approach zero in the infrared and we obtain

$$M^u_r = -\frac{y}{2\hat{h}} q_r \tilde{q}^u\,. \tag{4.16}$$

Substituting this into the superpotential gives us a remarkable result for the low-energy superpotential

$$W_L = -\frac{y^2}{4\hat{h}} q_r \tilde{q}^u q_u \tilde{q}^r \equiv \tilde{h} q_r \tilde{q}^u q_u \tilde{q}^r\,, \tag{4.17}$$

which has exact the same form as the superpotential of the original theory with $\tilde{h} = -y^2/4\hat{h} \sim 1/h$

4.2.1 Resolved conifold

Now let us come back to the dealing with the singularity. As previously said, resolution corresponds to a blow up of the S^2 at the bottom of the conifold. It is equivalent to deforming the modulus constraint from the section 4.1 into

$$|\, b_1 \,|^2 + |\, b_2 \,|^2 - |\, a_1 \,|^2 - |\, a_2 \,|= u^2 \,, \tag{4.18}$$

where $u \in \mathbb{R}$ is a parameter that controls resolution. In the dual theory a $u \neq 0$ corresponds to a particular choice of a vacuum. The metric is the given by [70]:

$$\begin{aligned} ds_6^2 &= K^{-1}dr^2 + \frac{1}{9}Kr^2\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \\ &\quad + \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\right) + \frac{1}{6}\left(r^2 + 6u^2\right)\left(d\theta_2^2 + \sin^2\theta^2 d\phi_2^2\right)\,, \end{aligned} \tag{4.19}$$

where $K = \frac{r^2+9u^2}{r^2+6u^2}$. Hence we see that the two $SU(2)$'s are not interchangeable any more. On the gauge theory side introduction of the resolution parameter corresponds to a particular choice of vacuum. One can define an operator $\mathcal{U}$:

$$\mathcal{U} = \frac{1}{N} Tr\left(B_1^\dagger B_1 + B_2^\dagger B_2 - A_1^\dagger A_1 - A_2^\dagger A_2\right)\,, \tag{4.20}$$

which corresponds to u. With this definition the singular conifold corresponds to gauge theory vacua where $<\mathcal{U}>= 0$, whereas warped resolved conifold correspond to vacua with $<\mathcal{U}>\neq 0$. In this case we have to give VEV's to the bifundamental fields so that $U(1)_B$ will be broken. There are several ways to do this. One can distribute D3-branes evenly on the resolved S^2 [70], which corresponds to assuming for the independence of the warp factor on θ_i, i.e. $h = h(\rho)$. In this case the symmetry of the theory is still $SU(2) \times SU(2) \times U(1)_R$. However, it has been shown for this case [70] that the 10-dim metric becomes singular as $\rho \to 0$, as opposed to the 10-dim metric of the unresolved conifold + Minkowski background. While D3-branes in Minkowski background placed on the tip of the conifold look like $AdS \times T^{1,1}$ in IR, resolved conifold with branes "smeared" on the tip gets a curvature singularity (Ricci tensor becomes singular in this case). Another possibility however is to put D3-branes localized on some particular point of the resolved S^2 (for example north pole). For this case, it has been shown [71], that the warp factor gets an θ_i dependency and remains finite in IR. On the field theory side it corresponds to giving VEV's to just one field, which breaks the symmetry down to $SU(2) \times U(1) \times U(1)_R$. However, in the IR gauge theory flows to the $N = 4\, SU(N)$ SYM theory, as evidenced by the appearance of an $AdS_5 \times S_5$ throat near D3-branes.

4.3 Wrapped branes and the deformed conifold

4.3.1 RG flow

Studies of the duality of the IIB string theory on the $AdS_5 \times T^{1,1}$ and the corresponding field theory have led also to studies of the branes wrapped around the cycles of the conifold and attempts to identify these states in the field theory [27]. For our purposes we are interested in introducing so called fractional D3-branes (wrapped D5-branes). The addition of M of such branes at the singular point changes the gauge group to $SU(N+M) \times SU(N)$. In the dual supergravity background the wrapped D5-branes serve as the source for the magnetic RR-form flux through the S^3 of the $T^{1,1}$. Hence in addition to the N units of the 5-form flux we obtain M units of the 3-form flux [72]

$$\frac{1}{4\pi^2\alpha'}\int_{S^3} F_3 = M\,, \qquad \frac{1}{(4\pi^2\alpha')^2}\int_{T^{1,1}} F_5 = N\,. \tag{4.21}$$

The coefficients above follow from the quantization rule for the Dp-brane tension

$$\int_{S^{8-p}} \star F_{p+2} = 2\kappa^2 \tau_p N\,, \tag{4.22}$$

with

$$\tau_p = \frac{\sqrt{\pi}}{\kappa}(4\pi^2\alpha')^{(3-p)/2} \tag{4.23}$$

and $\kappa = 8\pi^{7/2} g_s \alpha'^2$ is the 10-dimensional gravitational constant. The corresponding SUGRA solution - the warped conifold - was constructed in [30].
It is useful to introduce the following basis of the 1-forms on the compact space [73]:

$$\begin{aligned} g^1 &= \frac{e^1 - e^3}{\sqrt{2}}\,, \qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,, \qquad g^3 = \frac{e^1 + e^3}{\sqrt{2}} \\ g^4 &= \frac{e^2 + e^4}{\sqrt{2}}\,, \qquad g^5 = e^5\,. \end{aligned} \tag{4.24}$$

where the

$$\begin{aligned} e^1 &\equiv -\sin\theta_1 d\phi_1\,, \qquad e^2 \equiv d\theta_1\,, \\ e^3 &\equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2\,, \\ e^4 &\equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,, \\ e^5 &\equiv d\psi + \cos\theta_1 d\phi_1 + \cos\theta_3 d\phi_2\,. \end{aligned} \tag{4.25}$$

are the 1-forms on S^2 and S^3. In this basis the Einstein metric on $T^{1,1}$ becomes

$$ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2 \ . \tag{4.26}$$

Furthermore we have following expressions for the NS-NS 2-form and RR 3-flux

$$F_3 = \frac{M\alpha'}{2}\omega_3 \ , \qquad\qquad B_2 = \frac{3g_s M\alpha'}{2}\omega_2 \ln(r/r_0) \ , \tag{4.27}$$

$$H_3 = dB_2 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2 \ , \tag{4.28}$$

where

$$\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) \tag{4.29}$$

and

$$\omega_3 = \frac{1}{2} g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4) \ . \tag{4.30}$$

For the Hodge duals with respect to the metric ds^2_6 holds

$$g_s \star_6 F_3 = H_3 \, , \qquad g_s F_3 = - \star_6 H_3 \, , \tag{4.31}$$

so that the complex 3-form G_3 (2.10) satisfies the self-duality condition

$$\star_6 G_3 = i G_3 \, . \tag{4.32}$$

Furthermore, (4.31) leads to

$$g_s^2 F_3^2 = H_3^2 \, , \tag{4.33}$$

which implies that dilaton is constant, i.e. $\Phi = 0$. RR-scalar vanishes as well, due to $F_{3\mu\nu\lambda} H_3^{\mu\nu\lambda} = 0$.

The 10-d metric found in [30] has the structure of a “warped product” of $\mathbb{R}^{3,1}$ and the conifold:

$$ds^2_{10} = h^{-1/2}(r) dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 ds^2_{T^{1,1}}) \ . \tag{4.34}$$

The warp factor h can be determined from the trace of the Einstein equation and can be shown to be

$$h(r) = \frac{27\pi(\alpha')^2[g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4]}{4r^4} \tag{4.35}$$

with $a = 3/(2\pi)$.

The crucial feature of the Klebanov-Tseytlin background is that the 5-form $\tilde{F}_5$ obtains a radial dependence, due to

$$\tilde{F}_5 = F_5 + B_2 \wedge F_3 \, , \qquad F_5 = dC_4 \, , \tag{4.36}$$

and $\omega_2 \wedge \omega_3 = 54\text{vol}(T^{1,1})$. So that it can be written in the following way

$$\tilde{F}_5 = \mathcal{F}_5 + \star\mathcal{F}_5 \, , \qquad \mathcal{F}_5 = 27\pi\alpha'^2 N_{eff}(r)\text{vol}(\mathrm{T}^{1,1}) \, , \tag{4.37}$$

and

$$N_{eff}(r) = N + \frac{3}{2\pi} g_s M^2 \ln(r/r_0) \, . \tag{4.38}$$

So we see that the 5-flux in this solution is not conserved. Even if it is present at the UV scale $r = r_0$ it may completely disappear when we reach a scale where $N_{eff} = 0$.

Now let us see what the above expressions for the forms imply for the gauge couplings. For the two gauge couplings of the type $SU(N_1) \times SU(N_2)$ theory has been shown [23, 26]

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^{\Phi}} \, , \tag{4.39}$$

$$\left[\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2}\right] g_s e^{\Phi} = \frac{1}{2\pi\alpha'}\left(\int_{S^2} \mathcal{B}_2\right) - \pi(\mod 2m) \, . \tag{4.40}$$

Now, for the cases where $N_1 = N_2 = N$, the quantization condition on H_3 demands that $\frac{1}{2\pi\alpha'}\left(\int_{S^2} \mathcal{B}_2\right)$ is a periodic variable with period 2π. This is no longer the case for $N_1 \neq N_2$ and this is crucial for the cascade phenomenon, as we will see in a moment.

Now, we remember, that the 5-dimensional radial coordinate defines in the gauge/string duality the RG scale of the dual gauge theory [1, 2, 3, 74, 75]. One way to establish the precise relation is to identify the field theory energy scale Λ with the energy of a stretched string ending on a probe brane placed at some radius r. For the metrics of the form 4.34 this translates into

$$\Lambda \sim r \, . \tag{4.41}$$

Let us consider what does introduction of the M fractional branes implies for the two gauge couplings of the resulting $SU(N+M) \times SU(N)$ theory. The dilaton is still constant, as remarked before, so that β-function for the $\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2}$ still vanishes. However, B_2 has now a radial dependence and substituting (4.27) into (4.40) leads to

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M \ln(r/r_s) + \text{const} \; . \tag{4.42}$$

Since $\ln(r/r_s) = \ln(\Lambda/\mu)$, (4.42) implies a logarithmic running of $\frac{1}{g_1^2} - \frac{1}{g_2^2}$ in the $SU(N+M)\times SU(N)$ gauge theory. As a consistency check one can compare this SUGRA result with the Shifman-Vainshtein β-functions [76]

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_1^2} = 3(N+M) - 2N(1-\gamma) \; , \tag{4.43}$$

$$\frac{d}{d\log(\Lambda/\mu)}\frac{8\pi^2}{g_2^2} = 3N - 2(N+M)(1-\gamma) \; , \tag{4.44}$$

where γ is the anomalous dimension of operators $\mathrm{Tr}A_iB_j$. The conformal invariance of the field theory for $M=0$, and symmetry under $M \to -M$, require that $\gamma = -\frac{1}{2} + O[(M/N)^{2n}]$ where n is a positive integer [24]. For the difference of the two equations then holds

$$\begin{aligned} \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} &= M\ln(\Lambda/\mu)[3+2(1-\gamma)] \\ &= 6M\ln(\Lambda/\mu)(1+O[(M/N)^{2n}]) \, , \end{aligned} \tag{4.45}$$

which is in a remarkable agreement with the result (4.42) found on the SUGRA side. This constitutes a geometrical explanation of a field theory β-function, including its normalization.

Let us summarize the above results. The addition of the M fractional branes at the singularity of the conifold changes the gauge group to $SU(N+M)\times SU(N)$. We still have the four chiral superfields A_i and B_j, which now live in the $(N+M,\bar{N})$ representation and its conjugate and we still have the quartic superpotential. However, the theory is no longer conformal. Instead, as we have seen above, the relative coupling runs logarithmically. It was first pointed out in [28], where the supergravity equations were solved to the leading order in M/N and then confirmed in [30], where the solution was completed to all orders. Also the D3-brane charge, the 5-form flux decreases logarithmically. This is due to the fact that the $\int_{S^2} B_2$ is no longer a periodic variable, as it was in SUGRA solution dual to the $SU(N)\times SU(N)$. Instead, as the B_2 flux goes through a period, $N_{eff} \to N_{eff} - M$, which results in decrease of the 5-form flux (4.37) by M units. However, there is no cutoff at small radii for the logarithm in the solution, hence the D3-brane charge eventually becomes negative and the metric becomes singular.

In [30], it was conjectured that this solution of conifold with fractional D3-branes is dual to the flow in which the gauge group factors repeatedly drop in size by M units. Indeed, we observe that $1/g_1^2$ and $1/g_2^2$ flow in different directions and from (4.43) we see that there is a scale where the $SU(N+M)$ coupling, g_1, diverges. In order to continue past this infinite coupling we need to perform the Seiberg duality, described in the Section 4.2. The $SU(N+M)$ gauge factor has

2N flavors in the fundamental representation and after undergoing the Seiberg transformation this becomes an $SU(2N - [N + M]) = SU(N - M)$ gauge group (4.11). So that we end up with $SU(N) \times SU(N - M)$ theory. Hence, the theory is obviously self similar under the Seiberg transformation and we can perform it many times, resulting in the so-called "duality cascade"; however, as this theory flow to IR, the cascade must stop, since negative N would be unphysical. The singularity of the KT solution (4.34) gives us a hint that it has to be modified in IR. In fact it has been suggested [24] that by a suitable choice of the N and M we will finally land by the gauge group $SU(2M) \times SU(M)$ and the strong dynamics of this theory would resolve the naked singularity in the KT metric. The flow then becomes an infinite series of Seiberg duality transformations.

4.3.2 Deformation of the conifold

As mentioned in the section 4.1, the deformation of a conifold corresponds to a blow up of the S^3, which is achieved by [24]:

$$\sum_{i=1}^{4} z_i^2 = \epsilon^2 \, . \tag{4.46}$$

$$ds_{10}^2 = h^{-1/2}(\tau) dx_n dx_n + h^{1/2}(\tau) ds_6^2 \, , \tag{4.47}$$

where ds_6^2 is the metric of the deformed conifold (4.48) [68, 73]. The latter is diagonal in the basis (4.24):

$$ds_6^2 = \frac{1}{2}\varepsilon^{4/3} K(\tau) \Bigg[\frac{1}{3K^3(\tau)} (d\tau^2 + (g^5)^2) + \cosh^2\left(\frac{\tau}{2}\right) [(g^3)^2 + (g^4)^2] \\ + \sinh^2\left(\frac{\tau}{2}\right) [(g^1)^2 + (g^2)^2] \Bigg] \, , \tag{4.48}$$

with

$$K(\tau) = \frac{(\sinh(2\tau) - 2\tau)^{1/3}}{2^{1/3}\sinh\tau} \, . \tag{4.49}$$

For large τ one can introduce another radial coordinate r via

$$r^2 = \frac{3}{2^{5/3}} \varepsilon^{4/3} e^{2\tau/3} \, , \tag{4.50}$$

and in terms of this radial coordinate the metric acquires the form of the usual conifold metric $ds_6^2 \to dr^2 + r^2 ds_{T^{1,1}}^2$. At $\tau = 0$ the angular metric degenerates into the metric of a round S^3

[68, 73]

$$d\Omega_3^2 = \frac{1}{2}\varepsilon^{4/3}(2/3)^{1/3}[\frac{1}{2}(g^5)^2 + (g^3)^2 + (g^4)^2] \ , \tag{4.51}$$

and the additional two directions, corresponding to the S^2 fibered over the S^3, shrink as

$$\frac{1}{8}\varepsilon^{4/3}(2/3)^{1/3}\tau^2[(g^1)^2 + (g^2)^2] \ . \tag{4.52}$$

The simplest ansatz for the 2-form fields is

$$\begin{aligned} F_3 &= \frac{M\alpha'}{2}\left\{g^5 \wedge g^3 \wedge g^4 + d[F(\tau)(g^1 \wedge g^3 + g^2 \wedge g^4)]\right\} \\ &= \frac{M\alpha'}{2}\left\{g^5 \wedge g^3 \wedge g^4(1-F) + g^5 \wedge g^1 \wedge g^2 F \right. \\ &\qquad \left. + F' d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right\} \ , \end{aligned} \tag{4.53}$$

with $F(0) = 0$ and $F(\infty) = 1/2$, and

$$B_2 = \frac{g_s M\alpha'}{2}[f(\tau)g^1 \wedge g^2 + k(\tau)g^3 \wedge g^4] \ , \tag{4.54}$$

$$\begin{aligned} H_3 = dB_2 &= \frac{g_s M\alpha'}{2}\left[d\tau \wedge (f' g^1 \wedge g^2 + k' g^3 \wedge g^4) \right. \\ &\qquad \left. + \frac{1}{2}(k-f)g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right] \ . \end{aligned} \tag{4.55}$$

The self-dual 5-form field strength may be again decomposed as $\tilde{F}_5 = \mathcal{F}_5 + \star\mathcal{F}_5$. We have

$$\mathcal{F}_5 = B_2 \wedge F_3 = \frac{g_s M^2(\alpha')^2}{4}\ell(\tau)g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \ , \tag{4.56}$$

where

$$\ell = f(1-F) + kF \ , \tag{4.57}$$

and

$$\star\mathcal{F}_5 = 4g_s M^2(\alpha')^2\varepsilon^{-8/3}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \frac{\ell(\tau)}{K^2 h^2 \sinh^2(\tau)} \ . \tag{4.58}$$

Note, that for large-r the expressions for the 2-forms will coincide with corresponding expressions for the KT background.

Solving the first order equations for the ansatz we can find the expression for the warp factor [24, 72] and show that for small τ the 10-dimensional geometry is approximately $\mathbb{R}^{3,1}$ times the deformed conifold

$$
\begin{aligned}
ds^2_{10} \quad \rightarrow \quad & \frac{\varepsilon^{4/3}}{2^{1/3} a_0^{1/2} g_s M \alpha'} dx_n dx_n + a_0^{1/2} 6^{-1/3} (g_s M \alpha') \Bigg\{ \frac{1}{2} d\tau^2 + \frac{1}{2} (g^5)^2 \\
& + (g^3)^2 + (g^4)^2 + \frac{1}{4} \tau^2 [(g^1)^2 + (g^2)^2] \Bigg\} ,
\end{aligned} \tag{4.59}
$$

where $a_0 = 0.781805$ is an integration constant. The KS solution is $SU(2) \times SU(2) (\simeq SO(4))$ symmetric and also has a $\mathbb{Z}_2$ symmetry $\mathcal{I}$, which exchanges (θ_1, ϕ_1) and (θ_2, ϕ_2) accompanied by changing the signs of the 3-form fields.

This theory exhibits a number of interesting effects in the SUGRA background, which translate on the gauge theory side for example into confinement; existence of glueballs and baryons, with the mass scale set by a dimensional transmutation; existence of a gluino condensate that breaks the $\mathbb{Z}_{2M}$ chiral symmetry down to $\mathbb{Z}_2$. The most important for us from these features is the confinement, let us briefly describe how it arises. We can see it when we inspect the formula for the metric (4.59) at small τ. The function multiplying the factor $dx_n dx_n$ approaches a constant at small τ, as opposed to the AdS_5 metric, where it vanishes, or to the singular conifold, where it blows up. This implies confinement, because the chromo-electric flux tube, described by a fundamental string at $\tau = 0$ has tension

$$
T_s = \frac{1}{2\pi \alpha' h(0)} , \tag{4.60}
$$

which is in this case well defined and is given by

$$
T_s = \frac{1}{2^{4/3} a_0 \pi} \frac{\epsilon^{4/3}}{(\alpha')^2 g_s M} \tag{4.61}
$$

where ϵ is the parameter controlling the deformation of the conifold.

5 The Method of Pertubative Holographic Renormalization

Now let us finally turn to the main part of the work. We will proceed as follows. First we recall main features of the bulk dynamics, then we introduce the method of pertubative holographic renormalization, test this method on the known aAdS examples (GPPZ and Coulomb branch flow discussed earlier) and finally apply it to the KS case.

5.1 Bulk dynamics

First of all we will review the equations governing the dynamics of the bulk fields [46, 50]. We consider the systems that are called fake SUGRA in d+1 dimensions [42]. Corresponding action is of the form

$$S = \int d^{d+1}x\sqrt{g}\left[-\frac{1}{4}R + \frac{1}{2}\mathcal{G}_{ab}g^{MN}\partial_M\phi^a\partial_N\phi^b + V(\phi)\right] + S_b \tag{5.1}$$

where $M, N = 0, 1, ...d$ and the potential $V(\phi)$ is given in terms of a superpotential $W(\phi)$ by

$$V(\phi) = \frac{1}{2}G^{ab}W_aW_b - \frac{d}{d-1}W^2 \tag{5.2}$$

S_b stands for the boundary terms in the action. We will turn to them later on, as they are not affecting bulk dynamics. Field indices are covariantly lowered and raised with the sigma-model metric G_{ab} and its inverse G^{ab} respectively, $W_a = \partial_a W = \frac{\partial W(\phi)}{\partial\phi^a}$. Covariant derivatives with respect to the fields are denoted by D_a or by a "|" preceding the index, i.e $W_{a|b} = D_bW_a = \partial_bW_a - \mathcal{G}^c_{ab}$, where $\mathcal{G}^c_{ab}$ is the Christoffel symbol for the metric G_{ab}.

As discussed in Section 3.4, holographic renormalization group flows are described by domain wall backgrounds of the form

$$ds^2 = dr^2 + e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu\,, \qquad \phi^a = \bar{\phi}^a(r)\,, \tag{5.3}$$

with $\mu, \nu = 1, ...d$, which satisfy the BPS equations

$$\partial_r A = -\frac{2}{d-1}W(\bar{\phi})\,, \qquad \partial_r\bar{\phi}^a = W^a(\bar{\phi})\,. \tag{5.4}$$

$\bar{\phi}$ denotes the background value of the field.

For our purposes we want to describe fluctuations around such a domain wall in terms of gauge invariant variables. In order to do so we first rewrite the metric in the radial-sliced form

$$ds^2 = (n^2 + n_in^i)dr^2 + 2n_idrdx^i + g_{ij}dx^idx^j\,, \tag{5.5}$$

where g_{ij} is the induced metric on the hypersurfaces of constant r and n and n^i are the lapse and shift vector, respectively. In the next step we expand the radially-sliced metric around the background

$$\begin{aligned} g_{ij} &= \mathrm{e}^{2A(r)}(\eta_{ij} + h_{ij})\,, \\ n_i &= \nu_i\,, \\ n &= 1 + \nu\,, \end{aligned} \tag{5.6}$$

where h_{ij},ν_i and ν are small fluctuations. In following the indices will be raised and lowered with the flat metric η_{ij}. Now we need to isolate the physical degrees of freedom among the fluctuations $\{h^i_j, \nu^i, \nu, \varphi^a\}$. This can be done in different ways. One can remove the redundancy following from diffeomorphisms by partially gauge fixing, this can, however, create problems in coupled systems. We will choose the other way, namely the gauge invariant approach. In order to obtain the equations of motion in gauge invariant form, we first consider the effect of diffeomorphisms on the fluctuation fields. We consider following diffeomorphism

$$x^\mu = \exp_{x'}[\xi(x')]^\mu = x'^\mu + \xi^\mu(x') - \frac{1}{2}\Gamma^\mu_{\nu\rho}(x')\xi^\nu(x')\xi^\rho(x') + \cdots , \tag{5.7}$$

where ξ is an infinitesimally small parameter. The use of the exponential map implies that also transformation laws for the fields can be written in covariant way (the functions $\xi^\mu(x')$ are viewed as the components of a vector field). We have then for the scalar field transformation

$$\delta\phi = \xi^\mu \partial_\mu \phi + \frac{1}{2}\xi^\mu \xi^\nu \nabla_\mu \partial_\nu \phi + \cdots , \tag{5.8}$$

and for the transformation of a covariant tensor of rank two

$$\begin{aligned} \delta E_{\mu\nu} \;=\; & \xi^\lambda \nabla_\lambda E_{\mu\nu} + (\nabla_\mu \xi^\lambda)(E_{\lambda\nu} + \xi^\rho \nabla_\rho E_{\lambda\nu}) + (\nabla_\nu \xi^\lambda)(E_{\mu\lambda} + \xi^\rho \nabla_\rho E_{\mu\lambda}) \\ & + (\nabla_\mu \xi^\lambda)(\nabla_\nu \xi^\rho) E_{\lambda\rho} + \frac{1}{2}\xi^\rho \xi^\lambda (\nabla_\rho \nabla_\lambda E_{\mu\nu} - R^\sigma{}_{\lambda\mu\rho} E_{\sigma\nu} - R^\sigma{}_{\lambda\nu\rho} E_{\mu\sigma}) \\ & + \cdots . \end{aligned} \tag{5.9}$$

For the metric tensor $g_{\mu\nu}$, (5.9) simplifies to

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + (\nabla_\mu \xi^\lambda)(\nabla_\nu \xi_\lambda) - R_{\mu\lambda\nu\rho}\xi^\lambda \xi^\rho + \cdots . \tag{5.10}$$

in the equations above, we have included the 2nd order in ξ in order to demonstrate the covariance of the transformation laws. For our purposes, we will only need the linear terms.

Now, to proceed we split the fake supergravity fields into background and fluctuations. The prescription for the metric fields is given by (5.6) and for the scalar field is given by the exponential map

$$\phi^a = \exp_{\bar\phi}(\varphi)^a \equiv \bar\phi^a + \varphi^a + \frac{1}{2}\mathcal{G}^a_{bc}\varphi^b \varphi^c + \cdots . \tag{5.11}$$

Then to the lowest oder we obtain from (5.8) and (5.10) following expressions for the fluctuations

$$\begin{aligned} \delta\varphi^a &= W^a \xi^r + \mathcal{O}(f) , \\ \delta\nu &= \partial_r \xi^r + \mathcal{O}(f) , \\ \delta\nu^i &= \partial^i \xi^r + \mathrm{e}^{2A}\, \partial_r \xi^i + \mathcal{O}(f) , \\ \delta h^i_j &= \partial_j \xi^i + \partial^i(\eta_{jk}\xi^k) - \frac{4}{d-1} W \delta^i_j \xi^r + \mathcal{O}(f) . \end{aligned} \tag{5.12}$$

$\mathcal{O}(f^n)$ stands for the terms of order n in the fluctuations $\{\varphi^a, h_{ij}, \nu_i, \nu\}$. Furthermore, let us decompose h^i_j in the following way

$$h^i_j = {h^{TT}}^i_j + \partial^i \epsilon_j + \partial_j \epsilon^i + \frac{\partial^i \partial_j}{\Box} H + \frac{1}{d-1} \delta^i_j h \ , \tag{5.13}$$

where ${h^{TT}}^i_j$ stands for the traceless transverse part, and ϵ^i is a transverse vector. Then we obtain from (5.12)

$$\begin{aligned} \delta {h^{TT}}^i_j &= \mathcal{O}(f) \ , \\ \delta \epsilon^i &= \Pi^i_j \xi^j + \mathcal{O}(f) \ , \\ \delta H &= 2\partial_i \xi^i + \mathcal{O}(f) \ , \\ \delta h &= -4W\xi^r + \mathcal{O}(f) \ . \end{aligned} \tag{5.14}$$

Π^i_j is the transverse projector

$$\Pi^i_j = \delta^i_j - \frac{1}{\Box} \partial^i \partial_j \ . \tag{5.15}$$

As mentioned above, we would like to work with gauge-invariant variables. Using the transformation laws (5.12) and (5.14) we can construct following gauge-invariant combinations to the lowest order in fluctuations

$$\mathfrak{a}^a = \varphi^a + W^a \frac{h}{4W} + \mathcal{O}(f^2) \ , \tag{5.16}$$

$$\mathfrak{b} = \nu + \partial_r \left(\frac{h}{4W} \right) + \mathcal{O}(f^2) \ , \tag{5.17}$$

$$\mathfrak{c} = \mathrm{e}^{-2A}\, \partial_i \nu^i + \mathrm{e}^{-2A} \Box \frac{h}{4W} - \frac{1}{2} \partial_r H + \mathcal{O}(f^2) \ , \tag{5.18}$$

$$\mathfrak{d}^i = \mathrm{e}^{-2A}\, \Pi^i_j \nu^j - \partial_r \epsilon^i + \mathcal{O}(f^2) \ , \tag{5.19}$$

$$\mathfrak{e}^i_j = {h^{TT}}^i_j + \mathcal{O}(f^2) \ . \tag{5.20}$$

The variables $\mathfrak{c}$ and $\mathfrak{d}^i$ both arise from $\delta \nu^i$, which has been split into the longitudinal and transverse parts. Note that this choice of the gauge invariant fields is not unique, since any combination of them will be gauge invariant as well.

The fluctuations around the domain wall (5.4) are described by the traceless transversal metric fluctuations,$\mathfrak{e}^i_j$, and the scalar fluctuations $\mathfrak{a}^a$, which satisfy the following linearized equations of motion

$$\left[\left(D_r + M - \frac{2d}{d-1} W \right) (D_r - M) + e^{-2A} \Box \right] \mathfrak{a} = 0 \tag{5.21}$$

and

$$\left[\left(\partial_r - \frac{2d}{d-1}W\right)\partial_r + e^{-2A}\Box\right]\mathfrak{e}^i_j = 0. \tag{5.22}$$

M denotes the matrix

$$M^a_b = W^a_{|b} - \frac{W^a W_b}{W}\,, \tag{5.23}$$

and D_r stands for the background covariant derivative

$$D_r\mathfrak{a}^a = \partial_r\mathfrak{a}^a + \mathcal{G}^a_{bc}W^b\mathfrak{a}^c \tag{5.24}$$

In following we will concentrate on the scalar field equation. We will denote the number of the scalar fields by n_s and assume the existence of a set of $2n_s$ independent solutions of (5.21), which are defined as power series in k^2 in momentum space, with r-dependent coefficients which are more and more suppressed with increasing powers of k^2. Moreover, the leading term in each solution should be independent of k^2. In position space, k^2 translates into operator $-\Box$. Furthermore, one can divide this set of solutions into two subsets, n_s asymptotically dominant solutions $\hat{\mathfrak{a}}_i$ $(i = 1, ..., n_s)$ and n_s sub-dominant solutions $\check{\mathfrak{a}}_i$, with respect to their behavior at large r. When we include the field index, $\hat{\mathfrak{a}}^a_i$ and $\check{\mathfrak{a}}^a_i$ become $n_s \times n_s$ matrices. The regularity condition in the bulk allows only for n_s independent regular combinations of the asymptotic basis solutions. Thus, we shall decompose a general regular solution of (5.21) into

$$\mathfrak{a}^a(r,x) = \hat{\mathfrak{a}}^a_i(r,-\Box_x)\mathfrak{s}_i(x) + \check{\mathfrak{a}}^a_i(r,-\Box_x)\mathfrak{r}_i(x)\,, \tag{5.25}$$

where $\mathfrak{s}_i$ and $\mathfrak{r}_i$ are the source and response coefficients, respectively, and $\Box_x = \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}$. The bulk regularity condition uniquely determines the functional dependences of the responses $\mathfrak{r}_i$ on the sources $\mathfrak{s}_i$ and gives rise to the non-local information for the two-point functions of the dual operators.

5.2 Pertubative Holographic Renormalization

5.2.1 Scalar 2-point functions

Here we will present the method to obtain finite, renormalized two-point functions for the QFT operators that are dual to the bulk scalar fields. Our starting point is an action that is quadratic in the fluctuations and encodes the bulk field equations (5.21)

$$S = \frac{1}{2}\int \mathrm{d^{d+1}x\; e^{dA}}\left\{[(\mathrm{D_r - M})\mathfrak{a}]\cdot[(\mathrm{D_r-M})\mathfrak{a}] + \mathrm{e^{-2A}}\partial_\mu\mathfrak{a}\cdot\partial^\mu\mathfrak{a}\right\} + \frac{1}{2}\int \mathrm{d^dx\; e^{dA}}\;\mathfrak{a}\cdot\mathrm{U}\cdot\mathfrak{a}\,, \tag{5.26}$$

with some symmetric counterterm matrix U, which is a local operator to be specified later. $\cdot$ denotes the inner product in field space, i.e $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a}^a G_{ab} \mathfrak{b}^b$. The bulk integral in (5.26) is taken with a cutoff r_0, where the boundary terms are evaluated. Hence the variation of the on-shell action with respect to a variation of the boundary value $\mathfrak{a}^a(r_0)$ is given by

$$\frac{\delta S_{on-sh}}{\delta \mathfrak{a}^a} = \mathrm{e}^{dA}(D_r - M + U)\mathfrak{a}_a \ , \tag{5.27}$$

where the right-had side is evaluated at the cut-off $r = r_0$. Now let us turn to the counterterm matrix. We define it as

$$U_{ab} = M_{ab} - \frac{1}{2}\left[(D_r\hat{\mathfrak{a}})_{ia}(\hat{\mathfrak{a}}^{-1})_{ib} + (D_r\hat{\mathfrak{a}})_{ib}(\hat{\mathfrak{a}}^{-1})_{ia}\right] \ , \tag{5.28}$$

with $(\hat{\mathfrak{a}}^{-1})_{ib}$ being the inverse of the matrix $\hat{\mathfrak{a}}_i^a$, defined in momentum space as a series in k^2. There is one important subtlety about this definition. The counterterms in (5.26) need to be local in the fields which means that in momentum space U_{ab} should be a polynomial in k^2 (polynomial in $-\Box$ in position space, respectively). The assumptions that we made in previous section for the solutions to the equations of motion imply indeed that U_{ab} is a series in k^2. However, we also assumed that the coefficients of the series $\hat{\mathfrak{a}}$ with increasing powers k^2 are suppressed for large r due to the factor $e^{-2A(r)}$. This means that we can truncate the series in (5.26) to some polynomial, since the neglected terms vanish in the large-r limit. So strictly speaking the counterterm operator U_{ab} in (5.26) is a polynomial truncation of (5.28).

Now let us define following matrices, which will prove useful in a moment.

$$\begin{aligned} \tilde{Z}_{ij} &= \mathrm{e}^{dA}\left[(D_r\hat{\mathfrak{a}})_i \cdot \hat{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r\hat{\mathfrak{a}})_j\right] \ , \\ Z_{ij} &= \mathrm{e}^{dA}\left[(D_r\hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r\check{\mathfrak{a}})_j\right] \ , \\ z_{ij} &= \mathrm{e}^{dA}\left[(D_r\check{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \check{\mathfrak{a}}_i \cdot (D_r\check{\mathfrak{a}})_j\right] \ . \end{aligned} \tag{5.29}$$

One can show, using the scalar field equations, that all these matrices are independent of r (see Appendix A). This means that z_{ij} should be identically zero, as the sub-dominant solutions vanish fast asymptotically. In general it is not necessarily the case. If there are two or more bulk scalars with mass $m^2 = 2(2-d)$ (which would be dual to the $\Delta = 2$ operator in the aAdS case) and the background is not aAdS, then one has to check whether z_{ij} really vanishes. We will assume in following that $z_{ij} = 0$ as it simplifies the final result. Moreover for the examples in question it is a safe assumption, since in KS background we do not have any $\Delta = 2$ operators, and GPPZ and Coulomb branch backgrounds are aAdS. The matrices in (5.29) are also functions of k^2 in momentum space, or respectively of $-\Box$ in position space.

Substitution of the decomposition (5.25) into (5.27) gives for the linear term of the exact 1-point function in momentum space

$$\langle\mathcal{O}_i(k)\rangle_1 = -\lim_{r\to\infty} \mathrm{e}^{dA(r)} \left[\hat{\mathfrak{a}}_i + \check{\mathfrak{a}}_j \frac{\partial \mathfrak{r}_j}{\partial \mathfrak{s}_i}(k)\right] \cdot (D_r - M + U) \left[\hat{\mathfrak{a}}_l \mathfrak{s}_l(k) + \check{\mathfrak{a}}_l \mathfrak{r}_l(k)\right] \ . \tag{5.30}$$

Here, the dependence of $\mathfrak{a}$'s on r and k^2 was omitted for brevity and the subscript 1 on the left side means than there are only terms linear in fluctuations.

Now, after substituting (5.28) for U in (5.30) and using the matrices (5.29) we obtain

$$\langle\mathcal{O}_i(k)\rangle_1 \quad = \quad Z_{ij}\mathfrak{r}_j + \frac{1}{2}\tilde{Z}_{ij}\mathfrak{s}_j + \frac{1}{2} z_{jk} \frac{\partial \mathfrak{r}_j}{\partial \mathfrak{s}_i}\mathfrak{r}_k \tag{5.31}$$

$$+\frac{1}{2}\lim_{r\to\infty}\left[(\hat{\mathfrak{a}}^{-1})_l \cdot \check{\mathfrak{a}}_k\right]\left(\tilde{Z}_{li}\mathfrak{r}_k + \frac{\partial \mathfrak{r}_k}{\partial \mathfrak{s}_i}\tilde{Z}_{lj}\mathfrak{s}_j + \frac{\partial \mathfrak{r}_j}{\partial \mathfrak{s}_i} Z_{lj}\mathfrak{r}_k + \frac{\partial \mathfrak{r}_k}{\partial \mathfrak{s}_i} Z_{lj}\mathfrak{r}_j\right) \ . \tag{5.32}$$

Furthermore, we see that the third term in the equation above vanishes, as $z_{ij} \equiv 0$, and the last term, the only term with cut-off dependence, also vanishes in the large-r limit, because $(\hat{\mathfrak{a}}^{-1})_l \cdot \check{\mathfrak{a}}_k$ vanishes. Thus, we end up with a rather simple result

$$\langle\mathcal{O}_i\rangle_1 = Z_{ij}\mathfrak{r}_j + \frac{1}{2}\tilde{Z}_{ij}\mathfrak{s}_j \ , \tag{5.33}$$

which is valid both in momentum and in position space. From this we obtain the connected 2-point function to be

$$\langle\mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = Z_{ik}(-\Box_x)\frac{\delta \mathfrak{r}_k(x)}{\delta \mathfrak{s}_j(y)} + \frac{1}{2}\tilde{Z}_{ij}(-\Box_x)\delta(x-y) \ . \tag{5.34}$$

Notice, that both 1-point function and 2-point function are finite in the limit $r_0 \to \infty$, since the matrices Z_{ij} and $\tilde{Z}_{ij}$ are independent on r.

It is more practical to use 2-point function in its momentum-space form. In order to obtain it from (5.34) we set $y = 0$ by translational scale invariance and perform the Fourier transform

$$\int \mathrm{d}^d x\, e^{ikx} \langle\mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = Z_{ik}(k^2)\frac{\partial \mathfrak{r}_k}{\partial \mathfrak{s}_j}(k) + \frac{1}{2}\tilde{Z}_{ij}(k^2) \, . \tag{5.35}$$

In following we will often use the momentum space representation and omit the argument k. When we refer to the two-point function in momentum space $\langle\mathcal{O}_i\mathcal{O}_j\rangle$, we mean the (5.35).

5.2.2 VEV's

Here we will only made few comments about VEV's, as at the moment our gauge invariant approach does not allow for a systematic derivation of the VEV's yet. We see that equation (5.31) only provides the part of the one-point function linear in the fluctuations. However we can make and interesting observation. The scalar equations (5.21) have the zero mode solution

$$\bar{\mathfrak{a}} = G_{ab}\frac{W_b}{W} \,, \tag{5.36}$$

which only depends on the radial variable r. This solution, like any other fluctuation has also the decomposition as in (5.25) with $k^2 = 0$. We will see in section (5.3) that the response function $\mathfrak{r}$ of this zero mode solution encodes the VEV's for the GPPZ and the CB flows. It is still remains to see how general this feature is. In section (5.4) we will discuss the VEV's in KS theory by analyzing the response functions of the corresponding zero mode solution.

5.2.3 Scheme Dependence

In this section we would like to discuss the scheme dependence of the contact terms in the two-point functions form the bulk point of view. We will work in the momentum space and we will omit all functional arguments for the sake of brevity.

We start with the decomposition (5.25) of the regular solution to the bulk fiend equations. The definition of the dominant and sub-dominant solutions is not unique. Our assumption that $\hat{\mathfrak{a}}$ and $\check{\mathfrak{a}}$ are series in k^2 and the fact that all sub-dominant solutions are negligible with respect to dominant ones in the large-r limit still leaves room for the following change of basis

$$\hat{\mathfrak{a}}'_i = \Lambda_{ij}\hat{\mathfrak{a}}_j + \lambda_{ij}\check{\mathfrak{a}}_j \,, \qquad \check{\mathfrak{a}}'_i = \mu_{ij}\check{\mathfrak{a}}_j \,, \tag{5.37}$$

Λ_{ij}, λ_{ij} and μ_{ij} are non-degenerate matrices, polynomial in k^2. Under this change of basis $\mathbb{Z}_{ij}$ and $\tilde{Z}_{ij}$ become

$$\begin{aligned} \tilde{Z}'_{ij} &= \Lambda_{ik}\Lambda_{jl}\tilde{Z}_{kl} + \left(\Lambda_{ik}\lambda_{jl} - \Lambda_{jk}\lambda_{il}\right) Z_{kl} \ , \\ Z'_{ij} &= \Lambda_{ik}\mu_{jl}Z_{kl} \ , \end{aligned} \tag{5.38}$$

and the source and response coefficients transform to

$$\mathfrak{s}'_i = \mathfrak{s}_j(\Lambda^{-1})_{ji} \ , \qquad \mathfrak{r}'_i = \left[\mathfrak{r}_j - \mathfrak{s}_l(\Lambda^{-1})_{lk}\lambda_{kj}\right](\mu^{-1})_{ji} \ . \tag{5.39}$$

Inserting these transformations into (5.35), we obtain the connected two-point functions of the operators $\mathcal{O}'_i$ coupling to the sources $\mathfrak{s}'_i$,

$$\left\langle \mathcal{O}'_i\mathcal{O}'_j \right\rangle = \Lambda_{ik}\Lambda_{jl}\left\langle \mathcal{O}_k\mathcal{O}_l \right\rangle - \frac{1}{2}\left(\Lambda_{ik}\lambda_{jl} + \Lambda_{jk}\lambda_{il}\right) Z_{kl} \tag{5.40}$$

So we see that the matrix Λ_{ij} rotates the basis of the operators, while λ_{ij} changes the contact terms, which corresponds to a change of the renormalization scheme.

Now there are several possible restrictions on the choice of the transformation matrices. First of all, we know that in QFT operators are often characterized by their scaling dimension Δ, which is renormalization scheme dependent. Under renormalization they undergo operator mixing, so that an operator of the given dimension, defined at a certain renormalization scale, is in general composed of the operators of equal and lower dimensions, defined at a larger renormalization scheme. This is however not an unique composition, as operators of equal dimension and otherwise equal quantum numbers can be arbitrarily combined to equivalent combinations. This ambiguity is naturally reflected in the approach at hand. Ordering the dominant asymptotic solutions according to their asymptotic behavior in descending order, it is natural to choose Λ in the upper triangular form, so that each dominant solution gets modified only by solutions of equal or weaker asymptotic behavior. Same applies for the matrix μ.

Another restriction on the redefinition of the solutions could come from the fact that the lowest order terms in a near boundary expansion of the dominant solutions typically have a definite correlation between powers of e^{-r} and powers of k^2. This is well known, for example, for the in the aAdS case with a single scalar field. Similar correlation is noticeable in the KS case, see (Appendix C). We will refer to a choice of dominant solutions respecting this correlation as a "natural" choice.

Furthermore, it is reasonable to assume that Λ_{ij} and/or μ_{ij} can be chosen in such way, that $Z'_{ij} = \delta_{ij}$. Possible problem with this choice could be that the matrices needed to achieve this are non-polynomial in k^2. In KS case the choice $Z'_{ij} = \delta_{ij}$ is indeed possible. Starting with such a choice, a further change of basis using just λ_{ij} would lead to

$$\tilde{Z}'_{ij} = \tilde{Z}_{ij} + \lambda_{ji} - \lambda_{ij} \tag{5.41}$$

implying that one can achieve $\tilde{Z}'_{ij} = 0$ by a suitable choice of λ_{ij}, although this choice is not unique.

5.3 AAdS Examples

In this section we will apply the method introduced above to the known examples of GPPZ and Coulomb branch flows and compare our results to the known results, which we summarized in the section (3.4). Before we start, we remember that considering the AAdS examples we used a different radial coordinate, namely $\rho = \mathrm{e}^{-2r}$. Furthermore, matrices simplify to 1-dim quantities and covariant derivative to ordinary derivative, as we have only one scalar under consideration. Taking all this into account (5.27) becomes

$$\frac{\delta S_{on-sh}}{\delta \mathfrak{a}^a} = \mathrm{e}^{dA}(-2\rho\partial_\rho - M + U)\mathfrak{a}_a \ , \tag{5.42}$$

counterterm "matrix" U (5.28) is given by

$$U = M + 2\rho(\partial_\rho \hat{\mathfrak{a}})(\hat{\mathfrak{a}})^{-1}\,, \tag{5.43}$$

and for the matrices $\tilde{Z}_{ij}$ and Z_{ij} from (5.29) holds

$$\begin{aligned} \tilde{Z} &= 0\,, \\ Z &= \mathrm{e}^{dA}\, 2\rho \left[\hat{\mathfrak{a}}(\partial_\rho \check{\mathfrak{a}}) - (\partial_\rho \hat{\mathfrak{a}})\check{\mathfrak{a}}\right] . \end{aligned} \tag{5.44}$$

The linear term of the exact 1-point function is then again given by (5.33) with $\tilde{Z}_{ij}$ set to zero. However, at this point we would like to note, that in Section 3 we used the definition for the scalar function usual for the literature on holographic renormalization [36, 63], whereas our definition in (5.27) exhibits an additional "-" sign. Hence in order to compare the results below to the results of the Section 3, we have to account for this extra "-" and use following expression for the 1-point function

$$\langle \mathcal{O} \rangle_1 = -Z\mathfrak{r} \tag{5.45}$$

5.3.1 Coulomb Branch Flow

Let us start with the Coulomb Branch flow. The necessary formulas for superpotential and domain wall solution were already given in section (3.4.1), here we will only remind that the bulk scalar field has the following asymptotic expansion

$$\phi(r) = \phi_0\, \rho \log \rho + \tilde{\phi}_0 \rho + \ldots\,, \tag{5.46}$$

where ϕ_0 and $\tilde{\phi}_0$ are the two independent coefficients. The background solution is given by

$$\mathrm{e}^{\sqrt{6}\bar{\phi}} = 1 - l^2\rho + \mathcal{O}(\rho^2)\,, \quad e^{2A} = \frac{1}{\rho}\ . \tag{5.47}$$

We remember, that at the boundary the scalar field vanishes at the rate

$$\bar{\phi} = -\frac{1}{\sqrt{6}} l^2 \rho \tag{5.48}$$

which implies

$$\bar{\phi}_0 = 0\,, \qquad \bar{\tilde{\phi}}_0 = -\frac{l^2}{\sqrt{6}}\,. \tag{5.49}$$

From the section 3.4.1 we know, that the exact one-point function of the corresponding operator is given by

$$\langle \mathcal{O} \rangle = 2(\tilde{\phi}_0 + u_2 \phi_0)\,, \tag{5.50}$$

where the second term, involving the scheme dependent constant u_2 is coming from the finite counterterm proportional to

$$\frac{\phi^2}{r^2}|_{r=r_0}\,. \tag{5.51}$$

Linearizing around the background, $\phi = \bar{\phi} + \varphi$ leads to

$$\langle \mathcal{O} \rangle = -\frac{2l^2}{\sqrt{6}} + 2\tilde{\varphi}_0 + 2u_2\varphi\,. \tag{5.52}$$

The first term on the right hand side is the finite VEV of the dual $\Delta = 2$ operator and is independent of the renormalization scheme. In order to apply the method of section 5.2, we need to relate the field fluctuation φ and the gauge-invariant variable $\mathfrak{a}$ introduced in the section 5.1. We recall that fields $\mathfrak{a}$ and $\mathfrak{b}$ are given by

$$\mathfrak{a} = \varphi + W' \frac{h}{4W}\,,\ \mathfrak{b} = \nu + \partial_r \left(\frac{h}{4W}\right)\,, \tag{5.53}$$

with $W' = dW/d\phi$ and $\nu = 0$ in the orthonormal gauge. Moreover they are related on-shell by [50]

$$\mathfrak{b} = -\frac{W'}{W}\mathfrak{a}\,. \tag{5.54}$$

Using these relations in order to express linear terms in (5.52) in gauge invariant variables, we obtain

$$\mathfrak{a} = \varphi + \mathcal{O}(\rho^3)\,. \tag{5.55}$$

Using the definition

$$\mathfrak{a} = \mathfrak{a}_0\, \rho \log \rho + \tilde{\mathfrak{a}}_0 \rho + \ldots\ , \tag{5.56}$$

the one-point function in gauge invariant variables reads

$$\langle \mathcal{O} \rangle = -\frac{2l^2}{\sqrt{6}} + 2\tilde{\mathfrak{a}}_0 + 2u_2 \mathfrak{a}_0\ . \tag{5.57}$$

Now again, let us consider dominant and sub-dominant solutions

$$\hat{\mathfrak{a}} = \rho \log \rho + \tilde{\alpha}\rho\,, \qquad \check{\mathfrak{a}} = \rho\,. \tag{5.58}$$

Rewriting of the (5.56) in this basis yields us source and response coefficients

$$\mathfrak{s} = \mathfrak{a}_0\,, \qquad \mathfrak{r} = \tilde{\mathfrak{a}}_0 + \tilde{\alpha}\mathfrak{a}_0\,. \tag{5.59}$$

Moreover, (5.29) results in $\tilde{Z} = 0$ and $Z = -2$. Hence, (5.45) yields

$$\langle \mathcal{O} \rangle_1 = 2(\tilde{\mathfrak{a}}_0 - \tilde{\alpha}\mathfrak{a}_0)\ . \tag{5.60}$$

Comparison with the part of (5.57) which is linear in fluctuations implies that $\tilde{\alpha} = -u_2$.
The counterterm "matrix" U (5.28) obtained from the basis (5.58) is

$$U = \frac{2}{\log \rho} - \frac{2\tilde{\alpha}}{(\log \rho)^2} + \mathcal{O}((\log \rho)^{-3})\,. \tag{5.61}$$

So we see that we indeed obtain the usual logarithmically divergent counterterm and a scheme dependent finite contribution.
Finally, let us consider the zero mode $\bar{\mathfrak{a}} = \frac{W'}{W}$. In CB case it is given by

$$\frac{W'}{W} = -\frac{4}{3}\frac{l^2}{\sqrt{6}}\rho + \mathcal{O}(\rho^3)\,. \tag{5.62}$$

From this, using our definition for $\mathfrak{a}$ (5.56) we read off $\bar{\tilde{\mathfrak{a}}}_0 = -\frac{4}{3}\frac{l^2}{\sqrt{6}}$ and $\bar{\mathfrak{a}}_0 = 0$, and thus, the response is (5.59)

$$\bar{\mathfrak{r}} = -\frac{4}{3}\frac{l^2}{\sqrt{6}}\ . \tag{5.63}$$

Comparison with (5.52) shows that in the CB flow, the response of the zero mode gives the VEV only up to and overall factor, but it is non-vanishing independently of the scheme.

5.3.2 GPPZ flow

Again, the relevant formulas for superpotential and domain wall solution are given in section (3.4.2). We remember that the asymptotic expansion of the scalar field reads

$$\phi(r) = \phi_0\rho^{1/2} + \psi_2\rho^{3/2}\log\rho + \phi_2\rho^{3/2} + \cdots\,, \tag{5.64}$$

where ϕ_0 and ϕ_2 are independent coefficients, and $\psi_2 = -\frac{1}{4}\Box\phi_0$ (3.82) (we omit the terms depending on curvature). The background solution satisfies

$$e^{2\bar{\phi}/\sqrt{3}} = \frac{1+\rho^{1/2}}{1-\rho^{1/2}}\,, \tag{5.65}$$

which implies

$$\bar{\phi}_0 = \sqrt{3}\,, \bar{\psi}_2 = 0\,, \bar{\phi}_2 = \frac{1}{\sqrt{3}}\,. \tag{5.66}$$

The exact one-point function of the operator $\mathcal{O}$ coupling to the source ϕ_0 is given by equation (3.86) of the section 3.4.2

$$\langle\mathcal{O}\rangle = -2\phi_2 + \left(m_0 + \frac{1}{2}\right)\Box\phi_0 - \frac{u_4}{6}\phi_0^3\,. \tag{5.67}$$

The two scheme-dependent coefficients m_0 and u_4 come from the addition of the finite counterterms

$$\int \mathrm{d}^4x\sqrt{g}\left(\frac{u_4}{4!}\phi^4 + \frac{1}{2}m_0\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\,. \tag{5.68}$$

SUSY requires u_4 = -4/3, but m_0 remains undetermined by this condition (we remind the reader that in Section 3.4.2 the coefficients were already chosen appropriately).

Let us consider an arbitrary renormalization scheme. Linearizing around the background $\phi = \bar{\phi} + \varphi$ and switching to the momentum space leads to

$$\langle\mathcal{O}\rangle = -\frac{\sqrt{3}}{2}\left(u_4 + \frac{4}{3}\right) - \left[2(\varphi_2 - \varphi_0) + \frac{3}{2}\left(u_4 + \frac{4}{3}\right)\varphi_0 + \left(m_0 + \frac{1}{2}\right)k^2\varphi_0\right]\,. \tag{5.69}$$

The first term on the right-hand-side is the scheme dependent VEV, which vanishes if we impose the SUSY condition, and the second term encodes the two-point function.

Again we use (5.53) and (5.54) in order to relate $\mathfrak{a}$ and the fluctuations and arrive at the following form for the leading behavior of the gauge-invariant field $\mathfrak{a}$

$$\mathfrak{a}(r) = \varphi_0\rho^{1/2} + \frac{1}{4}k^2\varphi_0\,\rho^{3/2}\log\rho + (\varphi_2 - \mathfrak{a}_0)\rho^{3/2} + \cdots\,, \tag{5.70}$$

and from here we read off $\varphi_0 = \mathfrak{a}_0$ and $\varphi_2 = \mathfrak{a}_2 + \mathfrak{a}_0$. Thus the one-point function (5.69) becomes

$$\langle\mathcal{O}\rangle = -\frac{\sqrt{3}}{2}\left(u_4 + \frac{4}{3}\right) - \left[2\mathfrak{a}_2 + \frac{3}{2}\left(u_4 + \frac{4}{3}\right)\mathfrak{a}_0 + \left(m_0 + \frac{1}{2}\right)k^2\mathfrak{a}_0\right]\,. \tag{5.71}$$

Now let us compare this equation with results from the previous section. First of all let us define the dominant and sub-dominant solutions as follows

$$\hat{\mathfrak{a}} = \rho^{1/2} + \rho^{3/2}\log\rho\frac{1}{4}k^2 + \rho^{3/2}\alpha_2 + \cdots\ , \qquad \check{\mathfrak{a}} = \rho^{3/2} + \cdots\ , \tag{5.72}$$

where the coefficient α_2 is yet to be determined. From these we can calculate the counterterm "matrix" to be

$$U = \frac{1}{2}k^2\rho\log\rho + \rho\left(\frac{k^2}{2} + 2\alpha_2\right) + \cdots\ , \tag{5.73}$$

eclipses stand for terms that can be neglected. Notice, that the first term on the right-hand side agrees with the standard logarithmic counterterm and the second term gives finite contributions. Furthermore, (5.29) yields $\tilde{Z} = 0$ and $Z = 2$. Expressing (5.70) in the basis (5.72) leads to following formulas for source and response coefficients

$$\mathfrak{s} = \mathfrak{a}_0 \qquad \mathfrak{r} = \mathfrak{a}_2 - \alpha_2\mathfrak{a}_0\ , \tag{5.74}$$

and so (5.45) yields

$$\langle\mathcal{O}\rangle_1 = -2\left(\mathfrak{a}_2 - \alpha_2\mathfrak{a}_0\right)\ . \tag{5.75}$$

Now, comparison of this result with the linear term of (5.71) yields and agreement in the non-local term containing $\mathfrak{a}_2$ and we determine α_2 to be

$$\alpha_2 = -\frac{3}{4}\left(u_4 + \frac{4}{3}\right) - \frac{1}{4}(2m_0 + 1)k^2\ . \tag{5.76}$$

This result states explicitly the relation between the choice of the dominant basis and the renormalization scheme. Note that only $\alpha_2 \sim (k^2)^n$ for $n = 0$ or $n = 1$ corresponds to a change of the scheme. In connection with our discussion, we can call these k^2-dependences of redefinitions of the dominant solutions also "natural" choices. In SUSY scheme, $\alpha_2 = 0$ with $m = -\frac{1}{2}$.

To conclude this section let us check that VEV is indeed encoded in the response coefficient of the zero-mode $\bar{\mathfrak{a}} = W'/W = \frac{2}{\sqrt{3}}\rho^{1/2}$. Considering the decomposition (5.25) this implies $\bar{\mathfrak{a}}_0 = 2/\sqrt{3}$ and $\bar{\mathfrak{a}}_2 = 0$, and from (5.74) and (5.76) with $k^2 = 0$ for the zero-mode we obtain

$$\bar{\mathfrak{r}} = -\frac{2}{\sqrt{3}}\alpha_2 = \frac{\sqrt{3}}{2}\left(u_4 + \frac{4}{3}\right)\ . \tag{5.77}$$

The right-hand side coincides with the constant term in (5.71) up to the "-"-sign that comes from our 1-point function definition. Hence, in any renormalization scheme, the response coefficient of the background mode is just the - VEV. In the SUSY schemes, it vanishes, as expected.

5.4 KS System

Finally we would like to apply the formalism of the section 5.2 to the case of Klebanov-Strassler theory. We will introduce the relevant facts about background first.

5.4.1 KS Background

The effective 5-d model describing the bulk dynamics of the KS system contains seven scalar fields.We restrict ourselves to the $J^{PC} = 0^{++}$ scalar sector, where C denotes the quantum number under the $\mathbb{Z}_2$ charge conjugation symmetry of the KS theory, cf. [77, 78]. Additional scalar fluctuations with $J^{PC} = 0^{+-}$ and $J^{PC} = 0^{--}$ were discussed in [77, 78, 40].

We shall use the Papadopoulos-Tseytlin [45] variables $(x, p, y, \Phi, b, h_1, h_2)$. The dual operators have dimensions $\Delta = 8, 7, 6$ and twice 4 and 3 each. Now, strictly speaking this statement is not quite correct, as in contrast to aAdS settings, the KS system has no UV conformal fixed point, where the operator dimensions can be fixed. However, the deviation from aAdS behavior is quite mild, such that the asymptotic solutions behave nearly as if the dual operators had definite dimensions. This can be seen explicitly by inspecting the asymptotic solutions given in Appendix C. Their exponential τ-dependence ($\mathrm{e}^{(\Delta-4)\tau/3}$ for $\hat{\mathfrak{a}}$ and $\mathrm{e}^{-\Delta\tau/3}$ for $\check{\mathfrak{a}}$) is what one would expect for a solution dual to an operator of dimension Δ [the KS radial variable τ will be introduced momentarily in (5.80)]. Thus, we still regard the concept of dimension as useful for distinguishing the different asymptotic solutions.

The sigma-model metric is given by

$$\begin{aligned} G_{ab}\partial_M\phi^a\partial^M\phi^b \quad &= \partial_M x\partial^M x + 6\partial_M p\partial^M p + \frac{1}{2}\partial_M y\partial^M y + \frac{1}{4}\partial_M\Phi\partial^M\Phi + \frac{P^2}{2}\,\mathrm{e}^{\Phi-2x}\,\partial_M b\partial^M b + \\ &+\frac{1}{4}\,\mathrm{e}^{-\Phi-2x}\left[\mathrm{e}^{-2y}\,\partial_M(h_1-h_2)\partial^M(h_1-h_2) + \mathrm{e}^{2y}\,\partial_M(h_1+h_2)\partial^M(h_1+h_2)\right] \ , \end{aligned} \tag{5.78}$$

and the superpotential reads

$$W = -\frac{1}{2}\left(\mathrm{e}^{-2p-2x} + \mathrm{e}^{4p}\cosh y\right) + \frac{1}{4}\,\mathrm{e}^{4p-2x}\left[Q + 2P(bh_2 + h_1)\right] \ . \tag{5.79}$$

Here, Q and P are constants related to the number of D3-branes and wrapped D5-branes, respectively. It is useful to introduce the KS radial variable τ by

$$\partial_r = \mathrm{e}^{4p}\,\partial_\tau \ . \tag{5.80}$$

In terms of τ, the KS background solution of 5.4 is given by

$$\Phi = \Phi_0 \ , \tag{5.81}$$

$$\mathrm{e}^y = \tanh(\tau/2) \ , \tag{5.82}$$

$$b = -\frac{\tau}{\sinh\tau} \ , \tag{5.83}$$

$$h_1 = -\frac{Q}{2P} + P\,\mathrm{e}^{\Phi_0}\coth\tau(\tau\coth\tau - 1) \ , \tag{5.84}$$

$$h_2 = P\,\mathrm{e}^{\Phi_0}\,\frac{\tau\coth\tau - 1}{\sinh\tau} \ , \tag{5.85}$$

$$\frac{2}{3}\,\mathrm{e}^{6p+2x} = \coth\tau - \frac{\tau}{\sinh^2\tau} \ , \tag{5.86}$$

$$\mathrm{e}^{2x/3-4p} = 2P^2\,\mathrm{e}^{\Phi_0}\,3^{-2/3}h(\tau)\sinh^{4/3}\tau \ , \tag{5.87}$$

with

$$h(\tau) = \int_\tau^\infty \mathrm{d}\vartheta\,\frac{\vartheta\coth\vartheta - 1}{\sinh^2\vartheta}\,[2\sinh(2\vartheta) - 4\vartheta]^{1/3} \ . \tag{5.88}$$

Moreover, the warp factor satisfies

$$\mathrm{e}^{-2A} \sim \mathrm{e}^{4p}\left(\mathrm{e}^{-2x}\sinh\tau\right)^{2/3}h(\tau) \ , \tag{5.89}$$

with a proportionality factor that sets the momentum scale.

The Klebanov Tseytlin background solution is somewhat simpler, because there $y = b = h_2 = 0$, but it has a singularity. For the KT background solutions of the other fields see [46].

The sigma model covariant fluctuations around the KS background can be formed by using (5.16) from the fields

$$\varphi^a = (\delta x, \delta p, \delta h_1, \delta\Phi, \delta y, \delta b, \delta h_2)^T \ . \tag{5.90}$$

and fulfill the field equations (5.21). All scalars appear to be coupled in the bulk, but to the leading order it is possible to decouple a 4×4 set of fields from the 3×3 set [39]. After changing to τ we find for (5.21) (we omit all indices for simplicity)

$$\left[(\partial_\tau - M)(\partial_\tau - N) - k^2\,\mathrm{e}^{-2A-8p}\right]\mathfrak{a} = 0 \tag{5.91}$$

with

$$\begin{aligned} M^a_{\ b} &= -N^a_{\ b} - K^a_{\ b} - 2\,\mathrm{e}^{-2x-6p}\,\delta^a_b\,, \\ N^a_{\ b} &= \mathrm{e}^{-4p}\left(\partial_b W^a - \frac{W^a W^b}{W}\right)\,, \\ K^a_{\ b} &= 2\,\mathrm{e}^{-4p}\,\mathcal{G}^a_{\ bc} W^c\,. \end{aligned} \tag{5.92}$$

$$\tag{5.93}$$

In order to fix the momentum scale we define

$$I(\tau) = \frac{h(\tau)}{h(0)} \tag{5.94}$$

and choose the integration constants in (5.88) in such way that the warp factor becomes

$$\mathrm{e}^{-2A-8p} = \left(\mathrm{e}^{-6p-2x}\sinh\tau\right)^{2/3} I(\tau)\,. \tag{5.95}$$

The matrices M and N a priory do depend on the constants P and Φ_0. However we can perform a linear transformation which removes this dependence from (5.91). This means that this constants only affect the spectrum by an overall change of the momentum scale, which is not visible in the effective 5-d theory. For the fluctuations vector the transformation acts like $\varphi' = R\varphi$ with

$$\varphi^a = \left(\delta x, \delta p, \frac{\delta h_1}{P\,\mathrm{e}^{\Phi_0}}, \delta\Phi, \delta y, \delta b, \frac{\delta h_2}{P e^{\Phi_0}}\right)^T\,. \tag{5.96}$$

and the matrices are rotated as $M' = RMR^{-1}$. In the following we are using the rescaled fields and drop the primes, gauge invariant fluctuations also will be formed form the rescaled fields. For the explicit form of the rotated matrices see the Appendix B.
Now, let us consider the asymptotic (large-τ) behavior of the solutions of (5.91). We can expand the martices and the warp term in powers of $\mathrm{e}^{-\tau}$

$$K = K^{(0)} + \mathrm{e}^{-\tau}\,K^{(1)} + \mathcal{O}(\mathrm{e}^{-2\tau})\,, \qquad N = N^{(0)} + \mathrm{e}^{-\tau}\,N^{(1)} + \mathcal{O}(\mathrm{e}^{-2\tau})\,, \tag{5.97}$$

where the coefficients can contain only rational functions of τ, but no exponentials. For simplicity we will write the matrices in block form

$$K = \begin{pmatrix} K_{4\times4} & K_{4\times3} \\ K_{3\times4} & K_{3\times3} \end{pmatrix}\,, \tag{5.98}$$

and similarly for N. Then we have for the matrices from (5.97)

$$K^{(0)}_{4\times4} = \begin{pmatrix} 0 & 0 & \frac{2}{3(\tau-1/4)} & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -\frac{1}{\tau-1/4} & -1 \\ 0 & 0 & \frac{4}{3(\tau-1/4)} & 0 \end{pmatrix} , \tag{5.99}$$

$$K^{(0)}_{3\times3} = \begin{pmatrix} 0 & 0 & -\frac{4}{3(\tau-1/4)} \\ 0 & -\frac{1}{\tau-1/4} & 0 \\ 2 & 0 & -\frac{1}{\tau-1/4} \end{pmatrix} , \tag{5.100}$$

$$K^{(0)}_{4\times3} = K^{(0)}_{3\times4} = 0 , \tag{5.101}$$

$$N^{(0)}_{4\times4} = \begin{pmatrix} -\frac{1}{\tau+1/4} & -\frac{4\tau-1}{\tau+1/4} & -\frac{2}{3(\tau+1/4)} & 0 \\ -\frac{2(\tau-1/4)}{3(\tau+1/4)} & -\frac{2(\tau+5/4)}{3(\tau+1/4)} & \frac{2}{9(\tau+1/4)} & 0 \\ \frac{1}{\tau+1/4} & \frac{4\tau-1}{\tau+1/4} & \frac{2}{3(\tau+1/4)} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{5.102}$$

$$N^{(0)}_{3\times3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} , \tag{5.103}$$

$$N^{(0)}_{4\times3} = N^{(0)}_{3\times4} = 0 , \tag{5.104}$$

and

$$K^{(1)}_{4\times3} = \begin{pmatrix} 0 & \frac{4(\tau-1)}{3(\tau-1/4)} & -\frac{4\tau}{3(\tau-1/4)} \\ 0 & 0 & 0 \\ -4(\tau-2) & 0 & 4 \\ 0 & -\frac{8(\tau-1)}{3(\tau-1/4)} & -\frac{8\tau}{3(\tau-1/4)} \end{pmatrix} , \tag{5.105}$$

$$K^{(1)}_{3\times4} = \begin{pmatrix} 0 & 0 & \frac{8\tau}{3(\tau-1/4)} & 0 \\ -4(\tau-1) & 0 & 0 & 2(\tau-1) \\ 4(\tau-2) & 0 & 4 & 2(\tau-2) \end{pmatrix} , \tag{5.106}$$

$$K^{(1)}_{4\times4} = K^{(1)}_{3\times3} = 0 , \tag{5.107}$$

$$
\begin{aligned}
N^{(1)}_{4\times 3} &= \begin{pmatrix} \frac{1}{\tau+1/4} & -\frac{4(\tau-1)}{3(\tau+1/4)} & \frac{4\tau}{3(\tau+1/4)} \\ \frac{2(\tau-1/4)}{3(\tau+1/4)} & \frac{4(\tau-1)}{9(\tau+1/4)} & -\frac{4\tau}{9(\tau+1/4)} \\ \frac{4\tau^2-5\tau-5/2}{\tau+1/4} & \frac{16\tau-1}{3(\tau+1/4)} & -\frac{4\tau}{3(\tau+1/4)} \\ 0 & 0 & 0 \end{pmatrix} , && (5.108) \\
N^{(1)}_{3\times 4} &= \begin{pmatrix} \frac{2}{\tau+1/4} & \frac{8(\tau-1/4)}{\tau+1/4} & \frac{4}{3(\tau+1/4)} & 0 \\ \frac{2(\tau-1)}{\tau+1/4} & \frac{8(\tau-1)(\tau-1/4)}{\tau+1/4} & \frac{4(\tau-1)}{3(\tau+1/4)} & -2(\tau-1) \\ -\frac{2(\tau-2)}{\tau+1/4} & -\frac{8(\tau-2)(\tau-1/4)}{\tau+1/4} & -\frac{4(\tau-2)}{3(\tau+1/4)} & -2(\tau-2) \end{pmatrix} , && (5.109) \\
N^{(1)}_{4\times 4} &= N^{(1)}_{3\times 3} = 0 \; . && (5.110)
\end{aligned}
$$

The transformation $\mathfrak{a} \to \mathfrak{a}'$ has brought the matrices into this handy block form. We also need

$$
\mathrm{e}^{-2x-6p} = \frac{2}{3} + \mathcal{O}(\mathrm{e}^{-2\tau}) \qquad (5.111)
$$

as well as

$$
\mathrm{e}^{-2A-8p} = \frac{3^{1/3}}{h(0)} \left(\tau - \frac{1}{4}\right) \mathrm{e}^{-2\tau/3} \left[1 + \mathcal{O}(\mathrm{e}^{-2\tau})\right] \; . \qquad (5.112)
$$

Note, that since (5.112) is asymptotically suppressed, the leading order of the asymptotic solutions is independent of the momentum k. It is worth to mention that the leading-order terms of these expressions coincide with the respective quantities evaluated in the Klebanov-Tseytlin background [30].

The asymptotic UV solutions are now found by iteratively solving the equations

$$
\begin{aligned}
\left(\partial_\tau - N^{(0)}\right) \phi^{(n)} &= \psi^{(n)} + \mathrm{e}^{-\tau}\, N^{(1)} \phi^{(n-1)} \; , && (5.113) \\
\left(\partial_\tau - M^{(0)}\right) \psi^{(n)} &= \beta \left(\tau - \frac{1}{4}\right) \mathrm{e}^{-2\tau/3}\, \phi^{(n-1)} + \mathrm{e}^{-\tau}\, M^{(1)} \psi^{(n-1)} \; , && (5.114)
\end{aligned}
$$

where $\beta = 3^{1/3}k^2/h(0)$, and we set $\phi^{(-1)} = \psi^{(-1)} = 0$. The solutions $\phi^{(0)}$ are the leading order terms of the asymptotic solutions. The asymptotic solutions are collected in appendix C.

One can notice a pattern in the solutions of the appendix. Let us consider the two groups of scalars consisting, on the one hand, of x, p, h_1 and Φ, and on the other hand, of y, b and h_2, or more precisely, the gauge invariant scalars built on them according to (5.16). In [39], these two sets of scalars were called the "glueball sector" and the "gluinoball sector", respectively. In the KT background, the scalars in the gluinoball sector are *inert*, i.e. their background solutions are identically zero, and consequently any terms coupling the two sectors are absent. This eventually leads to the singularity in the IR, which is resolved in the KS background by taking into account the backreaction on the gluinoball sector. Nevertheless, the UV decoupling is also apparent in the asymptotic solutions of the appendix. The dominant solutions $\hat{\mathfrak{a}}_1, \hat{\mathfrak{a}}_3, \hat{\mathfrak{a}}_4$ and $\hat{\mathfrak{a}}_5$ and the subdominant solutions $\check{\mathfrak{a}}_1, \check{\mathfrak{a}}_3, \check{\mathfrak{a}}_4$

and $\check{\mathfrak{a}}_5$, which are related to the operators of dimensions $\Delta = 8, 6$ and 4, only have the first four components excited at leading (and next-to-leading) order. These four components correspond exactly to the scalars of the glueball sector. The mixing only appears at order $\mathrm{e}^{-\tau}$ relative to the leading order, as is to be expected from the asymptotic expansion of the equations of motion, cf. section 5.4 in [39]. Similarly, the dominant solutions $\hat{\mathfrak{a}}_2, \hat{\mathfrak{a}}_6$ and $\hat{\mathfrak{a}}_7$ and the subdominant solutions $\check{\mathfrak{a}}_2, \check{\mathfrak{a}}_6$ and $\check{\mathfrak{a}}_7$, which are related to operators of dimensions $\Delta = 7$ and 3, only have the last three components excited at leading (and next-to-leading) order. These correspond to the scalars in the gluinoball sector.

5.5 Holographic Renormalization

We are now ready to apply the formalism of section 5.2 to the case of the KS system. In the following discussion, we often restrict ourselves to the $\Delta \leq 4$ operators in order to simplify the calculations. It is sufficient though for the discussion of all the general features of a system with several coupled scalars. Regardless this simplification calculations in question are still very involved and were performed by using MAPLE.

In order to discuss the issue of scheme dependence, we allow for redefinitions of the dominant asymptotic solutions given in appendix C.1 with the subdominant solutions of appendix C.2.In doing so we also restrict to the $\Delta \leq 4$ operators and only modify the corresponding dominant solutions according to

$$\hat{\mathfrak{a}}'_i = \hat{\mathfrak{a}}_i + \lambda_{ij}\check{\mathfrak{a}}_j \tag{5.115}$$

with $i, j = 4, 5, 6, 7$.

Using the asymptotic solutions listed in the appendix C we can calculate the matrix Z from (5.29)

$$Z_{ij} = 3^{1/3} P^4\, \mathrm{e}^{2\phi_0} \begin{pmatrix} -\frac{80}{3} & 0 & \frac{5}{4}\beta & 0 & -\frac{2531}{192}\beta^2 & -\frac{19}{4}\beta & \frac{419}{80}\beta \\ 0 & -\frac{2}{9} & 0 & 0 & \frac{8}{3} & \frac{3337}{11520}\beta^2 & \frac{6913}{3200}\beta^2 \\ 0 & 0 & \frac{20}{9} & \frac{737}{120}\beta & -\frac{4439}{600}\beta & -\frac{76}{9} & -\frac{7}{15} \\ 0 & 0 & 0 & -\frac{4}{9} & \frac{10}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{9} & \frac{5}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{9} \end{pmatrix}, \tag{5.116}$$

where we have introduced the abbreviation

$$\beta = \frac{3^{1/3}}{h(0)} k^2 \ . \tag{5.117}$$

Note that Z does not depend on the λ_{ij}, i.e. it is scheme independent as is expected from 5.38. As discussed in section 5.2.3, it is also possible to redefine the dominant solutions by other dominant ones. In particular, to a dominant solution of dimension Δ, one could add other dominant solutions of dimensions smaller than or equal to Δ. This would amount to an upper triangular matrix Λ, cf. (5.37). One can check that using $\mu_{ij} = \delta_{ij}$ and

$$\Lambda_{ij} = \left(3^{1/3} P^4 \, e^{2\phi_0}\right)^{-1} \begin{pmatrix} -\frac{3}{80} & 0 & \frac{27}{1280}\beta & \frac{59697}{204800}\beta^2 & -\frac{3051207}{2048000}\beta^2 & 0 & \frac{297}{640}\beta \\ 0 & -\frac{9}{2} & 0 & 0 & 27 & -\frac{30033}{5120}\beta^2 & \frac{2990637}{102400}\beta^2 \\ 0 & 0 & \frac{9}{20} & \frac{19899}{3200}\beta & -\frac{257769}{32000}\beta & -\frac{171}{10} & \frac{8739}{400} \\ 0 & 0 & 0 & -\frac{9}{4} & \frac{45}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9}{2} & \frac{45}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{4} \end{pmatrix} \tag{5.118}$$

in (5.38), which also rescales all operators, one can indeed obtain

$$Z'_{ij} = \delta_{ij} \ . \tag{5.119}$$

for KS background, as mentioned in section (5.2.3). The appearance of the β-factors in (5.118) leads to a "natural" form of Λ_{ij}, according to the discussion in section 5.2.3, as it ensures that the structure of the dominant solutions of appendix C.1 stays intact, i.e. also after the redefinition the same combinations of β and e^{τ} appear as before.

Now let us consider the matrix $\tilde{Z}$ from (5.29). The asymptotic solutions in the appendix have been chosen in such way that the submatrix of $\tilde{Z}$ involving only the $\Delta \leq 4$ operators vanishes identically. To obtain the other components, one would have to calculate more sub-leading terms in the dominant asymptotic solutions, but as stated in section 5.2.3, one can always choose a basis such that $\tilde{Z}_{ij} = 0$. This statement holds also after the operator redefinition given by (5.118). Allowing for scheme dependence in the rotated basis, one would find from (5.38)

$$\tilde{Z}'_{ij} = \lambda_{ji} - \lambda_{ij} \ . \tag{5.120}$$

Calculating the two-point functions of the dual operator using the (5.35) is more involved.
Let us consider the countertem matrix U_{ab}. For simplicity reasons and we only give the leading terms in an expansion in $\epsilon = e^{-2\tau/3}$. For $\lambda_{ij} = 0$, it is given by

$$U_{ab} = 2^{1/3} \left(\frac{e^{-\phi_0}}{P^2(4\tau - 1)}\right)^{2/3} \begin{pmatrix} U_{4\times 4} & U_{4\times 3} \\ U_{3\times 4} & U_{3\times 3} \end{pmatrix} , \tag{5.121}$$

with the submatrices

$$U_{4\times4} = \begin{pmatrix} -\frac{32}{15} & -\frac{32}{5} & -\frac{9}{640}(32\tau^2+148\tau-873)\beta^2\epsilon^2 & -\frac{9\beta\epsilon}{20} \\ -\frac{32}{5} & -\frac{96}{5} & 3\beta\epsilon & -\frac{117}{20}\beta\epsilon \\ -\frac{9}{640}(32\tau^2+148\tau-873)\beta^2\epsilon^2 & 3\beta\epsilon & -\beta\epsilon & \frac{3}{2}\beta\epsilon \\ -\frac{9}{20}\beta\epsilon & -\frac{117}{20}\beta\epsilon & \frac{3}{2}\beta\epsilon & -\frac{3(4\tau+17)\beta\epsilon}{16} \end{pmatrix},$$

$$U_{3\times4} = U_{4\times3}^T = \frac{\epsilon^{3/2}}{4\tau+5}\begin{pmatrix} \frac{4}{5}(28\tau-31) & \frac{128}{5}(2\tau+1) & \frac{16}{3} & \mathcal{O}(\epsilon) \\ \frac{16}{15}(2\tau+19) & \frac{256}{15}(\tau+2) & -\frac{16}{9} & \mathcal{O}(\epsilon) \\ -\frac{16}{15}(2\tau+19) & -\frac{256}{15}(\tau+2) & \frac{16}{9} & \mathcal{O}(\epsilon) \end{pmatrix},$$

$$U_{3\times3} = \frac{1}{4\tau+5}\begin{pmatrix} -\frac{2}{3}(4\tau+17) & \frac{8}{3} & -\frac{8}{3} \\ \frac{8}{3} & -\frac{8}{9} & \frac{8}{9} \\ -\frac{8}{3} & \frac{8}{9} & -\frac{8}{9} \end{pmatrix}. \tag{5.122}$$

The entries of $U_{3\times4}$ and $U_{4\times3}$ lead to mixings between the fields in the glueball and gluinoball sectors. When considering non-vanishing λ_{ij}, one notices that they are all scheme dependent.

In general, the scheme dependent terms should only lead to finite contributions to the action. We have checked this explicitly for the $\Delta \leq 4$ operators. Using the counterterm matrix with $\lambda_{ij} \neq 0$ and considering non-vanishing sources only for the operators with $\Delta \leq 4$, we find

$$\mathrm{e}^{4A}\,\mathfrak{a}\cdot U\cdot\mathfrak{a} = \sum_{i,j=4}^{7}\mathfrak{s}_i\Big(V_{ij}^{(1)}(\lambda_{kl}) + \epsilon^{-1}V_{ij}^{(2)} + V_{ij}^{(3)}\Big)\mathfrak{s}_j \tag{5.123}$$

with

$$V^{(1)}|_{4-7} = \frac{1}{9}3^{1/3}P^4\,\mathrm{e}^{2\phi_0}\times$$
$$\begin{pmatrix} 10\lambda_{45}-4\lambda_{44} & 2\lambda_{45}-2\lambda_{54}+5\lambda_{55} & \frac{5}{2}\lambda_{47}+5\lambda_{65}-2\lambda_{64}-\lambda_{46} & 2\lambda_{47}-2\lambda_{74}+5\lambda_{75} \\ 2\lambda_{45}-2\lambda_{54}+5\lambda_{55} & 4\lambda_{55} & \frac{5}{2}\lambda_{57}-\lambda_{56}+2\lambda_{65} & 2\lambda_{57}+2\lambda_{75} \\ \frac{5}{2}\lambda_{47}+5\lambda_{65}-2\lambda_{64}-\lambda_{46} & \frac{5}{2}\lambda_{57}-\lambda_{56}+2\lambda_{65} & 5\lambda_{67}-2\lambda_{66} & \frac{5}{2}\lambda_{77}+2\lambda_{67}-\lambda_{76} \\ 2\lambda_{47}-2\lambda_{74}+5\lambda_{75} & 2\lambda_{57}+2\lambda_{75} & \frac{5}{2}\lambda_{77}+2\lambda_{67}-\lambda_{76} & 4\lambda_{77} \end{pmatrix}. \tag{5.124}$$

These finite terms are analogous to the finite quartic counterterm in the GPPZ flow and the finite quadratic counterterm in the CB flow, cf. (5.68) and (5.51), respectively, after expanding them to quadratic order in the fluctuations.

In addition to these finite terms there are also divergent contributions which are either linearly diverging in $\epsilon = \mathrm{e}^{-2\tau/3}$ or logarithmically. These are given by

$$V^{(2)}|_{4-7} = \frac{1}{9} 3^{1/3} P^4 \, \mathrm{e}^{2\phi_0} \times$$
$$\begin{pmatrix} -\frac{3}{2}(\tau^2 - 3\tau + 5)\beta & -\frac{3}{8}(4\tau - 7)\beta & 0 & 0 \\ -\frac{3}{8}(4\tau - 7)\beta & -3\frac{2\tau+1}{4\tau+1}\beta & 0 & 0 \\ 0 & 0 & -\frac{1}{4}(16\tau^2 + 28\tau + 19) & -4(2\tau + 1) \\ 0 & 0 & -4(2\tau + 1) & -16 \end{pmatrix} \tag{5.125}$$

and

$$V^{(3)}|_{4-7} = \frac{1}{9} 3^{1/3} P^4 \, \mathrm{e}^{2\phi_0} \begin{pmatrix} V^{(3,\Delta=4)} & 0_{2\times 2} \\ 0_{2\times 2} & V^{(3,\Delta=3)} \end{pmatrix} , \tag{5.126}$$

with

$$V^{(3,\Delta=4)} = \beta^2 \begin{pmatrix} \frac{9}{16}\tau^4 + \frac{75}{16}\tau^3 + \frac{189}{32}\tau^2 - \frac{5355}{128}\tau + \frac{38979}{256} & \frac{3}{256}\frac{256\tau^4+2368\tau^3+4368\tau^2-5664\tau-1821}{4\tau+1} \\ \frac{3}{256}\frac{256\tau^4+2368\tau^3+4368\tau^2-5664\tau-1821}{4\tau+1} & \frac{9}{64}\frac{32\tau^3+424\tau^2+916\tau+371}{4\tau+1} \end{pmatrix} ,$$
$$V^{(3,\Delta=3)} = \beta \begin{pmatrix} \frac{3}{2}\tau^4 + \frac{45}{2}\tau^3 + \frac{477}{4}\tau^2 + \frac{1959}{8}\tau - \frac{1239}{32} & 4\tau^3 + 51\tau^2 + \frac{975}{4}\tau - \frac{831}{16} \\ 4\tau^3 + 51\tau^2 + \frac{975}{4}\tau - \frac{831}{16} & 12\tau^2 + 156\tau + 213 \end{pmatrix} .$$

Note that the linear divergences are momentum independent for the $\Delta = 3$ operators and proportional to k^2 for the $\Delta = 4$ operators. Furthermore, the logarithmically divergent terms are proportional to k^2 for the $\Delta = 3$ operators and proportional to k^4 for the $\Delta = 4$ operators. All this is very similar to the aAdS case. There is, however, a difference in the fact that the logarithms appear in a much more complicated way, and they are even present in the linearly divergent terms. Although some of this may be an artifact of the choice of radial variable, this is consistent with the fact that the KS theory has no UV conformal fixed point.

As mentioned above, all the entries of $U_{3\times 4}$ and $U_{4\times 3}$ are scheme dependent and thus only contribute finite terms to the renormalized action. This implies that one could have determined all the divergent terms for the glueball-sector and the gluinoball-sector separately. In other words, one can renormalize the KT theory without embedding it into the KS theory. This is plausible, as the KT background is a good approximation to the KS background in the asymptotic region, and the field theory divergences are UV divergences. Indeed, one can derive the diagonal components of the counterterms, i.e. $U_{4\times 4}$ and $U_{3\times 3}$, by setting the last three components of the dominant solutions $\hat{\mathfrak{a}}_1, \hat{\mathfrak{a}}_3, \hat{\mathfrak{a}}_4$ and $\hat{\mathfrak{a}}_5$ to zero when using (5.28), as well as the first four components of the dominant solutions $\hat{\mathfrak{a}}_2, \hat{\mathfrak{a}}_6$ and $\hat{\mathfrak{a}}_7$. As explained at the end of section 5.4.1, this corresponds to decoupling the glueball from the gluinoball sector.

Finally, we would like to comment on the issue of VEVs. We have shown, that for the aAdS cases, the response function of the background fluctuation W_ϕ/W encodes the VEV (up to an overall factor for the CB flow). We would like to see how this carries over to the case of KS. In order to derive the VEV from first principles, one would need the exact form of the counterterms linear in the fluctuations, which we have not determined yet. Thus, we can only take the cases of GPPZ and CB as encouraging examples and calculate, in analogy, the response coefficients of W^a/W. Zero mode in this case reads

$$\frac{W^a}{W} = \frac{4}{4\tau+1}\begin{pmatrix} -1 \\ 1/3 \\ -2(\tau-1/4) \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \frac{4\,\mathrm{e}^{-\tau}}{4\tau+1}\begin{pmatrix} \mathbf{0}_4 \\ -4\tau+1 \\ (-4\tau+1)(\tau-1) \\ (\tau-2)(4\tau-1) \end{pmatrix} + \mathcal{O}(\mathrm{e}^{-2\tau}) \ . \tag{5.127}$$

Comparing this with the asymptotic solutions of the appendix we obtain

$$\frac{W^a}{W} = -2\hat{\mathfrak{a}}_5 - 4\check{\mathfrak{a}}_7 + 2\check{\mathfrak{a}}_6 \ . \tag{5.128}$$

This result suggests the interpretation that a combination of the two $\Delta = 3$ operators has a VEV, which is in agreement with the field theory expectation of a condensate of the gluino bilinear [24, 79]. However, this statement is again scheme dependent. The redefinition (5.115) leads to

$$\mathfrak{a} = \mathfrak{s}_i\hat{\mathfrak{a}}_i + \mathfrak{r}_i\check{\mathfrak{a}}_i = \mathfrak{s}_i\hat{\mathfrak{a}}'_i + (\mathfrak{r}_i - \mathfrak{s}_j\lambda_{ji})\check{\mathfrak{a}}_i \ , \tag{5.129}$$

and applying this to $\frac{W^a}{W}$ results in

$$\frac{W^a}{W} = -2\hat{\mathfrak{a}}_5 + (-4+2\lambda_{57})\check{\mathfrak{a}}_7 + (2+2\lambda_{56})\check{\mathfrak{a}}_6 + 2\lambda_{55}\check{\mathfrak{a}}_5 + 2\lambda_{54}\check{\mathfrak{a}}_4 \ . \tag{5.130}$$

Let us apply the "naturalness" criterion on the form of the λ_{ij} described in section 5.2.3. It would give $\lambda_{55}, \lambda_{54} \sim \beta^2$, but $\beta = 0$ in (5.130), so that the coefficients of $\check{\mathfrak{a}}_4$ and $\check{\mathfrak{a}}_5$ vanish. The coefficients of the $\check{\mathfrak{a}}_6$ and $\check{\mathfrak{a}}_7$ belonging to the $\Delta = 3$ operators are more subtle, because the $\mathrm{e}^{-\tau}$ term in $\hat{\mathfrak{a}}_5$ is independent of β. On physical grounds we expect that there should be a natural scheme in which the VEV s for the $\Delta = 3$ operators are not both vanishing simultaneously, cf. [24, 79]. It still remains to understand how to determine such a preferred scheme, which might amount to extending the "naturalness" criterion of section 5.2.3 or to finding an equivalent of the supersymmetric scheme in the GPPZ flow. In this respect, it is interesting to notice that the $e^{-\tau}$ term of $\hat{\mathfrak{a}}_5$ in (C.5) can be written as $e^{-\tau}/(4\tau+1) \times (\mathbf{0}_4, -4, 5, -9)^T - 2\check{\mathfrak{a}}_6$. This suggests that the analog of the supersymmetric scheme in the GPPZ flow (which amounts to having a vanishing contribution of the sub-dominant solution to the dominant one, i.e. $\alpha_2 = 0$ in (5.72)) might be given by choosing $\lambda_{56} = 2$ and $\lambda_{57} = 0$ in (5.130). It yet remains to make this argument more precise. In order to check that the VEV is indeed encoded in the response of the zero mode, it would be interesting to calculate the VEV independently using the linear terms of the action, but for this one would need the linear counterterms, as mentioned above.

A Z-matrices

In this Appendix we will demonstrate that the Z-matrices defined in (5.29), are independent of r. We will show this on the example of Z_{ij}, as it is the most general defined matrix of the three, for $\tilde{Z}_{ij}$ and z_{ij} the calculations are analog. First of all we write down Z_{ij} again

$$Z_{ij} = \mathrm{e}^{dA} \left[(D_r \hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r \check{\mathfrak{a}})_j \right] \quad . \tag{A.1}$$

Next, we remember that the equation of motion for $\mathfrak{a}$ is given by (5.21)

$$\left[\left(D_r + M - \frac{2d}{d-1} W \right) (D_r - M) + e^{-2A} \Box \right] \mathfrak{a} = 0 \,, \tag{A.2}$$

and M is given by

$$M^a_{\ b} = W^a_{\ |b} - \frac{W^a W_b}{W} \,, \tag{A.3}$$

and D_r is the background covariant derivative

$$D_r \mathfrak{a}^a = \partial_r \mathfrak{a}^a + \mathcal{G}^a_{bc} W^b \mathfrak{a}^c \,. \tag{A.4}$$

Now let us expand the brackets and change to the momentum space

$$\begin{aligned} \left[D_r^2 + M \cdot D_r - \frac{2d}{d-1} W D_r - (D_r \cdot M) - M \cdot D_r - M^2 - \frac{2d}{d-1} W M - e^{-2A} k^2 \right] \mathfrak{a} &= \\ \left[D_r^2 - \frac{2d}{d-1} W D_r - (D_r \cdot M) - M^2 - \frac{2d}{d-1} W M - e^{-2A} k^2 \right] \mathfrak{a} &= 0 \,, \end{aligned} \tag{A.5}$$

where $(D_r \cdot M)$ means that the derivative acts only on M.
The independence of Z_{ij} on r implies that

$$\partial_r Z_{ij} = 0 \,. \tag{A.6}$$

Using (5.4) we obtain

$$\partial_r Z_{ij} = -\frac{2dW}{d-1} \mathrm{e}^{dA} \left[(D_r \hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r \check{\mathfrak{a}})_j \right] + \mathrm{e}^{dA} \partial_r \left[(D_r \hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r \check{\mathfrak{a}})_j \right] \,. \tag{A.7}$$

Now we remember that $\mathfrak{a}_i \cdot \mathfrak{a}_j$ stands for $\mathfrak{a}_i^a G_{ab} \mathfrak{a}_j^b$, where G_{ab} is the sigma-model metric. Hence

$$\begin{aligned}
\partial_r Z_{ij} \quad &\sim \quad -\frac{2dW}{d-1}\left[(D_r\hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r\check{\mathfrak{a}})_j\right] && \text{(A.8)}\\
&\quad +\partial_r (D_r\hat{\mathfrak{a}})_i^a G_{ab}\check{\mathfrak{a}}_j^b + (D_r\hat{\mathfrak{a}})_i^a \partial_r G_{ab}\check{\mathfrak{a}}_j^b + (D_r\hat{\mathfrak{a}})_i^a G_{ab}\partial_r\check{\mathfrak{a}}_j^b \\
&\quad -\partial_r \hat{\mathfrak{a}}_i^a G_{ab}(D_r\check{\mathfrak{a}})_j^b - \hat{\mathfrak{a}}_i^a \partial_r G_{ab}(D_r\check{\mathfrak{a}})_j^b - \hat{\mathfrak{a}}_i^a G_{ab}\partial_r(D_r\check{\mathfrak{a}})_j^b\,. && \text{(A.9)}
\end{aligned}$$

Let us forget about the first line in (A.8) for now and consider the other terms more closely. Using the definition of the covariant derivative, the product rule $\partial_r G_{ab} = \partial_c G_{ab}\partial_r\bar{\phi}^c = \partial_c G_{ab} W^c$ and $\partial_c G_{ab} = \mathcal{G}_{abc} + \mathcal{G}_{bca}$ we arrive at following expressions

$$\begin{aligned}
\partial_r (D_r\hat{\mathfrak{a}})_i^a G_{ab}\check{\mathfrak{a}}_j^b \quad &= \quad (D_r^2\hat{\mathfrak{a}})_i^a G_{ab}\check{\mathfrak{a}}_j^b - \mathcal{G}^a_{cd} W^c (D_r\hat{\mathfrak{a}})_i^d G_{ab}\check{\mathfrak{a}}_j^b \\
&= (D_r^2\hat{\mathfrak{a}})_i^a G_{ab}\check{\mathfrak{a}}_j^b - (D_r\hat{\mathfrak{a}})_i^d \mathcal{G}_{bcd} W^c \check{\mathfrak{a}}_j^b && \text{(A.10)}\\
(D_r\hat{\mathfrak{a}})_i^a \partial_r G_{ab}\check{\mathfrak{a}}_j^b \quad &= \quad (D_r\hat{\mathfrak{a}})_i^a \partial_c G_{ab} W^c \check{\mathfrak{a}}_j^b = (D_r\hat{\mathfrak{a}})_i^a (\mathcal{G}_{abc} + \mathcal{G}_{bca}) W^c \check{\mathfrak{a}}_j^b && \text{(A.11)}\\
(D_r\hat{\mathfrak{a}})_i^a G_{ab}\partial_r\check{\mathfrak{a}}_j^b \quad &= \quad (D_r\hat{\mathfrak{a}})_i^a G_{ab} D_r\check{\mathfrak{a}}_j^b - (D_r\hat{\mathfrak{a}})_i^a G_{ab}\mathcal{G}^b_{cd} W^c \check{\mathfrak{a}}_j^d \\
&= (D_r\hat{\mathfrak{a}})_i^a G_{ab} D_r\check{\mathfrak{a}}_j^b - (D_r\hat{\mathfrak{a}})_i^a \mathcal{G}_{acd} W^c \check{\mathfrak{a}}_j^d\,, && \text{(A.12)}
\end{aligned}$$

and similar expressions for the terms from the 3rd line in (A.8) with the "-"-sign. We observe, that the first term in (A.10) and its counterpart will give us

$$(D_r^2\hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r^2\check{\mathfrak{a}})_j\,. \tag{A.13}$$

The first term in (A.12) will be canceled by its counterpart, and the rest will cancel each other when we rename the dummy indices appropriately and remember that Christoffel symbols are symmetric under the exchange of 2nd and 3rd indices. Putting all together we arrive at

$$\begin{aligned}
\partial_r Z_{ij} \quad &\sim \quad -\frac{2dW}{d-1}\left[(D_r\hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r\check{\mathfrak{a}})_j\right] + \left[(D_r^2\hat{\mathfrak{a}})_i \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot (D_r\check{\mathfrak{a}})_j\right] \\
&= \quad \left[\left(D_r^2 - \frac{2dW}{d-1}D_r\right)\hat{\mathfrak{a}}_i\right] \cdot \check{\mathfrak{a}}_j - \hat{\mathfrak{a}}_i \cdot \left[\left(D_r^2 - \frac{2dW}{d-1}D_r\right)\check{\mathfrak{a}}_j\right]\,. && \text{(A.14)}
\end{aligned}$$

Using the equation of motion we obtain

$$\begin{aligned}
\partial_r Z_{ij} \quad &\sim \quad \left[(D_r \cdot M) + M^2 + \frac{2d}{d-1}WM + e^{-2A}k^2\right]\hat{\mathfrak{a}}_i \cdot \check{\mathfrak{a}}_j \\
&\qquad -\hat{\mathfrak{a}}_i \cdot \left[(D_r \cdot M) + M^2 + \frac{2d}{d-1}WM + e^{-2A}k^2\right]\check{\mathfrak{a}}_j\,, && \text{(A.15)}
\end{aligned}$$

and this indeed vanishes due to the fact that M is symmetric (remember that in $(D_r \cdot M)$ acts only on M).

B KS matrices

We present here the explicit expressions for the 7×7 matrices appearing in Section 5.4.1. In order to shorten the formulae, we will introduce a number of abbreviations. First,

$$c = \cosh y = \coth \tau \ , \qquad s = \sinh y = -(\sinh \tau)^{-1} \ , \tag{B.1}$$

where y denotes the background field of Section 5.4.1. Second, we introduce

$$B_1 = \tau c - 1 \ , \qquad B_2 = \tau s^2 - c \ , \tag{B.2}$$

and

$$A_1 = h(\tau)\,(4sB_2)^{-1/3} = h(\tau) \sinh \tau \,(2 \sinh 2\tau - 4\tau)^{-1/3} \ , \tag{B.3}$$

$$A_2 = -A_1 \left(cB_2 - \frac{2}{3} \right) - \frac{1}{2} s B_1 B_2 \ . \tag{B.4}$$

Let us consider the behavior for small and large τ of A_1 and A_2. As $h(0)$ is a finite, positive constant, one obtains

$$A_1(0) = \frac{1}{2} 3^{1/3} h(0) \ , \qquad A_2(0) = \frac{4}{3} A_1(0) \ . \tag{B.5}$$

For large τ, starting from

$$h(\tau) \approx 3\, \mathrm{e}^{-4\tau/3} \left(\tau - \frac{1}{4} \right) \ , \tag{B.6}$$

one obtains

$$A_1(\tau) \approx \frac{3}{2}\, \mathrm{e}^{-\tau} \left(\tau - \frac{1}{4} \right) \ , \qquad A_2(\tau) \approx \frac{3}{2}\, \mathrm{e}^{-\tau} \left(\tau + \frac{1}{4} \right) \ . \tag{B.7}$$

With the abbreviations (B.1)–(B.4), the (rotated) matrices $K^a_{\ b} = 2\, \mathrm{e}^{-4p}\, \mathcal{G}^a_{bc} W^c$ and $N^a_{\ b}$ are given by

$$K = \begin{pmatrix} 0 & 0 & \frac{s}{2A_1B_2} & 0 & 0 & -\frac{s^2B_1}{2A_1B_2} & \frac{s^2\tau}{2A_1B_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2(1+2cB_2) & 0 & \frac{2(A_2+cA_1B_2)}{A_1B_2} & 2cB_2+1 & 2s(\tau+2B_2) & 0 & -2s \\ 0 & 0 & \frac{s}{A_1B_2} & 0 & 0 & \frac{s^2B_1}{A_1B_2} & \frac{s^2\tau}{A_1B_2} \\ 0 & 0 & -\frac{s^2\tau}{A_1B_2} & 0 & 0 & 0 & -\frac{s}{A_1B_2} \\ 2sB_1 & 0 & 0 & -sB_1 & 0 & \frac{2(A_2+cA_1B_2)}{A_1B_2} & 0 \\ -2s(\tau+2B_2) & 0 & -2s & -s(\tau+2B_2) & -2(1+2cB_2) & 0 & \frac{2(A_2+cA_1B_2)}{A_1B_2} \end{pmatrix} ,$$

(B.8)

$$
N = \begin{pmatrix}
-\frac{2c(A_2+cA_1B_2)}{A_2} & -\frac{4cA_1}{A_2} & \frac{cs}{2A_2} & 0 & -\frac{s(A_2+cA_1B_2)}{A_2} & -\frac{cs^2B_1}{2A_2} & \frac{cs^2\tau}{2A_2} \\
-\frac{2cA_1}{3A_2} & -\frac{sB_1+2cA_1}{A_2} & \frac{s}{6A_2B_2} & 0 & -\frac{sA_1}{3A_2} & -\frac{s^2B_1}{6A_2B_2} & \frac{s^2\tau}{6A_2B_2} \\
\frac{-2(1+2cB_2)(A_2+cA_1B_2)}{A_2} & -\frac{4A_1(1+2cB_2)}{A_2} & \frac{s(1+2cB_2)}{2A_2} & -(1+2cB_2) & -\frac{2sA_2(\tau+2B_2)+sA_1B_2(1+2cB_2)}{A_2} & -\frac{s^2B_1(1+2cB_2)+4csA_2}{2A_2} & \frac{s^2\tau(1+2cB_2)}{2A_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2s(A_2+cA_1B_2)}{A_2} & -\frac{4sA_1}{A_2} & \frac{s^2}{2A_2} & 0 & -\frac{s^2A_1B_2}{A_2} - c & -\frac{s^3B_1}{2A_2} & \frac{s^3\tau}{2A_2} \\
-\frac{2sB_1(A_2+cA_1B_2)}{A_2} & -\frac{4sA_1B_1}{A_2} & \frac{s^2B_1}{2A_2} & sB_1 & -\frac{s^2A_1B_1B_2}{A_2} & -\frac{s^3B_1^2}{2A_2} & \frac{s^3\tau B_1}{2A_2} + 1 \\
\frac{2s(\tau+2B_2)(A_2+cA_1B_2)}{A_2} & \frac{4sA_1(\tau+2B_2)}{A_2} & -\frac{s^2(\tau+2B_2)}{2A_2} & s(\tau+2B_2) & \frac{2A_2(1+2cB_2)+s^2A_1B_2(\tau+2B_2)}{A_2} & \frac{s^3B_1(\tau+2B_2)+2A_2(2s^2+1)}{2A_2} & -\frac{s^3\tau(\tau+2B_2)}{2A_2}
\end{pmatrix} , \tag{B.9}
$$

The matrix M is given by

$$
M = -N - K + \frac{4}{3B_2}\mathbb{I} , \tag{B.10}
$$

where $\mathbb{I}$ denotes the 7×7 unit matrix.

Finally, we also need the sigma-model metric for the rotated fluctuation fields. It transforms as $G' = (R^{-1})^T G R^{-1}$, where R is the linear transformation matrix that leads to (5.96), and the superscript T denotes the transpose. Explicitly, we find

$$
G' = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2}P^2\, \mathrm{e}^{\Phi_0-2x}\cosh(2y) & 0 & 0 & 0 & \frac{1}{2}P^2\, \mathrm{e}^{\Phi_0-2x}\sinh(2y) \\
0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}P^2\, \mathrm{e}^{\Phi_0-2x} & 0 \\
0 & 0 & \frac{1}{2}P^2\, \mathrm{e}^{\Phi_0-2x}\sinh(2y) & 0 & 0 & 0 & \frac{1}{2}P^2\, \mathrm{e}^{\Phi_0-2x}\cosh(2y)
\end{pmatrix} , \tag{B.11}
$$

where for x and y one should substitute the respective backgound solutions.

C Asymptotic KS Solutions

C.1 Dominant Solutions

The dominant asymptotic solutions, up to order $\mathrm{e}^{-5\tau/3}$ relative to the leading term, are (in momentum space)

$$\hat{\mathfrak{a}}_1 = \frac{\mathrm{e}^{4\tau/3}}{4\tau+1}\begin{pmatrix}-12\\4\\12\\0\\\mathbf{0}_3\end{pmatrix} + \frac{9\beta}{32(4\tau+1)}\,\mathrm{e}^{2\tau/3}\begin{pmatrix}6(5+4\tau)\\-(9+4\tau)\\-6(5+4\tau)\\0\\\mathbf{0}_3\end{pmatrix} + \frac{24}{4\tau+1}\,\mathrm{e}^{\tau/3}\begin{pmatrix}\mathbf{0}_4\\1\\\tau-1\\2-\tau\end{pmatrix}$$
$$+\frac{27\beta^2}{256(4\tau+1)}\begin{pmatrix}-24\tau^2-48\tau-63/2\\8\tau+9\\24\tau^2+48\tau+63/2\\0\\\mathbf{0}_3\end{pmatrix} - \frac{27\beta(4\tau+5)}{8(4\tau+1)}\,\mathrm{e}^{-\tau/3}\begin{pmatrix}\mathbf{0}_4\\1\\\tau-1\\2-\tau\end{pmatrix}\,,\check{\mathfrak{a}} \quad \text{(C.1)}$$

$$\hat{\mathfrak{a}}_2 = \mathrm{e}^{\tau}\begin{pmatrix}\mathbf{0}_4\\0\\1\\1\end{pmatrix} + \frac{9\beta}{32}\,\mathrm{e}^{\tau/3}\begin{pmatrix}\mathbf{0}_4\\2\\2-2\tau\\-1-2\tau\end{pmatrix} + \begin{pmatrix}2\\-2/3\\4\tau-2\\0\\\mathbf{0}_3\end{pmatrix} - \frac{9\beta^2}{256}\,\mathrm{e}^{-\tau/3}\begin{pmatrix}\mathbf{0}_4\\8\tau^2-30\tau+45\\-6\tau^2-39\tau+243/2\\6\tau^2+6\tau-81\end{pmatrix}$$
$$+\frac{\beta}{16(4\tau+1)}\,\mathrm{e}^{-2\tau/3}\begin{pmatrix}-192\tau^2-840\tau-57\\112\tau^2+214\tau-1/2\\-576\tau^3-1824\tau^2+1380\tau+309\\4(4\tau+1)(54\tau-153)\\\mathbf{0}_3\end{pmatrix}\,, \quad \text{(C.2)}$$

$$
\begin{aligned}
\hat{\mathfrak{a}}_3 = {} & \frac{\mathrm{e}^{2\tau/3}}{4\tau+1}\begin{pmatrix} 4\tau+13 \\ 2\tau-7/2 \\ 12\tau-9 \\ 0 \\ \mathbf{0}_3 \end{pmatrix} - \frac{\beta}{32}\begin{pmatrix} 36\tau+63 \\ 8\tau-18 \\ 24\tau^2+12\tau-279/2 \\ 72\tau+42 \\ \mathbf{0}_3 \end{pmatrix} - \frac{\mathrm{e}^{-\tau/3}}{4\tau+1}\begin{pmatrix} \mathbf{0}_4 \\ 64\tau^2-104\tau-6 \\ 120\tau^2-246\tau-99 \\ -24\tau^2+174\tau+99 \end{pmatrix} \\
& + \frac{3\,\mathrm{e}^{-2\tau/3}\,\beta^2}{256(4\tau+1)}\begin{pmatrix} \frac{3}{4}(448\tau^3+912\tau^2-292\tau-3203) \\ -\frac{1}{8}(320\tau^3+1008\tau^2-1148\tau-6505) \\ 3(192\tau^4+80\tau^3-2411\tau-1092\tau^2+236) \\ 9(47+80\tau+16\tau^2)(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix} \\
& + \frac{\mathrm{e}^{-\tau}\,\beta}{32}\begin{pmatrix} \mathbf{0}_4 \\ 8(32\tau^2+180\tau+39)\tau \\ 4(16\tau^3+216\tau^2+54\tau+345)\tau \\ -(64\tau^4+608\tau^3-1752\tau^2-612\tau-1173) \end{pmatrix}, \qquad \text{(C.3)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathfrak{a}}_4 = {} & \begin{pmatrix} 1/2 \\ -1/6 \\ \tau-1 \\ 1 \\ \mathbf{0}_3 \end{pmatrix} + \beta\,\mathrm{e}^{-2\tau/3}\begin{pmatrix} -\frac{3}{32}\frac{5+32\tau^2+140\tau}{4\tau+1} \\ \frac{7}{32}\frac{-1+8\tau^2+14\tau}{4\tau+1} \\ -\frac{3}{16}\frac{48\tau^3+128\tau^2-115\tau-22}{4\tau+1} \\ \frac{9}{16}(4\tau-15) \\ \mathbf{0}_3 \end{pmatrix} + \mathrm{e}^{-\tau}\begin{pmatrix} \mathbf{0}_4 \\ -1 \\ -\tau+\frac{1}{2} \\ 3\tau-\frac{7}{2} \end{pmatrix} \\
& + \beta^2\frac{3\,\mathrm{e}^{-4\tau/3}}{512(4\tau+1)}\begin{pmatrix} 56\tau^4+1340\tau^3+\frac{9387}{2}\tau^2+\frac{81873}{8}\tau+\frac{332787}{64} \\ -104\tau^4-692\tau^3-\frac{1593}{2}\tau^2-\frac{66891}{8}\tau-\frac{193425}{64} \\ 480\tau^5+4416\tau^4+7212\tau^3+\frac{89109}{2}\tau^2-\frac{416403}{64}-\frac{28917}{8}\tau \\ (4\tau+1)(14067-6012\tau+120\tau^2-384\tau^3-32\tau^4) \\ \mathbf{0}_3 \end{pmatrix} \\
& + \beta\frac{3\,\mathrm{e}^{-5\tau/3}}{4\tau+1}\begin{pmatrix} \mathbf{0}_4 \\ -\frac{1}{8}(26+95\tau+200\tau^2+48\tau^3) \\ \frac{1}{128}(3365+13956\tau+6864\tau^2+1600\tau^3) \\ -\frac{1}{128}(3379+14780\tau+13424\tau^2+1216\tau^3) \end{pmatrix}, \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathfrak{a}}_5 = {}& \frac{4}{4\tau+1}\begin{pmatrix} 1/2 \\ -1/6 \\ \tau - 1/4 \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \beta\frac{\mathrm{e}^{-2\tau/3}}{4\tau+1}\begin{pmatrix} -\frac{3}{2}(11+2\tau) \\ \frac{7}{16}(13+4\tau) \\ -\frac{3}{16}(-31+48\tau^2+224\tau) \\ \frac{9}{2}(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix} + \frac{\mathrm{e}^{-\tau}}{4\tau+1}\begin{pmatrix} \mathbf{0}_4 \\ -4 \\ 3-8\tau \\ -7+8\tau \end{pmatrix} \\
& + \frac{3\,\mathrm{e}^{-4/3\tau}\,\beta^2}{256(4\tau+1)}\begin{pmatrix} \frac{3}{16}(256\tau^3+6816\tau^2+28776\tau+26771) \\ -\frac{1}{16}(1280\tau^3+16032\tau^2+24168\tau+25059) \\ \frac{3}{16}(2048\tau^4+31488\tau^3+67872\tau^2+53640\tau-10347) \\ -(4\tau+1)(783+1044\tau+696\tau^2+32\tau^3) \\ \mathbf{0}_3 \end{pmatrix} \\
& + \frac{3\beta\,\mathrm{e}^{-5/3\tau}}{32(4\tau+1)}\begin{pmatrix} \mathbf{0}_4 \\ -8(-17+106\tau+24\tau^2) \\ 53+2016\tau+496\tau^2 \\ -208\tau^2+2912\tau-41 \end{pmatrix} ,
\end{aligned}
\tag{C.5}
$$

$$
\begin{aligned}
\hat{\mathfrak{a}}_6 = {}& \mathrm{e}^{-\tau/3}\begin{pmatrix} \mathbf{0}_4 \\ 2\tau+1 \\ 3\tau+3/2 \\ 9/4 \end{pmatrix} + \beta\,\mathrm{e}^{-\tau}\begin{pmatrix} \mathbf{0}_4 \\ -\frac{1}{16}(-327+16\tau^3+132\tau^2+498\tau) \\ -\frac{1}{8}\tau^2(2\tau^2+32\tau+177) \\ \frac{1}{32}(8\tau^4+96\tau^3+360\tau^2-1353\tau-18) \end{pmatrix} \\
& + \frac{\mathrm{e}^{-4/3\tau}}{4\tau+1}\begin{pmatrix} \frac{3}{4}(16\tau^2-32\tau-37) \\ \frac{1}{4}(16\tau^2+40\tau+37) \\ -\frac{3}{2}(16\tau^3+48\tau^2+27\tau-10) \\ 0 \\ \mathbf{0}_3 \end{pmatrix} \\
& + \beta^2\,\mathrm{e}^{-5/3\tau}\,\frac{9}{1024}\begin{pmatrix} \mathbf{0}_4 \\ -(166029+101586\tau+26904\tau^2+3136\tau^3+160\tau^4) \\ -\frac{1}{128}(-68652765-25230456\tau-2624160\tau^2+353024\tau^3+72192\tau^4+4096\tau^5) \\ \frac{1}{128}(-37335432\tau-5324640\tau^2+91392\tau^3+65024\tau^4+4096\tau^5-89060931) \end{pmatrix} \\
& + \frac{\mathrm{e}^{-2\tau}\,\beta}{4\tau+1}\begin{pmatrix} -\frac{8}{5}\tau^5-\frac{111}{5}\tau^4-\frac{2411}{25}\tau^3+\frac{47601}{250}\tau^2+\frac{5167071}{40000}\tau+\frac{114183051}{800000} \\ -\frac{2}{15}\tau^5-\frac{21}{10}\tau^4-\frac{1753}{150}\tau^3-\frac{1367}{250}\tau^2+\frac{1963411}{80000}\tau-\frac{55918909}{1600000} \\ \frac{58}{5}\tau^5+\frac{1127}{10}\tau^4+\frac{11511}{25}\tau^3-\frac{359529}{1000}\tau^2+\frac{21641679}{40000}\tau+\frac{32591949}{800000} \\ -\frac{9}{4}(127+37\tau+2\tau^2)(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix} ,
\end{aligned}
\tag{C.6}
$$

and

$$
\hat{\mathfrak{a}}_7 = \mathrm{e}^{-\tau/3}\begin{pmatrix}\mathbf{0}_4\\ 4\\ 9\\ -3\end{pmatrix} + \frac{\beta\,\mathrm{e}^{-\tau}}{16}\begin{pmatrix}\mathbf{0}_4\\ 246-48\tau^2-480\tau\\ -16\tau^3-336\tau^2-1674\\ 1647+16\tau^3+288\tau^2-564\tau\end{pmatrix} + \frac{\mathrm{e}^{-4\tau/3}}{4\tau+1}\begin{pmatrix}2(20\tau-19)\\ 2(4\tau+9)\\ -6(8\tau^2+22\tau-3)\\ 0\\ \mathbf{0}_3\end{pmatrix}
$$

$$
+\frac{\beta^2\,\mathrm{e}^{-5\tau/3}}{8192}\begin{pmatrix}\mathbf{0}_4\\ -6448032-3279744\tau-746496\tau^2-36864\tau^3\\ 13447863+2810376\tau-88992\tau^2-205056\tau^3-9216\tau^4\\ -5817528\tau-368928\tau^2+191232\tau^3+9216\tau^4-19368153\end{pmatrix}
$$

$$
+\frac{\beta\,\mathrm{e}^{-2\tau}}{4\tau+1}\begin{pmatrix}-\frac{32}{5}\tau^4-\frac{3032}{25}\tau^3+\frac{18681}{125}\tau^2+\frac{131688341}{101250}\tau+\frac{764839421}{2025000}\\ -\frac{8}{15}\tau^4-\frac{256}{25}\tau^3-\frac{4079}{250}\tau^2+\frac{89944303}{101250}\tau+\frac{306325301}{1518750}\\ \frac{192}{5}\tau^4+\frac{12832}{25}\tau^3-\frac{38931}{125}\tau^2-\frac{10181983}{50625}\tau-\frac{175564421}{2025000}\\ -\frac{9}{2}(25+2\tau)(4\tau+1)\\ \mathbf{0}_3\end{pmatrix}. \quad \text{(C.7)}
$$

C.2 Subdominant Solutions

The subdominant asymptotic solutions, up to and including terms of order $\mathrm{e}^{-8\tau/3}$, are

$$\check{\mathfrak{a}}_1 = \frac{\mathrm{e}^{-8\tau/3}}{30(4\tau+1)} \begin{pmatrix} 3(160\tau^2 - 172\tau + 1) \\ -(160\tau^2 + 308\tau + 121) \\ -6(260\tau^2 - 107\tau - 16) \\ -450(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix}, \tag{C.8}$$

$$\check{\mathfrak{a}}_2 = \mathrm{e}^{-7\tau/3} \begin{pmatrix} \mathbf{0}_4 \\ 1/2 \\ -\frac{3}{50}(5\tau+4) \\ -\frac{3}{100}(10\tau - 17) \end{pmatrix}, \tag{C.9}$$

$$\check{\mathfrak{a}}_3 = \frac{\mathrm{e}^{-2\tau}}{4\tau+1} \begin{pmatrix} 4\tau + 1/5 \\ 2\tau + 23/30 \\ -4\tau - 1/5 \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \frac{3\beta}{160(4\tau+1)}\,\mathrm{e}^{-8\tau/3} \begin{pmatrix} 80\tau^2 + 144\tau + 5 \\ \frac{8}{3}(20\tau^2 + 36\tau + 11) \\ -(80\tau^2 + 144\tau + 5) \\ 0 \\ \mathbf{0}_3 \end{pmatrix}, \tag{C.10}$$

$$\begin{aligned} \check{\mathfrak{a}}_4 = {} & \frac{\mathrm{e}^{-4\tau/3}}{4\tau+1} \begin{pmatrix} 3 \\ -1 \\ 12\tau \\ -4(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix} + \frac{3\beta}{20(4\tau+1)}\,\mathrm{e}^{-2\tau} \begin{pmatrix} -(12\tau^2 - 51\tau - 23) \\ -\frac{1}{24}(144\tau^2 - 56\tau + 53) \\ 2(56\tau^2 + 17\tau - 4) \\ -5(4\tau+1)^2 \\ \mathbf{0}_3 \end{pmatrix} \\ & + \frac{3}{25(4\tau+1)}\,\mathrm{e}^{-7\tau/3} \begin{pmatrix} \mathbf{0}_4 \\ -50 \\ -80\tau^2 - 74\tau + 49 \\ -80\tau^2 + 226\tau - 51 \end{pmatrix} \\ & + \frac{9\beta^2}{10240(4\tau+1)}\,\mathrm{e}^{-8\tau/3} \begin{pmatrix} -1152\tau^3 + 2720\tau^2 + 7122\tau + 1763 \\ -384\tau^3 - 992\tau^2/3 + 650\tau + 361/3 \\ 8256\tau^3 + 10444\tau^2 - 1392\tau - 4169/4 \\ -3(320\tau^2 + 200\tau - 343)(4\tau+1) \\ \mathbf{0}_3 \end{pmatrix}, \end{aligned} \tag{C.11}$$

$$\check{\mathfrak{a}}_5 = \mathrm{e}^{-4\tau/3}\begin{pmatrix}1\\-1\\6\tau-3\\-4\tau+9\\\mathbf{0}_3\end{pmatrix} + \frac{3\beta}{800}\,\mathrm{e}^{-2\tau}\begin{pmatrix}-80\tau^2-32\tau-291\\-40\tau^2-356\tau-5807/6\\1680\tau^2+2732\tau-1634\\-50(4\tau-15)(4\tau+9)\\\mathbf{0}_3\end{pmatrix}$$

$$+\frac{3}{250}\,\mathrm{e}^{-7\tau/3}\begin{pmatrix}\mathbf{0}_4\\2000\tau/3\\-400\tau^2+260\tau-547\\-400\tau^2+2260\tau-297\end{pmatrix}$$

$$+\frac{\beta^2}{204800}\,\mathrm{e}^{-8\tau/3}\begin{pmatrix}-34560\tau^3-111024\tau^2+259556\tau-497927/2\\-11520\tau^3-104688\tau^2-1489028\tau/3-2917225/6\\470880\tau^3+1839672\tau^2-135995\tau-766688\\-3(57600\tau^3-36000\tau^2-1110600\tau-1043879)\\\mathbf{0}_3\end{pmatrix}, \quad \text{(C.12)}$$

$$\check{\mathfrak{a}}_6 = \mathrm{e}^{-\tau}\begin{pmatrix}\mathbf{0}_4\\0\\1\\-1\end{pmatrix} + \frac{9\beta}{8}\,\mathrm{e}^{-5\tau/3}\begin{pmatrix}\mathbf{0}_4\\1\\\tau+1/8\\-\tau+5/8\end{pmatrix} + \mathrm{e}^{-2\tau}\begin{pmatrix}-3\\-13/6\\1\\0\\\mathbf{0}_3\end{pmatrix}$$

$$+\frac{3\beta^2}{640}\,\mathrm{e}^{-7\tau/3}\begin{pmatrix}\mathbf{0}_4\\195\tau+2989/12\\\frac{1}{50}(3150\tau^2+1505\tau-5591)\\-\frac{1}{100}(11700\tau^2-5260\tau-28993)\end{pmatrix}$$

$$+\frac{3\beta}{400(4\tau+1)}\,\mathrm{e}^{-8\tau/3}\begin{pmatrix}-3(40\tau^2+682\tau+69)\\-(4\tau+11/2)(140\tau+57)\\-3(620\tau^2-709\tau-117)\\0\\\mathbf{0}_3\end{pmatrix} \quad \text{(C.13)}$$

and

$$\check{\mathfrak{a}}_7 = \mathrm{e}^{-\tau}\begin{pmatrix}\mathbf{0}_4\\1\\\tau\\1-\tau\end{pmatrix} + \frac{9\beta}{16}\,\mathrm{e}^{-5\tau/3}\begin{pmatrix}\mathbf{0}_4\\4\tau+11\\2\tau^2+\tau-12\\-2\tau^2+\tau+85/4\end{pmatrix} + \frac{\mathrm{e}^{-2\tau}}{20}\begin{pmatrix}32\tau-29\\\frac{1}{6}(16\tau+37)\\-7(16\tau-7)\\0\\\mathbf{0}_3\end{pmatrix}$$

$$+\frac{27\beta^2}{6400}\,\mathrm{e}^{-7\tau/3}\begin{pmatrix}\mathbf{0}_4\\300\tau^2+1625\tau+627\\80\tau^3-33\tau^2-1131\tau-13649/10\\-\frac{3}{40}(1600\tau^3+2440\tau^2-35920\tau-27383)\end{pmatrix}$$

$$+\frac{\beta}{16000(4\tau+1)}\,\mathrm{e}^{-8\tau/3}\begin{pmatrix}3(35200\tau^3-27760\tau^2-316608\tau+42739)\\4(800\tau^3+4240\tau^2-15228\tau-33961)\\-3(169600\tau^3+219680\tau^2-705996\tau-67973)\\0\\\mathbf{0}_3\end{pmatrix}, \quad \text{(C.14)}$$

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