Dark Energy and Bouncing Universe from k-fields

Jin U Kang

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Jin U Kang
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Erstgutachter: Prof. Dr. Viatcheslav Mukhanov
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to my parents

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Zusammenfassung

Diese Arbeit beschäftigt sich mit einigen kosmologischen Implikationen der K-Felder, welche allgemeine Skalarfelder mit nicht-kanonischen kinetischen Termen in der Wirkung sind.

Wir benutzen sogenannte K-Essenzkosmologien zur Erklärung der derzeit beobachteten beschleunigten Ausdehnung des Universums. In diesen Kosmologien müssen keine Anfangsbedingungen von Hand vorgegeben werden, da das K-Essenzfeld ϕ, welches die Inflation treibt, aufgrund seiner Dynamik zu einem Attraktor im Phasenraum gezogen wird. Damit wird das Koinzidenzproblem gelöst. Bisher wurden diese Kosmologien aufgrund der Einsicht, dass alle K-Essenzkosmologien mit Lagrangedichte $p = L(X)/\phi^2$ notwendigerweise eine Epoche überlichtschneller Propagation des K-Essenzfelds enthalten, als unphysikalisch angesehen. Dies wurde als "no-go"-Theorem der K-Essenzkosmologie bezeichnet. Wir untersuchen intensiv kosmologische Lösungen (sog. Tracker) mit einem K-Essenzfeld und zusätzlicher Materie, die im Phasenraum zum Attraktor fließen, und klassifizieren alle Lagrangedichten der Form $p = K(\phi)L(X)$, welche asymptotisch stabile Tracker-Lösungen zulassen. Mit Hilfe dieser Klassifikation sind wir in der Lage, Modelle zu identifizieren, welche die derzeitige beschleunigte Ausdehnung des Universums beschreiben und gleichzeitig das Koinzidenzproblem lösen. Das Problem der überlichtschnellen Ausbreitung besteht allerdings auch in diesen Modellen weiterhin. Wir zeigen aber, dass die überlichtschnelle Epoche nicht zu Kausalitätsverletzungen führt. Weiterhin diskutieren wir allgemein die Implikationen überlichtschneller Signalausbreitung auf mögliche Kausalitätsverletzungen in lorentzinvarianten Feldtheorien.

Eine weitere Anwendung von K-Feldern wird im Zusammenhang mit den so genannten neuen ekpyrotischen Szenarien beschrieben. In einem Modell, bei dem das ekpyrotische/zyklische Szenario mit der Theorie der Geistkondensate und dem Kurvaton-Mechanismus zur Produktion adiabatischer Störungen der Metrik verbunden ist, versucht man das Singularitätsproblem des Urknalls durch Verletzung der Nullenergiebedingung zu lösen. Die Lagrangedichte dieser Theorie enthält, genau wie auch das Geistkondensatmodell, einen Term mit höheren Ableitungen, welcher die Funktion hat das Vakuum der Theorie zu stabilisieren. Wir finden, dass dieser Term die Dynamik der kosmischen Entwicklung beeinflusst. Für die Quantisierung muss man jedoch aufgrund dieses Terms ein neues Geistfeld einführen, welches zu einer katastrophalen Vakuuminstabilität führt. Wir erklären, wieso dieser gefährliche Term selbst bei niedrigen Energien und Impulsen nicht als kleine Korrektur betrachtet werden kann, und demonstrieren die Probleme, welche bei der Konstruktion

einer UV-Vervollständigung der Theorie auftreten.

Zuletzt betrachten wir ein neues Modell in der Stringtheorie, indem versucht wird die Urknallsingularität durch die Erzeugung einer instabilen (nicht-BPS) Bran aufzulösen. Diese instabile Bran wird erzeugt wenn das Universum am Umkehrpunkt eine von der Stringlänge dominierte Phase durchschreitet bevor es wieder expandiert und die Bran zerfällt. Der Vorteil unseres Modells ist, dass sowohl die Krümmung als auch die Ableitung des Dilaton-Feldes während der gesammten Entwicklung durch den Umkehrpunkt hindurch klein bleiben (im Vergleich zur Stringeinheit). Somit ist es gerechtfertigt sich der perturbativen Stringtheorie und der einfachsten Wirkung für die bei niedrigen Energien relevanten Felder zu bedienen. Ausserdem wird eine Feineinstellung von Parametern vermieden und es tritt keine Verletzung der "Null Energie Bedingung" auf.

Zusammenfassend wurden einige Szenarien mit K-Feldern untersucht, welche von Interesse im derzeitigen Forschungsbereich der Kosmologie sind. Zukünfitige Arbeit könnte diejenigen Modelle mit K-Feldern aussuchen, welche die bedeutenden Probleme der modernen Kosmologie lösen.

Abstract

In this thesis we consider some cosmological implications of k-fields, which are general scalar fields with non-canonical kinetic terms in the action.

Cosmological scenarios with k-essence are invoked in order to explain the observed late-time acceleration of the universe. These scenarios avoid the need for fine-tuned initial conditions (the "coincidence problem") because of the attractor-like dynamics of the k-essence field ϕ. It was recently shown that all k-essence scenarios with Lagrangians $p = L(X)\phi^{-2}$, where $X \equiv \frac{1}{2}\phi_{,\mu}\phi^{,\mu}$, necessarily involve an epoch where perturbations of ϕ propagate faster than light (the "no-go theorem"). We carry out a comprehensive study of attractor-like cosmological solutions ("trackers") involving a k-essence scalar field ϕ and another matter component. The result of this study is a complete classification of k-essence Lagrangians that admit asymptotically stable tracking solutions, among all Lagrangians of the form $p = K(\phi)L(X)$. Using this classification, we select the class of models that describe the late-time acceleration and avoid the coincidence problem through the tracking mechanism. An analogous "no-go theorem" still holds for this class of models, indicating the existence of a superluminal epoch. In the context of k-essence cosmology, the superluminal epoch does not lead to causality violations. We discuss the implications of superluminal signal propagation for possible causality violations in Lorentz-invariant field theories.

Another application of k-fields was made in the new ekpyrotic scenario that attempts to solve the big-bang singularity problem by involving violation of the null energy condition in a model which combines the ekpyrotic/cyclic scenario with the ghost condensate theory and the curvaton mechanism of production of adiabatic perturbations of metric. The Lagrangian of this theory, as well as of the ghost condensate model, contains a term with higher derivatives, which was added to the theory to stabilize its vacuum state. We find that this term may affect the dynamics of the cosmological evolution. Moreover, after a proper quantization, this term results in the existence of a new ghost field with negative energy, which leads to a catastrophic vacuum instability. We explain why one cannot treat this dangerous term as a correction valid only at small energies and momenta below some UV cut-off, and demonstrate the problems arising when one attempts to construct a UV completion of this theory.

Finally, we consider a novel scenario in string theory that attempts to resolve the big-bang singularity through the creation of an unstable (non-BPS) brane as the universe bounces through a string size regime before expanding again as the brane decays. The nice feature in our scenario is that the curvature as well as the derivative of the dilaton remain small (in string units) through the bounce. Thus we are justified in using perturbative string theory and the simplest low energy effective action for these fields. In addition no fine tuning is required and no violation of the NEC occurs in our model.

In summary, we have investigated several scenarios involving k-fields that are of interest in current research in cosmology. Further work may select the models with k-fields that could solve the outstanding problems of modern cosmology.

1 Introduction and Discussion

Scalar fields with non-standard kinetic terms have remarkable implications for cosmology. Throughout this thesis we will refer to general scalar fields with non-standard kinetic terms as k-fields. The advent of k-fields traces back to the thirties in the last century. It was Born and Infeld [1] who introduced this kind of fields for the first time to avoid the infinite self energy of electron, and their ideas were further developed by Heisenberg [2] and Dirac [3] later. Due to the non-linearity coming from non-standard kinetic terms, the phenomenology of k-fields is very rich and has attracted considerable interest. The cosmological application of k-fields is the main subject of this thesis.

From the view point of the top-down approach, the main motivation of k-fields comes from the fact that non-standard kinetic terms generically arise in the low energy effective field theories of fundamental physics such as string theory. The most general form of the effective action of a fundamental theory contains all the terms that are allowed by the required symmetries, and the low energy action reduces to the form of k-fields when one restricts the action to the first derivative terms [4]. Therefore, it is quite generic to encounter the k-fields when one deals with effective field theory. Some examples of k-fields theories originating from string theory includes dilaton in Einstein frame [5], DBI action for D-brane [6, 7, 8], and tachyons on a non-BPS D-brane [9, 10].

The cosmology of k-fields was first studied in the context of inflation [11]. Inflation driven by k-fields was called k-inflation. There were also variants of k-inflation such as ghost condensate scenario [12] and ghost inflation [13]. Traditional inflationary models rely on flat potentials that satisfy the so called *slow-roll conditions*, while an accelerating expansion can take place due to the nontrivial kinetic term in the models of k-inflation. Even without any potential term it is possible to have de Sitter phase as a attractor solution. In contrast to the slow-roll inflation, the inflationary phase can arise even if the time derivative of the background field is large. Because the speed of sound is a free parameter of the k-field theory, it is possible to have scenario where the tensor-to-scalar ratio is much enhanced compared to what is naively expected in simple inflationary model [14]. Furthemore, in the slow-roll models the non-Gaussianity is highly suppressed by slow-roll condition, whereas one can get significant amount of non-Gaussianity in the theory of the scalar field with non-canonical kinetic term [15].

After the k-inflation was proposed, this theory was applied to dark energy, and this scenario was called k-essence scenario [16, 17, 18]. Quintessence relies on the potential energy of scalar fields to explain the late time acceleration of the universe, whereas k-essence gives rise to the accelerated expansion out of the modifications to the kinetic energy.

On the other hand, the k-fields have been used to resolve the initial big-bang singularity [19, 20, 21] by invoking a transition from contracting universe to expanding one. This transition is called *bounce*. In the context of the scalar field theory with the canonical kinetic term the bounce can not arise due to null energy condition, and therefore the scalar field with non-canonical kinetic term has been invoked for the bounce dynamics. Since the energy scale in this regime is supposed to be high, the resolution of the big-bang singularity has been a natural playground for the k-fields which is supposed to originate from fundamental physics.

In this thesis we consider the applications of k-fields to the dark energy and bouncing cosmology.

Cosmological scenarios involving a scalar field known as k-essence [16, 17, 18] are intended to explain the late-time acceleration of the universe (see Ref. [22] for a recent review of dynamical models of dark energy). An important motivation behind the k-essence scenarios is to avoid the fine-tuning of the initial conditions for the scalar field (the "coincidence problem"). The effective Lagrangian $p(X, \phi)$ describing the dynamics of the scalar field ϕ consists of a noncanonical kinetic term,

$$p(X, \phi) = K(\phi)L(X), \quad X \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{1.1}$$

where $K(\phi)$ and $L(X)$ are functions determined by the underlying fundamental theory. One considers the evolution of the field ϕ coupled to gravity in a standard homogeneous cosmology in the presence of matter. With a suitable choice of the Lagrangian, the evolution of ϕ during radiation domination quickly drives the system into a region in phase space where the k-essence field ϕ has a nearly constant equation of state with $w_\phi = \frac{1}{3}$, mimicking radiation. Thus the energy density ε_ϕ of k-essence approaches a constant fraction of the energy density ε_m of the radiation. This behavior of k-essence ($w_\phi \to$ const and $\varepsilon_\phi/\varepsilon_{\text{tot}} \to$ const, where $\varepsilon_{\text{tot}} \equiv \varepsilon_\phi + \varepsilon_m$) is called tracking, and the solution with $w_\phi \approx$ const is called a tracker solution.

The parameters of the Lagrangian can be adjusted such that the energy density in k-essence during the radiation era is small ($\varepsilon_m \approx \varepsilon_{\text{tot}}$), so that the standard cosmological evolution is not significantly altered. After the onset of dust domination ($w_m = 0$), the energy density in k-essence quickly becomes negligible and the evolution leaves the radiation tracker. A tracking solution with $w_\phi = 0$ does not exist (due to a particular choice of the Lagrangian), and instead the k-essence is driven to a tracking regime with $w_\phi \approx$ const < 0. Since $w_\phi < w_m$, the k-essence will eventually dominate the energy density of the dust component. The precise value of w_ϕ in that regime can be parametrically adjusted to fit the currently observed data; in particular, values $w_\phi \approx -1$ can be achieved.[1]

[1]We note that the "phantom" values $w_\phi < -1$ cannot be reached in this single-field model; see e.g. [23, 24].

In our terminology, a "tracking solution" is a solution for which w_ϕ approaches a fixed value, whether or not this value is equal to the equation of state parameter w_m of the dominant matter component. It is essential that the tracker solutions are stable attractors for nearby solutions. Because of this property, the field ϕ is driven into the tracker regime in the phase space with fixed values of w_ϕ and the ratio $\varepsilon_\phi/\varepsilon_{\mathrm{tot}}$, for a wide range of initial conditions for ϕ. To construct a viable k-essence model, it is important to choose a Lagrangian $p(X,\phi)$ for which stable tracker solutions exist within the radiation- and dust-dominated cosmological eras.

Previous works concerning the dynamics of k-essence either assumed a specific form of the Lagrangian, for instance [18]

$$p(X,\phi) = \frac{L(X)}{\phi^2}, \tag{1.2}$$

or imposed ad hoc restrictions on the Lagrangian with the purpose of deriving exact solutions (e.g. [26]). In particular, it was assumed that $w_\phi = \mathrm{const}$ is an exact solution of the equations of motion. However, the physically necessary requirement is weaker: namely, one merely needs that w_ϕ should *approach* a constant value asymptotically at late times. The existence of an *exact* solution $w_\phi = \mathrm{const}$ is not necessary. With this weaker requirement, a much wider range of Lagrangians enters the consideration.

In Chapter 2, we restrict our attention to Lagrangians of the "factorized" form (1.1) but do not impose further restrictions on the Lagrangians; neither do we require the existence of analytic exact solutions, or of solutions with $w_\phi = \mathrm{const}$. It is only assumed that the cosmological scenario is realized with $\dot\phi > 0$ and that ϕ reaches arbitrarily large values. Our results can be viewed as a comprehensive extension of previous studies of attractor behavior in k-essence cosmology (e.g. [27, 28, 26]). We determine the class of Lagrangians $p(X,\phi)$ that admit stable tracking regimes in which $w_\phi \to \mathrm{const}$, for a given value of w_m. The possible asymptotic values of w_ϕ and $\varepsilon_\phi/\varepsilon_{\mathrm{tot}}$ are derived in each case.

The form (1.1) is sufficiently general to reproduce an observationally measured cosmological history [29] and covers many interesting cases, such as k-essence with purely kinetic term [30] or the "kinetic quintessence" [16]. Factorized Lagrangians have been the main focus of attention in the study of k-essence (see e.g. [31, 32, 33]). More generally, Lagrangians of the form

$$p(X,\phi) = \left[K_1(\phi)X^{n_1} + K_2(\phi)X^{n_2}\right]^{n_3}, \tag{1.3}$$

where n_1, n_2, n_3 are constants, can be reduced to the Lagrangian (1.1) by a suitable redefinition of the field ϕ. Our analysis will also apply to Lagrangians that have the asymptotic form $p \approx K(\phi)L(X)$ for $\phi \to \infty$ and for which only the large-ϕ regime is cosmologically relevant. Nonfactorizable Lagrangians, such as those studied in Refs. [34, 35, 36, 37, 26], require a separate consideration which we do not attempt here.

"Phantom" models such as that of Ref. [25] cannot describe the tracking behavior of k-essence since in these models $w_\phi < -1$ at all times.

Recently, it was shown that the scenarios of k-essence cosmology with Lagrangians of the form (1.2) necessarily include an epoch when perturbations in the k-essence field propagate faster than light (the "no-go theorem" [38]). It is well known that superluminal propagation of perturbations opens the possibility of causality violations, although causality is actually preserved in many cases. This issue has been a subject of some debate, see e.g. the discussion in Refs. [39, 40, 41, 42, 43, 44, 45, 46]. One of the motivations for the present work is to determine whether the "no-go theorem," derived for a restricted class of k-essence Lagrangians, still holds in scenarios with more general Lagrangians.

To answer this question, we performed an exhaustive analysis of all the possibilities for the existence of stable tracking solutions in ghost-free k-essence theories with positive energy density (the complete list of physical restrictions is given in Sec. 2.1). We considered the cosmological evolution of a scalar k-essence field ϕ coupled through gravity to a matter component having a fixed equation-of state parameter w_m. In this context, we enumerated all Lagrangians of the form (1.1) that admit attractor solutions with $w_\phi \to$ const and $\varepsilon_\phi/\varepsilon_{\text{tot}} \to$ const at late times (Sec. 2.4.1). Since our task is to determine the entire class of theories admitting a certain asymptotic behavior, numerical calculations could not be used. The analytic method used for the asymptotic analysis of the dynamical evolution is outlined at the beginning of Appendix A, where all the calculations are presented in detail. This method is similar to that developed in Ref. [47] for the analysis of attractors in models of k-inflation.

Armed with the complete enumeration of stable trackers, we then select the Lagrangians capable of providing a subdominant tracker solution during the radiation era and an asymptotically dominant tracker solution during the dust era. We show that the only appropriate class of Lagrangians consists of functions $p(X, \phi)$ of the form

$$p(X, \phi) = \frac{1 + K_0(\phi)}{\phi^2} L(X), \quad \lim_{\phi \to \infty} K_0(\phi) = 0. \tag{1.4}$$

Since the dynamical evolution drives ϕ towards very large values, these Lagrangians are practically indistinguishable from the Lagrangians of the form (1.2). Then one can prove, similarly to Ref. [38], that the cosmological evolution necessarily includes an epoch where perturbations of the k-essence field ϕ propagate with a superluminal speed. Thus, we prove the "no-go theorem" starting from a much wider initial class of k-essence Lagrangians.

In Sec. 2.2 we discuss the implications of superluminal signal propagation for causality. The cosmological scenario of k-essence does not exhibit any causality violations at the classical level, despite the presence of superluminal signals. Preservation of causality in a general configuration of k-essence field can be viewed as a potential problem, on the same footing as the chronology protection problem in General Relativity [48].

In Chapter 3 we turn our attention to a study of the so called "new ekpyrotic cosmology".

After more than 25 years of its development, inflationary theory gradually becomes a standard cosmological paradigm. It solves many difficult cosmological problems and makes several predictions, which are in a very good agreement with observational data. There were many attempts to propose an alternative to inflation. In general, this could be a very healthy tendency. If one of these attempts will succeed, it will be of great importance. If none of them are successful, it will be an additional demonstration of the advantages of inflationary cosmology. However, since the stakes are high, we are witnessing a growing number of premature announcements of success in developing an alternative cosmological theory.

An instructive example is given by the ekpyrotic scenario [49]. The authors of this scenario claimed that it can solve all cosmological problems without using the stage of inflation. However, the original ekpyrotic scenario did not work. It is sufficient to say that the large mass and entropy of the universe remained unexplained, instead of solving the homogeneity problem this scenario only made it worse, and instead of the big bang expected in [49], there was a big crunch [50, 51].

Soon after that, the ekpyrotic scenario was replaced by the cyclic scenario, which used an infinite number of periods of expansion and contraction of the universe [52]. Unfortunately, the origin of the scalar field potential required in this model, as well as in [49], remains unclear, and the very existence of the cycles postulated in [52] has not been demonstrated. When this scenario was analyzed using the particular potential given in [52] and taking into account the effect of particle production in the early universe, a very different cosmological regime was found [53, 54].

The most difficult of the problems facing this scenario is the problem of the cosmological singularity. Originally there was a hope that the cosmological singularity problem will be solved in the context of string theory, but despite the attempts of the best experts in string theory, this problem remains unsolved [55, 56, 57]. Recently there were some developments in the analysis of this problem using the AdS/CFT correspondence [58], but the results rely on certain conjectures and apply only to five dimensional space. As the authors admit, "precise calculations are currently beyond reach" for the physically interesting four dimensional space-time. This issue was previously studied in [59], where it was concluded that "In our study of the field theory evolution, we find no evidence for a bounce from a big crunch to a big bang."

In this thesis we will discuss the recent development of this theory, called 'the new ekpyrotic scenario' [19, 21, 20, 60], which created a new wave of interest in the ekpyrotic/cyclic ideas. This is a rather complicated scenario, which attempts to solve the singularity problem by involving violation of the null energy condition (NEC) in a model which combines the ekpyrotic scenario [49] with the ghost condensate theory [12, 61] and the curvaton mechanism of production of adiabatic perturbations of metric [62, 63, 64, 65, 66, 67].

Usually the NEC violation leads to a vacuum instability, but the authors of [19, 21, 20, 60] argued that the instability occurs only near the bounce, so it does not have enough time to fully develop. The instability is supposed to be dampened by higher derivative terms of the type $-(\Box\phi)^2$ (the sign is important, see below), which were added to the action of the ghost condensate in [12, 61]. These terms are absolutely essential in the new ekpyrotic theory for stabilization of the vacuum against the gradient and Jeans instabilities near the bounce.

However, these terms are quite problematic. Soon after introducing them, the authors of the ghost condensate theory, as well as several others, took a step back and argued that these terms cannot appear in any consistent theory, that the ghost condensate theory is ultraviolet-incomplete, that theories of this type lead to violation of the second law of thermodynamics, allow construction of a *perpetuum mobile* of the 2nd kind, and therefore they are incompatible with basic gravitational principles [40, 42, 68, 69].

These arguments did not discourage the authors of the new ekpyrotic theory and those who followed it, so we decided to analyze the situation in a more detailed way. First of all, we found that the higher derivative terms were only partially taken into account in the investigation of perturbations, and were ignored in the investigation of the cosmological evolution in [19, 21, 20, 60]. Therefore the existence of consistent and stable bouncing solutions postulated in the new ekpyrotic scenario required an additional investigation. We report the results of this investigation in Section 3.5.

More importantly, we found that these additional terms lead to the existence of *new ghosts*, which have not been discussed in the ghost condensate theory and in the new ekpyrotic scenario [19, 21, 20, 60, 12, 61]. In order to distinguish these ghosts from the relatively harmless condensed ghosts of the ghost condensate theory, we will call them *ekpyrotic ghosts*, even though, as we will show, they are already present in the ghost condensate theory. These ghosts lead to a catastrophic vacuum instability, quite independently of the cosmological evolution. In other words, the new ekpyrotic scenario, as well as the ghost condensate theory, appears to be physically inconsistent. But since the new ekpyrotic scenario, as different from the ghost condensate model, claims to solve the fundamental singularity problem by justifying the bounce solution, the existence of the ekpyrotic ghosts presents a much more serious problem for the new ekpyrotic scenario with such an ambitious goal. We describe this problem in Sections 3.1, 3.2, 3.3, 3.4, and 3.6.

Finally, in Appendix B we discuss certain attempts to save the new ekpyrotic scenario. One of such attempts is to say that this scenario is just an effective field theory which is valid only for sufficiently small values of frequencies and momenta. But then, of course, one cannot claim that this theory solves the singularity problem until its consistent UV completion with a stable vacuum is constructed. For example, we will show that if one simply ignores the higher derivative terms for frequencies and momenta above a certain cutoff, then the new ekpyrotic scenario fails to work because of the vacuum instability which is even much stronger than the ghost-related instability. We will also describe a possible procedure which may provide a consistent UV completion of the theory with higher derivative terms of the type $+(\Box\phi)^2$. Then we explain why this procedure fails for the ghost condensate and the new ekpyrotic theory where the sign of the higher derivative term must be negative.

In Chapter 4 we construct bouncing universe scenarios involving the creation and annihilation of a non-BPS D9-brane in type IIA superstring theory.

The resolution of the big-bang singularity is not only an important open problem in standard cosmology (see e. g. [70, 71] for a review and references therein), but also a natural playground for string theory since quantum gravity corrections are expected to be relevant in this regime. In the ekpyrotic scenario [72, 73] and a refined version, the cyclic universe [74], the hot big bang is the result of the collision of two branes. Explicit cyclic models have been suggested as effective four-dimensional models inspired from heterotic M-theory. The bounce in the ekpyrotic scenario occurs at a real curvature singularity and thus does not solve the singularity problem. As mentioned above the new ekpyrotic scenario [21] realizes an explicit bounce dynamics by addition of a ghost condensate but suffers from a vacuum instability problem (see [75]), and more generally, due to the necessary violation of the null energy condition (NEC) during a bounce, phenomenological models producing a bounce often suffer from the problem of introducing matter with negative energy density, i. e. ghosts.

There are other ways to address the big-bang singularity problem in string theory[2]. The pre-big bang scenario (see [79] for a review) was motivated by the fact that the tree-level equations of motion of string theory are not only symmetric under time reflection $t \mapsto -t$ but also symmetric under the scale-factor duality transformation $a \mapsto \frac{1}{a}$ with an appropriate transformation of the dilaton. The 'post-big bang' solution of standard cosmology with decelerated expansion defined for positive times is by these dualities connected to an inflationary 'pre-big bang' solution for negative times. In this way the cosmic evolution is extended to times prior to the big bang in a self-dual way but the solution is still singular. One can obtain regular self-dual solutions by tuning a suitable potential for the dilaton. But albeit a potential of this form might be the result of higher-loop quantum corrections, its form has not been derived from string theory.

In Chapter 4 we consider a novel scenario in string theory where a bounce occurs in the string frame due to the creation of an unstable (non-BPS) brane as the universe bounces through a string size regime before expanding as the brane decays. Our solutions effectively interpolate either between a contracting and an expanding pre-big bang solution or between a contracting and an expanding post-big bang solution. The future (past) singularity of the pre (post)-big bang solution is not resolved in this scenario. The nice feature in our model is that the curvature as well as the dilaton and its derivative remain small (in string units) through the bounce so that referring to perturbative string theory and the simplest low energy effective action for these fields is justified. In addition no fine tuning is required and no violation of the NEC occurs in our model. On the other hand, we do not address issues such as dilaton and moduli stabilization. It is important to consider

[2] Alternative suggestions avoiding the problem of introducing matter with negative energy density include loop quantum cosmology [76] and matrix models (e. g. [77, 78]).

these issues if one wishes to embed this model into late time standard cosmology. However, these issues are not relevant during the string scale regime where the bounce occurs. We should stress that our string frame bounce solutions describe monotonically contracting or expanding geometries in the Einstein frame. A bounce in the Einstein frame may occur upon stabilizing the dilaton asymptotically. However, this entails violating the NEC or, alternatively, allowing the gravitational coupling to change sign in the string frame. We will discuss a model where the latter effect occurs.

This thesis is based on the following publications of the author:

- J. U Kang, V. Vanchurin and S. Winitzki,
  Phys. Rev. D **76**, 083511 (2007) [arXiv:0706.3994 [gr-qc]].
- R. Kallosh, J. U Kang, A. Linde and V. Mukhanov,
  JCAP **0804**, 018 (2008) [arXiv:0712.2040 [hep-th]].
- P. H. v. Loewenfeld, J. U Kang, N. Moeller and I. Sachs,
  arXiv:0906.3242 [hep-th].

2 Attractor scenarios and superluminal signals in k-essence cosmology

In this chapter we study attractor-like cosmological solutions ("trackers") involving a k-essence scalar field ϕ and another matter component, following our paper [80]. One of the main purposes of this chapter is a complete classification of k-essence Lagrangians that admit asymptotically stable tracking solutions, among all Lagrangians of the form $p = K(\phi)L(X)$. This chapter consists of four sections. In Section 2.1 we consider the physical restrictions on Lagrangians and solutions, and the implications of superluminal signal propagation for causality is discussed in Section 2.2. In Section 2.3 we present the equations of motion, and the detailed derivation of the asymptotically stable solutions is given in Appendix A. Finally in Section 2.4 we determine the entire class of theories admitting a certain asymptotic behavior and select the viable scenarios for k-essence cosmology.

2.1 Physical restrictions on Lagrangians and solutions

In this section we consider some physically necessary restrictions on the possible Lagrangians $p(X, \phi)$ and solutions $\phi(t)$.

The main physical context for k-essence scenarios is the evolution of the k-essence field on the background of a matter component with a fixed equation of state parameter w_m. The energy density of k-essence is not necessarily dominant during this evolution. Since k-essence scenarios are proposed as an explanation of the dark energy, we do not consider the case $w_m = -1$ (during primordial inflation, one must also have $w_m > -1$ due to the necessity of the graceful exit). However, we leave open the possibility $w_m < -1$.

An important requirement for a field theory is stability. A theory for a field ϕ is stable and ghost-free if the energy density ε_ϕ is positive, the speed of sound c_s is real (not imaginary), i.e. $c_s^2 > 0$, and the Lagrangian for linear perturbations has a hyperbolic signature and a positive sign at the kinetic term. The speed of sound for perturbations on a given background is given by [81]

$$c_s^2 = \frac{p_{,v}}{v p_{,vv}}. \tag{2.1}$$

To obtain the leading terms of the Lagrangian for the perturbations, one writes a perturbed solution as $\phi = \phi_0(t) + \chi(t, \mathbf{x})$ and expands the Lagrangian $p(X, \phi)$ to second order in χ;

the Lagrangian $p(X, \phi)$ is assumed to be an analytic function of X at $X = 0$. The relevant terms are those quadratic in the derivatives of χ,

$$p(X, \phi) = p_{,X}\frac{1}{2}\chi_{,\mu}\chi^{,\mu} + \frac{1}{2}p_{,XX}\chi_{,\mu}\chi_{,\nu}\phi_0^{;\mu}\phi_0^{;\nu} + ... \\ \equiv \frac{1}{2}G^{\mu\nu}\chi_{,\mu}\chi_{,\nu} + ... \tag{2.2}$$

It follows that linear perturbations χ propagate in the effective metric

$$G^{\mu\nu} \equiv p_{,X}g^{\mu\nu} + p_{,XX}\phi_0^{;\mu}\phi_0^{;\nu}. \tag{2.3}$$

The no-ghost requirement is that the metric $G^{\mu\nu}$ should have the same signature as $g^{\mu\nu}$. Regardless of whether the 4-gradient $\phi_0^{;\mu}$ is spacelike or timelike,[1] the resulting conditions are [82]

$$p_{,X} = \frac{1}{v}p_{,v} > 0, \quad p_{,X} + 2Xp_{,XX} = p_{,vv} > 0. \tag{2.4}$$

In the cosmological context, the field ϕ is a function of time t only; in standard k-essence scenarios that we are presently considering, $\phi(t)$ grows monotonically with t. Hence, $\phi_0^{;\mu}$ is timelike and the velocity $v \equiv \dot{\phi}$ is positive,

$$v \equiv \frac{d\phi}{dt} = \sqrt{2X} > 0, \quad \frac{\partial}{\partial X} = \frac{1}{v}\frac{\partial}{\partial v}. \tag{2.5}$$

We conclude that a physically reasonable cosmological solution should satisfy (for $v > 0$) the conditions

$$vp_{,v} - p > 0, \quad p_{,v} > 0, \quad p_{,vv} > 0. \tag{2.6}$$

It follows that $p(v, \phi)_{v=0} \leq 0$ and that $p(v, \phi)$ is a convex, monotonically growing function of v at fixed ϕ (at least for values of ϕ and v relevant in a cosmological scenario). For factorized Lagrangians $p(v, \phi) = K(\phi)Q(v)$, we find that $Q(v)$ must be a convex, monotonically growing function of v with $Q(0) \leq 0$, and also $Q'(v) > 0$ and $Q''(v) > 0$ for all values of $v > 0$ that are relevant in a given cosmological scenario.

Finally, we assume that $K(\phi)$ has monotonic behavior at $\phi \to \infty$.

2.2 Superluminal signals and causality

One of the results of this work is a conclusion that every k-essence scenario based on attractor behavior and a Lagrangian of the form (1.1) will include an epoch where the perturbations of the k-essence field propagate superluminally. It is therefore pertinent to discuss the possibility of causality violations in the presence of superluminal signals.

[1] If $\phi_0^{;\mu}$ is null, the metric $G^{\mu\nu}$ will have the correct signature if $p_{,X} > 0$ and $p_{,XX} > 0$.

We first consider small perturbations $\phi_0 + \delta\phi$ of an arbitrary background solution $\phi_0(x)$ in a Lorentz-invariant, nonlinear field theory. To first order, the evolution of $\delta\phi$ is described by a linear equation of the form

$$G^{\mu\nu}[\phi_0]\nabla_\mu\nabla_\nu\delta\phi + B^\mu[\phi_0]\nabla_\mu\delta\phi + C[\phi_0]\delta\phi = 0, \tag{2.7}$$

where the coefficients $G^{\mu\nu}$, B^μ, and C are determined by the Lagrangian and depend on the background solution ϕ_0. Unless Eq. (2.7) is hyperbolic (the matrix $G^{\mu\nu}$ having signature $+---$ or equivalent), the theory will trivially violate causality: an initial-value (Cauchy) problem will be ill-posed in any reference frame, and the evolution of perturbations will be physically unpredictable. Therefore, it is necessary to require that $G^{\mu\nu}$ have a hyperbolic signature. Background solutions ϕ_0 that lead to a parabolic or an elliptic signature of $G^{\mu\nu}$ even in a small spacetime domain must be avoided as pathological. The cosmological solution $\phi_0(t)$ used in k-essence scenarios will be well-behaved if the conditions (2.6) hold. Below we assume that $G^{\mu\nu}$ has signature $+---$.

Within a sufficiently small spacetime domain, we may regard $G^{\mu\nu}$, B^μ, and C as constants. Then it is straightforward to derive the dispersion relation

$$G^{\mu\nu}k_\mu k_\nu + \mathrm{i}B^\mu k_\mu + C = 0 \tag{2.8}$$

for plane wave perturbations $\delta\phi(x) \propto \exp[\mathrm{i}k_\mu x^\mu]$. In order to send information ("signals" or "sounds") by means of a perturbation $\delta\phi(x)$, one needs to create a wave front, that is, a perturbation with an extremely short wavelength and a high frequency. Thus, wave fronts propagate along wave vectors k_μ determined by the leading term in Eq. (2.8),

$$G^{\mu\nu}k_\mu k_\nu = 0. \tag{2.9}$$

Any wave packet consisting of a superposition of plane waves will propagate behind the wave front. Therefore, a 4-vector u^μ of signal velocity must lie within the sound cone,

$$G^{\mu\nu}u_\mu u_\nu > 0. \tag{2.10}$$

Since the "sound metric" $G^{\mu\nu}$ is determined by the local behavior of the background solution $\phi_0(x)$, the sound cone may have an arbitrary relationship with the lightcone $g^{\mu\nu}u_\mu u_\nu = 0$ determined by the spacetime metric $g^{\mu\nu}$. Thus, in some theories the sound signal worldlines may be timelike, null, or even spacelike depending on the spatial direction of their propagation.

The speed of sound waves is therefore direction-dependent. The background tensor $G^{\mu\nu}$ determines (a class of) preferred reference frames where $G^{\mu\nu}$ is diagonal. Propagation of sound is most conveniently described in terms of sound speeds in different directions in a preferred frame. In this sense, one may say that a dynamical Lorentz violation takes place for sound waves, although the full theory (that includes the tensor $G^{\mu\nu}$ as a dynamical quantity) of course remains Lorentz-invariant.

In the context of k-essence cosmology, the sound metric $G^{\mu\nu}$ is given by Eq. (2.3). Preferred frames are those where the t axis coincides with the cosmological time. So in the preferred frames $\phi_0 = \phi_0(t)$ is a function of time only, and the dispersion relation is

$$\omega^2 = c_s^2 \left|\mathbf{k}\right|^2 , \tag{2.11}$$

where $\mathbf{k}$ is the 3-dimensional wave vector and c_s is the (direction-independent) speed of sound defined by Eq. (2.1). We show that the considerations of the "no-go theorem" [38] hold for those Lagrangians of the form (1.1) that admit scenarios of tracking k-essence. By virtue of this theorem, there exists an epoch with $c_s^2 > 1$. During this epoch (which may be quite short [38]), it is possible to send signals along spacelike worldlines.

If spacelike sound signals propagated in arbitrary *spacetime* directions, one could easily create closed worldlines made of signals, called "closed signal curves" (CSCs) in Ref. [46]. This would open a Pandora's box of classical time travel paradoxes, also violating the unitarity of quantum theory (see e.g. [83, 84, 85, 25]). However, the allowed sound signal directions are only those within the sound cone (2.10). This limitation precludes the possibility of constructing CSCs within a small domain where $G^{\mu\nu} \approx$ const. This can be shown as follows. Diagonalizing the tensor $G^{\mu\nu}$ within that domain, one finds a preferred reference frame $\{t, x, y, z\}$ where sound signals (whether spacelike, null, or timelike) always propagate in the positive direction along the t axis. Signals sent by conventional means also propagate in the positive t direction. Since the local coordinates $\{t, x, y, z\}$ are valid within the entire domain where $G^{\mu\nu} \approx$ const, no CSCs are possible within that domain.

It is straightforward to see that no causality violations through CSCs can occur in k-essence cosmology. Since $\dot{\phi}_0 > 0$ at all times, the 4-vector $\nabla_\mu \phi_0$ is everywhere timelike and selects a global preferred reference frame. (Even if $\dot{\phi}_0 = 0$ momentarily, the preferred frame is still selected by continuity.) In this reference frame, the sound waves propagate in the direction of increasing coordinate t. Hence, there exists a global foliation of the entire spacetime by spacelike hypersurfaces of equal t. Any sound signals (whether spacelike, null, or timelike), as well as any signals sent by conventional means, will traverse these hypersurfaces in the direction of increasing t. It follows that CSCs cannot occur, either locally or globally.

Similar conclusions were reached in models of inflation having $c_s^2 > 1$ [14] as well as in situations involving a k-essence field on a black hole background [41, 44]. By itself, a superluminal speed of sound does not automatically lead to CSCs or causality violations.

In certain field theories, one can construct backgrounds $\phi_0(x)$ where CSCs are possible; a notable example is given in Ref. [40]. However, such backgrounds are artificial in the sense that they require an ad hoc configuration of the field $\phi_0(x)$. It remains to be seen whether such causality-violating backgrounds can occur as a result of the dynamical evolution of the field $\phi_0(x)$ in a cosmological context.

The problem of causality violation by CSCs is similar to the problem of closed timelike curves (CTCs) occurring in General Relativity [48]. It is difficult to find a metric $g_{\mu\nu}$

that is initially well-behaved but admits CTCs as a result of dynamical evolution (one such example is given in Ref. [86]). Hawking's "chronology protection conjecture" states that such spacetimes containing CTCs will be always unstable due to quantum effects; but it remains an open conjecture [48]. Similar considerations apply to CSCs occurring in nonlinear field theories. It is possible that CSCs will always lead to quantum instabilities due to a similar "chronology protection" mechanism. Further work is needed to resolve this intriguing question.

2.3 Equations of motion

We begin by writing the well-known evolution equations for k-essence cosmology in a convenient set of variables. The equations in this section will be used at various points in the following analysis.

We consider a spatially flat FRW universe with the metric

$$g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left[dx^2 + dy^2 + dz^2\right], \tag{2.12}$$

where $a(t)$ is the scale factor. In the epoch of interest, the universe contains the dynamical k-essence field $\phi(t)$ and a matter component with energy density ε_m and pressure p_m. The matter component can be approximately treated as nondynamical in the sense that its equation of state is fixed,

$$w_m \equiv \frac{p_m}{\varepsilon_m} = \text{const.} \tag{2.13}$$

The energy-momentum tensor of the field ϕ is that of a perfect fluid with pressure $p(X,\phi)$ and energy density

$$\varepsilon_\phi = 2Xp_{,X} - p. \tag{2.14}$$

Here and below we denote partial derivatives by a comma, so $p_{,X} \equiv \partial p/\partial X$. We introduce the velocity $v \equiv \dot{\phi}$ as shown by Eq. (2.5). Note that the Lagrangian $p(X,\phi)$ is an analytic function of X and thus an analytic function of v^2.

The equation of state parameter for k-essence, w_ϕ, is defined by

$$w_\phi \equiv \frac{p(X,\phi)}{\varepsilon_\phi} = \frac{p}{vp_{,v} - p}. \tag{2.15}$$

A factorizable Lagrangian (1.1) is expressed as a function of v and ϕ as follows,

$$p(X,\phi) = K(\phi)Q(v), \quad Q(v) \equiv L(X). \tag{2.16}$$

For a Lagrangian of this form, w_ϕ is a function of v only,

$$w_\phi(v) = \frac{Q}{vQ' - Q}, \tag{2.17}$$

since the energy density factorizes,

$$\varepsilon_\phi = K(\phi)\tilde{\varepsilon}_\phi(v), \quad \tilde{\varepsilon}_\phi(v) \equiv vQ'(v) - Q(v). \tag{2.18}$$

We assume that the functions K and Q in Eq. (2.16) are chosen such that $K(\phi) > 0$.

The cosmological evolution is described by the equations of motion for $\phi(t)$, $\varepsilon_m(t)$, and $a(t)$,

$$\frac{\dot{a}}{a} \equiv H = \kappa\sqrt{\varepsilon_\phi + \varepsilon_m}, \qquad \kappa^2 \equiv \frac{8\pi G}{3}, \tag{2.19}$$

$$\frac{d}{dt}\left(p_{,v}(v,\phi)\right) \equiv \ddot{\phi} p_{,vv} + \dot{\phi} p_{,\phi v} = -3Hp_{,v} + p_{,\phi}, \tag{2.20}$$

$$\dot{\varepsilon}_m = -3H\left(\varepsilon_m + p_m\right) = -3H\left(1 + w_m\right)\varepsilon_m. \tag{2.21}$$

The equation of motion for the field ϕ can be also rewritten as a conservation law,

$$\dot{\varepsilon}_\phi = -3H\left(\varepsilon_\phi + p(X,\phi)\right) = -3H\left(1 + w_\phi\right)\varepsilon_\phi. \tag{2.22}$$

The total energy density $\varepsilon_{\text{tot}} \equiv \varepsilon_\phi + \varepsilon_m$ satisfies the equation

$$\dot{\varepsilon}_{\text{tot}} = -3H\varepsilon_{\text{tot}}\left[(1 + w_m) + \frac{\varepsilon_\phi}{\varepsilon_{\text{tot}}}\left(w_\phi - w_m\right)\right]. \tag{2.23}$$

Since the equations of motion (2.19)–(2.21) do not depend explicitly on time, and since $\phi(t)$ is monotonic in t, we may use the value of ϕ as the time variable instead of t. Then we obtain a closed system of two first-order equations for $v(\phi)$ and $\varepsilon_m(\phi)$,

$$\frac{dv(\phi)}{d\phi} = -\frac{vp_{,v\phi} - p_{,\phi} + 3\kappa p_{,v}\sqrt{\varepsilon_m + vp_{,v} - p}}{vp_{,vv}}, \tag{2.24}$$

$$\frac{d\varepsilon_m}{d\phi} = -\frac{3\kappa\left(1 + w_m\right)\varepsilon_m}{v}\sqrt{\varepsilon_m + vp_{,v} - p}. \tag{2.25}$$

We will make extensive use of the auxiliary quantity R defined by

$$R \equiv \frac{\varepsilon_m}{\varepsilon_\phi + \varepsilon_m}. \tag{2.26}$$

Since energy densities ε_ϕ and ε_m are always positive, the ratio R always remains between 0 and 1. The equation of motion for $R(\phi)$ is straightforwardly derived from Eqs. (2.21)–(2.22) and can be written as

$$\frac{dR}{d\phi} = -\frac{3H}{v}R(1-R)\left(w_m - w_\phi(v,\phi)\right). \tag{2.27}$$

We may reformulate the equations of motion (2.24)–(2.25) as a closed system of equations involving only the variables $v(\phi)$ and $R(\phi)$. Since

$$\varepsilon_\phi + \varepsilon_m = \frac{\varepsilon_\phi}{1-R} = \frac{vp_{,v} - p}{1-R}, \tag{2.28}$$

we obtain

$$\frac{dv}{d\phi} = -\frac{1}{vp_{,vv}}\left[vp_{,v\phi} - p_{,\phi} + 3\kappa p_{,v}\sqrt{\frac{vp_{,v}-p}{1-R}}\right], \tag{2.29}$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v}R\sqrt{1-R}\sqrt{vp_{,v}-p}\left(w_m - \frac{p}{vp_{,v}-p}\right). \tag{2.30}$$

For Lagrangians of the form (2.16), these equations are rewritten as

$$\frac{dv}{d\phi} = -c_s^2(v)\left[\frac{(\ln K)_{,\phi}\, v}{1+w_\phi(v)} + 3\kappa\sqrt{\frac{K(\phi)\tilde{\varepsilon}_\phi(v)}{1-R}}\right], \tag{2.31}$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v}R\sqrt{1-R}\sqrt{K(\phi)\tilde{\varepsilon}_\phi(v)}\,(w_m - w_\phi(v)). \tag{2.32}$$

Here $\tilde{\varepsilon}_\phi(v)$, $c_s^2(v)$, and $w_\phi(v)$ are understood as fixed functions of v,

$$\tilde{\varepsilon}_\phi(v) \equiv vQ' - Q, \quad c_s^2(v) \equiv \frac{Q'}{vQ''}, \quad w_\phi(v) \equiv \frac{Q(v)}{\tilde{\varepsilon}_\phi(v)}, \tag{2.33}$$

determined by the given Lagrangian $p(v,\phi) = Q(v)K(\phi)$. These functions satisfy the following equations,

$$\frac{d}{dv}\tilde{\varepsilon}_\phi(v) = \frac{1+w_\phi(v)}{vc_s^2(v)}\tilde{\varepsilon}_\phi(v), \tag{2.34}$$

$$\frac{d}{dv}w_\phi(v) = \frac{1+w_\phi(v)}{v}\left[1 - \frac{w_\phi(v)}{c_s^2(v)}\right]. \tag{2.35}$$

2.4 Viable Lagrangians for tracking solutions

The detailed analysis of asymptotically stable solutions is given in Appendix A. Each asymptotically stable solution is characterized by the asymptotic values of $v = \dot{\phi}$ and $R = \varepsilon_m/\varepsilon_{\mathrm{tot}}$, considered as functions of ϕ:

$$v_0 \equiv \lim_{\phi\to\infty} v(\phi), \quad R_0 \equiv \lim_{\phi\to\infty}\frac{\varepsilon_m(\phi)}{\varepsilon_{\mathrm{tot}}(\phi)}. \tag{2.36}$$

As a summary of the results, we list all of the possibilities, together with the requirements on the Lagrangian $p = K(\phi)Q(v)$ and the allowed values of v_0, R_0, w_m, and $w_\phi(v_0)$. [Note that the function $K(\phi)$ can be always multiplied by a constant, to be absorbed in $Q(v)$.] The requirements listed are necessary and sufficient conditions for the asymptotic stability of tracker solutions. The applicability of these tracker scenarios to k-essence cosmology is analyzed in subsections 2.4.2, 2.4.3, and 2.4.4.

2.4.1 Tracker solutions

Case 1. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{2.37}$$

The value v_0 is determined from $w_\phi(v_0) = w_m$, and then R_0 is given by

$$R_0 = 1 - \frac{9\kappa^2 Q'(v_0)^2}{4\tilde{\varepsilon}_\phi(v_0)}. \tag{2.38}$$

This value of R_0 must satisfy $0 < R_0 < 1$ (the possibility $R_0 = 0$ is equivalent to case 2). The conditions

$$v_0 \neq 0,\ c_s^2(v_0) \neq 0,\ |w_m| < 1,\ w_m < c_s^2(v_0),\ \tilde{\varepsilon}_\phi(v_0) \neq 0 \tag{2.39}$$

must hold.

Case 2. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{2.40}$$

The value v_0 is determined from

$$3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)} = \frac{2v_0}{1 + w_\phi(v_0)} \tag{2.41}$$

and must satisfy $v_0 \neq 0$. The following conditions must hold,

$$w_\phi(v_0) < w_m,\ |w_\phi(v_0)| < 1,\ c_s^2(v_0) \neq 0. \tag{2.42}$$

The tracker solution has $R_0 = 0$ (k-essence dominates at late times).

Case 3. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi\to\infty} \frac{\ln K_0(\phi)}{\ln\phi} = 0, \tag{2.43}$$

i.e. the function K_0 either tends to a constant, or grows or decays slower than any power of ϕ at $\phi \to \infty$. This condition determines the value of α. This value of α must satisfy

$$2 < \alpha < 1 + \frac{2}{1 + w_m}. \tag{2.44}$$

The interval for α is nonempty if

$$|w_m| < 1. \tag{2.45}$$

The value of α determines v_0 by

$$\alpha = 2\frac{1 + w_\phi(v_0)}{1 + w_m}. \tag{2.46}$$

The resulting value of v_0 must satisfy the conditions

$$v_0 \neq 0,\ c_s^2(v_0) > w_\phi(v_0) > w_m, c_s^2(v_0) \neq 0,\ \tilde{\varepsilon}_\phi(v_0) \neq 0. \tag{2.47}$$

The tracker solution has $R_0 = 1$ (k-essence is negligible).

Case 4. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \tag{2.48}$$

where the function K_0 must satisfy

$$\lim_{\phi\to\infty} K_0(\phi) = 0, \quad \lim_{\phi\to\infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0, \tag{2.49}$$

i.e. $K_0(\phi)$ decays slower than any power of ϕ at $\phi \to \infty$. The value of v_0 is determined from the conditions

$$w_\phi(v_0) = w_m, \quad |w_m| < 1. \tag{2.50}$$

The following conditions must then hold,

$$v_0 \neq 0,\ c_s^2(v_0) > w_m,\ \tilde{\varepsilon}_\phi(v_0) \neq 0,\ c_s^2(v_0) \neq 0. \tag{2.51}$$

The tracker solution has $R_0 = 1$ (k-essence is negligible).

Case 5. The function $K(\phi)$ decays slower than $\phi^{-\alpha}$ (or grows), where

$$\alpha \equiv \frac{2}{1 + w_m}, \quad -1 < w_m < 0. \tag{2.52}$$

More precisely,

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.53}$$

The value of v_0 is determined as a root of $Q(v_0) = 0$ and $Q'(v_0) = 0$, i.e. we must have a Taylor expansion near $v = v_0$ of the form

$$Q(v) = \frac{Q_0}{nv_0}(v - v_0)^n, \quad n \geq 2,\ Q_0 > 0. \tag{2.54}$$

Then the tracker solution has $R_0 = 1$ (k-essence is negligible) and $w_\phi(v_0) = 0$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 6. The function $Q(v)$ has an expansion at $v = 0$ of the form

$$Q(v) = Q_1 v^n + o(v^n), \quad Q_1 > 0,\ n > 2. \tag{2.55}$$

This determines the value of n. The function $K(\phi)$ decays slower than $\phi^{-\alpha}$ (or grows), where

$$\alpha \equiv \frac{2n}{(n-1)(1+w_m)}. \tag{2.56}$$

More precisely,

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.57}$$

The condition

$$-\frac{n-3}{n-1} < w_m < \frac{1}{n-1} \tag{2.58}$$

must hold. Then the tracker solution has $R_0 = 1$ (k-essence is negligible), $v_0 = 0$, $w_\phi(v_0) = \frac{1}{n-1}$, and $c_s^2(v_0) = \frac{1}{n-1}$.

Case 7. The function $K(\phi)$ has the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \tag{2.59}$$

where the function $K_0(\phi)$ is such that

$$\lim_{\phi\to\infty} K_0(\phi) > \frac{1}{9\kappa^2 Q_1} \quad \text{or} \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.60}$$

The function $Q(v)$ has an expansion at $v = 0$ of the form

$$Q(v) = Q_1 v^2 + o(v^2), \quad Q_1 > 0. \tag{2.61}$$

We must have $w_m > 1$. The tracker solution has $R_0 = 0$ (k-essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2(v_0) = 1$.

Case 8. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.62}$$

The value of v_0 is determined from $Q'(v_0) = 0$, $Q(v_0) < 0$. More precisely, we have an expansion near $v = v_0$,

$$Q(v) = Q_0 + Q_2(v - v_0)^n, \quad Q_0 < 0, \quad n \geq 2. \tag{2.63}$$

We must have $v_0 \neq 0$ and $w_m > -1$. The tracker solution has $R_0 = 0$ (k-essence dominates) and $w_\phi(v_0) = -1$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 9. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.64}$$

The value of v_0 is determined from $Q'(v_0) = 0$, $Q(v_0) = 0$. More precisely, we have an expansion near $v = v_0$,

$$Q(v) = Q_1(v - v_0)^n, \quad n \geq 2. \tag{2.65}$$

We must have $w_m > 0$ and $v_0 \neq 0$. The tracker solution has $R_0 = 0$ (k-essence dominates) and $w_\phi(v_0) = 0$. The value of v must be above v_0 at all times (or else $c_s^2 < 0$).

Case 10. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.66}$$

The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = -Q_0 + Q_1 v^n, \quad Q_0 > 0,\ n \geq 2. \tag{2.67}$$

We must have $w_m > -1$. The tracker solution has $R_0 = 0$ (k-essence dominates), $v_0 = 0$, and $w_\phi(v_0) = -1$.

Case 11. The function $K(\phi)$ decays slower than ϕ^{-2} or grows,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.68}$$

The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = Q_1 v^n + o(v^n), \quad Q_1 > 0,\ n > 2. \tag{2.69}$$

This determines the value of n. The condition

$$w_m > \frac{1}{n-1} \tag{2.70}$$

must hold. The tracker solution has $R_0 = 0$ (k-essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2 = \frac{1}{n-1}$.

Case 12. The function $Q(v)$ must have an expansion near $v = 0$ of the form

$$Q(v) = Q_1 v^n + Q_2 v^{n+p}, \quad Q_1 > 0,\ n > 2,\ p > 0. \tag{2.71}$$

This determines the values of n and p. The function $K(\phi)$ must be of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \tag{2.72}$$

where $K_0(\phi)$ must satisfy

$$\lim_{\phi\to\infty} K_0(\phi) = \infty, \quad \int^{\infty} \frac{d\phi}{\phi} K_0^{-\frac{p}{n-2}}(\phi) = \infty. \tag{2.73}$$

(The function $K_0(\phi)$ grows slower than $(\ln\phi)^{(n-2)/p}$.) We must have $w_m = \frac{1}{n-1}$. The tracker solution has $R_0 = 0$ (k-essence dominates), $v_0 = 0$, and $w_\phi(v_0) = c_s^2 = \frac{1}{n-1}$.

2.4.2 Radiation-dominated era

We now select Lagrangians that admit tracker solutions during radiation domination, $w_m = \frac{1}{3}$. In order not to violate the nucleosynthesis bound, the energy density of k-essence must be subdominant throughout the radiation era [18],

$$R_0 \gtrsim 0.99. \tag{2.74}$$

Admissible trackers may have a value R_0 within the range $0.99 \lesssim R_0 < 1$, or $R_0 = 1$. A solution with $0 < R_0 < 1$ is only possible with Lagrangians given by case 1,

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{2.75}$$

We denote by v_r the asymptotic value of v during the radiation era. Possible values of v_r are determined from $w_\phi(v_r) = \frac{1}{3}$, and v_r must satisfy

$$c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \tag{2.76}$$

The corresponding value of R_0 must respect the bound (2.74),

$$R_0 = 1 - \left.\frac{9\kappa^2 Q'^2}{4\tilde{\varepsilon}_\phi}\right|_{v=v_r} \gtrsim 0.99. \tag{2.77}$$

Solutions with $R_0 = 1$ and $w_m = \frac{1}{3}$ are possible in cases 3, 4, and 6. The first set of solutions is given by

$$K(\phi) = \frac{K_0(\phi)}{\phi^\alpha}, \quad \lim_{\phi\to\infty} \frac{\ln K_0(\phi)}{\ln\phi} = 0, \tag{2.78}$$

where $2 < \alpha < \frac{5}{2}$. Admissible functions $K_0(\phi)$ decay or grow slower than any power of ϕ, e.g. $K_0(\phi) \propto (\ln \phi)^\beta$. Admissible values of v_r are determined from the conditions

$$w_\phi(v_r) = \frac{2\alpha}{3} - 1, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \tag{2.79}$$

The second set of Lagrangians is

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0, \quad \lim_{\phi\to\infty} \frac{\ln K_0(\phi)}{\ln \phi} = 0. \tag{2.80}$$

The possible values of v_r are determined from $w_\phi(v_r) = \frac{1}{3}$, and the following conditions must be also satisfied,

$$c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \quad v_r \neq 0. \tag{2.81}$$

The third set of admissible Lagrangians is described by case 6 with $n = 3$, namely

$$K(\phi) = \frac{K_0(\phi)}{\phi^{9/4}}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty, \tag{2.82}$$

$$Q(v) = Q_1 v^3 + o(v^3), \quad Q_1 > 0. \tag{2.83}$$

In this case, $v_r = 0$. The solution of case 6 with $n \geq 4$ cannot be used since the condition (2.58) cannot be satisfied with $w_m = \frac{1}{3}$.

2.4.3 Dust-dominated era

We now select the tracker solutions that exist for $w_m = 0$. In order to describe the late-time domination of k-essence, we must look for solutions with $w_\phi < -\frac{1}{3}$ and $R_0 = 0$. The possible trackers are cases 2, 8, and 10.

In case 2, the Lagrangian must satisfy

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{2.84}$$

We denote by v_d the asymptotic value of v during the dust era. The admissible values of $v_d \neq 0$ are determined from

$$3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_d)} = \frac{2v_d}{1 + w_\phi(v_d)}. \tag{2.85}$$

In addition, the following conditions must be satisfied:

$$-1 < w_\phi(v_d) < 0, \quad c_s^2(v_d) \neq 0. \tag{2.86}$$

The second set of Lagrangians is for cases 8 and 10,

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty. \tag{2.87}$$

This condition for $K(\phi)$ is satisfied, for example, by $K(\phi) \propto \phi^\alpha$ with $\alpha > -2$. The value v_d must be such that

$$Q(v_d) < 0, \quad Q'(v_d) = 0, \tag{2.88}$$

while we may have either $v_d \neq 0$ or $v_d = 0$.

2.4.4 Viable scenarios

Having listed all the Lagrangians that admit desired solutions in the radiation- and dust-dominated eras, it remains to determine the overlap between these classes of Lagrangians. By comparing the requirements on the functions $K(\phi)$ and $Q(v)$, we find only two possibilities for trackers in the radiation/dust era: case 1/case 2 and case 6/case 8.

The first set of Lagrangians (case 1/case 2) is

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{2.89}$$

In the radiation era, the asymptotic value of v is given by $v_r \neq 0$ such that

$$w_\phi(v_r) = \frac{1}{3}, \quad c_s^2(v_r) > \frac{1}{3}, \quad \tilde{\varepsilon}_\phi(v_r) \neq 0, \tag{2.90}$$

and the dust attractor is given by $v_d \neq 0$ such that Eqs. (2.85)–(2.86) hold. These Lagrangians describe the well-known scenario [17] where the k-essence tracks radiation during the radiation era and eventually starts to dominate in the dust era. The function $Q(v)$ must be chosen to satisfy the conditions of cases 1 and 2. Additionally, one must exclude the possibility of a dust tracker (case 1, $w_m = 0$) by adjusting $Q(v)$ such that the conditions of case 1 are not satisfied for $w_\phi(v_0) = w_m = 0$ [18].

The second set of Lagrangians is described by case 6/case 8. The function $K(\phi)$ is of the form

$$K(\phi) = \frac{K_0(\phi)}{\phi^2}, \quad \lim_{\phi\to\infty} K_0(\phi) = \infty.$$

The function $Q(v)$ must be such that

$$Q(v) = Q_1 v^3 + o(v^3), \quad Q_1 > 0. \tag{2.91}$$

Then the asymptotic values of v are $v_r = 0$ in the radiation era (where $w_\phi \approx \frac{1}{2}$) and $v_d \neq 0$ in the dust era (where $w_\phi \approx -1$). The value v_d must be a root of $Q'(v)$ such that

$$Q(v_d) < 0, \quad Q'(v_d) = 0. \tag{2.92}$$

This scenario, however, has a fatal flaw. The attractor of case 8 requires that $v > v_d$ at all times, while the attractor of case 6 is realized at very small $v \approx 0$. Therefore, a transition from the first attractor to the second will necessarily involve values of $v < v_d$ for which the theory is unstable since $c_s^2(v) < 0$. Hence, this scenario must be discarded.

Thus we conclude that successful models of k-essence are produced only by Lagrangians described by Eq. (2.89) under the conditions of case 1 and case 2.

2.4.5 The existence of a superluminal epoch

We have shown that the only viable k-essence scenario is described by case 1/case 2 of Sec. 2.4.1. Now we demonstrate that in these scenarios $c_s^2(v_*) > 1$ for some value v_* that is reached during the dust-dominated epoch. The argument is similar to that in Ref. [38].

Since $Q(v_r) > 0$ and $Q(v_d) < 0$, while $Q(v)$ is a monotonically growing function of v, we must have $v_d < v_r$. In both scenarios of case 1 and case 2, the asymptotic fraction of the energy density R_0 is equal to a certain function F of v_0,

$$R_0 = F(v_0) \equiv 1 - \frac{9\kappa^2}{4} \frac{Q'^2}{vQ' - Q}\bigg|_{v=v_0}. \tag{2.93}$$

In case 2, $F(v_0) = 0$ due to Eq. (2.41); therefore, we may describe both cases 1 and 2 by a single function $F(v_0)$. We note that $\tilde{\varepsilon}_\phi(v) = vQ'(v) - Q(v)$ is a monotonically growing function of v because

$$\frac{d}{dv}\tilde{\varepsilon}_\phi(v) = vQ''(v) > 0 \quad \text{for } v > 0. \tag{2.94}$$

Since $\tilde{\varepsilon}_\phi(v) > 0$ for every relevant value of v, it follows that $F(v)$ is a continuous function for these v. For a successful model of k-essence, the radiation tracker must have $F(v_r) \gtrsim 0.99$ and the dust tracker must have $F(v_d) = 0$. During the evolution from the first tracker to the second, the value of v must traverse the interval $[v_d, v_r]$. The condition $F(v_d) < F(v_r)$ implies (due to the continuity of F) that there exists a value $v_1 \in [v_d, v_r]$ such that $F'(v_1)$ is positive:

$$F'(v_1) = -\frac{9\kappa^2 Q' Q''}{2(vQ' - Q)^2}\left[\frac{vQ'}{2} - Q\right]\bigg|_{v=v_1} > 0. \tag{2.95}$$

Since $Q' > 0$, $Q'' > 0$, and $\tilde{\epsilon}_\phi = vQ' - Q > 0$ for all $v \in [v_d, v_r]$, we can simplify this condition to

$$\frac{vQ'}{2} - Q\bigg|_{v=v_1} < 0, \tag{2.96}$$

or equivalently to

$$w_\phi(v_1) = \frac{Q}{vQ' - Q}\bigg|_{v=v_1} > 1. \tag{2.97}$$

The equation of state parameter $w_\phi(v)$ is a continuous function of v that satisfies

$$0 = w_\phi(v_d) < 1 < w_\phi(v_1). \tag{2.98}$$

Hence, there exists a value $v_* \in [v_d, v_1]$ such that $w_\phi(v_*) > 1$ and $w'_\phi(v_*) > 0$.

Finally, we show that $c_s^2(v_*) > 1$ follows from the conditions $w_\phi(v_*) > 1$ and $w'_\phi(v_*) > 0$. According to Eq. (2.35), we have

$$w'_\phi(v) = \left.\frac{(1 + w_\phi)\,(c_s^2 - w_\phi)}{v c_s^2}\right|_{v=v_*} > 0. \tag{2.99}$$

Therefore

$$c_s^2(v_*) > w_\phi(v_*) > 1. \tag{2.100}$$

Since $c_s^2(v)$ is a continuous function, this demonstrates the existence of an interval of values of v within $[v_d, v_r]$ where $c_s^2(v) > 1$. This superluminal epoch occurs during the dust-dominated era.

3 New Ekpyrotic Ghost

In this chapter we consider the new ekpyrotic scenario, following our paper [75]. The main task of this study is to identify the physical implications of a term with higher derivatives, which was added to the theory to stabilize its vacuum state. This chapter consists of six sections. In Section 3.1 we review the basic scenario of the ghost condensate and new ekpyrosis. In Sections 3.2 and 3.3 by means of proper quantizations we show that the higher derivative term results in the existence of a new ghost field with negative energy, which leads to a catastrophic vacuum instability. In Section 3.4 we present the energy-momentum tensor and equations of motion for the cosmological evolution, and the numerical investigation on the bounce dynamics is given in Section 3.5. Finally we conclude in the last section. In Appendix B we show that the ghost in the ghost condensate theory and the new ekpyrotic scenario can not be removed by field redefinitions and adding other degree of freedom in the effective UV theory.

3.1 Ghost condensate and new ekpyrosis: The basic scenario

The full description of the new ekpyrotic scenario is pretty involved. It includes two fields, one of which, ϕ, is responsible for the ekpyrotic collapse, and another one, χ, is responsible for generation of isocurvature perturbations, which eventually should be converted to adiabatic perturbations. Both fields must have quite complicated potentials, which can be found e.g. in [60]. For the purposes of our discussion it is sufficient to consider a simplified model containing only one field, ϕ. The simplest version of this scenario can be written as follows:

$$L = \sqrt{g}\left[M^4 P(X) - \frac{1}{2}\left(\frac{\Box\phi}{M'}\right)^2 - V(\phi)\right], \tag{3.1}$$

where $X = \frac{(\partial\phi)^2}{2m^4}$ is dimensionless. $P(X)$ is a dimensionless function which has a minimum at $X \neq 0$. The first two terms in this theory represent the theory of a ghost condensate, the last one is the ekpyrotic potential. This potential is very small and very flat at large ϕ, so for large ϕ this theory is reduced to the ghost condensate model of [12, 61].

The ghost condensate state corresponds to the minimum of $P(X)$. Without loss of generality one may assume that this minimum occurs at $X = 1/2$, i.e. at $\partial_i\phi = 0$, $\dot{\phi} = -m^2$, so that $\phi = -m^2 t$. As a simplest example, one can consider a function which looks as follows in the vicinity of its minimum:

$$P(X) = \frac{1}{2}(X - 1/2)^2 \ . \tag{3.2}$$

The term $-\frac{1}{2}\left(\frac{\Box\phi}{M'}\right)^2$ was added to the Lagrangian in [12] for stabilization of the fluctuations of the field ϕ in the vicinity of the background solution $\phi(t) = -m^2 t$; more about it later.

This theory was represented in several different ways in [12, 19, 21, 20, 60], where a set of parameters such as K and $\bar{M} = M^2/M'$ was introduced. The parameter K can always be absorbed in a redefinition of M; in our notation, $K = 1$.

The equation for the homogeneous background can be represented as follows:

$$\partial_t \left[a^3 \left(P_{,X}\, \dot{\phi} + \frac{\partial_t(\ddot{\phi} + 3H\dot{\phi})}{m_g^2} \right) \right] = -\frac{a^3 V_{,\phi}\, m^4}{M^4} \ , \tag{3.3}$$

where we introduced the notation

$$m_g = \frac{M' M^2}{m^2} \ . \tag{3.4}$$

The meaning of this notation will be apparent soon.

The complete equation describing the dependence on the spatial coordinates is

$$\partial_t \left[a^3 \left(P_{,X}\, \partial_t\phi + \frac{\partial_t(\Box\phi)}{m_g^2} \right) \right] - \partial_i \left[a \left(P_{,X}\, \partial_i\phi + \frac{\partial_i(\Box\phi)}{m_g^2} \right) \right] = -\frac{a^3 V_{,\phi}\, m^4}{M^4}. \tag{3.5}$$

Instead of solving these equations, the authors of [19, 21, 20, 60] analyzed (though not solved) equation (3.3) ignoring the higher derivative term $\partial_t(\ddot{\phi} + 3H\dot{\phi})/m_g^2$, assuming that it is small. Then they analyzed equation (3.5), applying it to perturbations, ignoring the term $\partial_t(\Box\phi)/m_g^2$, but keeping the term $\partial_i(\Box\phi)/m_g^2$, assuming that it is large. Our goal is to see what happens if one performs an investigation in a self-consistent way.

In order to do this, let us temporarily assume that the higher derivative term is absent, which corresponds to the limit $m_g \to \infty$. In this case our equation for ϕ reduces to the equation used in [19, 21, 20, 60]

$$\partial_t [a^3 P_{,X}\, \dot{\phi}] = -a^3 V_{,\phi}\, m^4/M^4 \ . \tag{3.6}$$

One of the Einstein equations, in the same approximation, is

$$\dot{H} = -\frac{1}{2}(\varepsilon + p) = -M^4 P_{,X}\ X = -M^4 X(X - 1/2) \ , \tag{3.7}$$

where ε is the energy density and p is the pressure. (We are using the system of units where $M_p^2 = (8\pi G)^{-1} = 1$.)

The null energy condition (NEC) requires that $\varepsilon + p \geq 0$, and $\dot{H} \leq 0$. Therefore a collapsing universe with $H < 0$ cannot bounce back unless NEC is violated. It implies that the bounce can be possible only if $P_{,X}$ becomes negative, $P_{,X} < 0$, i.e. the field X should become smaller than 1/2.

It is convenient to represent the general solution for $\phi(t)$ as

$$\phi(t) = -m^2 t + \pi_0(t) + \pi(x_i, t) \ , \tag{3.8}$$

where $\pi_0(t)$ satisfies equation

$$\ddot{\pi}_0 + 3H\dot{\pi}_0 = -\frac{m^4}{M^4} V_{,\phi} \ . \tag{3.9}$$

In this case one can show that the perturbations of the field $\pi(x_i, t)$ have the following spectrum at small values of $P_{,X}$:

$$\omega^2 = P_{,X}\, k^2 \ . \tag{3.10}$$

This means that $P_{,X}$ plays in this equation the same role as the square of the speed of sound. For small $P_{,X}$, one has

$$c_s^2 = P_{,X} \ . \tag{3.11}$$

The ghost condensate point $P_{,X} = 0$, which separates the region where NEC is satisfied and the region where it is violated, is the point where the perturbations are frozen. The real disaster happens when one crosses this border and goes to the region with $P_{,X} < 0$, which corresponds to $c_s^2 < 0$. In this area the NEC is violated, and, simultaneously, perturbations start growing exponentially,

$$\pi_k(t) \sim \exp(\sqrt{|c_s^2|}\, |k|\, t) \sim \exp(\sqrt{|P_{,X}|}\, |k|\, t) \ . \tag{3.12}$$

This is a disastrous gradient instability, which is much worse than the usual tachyonic instability. The tachyonic instability develops as $\exp(\sqrt{m^2 - k^2}\, t)$, so its rate is limited by the tachyonic mass, and it occurs only for $k^2 < m^2$. Meanwhile the instability (3.12) occurs at all momenta k, and the rate of its development exponentially grows with the growth of k. This makes it abundantly clear how dangerous it is to violate the null energy condition.

That is why it was necessary to add higher derivative terms of the type of $-\frac{1}{2}\left(\frac{\Box\phi}{M'}\right)^2$ to the ghost condensate Lagrangian [12]: The hope was that such terms could provide at least some partial protection by changing the dispersion relation.

Since we are interested mostly in the high frequency effects corresponding to the rapidly developing instability, let us ignore for a while the gravitational effects, which can be achieved by taking $a(t) = 1$, $H = 0$. In this case, the effective Lagrangian for perturbations π of the field ϕ in a vicinity of the minimum of $P(X)$ (i.e. for small $|P_{,X}|$) is

$$L = \frac{M^4}{m^4}\left[\frac{1}{2}\dot{\pi}^2 - \frac{1}{2}P_{,X}\,(\nabla\pi)^2 - \frac{1}{2m_g^2}(\Box\pi)^2\right] \ . \tag{3.13}$$

The equation of motion for the field π is

$$\ddot{\pi} - P_{,X}\,\nabla^2\pi + \frac{1}{m_g^2}\Box^2\pi = 0 \ . \tag{3.14}$$

At small frequencies ω, which is the case analyzed in [12], the dispersion relation corresponding to this equation looks as follows:

$$\omega^2 = P_{,X}\,k^2 + \frac{k^4}{m_g^2} \ . \tag{3.15}$$

This equation implies that the instability occurs only in some limited range of momenta k, which can be made small if the parameter m_g is sufficiently small and, therefore, the higher derivative therm is sufficiently large. This is the one of the main assumptions of the new ekpyrotic scenario: If the violation of the NEC occurs only during a limited time near the bounce from the singularity, one can suppress the instability by adding a sufficiently large term $-\frac{1}{2m_g^2}(\Box\pi)^2$. (This term must have negative sign, because otherwise it does not protect us from the gradient instability. This will be important for the discussion in Appendix.)

Note that one cannot simply add the higher derivative term and take it into account only up to some cut-off $\omega^2, k^2 < \Lambda^2$. For example, if we "turn on" this term only at $k^2 < \Lambda^2$, it is not going to save us from the gradient instability which occurs at $\omega^2 = P_{,X}\,k^2$ for all unlimitedly large k in the region where the NEC is violated and $P_{,X} < 0$.

There are several different problems associated with this scenario. First of all, in order to tame the instability during the bounce one should add a sufficiently large term $-\frac{1}{2}\left(\frac{\Box\phi}{M'}\right)^2$, which leads to the emergence of the term $\frac{1}{2m_g^2}(\Box\pi)^2$ in the equation for π. But if this term is large, then one should not discard it in the equations for the homogeneous scalar field and in the Einstein equations, as it was done in [19, 21, 20, 60].

The second problem is associated with the way the higher derivative terms were treated in [12, 19, 21, 20, 60]. The dispersion relation studied there was incomplete. The full dispersion relation for the perturbations in the theory (3.13), (3.14) is

$$\omega^2 = P_{,X}\,k^2 + \frac{(\omega^2 - k^2)^2}{m_g^2} \ . \tag{3.16}$$

This equation coincides with eq. (3.15) in the limit of small ω studied in [12, 19, 21, 20, 60]. However, this equation has two different branches of solutions, which we will present, for simplicity, for the case $P_{,X} = 0$ corresponding to the minimum of the ghost condensate potential $P(X)$:

$$\omega = \pm\, \omega_i\ , \qquad i = 1, 2, \tag{3.17}$$

where

$$\omega_1 = \frac{1}{2}\left(\sqrt{m_g^2 + 4k^2}\, -\, m_g\right), \tag{3.18}$$

$$\omega_2 = \frac{1}{2}\left(\sqrt{m_g^2 + 4k^2}\, +\, m_g\right). \tag{3.19}$$

At high momenta, for $k^2 \gg m_g^2$, the spectrum for all 4 solutions is nearly the same

$$\omega \approx \pm|k|\ . \tag{3.20}$$

At small momenta, for $k^2 \ll m_g^2$, one has two types of solutions: The lower frequency solution, which was found in [12], is

$$\omega = \pm\, k^2/m_g\ . \tag{3.21}$$

But there is also another, higher frequency solution,

$$\omega = \pm\, m_g\ . \tag{3.22}$$

The reason for the existence of an additional branch of solutions is very simple. Equation for the field ϕ in the presence of the term with the higher derivatives is of the fourth order. To specify its solutions it is not sufficient to know the initial conditions for the field and its first derivative, one must know also the initial conditions for the second and the third derivatives. As a result, a single equation describes two different degrees of freedom.

To find a proper interpretation of these degrees of freedom, one must perform their quantization. This will be done in the next two sections. As we will show in these sections, the lower frequency solution corresponds to normal particles with positive energy $\omega = +\omega_1(k)$, whereas the higher frequency solution corresponds to ekpyrotic ghosts with negative energy $-\omega_2(k)$. The quantity $-m_g$ has the meaning of the ghost mass: it is given by the energy $\omega = -\omega_2(k)$ at $k = 0$ and it is negative.

3.2 Hamiltonian quantization

We see that our equations for ω have two sets of solutions, corresponding to states with positive and negative energy. As we will see now, some of them correspond to normal particles, some of them are ghosts. We will find below that the Hamiltonian based on the classical Lagrangian in eq. (3.13) is

$$H_{quant} = \int \frac{d^3k}{(2\pi)^3}\left(\omega_1 a_k^\dagger a_k - \omega_2 c_k^\dagger c_k\right)\ . \tag{3.23}$$

The expressions for ω_1 and ω_2 will be presented below for the case of generic c_s^2, for $c_s^2 = 0$ they are given in eqs. (3.18) and (3.19). Both ω_1 and ω_2 are positive, therefore $a_k^\dagger$ and a_k are creation/annihilation operators of normal particles whereas $c_k^\dagger$ and c_k are creation/annihilation operators of ghosts.

We will perform the quantization starting with the Lagrangian in eq. (3.13), with an arbitrary speed of sound, $c_s^2 = P_{,X}$. The case $c_s = 1$ is the Lorentz invariant Lagrangian. The case $c_s^2 = P_{,X} = 0$ is the case considered in the previous section and appropriate to the ghost condensate and the new ekpyrotic scenario at the minimum of $P(X)$.

By rescaling the field $\pi \to \frac{M^2}{m^2}\pi$ we have

$$L = \frac{1}{2}\left(\partial_\mu \pi \partial^\mu \pi + \left(c_s^2 - 1\right)\pi\Delta\pi - \frac{1}{m_g^2}(\Box\pi)^2\right). \tag{3.24}$$

This is the no-gravity theory considered in the previous section. Note that the ghost condensate set up is already build in, the negative kinetic term for the original ghost is eliminated by the condensate. The existence of higher derivatives was only considered in [19, 21, 20, 60, 12] as a 'cure' for the problem of stabilizing the system after the *original* ghost condensation. As we argued in the previous section, this 'cure' brings in a *new ghost*, which remained unnoticed in [19, 21, 20, 60, 12]. In this section, as well as in the next one, we will present a detailed derivation of this result. Because of the presence of higher derivatives in the Lagrangian, the Hamiltonian quantization of this theory is somewhat nontrivial. It can be performed by the method invented by Ostrogradski [87].

Thus we start with the rescaled eq. (3.13)

$$L = \frac{1}{2}\left[\dot{\pi}^2 - c_s^2(\nabla_x\pi)^2 - \frac{1}{m_g^2}(\Box\pi)^2\right] . \tag{3.25}$$

The equation of motion for the field π is

$$\ddot{\pi} - c_s^2\nabla_x^2\pi + \frac{1}{m_g^2}\Box^2\pi = 0 . \tag{3.26}$$

If the Lagrangian depends on the field π and on its first and second time derivatives, the general procedure is the following. Starting with $L = L(\pi, \dot{\pi}, \ddot{\pi})$, one defines 2 canonical degrees of freedom, (q_1, p_1) and (q_2, p_2):

$$\begin{aligned} q_1 &\equiv \pi , \qquad p_1 = \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_1} , \\ q_2 &\equiv \dot{\pi} , \qquad p_2 = \frac{\partial L}{\partial \ddot{q}_1} . \end{aligned} \tag{3.27}$$

The canonical Hamiltonian is

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - L(q_1, q_2, \dot{q}_1, \dot{q}_2) . \tag{3.28}$$

The canonical Hamiltonian equations of motion,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \, , \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \, , \qquad i = 1, 2 \, , \tag{3.29}$$

are standard; they exactly reproduce the Lagrangian equation of motion (3.26). The quantization procedure requires promoting the Poisson brackets to commutators which allows to identify the spectrum. There are many known examples of the Ostrogradski procedure of derivation of the canonical Hamiltonian, see for example [88, 89].

The Hamiltonian density constructed by the Ostrogradski procedure for the Lagrangian (3.25) is

$$H_{cl}(\mathbf{x}, t) = \frac{1}{2}[p_1^2 - (p_1 - q_2)^2 - m_g^2 \left(p_2 - \frac{1}{m_g^2}\nabla_x^2 q_1\right)^2 + c_s^2(\nabla_x q_1)^2 + \frac{1}{m_g^2}\left(\nabla_x^2 q_1\right)^2] \, . \tag{3.30}$$

The next step in quantization is to consider the ansatz for the solution of classical equations of motion in the form

$$q_1(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \left[\frac{f_k^1}{\sqrt{2\omega_1}} \, e^{-ik_1 x} + \frac{f_k^2}{\sqrt{2\omega_2}} \, e^{ik_2 x} + \mathrm{cc}\right] , \tag{3.31}$$

where $k_1 \, x \equiv \omega_1(k) \, t - \mathbf{k} \, \mathbf{x}$, and $k_2 \, x \equiv \omega_2(k) \, t - \mathbf{k} \, \mathbf{x}$. We impose the Poisson brackets

$$\{q_i, p_j\} = \delta_{ij} \tag{3.32}$$

and promote them to commutators of the type

$$[q_i(\mathbf{x}, t), p_j(\mathbf{x}', t)] = i\delta_{ij}\delta^3(\mathbf{x} - \mathbf{x}') \, . \tag{3.33}$$

This quantization condition requires to promote the solution of the classical equation (3.31) to the quantum operator form, where

$$f_k^1 = a_k \frac{m_g}{\sqrt{\omega_2^2 - \omega_1^2}}, \qquad f_k^2 = c_k \frac{m_g}{\sqrt{\omega_2^2 - \omega_1^2}} \, , \tag{3.34}$$

and we impose normal commutation relation both on particles with creation and annihilation operators $a^\dagger$ and a and ghosts, $c^\dagger$ and c:

$$[a_k, a_{k'}^\dagger] = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \, , \qquad [c_k, c_{k'}^\dagger] = (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}') \, . \tag{3.35}$$

Here

$$\omega_1(k^2; m_g, c_s^2) = \left(k^2 + \frac{m_g^2}{2} - \sqrt{k^2 m_g^2(1 - c_s^2) + \frac{m_g^4}{4}}\right)^{1/2} \tag{3.36}$$

and

$$\omega_2(k^2; m_g, c_s^2) = \left(k^2 + \frac{m_g^2}{2} + \sqrt{k^2 m_g^2(1 - c_s^2) + \frac{m_g^4}{4}}\right)^{1/2} \tag{3.37}$$

The Hamiltonian operator acquires a very simple form

$$H_{quant} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\left(\omega_1(a_k^\dagger a_k + a_k a_k^\dagger) - \omega_2(c_k^\dagger c_k + c_k c_k^\dagger)\right) = \int \frac{d^3k}{(2\pi)^3}\left(\omega_1 a_k^\dagger a_k - \omega_2 c_k^\dagger c_k + C\right) . \tag{3.38}$$

Here the infinite term C is equal to $\frac{(2\pi)^3}{2}(\omega_1 - \omega_2)\delta^3(0)$. It represents the infinite shift of the vacuum energy due to the sum of all modes of the zero-point energies and is usually neglected in quantum field theory. Apart from this infinite c-number this is an expression promised in eq. (3.23).

We now define the vacuum state $|0\rangle$ as the state which is annihilated both by the particle as well as by the ghosts annihilation operators, $a_k|0\rangle = c_k|0\rangle = 0$. Thus the energy operator acting on a state of a particle has positive value and on a state of a ghost has the negative value[1]

$$H_{quant}\; a_k^\dagger|0\rangle = \omega_1(k)\; a_k^\dagger|0\rangle \;, \qquad H_{quant}\; c_k^\dagger|0\rangle = -\omega_2(k)\; c_k^\dagger|0\rangle \;. \tag{3.39}$$

This confirms the physical picture outlined at the end of the previous section.

3.3 Lagrangian quantization

The advantage of the Hamiltonian method is that it gives an unambiguous definition of the quantum-mechanical energy operator, which is negative for ghosts. This is most important for our subsequent analysis of the vacuum instability in the new ekpyrotic scenario. However, it is also quite instructive to explain the existence of the ghost field in new ekpyrotic scenario using the Lagrangian approach. The Lagrangian formulation is very convenient for coupling of the model to gravity.

Using Lagrangian multiplier, one can rewrite eq. (3.24) as

$$L = \frac{1}{2}\left(\partial_\mu\pi\partial^\mu\pi + \left(c_s^2 - 1\right)\pi\Delta\pi - \frac{B^2}{m_g^2}\right) + \lambda\left(B - \Box\pi\right). \tag{3.40}$$

Variation with respect to B gives $\lambda = \frac{B}{m_g^2}$. After substituting λ in (3.40) and skipping total derivative we obtain

$$\begin{aligned} L &= \frac{1}{2}\left(\partial_\mu\pi\partial^\mu\pi + \left(c_s^2 - 1\right)\pi\Delta\pi + \frac{B^2}{m_g^2}\right) + \frac{1}{m_g^2}\partial_\mu B\,\partial^\mu\pi \\ &= \frac{1}{2}\partial_\mu\left(\pi + \frac{2B}{m_g^2}\right)\partial^\mu\pi + \frac{B^2}{2m_g^2} + \frac{1}{2}\left(c_s^2 - 1\right)\pi\Delta\pi \;. \end{aligned} \tag{3.41}$$

[1] One could use an alternative way of ghost quantization, by changing the sign of their commutator relations. In this case the ghosts would have positive energy, but this would occur at the expense of introducing a nonsensical notion of negative probabilities.

Introducing new variables σ, ξ according to

$$\begin{aligned}\sigma + \xi &= \pi + \frac{2B}{m_g^2},\\ \sigma - \xi &= \pi \ ,\end{aligned}$$

and substituting $\pi = \sigma - \xi$, $B = \Box\pi = m_g^2\xi$ in (3.41) we obtain

$$L = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - \partial_\mu\xi\partial^\mu\xi + m_g^2\xi^2\right) + \frac{1}{2}\left(c_s^2 - 1\right)(\sigma - \xi)\Delta(\sigma - \xi) \ . \tag{3.42}$$

In the case $c_s^2 = 1$ we have two decoupled scalar fields: massive with negative kinetic energy and massless with positive kinetic energy.

$$L_{c_s^2=1} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - \partial_\mu\xi\partial^\mu\xi + m_g^2\xi^2\right) \ . \tag{3.43}$$

For the homogeneous field ($k = 0$ mode) the Lagrangian does not depend on c_s^2 and is reduced to

$$L_{k=0} = \frac{1}{2}\left(\dot{\sigma}^2 - \dot{\xi}^2 + m_g^2\xi^2\right) \ . \tag{3.44}$$

The relation between the $k = 0$ mode of the original field π, the normal field σ and the ghost field ξ is

$$\sigma = \pi + \xi \ , \qquad \xi = \frac{\ddot{\pi}}{m_g^2} \ . \tag{3.45}$$

When $c_s^2 \neq 1$ and $k \neq 0$ these fields still couple. To diagonalize the Lagrangian in eq. (3.24) and decouple the oscillators we have to go to normal coordinates, similar to the case of the classical mechanics of coupled harmonic oscillators. For that we need to solve the eigenvalue problem and find the eigenfrequencies of the oscillators. Let us consider the modes with the wavenumbers $\mathbf{k}$. For such modes we can perform the following change of variables

$$\sigma_{\mathbf{k}} \equiv \frac{\ddot{\pi}_{\mathbf{k}} + \omega_2^2\pi_{\mathbf{k}}}{m_g\sqrt{\omega_2^2 - \omega_1^2}} \ , \qquad \xi_{\mathbf{k}} \equiv \frac{\ddot{\pi}_{\mathbf{k}} + \omega_1^2\pi_{\mathbf{k}}}{m_g\sqrt{\omega_2^2 - \omega_1^2}} \ , \tag{3.46}$$

where ω_1, ω_2 are defined in eqs. (3.36), (3.37). In the special case of $c_s^2 = 0$ the answers for ω_1, ω_2 simplify and are shown in eqs. (3.18) and (3.19). After a change of variables we find for these modes in the momentum space

$$\tilde{L}_{c_s^2} = \frac{1}{2}\left(\dot{\sigma}_{\mathbf{k}}\dot{\sigma}_{-\mathbf{k}} - \omega_1^2\,\sigma_{\mathbf{k}}\sigma_{-\mathbf{k}} - \dot{\xi}_{\mathbf{k}}\dot{\xi}_{-\mathbf{k}} + \omega_2^2\,\xi_{\mathbf{k}}\xi_{-\mathbf{k}}\right). \tag{3.47}$$

The modes of σ are normal, and the modes of ξ are ghosts. Using this Lagrangian, one can easily confirm the final result of the Hamiltonian quantization given in the previous section.[2] The classical mode $\sigma_{\mathbf{k}}$ is associated with creation/annihilation operators $a_k^\dagger, a_k$ of normal particles after quantization and the classical mode $\xi_{\mathbf{k}}$ is associated with creation/annihilation operators of ghosts particles $c_k^\dagger, c_k$ after quantization. A quantization of the theory in eq. (3.47) leads to the Hamiltonian in eq. (3.23).

[2]After we finished this paper, we learned that the Lagrangian quantization of the ghost condensate

3.4 Energy-momentum tensor and equations of motion

First, we will compute the energy-momentum tensor (EMT) of the Lagrangian (3.42) using the Noether procedure:

$$T^{\mu}_{\nu} = \frac{\partial L}{\partial(\partial_\mu \varphi)}\partial_\nu \varphi - L\,\eta^{\mu}_{\nu},$$

where η^{μ}_{ν} is Minkowski metric. We find

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\sigma\partial_\nu\sigma - \partial_\mu\xi\partial_\nu\xi + \eta_{\mu i}(c_s^2-1)\partial^i(\sigma-\xi)\partial_\nu(\sigma-\xi) \\ &\quad -\eta_{\mu\nu}\left(\frac{1}{2}(\partial_\alpha\sigma\partial^\alpha\sigma - \partial_\alpha\xi\partial^\alpha\xi + m_g^2\xi^2) + \frac{1}{2}(c_s^2-1)\partial_i(\sigma-\xi)\partial^i(\sigma-\xi)\right). \end{aligned} \tag{3.48}$$

The energy density is

$$\varepsilon = T_{00} = \frac{1}{2}[\dot{\sigma}^2 + (\partial_i\sigma)^2 - \dot{\xi}^2 - (\partial_i\xi)^2 - m_g^2\xi^2 + (c_s^2-1)(\partial_i(\sigma-\xi))^2].$$

For the homogeneous field ($k = 0$ mode), the energy density can be split into two parts, i.e. a normal field part and an ekpyrotic ghost field part:

$$\varepsilon = \varepsilon_\sigma + \varepsilon_\xi,$$

where

$$\varepsilon_\sigma = \frac{1}{2}\dot{\sigma}^2 > 0, \qquad \varepsilon_\xi = -\frac{1}{2}\dot{\xi}^2 - \frac{1}{2}m_g^2\xi^2 < 0\ .$$

Thus the energy of the ghost field ξ is negative.

Up to now we have turned off gravity. In the presence of gravity, the energy-momentum tensor of the full Lagrangian (1) in Sec. 2 is calculated by varying the action with respect to the metric:

$$\begin{aligned} T_{\mu\nu} &= g_{\mu\nu}\left[-M^4P(X) - \frac{(\Box\phi)^2}{2M'^2} + V(\phi) - \frac{\partial_\alpha(\Box\phi)\partial^\alpha\phi}{M'^2}\right] \\ &\quad +M^4m^{-4}P_{,X}\,\partial_\mu\phi\partial_\nu\phi + M'^{-2}(\partial_\mu(\Box\phi)\,\partial_\nu\phi + \partial_\nu(\Box\phi)\partial_\mu\phi) \\ &\equiv g_{\mu\nu}\left[-M^4P(X) - \frac{M'^2Y^2}{2} + V(\phi) - \partial_\alpha Y\,\partial^\alpha\phi\right] \\ &\quad +M^4m^{-4}P_{,X}\,\partial_\mu\phi\partial_\nu\phi + \partial_\mu Y\,\partial_\nu\phi + \partial_\nu Y\,\partial_\mu\phi\ , \end{aligned} \tag{3.49}$$

scenario was earlier performed by Aref'eva and Volovich for the case $c_s^2 = P_{,X} = 0$ [90], and they also concluded that this scenario suffers from the existence of ghosts. When our works overlap, our results agree with each other. We use the Lagrangian approach mainly to have an alternative derivation of the results of the Hamiltonian quantization. The Hamiltonian approach clearly establishes the energy operator and the sign of its eigenvalues, which is necessary to have an unambiguous proof that the energy of the ghosts is indeed negative and the ghosts do not disappear at non-vanishing $c_s^2 = P_{,X} \neq 0$ when the ekpyrotic universe is out of the ghost condensate minimum.

where

$$Y \equiv M'^{-2}\Box\phi \ . \tag{3.50}$$

From this, for a homogeneous, spatially flat FRW space time we have the energy density

$$\varepsilon = M^4(2P_{,X}\ X - P(X)) + V(\phi) - \frac{M'^2Y^2}{2} + \dot{Y}\,\dot{\phi} \tag{3.51}$$

and the pressure

$$p = M^4P(X) - V(\phi) + \frac{M'^2Y^2}{2} + \dot{Y}\,\dot{\phi}\ , \tag{3.52}$$

so that

$$\dot{H} = -\frac{1}{2}(\varepsilon + p) = -M^4P_{,X}\ X - \dot{Y}\,\dot{\phi}\ . \tag{3.53}$$

Note that in the homogeneous case in the absence of gravity the ekpyrotic ghost field ξ as defined in eq. (3.45) is directly proportional to the field Y:

$$\xi = \frac{m^2}{M^2}\ Y\ . \tag{3.54}$$

The closed equations of motion, which we used for our numerical analysis, are obtained as follows:

$$\begin{aligned}
\ddot{\phi}\,(P_{,X} + 2XP_{,XX}) + 3H\,P_{,X}\ \dot{\phi} + \frac{m^4}{M^4}(\ddot{Y} + 3H\dot{Y}) \quad &= \quad -V_{,\phi}\,m^4/M^4\ , \\
\dot{H} = -M^4P_{,X}\ \frac{\dot{\phi}^2}{2m^4} - \dot{Y}\,\dot{\phi}\ , \\
M'^{-2}(\ddot{\phi} + 3H\dot{\phi}) = Y\ .
\end{aligned} \tag{3.55}$$

Here $X = \dot{\phi}^2/2m^4$.

In these equations the higher derivative corrections appear in the terms containing the derivatives of Y. The last of these equations shows that $Y \to 0$ in the limit $M' \to \infty$ (i.e. $m_g \to \infty$), and then the dynamics reduces to one with no higher derivative corrections.

The closed equations of motion for π coupled to gravity are obtained by expanding (3.55) and linearizing with respect to π and Y. Then we get

$$\begin{aligned}
\ddot{\pi} + 3H\,\dot{\pi} + \frac{m^4}{M^4}\,(\ddot{Y} + 3H\,\dot{Y}) = -V_{,\phi}\,m^4/M^4\ , \\
\ddot{\pi} + 3H\,\dot{\pi} = M'^2\,Y + 3H\,m^2\ , \\
\dot{H} = \frac{M^4\dot{\pi}}{2m^2} + \dot{Y}\,m^2\ .
\end{aligned} \tag{3.56}$$

3.5 On reality of the bounce and reality of ghosts

Using the equations derived above, we performed an analytical and numerical investigation of the possibility of the bounce in the new ekpyrotic scenario. We will not present all of the details of this investigation here since it contains a lot of material which may distract the reader from the main conclusion of our paper, discussed in the next section: Because of the existence of the ghosts, this theory suffers from a catastrophic vacuum instability. If this is correct, any analysis of classical dynamics has very limited significance. However, we will briefly discuss our main findings here, just to compare them with the expectations expressed in [19, 21, 20, 60].

Our investigation was based on the particular scenario discussed in [21, 60] because no explicit form of the full ekpyrotic potential was presented in [19, 20]. The authors of [21, 60] presented the full ekpyrotic potential, but they did not fully verify the validity of their scenario, even in the absence of the higher derivative terms.

Before discussing our results taking into account higher derivatives, let us remember several constraints on the model parameters which were derived in [19, 21, 20, 60]. We will represent these constraints in terms of the ghost condensate mass instead of the parameter M', for $K = 1$. In this case the stability condition (7.19) in [21] (see also [19, 20]) reads:

$$\frac{|\dot{H}|}{|H|} \lesssim \frac{M^4}{m_g} \lesssim |H| \ . \tag{3.57}$$

It was assumed in [21] that the bounce should occur very quickly, during the time $\Delta t \lesssim |H_0|^{-1} \sim 1/\sqrt{p|V_{min}|}$. Here H_0 is the Hubble constant at the end of the ekpyrotic state, just before it start decreasing during the bounce, $p \sim 10^{-2}$, and V_{min} is the value of the ekpyrotic potential in its minimum. During the bounce one can estimate $\Delta H \sim |H_0| \sim \dot{H}\Delta t \lesssim \frac{\dot{H}}{|H_0|}$ because we assume, following [21], that $\Delta t < |H_0|^{-1}$, and we assume an approximately linear change of H from $-|H_0|$ to $|H_0|$. This means that $\frac{\dot{H}}{|H_0|} \gtrsim |H_0|$. In this case the previous inequalities become quite restrictive,

$$|H_0| \lesssim \frac{M^4}{m_g} \lesssim |H_0| \ . \tag{3.58}$$

This set of inequalities requires that the stable bounce is not generic; it can occur only for a fine-tuned value of the ghost mass,

$$m_g \sim \frac{M^4}{|H_0|} \sim \frac{M^4}{\sqrt{p|V_{min}|}} \ . \tag{3.59}$$

The method of derivation of these conditions required an additional condition to be satisfied, $|H_0| \sim \sqrt{p|V_{min}|} \ll M^2$, see Eqs. (8.8) and (8.17) of Ref. [21]. This condition is satisfied for

$$m_g \gg M^2 \ . \tag{3.60}$$

Whereas the condition (3.59) seems necessary in order to avoid the development of the gravitational instability and the gradient instability during the bounce for $m_g \gg M^2$, it is not sufficient, simply because the very existence of the bounce may require m_g to be very much different from its fine-tuned value $m_g \sim \frac{M^4}{\sqrt{p|V_{min}|}}$. Indeed, our investigation of the

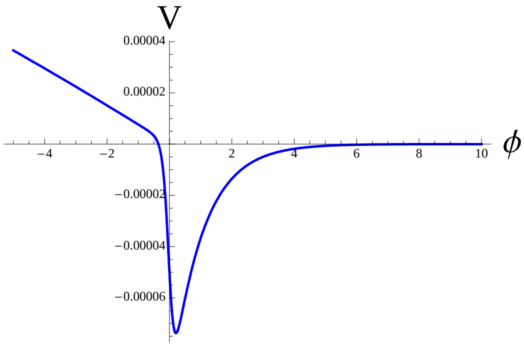


Figure 3.1: The "new ekpyrotic potential," see Fig. 3 in [21] and Fig. 6 in [60]. The cosmological evolution in this model results in a universe with a permanently growing rate of expansion after the bounce, which is unacceptable.

cosmological evolution in this model shows that generically the bounce does not appear at all, or one encounters a singular behavior of $\ddot{\phi}$ because of the vanishing of the term $P_{,X} + 2XP_{,XX}$ in (3.55), or one finds an unstable bounce, or the bounce ends up with an unlimited growth of the Hubble constant, like in the Big Rip scenario [91, 92]. Finding a proper potential leading to a desirable cosmological evolution requires a lot of fine-tuning, in addition to the fine-tuning already described in [21, 60].

For example, the bounce in the model with the "new ekpyrotic potential" described in [21, 60] and shown in Fig. 3.1 results in a universe with a permanently growing rate of expansion after the bounce, which would be absolutely different from our universe. To avoid this disaster, one must bend the potential, to make it approaching the value corresponding to the present value of the cosmological constant, see Fig. 3.2. This bending should not be too sharp, and it should not begin too early, since otherwise the universe bounces back and ends up in the singularity. Fig. 3.3 shows the bouncing solution in the theory with this potential.

Our calculations clearly demonstrate the reality of the ekpyrotic ghosts, see Fig. 3.4, which shows the behavior of the ghost-related field $Y = \frac{M^2}{m^2}\xi$ near the bounce. The oscillations shown in Fig. 3.4 represent the ghost matter with negative energy, which was generated during the ekpyrotic collapse. We started with initial conditions $Y = \dot{Y} = 0$, i.e. in the vacuum without ghosts, and yet the ghost-related field Y emerged dynamically. It oscillates with the frequency which is much higher than the rate of the change of the average value of the field ϕ.

This shows that the ekpyrotic ghost is not just a mathematical construct or a figment of imagination, but a real field. We have found that the amplitude of the oscillations of

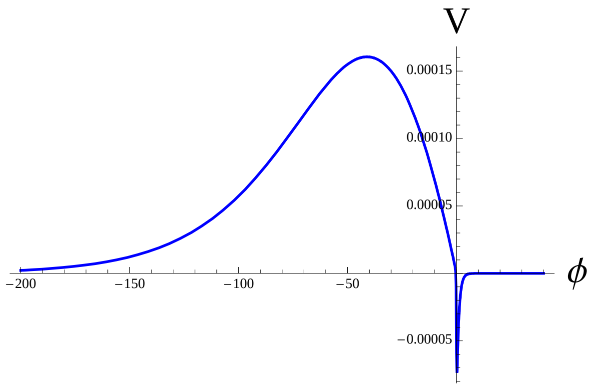


Figure 3.2: An improved potential which leads to a bounce followed by a normal cosmological evolution. We do not know whether this extremely fine-tuned potential can be derived from any realistic theory.

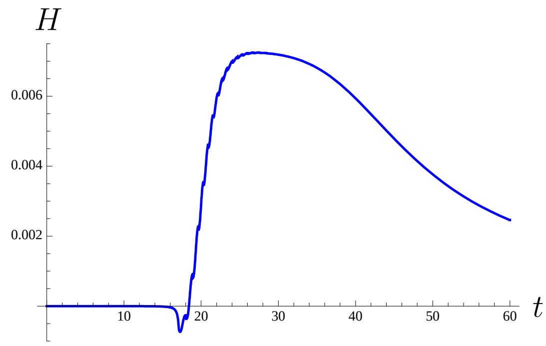


Figure 3.3: The behavior of the Hubble constant $H(t)$ near the bounce, which occurs near $t = 18$. To verify the stability of the universe during the bounce, one would need to perform an additional investigation taking into account the ghost field oscillations shown in Fig. 3.4.

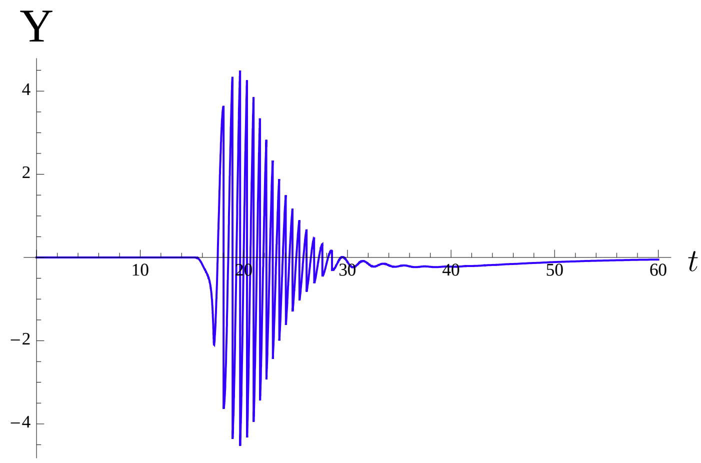


Figure 3.4: Ekpyrotic ghost field oscillations.

the ghost field is very sensitive to the choice of initial conditions; it may be negligibly small or very large. Therefore in the investigation of the cosmological dynamics one should not simply consider the universe filled with scalar fields or scalar particles. The universe generically will contain normal particles and ghost particles and fields with negative energy. The ghost particles will interact with normal particles in a very unusual way: particles and ghosts will run after each other with ever growing speed. This regime is possible because when the normal particles gain energy, the ghosts loose energy, so the acceleration regime is consistent with energy conservation. This unusual instability, which is very similar to the process to be considered in the next section, can make it especially difficult to solve the homogeneity problem in this scenario.

3.6 Ghosts, singularity and vacuum instability

It was not the goal of the previous section to prove that the ghosts do not allow one to solve the singularity problem. They may or may not spoil the bounce in the new ekpyrotic scenario. However, in general, if one is allowed to introduce ghosts, then the solution of the singularity problem becomes nearly trivial, and it does not require the ekpyrotic scenario or the ghost condensate.

Indeed, let us consider a simple model describing a flat collapsing universe which contains a dust of heavy non-relativistic particles with initial energy density ρ_M, and a gas of ultra-relativistic ghosts with initial energy density $-\rho_g < 0$. Suppose that at the initial moment $t = 0$, when the scale factor of the universe was equal to $a(0) = 1$, the energy density was dominated by energy density of normal particles, $\rho_M - \rho_g > 0$. The absolute value of the ghost energy density in the collapsing universe grows faster than the energy of the non-relativistic matter. The Friedmann equation describing a collapsing universe is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho_M}{a^3} - \frac{\rho_g}{a^4}\ . \tag{3.61}$$

In the beginning of the cosmological evolution, the universe is collapsing, but when the scale factor shrinks to $a_{\rm bounce} = \frac{\rho_g}{\rho_M}$, the Hubble constant vanishes, and the universe bounces back, thus avoiding the singularity.

Thus nothing can be easier than solving the singularity problem once we invoke ghosts to help us in this endeavor, unless we are worried about the gravitational instability problem mentioned in the previous section. Other examples of the situations when ghosts save us from the singularity can be found, e.g. in [93], where the authors not only study a way to avoid the singularity with the help of ghosts, but even investigate the evolution of metric perturbations during the bounce. So what can be wrong with it?

A long time ago, an obvious answer would be that theories with ghosts lead to negative probabilities, violate unitarity and therefore do not make any sense whatsoever. Later on, it was realized that if one treats ghosts as particles with negative energy, then problems with unitarity are replaced by the problem of vacuum stability due to interactions between ghosts and normal particles with positive energy, see, e.g. [94, 95, 96, 97, 98, 99, 100]. Indeed, unless the ghosts are hidden in another universe [95], nothing can forbid creation of pairs of ghosts and normal particles under the condition that their total momentum and energy vanish. Since the total energy of ghosts is negative, this condition is easy to satisfy.

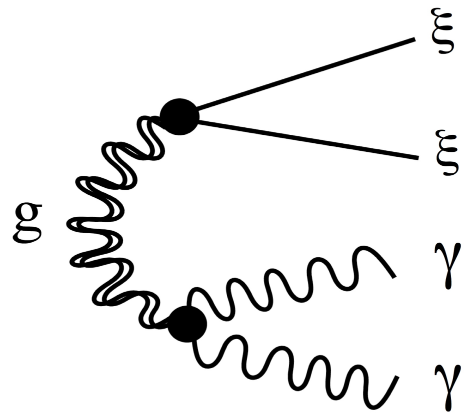


Figure 3.5: Vacuum decay with production of ghosts ξ and usual particles γ interacting with each other by the graviton exchange.

There are many channels of vacuum decay; the simplest and absolutely unavoidable one is due to the universal gravitational interaction between ghosts and all other particles, e.g. photons. An example of this interaction was considered in [97], see Fig. 3.5. Nothing can forbid this process because it does not require any energy input: the positive energy of normal particles can be compensated by the negative energy of ghosts.

An investigation of the rate of the vacuum decay in this process leads to a double-divergent result. First of all, there is a power-law divergence because nothing forbids creation of particles with indefinitely large energy. In addition, there is also a quadratic divergence in the integral over velocity [97, 99]. This leads to a catastrophic vacuum decay.

Of course, one can always argue that such processes are impossible or suppressed because of some kind of cutoff in momentum space, or further corrections, or non-local interactions. However, the necessity of introducing such a cut-off, or additional corrections to corrections, after introducing the higher derivative terms which were supposed to work as a cutoff in the first place, adds a lot to the already very high price of proposing an alternative to inflation: First it was the ekpyrotic theory, then the ghost condensate and curvatons, and finally - ekpyrotic ghosts with negative energy which lead to a catastrophic vacuum instability. And if we are ready to introduce an ultraviolet cutoff in momentum space, which corresponds to a small-scale cutoff in space-time, then why would we even worry about the singularity problem, which is supposed to occur on an infinitesimally small space-time scale?

In fact, this problem was already emphasized by the authors of the new ekpyrotic scenario, who wrote [21]:

"But ghosts have disastrous consequences for the viability of the theory. In order to regulate the rate of vacuum decay one must invoke explicit Lorentz breaking at some low scale [97]. In any case there is no sense in which a theory with ghosts can be thought as an effective theory, since the ghost instability is present all the way to the UV cut-off of the theory."

We have nothing to add to this characterization of their own model.

4 Bouncing Universe and Non-BPS Brane

In this chapter we describe bouncing universe scenarios involving the creation and annihilation of a non-BPS D9-brane in type IIA superstring theory, following our paper [101]. This chapter consists of six sections. The model we employ is described in Section 4.1 where we present the effective actions and the equations of motions for the metric, dilaton and tachyon from a non-BPS brane, which makes the bounce possible. In Section 4.2 we show that our model has no ghost by considering the null energy condition in the Einstein frame, and some features of string frame bounce scenarios are studied in relation to the Einstein frame. In Section 4.3, the asymptotic behavior of the solutions is analyzed and its qualitative similarity to pre-big bang scenario is clarified. The numerical determination of the global bounce solution is presented in Section 4.4. In Section 4.5 we present a simple model that resolves the asymptotic curvature singularity in the string frame. Finally we conclude in the last section.

4.1 The model

We consider a non-BPS space-filling D9-brane in type IIA superstring theory. The details of the compactification will not play a role here. Concretely we consider the lowest order effective action for the metric and dilaton in the string frame as well as an effective action for the open tachyonic mode of the non-BPS D-brane. For now, the only assumption being made for the tachyon action is that only the first derivatives of the tachyon appear in it. The ansatz for the gravitational action is justified provided the dilaton and metric are slowly varying in string units. We write

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}\left(R + 4\,\partial_\mu\Phi\partial^\mu\Phi\right) + S_{\mathrm{T}} \tag{4.1}$$

$$\text{with}\quad S_{\mathrm{T}} = \int \mathrm{d}^{10}x\,\sqrt{-g}\,\mathrm{e}^{-\Phi}L(T,\partial_\mu T\partial^\mu T), \tag{4.2}$$

where Φ is the dilaton, T is the tachyon, and $\kappa_{10}^2 = 8\pi G_{10}$ with G_{10} the ten-dimensional Newton constant. We use the signature $(-,+,\ldots,+)$ for the metric. With these conventions, the matter energy-momentum tensor is given by

$$T^{\mu}_{\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{T}}(T, \partial_\rho T \partial^\rho T, \Phi)}{\delta g^{\nu}_{\mu}}. \tag{4.3}$$

For a Lagrangian minimally coupled to gravity, the metric appears only in $(\partial T)^2 = \partial_\rho T \partial^\rho T$, and we can write the energy-momentum tensor as

$$T^{\mu}_{\nu} = 2\, \mathrm{e}^{-\Phi} \frac{\partial L(T, (\partial T)^2)}{\partial\left((\partial T)^2\right)} \partial^\mu T \partial_\nu T - \delta^{\mu}_{\nu}\, \mathrm{e}^{-\Phi} L \tag{4.4}$$

In a homogeneous isotropic universe that we will consider, all fields are assumed to depend only on time; the energy density ϵ and pressure p are given by

$$\epsilon = T^0_0 \quad , \qquad p = -T^i_i \text{ (no sum) } = \mathrm{e}^{-\Phi} L. \tag{4.5}$$

We are now ready to make an ansatz for the metric. We take a four-dimensional spatially flat FRW spacetime times a six-torus characterized by a single modulus σ. Namely

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\, \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j + \mathrm{e}^{2\sigma(t)}\, \delta_{IJ} \mathrm{d}x^I \mathrm{d}x^J, \tag{4.6}$$

where the lower-case Latin indices run over the three uncompactified space coordinates, while the upper-case Latin indices label the six compactified dimensions. We further simplify the problem by restricting to the case where $a(t) = \mathrm{e}^{\sigma(t)}$. Although equality of the two scale factors is phenomenologically not satisfying at late times it may be assumed near the cosmological bounce which is the prime focus of this paper. In particular, we will not address the important problem of moduli- and dilaton stabilization required to connect to the standard cosmology at late times. With the ansatz (4.6) the Einstein equations are now effectively isotropic. And in particular the relations (4.5) can be used. From now on the indices i, j include I, J.

The equations of motion for g^0_0, g^i_i and Φ are then given by the following first three equations

$$\begin{aligned}
72H^2 - 36H\dot{\Phi} + 4\dot{\Phi}^2 - 2\kappa_{10}^2\, \mathrm{e}^{2\Phi}\, \epsilon &= 0, &\quad (4.7)\\
2\ddot{\Phi} - 8\dot{H} + 16H\dot{\Phi} - 2\dot{\Phi}^2 - 36H^2 - \kappa_{10}^2\, \mathrm{e}^{2\Phi}\, p &= 0, &\quad (4.8)\\
2\ddot{\Phi} + 18H\dot{\Phi} - 2\dot{\Phi}^2 - 9\dot{H} - 45H^2 - \frac{\kappa_{10}^2}{2}\, \mathrm{e}^{2\Phi}\, p &= 0, &\quad (4.9)\\
\dot{\epsilon} + 9H(\epsilon + p) - \dot{\Phi}\, p &= 0, &\quad (4.10)
\end{aligned}$$

and the last equation follows from the generalized conservation law $\nabla_\mu T^{\mu}_{\nu} = (\partial_\nu \Phi)\, \mathrm{e}^{-\Phi}\, L$.

In the case where $\epsilon = 0$ and $p = 0$, these equations allow exact solutions, namely $H = \pm\frac{1}{3t}$, $\dot{\Phi} = \frac{-1\pm 3}{2t}$. These are the special cases of the pre -big bang ($t < 0$) and post-big bang ($t > 0$) solutions, which respect the time reflection symmetry ($t \mapsto -t$) and scale-factor duality symmetry ($a \mapsto a^{-1}$) (see for example [102]).

Let us explore the possibility of the bounce. Subtracting Eq. (4.9) from Eq. (4.8) we find

$$\dot{H} + 9H^2 - 2H\dot{\Phi} - \frac{\kappa_{10}^2}{2}\,\mathrm{e}^{2\Phi}\,p = 0\,. \tag{4.11}$$

This is an important equation. It tells us that a necessary condition to have a bounce,

$$H = 0 \qquad \text{and} \qquad \dot{H} > 0\,, \tag{4.12}$$

is that the tachyon pressure p must be *positive*. This is a tight constraint for a scalar field action; the Born-Infeld action for instance, which is an often used ansatz as a higher derivative scalar field action, gives a pressure that is always negative.

Furthermore, assuming that H is negative during a contracting phase with growing dilaton and a negative pressure, (4.11) implies $\dot{H} < 0$, i. e. accelerated contraction. On the other hand, for positive equation of state for the scalar field, $w > 0$, the conservation equation in (4.10) gives a growing pressure p in the contracting phase so that a "turn around" $\dot{H} = 0$ is compatible with (4.11).

Of course, a Born-Infeld action for the tachyonic sector of non-BPS branes is not in any way suggested by string theory. On the other hand, within the restriction to first derivative actions it is possible to derive an approximate effective action from string theory for the open string tachyon of an unstable brane. This action, constructed in [10] and further studied in [103] is given by

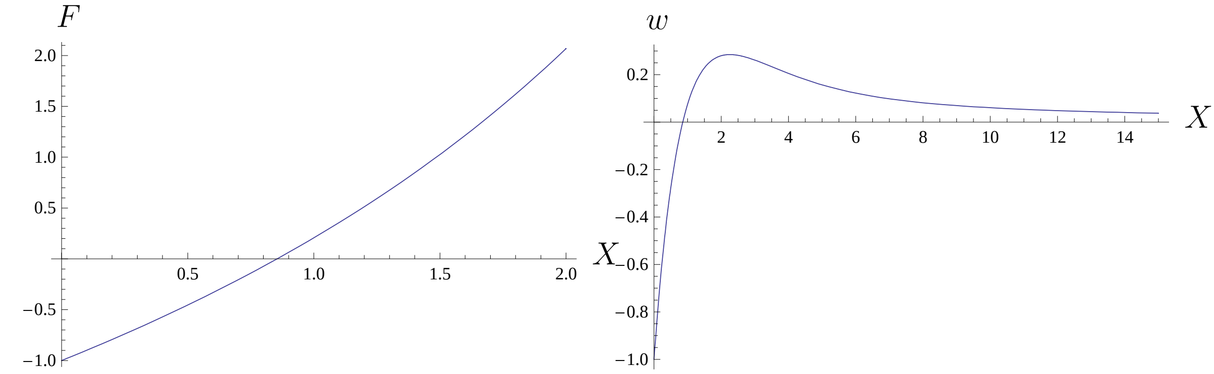
$$L = -\sqrt{2}\,\tau_9\,\mathrm{e}^{-\frac{T^2}{2\alpha'}}\left(\mathrm{e}^{-(\partial T)^2} + \sqrt{\pi(\partial T)^2}\;\mathrm{erf}\left(\sqrt{(\partial T)^2}\right)\right), \tag{4.13}$$

where τ_9 is the tension of a BPS 9-brane, and therefore $\sqrt{2}\,\tau_9$ is the tension of a non-BPS 9-brane [104]. Let us shortly summarize how this action was constructed. First, setting $(\partial T)^2$ to zero, we see that the potential is given by

$$V(T) = \sqrt{2}\,\tau_9\,\mathrm{e}^{-\frac{T^2}{2\alpha'}}\,. \tag{4.14}$$

This is the exact potential for the open string tachyon potential found in boundary superstring field theory [105, 106, 107]. The locations of the minima of $V(T)$ are at $T = \pm\infty$. At these values the energy is degenerate with the closed string vacuum which means that the non-BPS brane is absent. The construction of the full action (4.13) is based on the observation that the tachyon kink $T(x) = \chi\,\sin(x/\sqrt{2\alpha'})$, where x is one of the spatial world volume coordinates, is an exactly marginal deformation of the underlying boundary conformal field theory [108, 109] and thus should be a solution of the equations of motion obtained from (4.13). It turns out that this requirement determines uniquely the action once the potential has been chosen. Furthermore, it follows by analytic continuation that Sen's rolling tachyon solution [110]

$$T(t) = A\,\sinh(t/\sqrt{2\alpha'}) + B\,\cosh(t/\sqrt{2\alpha'})$$



(a) X-dependence of the pressure $F(X)$.

(b) Equation of state w of the tachyon as a function of X.

is also a solution of the action (4.13) for all values of A and B. In this dynamical decay (or creation) of the non-BPS brane the energy is conserved. The asymptotic state for large positive (or negative) times has been argued to be given by "tachyon matter" - essentially cold dust made from very massive closed string states. Let us now briefly explain how this action can allow a positive pressure, or equivalently a positive Lagrangian [103]. This follows from the fact that it is real and continuous also for negative values of $(\partial T)^2$. Indeed if we write $-(\partial T)^2 = X$ (note that $X = \dot{T}^2$ in the homogeneous case), we can see that

$$\sqrt{-\pi X}\ \mathrm{erf}(\sqrt{-X}) = -2\sqrt{X}\int_0^{\sqrt{X}} \mathrm{e}^{s^2}\,\mathrm{d}s\,. \tag{4.15}$$

This is negative and grows in absolute value faster than the first term e^X in the Lagrangian; so for positive enough X (for negative enough $(\partial T)^2$), the Lagrangian, and thus the pressure, is always positive. This can be seen from Fig. 4.1(a), where we show the X - dependence of the pressure, $F(X) \equiv -\left(\mathrm{e}^X - 2\sqrt{X}\int_0^{\sqrt{X}}\mathrm{e}^{s^2}\,\mathrm{d}s\right)$. In terms of F the Lagrangian can be expressed as $L = \sqrt{2}\tau_9\mathrm{e}^{-\frac{T^2}{2\alpha'}}F(X)$. From the figure it is clear that $\mathrm{d}F/\mathrm{d}X > 0$ (this is also clear from $\frac{\mathrm{d}F}{\mathrm{d}X} = \frac{1}{\sqrt{X}}\int_0^{\sqrt{X}}\mathrm{e}^{s^2}\,\mathrm{d}s > 0$). From now on we work in the unit system with $\alpha' = 1/2$. The energy density and pressure of the tachyon are

$$\epsilon = \sqrt{2}\tau_9\mathrm{e}^{-\Phi}\mathrm{e}^{-T^2+\dot{T}^2} \quad , \qquad p = \sqrt{2}\tau_9\mathrm{e}^{-\Phi}\mathrm{e}^{-T^2}F(\dot{T}^2)\,. \tag{4.16}$$

Note that the equation of state $w = \epsilon/p$ depends only on $\dot{T}^2$ as shown in Fig. 4.1(b). In particular, $w \to 0$ as $\dot{T}^2 \to \infty$, while $w \to -1$ as $\dot{T}^2 \to 0$.

4.2 Einstein frame and null energy condition

In this section we show that our model has no ghost and satisfies NEC in the Einstein frame. This is important because there might be a pathology due to an instability coming from NEC violation. On the other hand, the consideration on NEC will help us draw more general conclusions concerning the bounce scenarios.

The action (4.1)-(4.2) in the string frame with L given in (4.13) is expressed in the Einstein frame by means of a conformal transformation $g_{\mu\nu} = \tilde{g}_{\mu\nu}\, \mathrm{e}^{\frac{\Phi}{2}}$ (where $\tilde{g}_{\mu\nu}$ is the metric in the Einstein frame). This yields

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{-\tilde{g}}\left(\tilde{R} - \frac{(\tilde{\nabla}\Phi)^2}{2}\right) + S_\mathrm{T} \tag{4.17}$$

$$\text{with}\quad S_\mathrm{T} = \int \mathrm{d}^{10}x\,\sqrt{-\tilde{g}}\,\mathrm{e}^{\frac{3\Phi}{2}}\,L(T,\tilde{X}\mathrm{e}^{-\frac{\Phi}{2}})\,, \tag{4.18}$$

where $\tilde{R}$ and $\tilde{\nabla}$ are the scalar curvature and the covariant derivative associated with $\tilde{g}$, and $\tilde{X} = -(\tilde{\nabla}T)^2 = X\mathrm{e}^{\frac{\Phi}{2}}$.

The relevant quantity for verifying classical stability and nonexistence of ghost is the sign of the slope of the kinetic term with respect to the first derivative of the fields, which should be positive. If it is negative, it signals the existence of a ghost (quantum mechanical vacuum instability) and also that the squared speed of sound is negative (classical instability) [75]. Since the dilaton has the correct sign for the kinetic term in the Einstein frame, the only possible source of ghost is the tachyon Lagrangian. Therefore, one should only check the sign of $\frac{\mathrm{d}L}{\mathrm{d}\tilde{X}}$. We have that $\frac{\mathrm{d}L}{\mathrm{d}\tilde{X}} = \frac{\mathrm{d}L}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\tilde{X}} \sim \mathrm{e}^{\frac{\Phi}{2}}\frac{\mathrm{d}F}{\mathrm{d}X} = \frac{\mathrm{e}^{\frac{\Phi}{2}}}{\sqrt{X}}\int_0^{\sqrt{X}} \mathrm{e}^{s^2}\,\mathrm{d}s > 0$. So there is no ghost and therefore no violation of NEC in our model.

Now we turn to the issue of the bounce. In the spatially flat FRW spacetime we are considering, the Hubble parameter H_E in the Einstein frame is related to that in the string frame by $H_\mathrm{E} = H - \dot{\Phi}/4$. Since the null energy condition is not violated in our model, the Hubble parameter in the Einstein frame monotonously decreases, so that the bounce cannot arise in the Einstein frame. If the dilaton were frozen, it would be impossible to have a bounce in the string frame as well because the two frames would be trivially related (in particular $H_\mathrm{E} = H$). This is why the running dilaton is crucial for the string frame bounce. Now we make some remarks on the bounce scenario in the string frame. We assume that the bounce arises as an interpolation between two out of four different phases (i. e. contracting or expanding pre/post-big bang phases) in the pre-big bang scenario. Four transitions are then possible, namely from the contracting pre-big bang phase to expanding pre/post-big bang, or from contracting post-big bang to expanding post/pre-big bang. But the transition from pre-big bang to post-big bang cannot happen in our model because the NEC in the corresponding Einstein frame is not violated. To show this, we note that in the pre-big bang scenario with $S_\mathrm{T} = 0$ the solutions are

$$H = \frac{n}{t-t_0}\quad,\qquad \dot{\Phi} = \frac{9n-1}{2(t-t_0)}\quad\text{with}\quad n = \pm\frac{1}{3}\,. \tag{4.19}$$

Here, $t < t_0$ corresponds to the pre-big bang phase, and $t > t_0$ to the post-big bang phase. It then follows that $H_\mathrm{E} = \frac{1-n}{8(t-t_0)}$ is negative for pre-big bang solutions and positive for post-big bang solutions, meaning that pre/post-big bang solutions correspond to a contracting/expanding universe in the Einstein frame, regardless of whether the universe is contracting or expanding in the string frame. This means that a transition from pre-big bang to post-big bang in the string frame corresponds to a bounce in the Einstein frame, which is impossible unless the NEC is violated. Therefore, under the assumption mentioned above, the only possible string frame bounce scenarios in our model are the interpolations either between two pre-big bang phases or between two post-big bang phases.

This argument can be extended to the case where the asymptotic behavior of the solutions in the string frame is qualitatively, but not exactly, in agreement with the pre-big bang scenario, This is the case in our model, as we will see in the next section. In conclusion, if one is given a model that does not violate the NEC and if one knows the asymptotic boundary conditions of the solution in the string frame, one can then predict the possible bouncing scenarios in the string frame by looking at the corresponding Einstein frame. An example will be given in the next section.

4.3 Asymptotic analysis

In this section we will try to obtain approximate analytic solutions. This will provide us with the asymptotic boundary conditions for the numerical solutions in the next section. We emphasize here that by "asymptotic" we mean $t \to -\infty$, or t approaching a pole t_0, at which H diverges, as in the pre-big bang scenario. Similarly the present analysis applies to $t \to \infty$, or t approaching a pole t_0, at which H diverges as in a post-big bang scenario

To simplify our analysis, we will assume that in either of these limits, the tachyon behaves like dust, i. e. $p = w(t)\epsilon$ with $w(t) \to 0$. This is equivalent to claiming that $|\dot{T}| \to \infty$ asymptotically because $p \propto \epsilon/(\dot{T})^2$ when $|\dot{T}| \to \infty$ (see [103] for details). To justify this assumption, we look at the tachyon equation of motion following from (4.13). We have

$$\ddot{T} + \left(9H - \dot{\Phi}\right) D(\dot{T}) - T = 0\,, \tag{4.20}$$

where the function $D(y) = \mathrm{e}^{-y^2} \int_0^y \mathrm{e}^{s^2} \mathrm{d}s$ is known as the *Dawson integral*. This function is an odd function, and thus vanishes at $y = 0$. We will also use the fact that $D(y) = \frac{1}{2y} + \mathcal{O}(y^{-3})$ for $|y| \to \infty$. For $|t| \to \infty$, we will see that $(9H - \dot{\Phi})$ tends to zero. We can thus ignore the second term in the equation of motion[1] (4.20). Thus $\ddot{T} = T$ and T, as well as $\dot{T}$, will grow exponentially; and the pressure will thus vanish for $|t| \to \infty$. We emphasize that since the pressure vanishes *exponentially fast*, we can simply remove it from the asymptotic equations of motion because it will always be dominated by the other terms that will vanish with a power law.

[1]Note that we ignore the possibility $T \to 0$ in the infinite past or future; we are only interested in the cases where the D-brane is absent in these limits.

When t approaches a pole t_0 the analysis is slightly different because there the term $(9H-\dot{\Phi})$ diverges (unless $\dot{\Phi} \sim 9H$, but we will see that this does not happen on our numerical solution). We will further assume that this term diverges at least as fast as $\frac{1}{t-t_0}$. Assuming that $\dot{T}$ is finite at t_0 this then implies that either T or $\ddot{T}$ will diverge like $\frac{1}{t-t_0}$ or faster, in contradiction with a finite $\dot{T}$. We cannot exclude the case where $\dot{T}(t_0)$ vanishes in such a way that it precisely cancels the divergence in $(9H-\dot{\Phi})$. In that case the second term of the equation of motion could be regular at t_0, and thus T and $\ddot{T}$ could be regular there as well. We will nevertheless ignore this possibility because $\dot{T}(t_0)$ corresponds to very particular initial conditions. For generic initial conditions we will therefore have that $|\dot{T}| \to \infty$ when $t \to t_0$. This then justifies our claim that we can ignore the pressure for the asymptotic analysis. An immediate consequence of Eq. (4.10) is then that $\epsilon \sim a^{-9}$.

We now will proceed by analyzing the system of equations (4.7-4.10) assuming that either $\dot{\Phi} \propto H$, $|\dot{\Phi}| \ll |H|$ or $|\dot{\Phi}| \gg |H|$ asymptotically.

i) Let us first consider the possibility $\dot{\Phi} \propto H$. In that case (4.8) and (4.9) imply that either $|\dot{H}| \gg H^2$ or $\dot{H} \propto H^2$. In the first case we get $\dot{\Phi} \simeq 5H$ and then (4.7) implies that

$$-8H^2 = 2\kappa_{10}^2 e^{2\Phi}\epsilon \tag{4.21}$$

i.e. negative energy. We thus exclude that possibility. In the second case from (4.8-4.9) we obtain the solutions of the pre-big bang scenario, i. e. Eq. (4.19). For consistency, we must verify that these solutions satisfy the constraint (4.7). This is the case only when the energy density is subdominant compared to the other terms in this equation. By using (4.19), one can see that $e^{2\Phi}\epsilon$ goes like $\frac{1}{|t-t_0|}$ while the other terms behave like $\frac{1}{(t-t_0)^2}$, so the energy is subdominant as $t \to t_0$. Thus the solutions can be approximated to those of the pre-big bang scenario near the pole, t_0. At the same time we see that this possibility is excluded for $|t| \to \infty$.

ii) For $|\dot{\Phi}| \ll |H|$ Eqs. (4.8) and (4.9) imply $27H^2 + 5\dot{H} = 0$. On the other hand, setting $p = 0$ in Eq. (4.11) gives us $\dot{H} + 9H^2 = 0$, a clear contradiction. Thus, $|\dot{\Phi}| \ll |H|$ is excluded.

iii) We are thus left with the sole possibility $|\dot{\Phi}| \gg |H|$. In that case (4.7) implies

$$2\dot{\Phi}^2 = \kappa_{10}^2 e^{2\Phi}\epsilon, \tag{4.22}$$

and (4.8) together with (4.9) imply $-\ddot{\Phi} + \dot{\Phi}^2 = 0$, which gives $\Phi = -\log(|t - t_0|)$. With $p = 0$, Eq. (4.11) then implies $\dot{H} - 2H\dot{\Phi} = 0$, which gives $H = \frac{h}{(t-t_0)^2}$, where h is some constant. This is consistent with Eq. (4.22) only for $|t| \to \infty$ because $e^{2\Phi}\epsilon \sim \frac{1}{t^2}$ as $|t| \to \infty$.

To summarize, we find that the only consistent asymptotic solution for $|t| \to \infty$ is given by

$$\Phi \simeq -\log(|t|) , \qquad H \simeq \frac{h}{t^2}. \tag{4.23}$$

and (4.19) for $t \to t_0$. Note that $H_{\mathrm{E}} \simeq -\frac{\dot{\Phi}}{4}$ for (4.23).

Now we are in the position to predict the possible string frame bounce scenarios. Following the same logic as in the last part of the previous section concerning the NEC in the Einstein frame, one can show that the only possible bounce scenario is the transition either from pre-big bang-like solution[2] to pre-big bang solution or from post-big bang solution to post-big bang-like solution. This is because the other transitions correspond to a bounce in Einstein frame, which is excluded in our model which satisfies the NEC. To explain this in more detail, we consider the case where we start with contracting pre-big bang-like phase. For large negative times our bounce solution is in agreement with the pre-big bang-like solution with accelerated contraction of the universe and growing dilaton. Then the universe goes through a bounce and for $t \to t_0$ it approaches a pre-big bang solution with accelerated expansion and growing dilaton. The Hubble parameter in the Einstein frame remains negative and keeps decreasing. As will be seen in the next section, the numerical solutions are in good agreement with this picture.

4.4 Numerical results

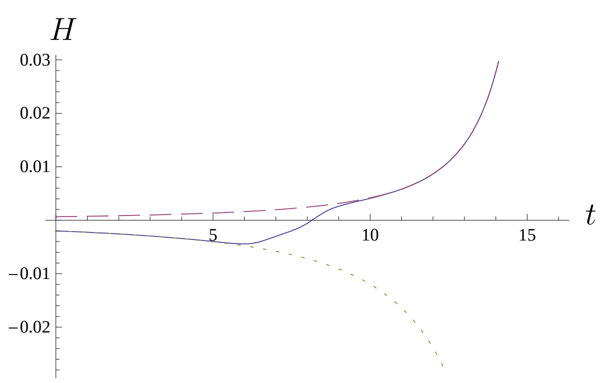
In this section, we numerically solve Eqs. (4.7-4.10) to obtain global solutions. In what follows we set $2\sqrt{2}\kappa_{10}^2\tau_9 = 1$ (this can always be achieved by adding a suitable constant to the dilaton), so that

$$2\kappa_{10}^2\epsilon = \mathrm{e}^{-\Phi}\mathrm{e}^{-T^2+\dot{T}^2} \tag{4.24}$$

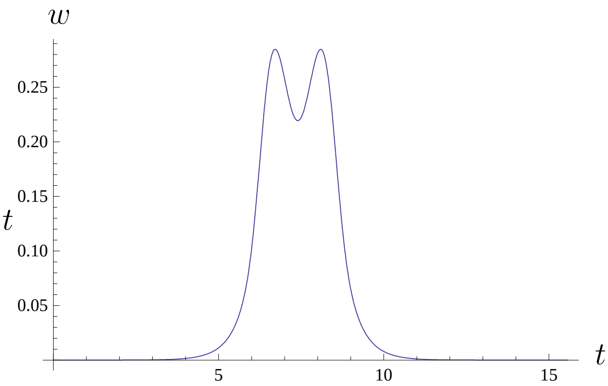
$$2\kappa_{10}^2\, p = -\mathrm{e}^{-\Phi}\mathrm{e}^{-T^2}\left(\mathrm{e}^{\dot{T}^2} - 2\sqrt{\dot{T}^2}\int_0^{\sqrt{\dot{T}^2}} \mathrm{e}^{s^2} ds\right). \tag{4.25}$$

With this setup, we performed the numerical analysis and found a family of bounce solutions. For example, Figs. 4.1 show a bounce solution with the initial conditions, $\Phi(0) = -5$, $\dot{\Phi}(0) = 0.05$, $T(0) = 1000$ and $H(0) = -0.002$. The graph in Fig. 4.1(c)a shows the evolution of the Hubble parameter (solid line). The bounce takes place near $t = 8$. The bouncing solution can be seen as a transition from the contracting pre-big bang-like phase (short dashed line) to the expanding one (long dashed line). Both asymptotic solutions are obtained by setting $p = 0$ in the equations of motions since the pressure is negligible in the far future and past (see Fig. 4.1(d)b). The 'double bump' feature of the equation of state can be understood by noting that as the non-BPS brane builds up $|\dot{T}|$ decreases and thus w increases from zero as explained in section 2. Then as T reaches the top of the potential $|\dot{T}|$ becomes small and consequently w decreases again. Indeed for $\dot{T} = 0$ the equation of state is that of a cosmological constant.

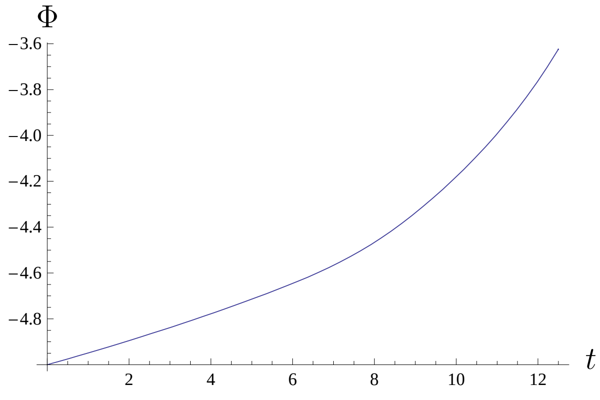
[2]Note that what we refer to as pre/post-big bang-like solution is given in Eq. (4.23) with negative/positive time



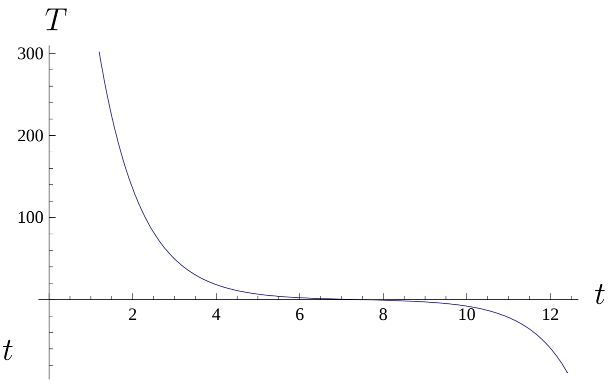
(c) Hubble parameter H (solid line)



(d) equation of state w



(e) dilaton Φ

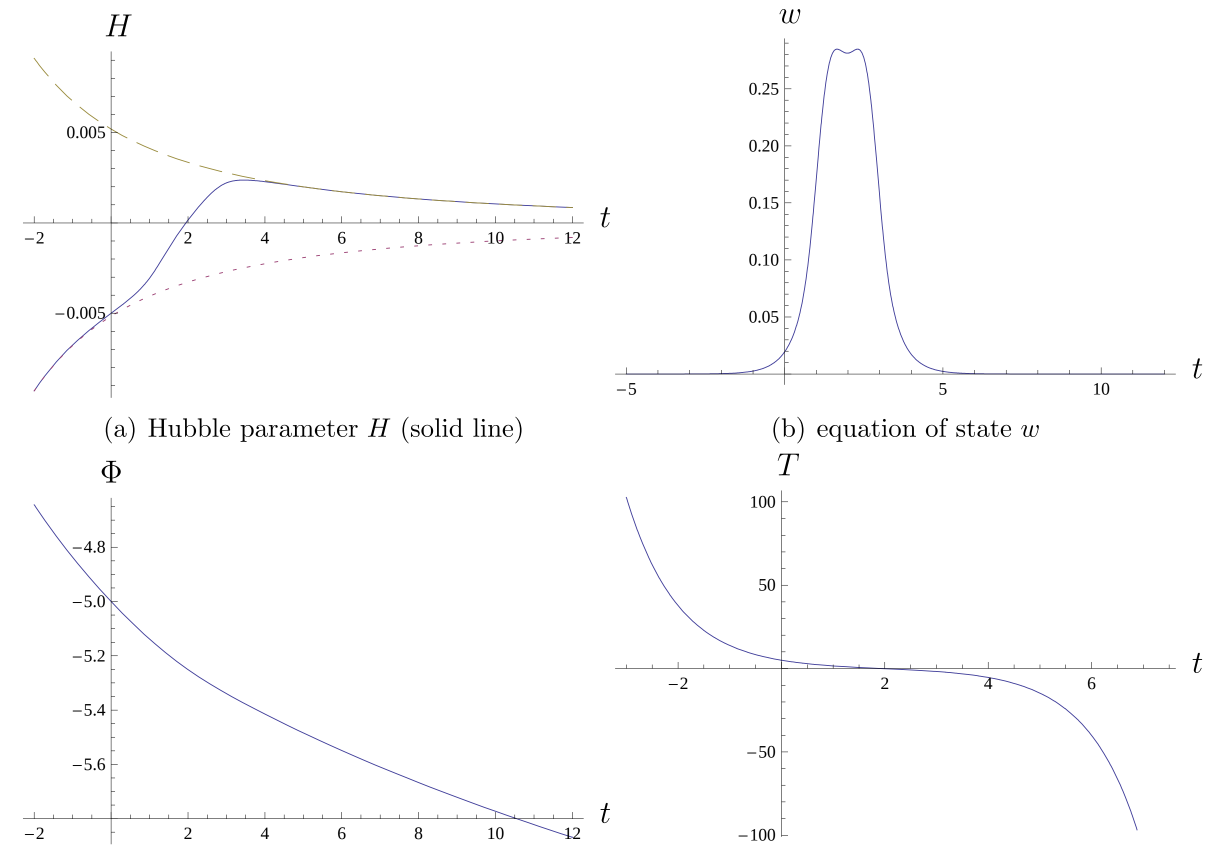


(f) tachyon T

Figure 4.1: Bouncing numerical solution with the initial conditions $\Phi(0) = -5$, $\dot{\Phi}(0) = 0.05$, $T(0) = 1000$ and $H(0) = -0.002$.

We found that a broad range of the initial conditions are allowed for the bounce, so there is no fine-tuning problem. For instance, $T(0) = \pm 10^4$ and $T(0) = \pm 100$ (keeping the same initial conditions as above for the other variables) gives bounce solutions with essentially the same behavior. Note that the large initial values for T do not represent a fine tuning. Rather it reflects the condition that the non-BPS brane is absent at very early times. We see that the asymptotic behavior of this family of solutions is similar to the expanding pre-big bang case, in which the Hubble parameter blows up. This agrees with the results of the previous section.

If one changes the sign of $\dot{\Phi}(0)$, a very different kind of bounce is obtained. For instance, Figs. 4.2 show a bounce solution with the initial conditions, $\Phi(0) = -5$, $\dot{\Phi}(0) = -0.15$, $T(0) = 5$ and $H(0) = -0.005$. The graph in Fig. 4.2(a) shows the evolution of the Hubble parameter H. The bounce takes place near $t = 2$, and the universe smoothly evolves to the standard cosmological regime, where the Hubble parameter and the dilaton decreases with time (Fig. 4.2(c)). But going back in time further, we found a singularity where the Hubble parameter blows up. This solution can be seen as a transition from the contracting



(a) Hubble parameter H (solid line)

(b) equation of state w

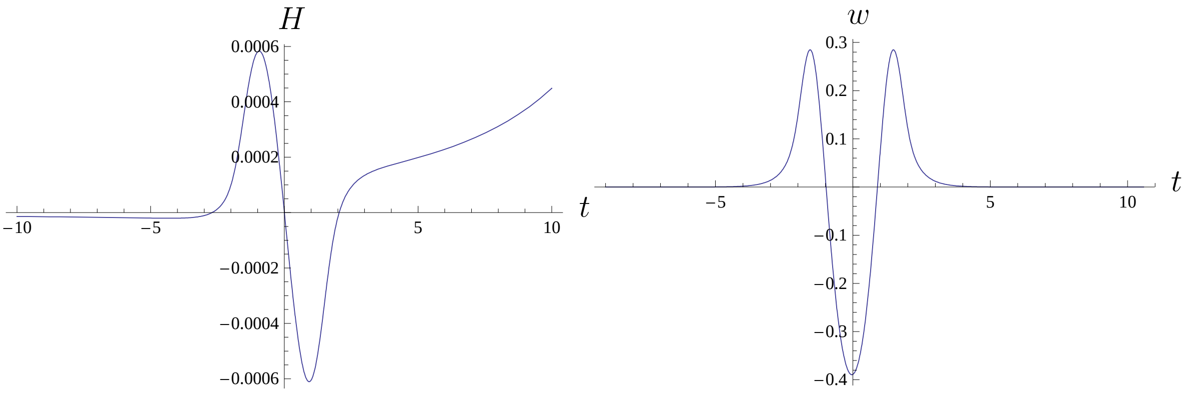
(c) dilaton Φ

(d) tachyon T

Figure 4.2: Bouncing, numerical solution with non-singular future using initial conditions $\Phi(0) = -5$, $\dot{\Phi}(0) = -0.15$, $T(0) = 5$ and $H(0) = -0.005$.

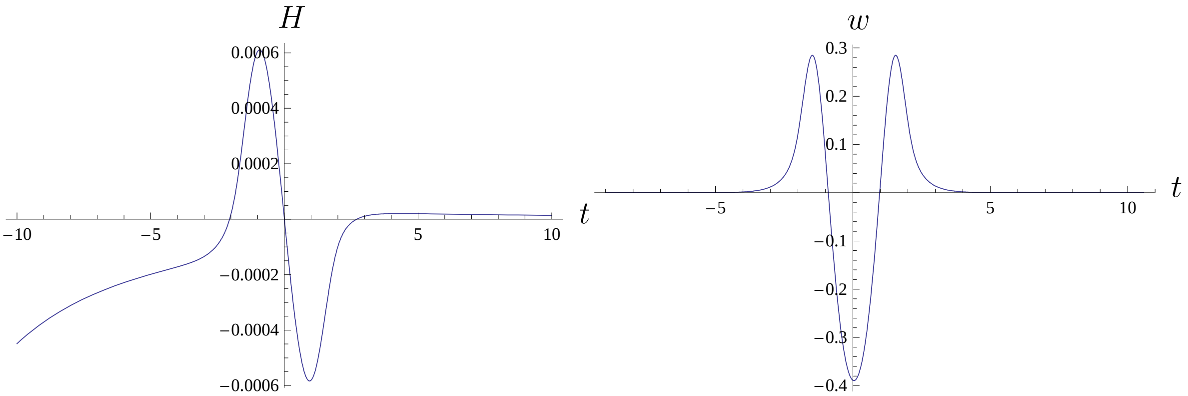
post-big bang phase (short dashed line) to the expanding post-big bang-like phase (long dashed line). Both asymptotic solutions are obtained by setting $p = 0$ in the equations of motion since the pressure is negligible in the far future and past (see Fig. 4.2(b)).

In addition, we found oscillatory solutions, in which a double bounce takes place (see Figs. 4.3). The solution in Figs. 4.4 can be seen as a time reflected one of the solution of Figs. 4.3. Which solution is obtained depends on the sign of $\dot{\Phi}(0)$. In both cases the asymptotic behaviors towards the curvature singularity are analogous to the non-oscillatory cases mentioned above. The evolutions of the equation of state are shown in Figs. 4.3(b) and 4.4(b). Note that the negative equation of state implies small $|\dot{T}|$ (see Fig. 4.1(b)), and this means that the oscillatory solutions can arise if the speed of the tachyon is small around the top of the tachyon potential (i. e. near $T = 0$). Alternatively this kind of solution can be obtained when we arrange $|\dot{\Phi}|$ to be small enough around the top of the tachyon potential (see Fig. 4.5). The variation of $\dot{\Phi}(0)$ with the other initial conditions fixed (in this case at $\Phi(0) = -5$, $H(0) = 0$, and $T(0) = 0$) shows that decreasing $\dot{\Phi}(0)$ gives rise to a transition from single bounce to cyclic bounce.



(a) Hubble parameter H (b) equation of state w

Figure 4.3: Oscillatory, numerical solution with initial conditions $\Phi(0) = -5$, $H(0) = 0$, $\dot{\Phi}(0) = 0.05$ and $T(0) = 0$.



(a) Hubble parameter H (b) equation of state w

Figure 4.4: Oscillatory, numerical solution with initial conditions $\Phi(0) = -5$, $H(0) = 0$, $\dot{\Phi}(0) = -0.05$ and $T(0) = 0$.

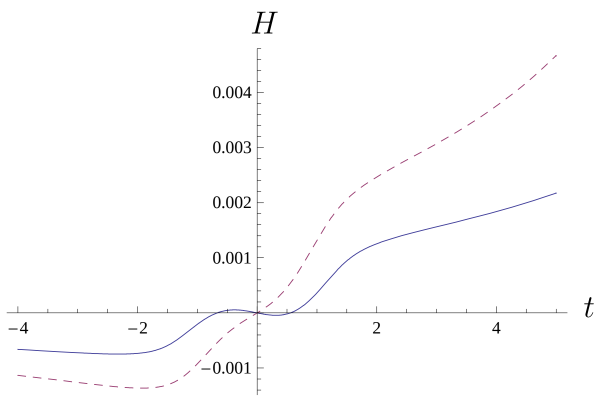
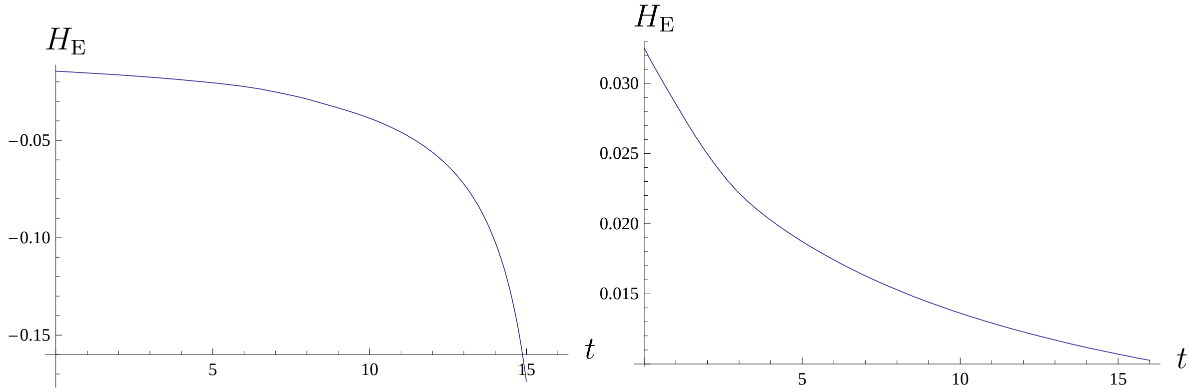


Figure 4.5: Hubble parameter H for $\dot{\Phi}(0) = 0.06$ (solid line) and $\dot{\Phi}(0) = 0.072$ (dashed line).



(a) Hubble parameter H_{E} in the Einstein frame corresponding to Fig. 4.1(c)

(b) Hubble parameter H_{E} in the Einstein frame corresponding to Fig. 4.2(a)

Figure 4.6: Hubble parameter in Einstein frame

In conclusion, we obtained bounce solutions, where the universe smoothly evolves from the contracting phase to the expansion. The bounce scenario that we here found can be classified into the following two cases. Note that these are exactly in agreement with our predictions on the bounce scenarios in the previous section:

1. Transition from accelerating contraction (the contracting pre-big bang-like phase) to accelerating expansion (the pre-big bang inflation): In this case the dilaton grows up, and if the speed of the tachyon (or dilaton) is small enough near the maximum of the tachyon potential, a double bounce can take place (Figs. 4.3) before the universe evolves to the pre-big bang phase.

2. Transition from decelerating contraction (the contracting post-big bang phase) to decelerating expansion (post-big bang like phase): In this case the dilaton decays, and a double bounce can also take place (Figs. 4.4) under the same condition mentioned above.

In all cases the tachyon rolls over the top of the potential in the course of its evolution, and the bounce seems to happen when the tachyon reaches around the top of the potential. The pressure is important only around the bounce, and negligible (dust) asymptotically.

Our string frame bounce solutions correspond to monotonously contracting (i. e. $H_{\mathrm{E}} = H - \frac{\dot{\Phi}}{4} < 0$) or expanding geometries ($H_{\mathrm{E}} > 0$) in the Einstein frame (see Figs. 4.6(a)-4.6(b)), meaning that there is no bounce in the Einstein frame for our solutions. This is the fact that our model does not violate the NEC in the Einstein frame.

The numerical results presented in this section verify the results on the asymptotics in the previous section. Interestingly, as far as the dilaton Φ is concerned the asymptotic solution obtained here agrees qualitatively with the global numerical solution. In other words the dynamics of the dilaton is not much affected by the presence of the non-BPS brane and is qualitatively similar to that of the pre-big bang scenario. Concerning $H(t)$ things are different: See Fig. 4.1(c) for example. For large negative times $H(t)$ is well described by the asymptotic solution described in Eq. (4.23). Then near the bounce which takes place at $t \simeq 0$ [3] the non-BPS brane affects $H(t)$ significantly. Then for $t \to t_0 > 0$ (near the pole), where the pre big-bang singularity occurs, the Hubble constant is well described by the solution in the pre-big bang scenario. This can be understood from the fact that the brane has already decayed for $t \to t_0$.

4.5 Nonsingular solutions – Example

As we have seen in the previous section, there is a singularity either in the future or in the past, depending on the sign of $\dot{\Phi}$, and we expect that this singularity may be resolved in the same way as in pre-big bang scenarios, e. g. relying on α' corrections or quantum loop corrections or alternatively using a dilaton potential (see [102] for some explicit examples). As a matter of fact, resolving this kind of asymptotic singularities is less difficult than the big-bang singularity. In this section we will give an example of resolution of this asymptotic singularity.

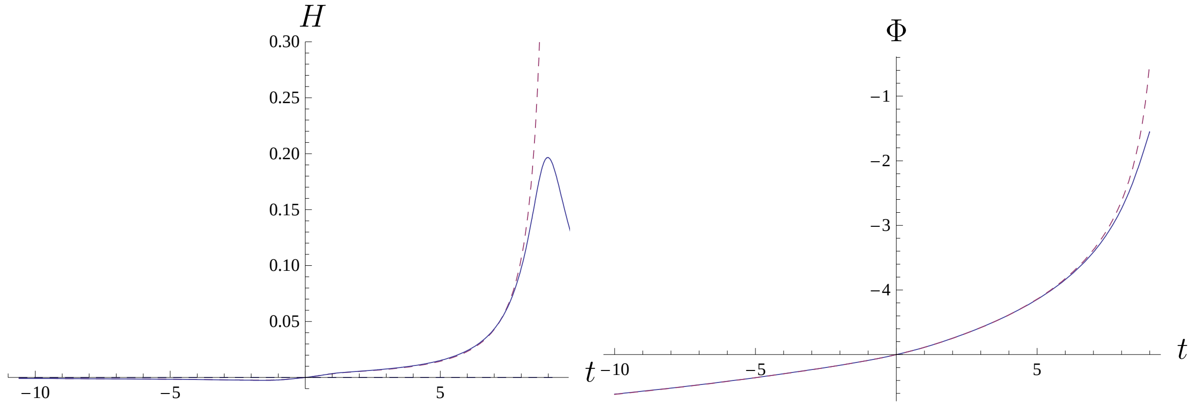
Near the singularity the curvature and the dilaton blow up, and this suggests that a potential term of the form $R\,\mathrm{e}^{\Phi}$ in the Lagrangian may smooth out the singularity. Here the coupling between the Ricci scalar and the dilaton is introduced because the singularity appears both in the curvature and dilaton. Such a term is quite likely to appear as α' correction in the open string sector, since it has the form of a tree level correction in the open string coupling constant and R is the natural invariant built from background metric derivatives[4].

Thus as an example, in which such an additional term may resolve the singularities, we study the dynamics of the system where the action (4.1) is supplemented by a potential

$$\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}\,x\sqrt{-g}\mathrm{e}^{-2\Phi}RV(\Phi). \tag{4.26}$$

[3]In fact one can always arrange the bounce to take place at $t = 0$ by shifting the time variable.

[4]A similar potential has been motivated in the context of string gas cosmology in [111] as a Casimir-type potential.



(a) Hubble parameter H (b) dilaton Φ

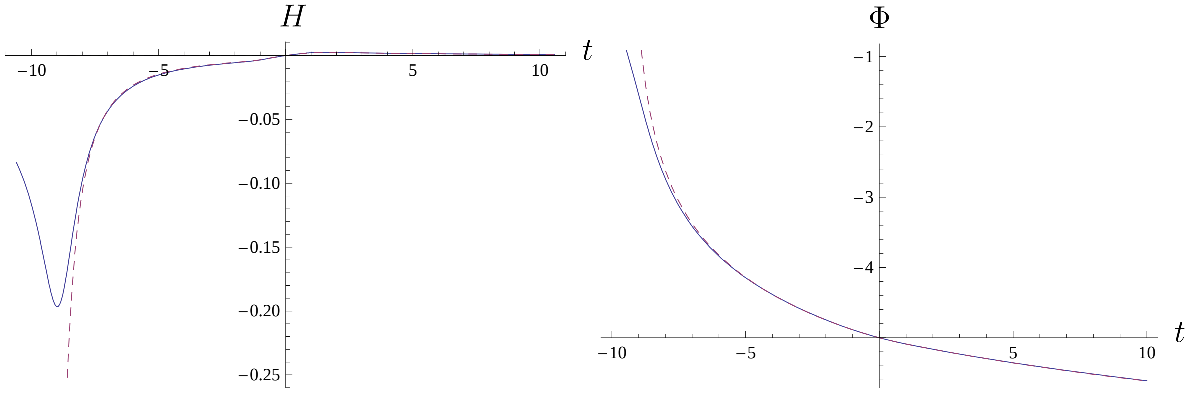
Figure 4.7: Numerical solution with growing dilaton (solid lines: with additional term, dashed lines: without additional term).

The equations of motion then take the form

$$
\begin{aligned}
72H^2\left(1+V(\Phi)\right)+4\dot{\Phi}^2-36H\dot{\Phi}\left(1+V(\Phi)-\frac{V'(\Phi)}{2}\right)-2\kappa_{10}^2\mathrm{e}^{2\Phi}\epsilon &= 0 \quad (4.27)\\
2\ddot{\Phi}-2\dot{\Phi}^2+18H\dot{\Phi}-\left(9\dot{H}+45H^2\right)\left(1+V(\Phi)-\frac{V'(\Phi)}{2}\right)-\kappa_{10}^2\mathrm{e}^{2\Phi}p/2 &= 0 \quad (4.28)\\
(\ddot{\Phi}+8H\dot{\Phi})\left(V'(\Phi)-2V(\Phi)-2\right)+\dot{\Phi}^2\left(V''(\Phi)-4V'(\Phi)+4V(\Phi)+2\right)+ &\\
\left(1+V(\Phi)\right)(36H^2+8\dot{H})+\kappa_{10}^2\mathrm{e}^{2\Phi}p &= 0 \quad (4.29)\\
\dot{\epsilon}+9H(\epsilon+p)-\dot{\Phi}p &= 0 \quad (4.30)
\end{aligned}
$$

We require that the additional term should not spoil the bounce, namely this term is important only when the curvature becomes very big. For concreteness we choose $V = -\mathrm{e}^{\Phi+5}/40$. Using the same setup as in the previous section, we perform the numerical analysis.

First, let us consider the case in which the dilaton grows; in this case we faced a future singularity. We found that the addition of a potential (4.26) can resolve the future curvature singularity. This is shown in Figs. 4.7, where the dashed (solid) curve corresponds to the case without (with) the additional term. Here the initial conditions are chosen such that the bounce takes place at $t = 0$ (in other words we impose the initial conditions at the bounce and extrapolate in both directions in time). In the case without the additional term, there is a singularity, while in the other case the universe evolves to the standard cosmological regime where the Hubble parameter decreases. As can be seen from the plot, the dynamics are almost the same in both cases before the Hubble parameter gets significantly big, so that the bounce is not spoiled. Once the Hubble parameter is large enough, the additional term smooths out the singularity.



(a) Hubble parameter H

(b) dilaton Φ

Figure 4.8: Numerical solution with decreasing dilaton (solid lines: with additional term, dashed lines: without additional term).

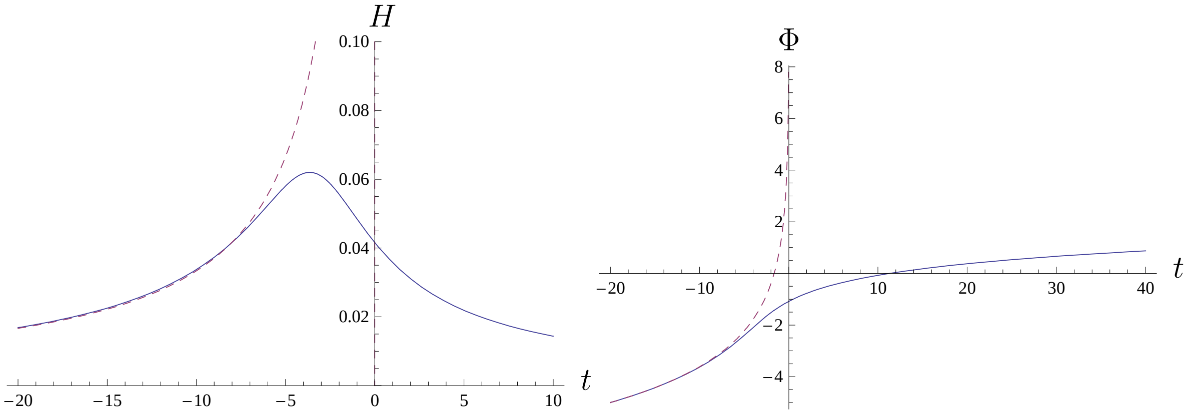
Now we turn to the case in which the dilaton decreases. In the previous section we have seen that in this case there is a past singularity. With the help of the additional term mentioned above, we found that this singularity can be resolved as well. This is shown in Figs. 4.8. The mechanism of resolving the singularity is analogous to the case of growing dilaton that we have seen above. What is interesting is that this solution corresponds to the time reflected version of the case of growing dilaton since $H \mapsto -H$ and $\dot{\Phi} \mapsto -\dot{\Phi}$ under the time reflection, $t \mapsto -t$.

In both cases, the dilaton dynamics still has a singularity either in the future or in the past. This is in contrast to the case where the tachyon sector is absent (i. e. $\epsilon = 0$ and $p = 0$). (see Figs. 4.9, where the pre-big bang singularity is resolved not only in H, but also in Φ). Since the singular behavior of our solutions is the same as in the pre-big bang scenario, we expect that in principle the dilaton singularity can be resolved as in Figs. 4.9, but this may require fine-tuning.

To sum up, we have shown an explicit example in which an additional term that might arise from higher order corrections can resolve the curvature singularity without affecting the bounce dynamics, though the singularity in dilaton has not been resolved.

4.6 Discussion

We suggested bounce scenarios, in which a non-BPS space-filling D9-brane in type IIA superstring theory drives a bounce of the scale factor in the string frame. We employed the lowest order effective action for the metric and dilaton in the string frame as well as an effective action for the open tachyonic mode of the non-BPS D-brane. The positivity of the pressure of the tachyon field is responsible for the bounce, which is why the DBI



(a) Hubble parameter H (b) dilaton Φ

Figure 4.9: Numerical solution with tachyon sector absent (solid lines: with the potential given in (4.26), dashed lines: pre-big bang solution).

action, for instance, can not drive the bounce. The curvature as well as the time derivative of the dilaton remain small during the bounce. In other words, the gravitational sector is entirely classical.

Asymptotically our bounce solutions look like pre-big bang or post-big bang solutions, with singular behavior of the curvature and the dilaton. The asymptotic string frame curvature singularity can be resolved by adding a phenomenological potential, $\propto R\mathrm{e}^{-\Phi}$, which may or may not result from α' corrections in the open string sector. It would be desirable to determine the sign and the precise numerical value of the proportionality coefficient. With our choice of the sign the gravitational coupling changes sign in the string frame. This results in a bounce in the Einstein frame at some time after the bounce has taken place in the string frame without violating the null energy condition. An interesting observation is that while our phenomenological potential stabilizes the dilaton within the perturbative regime it fails to do so once the tachyonic sector is included. An obvious question is then whether a modified potential exists which stabilizes the dilaton in our model, and if so, whether it can be derived from string theory. We should also mention that throughout this paper we assumed the isotropy in 9-dimensional space (modulo compactification, the details of which did not play a role here) during the era of the bounce. For phenomenological reasons it may be preferable to consider scenarios with a different dynamics for the scale factor of the internal dimensions. In particular, orbifold compactifications are interesting since they are accompanied by a reduction of supersymmetry. A preliminary analysis shows that for a T^6/Z_2 orbifold the asymptotic solutions in the far past are not modified qualitatively. The numerical evolution of the global solution requires more work, however. We hope to report on this issue in a future publication.

A Asymptotically stable solutions

The standard analysis of the k-essence trackers (e.g. [18]) involves several simplifying but restrictive assumptions concerning the behavior of the solutions. A wider range of k-essence models will be obtained if some of these assumptions are lifted. Let us therefore characterize the desired features of the cosmological evolution of k-essence in a general manner.

Scenarios of k-essence are based on the assumption that the field ϕ has an almost constant equation of state parameter (w_ϕ) during a cosmologically long epoch while another matter component dominates the energy density of the universe. Eventually, the k-essence itself becomes dominant and plays the role of "dark energy," again with an approximately constant w_ϕ. It is important that the solution curves serve as attractors for all neighbor solutions. In that case, the value of w_ϕ at late times is essentially independent of the initial conditions.

When the radiation-dominated epoch gives way to the epoch of dust domination, the behavior of k-essence will change in a model-dependent way. However, it is technically convenient to study the behavior of k-essence under the assumption that the dominant matter component has a fixed equation of state for all time. Then the existence of tracker solutions will be found by studying the asymptotic behavior of the solutions at $t \to \infty$ (equivalently, at $\phi \to \infty$). This is the approach taken here.

The evolution of k-essence together with a single matter component is described by the equations of motion (EOM) shown above as Eqs. (2.24)–(2.25) in terms of the variables $\{v(\phi), \varepsilon_m(\phi)\}$. We call a solution $\{v(\phi), \varepsilon_m(\phi)\}$ *asymptotically stable* if $w_\phi(\phi)$ tends to a constant at $\phi \to \infty$ and if *all* neighbor solutions (at least within a finite domain of attraction) also approach the same value of w_ϕ. In this section, we restrict our attention to asymptotically stable solutions with one matter component. Since reasonable values of w_ϕ are within the interval $[-1, 1]$, it is justifiable to ignore solutions where w_ϕ tends to infinity at late times. In principle, one could also have solutions where $w_\phi(\phi)$ oscillates without reaching any limit as $\phi \to \infty$, but such solutions are of little physical interest since the value of w_ϕ at the end of a given cosmological epoch will then be largely unpredictable. Since in the models under consideration w_ϕ is a function of v only, solutions $v(\phi)$ that oscillate without reaching any limit are also excluded. Applicability of the effective field theory requires that the derivatives of ϕ remain bounded; thus $v = \dot{\phi}$ cannot diverge to infinity as $\phi \to \infty$ and must also tend to a constant value, $v(\phi) \to v_0 < \infty$.

There may also exist solutions with initially negligible but growing ratio $\varepsilon_\phi/\varepsilon_m$. Such solutions may have a stable behavior with an almost constant w_ϕ for a finite (but very long) time, until the energy density of k-essence starts to dominate. We do not consider such "transient attractors" in the present paper.

Our main task is to deduce the possible k-essence Lagrangians $p(X, \phi)$ that admit physically meaningful asymptotically stable solutions. We consider only Lagrangians that have a factorized form (2.16).[1] We assume that the matter component has a constant equation of state parameter w_m such that $w_m \neq -1$.

It will be convenient to use also the auxiliary variable $R(\phi)$ satisfying the EOM (2.32). Since the values of R are limited to the interval $[0, 1]$, any asymptotically stable solution will necessarily approach a constant value, $R(\phi) \to R_0$ as $\phi \to \infty$. The possible values of R_0 and v_0 are yet to be determined; the cases when R_0 or v_0 assume critical values ($R_0 = 0$, $R_0 = 1$, $v_0 = 0$) will need to be treated separately.

The general method of analysis is the following. We have a system of nonlinear EOM parameterized by a pair of functions $K(\phi), Q(v)$; the general solution of the EOM is not available in closed form. Our purpose is to determine the functions $K(\phi), Q(v)$ for which a solution of the EOM exists with the asymptotic stability property. We first *assume* the existence of an asymptotically stable solution $\{v(\phi), \varepsilon_m(\phi)\}$ and derive the *necessary* conditions on the functions $K(\phi)$ and $Q(v)$ that admit such solutions (perhaps in more convenient variables, such as $\{v(\phi), R(\phi)\}$). At this step, there will be many cases corresponding to different asymptotic behavior of $v(\phi)$ and $R(\phi)$. For instance, $v(\phi)$ may tend either to a nonzero constant or to zero, etc. In each case, we then obtain the *general* solution of the EOM (with two integration constants) near the assumed stable solution (e.g. $v(\phi) = v_0 - A(\phi)$, with $A(\phi)$ very small). At this point, it is possible to make simplifying assumptions because we only consider the solutions in the asymptotic limit $\phi \to \infty$ and infinitesimally close to an assumed trajectory. We then investigate whether the general solution is attracted to the assumed stable solution. In this way, we either obtain *sufficient* conditions for the existence of a stable solution of an assumed type, or conclude that no stable solution exists in a given case. After enumerating all the cases, we will thus obtain necessary and sufficient conditions on $K(\phi)$ and $Q(v)$ for every possible type of stable tracking behavior.

Let us begin by drawing some general consequences about the asymptotic behavior of stable solutions at $\phi \to \infty$. Rewriting Eq. (2.25) as

$$\frac{d\varepsilon_m^{-1/2}(\phi)}{d\phi} = \frac{3\kappa\,(1 + w_m)}{2v(\phi)\sqrt{R(\phi)}}, \tag{A.1}$$

[1] Since the analysis uses only the properties of the Lagrangian in the asymptotic limit $\phi \to \infty$, our results will apply to more general Lagrangians that have the form $p = K(\phi)L(X)$ asymptotically at large ϕ and fixed X.

and noting that the right-hand side of Eq. (A.1) is bounded away from zero, we conclude that $\varepsilon_m(\phi)$ decays either as ϕ^{-2} or faster at $\phi \to \infty$, depending on whether $v(\phi)\sqrt{R(\phi)}$ tends to zero at large ϕ. In the following subsections, we consider all the possible cases.

Based on the motivation for introducing k-essence, we have assumed that $w_m \neq -1$. According to Eq. (A.1), for $w_m < -1$ (phantom matter) the energy density ε_m will satisfy the differential inequality

$$\frac{d}{dt}\varepsilon_m^{-1/2} = -\frac{3\kappa\,|1+w_m|}{2\sqrt{R(\phi)}} < -C_1, \tag{A.2}$$

where C_1 is a positive constant. Thus, $\varepsilon_m(t)$ will reach infinity in finite time regardless of the behavior of $R(\phi)$ and $v(\phi)$. However, this time can be quite long and the phantom behavior might be only a temporary phenomenon. Therefore, we will use the property $w_m \neq -1$ but avoid assuming that $w_m > -1$.

In the analysis below, we will also use the following elementary facts:

a) If a function $F(\phi)$ monotonically goes to a constant at $\phi \to \infty$, then $F'(\phi)$ decays faster than ϕ^{-1}. This is easily established using the identity

$$F(0) - \lim_{\phi\to\infty} F(\phi) = -\int_0^\infty F'(\phi)d\phi < \infty, \tag{A.3}$$

which means that $F'(\phi)$ is integrable at $\phi \to \infty$. Hence, $F'(\phi)$ decays faster than ϕ^{-1} at $\phi \to \infty$.

b) If a function $F(\phi)$ is monotonic, then $F'(\phi) \to 0$ if and only if

$$\lim_{\phi\to\infty} \frac{F(\phi)}{\phi} = 0. \tag{A.4}$$

This statement follows from the L'Hopital's rule in case $F(\phi) \to \infty$, and is trivial in case $F(\phi)$ has a finite limit at $\phi \to \infty$.

A.1 Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 \neq 1$, main case

According to Eq. (A.1), the asymptotic behavior $\varepsilon_m \propto \phi^{-2}$ is possible only if $v(\phi)\sqrt{R(\phi)}$ stays bounded away from zero as $\phi \to \infty$, in other words if $v_0 \neq 0$ and $R_0 \neq 0$. We also assume $R_0 \neq 1$, meaning that the energy density of k-essence tracks the matter component; thus $\varepsilon_\phi(\phi) \propto \phi^{-2}$ as well. It follows that $H(\phi) \propto \sqrt{\varepsilon_\phi(\phi)} \propto \phi^{-1}$, and then Eq. (2.27) yields

$$\frac{dR(\phi)}{d\phi} \propto \frac{w_m - w_\phi(\phi)}{\phi}. \tag{A.5}$$

Since $R(\phi) \to \text{const}$, the derivative $dR/d\phi$ must decay faster than ϕ^{-1} as $\phi \to \infty$. Hence $w_\phi(\phi) \to w_m$ as $\phi \to \infty$. This is the standard tracker behavior: the equation of state parameters of k-essence and matter become almost equal at late times.

By assumption, at large ϕ the Lagrangian is factorized, $p = K(\phi)Q(v)$, and then we have

$$w_m = w_\phi(v_0) = \frac{Q(v_0)}{v_0 Q'(v_0) - Q(v_0)}. \tag{A.6}$$

This algebraic equation determines the possible values of v_0 for a given w_m. (Tracker solutions of this type are impossible if this equation has no roots.) The property $\varepsilon_\phi(\phi) \propto \phi^{-2}$ becomes

$$\varepsilon_\phi(\phi) = K(\phi)\left(vQ' - Q\right) \propto \phi^{-2}. \tag{A.7}$$

Generically one expects

$$\tilde{\varepsilon}_\phi(v_0) \equiv v_0 Q'(v_0) - Q(v_0) \neq 0, \tag{A.8}$$

and we temporarily make this additional assumption. Then we obtain

$$K(\phi) \propto \phi^{-2} \, \text{as} \, \phi \to \infty. \tag{A.9}$$

This is somewhat more general than the function $K(\phi) = \text{const} \cdot \phi^{-2}$ usually considered in k-essence models.

We may consider Lagrangians $p = K(\phi)Q(v)$ with the function $K(\phi)$ of the form

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \to \infty} K_0(\phi) = 0. \tag{A.10}$$

Let us now derive a sharp condition for the existence of an asymptotically stable solution $\{v(\phi), R(\phi)\}$ in this case. We use the ansatz

$$v(\phi) = v_0 - A(\phi), \quad R(\phi) = R_0 - B(\phi), \tag{A.11}$$

where by assumption the unknown functions $A(\phi), B(\phi)$ tend to zero at $\phi \to \infty$. After deriving and solving the equations for $A(\phi)$ and $B(\phi)$, we will need to verify this assumption.

Since the left-hand side of Eq. (2.31) is $-A'$, it tends to zero faster than ϕ^{-1}. On the other hand, assuming that $c_s^2(v_0) \neq 0$, we find that the right-hand side of Eq. (2.31) contains leading terms of order ϕ^{-1}, such as $(\ln K)_{,\phi}$ and $\sqrt{K}$. Hence, these terms must cancel, which entails

$$3\kappa \sqrt{\frac{\tilde{\varepsilon}_\phi(v_0)}{1 - R_0}} = \frac{2v_0}{1 + w_\phi(v_0)} = \frac{2v_0}{1 + w_m} = \frac{2\tilde{\varepsilon}_\phi(v_0)}{Q'(v_0)}. \tag{A.12}$$

Since v_0 is determined from Eq. (A.6), this condition fixes the value of R_0,

$$R_0 = 1 - \frac{9\kappa^2}{4}\frac{Q'^2(v_0)}{v_0 Q'(v_0) - Q(v_0)}. \tag{A.13}$$

The requirement that the values of R_0 be between 0 and 1 further restricts the possible functions $Q(v)$. Using Eq. (A.12), the condition $R_0 > 0$ can be expressed equivalently as

$$Q(v_0) < \frac{4}{9\kappa^2} v_0^2 \frac{w_m}{(1+w_m)^2}. \tag{A.14}$$

No tracker solution is possible if this condition is violated.

The equations for $A(\phi)$ and $B(\phi)$ are now found by linearizing the equations (2.31)–(2.32). For brevity, we rewrite these equations as

$$\frac{dv}{d\ln\phi} = -\Lambda_1(v)\frac{d\ln K}{d\ln\phi} - \Lambda_2(v)\sqrt{\frac{1+K_0(\phi)}{1-R}}, \tag{A.15}$$

$$\frac{dR}{d\ln\phi} = -\Lambda_3(v,R)\sqrt{1+K_0(\phi)}\left(w_m - w_\phi(v)\right), \tag{A.16}$$

where the auxiliary functions $\Lambda_1, \Lambda_2, \Lambda_3$ are defined by

$$\Lambda_1(v) \equiv \frac{c_s^2(v) v}{1+w_\phi(v)} = \frac{\tilde{\varepsilon}_\phi}{vQ''(v)}, \tag{A.17}$$

$$\Lambda_2(v) \equiv 3\kappa c_s^2(v)\sqrt{\tilde{\varepsilon}_\phi(v)}, \tag{A.18}$$

$$\Lambda_3(v,R) \equiv \frac{3\kappa}{v} R\sqrt{1-R}\sqrt{\tilde{\varepsilon}_\phi(v)} = \frac{R\sqrt{1-R}}{vc_s^2(v)}\Lambda_2(v). \tag{A.19}$$

Note that Eq. (A.12) is equivalent to

$$2\Lambda_1(v_0) = \frac{\Lambda_2(v_0)}{\sqrt{1-R_0}}. \tag{A.20}$$

Substituting the ansatz (A.11) into Eqs. (A.15)–(A.16), using the identity (A.20), and keeping only the leading linear terms, we find

$$\frac{dA}{d\ln\phi} = \left(\phi K_0' + K_0\right)\Lambda_1(v_0) - \alpha_0 A - \frac{\Lambda_1(v_0)B}{1-R_0}, \tag{A.21}$$

$$\frac{dB}{d\ln\phi} = \Lambda_3(v_0, R_0) w_\phi'(v_0) A, \tag{A.22}$$

where we defined the auxiliary constant α_0 by

$$\alpha_0 \equiv \frac{\Lambda_2'(v_0)}{\sqrt{1-R_0}} - 2\Lambda_1'(v_0) = \frac{1-w_m}{1+w_m}. \tag{A.23}$$

For the moment, we assume additionally that

$$w_\phi'(v_0) \equiv \left(\frac{Q}{vQ' - Q}\right)'_{v=v_0} = \left(1 - \frac{w_m}{c_s^2(v_0)}\right)\frac{1+w_m}{v_0} \neq 0. \tag{A.24}$$

Differentiating Eq. (A.22) with respect to $\ln\phi$ and substituting into Eq. (A.21), we find a closed second-order equation for $A(\phi)$,

$$\frac{d^2 B}{d\left(\ln\phi\right)^2} + \alpha_0 \frac{dB}{d\ln\phi} + \beta_0 B = \gamma_0 \left[\frac{dK_0}{d\ln\phi} + K_0\right], \tag{A.25}$$

where the constant coefficients β_0, γ_0 are defined by

$$\begin{aligned}\gamma_0 &\equiv \Lambda_1(v_0)\Lambda_3(v_0, R_0) w_\phi'(v_0)\\ &= 2\frac{c_s^2(v_0) - w_m}{1+w_m} w_m^2 R_0 \left(1 - R_0\right), \end{aligned}\tag{A.26}$$

$$\beta_0 \equiv \frac{\gamma_0}{1-R_0} = 2\frac{c_s^2(v_0) - w_m}{1+w_m} w_m^2 R_0. \tag{A.27}$$

The general solution of Eq. (A.25) is the sum of an inhomogeneous solution and the general solution of the homogeneous equation. Homogeneous solutions are stable if both roots $\lambda_{1,2}$ of the characteristic equation

$$\lambda^2 + \alpha_0 \lambda + \beta_0 = 0 \tag{A.28}$$

have negative real parts,

$$\mathrm{Re}\left(\lambda_1\right) < 0, \quad \mathrm{Re}\left(\lambda_2\right) < 0. \tag{A.29}$$

This will be the case if

$$\alpha_0 > 0, \quad \beta_0 > 0, \tag{A.30}$$

which is equivalent to the conditions

$$|w_m| < 1, \quad c_s^2(v_0) > w_m. \tag{A.31}$$

An inhomogeneous solution of Eq. (A.25) can be expressed as

$$B(\phi) = B_1(\phi)\phi^{\lambda_1} + B_2(\phi)\phi^{\lambda_2}, \tag{A.32}$$

$$B_1(\phi) \equiv \frac{\gamma_0}{\lambda_1 - \lambda_2}\int^\phi \phi^{-\lambda_1 - 1}\left(\phi K_0' + K_0\right) d\phi, \tag{A.33}$$

$$B_2(\phi) \equiv \frac{\gamma_0}{\lambda_2 - \lambda_1}\int^\phi \phi^{-\lambda_2 - 1}\left(\phi K_0' + K_0\right) d\phi. \tag{A.34}$$

Since the function $K_0(\phi)$ tends to zero at $\phi \to \infty$ by assumption, the inhomogeneous solution also tends to zero at $\phi \to \infty$ as long as the condition (A.29) holds. This is straightforward to show by assuming an upper bound

$$|\phi K_0' + K_0| < M \text{ for all } \phi > \phi_M, \tag{A.35}$$

where ϕ_M can be chosen for any $M > 0$. Then the inhomogeneous solution $B(\phi)$ is bounded for $\phi > \phi_M$ by

$$|B(\phi)| < \text{const} \cdot M + \text{const} \cdot \phi^{\lambda_1} + \text{const} \cdot \phi^{\lambda_2}, \tag{A.36}$$

which means that $B(\phi) \to 0$ at $\phi \to \infty$.

Under the same assumptions, the function $A(\phi)$ will have the same behavior at $\phi \to \infty$. We conclude that asymptotically stable solutions $\{v(\phi), R(\phi)\}$ approaching $\{v_0, R_0\}$ exist under the assumption $c_s^2(v_0) \neq 0$ and the further conditions (A.6), (A.8), (A.12), (A.24), and (A.31).[2] These conditions are similar to those derived in Ref. [18] under a more restrictive assumption $K(\phi) = \text{const} \cdot \phi^{-2}$. Let us now investigate whether these assumptions can be relaxed further.

A.2 Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 \neq 1$, marginal cases

The last assumption used in the derivation of the stability condition (A.31) was Eq. (A.24). If $c_s^2(v_0) = w_m$ while all the other assumptions hold, we have $w'_\phi(v_0) = 0$ and the equation (A.22) for $B(\phi)$ is modified. We may then rewrite Eqs. (A.21)–(A.22) as

$$\frac{dA}{d\ln\phi} = (\phi K_0' + K_0)\,\Lambda_1(v_0) - \alpha_0 A - \frac{\Lambda_1(v_0)B}{1-R_0}, \tag{A.37}$$

$$\frac{dB}{d\ln\phi} = O(A^2). \tag{A.38}$$

Differentiating the first equation with respect to $\ln\phi$, we obtain

$$\frac{d^2A}{d\,(\ln\phi)^2} = \phi\,(\phi K_0)''\,\Lambda_1(v_0) - \alpha_0\frac{dA}{d\ln\phi} + O(A^2). \tag{A.39}$$

The second-order terms $O(A^2)$ can be disregarded for the stability analysis. Since the characteristic equation

$$\lambda^2 + \alpha_0\lambda = 0 \tag{A.40}$$

has a zero root, the general solution $\{A(\phi), B(\phi)\}$ will not tend to zero at $\phi \to \infty$. Hence, no asymptotically stable solutions exist when the condition (A.31) is violated.

Another assumption, $c_s^2(v_0) \neq 0$, was used to derive Eq. (A.12) that determines the allowed value of R_0. Let us briefly consider the possibility $c_s^2(v_0) = 0$. (We note that $v \neq v_0$ on actual trajectories, so stability will hold as long as the trajectories $v(\phi)$ do not reach the regime $c_s^2(v) \leq 0$.) If

$$c_s^2(v_0) = \frac{Q'(v_0)}{v_0 Q''(v_0)} = 0, \tag{A.41}$$

[2] This is case 1 in Sec. 2.4.1.

then $Q'(v_0) = 0$ and the asymptotic equation of state is

$$w_\phi(v_0) = \frac{Q(v_0)}{v_0 Q'(v_0) - Q(v_0)} = -1 \tag{A.42}$$

as long as $Q(v_0) \neq 0$. However, we assumed a matter component with $w_m \neq -1$, and so we discard the possibility that $Q(v_0) \neq 0$. If, on the other hand, $Q(v_0) = 0$, then we must also have $\tilde{\varepsilon}_\phi(v_0) = 0$. Thus $c_s^2(v_0) \neq 0$ is justified given that $\tilde{\varepsilon}_\phi(v_0) \neq 0$.

Relaxing the assumption $\tilde{\varepsilon}_\phi(v_0) \neq 0$ requires some more work. If $\tilde{\varepsilon}_\phi(v_0) = 0$, then we cannot conclude that $K(\phi) \propto \phi^{-2}$ at $\phi \to \infty$; the function $K(\phi)$ remains undetermined even though we know that $\varepsilon_\phi(\phi) = K(\phi)\tilde{\varepsilon}_\phi(v) \propto \phi^{-2}$. The analysis after Eq. (A.6) needs to be modified as follows. The finiteness of w_ϕ,

$$w_\phi(v_0) = \lim_{v \to v_0} \frac{Q(v)}{\tilde{\varepsilon}_\phi(v)} < \infty, \tag{A.43}$$

requires that $Q(v_0) = 0$ and thus (since $v_0 \neq 0$) also $Q'(v_0) = 0$. In general, we may suppose that $Q(v)$ has an expansion

$$Q(v) = \frac{Q_0}{n v_0} (v - v_0)^n \left[1 + O(v - v_0)\right], \tag{A.44}$$

where Q_0 is a nonzero constant and $n \geq 2$. In this case we have the expansions

$$\tilde{\varepsilon}_\phi(v) = Q_0 (v - v_0)^{n-1} \left[1 + O(v - v_0)\right], \tag{A.45}$$

$$w_\phi(v) = \frac{v - v_0}{n v_0} \left[1 + O(v - v_0)\right], \tag{A.46}$$

$$c_s^2(v) = \frac{v - v_0}{(n-1)\, v_0} \left[1 + O(v - v_0)\right]. \tag{A.47}$$

It follows that $w_\phi(v_0) = 0$, so the only possibility for tracking is $w_m = 0$. Also, the only admissible solutions are those with $v(\phi) > v_0$, meaning that $A(\phi) < 0$ and $Q_0 > 0$. Let us now perform a stability analysis of these solutions. Substituting the ansatz (A.11) into Eqs. (2.31)–(2.32) and keeping only the leading terms in the perturbation variables $A(\phi)$ and $B(\phi)$, we obtain

$$\frac{dA}{d\phi} = \frac{(-A)}{n-1} \frac{K'}{K} + \frac{3\kappa}{(n-1)\, v_0} \sqrt{\frac{K(\phi) Q_0}{1 - R_0}} \, (-A)^{(n+1)/2}, \tag{A.48}$$

$$\frac{dB}{d\phi} = -\frac{3\kappa}{n v_0^2} R_0 \sqrt{1 - R_0} \sqrt{K(\phi) Q_0} \, (-A)^{(n+1)/2}. \tag{A.49}$$

For the purposes of a stability analysis, it is sufficient to note that Eq. (A.48) does not involve $B(\phi)$. One can solve Eq. (A.48) explicitly for $A(\phi)$ and find such $K(\phi)$ that the general solution for $A(\phi)$ tends to zero at $\phi \to \infty$; for instance, $K(\phi) \propto \phi^r$ with $r > -2$. However, the general solution for $B(\phi)$ is

$$B(\phi) = B_0 - \text{const} \cdot \int_{\phi_0}^{\phi} \sqrt{K(\phi)}\,(-A)^{(n+1)/2}\,d\phi, \tag{A.50}$$

where B_0 is an arbitrary integration constant. It follows that $B(\phi)$ will either diverge or tend to an arbitrary constant of integration at $\phi \to \infty$. Hence, the general perturbation will not tend to zero at large ϕ. We conclude that no asymptotically stable solutions exist when $\tilde{\varepsilon}_\phi(v_0) = 0$.

A.3 Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 = 1$, main case

We use the ansatz $R(\phi) = 1 - B(\phi)$, where the function $B(\phi)$ is positive and tends to zero monotonically as $\phi \to \infty$. Since $dR/d\phi > 0$, it follows from Eq. (2.27) that $w_m < w_\phi(v(\phi))$ for all sufficiently large ϕ. Thus, any asymptotically stable solutions will necessarily satisfy the condition

$$w_m \le w_\phi(v_0). \tag{A.51}$$

Since $R \to 1$ as $\phi \to \infty$, we have $\varepsilon_{\text{tot}}(\phi) \propto \varepsilon_m(\phi) \propto \phi^{-2}$, so we may write

$$\varepsilon_{\text{tot}}(\phi) \approx E_0 \phi^{-2}, \quad \phi \to \infty, \tag{A.52}$$

where E_0 is a nonzero constant. The value of E_0 can be related to other parameters by using Eq. (2.21), rewritten as

$$\frac{d \ln \varepsilon_m}{d \ln \phi} = -\frac{3\kappa}{v} \sqrt{\phi^2 \varepsilon_m}\,(1 + w_m)\,, \tag{A.53}$$

which yields, in the limit $\phi \to \infty$,

$$2 = \frac{3\kappa}{v_0} \sqrt{E_0}\,(1 + w_m)\,. \tag{A.54}$$

Expressing ε_{tot} through ε_ϕ, we have

$$E_0 \phi^{-2} \approx \varepsilon_{\text{tot}} = \frac{\varepsilon_\phi}{1 - R} = \frac{\tilde{\varepsilon}_\phi(v) K(\phi)}{B}; \tag{A.55}$$

hence

$$B(\phi) \approx \frac{\tilde{\varepsilon}_\phi(v) \phi^2 K(\phi)}{E_0}, \quad \phi \to \infty. \tag{A.56}$$

We now assume that $\tilde{\varepsilon}_\phi(v_0) \neq 0$; the case $\tilde{\varepsilon}_\phi(v_0) = 0$ will be considered later. If $\tilde{\varepsilon}_\phi(v_0) \neq 0$, it follows that

$$B(\phi) \approx \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0}\phi^2 K(\phi), \quad \phi \to \infty. \tag{A.57}$$

Rewriting Eq. (2.27) as

$$\frac{d\ln(1-R)}{d\ln\phi} = \frac{3\kappa R}{v}\sqrt{\phi^2\varepsilon_{\mathrm{tot}}}\,(w_m - w_\phi(v)) \tag{A.58}$$

and substituting Eqs. (A.52) and (A.57), we find for large ϕ

$$\frac{d\ln(\phi^2 K(\phi))}{d\ln\phi} \approx \frac{3\kappa\sqrt{E_0}}{v_0}(w_m - w_\phi(v)) = 2\frac{w_m - w_\phi(v)}{1+w_m}. \tag{A.59}$$

It is now clear that the possible asymptotic behavior of $K(\phi)$ at $\phi \to \infty$ depends on whether $w_\phi(v)$ tends to w_m at large ϕ, i.e. on whether or not $w_\phi(v_0) = w_m$. (We note that the value of v_0 is yet to be determined by the analysis that follows.)

Considering the interesting case $w_\phi(v_0) \neq w_m$, we find that the right-hand side of Eq. (A.59) tends to a negative constant as $\phi \to \infty$. Denoting that constant by $-\mu$, where

$$\mu \equiv 2\frac{w_\phi(v_0) - w_m}{1+w_m} > 0, \tag{A.60}$$

and integrating Eq. (A.59), we infer the following asymptotic behavior of $K(\phi)$,

$$K(\phi) \propto \phi^{-2-\mu}K_0(\phi), \quad \phi \to \infty, \tag{A.61}$$

where $K_0(\phi)$ is an auxiliary function that satisfies

$$\lim_{\phi\to\infty}\frac{d\ln K_0(\phi)}{d\ln\phi} = 0. \tag{A.62}$$

This condition is equivalent to

$$\lim_{\phi\to\infty}\frac{\ln K_0(\phi)}{\ln\phi} = 0. \tag{A.63}$$

Thus, the function $K_0(\phi)$ may go to a constant at large ϕ, or may grow or decay slower than any power of ϕ; examples of admissible functions $K_0(\phi)$ are

$$K_0(\phi) = (\ln\phi)^p; \quad K_0(\phi) = \exp\left(C_1(\ln\phi)^s\right), \quad |s| < 1. \tag{A.64}$$

With any such $K_0(\phi)$, solutions of the currently considered type are possible only for Lagrangians $p = K(\phi)Q(v)$ with

$$K(\phi) = \phi^{-2\alpha}K_0(\phi), \tag{A.65}$$

where

$$\alpha \equiv \frac{2+\mu}{2} = \frac{1+w_\phi(v_0)}{1+w_m} > 1. \tag{A.66}$$

For a given Lagrangian of this type, the possible values of v_0 are fixed by Eq. (A.66). If Eq. (A.66) is not satisfied for any such v_0, solutions of this type do not exist. The value $w_\phi(v_0)$ is determined by Eq. (A.66) as

$$w_\phi(v_0) = (1+w_m)\,\alpha - 1. \tag{A.67}$$

Since $w_m \neq -1$, we must have $w_\phi(v_0) \neq -1$ also.

It remains to investigate the asymptotic stability of the general solution. Since $B(\phi)$ must satisfy Eq. (A.56), we may write an ansatz

$$B(\phi) = \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0}\phi^2 K(\phi)\,(1+C(\phi))\,, \tag{A.68}$$

where $C(\phi)$ is a new perturbation variable. Hence, we substitute Eq. (A.65) together with the ansatz

$$v(\phi) = v_0 - A(\phi), \tag{A.69}$$

$$R(\phi) = 1 - \frac{\tilde{\varepsilon}_\phi(v_0)}{E_0}\phi^2 K(\phi)\,(1+C(\phi))\,, \tag{A.70}$$

$$\varepsilon_{\mathrm{tot}}(v,\phi) = E_0\phi^{-2}\frac{\tilde{\varepsilon}_\phi(v)}{\tilde{\varepsilon}_\phi(v_0)}\frac{1}{1+C(\phi)}, \tag{A.71}$$

into Eqs. (2.31) and (A.58). Using Eqs. (A.54), (A.60), and (A.65), we obtain at an intermediate step the equations

$$\begin{aligned}\frac{dA}{d\phi} &= \left[-\frac{2\alpha}{\phi} + (\ln K_0)'\right]\Lambda_1(v)\\ &\quad + \frac{\phi^{-1}}{\sqrt{1+C}}\frac{2v_0}{1+w_m}\frac{\Lambda_2(v)}{3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)}},\end{aligned} \tag{A.72}$$

$$\begin{aligned}\frac{1}{1+C}\frac{dC}{d\phi} &= \mu\phi^{-1} - (\ln K_0)'\\ &\quad - \frac{\mu\phi^{-1}}{\sqrt{1+C}}\frac{\Lambda_4(v)}{\Lambda_4(v_0)}\left[1+O(\phi^{-\mu})\right],\end{aligned} \tag{A.73}$$

where the functions $\Lambda_1(v)$ and $\Lambda_2(v)$ were defined by Eqs. (A.17)–(A.18), while the new auxiliary function $\Lambda_4(v)$ is defined by

$$\Lambda_4(v) \equiv 3\kappa\sqrt{\tilde{\varepsilon}_\phi(v)}\frac{w_\phi(v)-w_m}{v}. \tag{A.74}$$

In the present case, the identity

$$2\alpha\Lambda_1(v_0) = \frac{2v_0}{1+w_m}\frac{\Lambda_2(v_0)}{3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)}} \tag{A.75}$$

holds due to Eq. (A.66).

We now linearize Eqs. (A.72)–(A.73) with respect to the perturbation variables A and C. To simplify the linearized equations, we use Eqs. (2.35), (A.54), and the definition (A.66) of α. (We note that $\Lambda_1(v_0) \neq 0$; otherwise, we would have $c_s^2(v_0) = 0$, which contradicts the earlier assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$ and $w_\phi(v_0) \neq -1$.) After some algebra, we find (to the leading order)

$$\begin{aligned}\frac{dA}{d\ln\phi} &= 2\alpha\Lambda_2(v_0)\left(\frac{\Lambda_1(v)}{\Lambda_2(v)}\right)'_{v_0} A \\ &\quad + \Lambda_1(v_0)\left[\frac{d\ln K_0}{d\ln\phi} - \alpha C\right], \qquad (A.76)\\ \frac{dC}{d\ln\phi} &= -\frac{d\ln K_0}{d\ln\phi} + \frac{1}{2}\mu C + \mu\frac{\Lambda_4'(v_0)}{\Lambda_4(v_0)}A + \mu\frac{\tilde{\varepsilon}_\phi(v_0)}{E_0}\phi^{-\mu}. \qquad (A.77)\end{aligned}$$

This is an inhomogeneous linear system for $A(\phi)$ and $C(\phi)$. The analysis of the asymptotic stability is similar to that after Eq. (A.29). Since all the inhomogeneous terms are decaying at $\phi \to \infty$, it suffices to require that both the eigenvalues of the homogeneous system have negative real parts. For a homogeneous system of the form

$$\frac{dA}{d\ln\phi} = \beta_1 A + \beta_2 C, \tag{A.78}$$

$$\frac{dC}{d\ln\phi} = \gamma_1 A + \gamma_2 C, \tag{A.79}$$

the characteristic equation is

$$\lambda^2 - (\beta_1+\gamma_2)\,\lambda + (\beta_1\gamma_2 - \beta_2\gamma_1) = 0, \tag{A.80}$$

and the stability conditions are

$$\beta_1+\gamma_2 < 0, \quad \beta_1\gamma_2 - \beta_2\gamma_1 > 0. \tag{A.81}$$

Presently, the constants $\beta_1, \beta_2, \gamma_1, \gamma_2$ can be read off from Eqs. (A.76)–(A.77); simplifying, we obtain

$$\beta_1 = -\alpha\frac{1-w_\phi(v_0)}{1+w_\phi(v_0)}, \quad \beta_2 = -\alpha\frac{c_s^2(v_0)v_0}{1+w_\phi(v_0)}, \tag{A.82}$$

$$\begin{aligned}\gamma_1 &= \frac{\mu}{v_0 c_s^2(v_0)}\left[\frac{\left(c_s^2(v_0)-w_\phi(v_0)\right)(1+w_m)}{w_\phi(v_0)-w_m}\right.\\ &\quad\left. + \frac{1-w_\phi(v_0)}{2}\right], \quad \gamma_2 = \frac{\mu}{2}. \qquad (A.83)\end{aligned}$$

The stability conditions (A.81) can be simplified to

$$\frac{1-w_\phi(v_0)}{1+w_\phi(v_0)} > \frac{\mu}{2\alpha}, \quad \frac{c_s^2(v_0)-w_\phi(v_0)}{w_\phi(v_0)-w_m} > 0. \tag{A.84}$$

Since $w_\phi(v_0) > w_m$ for solutions of the present type, while $2\alpha = \mu + 2$, the stability conditions (together with the condition $w_\phi(v_0) > w_m$) are

$$w_m < w_\phi(v_0) < \frac{1}{1+\mu}, \quad c_s^2(v_0) > w_\phi(v_0). \tag{A.85}$$

Using Eq. (A.67), we can transform the first of these conditions into a condition for α:

$$1 < \alpha < \frac{1}{2} + \frac{1}{1+w_m}, \quad c_s^2(v_0) > w_\phi(v_0). \tag{A.86}$$

The first inequality above will define a nonempty interval of α only if $|w_m| < 1$. These are the final conditions for the asymptotic stability of the solutions obtained under the assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$, $w_\phi(v_0) \neq w_m$, and (A.66).[3]

A.4 Energy density $\varepsilon_m \propto \phi^{-2}$ and $R_0 = 1$, marginal cases

The analysis in the previous section used the assumptions $\tilde{\varepsilon}_\phi(v_0) \neq 0$ and $w_\phi(v_0) \neq w_m$. In this section we lift these assumption, in the reverse order used.

If $w_\phi(v_0) = w_m$ while $\tilde{\varepsilon}_\phi(v_0) \neq 0$, then we may continue the arguments starting with Eq. (A.59). Note that Eqs. (A.54) and (A.57) still hold. Since the right-hand side of Eq. (A.59) tends to zero at $\phi \to \infty$, it follows that

$$\lim_{\phi\to\infty} \frac{d\ln\left(\phi^2 K(\phi)\right)}{d\ln\phi} = 0. \tag{A.87}$$

This condition is equivalent to

$$\lim_{\phi\to\infty} \frac{\ln K(\phi)}{\ln\phi} = -2. \tag{A.88}$$

Also, according to Eq. (A.57) we can have $B(\phi) \to 0$ only if

$$\lim_{\phi\to\infty} \phi^2 K(\phi) = 0. \tag{A.89}$$

So the function $K(\phi)$ cannot have a power-law asymptotic other than ϕ^{-2}; more precisely, for any $\varepsilon > 0$ and for large enough ϕ we must have

$$K(\phi) < \phi^{-2+\varepsilon}, \quad K(\phi) > \phi^{-2-\varepsilon}, \quad \phi \to \infty. \tag{A.90}$$

[3] This is case 3 in Sec. 2.4.1.

However, a non-power law asymptotic behavior at $\phi \to \infty$ is still admissible, for instance $K(\phi) \propto \phi^{-2} (\ln \phi)^{-s}$, where $s > 0$ to allow $B(\phi) \to 0$ according to Eq. (A.57). Rather than assume a particular form of $K(\phi)$, we will perform the analysis for arbitrary $K(\phi)$ satisfying Eq. (A.88).

We again use the ansatz (A.68) to linearize Eqs. (2.31) and (A.58). After some algebra, we find (to the leading order)

$$\frac{dA}{d\ln\phi} = \left[-2 + \frac{d\ln(\phi^2 K)}{d\ln\phi}\right]\Lambda_1(v) + \frac{\Lambda_2(v)}{\sqrt{1+C}}\sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}}, \tag{A.91}$$

$$\frac{dC}{d\ln\phi} = -\frac{d\ln(\phi^2 K)}{d\ln\phi} - \frac{\Lambda_4(v)}{\sqrt{1+C}}\sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}}, \tag{A.92}$$

where the auxiliary functions $\Lambda_1(v)$, $\Lambda_2(v)$, and $\Lambda_4(v)$ were defined above by Eqs. (A.17), (A.18), and (A.74). Since $w_\phi(v_0) = w_m$ and $c_s^2(v_0) \neq 0$, we have the relationship,

$$2\Lambda_1(v_0) = \Lambda_2(v_0)\sqrt{\frac{E_0}{\tilde{\varepsilon}_\phi(v_0)}} \neq 0, \tag{A.93}$$

and then, assuming for the moment that $\Lambda_4'(v_0) \neq 0$, we can linearize Eqs. (A.91)–(A.92) as

$$\begin{aligned}\frac{dA}{d\ln\phi} &= 2\Lambda_2(v_0)\left(\frac{\Lambda_1(v)}{\Lambda_2(v)}\right)'_{v_0} A - C\Lambda_1(v_0) \\ &\quad + \frac{d\ln(\phi^2 K)}{d\ln\phi}\Lambda_1(v_0), \end{aligned}\tag{A.94}$$

$$\frac{dC}{d\ln\phi} = \frac{2v_0}{1+w_m}\frac{\Lambda_4'(v_0)}{3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)}}A - \frac{d\ln(\phi^2 K)}{d\ln\phi}. \tag{A.95}$$

The stability analysis proceeds as before, since all the inhomogeneous terms are decaying at $\phi \to \infty$. The resulting conditions are simplified to

$$\frac{1-w_m}{1+w_m} > 0, \quad \frac{c_s^2 - w_m}{1+w_m} > 0, \tag{A.96}$$

and further to

$$|w_m| < 1, \quad c_s^2 > w_m. \tag{A.97}$$

These conditions are the same as the standard stability conditions for a tracker solution. Under these conditions, a tracker solution with $w_\phi(v_0) = w_m$ exists as long as $K(\phi)$ satisfies Eqs. (A.88)–(A.89).[4]

[4]This is case 4 in Sec. 2.4.1.

Finally, we analyze the case $\tilde{\varepsilon}_\phi(v_0) = 0$. Since Eqs. (A.52), (A.54), and (A.56) still hold for an asymptotically stable solution, we are motivated to use the ansatz

$$v(\phi) = v_0 - A(\phi), \tag{A.98}$$

$$R(\phi) = 1 - B(v,\phi), \tag{A.99}$$

$$B(v,\phi) \equiv \frac{\tilde{\varepsilon}_\phi(v)}{E_0}\phi^2 K(\phi)\left(1 + C(\phi)\right). \tag{A.100}$$

We first derive the *exact* equations of motion for the variables $A(\phi), C(\phi)$ from Eqs. (2.31) and (A.58):

$$\frac{dA}{d\ln\phi} = \frac{v c_s^2(v)}{1 + w_\phi(v)}\frac{d\ln K}{d\ln\phi} + \frac{2v_0}{1+w_m}\frac{c_s^2(v)}{\sqrt{1+C}}, \tag{A.101}$$

$$\begin{aligned}\frac{d\ln(1+C)}{d\ln\phi} &= \frac{2}{v}A + \frac{2v_0}{v}\left(\frac{1}{\sqrt{1+C}} - 1\right)\\ &\quad - \frac{2v_0}{v}\frac{B(v,\phi)}{\sqrt{1+C(\phi)}}\frac{w_m - w_\phi(v)}{1+w_m}.\end{aligned} \tag{A.102}$$

Then the stability analysis consists of checking that the general solution involves functions $A(\phi), B(v,\phi), C(\phi)$ that decay as $\phi \to \infty$. Since $v_0 \neq 0$, the expansions (A.45)–(A.47) hold with $n \geq 2$; we note that $A < 0$ to guarantee $c_s^2 > 0$, and that $w_\phi(v_0) = 0$. The leading-order terms in Eq. (A.101) are

$$\frac{d\,|A|}{d\ln\phi} = -\frac{|A|}{n-1}\left(\frac{d\ln K(\phi)}{d\ln\phi} + \frac{2}{1+w_m}\right), \tag{A.103}$$

and the general solution is

$$|A| = A_0\left[\phi^{\frac{2}{1+w_m}}K(\phi)\right]^{-\frac{1}{n-1}}. \tag{A.104}$$

Since $n \geq 2$, solutions $A(\phi)$ decay at $\phi \to \infty$ as long as

$$K(\phi)\phi^{\frac{2}{1+w_m}} \to \infty, \quad \phi \to \infty. \tag{A.105}$$

The function $B(v,\phi)$ is then expressed as

$$B(v,\phi) = \frac{Q_0 A_0^{n-1}}{E_0}\phi^{\frac{2w_m}{1+w_m}}\left(1 + C(\phi)\right), \tag{A.106}$$

and its decay at $\phi \to \infty$ requires that $-1 < w_m < 0$. The leading terms of Eq. (A.102) are

$$\frac{dC}{d\ln\phi} = \frac{2}{v_0}A - C - \left(1 - \frac{1}{2}C\right)B(v,\phi)\frac{w_m}{1+w_m}.$$

Since the homogeneous solution $C(\phi) \propto \phi^{-1}$ decays as a power of ϕ, while the inhomogeneous terms all decay at $\phi \to \infty$, the general solution $C(\phi)$ will also decay at $\phi \to \infty$. Thus, we find a family of asymptotically stable solutions corresponding to a value v_0 such that Eq. (A.44) holds, in case $w_m < 0$ and for Lagrangians with $K(\phi)$ that either does not decay at large ϕ, or decays slower than $\phi^{-2/(1+w_m)}$.[5]

[5] This is case 5 in Sec. 2.4.1.

A.5 Domination by k-essence, $v_0 \neq 0$, main case

We now consider the case $R_0 = 0$. In this case, the matter component becomes subdominant at late times, so $\varepsilon_{\rm tot} \approx \varepsilon_\phi$ at $\phi \to \infty$. According to Eq. (2.22), we have at late times

$$\begin{aligned} \frac{d}{d\phi}\varepsilon_\phi(\phi) &= -\frac{3\kappa}{v}\sqrt{\varepsilon_{\rm tot}}\,(1+w_\phi)\,\varepsilon_\phi \\ &\approx -\frac{3\kappa}{v_0}\varepsilon_\phi^{3/2}\,(1+w_\phi(v))\,, \end{aligned} \tag{A.107}$$

thus the asymptotic behavior of $\varepsilon_\phi(\phi)$ depends on whether or not $w_\phi(v_0) = -1$. With $v_0 \neq 0$, one can have $w_\phi(v_0) = -1$ only if $Q'(v_0) = 0$, which entails

$$c_s^2(v_0) = \frac{1}{v_0}\lim_{v\to v_0}\frac{Q'(v)}{Q''(v)} = \frac{1}{v_0}\lim_{v\to v_0}\frac{1}{(\ln Q'(v))'} = 0. \tag{A.108}$$

Let us postpone the consideration of the case $c_s(v_0) = 0$; thus, presently we have $w_\phi(v_0) \neq -1$. In that case, the asymptotic behavior of $\varepsilon_\phi(\phi)$ and $\varepsilon_{\rm tot}(\phi)$ can be expressed as

$$\varepsilon_{\rm tot}(\phi) \approx \varepsilon_\phi(\phi) \approx E_0\phi^{-2}, \tag{A.109}$$

where the constant E_0 is given by

$$3\kappa\sqrt{E_0} = \frac{2v_0}{1+w_\phi(v_0)}, \tag{A.110}$$

due to Eq. (A.107). We use Eqs. (2.31)–(2.32) to describe asymptotically stable solutions. Since on such solutions $R(\phi)$ approaches zero while remaining positive, we must have $w_\phi(v) < w_m$ at late times. Computing the limit of Eq. (2.31) as $\phi \to \infty$ and using Eq. (A.110), we find

$$\begin{aligned} 0 = \lim_{\phi\to\infty}\frac{dv}{d\ln\phi} &= \lim_{\phi\to\infty}\phi c_s^2(v)\left[\frac{(\ln K)_{,\phi}\,v}{1+w_\phi(v)} + 3\kappa\sqrt{\varepsilon_{\rm tot}}\right] \\ &= \frac{v_0}{1+w_\phi(v_0)}\lim_{\phi\to\infty}c_s^2(v)\left[\frac{d\ln K(\phi)}{d\ln\phi}+2\right]. \end{aligned} \tag{A.111}$$

The right-hand side of Eq. (A.111) can vanish at $\phi \to \infty$ if, for instance, $c_s^2(v_0) = 0$. We postpone the consideration of the case $c_s^2(v_0) = 0$ and presently assume that $c_s^2(v_0) \neq 0$, which (together with $v_0 \neq 0$) also implies $\tilde{\varepsilon}_\phi(v_0) \neq 0$. Then $\varepsilon_\phi(\phi) \propto \phi^{-2}$ entails $K(\phi) \propto \phi^{-2}$ at $\phi \to \infty$; accordingly, the right-hand side of Eq. (A.111) vanishes at $\phi \to \infty$ due to

$$\lim_{\phi\to\infty}\frac{d\ln K}{d\ln\phi} = -2. \tag{A.112}$$

By absorbing a constant into $Q(v)$ if necessary, we may express $K(\phi)$ as

$$K(\phi) = \frac{1 + K_0(\phi)}{\phi^2}, \quad \lim_{\phi \to \infty} K_0(\phi) = 0. \tag{A.113}$$

This is the familiar form of the function $K(\phi)$, shown by Eq. (A.10) in Sec. A.1.

For these $K(\phi)$, the condition (A.110) becomes

$$3\kappa\sqrt{E_0} = 3\kappa\sqrt{\tilde{\varepsilon}_\phi(v_0)} = \frac{2v_0}{1 + w_\phi(v_0)}, \tag{A.114}$$

which is an equation for determining the admissible values of v_0. For these v_0, we linearize Eqs. (2.31)–(2.32) using the ansatz

$$v = v_0 - A(\phi), \quad R = B(\phi), \tag{A.115}$$

where $A(\phi), B(\phi)$ tend to zero as $\phi \to \infty$. The manipulations with Eq. (2.31) are the same as those in Sec. A.1; the result of the linearization is quite similar to Eq. (A.21) with $R_0 = 0$ and without the relationship $w_\phi(v_0) = w_m$,

$$\frac{dA}{d\ln\phi} = (\phi K_0' + K_0)\,\Lambda_1(v_0) - \frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)} A - \Lambda_1(v_0) B. \tag{A.116}$$

The linearized form of Eq. (2.32) is

$$\begin{aligned} \frac{dB}{d\ln\phi} &= -\frac{3\kappa\sqrt{E_0}}{v_0}\,(w_m - w_\phi(v_0))\,B \\ &= -2\frac{w_m - w_\phi(v_0)}{1 + w_\phi(v_0)} B. \end{aligned} \tag{A.117}$$

Since the equation for $B(\phi)$ does not involve $A(\phi)$, and since $w_m > w_\phi(v_0)$, all solutions $B(\phi)$ decay, and thus all solutions $A(\phi)$ also decay as long as

$$\frac{1 - w_\phi(v_0)}{1 + w_\phi(v_0)} > 0, \tag{A.118}$$

which is equivalent to $|w_\phi(v_0)| < 1$. Therefore, solutions are asymptotically stable under the conditions (A.114), $|w_\phi(v_0)| < 1$, $w_m > w_\phi(v_0)$, and $c_s(v_0) \neq 0$.[6]

A.6 Domination by k-essence, $v_0 \neq 0$, marginal cases

In this section we continue considering the case $R_0 = 0$, $v_0 \neq 0$, and examine the possibility that $c_s^2(v_0) = 0$. In that case, we have $Q'(v_0) = 0$ as well, which fixes admissible values of v_0. There are two further possibilities: either $Q(v_0) \neq 0$ or $Q(v_0) = 0$.

[6] This is case 2 in Sec. 2.4.1.

If $Q(v_0) \equiv Q_0 \neq 0$, then $Q(v)$ can be expanded about $v = v_0$ as

$$Q(v) = Q_0 + Q_1 (v - v_0)^n [1 + O(v - v_0)], \tag{A.119}$$

where $n \geq 2$. One readily obtains the expansions

$$\tilde{\varepsilon}_\phi(v) = -Q_0 + nv_0 Q_1 (v - v_0)^{n-1} [1 + O(v - v_0)], \tag{A.120}$$

$$w_\phi(v) = -1 + \frac{nv_0 Q_1}{-Q_0} (v - v_0)^{n-1} [1 + O(v - v_0)], \tag{A.121}$$

$$c_s^2(v) = \frac{1}{v_0} \frac{v - v_0}{n - 1} [1 + O(v - v_0)]. \tag{A.122}$$

It is clear that one must have $K(\phi) Q_0 < 0$ due to the positivity of the energy density. For convenience, let us assume that $K(\phi) > 0$ and $Q_0 < 0$. Substituting the expansions above into Eqs. (2.31)–(2.32) together with the ansatz (A.115) and neglecting the subleading terms, we obtain

$$\frac{dA}{d\phi} = \frac{1}{v_0} \frac{-A}{n-1} \left[\frac{|Q_0| (\ln K)_{,\phi}}{nQ_1 (-A)^{n-1}} + 3\kappa \sqrt{|Q_0| K(\phi)} \right], \tag{A.123}$$

$$\frac{dB}{d\phi} = -\frac{3\kappa}{v_0} B \sqrt{|Q_0| K(\phi)} (w_m + 1). \tag{A.124}$$

Since these equations are uncoupled in the leading order, the stability analysis is performed for each equation separately. Integrating Eq. (A.124), we find the general solution

$$B(\phi) = \exp \left[C_0 - \frac{3\kappa \sqrt{|Q_0|}}{v_0} (w_m + 1) \int^\phi \sqrt{K(\phi)} d\phi \right], \tag{A.125}$$

where C_0 is an integration constant. The general solution $B(\phi)$ will tend to zero if and only if $\int \sqrt{K(\phi)} d\phi$ diverges as $\phi \to \infty$ and $w_m > -1$. Let us temporarily denote

$$\chi(\phi) \equiv \int^\phi \sqrt{K(\phi)} d\phi, \quad \chi \to \infty \text{ as } \phi \to \infty. \tag{A.126}$$

Then we rewrite the first equation as

$$\frac{d}{d\chi} (-A)^{n-1} = -\frac{|Q_0|}{nv_0 Q_1} \frac{K'(\phi)}{K^{3/2}} - (-A)^{n-1} \frac{3\kappa}{v_0} \sqrt{|Q_0|}. \tag{A.127}$$

(Note that we must have $A < 0$ on solutions, due to the requirement of positivity of c_s^2.) The general solution $A(\chi)$ can now be written explicitly, but it suffices to observe that $A(\chi)$ will approach zero as $\chi \to \infty$ if and only if

$$\lim_{\phi \to \infty} \frac{K'(\phi)}{K^{3/2}} = -2 \lim_{\phi \to \infty} \frac{d}{d\phi} K^{-1/2} = 0. \tag{A.128}$$

This condition is equivalent to

$$\lim_{\phi\to\infty} \phi\sqrt{K(\phi)} = \infty. \tag{A.129}$$

Note that the condition (A.126) follows from that of Eq. (A.129). To verify this more formally, consider a function $K(\phi)$ such that $\int^\infty \sqrt{K(\phi)}d\phi < \infty$. Then $K^{1/2}(\phi)$ necessarily decays faster than ϕ^{-1} at $\phi \to \infty$, and so $K^{-1/2}$ grows faster than ϕ at $\phi \to \infty$. Such $K(\phi)$ cannot satisfy Eq. (A.129). Therefore it is sufficient to impose only the condition (A.129). This condition is satisfied, for instance, by functions $K(\phi) \propto \phi^s$ with $s > -2$. Thus, we conclude that the solution with $R_0 = 0$ is asymptotically stable under the condition (A.129) and assumptions $Q(v_0) \neq 0$, $Q'(v_0) = 0$.[7]

It remains to consider the case $R_0 = 0$, $Q(v_0) = Q'(v_0) = 0$. In that case, similarly to that discussed in Sec. A.2, we may use the expansions (A.44)–(A.47). It follows that $w_\phi(v_0) = 0$. With the ansatz $v(\phi) = v_0 - A(\phi)$, we find that $A(\phi) < 0$ on physically reasonable solutions. Then the leading terms of Eq. (2.31) are

$$\frac{d\,(-A)}{d\phi} = -\frac{(-A)}{n-1}\frac{K'}{K} - \frac{3\kappa\sqrt{Q_0}}{(n-1)\,v_0}\sqrt{K(\phi)}\,(-A)^{(n+1)/2}. \tag{A.130}$$

Since this equation is independent of B, it suffices to ensure that $A(\phi) \to 0$ as $\phi \to \infty$ and subsequently consider the general solution for $R(\phi)$. The general solution for $A(\phi)$ can be easily found by rewriting Eq. (A.130) as

$$\frac{d}{d\phi}\left[(-A)^{-(n-1)/2}\,K^{-1/2}\right] = \frac{3\kappa\sqrt{Q_0}}{2v_0}. \tag{A.131}$$

We find

$$(-A)^{(n-1)/2} = \frac{2v_0}{3\kappa\sqrt{Q_0}}\frac{1}{\phi-\phi_0}\frac{1}{\sqrt{K(\phi)}}, \tag{A.132}$$

where ϕ_0 is a constant of integration. It follows that $A(\phi) \to 0$ as $\phi \to \infty$ if $K(\phi)$ is such that $\phi^2 K(\phi) \to \infty$. Under this assumption, we find that

$$\varepsilon_\phi = K(\phi)\tilde{\varepsilon}_\phi(v) \propto \phi^{-2}, \quad \phi \to \infty, \tag{A.133}$$

as it should according to Eq. (A.109). Now we analyze the general solution for $R(\phi)$ Then the leading terms of Eq. (2.32) are

$$\frac{dR}{d\ln\phi} = -R\frac{3\kappa\sqrt{E_0}}{v_0}\left(w_m + \frac{A}{nv_0}\right) = -2R\left(w_m + \frac{A}{nv_0}\right), \tag{A.134}$$

where we used Eq. (A.110). If $w_m = 0$, the right-hand side above is always positive and (since R is always positive) the general solution for $R(\phi)$ cannot approach zero. If $w_m \neq 0$, the general solution for $R(\phi)$ is

$$R(\phi) \propto \phi^{-2w_m} \quad \text{as}\,\phi \to \infty. \tag{A.135}$$

[7] This is case 8 in Sec. 2.4.1.

It follows that the general solution $R(\phi) \to 0$ at $\phi \to \infty$ as long as $w_m > 0$. We conclude that an asymptotically stable solution exists in case $Q(v_0) = Q'(v_0) = 0$ if $w_m > 0$ and $\phi^2 K(\phi) \to \infty$ as $\phi \to \infty$. The admissible functions $K(\phi)$ are, for instance, $K(\phi) \propto \phi^s$ with $s > -2$.[8]

A.7 Slow motion ($v_0 = 0$), main case ($Q(0) \neq 0$)

Previously we have been assuming that $v_0 \neq 0$. Now we turn to the case $v_0 = 0$, which means that the velocity $\dot{\phi} \equiv v(\phi)$ of the field ϕ approaches zero, albeit sufficiently slowly so that ϕ still reaches arbitrarily large values at late times. We will now obtain the conditions for the existence of asymptotically stable solutions with $v(\phi) \to 0$ at $\phi \to \infty$.

The finiteness of the speed of sound at $v \to 0$,

$$\lim_{v\to 0} c_s^2(v) = \lim_{v\to 0} \frac{Q'(v)}{vQ''(v)} < \infty, \tag{A.136}$$

requires that $Q'(0) = 0$. Since the important quantity $\tilde{\varepsilon}_\phi(v) = vQ' - Q$ approaches $-Q(0)$ at late times, it is useful to distinguish two possibilities: $Q(0) \neq 0$ and (less generically) $Q(0) = 0$. In this section we consider the generic case, $Q(0) \equiv -Q_0 \neq 0$. Positivity of the energy density requires that $K(\phi)Q_0 > 0$, and we will choose $K(\phi) > 0$ and $Q_0 > 0$.

Under these assumptions, we may expand the function $Q(v)$ near $v = 0$ as

$$Q(v) = -Q_0 + Q_1 v^n \left[1 + O(v)\right], \tag{A.137}$$

where $n \geq 2$ is the lowest order of the nonvanishing derivative of $Q(v)$ at $v = 0$, and $Q_1 > 0$ because $Q(v)$ is a convex and monotonically growing function of v. Other relevant quantities are then expanded as

$$\tilde{\varepsilon}_\phi(v) = Q_0 + (n-1)\, Q_1 v^n \left[1 + O(v)\right], \tag{A.138}$$

$$w_\phi(v) = -1 + \frac{nQ_1}{Q_0} v^n \left[1 + O(v)\right], \tag{A.139}$$

$$c_s^2(v) = \frac{1}{n-1} \left[1 + O(v)\right]. \tag{A.140}$$

It follows that the only possible equation of state is $w_\phi(0) = -1$, indicating a possible de Sitter tracker solution.

The equations of motion (2.31)–(2.32) become (neglecting terms of order v)

$$\frac{dv}{d\phi} = -\frac{1}{n-1} \left[\frac{Q_0}{nQ_1 v^{n-1}} \frac{K'}{K} + 3\kappa \sqrt{\frac{K(\phi)Q_0}{1-R}} \right], \tag{A.141}$$

$$\frac{dR}{d\phi} = -\frac{3\kappa}{v} R\sqrt{1-R} \sqrt{K(\phi)Q_0}\, (w_m + 1)\,. \tag{A.142}$$

[8] This is case 9 in Sec. 2.4.1.

The first step is to investigate the possibility that $R(\phi) \to 1$ at large ϕ (we will find that this possibility cannot be realized). We note that for $w_m > -1$, the right-hand side of Eq. (A.142) always remains negative. Thus, for $w_m > -1$ the general solution $R(\phi)$ cannot tend to 1 at $\phi \to \infty$, regardless of the behavior of $K(\phi)$ and $v(\phi)$. In case $w_m < -1$, we need to do more work to establish that there are no asymptotically stable solutions with $R_0 = 1$.

Substituting the ansatz $R(\phi) = 1 - B(\phi)$ into Eq. (A.142) and assuming that $B \to 0$, we obtain (omitting terms of order v and B)

$$\frac{d\sqrt{B}}{d\phi} = -\frac{3\kappa}{2v}\sqrt{K(\phi)Q_0}\,|1 + w_m|\,. \tag{A.143}$$

Changing the variable from ϕ to χ defined by

$$\chi(\phi) \equiv \int^{\phi} \sqrt{K(\phi)}d\phi, \tag{A.144}$$

we find

$$\frac{d\sqrt{B}}{d\chi} = -\frac{3\kappa}{2v}\sqrt{Q_0}\,|1 + w_m|\,. \tag{A.145}$$

There are now two possibilities: either the integral in Eq. (A.144) diverges at $\phi \to \infty$, or it converges. Accordingly, either $\chi \to \infty$ or $\chi \to \chi_0 < \infty$ at $\phi \to \infty$. In case $\chi \to \infty$ at $\phi \to \infty$, we would have

$$\lim_{\chi\to\infty} \frac{d\sqrt{B}}{d\chi} = 0. \tag{A.146}$$

Since the right-hand side in Eq. (A.145) tends to infinity at $\phi \to \infty$, the case $\chi \to \infty$ is impossible. Thus, the integral in Eq. (A.144) must converge at $\phi \to \infty$. It follows that $K(\phi) \to 0$ faster than ϕ^{-2} at $\phi \to \infty$, and then we may express $K(\phi)$ through an auxiliary function $K_0(\phi)$ as

$$K(\phi) = \phi^{-2}K_0(\phi), \quad \lim_{\phi\to\infty} K_0(\phi) = 0. \tag{A.147}$$

Further, we rewrite Eq. (A.141) as

$$\begin{aligned}\frac{dv}{d\ln\phi} = \frac{1}{n-1}\Bigg[&\frac{Q_0}{nQ_1v^{n-1}}\left(2 - \frac{d\ln K_0}{d\ln\phi}\right)\\ &-3\kappa\sqrt{\frac{K_0(\phi)Q_0}{B}}\Bigg].\end{aligned} \tag{A.148}$$

By construction,

$$\lim_{\phi\to\infty}\left(2 - \frac{d\ln K_0}{d\ln\phi}\right) > 2 \tag{A.149}$$

(the limit might even be positive infinite if K_0 tends to zero sufficiently quickly). Hence, under the assumptions $v(\phi) \to 0$ and $B(\phi) \to 0$ we must have

$$\lim_{\phi\to\infty} \frac{Q_0}{nQ_1 v^{n-1}} \left(2 - \frac{d\ln K_0}{d\ln\phi}\right) = +\infty. \tag{A.150}$$

It then follows by taking the limit $\phi \to \infty$ of Eq. (A.148) that the two terms in the brackets must cancel while both approach infinity. Therefore, at large ϕ we must have the approximate relationship

$$\frac{v^{n-1}(\phi)}{\sqrt{B(\phi)}} \approx \frac{\sqrt{Q_0}}{3\kappa n Q_1} \frac{1}{\sqrt{K_0(\phi)}} \left(2 - \frac{d\ln K_0}{d\ln\phi}\right) \equiv M(\phi). \tag{A.151}$$

Due to Eq. (A.149), the auxiliary function $M(\phi)$ defined by Eq. (A.151) has the properties

$$\lim_{\phi\to\infty} M(\phi)\sqrt{K_0(\phi)} > \frac{2\sqrt{Q_0}}{3\kappa n Q_1}, \quad \lim_{\phi\to\infty} M(\phi) = +\infty. \tag{A.152}$$

(The first limit may be positive infinite.) Using the function $M(\phi)$, we may rewrite Eq. (A.148) as

$$\frac{dv}{d\ln\phi} = \frac{3\kappa\sqrt{K_0(\phi)Q_0}}{n-1} \left[\frac{M}{v^{n-1}} - \frac{1}{\sqrt{B}}\right]. \tag{A.153}$$

Expressing $\sqrt{B}$ through v using Eq. (A.151) and substituting the resulting expression for $\sqrt{B}$ into Eq. (A.143), we find

$$\begin{aligned}\frac{d}{d\ln\phi}\left[\frac{v^{n-1}}{M}\right] &= -\frac{3\kappa}{2v}\sqrt{K_0(\phi)Q_0}\,|1+w_m| \\ &= (n-1)\frac{v^{n-2}}{M}\frac{dv}{d\ln\phi} - v^{n-1}M^{-2}\frac{dM}{d\ln\phi}.\end{aligned} \tag{A.154}$$

Rewriting the last equation as

$$\frac{3\kappa}{2} M\sqrt{K_0Q_0}\,|1+w_m| = -\frac{n-1}{n}\frac{dv^n}{d\ln\phi} + v^n\frac{d\ln M}{d\ln\phi}, \tag{A.155}$$

we note that the left-hand side tends to a positive limit (or to a positive infinity) due to Eq. (A.152), while the term $dv^n/d\ln\phi$ tends to zero at $\phi \to \infty$ and can be neglected. Therefore, for large ϕ we obtain

$$v^n \approx \frac{3\kappa}{2}\sqrt{K_0(\phi)Q_0}\,|1+w_m|\,M^2\left[\frac{dM}{d\ln\phi}\right]^{-1}. \tag{A.156}$$

This relationship is sufficient for our purposes; we will now show that $v(\phi)$ cannot tend to zero at $\phi \to \infty$. If we assume that $v(\phi) \to 0$, we must have

$$\lim_{\phi\to\infty} \frac{\sqrt{K_0(\phi)}}{\frac{d}{d\ln\phi}M^{-1}} = 0. \tag{A.157}$$

Using Eq. (A.151), we transform this condition into

$$\lim_{\phi\to\infty} \left[\frac{\frac{1}{2}\left(\ln K_0\right)_{,\ln\phi}}{2-\left(\ln K_0\right)_{,\ln\phi}} + \frac{d}{d\ln\phi}\left[2-\frac{d\ln K_0}{d\ln\phi}\right]^{-1} \right] = \infty. \tag{A.158}$$

It is now straightforward to show that the condition (A.158) cannot be satisfied by a function $K_0(\phi)$ that tends to zero at $\phi \to \infty$. Since $(\ln K_0)' \leq 0$ for all ϕ, the function $\left(\ln K_0\right)_{,\ln\phi}$ tends to a nonpositive constant or to a negative infinity at $\phi \to \infty$. Hence, we obtain the bounds

$$-1 < \frac{\frac{1}{2}\left(\ln K_0\right)_{,\ln\phi}}{2-\left(\ln K_0\right)_{,\ln\phi}} < 0, \quad 0 < \left[2-\frac{d\ln K_0}{d\ln\phi}\right]^{-1} < \frac{1}{2}. \tag{A.159}$$

The derivative of a bounded function cannot have an infinite limit. Therefore the limit (A.158) cannot be infinite. Since the condition (A.158) cannot be satisfied, solutions with $v(\phi) \to 0$ and $B(\phi) \to 0$ do not exist under the present assumptions.

Having shown that $R_0 = 1$ is impossible, we assume $R_0 < 1$ in the rest of this section. Let us now consider the admissible behavior of $v(\phi)$ at large ϕ. It is convenient to change the independent variable from ϕ to χ defined by Eq. (A.144) and to rewrite Eq. (A.141) as

$$\frac{dv}{d\chi} = \frac{1}{n-1}\left[\frac{Q_0}{nQ_1 v^{n-1}}\left(\frac{2}{\sqrt{K}}\right)_{,\phi} - 3\kappa\sqrt{\frac{Q_0}{1-R}}\right]. \tag{A.160}$$

For an asymptotically stable solution, we need $v(\phi) \to 0$ while $v(\phi) > 0$. Therefore, $dv/d\phi$ (and therefore also $dv/d\chi$) must remain negative at large ϕ. Let us examine the condition under which the right-hand side of Eq. (A.160) might be negative at large ϕ.

We notice that the first term in the right-hand side of Eq. (A.160) contains a negative power of v multiplied by a nonnegative function $d(K^{-1/2})/d\phi$ and a positive constant. This term will diverge to positive infinity as $v \to 0$ unless $d(K^{-1/2})/d\phi$ tends to zero at large ϕ. On the other hand, the second term,

$$-3\kappa\sqrt{\frac{Q_0}{1-R}}, \tag{A.161}$$

tends to a negative constant at large ϕ. Thus, $dv/d\chi$ may become negative at large ϕ only when $d(K^{-1/2})/d\phi$ tends to zero at large ϕ. If $K(\phi)$ is such that $K'K^{-3/2} \to 0$, then $\chi(\phi) \equiv \int^{\phi} \sqrt{K(\phi)}d\phi$ diverges at $\phi \to \infty$; this was already shown in the previous section after Eq. (A.129). Let us therefore continue the analysis under the assumptions (A.129) and $R_0 < 1$, taking into account that $\chi \to \infty$ together with $\phi \to \infty$.

Rewriting Eq. (A.142) as

$$\frac{dR}{d\chi} = -\frac{3\kappa}{v} R\sqrt{1-R}\sqrt{Q_0}\left(w_m+1\right)\left[1+O(v)\right], \tag{A.162}$$

and noting that $w_m + 1 \neq 0$, we immediately see that $dR/d\chi \to 0$ can be realized only if $w_m + 1 > 0$ and either $R_0 = 0$ or $R_0 = 1$, where $R_0 \equiv \lim_{\phi\to\infty} R(\phi)$. Since we are assuming $R_0 < 1$, the only admissible value is $R_0 = 0$. Therefore, we now look for solutions of Eqs. (A.160)–(A.162) such that $v(\chi) \to 0$ and $R(\chi) \to 0$ as $\chi \to \infty$ (at the same time as $\phi \to \infty$).

Computing the limit of Eq. (A.160) as $\phi \to \infty$ and noting that $dv/d\chi \to 0$ on asymptotically stable solutions, we obtain the condition

$$\lim_{\phi\to\infty} \frac{Q_0}{nQ_1} \frac{1}{v^{n-1}} \left(\frac{2}{\sqrt{K}} \right)_{,\phi} = 3\kappa\sqrt{Q_0}. \tag{A.163}$$

The right-hand side above is a nonzero constant. Therefore it suffices to look for solutions $v(\phi)$ of the form

$$v^{n-1}(\phi) = \frac{\sqrt{Q_0}}{3\kappa n Q_1} \left(\frac{2}{\sqrt{K}} \right)_{,\phi} [1 + A(\phi)]\,, \tag{A.164}$$

where $A(\phi)$ is a new unknown function replacing $v(\phi)$. Solutions $v(\phi) \to 0$ will be asymptotically stable if the general solution for $A(\phi)$ tends to zero as $\phi \to \infty$. For brevity, we rewrite the ansatz (A.164), with the independent variable ϕ expressed through χ, as

$$v(\chi) = [(1 + A(\chi))\, W(\chi)]^{\frac{1}{n-1}}\,, \tag{A.165}$$

where $W(\chi)$ is a fixed function defined through

$$W(\chi)|_{\chi=\chi(\phi)} = \frac{\sqrt{Q_0}}{3\kappa n Q_1} \left(\frac{2}{\sqrt{K}} \right)_{,\phi}. \tag{A.166}$$

By assumption, we have $W(\chi) \to 0$ as $\chi \to \infty$. Substituting the ansatz (A.165) into Eqs. (A.160)–(A.162), we obtain, to the leading order in A and R,

$$\frac{dA}{d\chi} = -\frac{3\kappa\sqrt{Q_0}\left(\frac{1}{2}R + A\right) + (n-1)\left(W^{\frac{1}{n-1}}\right)_{,\chi}}{W^{\frac{1}{n-1}}}, \tag{A.167}$$

$$\frac{dR}{d\chi} = -3\kappa\sqrt{Q_0}\,(w_m + 1)\, R W^{-\frac{1}{n-1}}. \tag{A.168}$$

Since the equation for R does not contain A, the stability analysis can be performed first for $R(\chi)$ and then for $A(\chi)$ assuming that $R(\chi) \to 0$. It is convenient to replace the independent variable χ temporarily by

$$\psi(\chi) \equiv \int^{\chi} W^{-\frac{1}{n-1}}(\chi) d\chi. \tag{A.169}$$

Since $W(\chi) \to 0$ as $\chi \to \infty$, the new variable ψ grows to infinity together with χ. The new equations for $A(\psi)$ and $R(\psi)$ are

$$\frac{dA}{d\psi} = -3\kappa\sqrt{Q_0}\left(\frac{1}{2}R + A\right) - (n-1)\left(W^{\frac{1}{n-1}}\right)_{,\chi}, \tag{A.170}$$

$$\frac{dR}{d\psi} = -3\kappa\sqrt{Q_0}\,(w_m + 1)\, R. \tag{A.171}$$

It is clear that the general solution for $R(\psi)$ tends to zero if $w_m > -1$. The general solution for $A(\psi)$ is a sum of the general homogeneous solution (which tends to zero) and an inhomogeneous solution. The inhomogeneous terms are proportional to R and $\left(W^{1/(n-1)}\right)_{,\chi}$, both of which tend to zero at $\chi \to \infty$ ($\psi \to \infty$). Therefore the general solution for $v(\phi)$ and $R(\phi)$ is asymptotically stable under the current assumptions.[9]

A.8 Slow motion ($v_0 = 0$), marginal cases ($Q(0) = 0$)

Let us now turn to the case $Q(0) = 0$. In this case, we may expand the relevant quantities near $v = 0$ as follows,

$$Q(v) = Q_1 v^n \left[1 + O(v)\right], \tag{A.172}$$
$$\tilde{\varepsilon}_\phi(v) = (n-1)\, Q_1 v^n \left[1 + O(v)\right], \tag{A.173}$$
$$w_\phi(v) = \frac{1}{n-1}\left[1 + O(v)\right], \tag{A.174}$$
$$c_s^2(v) = \frac{1}{n-1}\left[1 + O(v)\right], \tag{A.175}$$

where $n \geq 2$ and $Q_1 > 0$. Using these expansions, we rewrite the equations of motion (2.31)–(2.32), in the leading order in v, as

$$\frac{dv}{d\phi} = -\frac{v}{n}\frac{K'}{K} - \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}\sqrt{\frac{K(\phi)}{1-R}}\, v^{\frac{n}{2}}, \tag{A.176}$$
$$\frac{dR}{d\phi} = -3\kappa R\sqrt{1-R}\sqrt{(n-1)\, Q_1 K(\phi)}\,\frac{w_m - w_\phi(v)}{v^{1-n/2}}. \tag{A.177}$$

The possible asymptotic values of equation of state parameter $w_\phi(0)$ are $1/(n-1)$ for $n \geq 2$; in particular, we can have $w_\phi(0) = \frac{1}{3}$, mimicking radiation, if $n = 4$. When $w_m = 1/(n-1)$, we may need to expand the term $w_m - w_\phi(v)$ to a higher nonvanishing order in v. For instance, assuming an expansion

$$Q(v) \equiv Q_1 v^n + Q_2 v^{n+p}\left[1 + O(v)\right], \tag{A.178}$$

where $n \geq 2$ and $p \geq 1$, we find

$$w_\phi(v) = \frac{Q(v)}{vQ'(v) - Q} = \frac{1 + O(v)}{n-1}\left[1 - \frac{pQ_2 v^p}{(n-1)\, Q_1}\right]. \tag{A.179}$$

Let us begin by considering the possible asymptotic value $R_0 = 1$ of $R(\phi)$ at $\phi \to \infty$; values $R_0 < 1$ will be considered subsequently. In case $R_0 = 1$, we write the ansatz

[9] This is case 10 in Sec. 2.4.1.

$R(\phi) = 1 - B(\phi)$ and transform Eq. (A.177) into

$$\frac{d\sqrt{B}}{d\phi} = \frac{3}{2}\kappa\sqrt{(n-1)\,Q_1 K(\phi)}v^{\frac{n}{2}-1}\left(w_m - w_\phi(v)\right). \tag{A.180}$$

The right-hand side of the equation above must be negative to allow $\sqrt{B(\phi)} \to 0$ at $\phi \to \infty$. This cannot happen if $w_m - w_\phi(0) > 0$. Thus, the only possibility for the existence of stable solutions is $w_\phi(v) > w_m$ for $v > 0$ (which does not exclude $w_\phi(0) = w_m$). Under the assumption $w_\phi(0) > w_m$, Eqs. (A.176) and (A.180) can be rewritten (again keeping only the leading-order terms) as

$$\frac{d}{d\phi}\ln\left(K^{1/n}v\right) = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}\sqrt{\frac{K(\phi)}{B(\phi)}}v^{\frac{n}{2}-1}, \tag{A.181}$$

$$\frac{d\ln B}{d\phi} = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}\left(1-(n-1)\,w_m\right)\sqrt{\frac{K(\phi)}{B(\phi)}}v^{\frac{n}{2}-1}. \tag{A.182}$$

In case $w_\phi(0) = w_m$, we assume the expansion (A.178) and use Eq. (A.179); then Eq. (A.182) is replaced by

$$\frac{d\ln B}{d\phi} = \frac{3\kappa p Q_2}{(n-1)^{3/2}\sqrt{Q_1}}\sqrt{\frac{K(\phi)}{B(\phi)}}v^{\frac{n}{2}-1+p}. \tag{A.183}$$

As in the case $w_\phi(0) \neq w_m$, stable solutions are possible only if the right-hand side of Eq. (A.183) is negative, i.e. if $Q_2 < 0$.

We now need to analyze the solutions of the systems (A.181)–(A.182) and (A.181), (A.183) by looking for such $K(\phi)$ that the general solutions $v(\phi)$ and $B(\phi)$ always tend to zero in the two cases.

The general solution of Eqs. (A.181)–(A.182) can be found by first noticing that

$$\frac{d}{d\phi}\left[\ln\left(K^{1/n}v\right) - \frac{\ln B}{1-(n-1)\,w_m}\right] = 0. \tag{A.184}$$

Hence we may express

$$B(\phi) = C_0\left[K^{1/n}v\right]^{1-(n-1)w_m}, \tag{A.185}$$

where $C_0 > 0$ is an integration constant. Then we substitute this $B(\phi)$ into Eq. (A.181) and obtain the following equation for the auxiliary function $u \equiv K^{1/n}v$,

$$\frac{du}{d\phi} = -F(\phi)u^s, \tag{A.186}$$

where we defined the auxiliary constant s and function $F(\phi)$ as

$$s \equiv \frac{n-1}{2}\left(1+w_m\right), \tag{A.187}$$

$$F(\phi) \equiv 3\kappa\sqrt{\frac{Q_1}{(n-1)\,C_0}}K^{1/n}(\phi) > 0. \tag{A.188}$$

Since by assumption $w_m < \frac{1}{n-1}$, the possible values of s are $s < \frac{n}{2}$. We are now looking for functions $K(\phi)$ such that both $v = u/F$ and $B \propto u^{n-2s}$ always tend to zero as $\phi \to \infty$; in other words, we require

$$\lim_{\phi\to\infty} \frac{u(\phi)}{F(\phi)} = 0, \quad \lim_{\phi\to\infty} u(\phi) = 0 \tag{A.189}$$

for the general solution $u(\phi)$. The general solution for $u(\phi)$ can be written as

$$u(\phi) = \begin{cases} \left[C_1 + (s-1)\int^\phi F(\phi)d\phi\right]^{1/(1-s)}, & s \neq 1, \\ \exp\left(C_1 - \int^\phi F(\phi)d\phi\right), & s = 1, \end{cases} \tag{A.190}$$

where C_1 is a constant of integration. If $s < 1$, the power $1/(1-s)$ is positive and so the general solution $u(\phi)$ does not tend to zero. If $s \geq 1$, the general solution $u(\phi)$ tends to zero in case $\int^\phi F(\phi)d\phi$ diverges as $\phi \to \infty$, and does not tend to zero if $\int^\phi F(\phi)d\phi$ converges. Therefore, the only possibility for a stable solution is $s \geq 1$ and $\int^\phi F(\phi)d\phi \to \infty$ as $\phi \to \infty$, or equivalently

$$w_m > -\frac{n-3}{n-1}; \quad \lim_{\phi\to\infty} \int^\phi K^{1/n}(\phi)d\phi = \infty. \tag{A.191}$$

It remains to examine the condition $u(\phi)/F(\phi) \to 0$ under these assumptions.

Since we already have $u \to 0$, the condition $u/F \to 0$ holds if $F(\phi)$ approaches a nonzero constant or infinity as $\phi \to \infty$. However, if

$$\lim_{\phi\to\infty} F(\phi) = \lim_{\phi\to\infty} K^{1/n}(\phi) = 0, \tag{A.192}$$

the condition $u/F \to 0$ is a nontrivial additional constraint on the function $K(\phi)$. This constraint can be expressed as a condition on $K(\phi)$ as follows. We find from Eq. (A.190) that

$$\frac{u}{F} \propto \begin{cases} \left[F^{s-1}\int^\phi F(\phi)d\phi\right]^{-1/(s-1)}, & s > 1, \\ \exp\left(-\ln F - \int^\phi F(\phi)d\phi\right), & s = 1. \end{cases} \tag{A.193}$$

The condition $u/F \to 0$ is then equivalent to

$$\lim_{\phi\to\infty} F^{s-1} \int^\phi F(\phi)d\phi = \infty, \quad s > 1; \tag{A.194}$$

$$\lim_{\phi\to\infty} \left(\ln F + \int^\phi F(\phi)d\phi\right) = \infty, \quad s = 1. \tag{A.195}$$

We note that the left-hand sides in Eqs. (A.194)–(A.195) depend monotonically on the growth of $F(\phi)$; more precisely, the terms under the limits become larger when we choose a faster-growing or slower-decaying function $F(\phi)$. Thus, it is clear that the conditions (A.194)–(A.195) will hold if $F(\phi)$ decays sufficiently slowly as $\phi \to \infty$ (or grows, but this case was already considered). With some choices of $F(\phi) = F_0(\phi)$, the limits in Eqs. (A.194)–(A.195) will be finite nonzero constants. We can easily determine such $F_0(\phi)$,

$$F_0(\phi) \propto \phi^{-1/s}, \quad s \geq 1,\ \phi \to \infty. \tag{A.196}$$

Hence, the limits (A.194)–(A.195) will be infinite when $F(\phi)$ decays slower than $\phi^{-1/s}$. The corresponding condition for $K(\phi)$ can be written as

$$\lim_{\phi\to\infty} \phi^{n/s} K(\phi) = \infty. \tag{A.197}$$

We can make this argument more rigorous by assuming the ansatz

$$K(\phi) = \phi^{-n/s} K_1(\phi), \tag{A.198}$$

where $K_1(\phi) > 0$ is an auxiliary function. Note that the function $F(\phi)$ is related to $K(\phi)$ by Eq. (A.188), which contains an arbitrary integration constant $C_0 > 0$. Thus we may write

$$F(\phi) = C_1 \phi^{-1/s} K_1^{1/n}(\phi), \tag{A.199}$$

where $C_1 > 0$ is an arbitrary constant. If

$$\lim_{\phi\to\infty} K_1(\phi) = \infty, \tag{A.200}$$

it means that $K_1(\phi)$ is larger than any constant at sufficiently large ϕ. Then we obtain lower bounds (for arbitrary constant $C_2 > 0$)

$$\int^{\phi} F(\phi) d\phi > C_2 \phi^{1-1/s}, \qquad s > 1; \tag{A.201}$$

$$\int^{\phi} F(\phi) d\phi > C_2 \ln \phi, \qquad s = 1, \tag{A.202}$$

and the conditions (A.194)–(A.195) hold. On the other hand, if

$$\lim_{\phi\to\infty} K_1(\phi) \equiv K_1^{(0)} < \infty, \tag{A.203}$$

we find

$$\int^{\phi} F(\phi) d\phi \approx C_2 \phi^{1-1/s}, \qquad s > 1; \tag{A.204}$$

$$\int^{\phi} F(\phi) d\phi \approx C_2 \ln \phi, \qquad s = 1, \tag{A.205}$$

where $C_2 > 0$ is an arbitrary constant. In that case, the conditions (A.194)–(A.195) cannot hold for arbitrary C_2. Therefore, the condition (A.197) is necessary and sufficient for Eqs. (A.194)–(A.195) to hold.

We conclude that an asymptotically stable solution exists for $v_0 = 0$, $Q(0) = 0$ with the expansion (A.178), when $R_0 = 1$, $w_m < \frac{1}{n-1}$, and the conditions (A.191) and (A.197) hold. We note that for $n = 2$ the condition $w_m < \frac{1}{n-1}$ contradicts the first condition in Eq. (A.191), so admissible solutions exist only for $n > 2$.[10]

The remaining case requires the analysis of Eqs. (A.181)–(A.183). The general solution of these equations cannot be obtained in closed form; however, we only need to analyze the asymptotic behavior at $\phi \to \infty$. So we will estimate the relative magnitude of different terms in these equations. Let us rewrite Eqs. (A.181)–(A.183) as

$$\frac{d\ln v}{d\phi} = -\frac{1}{n}\frac{K'}{K} - \tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}, \tag{A.206}$$

$$\frac{d\sqrt{B}}{d\phi} = -\tilde{Q}_2 K^{1/2} v^{n/2-1+p}, \tag{A.207}$$

where the auxiliary positive constants

$$\tilde{Q}_1 \equiv \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}, \quad \tilde{Q}_2 \equiv -\frac{3\kappa p Q_2}{(n-1)^{3/2}\sqrt{Q_1}} \tag{A.208}$$

were introduced for brevity. (Positivity of these constants is clearly necessary for the existence of asymptotically stable solutions.) Suppose that $v(\phi)$ and $B(\phi)$ are decaying solutions of Eqs. (A.206)–(A.207), and let us compare the magnitude of the terms in the right-hand side of Eq. (A.206) in the limit $\phi \to \infty$. There are only three possibilities: the first term dominates; the two terms have the same order; or the second term dominates. In other words, the limit of the ratio of the second term to the first,

$$q \equiv \lim_{\phi\to\infty} \frac{\tilde{Q}_1 K^{3/2} v^{n/2-1}}{K'\sqrt{B}}, \tag{A.209}$$

must be either zero, or finite but nonzero, or infinite. The value of q must be the same for every decaying solution $\{v(\phi), B(\phi)\}$ except perhaps for a discrete subset of solutions, which we may ignore for the purposes of stability analysis. In each of the three cases, Eqs. (A.206)–(A.207) are simplified and become amenable to asymptotic analysis in the limit $\phi \to \infty$. We will now consider these three possible values of q in turn.

If $q = 0$, we have at large ϕ

$$\frac{1}{n}\frac{K'}{K} \gg \tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}, \tag{A.210}$$

and thus only the first term is left in Eq. (A.206),

$$\frac{d\ln v}{d\phi} \approx -\frac{1}{n}\frac{K'}{K} \quad \Rightarrow \quad v \propto K^{-1/n}. \tag{A.211}$$

[10] This is case 6 in Sec. 2.4.1.

Decaying solutions have $v(\phi) \to 0$; so a necessary condition is $K^{-1/n}(\phi) \to 0$ at $\phi \to \infty$. With this $v(\phi)$, the condition (A.210) becomes

$$\left(K^{-1/n}\right)' \gg \frac{\tilde{Q}_1}{\sqrt{B}}. \tag{A.212}$$

However, this condition cannot be satisfied, since the left-hand side tends to zero at large ϕ while the right-hand side tends to infinity because $B \to 0$. Thus, decaying solutions $v(\phi), B(\phi)$ are impossible with $q = 0$.

If $q \neq 0$ and $|q| < \infty$, we consider q as an unknown constant that possibly depends on the solutions $v(\phi)$ and $B(\phi)$. At large ϕ, we have

$$q\frac{K'}{K} \approx \tilde{Q}_1 \frac{1}{\sqrt{B}} v^{n/2-1} K^{1/2}, \tag{A.213}$$

$$\frac{d \ln v}{d\phi} \approx -\frac{K'}{K}\left(\frac{1}{n} + q\right). \tag{A.214}$$

Therefore,

$$v(\phi) \propto K^{-q-1/n}(\phi) \to 0 \tag{A.215}$$

since we need a decaying solution $v(\phi)$. With this $v(\phi)$, Eq. (A.213) yields

$$\sqrt{B} \approx \frac{\tilde{Q}_1}{q} \frac{K^{1-(n/2-1)q+1/n}}{K'}. \tag{A.216}$$

For a decaying solution $B(\phi) \to 0$, we thus must have

$$\frac{1}{\sqrt{B}} \propto \frac{d}{d\phi}\left[K^{(n/2-1)q-1/n}\right] \to \infty \text{ as } \phi \to \infty, \tag{A.217}$$

and in particular

$$\left(\frac{n}{2} - 1\right) q - \frac{1}{n} \neq 0. \tag{A.218}$$

Substituting the expressions for $v(\phi)$ and $\sqrt{B(\phi)}$ into Eq. (A.207), we find

$$\begin{aligned}
&\frac{d\sqrt{B}}{d\phi} = -\tilde{Q}_2 K^{\frac{1}{2}-\left(\frac{n}{2}-1+p\right)\left(q+\frac{1}{n}\right)} \\
&= \frac{d}{d\phi}\frac{\tilde{Q}_1}{q}\frac{K^{1-\left(\frac{n}{2}-1\right)q+\frac{1}{n}}}{K'} \\
&= \frac{\tilde{Q}_1}{q} K^{-\left(\frac{n}{2}-1\right)q+\frac{1}{n}}\left[1 - \left(\frac{n}{2} - 1\right) q + \frac{1}{n} - \frac{KK''}{K'^2}\right].
\end{aligned} \tag{A.219}$$

This is now a closed equation for $K(\phi)$, which we may rewrite as

$$\begin{aligned}\left(\frac{K}{K'}\right)' &= 1 - \frac{KK''}{K'^2}\\ &= \left[\left(\frac{n}{2} - 1\right) q - \frac{1}{n}\right] - \frac{q\tilde{Q}_2}{\tilde{Q}_1} K^{-p\left(q+\frac{1}{n}\right)}.\end{aligned} \tag{A.220}$$

Due to the conditions (A.215), (A.218), and since $p > 0$, the right-hand side above tends to a nonzero limit as $\phi \to \infty$, namely

$$\left(\frac{K}{K'}\right)' \approx \left(\frac{n}{2} - 1\right) q - \frac{1}{n} \equiv \frac{1}{\alpha} \neq 0. \tag{A.221}$$

It follows that the only admissible form of the function $K(\phi)$ is

$$K(\phi) \propto \phi^\alpha, \quad \phi \to \infty. \tag{A.222}$$

However, this expression does not satisfy Eq. (A.217). Therefore, asymptotically stable solutions are impossible.

In the last case, $q = \infty$, we may disregard the first term in Eq. (A.206) and obtain

$$\frac{d \ln v}{d\phi} \approx -\tilde{Q}_1 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1}. \tag{A.223}$$

Then we can rewrite Eq. (A.207) as

$$\begin{aligned}\frac{d \ln \sqrt{B}}{d\phi} &= -\tilde{Q}_2 \frac{K^{1/2}}{\sqrt{B}} v^{n/2-1+p} = \frac{\tilde{Q}_2}{\tilde{Q}_1} v^p \frac{d \ln v}{d\phi}\\ &= \frac{\tilde{Q}_2}{p\tilde{Q}_1} \frac{d}{d\phi} v^p.\end{aligned} \tag{A.224}$$

This relationship between B and v can be integrated and yields

$$\sqrt{B} = \exp\left[C_1 + \frac{\tilde{Q}_2}{p\tilde{Q}_1} v^p\right], \tag{A.225}$$

where C_1 is a constant of integration. It follows that it is impossible to find simultaneously decaying solutions $v(\phi) \to 0$ and $B(\phi) \to 0$ at $\phi \to \infty$.

This concludes the consideration of the case $R_0 = 1$ and $w_m = \frac{1}{n-1}$, in which case there are no asymptotically stable solutions.

We now turn to the analysis of the case $R_0 < 1$. We first note that the leading terms of Eq. (A.176) do not contain R when $R \to R_0 < 1$. Therefore the stability analysis can be performed for $R(\phi)$ and $v(\phi)$ separately. Using the ansatz $R(\phi) = R_0 + B(\phi)$ and assuming a fixed solution $v(\phi)$, we find that the right-hand side of Eq. (A.177) is independent of $B(\phi)$ if $0 < R_0 < 1$. Therefore, general solutions $B(\phi)$ will not approach zero in case $R_0 \neq 0$. It remains to look for asymptotically stable solutions $v(\phi)$ and $R(\phi)$ in case $R_0 = 0$.

In case $R_0 = 0$, we begin by analyzing the asymptotic behavior of $v(\phi)$. Rewriting Eq. (A.176) as

$$\frac{du}{d\phi} = -\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} u^{\frac{n}{2}} K^{1/n}, \quad u \equiv K^{1/n} v, \tag{A.226}$$

we find the approximate general solutions (valid only for large ϕ)

$$u(\phi) = \exp\left[-3\kappa\sqrt{Q_1}\int_{\phi_0}^{\phi} K^{1/2} d\phi\right], \quad n = 2, \tag{A.227}$$

$$u(\phi) = \left[\frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}}\int_{\phi_0}^{\phi} K^{1/n} d\phi\right]^{-\frac{2}{n-2}}, \quad n > 2, \tag{A.228}$$

where ϕ_0 is an integration constant. The general solution $v = K^{-1/n}u$ should tend to zero as $\phi \to \infty$. We note that Eq. (A.226) is similar to Eq. (A.186) after the replacements

$$F(\phi) \equiv \frac{3\kappa\sqrt{Q_1}}{\sqrt{n-1}} K^{1/n}(\phi), \quad s \equiv \frac{n}{2}. \tag{A.229}$$

Therefore, we may use the conclusion obtained after Eq. (A.190), with the caveat that $F(\phi)$ is presently related to $K(\phi)$ uniquely, without an arbitrary proportionality factor. This was used to exclude the boundary case (A.203), which is presently still allowed. Thus the condition (A.197) obtained above,

$$\lim_{\phi\to\infty} \phi^{n/s} K(\phi) = \lim_{\phi\to\infty} \phi^2 K(\phi) = \infty, \tag{A.230}$$

is now merely a *sufficient* condition for the stability of the general solution $v(\phi)$. In the boundary case,

$$\lim_{\phi\to\infty} \phi^2 K(\phi) \equiv K_0, \quad 0 < K_0 < \infty, \tag{A.231}$$

we find

$$v(\phi) \propto \exp\left[\left(1 - 3\kappa\sqrt{Q_1 K_0}\right)\ln\phi\right], \quad n = 2, \tag{A.232}$$

$$v(\phi) \approx \text{const}, \quad n > 2. \tag{A.233}$$

Thus, the case (A.231) yields a stable solution for $v(\phi)$ when $n = 2$ and $3\kappa\sqrt{Q_1 K_0} > 1$. (The possibility $3\kappa\sqrt{Q_1 K_0} = 1$ is unphysical because it requires an infinitely precise fine-tuning of the parameters in the field Lagrangian.) Thus a sharp condition for the asymptotic stability of $v(\phi)$ is

$$K(\phi) \geq \frac{1}{9\kappa^2 Q_1}\phi^{-2} \quad \text{at } \phi \to \infty, \quad n = 2; \tag{A.234}$$

$$\lim_{\phi\to\infty} \phi^2 K(\phi) = \infty, \quad n > 2. \tag{A.235}$$

A weaker necessary condition is

$$\int^{\infty} K^{1/n}(\phi)d\phi = \infty. \tag{A.236}$$

Let us now consider the stability of the general solution for $R(\phi)$. It follows from Eq. (A.177) that

$$\frac{d\ln B}{d\phi} = -3\kappa\sqrt{(n-1)\,Q_1 K(\phi)}v^{\frac{n}{2}-1}\left(w_m - w_\phi(v)\right). \tag{A.237}$$

This equation integrates to

$$B(\phi) = B_0 \exp\left[-\text{const}\cdot\int^{\phi}\left(w_m - w_\phi(v)\right)K^{\frac{1}{2}}v^{\frac{n}{2}-1}d\phi\right], \tag{A.238}$$

where B_0 is an integration constant. The general solution for $B(\phi)$ will tend to zero as long as the integral in Eq. (A.238) diverges to a positive infinity at $\phi \to \infty$,

$$\int^{\infty}\left(w_m - w_\phi(v)\right)K^{\frac{1}{2}}v^{\frac{n}{2}-1}d\phi = \infty. \tag{A.239}$$

A necessary condition for that is $w_m \geq \frac{1}{n-1}$. Precise constraints on $K(\phi)$ for Eq. (A.239) can be obtained by considering the cases $n = 2$, $n \neq 2$, $w_m = \frac{1}{n-1}$, and $w_m > \frac{1}{n-1}$ separately.

If $w_m > \frac{1}{n-1}$, the condition (A.239) holds when

$$\int^{\infty} K^{\frac{1}{2}}v^{\frac{n}{2}-1}d\phi = \infty. \tag{A.240}$$

If $n = 2$, the above integral diverges due to the necessary condition (A.236). If $n > 2$, we use the solution (A.228), where $u = K^{1/n}v$, to obtain

$$\int^{\infty} K^{\frac{1}{2}}v^{\frac{n}{2}-1}d\phi = \int^{\infty} d\phi\left[\int_{\phi_0}^{\phi} K^{1/n}(\phi_1)d\phi_1\right]^{-1} K^{1/n}(\phi). \tag{A.241}$$

Temporarily introducing the auxiliary function

$$I(\phi) \equiv \int^{\phi} K^{1/n}(\phi)d\phi, \tag{A.242}$$

we note that $\lim_{\phi\to\infty} I(\phi) = \infty$ by Eq. (A.236). Therefore we express Eq. (A.241) through $I(\phi)$ and obtain

$$\begin{aligned}\int^{\infty} K^{\frac{1}{2}}v^{\frac{n}{2}-1}d\phi &= \int^{\infty}\frac{I'(\phi)}{I(\phi)}d\phi \\ &= \lim_{\phi\to\infty}\ln I(\phi) + \text{const} = \infty.\end{aligned} \tag{A.243}$$

Therefore, the general solution $B(\phi)$ tends to zero with $w_m > \frac{1}{n-1}$ for any $n \geq 2$ under the condition (A.236).

When $w_m = \frac{1}{n-1}$, it follows from Eq. (A.178) that the integrand in Eq. (A.238) acquires an additional factor proportional to v^p, where $p \geq 1$. Therefore the general solution for $B(\phi)$ will tend to zero only if

$$\int^{\infty} K^{1/2} v^{n/2-1+p} d\phi = \infty, \tag{A.244}$$

where we need to substitute $v(\phi) = K^{-1/n} u(\phi)$ and $u(\phi)$ as given by Eqs. (A.227)–(A.228).

Consider first the case $n = 2$; we will now show that the condition (A.244) is incompatible with the earlier condition (A.234). Using the solution (A.227), we can rewrite the condition (A.244) as

$$\int^{\infty} d\phi \, K^{(1-p)/2} \exp\left[-3p\kappa\sqrt{Q_1}\int_{\phi_0}^{\phi} K^{1/2} d\phi_1\right] = \infty. \tag{A.245}$$

By the condition (A.234), we have

$$K^{(1-p)/2} < \text{const} \cdot \phi^{p-1}, \tag{A.246}$$

$$\int_{\phi_0}^{\phi} K^{1/2}(\phi_1) d\phi_1 > \sqrt{K_0} \ln \phi + \text{const}. \tag{A.247}$$

Therefore the integral in Eq. (A.245) is bounded from above by

$$\text{const} \int^{\infty} d\phi \, \phi^{p-1-3p\kappa\sqrt{Q_1 K_0}} = \text{const} \int^{\infty} d\phi \, \phi^{-1-\alpha} < \infty, \tag{A.248}$$

where we temporarily denoted

$$\alpha \equiv \left(3\kappa\sqrt{Q_1 K_0} - 1\right) p > 0, \tag{A.249}$$

and so the condition (A.245) cannot hold.

It remains to consider the case $w_m = \frac{1}{n-1}$ and $n > 2$. Using Eq. (A.228), we rewrite the condition (A.244) as

$$\int^{\infty} K^{\frac{1-p}{n}}(\phi) \left[\int_{\phi_0}^{\phi} K^{1/n}(\phi_1) d\phi_1\right]^{-\frac{2p}{n-2}-1} d\phi = \infty. \tag{A.250}$$

According to Eq. (A.235), we must have

$$K(\phi) > C_0 \phi^{-2} \tag{A.251}$$

for any $C_0 > 0$ at large enough ϕ; thus $K(\phi)$ should decay slower than ϕ^{-2}. However, it is straightforward to verify that a power-law behavior

$$K(\phi) \propto \phi^{-2+\delta}, \quad \phi \to \infty,\ \delta > 0, \tag{A.252}$$

yields a convergent integral in Eq. (A.250). Therefore, the only possibility of having an asymptotically stable solution is to choose $K(\phi)$ such that it decays slower than ϕ^{-2} but faster than $\phi^{-2+\delta}$ for any $\delta > 0$. An example of an admissible choice of $K(\phi)$ is

$$K(\phi) \propto \phi^{-2} (\ln \phi)^{\alpha}, \quad \alpha > 0. \tag{A.253}$$

With this $K(\phi)$, we obtain the following asymptotic estimate at large ϕ,

$$\int_{\phi_0}^{\phi} K^{1/n}(\phi_1) d\phi_1 \propto \text{const} \cdot \phi^{1-2/n} (\ln \phi)^{\alpha/n}, \tag{A.254}$$

and so the integral (A.250) becomes, after some algebra,

$$\text{const} \int^{\infty} \phi^{-1} (\ln \phi)^{-\alpha \frac{p}{n-2}} d\phi = \infty \quad \text{if } \alpha \frac{p}{n-2} \leq 1. \tag{A.255}$$

Since the convergence of the integral in Eq. (A.250) monotonically depends on the growth properties of the function $K(\phi)$, it is clear that the condition (A.250) will also hold for functions $K(\phi)$ satisfying Eq. (A.235) but growing slower than those given in Eq. (A.253). However, the condition (A.250) may not hold for $K(\phi)$ growing faster than those in Eq. (A.253).

To investigate the admissible class of functions $K(\phi)$ more precisely, let us use the ansatz

$$K(\phi) = \phi^{-2} K_0(\phi), \tag{A.256}$$

where $K_0(\phi)$ is a function growing slower than any power of ϕ. Then we have an asymptotic estimate (for $n > 2$)

$$\int_{\phi_0}^{\phi} K^{1/n}(\phi_1) d\phi_1 \approx \text{const} \cdot \phi^{1-2/n} (K_0(\phi))^{1/n}, \tag{A.257}$$

and we can rewrite Eq. (A.244) as

$$\begin{aligned} &\int^{\infty} K^{\frac{1-p}{n}}(\phi) \left[\int_{\phi_0}^{\phi} K^{1/n}(\phi_1) d\phi_1 \right]^{-\frac{2p}{n-2}-1} d\phi \\ &= \text{const} \cdot \int^{\infty} \phi^{-1} [K_0(\phi)]^{-\frac{p}{n-2}} d\phi = \infty. \end{aligned} \tag{A.258}$$

Substituting $K_0 = \phi^2 K$ into Eq. (A.258), we find that the conditions (A.235) and (A.250) are equivalent to

$$\lim_{\phi \to \infty} \phi^2 K(\phi) = \infty, \quad \int^{\infty} \phi^{-1-\frac{2p}{n-2}} [K(\phi)]^{-\frac{p}{n-2}} d\phi = \infty. \tag{A.259}$$

The condition (A.235) guarantees the stability of $v(\phi)$, while Eq. (A.250) guarantees the stability of $B(\phi)$. Therefore, Eq. (A.259) is a sharp (necessary and sufficient) condition for the stability of the solution $\{v, B\}$.

A sufficient (but not a necessary) condition for the divergence of the integral in Eq. (A.258) is

$$\lim_{\phi\to\infty} (\ln\phi)^{-\frac{n-2}{p}} K_0(\phi) < \infty. \tag{A.260}$$

The corresponding sufficient condition for $K(\phi)$ is

$$\lim_{\phi\to\infty} \phi^2 K(\phi) = \infty, \quad \lim_{\phi\to\infty} \phi^2 (\ln\phi)^{-\frac{n-2}{p}} K(\phi) < \infty. \tag{A.261}$$

The sharp condition (A.259) cannot be restated in terms of the asymptotic behavior of $K(\phi)$ at $\phi \to \infty$, but of course one can check whether Eq. (A.259) holds for a given $K(\phi)$. The condition (A.259) specifies a rather narrow class of functions; however, we strive for generality and avoid prejudice regarding the possible Lagrangians.

In this section we have shown that asymptotically stable solutions exist with $v_0 = 0$ and $Q(0) = 0$ only in the following cases: (a) Asymptotic value $R_0 = 1$. Expansion (A.172) holds with $Q_1 > 0$, determining the value of n, which should be $n > 2$; $-\frac{n-3}{n-1} < w_m < \frac{1}{n-1}$ according to Eq. (A.191); and $K(\phi)$ satisfies Eq. (A.197), where s is defined by Eq. (A.187).[11] There are no stable solutions when $w_m = \frac{1}{n-1}$ and expansion (A.178) holds. (b) Asymptotic value $R_0 = 0$. Expansion (A.172) holds with $Q_1 > 0$, determining the value of $n \geq 2$; either $n = 2$, $w_m > 1$, and $K(\phi)$ satisfies Eq. (A.234)[12] or Eq. (A.235),[13] or $n > 2$, $w_m = \frac{1}{n-1}$, and $K(\phi)$ satisfies Eq. (A.259).[14]

[11] This is case 6 in Sec. 2.4.1.
[12] This is case 7 in Sec. 2.4.1.
[13] This is case 11 in Sec. 2.4.1.
[14] This is case 12 in Sec. 2.4.1.

B Exorcising ghosts?

After this paper was submitted, one of the authors of the new ekpyrotic scenario argued [112] that, according to [113], ghosts can be removed by field redefinitions and adding other degrees of freedom in the effective UV theory [113]. Let us reproduce this argument and explain why it does not apply to the ghost condensate theory and to the new ekpyrotic scenario.

Refs. [112, 113] considered a normal massless scalar field ϕ with Lagrangian density in $(-,+,+,+)$ signature.[1]

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 + \frac{a}{2m_g^2}(\Box\phi)^2 - V_{\rm int}(\phi)\ , \tag{B.1}$$

where $a = \pm 1$, and $V_{\rm int}$ is a self-interaction term. This theory is similar to the ghost condensate/new ekpyrotic theory in the case $a = -1$, $c_s = 1$, see eqs. (1) and (3.24). The sign of a is crucially important: the term $+\frac{1}{2m_g^2}(\Box\phi)^2$ would not protect this theory against the gradient instability in the region with the NEC violation.

Note that in notation of [112, 113], $m_g = \Lambda$, which could suggest that the ghost mass is a UV cut-off, and therefore there are no dangerous excitations with energies and momenta higher than m_g. However, this interpretation of the theory (B.1) would be misleading. Upon a correct quantization, this theory can be represented as a theory of two fields without the higher derivative non-renormalizable term $\frac{a}{2m_g^2}(\Box\phi)^2$, see Eq. (3.42). One can introduce the UV cut-off Λ when regularizing Feynman diagrams in this theory, but there is absolutely no reason to identify it with m_g; in fact, the UV cut-off which appears in the regularization procedure is supposed to be arbitrarily large, so the perturbations with frequencies greater than m_g should not be forbidden.

Moreover, as we already explained in Section 2, one cannot take the higher derivative term into account only up to some cut-off $\omega^2, k^2 < \Lambda^2$. If, for example, we "turn on" this term only at $k^2 < \Lambda^2$, it is not going to protect us from the gradient instability, which occurs at $\omega^2 = P_{,X}\, k^2$ for all indefinitely large k in the region where the NEC is violated and $P_{,X} < 0$. Note that this instability grows stronger for greater values of momenta k. Therefore if one wants to prove that the new ekpyrotic scenario does not lead to instabilities, one must

[1] In our paper we used the signature $(+,-,-,-)$, so some care should be taken when comparing the equations. Note that this does not change the sign of the higher derivative term $\sim a(\Box\phi)^2$; the ghost condensate/ekpyrotic theory corresponds to $a = -1$.

verify it for *all* values of momenta. Checking it for $\omega^2, k^2 < m_g^2$ is insufficient. Our results imply that if one investigates this model exactly in the way it is written now (i.e. with the term $-\frac{1}{2m_g^2}(\Box\phi)^2$), it does suffer from vacuum instability, and if we discard the higher derivative term at momenta greater that some cut-off, the instability becomes even worse. Is there any other way to save the new ekpyrotic scenario?

One could argue [112, 113] that the term $\frac{a}{2m_g^2}(\Box\phi)^2$ is just the first term in a sum of many higher derivative terms in an effective theory, which can be obtained by integration of high energy degrees of freedom of some extended physically consistent theory. In other words, one may conjecture that the theory can be made UV complete, and after that the problem with ghosts disappears. However, not every theory with higher derivatives can be UV completed. In particular, the possibility to do it may depend on the sign of the higher derivative term [40].

According to [113], the theory (B.1) is plagued by ghosts independently of the sign of the higher derivative term in the Lagrangian. One can show it by introducing an auxiliary scalar field χ and a new Lagrangian

$$\mathcal{L}' = -\tfrac{1}{2}(\partial\phi)^2 - a\,\partial_\mu\chi\partial^\mu\phi - \tfrac{1}{2}a\,m_g^2\,\chi^2 - V_{\rm int}(\phi)\ , \tag{B.2}$$

which reduces exactly to $\mathcal{L}$ once χ is integrated out. $\mathcal{L}'$ is diagonalized by the substitution $\phi = \phi' - a\chi$:

$$\mathcal{L}' = -\tfrac{1}{2}(\partial\phi')^2 + \tfrac{1}{2}(\partial\chi)^2 - \tfrac{1}{2}a\,m_g^2\,\chi^2 - V_{\rm int}(\phi',\chi)\ , \tag{B.3}$$

which clearly signals the presence of a ghost: χ has a wrong-sign kinetic term.

Then the authors of [113] identified χ as a tachyon for $a = -1$, suggesting that in this case χ has exponentially growing modes. However, this is not the case: due to the opposite sign of the kinetic term for the χ-field, the tachyon is at $a = +1$, not at $a = -1$. Indeed, because of the flip of the sign of the kinetic term for the field χ, its equation of motion has a solution $\chi \sim e^{\pm i(\omega t - \mathbf{kx})}$ with

$$-(\omega^2 - \mathbf{k}^2) = a\,m_g^2\ . \tag{B.4}$$

For the field with the normal sign of the kinetic term, the negative mass squared would mean exponentially growing modes. But the flip of the sign of the kinetic term performed together with the flip of the sign of the mass term does not lead to exponentially growing modes [94, 95]. Based on the misidentification of the negative mass of the field with the wrong kinetic terms as a tachyon, the authors choose to continue with the $a = +1$ case in eq. (B.1). Starting from this point, their arguments are no longer related to the ghost condensate theory and the new ekpyrotic theory, where $a = -1$. We will return to the case $a = -1$ shortly.

For the $a = 1$ case they argued that the situation is not as bad as it could seem. They proposed to use the scalar field theory eq. (B.1) at energies below m_g, and postulated that some new degree of freedom enters at $k > m_g$ and takes care of the ghost instability.

The authors describe this effect by adding a term $-(\partial\chi)^2$ to construct the high energy Lagrangian. For $V_{\rm int} = 0$ they postulate

$$\mathcal{L}_{\rm UV}^{a=1} \equiv \mathcal{L}' - (\partial\chi)^2 = -\tfrac{1}{2}(\partial\phi)^2 - \partial_\mu\chi\partial^\mu\phi - (\partial\chi)^2 - \tfrac{1}{2}\, m_g^2\, \chi^2 \tag{B.5}$$

and use the shift $\phi = \tilde{\phi} - \chi$ to get a simple form of a UV theory. This trick reverses the sign of the kinetic term of the field χ, and the ghost magically converts into a perfectly healthy scalar with mass m_g:

$$\mathcal{L}_{\rm UV}^{a=1} = -\tfrac{1}{2}(\partial\tilde{\phi})^2 - \tfrac{1}{2}(\partial\chi)^2 - \tfrac{1}{2}\, m_g^2\, \chi^2\ . \tag{B.6}$$

One may question validity of this procedure, but let us try to justify it by looking at the final result. Consider equations of motion for χ from eq. (B.5) and solve them by iteration in the approximation when $\Box \ll m_g^2$:

$$\chi = \frac{\Box\phi}{m_g^2} + 2\frac{\Box\chi}{m_g^2} \approx \frac{\Box\phi}{m_g^2} + 2\frac{\Box^2\phi}{m_g^4} + ... \tag{B.7}$$

Now replace χ in eq. (B.5) by its expression in terms of $\Box\phi$ as given in eq. (B.7). The result is our original Lagrangian (B.1), plus some additional higher derivative terms, which are small at $|\Box| \ll m_g^2$, i.e. at $|\omega^2 - k^2| \ll m_g^2$. Thus one may conclude that, for $a = 1$, the theory (B.1), which has tachyonic ghosts, may be interpreted as a low energy approximation of the UV consistent theory (B.6).

Now let us return to the ghost condensate/new ekpyrotic case. To avoid gradient instabilities in the ekpyrotic scenario, the sign of the higher derivative term in eq. (B.1) has to be negative, $a = -1$, see eq. (1) and also eq. (3.13) and the discussion below it. This means that one should start with eq. (B.1) with $a = -1$.

This theory is not tachyonic, but, as we demonstrated by performing its Hamiltonian quantization, it has ghosts, particles with negative energy, in its spectrum. Can we improve the situation by the method used above? Let us start with the same formula, $\mathcal{L}' - (\partial\chi)^2$, as in $a = +1$ case:

$$\mathcal{L}_{\rm UV}^{a=-1} \equiv \mathcal{L}' - (\partial\chi)^2 = -\tfrac{1}{2}(\partial\phi)^2 + \partial_\mu\chi\partial^\mu\phi - (\partial\chi)^2 + \tfrac{1}{2}\, m_g^2\, \chi^2 \tag{B.8}$$

and replace χ by the iterative solution of its equation of motion

$$\chi = \frac{\Box\phi}{m_g^2} - 2\frac{\Box\chi}{m_g^2} \approx \frac{\Box\phi}{m_g^2} - 2\frac{\Box^2\phi}{m_g^4} + ... \tag{B.9}$$

Thus, up to the terms which are small at $|\Box| \ll m_g^2$, the theory with the Lagrangian (B.8) does reproduce the model (B.1) with $a = -1$, up to higher order corrections in $|\Box|/m_g^2$. The theory (B.8) can be also written as

$$\mathcal{L}_{\rm UV}^{a=-1} = -\tfrac{1}{2}(\partial\tilde{\phi})^2 - \tfrac{1}{2}(\partial\chi)^2 + \tfrac{1}{2}\, m_g^2\, \chi^2\ . \tag{B.10}$$

where $\phi = \tilde{\phi} + \chi$. The sign of the kinetic term of both fields is normal, but the mass term still has the wrong sign, which leads to the tachyonic instability $\delta\chi \sim \exp\sqrt{m_g^2 - \mathbf{k}^2}\, t$. Therefore the cure for the ghost instability proposed in [112, 113] does not work for the case $a = -1$ of the ghost condensate/ekpyrotic scenario.

Moreover, the procedure described above is valid only for $|\omega^2 - \mathbf{k}^2| \ll m_g^2$. Meanwhile the gradient instability of the ekpyrotic theory in the regime of null energy condition violation ($c_s^2 < 0$) is most dangerous in the limit $\mathbf{k}^2 \to \infty$, where this procedure does not apply, see (3.12). This agrees with the general negative conclusion of Refs. [40, 42, 68, 69] with respect to the theories of this type.

In this Appendix we analyzed the Lorentz-invariant theory (B.1) because the argument given in [112, 113] was formulated in this context. The generalization of our results for the ghost condensate/new ekpyrotic case is straightforward. Indeed, our results directly follow from the correlation between the sign of the higher derivative term in (B.2) and the sign of the mass squared term in (B.3). One can easily verify that this correlation is valid independently of the value of c_s^2, i.e. at all stages of the ghost condensate/new ekpyrotic scenario.

To conclude, our statement that the ghost condensate theory and the new ekpyrotic scenario imply the existence of ghosts is valid for the currently available versions of these theories, as they are presented in the literature. In this Appendix we explained why the recent attempts to make the theories with higher derivatives physically consistent [112, 113] do not apply to the ghost condensate theory and the new ekpyrotic scenario. One can always hope that one can save the new ekpyrotic scenario and provide a UV completion of this theory in some other way, but similarly one can always hope that the problem of the cosmological singularity will be solved in some other way. Until it is done, one should not claim that the problem is already solved.

Bibliography

[1] M. Born and L. Infeld. Foundations of the new field theory. *Proc. Roy. Soc. Lond.*, A144:425–451, 1934.

[2] W. Heisenberg. Mesonenerzeugung als Stosswellenproblem. *Z. Phys.*, 133:65, 1952.

[3] Paul A. M. Dirac. An Extensible model of the electron. *Proc. Roy. Soc. Lond.*, A268:57–67, 1962.

[4] Steven Weinberg. Effective Field Theory for Inflation. *Phys. Rev.*, D77:123541, 2008.

[5] Thibault Damour and Viatcheslav F. Mukhanov. Inflation without slow roll. *Phys. Rev. Lett.*, 80:3440–3443, 1998.

[6] Curtis G. Callan and Juan Martin Maldacena. Brane dynamics from the Born-Infeld action. *Nucl. Phys.*, B513:198–212, 1998.

[7] G. W. Gibbons. Born-Infeld particles and Dirichlet p-branes. *Nucl. Phys.*, B514:603–639, 1998.

[8] G W Gibbons. Aspects of Born-Infeld theory and string / M-theory. *Rev. Mex. Fis.*, 49S1:19–29, 2003.

[9] Ashoke Sen. Tachyon matter. *JHEP*, 07:065, 2002.

[10] N. D. Lambert and I. Sachs. On higher derivative terms in tachyon effective actions. *JHEP*, 06:060, 2001.

[11] C. Armendariz-Picon, T. Damour, and Viatcheslav F. Mukhanov. k-Inflation. *Phys. Lett.*, B458:209–218, 1999.

[12] Nima Arkani-Hamed, Hsin-Chia Cheng, Markus A. Luty, and Shinji Mukohyama. Ghost condensation and a consistent infrared modification of gravity. *JHEP*, 05:074, 2004.

[13] Nima Arkani-Hamed, Paolo Creminelli, Shinji Mukohyama, and Matias Zaldarriaga. Ghost Inflation. *JCAP*, 0404:001, 2004.

[14] Viatcheslav F. Mukhanov and Alexander Vikman. Enhancing the tensor-to-scalar ratio in simple inflation. *JCAP*, 0602:004, 2006.

[15] Xingang Chen, Min-xin Huang, Shamit Kachru, and Gary Shiu. Observational signatures and non-Gaussianities of general single field inflation. *JCAP*, 0701:002, 2007.

[16] Takeshi Chiba, Takahiro Okabe, and Masahide Yamaguchi. Kinetically driven quintessence. *Phys. Rev.*, D62:023511, 2000.

[17] C. Armendariz-Picon, Viatcheslav F. Mukhanov, and Paul J. Steinhardt. A dynamical solution to the problem of a small cosmological constant and late-time cosmic acceleration. *Phys. Rev. Lett.*, 85:4438–4441, 2000.

[18] C. Armendariz-Picon, Viatcheslav F. Mukhanov, and Paul J. Steinhardt. Essentials of k-essence. *Phys. Rev.*, D63:103510, 2001.

[19] Paolo Creminelli, Markus A. Luty, Alberto Nicolis, and Leonardo Senatore. Starting the universe: Stable violation of the null energy condition and non-standard cosmologies. *JHEP*, 12:080, 2006.

[20] Paolo Creminelli and Leonardo Senatore. A smooth bouncing cosmology with scale invariant spectrum. *JCAP*, 0711:010, 2007.

[21] Evgeny I. Buchbinder, Justin Khoury, and Burt A. Ovrut. New Ekpyrotic Cosmology. *Phys. Rev.*, D76:123503, 2007.

[22] Edmund J. Copeland, M. Sami, and Shinji Tsujikawa. Dynamics of dark energy. *Int. J. Mod. Phys.*, D15:1753–1936, 2006.

[23] Alexander Vikman. Can dark energy evolve to the phantom? *Phys. Rev.*, D71:023515, 2005.

[24] Robert R. Caldwell and Michael Doran. Dark-energy evolution across the cosmological-constant boundary. *Phys. Rev.*, D72:043527, 2005.

[25] Jian-gang Hao and Xin-zhou Li. Phantom with Born-Infield type Lagrangian. *Phys. Rev.*, D68:043501, 2003.

[26] Daniele Bertacca, Sabino Matarrese, and Massimo Pietroni. Unified dark matter in scalar field cosmologies. *Mod. Phys. Lett.*, A22:2893–2907, 2007.

[27] Takeshi Chiba. Tracking k-essence. *Phys. Rev.*, D66:063514, 2002.

[28] Rupam Das, Thomas W. Kephart, and Robert J. Scherrer. Tracking quintessence and k-essence in a general cosmological background. *Phys. Rev.*, D74:103515, 2006.

[29] Hui Li, Zong-Kuan Guo, and Yuan-Zhong Zhang. Parametrization of K-essence and its kinetic term. *Mod. Phys. Lett.*, A21:1683–1690, 2006.

[30] Robert J. Scherrer. Purely kinetic k-essence as unified dark matter. *Phys. Rev. Lett.*, 93:011301, 2004.

[31] Michael Malquarti, Edmund J Copeland, Andrew R Liddle, and Mark Trodden. A new view of k-essence. *Phys. Rev.*, D67:123503, 2003.

[32] Hao Wei and Rong-Gen Cai. K-chameleon and the coincidence problem. *Phys. Rev.*, D71:043504, 2005.

[33] Alan D. Rendall. Dynamics of k-essence. *Class. Quant. Grav.*, 23:1557–1570, 2006.

[34] Eva Silverstein and David Tong. Scalar Speed Limits and Cosmology: Acceleration from D- cceleration. *Phys. Rev.*, D70:103505, 2004.

[35] Mohsen Alishahiha, Eva Silverstein, and David Tong. DBI in the sky. *Phys. Rev.*, D70:123505, 2004.

[36] Gianluca Calcagni and Andrew R. Liddle. Tachyon dark energy models: Dynamics and constraints. *Phys. Rev.*, D74:043528, 2006.

[37] Wei Fang, H. Q. Lu, and Z. G. Huang. Cosmologies with general non-canonical scalar field. *Class. Quant. Grav.*, 24:3799–3811, 2007.

[38] Camille Bonvin, Chiara Caprini, and Ruth Durrer. A no-go theorem for k-essence dark energy. *Phys. Rev. Lett.*, 97:081303, 2006.

[39] Stefano Liberati, Sebastiano Sonego, and Matt Visser. Faster-than-c signals, special relativity, and causality. *Annals Phys.*, 298:167–185, 2002.

[40] Allan Adams, Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, and Riccardo Rattazzi. Causality, analyticity and an IR obstruction to UV completion. *JHEP*, 10:014, 2006.

[41] E. Babichev, Viatcheslav F. Mukhanov, and A. Vikman. Escaping from the black hole? *JHEP*, 09:061, 2006.

[42] S. L. Dubovsky and S. M. Sibiryakov. Spontaneous breaking of Lorentz invariance, black holes and perpetuum mobile of the 2nd kind. *Phys. Lett.*, B638:509–514, 2006.

[43] Jean-Philippe Bruneton. On causality and superluminal behavior in classical field theories. Applications to k-essence theories and MOND-like theories of gravity. *Phys. Rev.*, D75:085013, 2007.

[44] Eugeny Babichev, Viatcheslav Mukhanov, and Alexander Vikman. Looking beyond the horizon. 2007.

[45] George Ellis, Roy Maartens, and Malcolm A. H. MacCallum. Causality and the speed of sound. *Gen. Rel. Grav.*, 39:1651–1660, 2007.

[46] Camille Bonvin, Chiara Caprini, and Ruth Durrer. Superluminal motion and closed signal curves. 2007.

[47] Ferdinand Helmer and Sergei Winitzki. Self-reproduction in k-inflation. *Phys. Rev.*, D74:063528, 2006.

[48] Matt Visser. The quantum physics of chronology protection. 2002.

[49] Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. The ekpyrotic universe: Colliding branes and the origin of the hot big bang. *Phys. Rev.*, D64:123522, 2001.

[50] Renata Kallosh, Lev Kofman, and Andrei D. Linde. Pyrotechnic universe. *Phys. Rev.*, D64:123523, 2001.

[51] Renata Kallosh, Lev Kofman, Andrei D. Linde, and Arkady A. Tseytlin. BPS branes in cosmology. *Phys. Rev.*, D64:123524, 2001.

[52] Paul J. Steinhardt and Neil Turok. Cosmic evolution in a cyclic universe. *Phys. Rev.*, D65:126003, 2002.

[53] Gary N. Felder, Andrei V. Frolov, Lev Kofman, and Andrei V. Linde. Cosmology with negative potentials. *Phys. Rev.*, D66:023507, 2002.

[54] Andrei Linde. Inflationary theory versus ekpyrotic / cyclic scenario. 2002.

[55] Hong Liu, Gregory W. Moore, and Nathan Seiberg. Strings in a time-dependent orbifold. *JHEP*, 06:045, 2002.

[56] Gary T. Horowitz and Joseph Polchinski. Instability of spacelike and null orbifold singularities. *Phys. Rev.*, D66:103512, 2002.

[57] Micha Berkooz and Dori Reichmann. A short review of time dependent solutions and space-like singularities in string theory. *Nucl. Phys. Proc. Suppl.*, 171:69–87, 2007.

[58] Neil Turok, Ben Craps, and Thomas Hertog. From Big Crunch to Big Bang with AdS/CFT. 2007.

[59] Thomas Hertog and Gary T. Horowitz. Holographic description of AdS cosmologies. *JHEP*, 04:005, 2005.

[60] Evgeny I. Buchbinder, Justin Khoury, and Burt A. Ovrut. On the Initial Conditions in New Ekpyrotic Cosmology. *JHEP*, 11:076, 2007.

[61] Nima Arkani-Hamed, Hsin-Chia Cheng, Markus A. Luty, Shinji Mukohyama, and Toby Wiseman. Dynamics of Gravity in a Higgs Phase. *JHEP*, 01:036, 2007.

[62] Andrei D. Linde and Viatcheslav F. Mukhanov. Nongaussian isocurvature perturbations from inflation. *Phys. Rev.*, D56:535–539, 1997.

[63] Kari Enqvist and Martin S. Sloth. Adiabatic CMB perturbations in pre big bang string cosmology. *Nucl. Phys.*, B626:395–409, 2002.

[64] David H. Lyth and David Wands. Generating the curvature perturbation without an inflaton. *Phys. Lett.*, B524:5–14, 2002.

[65] Takeo Moroi and Tomo Takahashi. Effects of cosmological moduli fields on cosmic microwave background. *Phys. Lett.*, B522:215–221, 2001.

[66] Jean-Luc Lehners, Paul McFadden, Neil Turok, and Paul J. Steinhardt. Generating ekpyrotic curvature perturbations before the big bang. *Phys. Rev.*, D76:103501, 2007.

[67] Andrew J. Tolley and Daniel H. Wesley. Scale-invariance in expanding and contracting universes from two-field models. *JCAP*, 0705:006, 2007.

[68] Christopher Eling, Brendan Z. Foster, Ted Jacobson, and Aron C. Wall. Lorentz violation and perpetual motion. *Phys. Rev.*, D75:101502, 2007.

[69] Nima Arkani-Hamed, Sergei Dubovsky, Alberto Nicolis, Enrico Trincherini, and Giovanni Villadoro. A Measure of de Sitter Entropy and Eternal Inflation. *JHEP*, 05:055, 2007.

[70] M. Novello and S. E. Perez Bergliaffa. Bouncing Cosmologies. *Phys. Rept.*, 463:127–213, 2008.

[71] Robert H. Brandenberger. String Gas Cosmology. 2008.

[72] Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. The ekpyrotic universe: Colliding branes and the origin of the hot big bang. *Phys. Rev.*, D64:123522, 2001.

[73] Justin Khoury, Burt A. Ovrut, Nathan Seiberg, Paul J. Steinhardt, and Neil Turok. From big crunch to big bang. *Phys. Rev.*, D65:086007, 2002.

[74] Paul J. Steinhardt and Neil Turok. Cosmic evolution in a cyclic universe. *Phys. Rev.*, D65:126003, 2002.

[75] Renata Kallosh, Jin U. Kang, Andrei Linde, and Viatcheslav Mukhanov. The New Ekpyrotic Ghost. *JCAP*, 0804:018, 2008.

[76] Abhay Ashtekar. Singularity Resolution in Loop Quantum Cosmology: A Brief Overview. 2008.

[77] Ben Craps, Savdeep Sethi, and Erik P. Verlinde. A Matrix Big Bang. *JHEP*, 10:005, 2005.

[78] Daniela Klammer and Harold Steinacker. Cosmological solutions of emergent noncommutative gravity. 2009.

[79] M. Gasperini and G. Veneziano. String Theory and Pre-big bang Cosmology. 2007.

[80] Jin U. Kang, Vitaly Vanchurin, and Sergei Winitzki. Attractor scenarios and superluminal signals in k-essence cosmology. *Phys. Rev.*, D76:083511, 2007.

[81] Jaume Garriga and Viatcheslav F. Mukhanov. Perturbations in k-inflation. *Phys. Lett.*, B458:219–225, 1999.

[82] C. Armendariz-Picon and Eugene A. Lim. Haloes of k-essence. *JCAP*, 0508:007, 2005.

[83] Christopher J. Fewster and C. G. Wells. Unitary of quantum theory and closed timelike curves. *Phys. Rev.*, D52:5773–5782, 1995.

[84] Seth Rosenberg. Testing causality violation on spacetimes with closed timelike curves. *Phys. Rev.*, D57:3365–3377, 1998.

[85] Allen Everett. Time travel paradoxes, path integrals, and the many worlds interpretation of quantum mechanics. *Phys. Rev.*, D69:124023, 2004.

[86] Amos Ori. Formation of closed timelike curves in a composite vacuum/dust asymptotically-flat spacetime. *Phys. Rev.*, D76:044002, 2007.

[87] M. Ostrogradski. *Memoires sur les equations differentielles relatives au probleme des isoperimetres.* Mem. Ac. St. Petersbourg, VI **4**, 1850.

[88] F. J. de Urries and J. Julve. Ostrogradski formalism for higher-derivative scalar field theories. *J. Phys.*, A31:6949–6964, 1998.

[89] H. A. Weldon. Quantization of higher-derivative field theories. *Annals Phys.*, 305:137–150, 2003.

[90] I. Ya. Aref'eva and I. V. Volovich. On the null energy condition and cosmology. *Theor. Math. Phys.*, 155:503–511, 2008.

[91] Brett McInnes. The dS/CFT correspondence and the big smash. *JHEP*, 08:029, 2002.

[92] Robert R. Caldwell, Marc Kamionkowski, and Nevin N. Weinberg. Phantom Energy and Cosmic Doomsday. *Phys. Rev. Lett.*, 91:071301, 2003.

[93] Yi-Fu Cai, Taotao Qiu, Robert Brandenberger, Yun-Song Piao, and Xinmin Zhang. On Perturbations of Quintom Bounce. *JCAP*, 0803:013, 2008.

[94] Andrei D. Linde. The Inflationary Universe. *Rept. Prog. Phys.*, 47:925–986, 1984.

[95] Andrei D. Linde. The universe multiplication and the cosmological constant problem. *Phys. Lett.*, B200:272, 1988.

[96] Sean M. Carroll, Mark Hoffman, and Mark Trodden. Can the dark energy equation-of-state parameter w be less than -1? *Phys. Rev.*, D68:023509, 2003.

[97] James M. Cline, Sangyong Jeon, and Guy D. Moore. The phantom menaced: Constraints on low-energy effective ghosts. *Phys. Rev.*, D70:043543, 2004.

[98] Roman V. Buniy and Stephen D. H. Hsu. Instabilities and the null energy condition. *Phys. Lett.*, B632:543–546, 2006.

[99] David E. Kaplan and Raman Sundrum. A symmetry for the cosmological constant. *JHEP*, 07:042, 2006.

[100] Roman V. Buniy, Stephen D. H. Hsu, and Brian M. Murray. The null energy condition and instability. *Phys. Rev.*, D74:063518, 2006.

[101] Philipp Hoffer v. Loewenfeld, Jin U Kang, Nicolas Moeller, and Ivo Sachs. Bouncing universe and non-BPS branes. 2009.

[102] Maurizio Gasperini. *Elements of string cosmology.* Cambridge Univ. Press, Cambridge, 1. publ. edition, 2007.

[103] N. D. Lambert and I. Sachs. Tachyon dynamics and the effective action approximation. *Phys. Rev.*, D67:026005, 2003.

[104] Ashoke Sen. Supersymmetric world-volume action for non-BPS D-branes. *JHEP*, 10:008, 1999.

[105] David Kutasov, Marcos Marino, and Gregory W. Moore. Remarks on tachyon condensation in superstring field theory. 2000.

[106] Per Kraus and Finn Larsen. Boundary string field theory of the DD-bar system. *Phys. Rev.*, D63:106004, 2001.

[107] Tadashi Takayanagi, Seiji Terashima, and Tadaoki Uesugi. Brane-antibrane action from boundary string field theory. *JHEP*, 03:019, 2001.

[108] Curtis G. Callan, Igor R. Klebanov, Andreas W. W. Ludwig, and Juan Martin Maldacena. Exact solution of a boundary conformal field theory. *Nucl. Phys.*, B422:417–448, 1994.

[109] Andreas Recknagel and Volker Schomerus. Boundary deformation theory and moduli spaces of D- branes. *Nucl. Phys.*, B545:233–282, 1999.

[110] Ashoke Sen. Rolling Tachyon. *JHEP*, 04:048, 2002.

[111] Robert H. Brandenberger, Andrew R. Frey, and Sugumi Kanno. Towards A Nonsingular Tachyonic Big Crunch. *Phys. Rev.*, D76:063502, 2007.

[112] Justin Khoury. New Ekpyrotic Cosmology and Non-Gaussianity. a talk at the conference. *The very early universe*, Cambridge, UK, 17-20 December, 2007.

[113] Paolo Creminelli, Alberto Nicolis, Michele Papucci, and Enrico Trincherini. Ghosts in massive gravity. *JHEP*, 09:003, 2005.

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