Holography in External Fields and in Time Dependent Backgrounds

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To Shuangyu and Felix

Zusammenfassung

Holographische Theorien beschreiben höherdimensionale physikalische Systeme durch eine niederdimensionale Theorie oder umgekehrt. In der Stringtheorie sind zwei derartige Situationen bekannt: Die Anti-de Sitter/Conformal Field Theorie-Korrespondenz (AdS/CFT-Korrespondenz), welche in Kapitel 2 näher erläutert wird, ermöglicht eine äquivalente Beschreibung von Stringtheorie in Räumen mit fünfdimensionaler Anti-de Sitter-Asymptotik durch Eichtheorien auf dem konformen Rand dieser Räume, insbesondere dem vierdimensionalen Minkowskiraum. Umgekehrt ermöglicht die Korrespondenz die Beschreibung von Eichtheorien bei starker Kopplung durch zehndimensionale Supergravitation. Andererseits beschreiben – wie in Kapitel 5 ausgeführt – quantenmechanische Modelle mit matrixwertigen Freiheitsgraden, sogenannte Matrixmodelle, holographisch die elfdimensionale $\mathcal{M}$-Theorie, eine aus der Stringtheorie folgenden Quantentheorie der elfdimensionalen $\mathcal{N} = 1$ Supergravitation. Diese Dissertation beschäftigt sich mit drei Verallgemeinerungen dieser beiden holographischen Modelle auf physikalische Systeme mit weniger Symmetrien.

Die AdS/CFT-Korrespondenz wurde, mit Hinblick auf eine zukünftige Anwendbarkeit zur Beschreibung der Infrarotdynamik von QCD, insbesondere auf Eichtheorien mit fundamentaler Materie (Quarks) erweitert. In Kapitel 3 wird eine dieser Erweiterungen, das $\mathcal{N} = 2$ supersymmetrische D3-D7-Modell, in konstanten äußeren elektrischen und magnetischen Feldern sowie bei endlicher Temperatur analysiert [1], wobei die Lorentzsymmetrie explizit gebrochen ist. Ich zeige, daß das Magnetfeld spontane chirale Symmetriebrechung induziert und mesonische Anregungen stabilisiert. Das elektrische Feld hingegen destabilisiert die Mesonen und induziert einen Metall-Isolator-Phasenübergang bei endlicher Quarkmasse.

In Kapitel 4 wird eine neue Deformation des D3-D7-Modells durch einen Fayet-Iliopoulos-Term vorgestellt [2]. Diese Deformation bricht Supersymmetrie, und erlaubt somit die Beschreibung nichtsupersymmetrischer Zustände in der Feldtheorie. Ich beschreibe eine exakte Abbildung zwischen nichtsupersymmetrischen Coulomb-Higgs-Zuständen der Feldtheorie, in welchen Teile der Eichsymmetrie spontan gebrochen sind, und nichtkommutativen Instantonkonfigurationen auf D7-Branen im $\mathrm{AdS}_5 \times \mathrm{S}^5$-Hintergrund. Diese verallgemeinerte AdS/CFT-Dualität wird verschiedenen Konsistenztests, insbesondere der globalen Symmetrien sowie der Supersymmetrie, unterworfen.

Kapitel 5 schließlich stellt basierend auf [3] ein aus phänomenologischen Annahmen hergeleitetes bosonisches $U(N)$ Matrixmodell im räumlich flachen Robertson-Walker-Universum vor. Die Herleitung erfolgt mit Hilfe der Matrixregularisierungsprozedur der Nambu-Goto-Wirkung einer bosonischen Membranen und basiert auf einer neuartigen Eichfixierungsmethode, welche die Lichtkegeleichung ersetzt. Am Ende von Kapitel 5 wird gezeigt, daß schon kurz nach dem Urknall, d.h. sobald sich das Universum auf die Größe einiger Planckslängen ausgedehnt hat, das Matrixmodell Lösungen besitzt, welche als glatte räumliche Geometrien interpretiert werden können. Abschnitt 5.3.1 enthält darüber hinaus eine in sich geschlossene Einführung in die Verbindung zwischen Matrixmodellen, Quantengravitation und $\mathcal{M}$-Theorie.

Eine erweiterte Einführung ist in Kapitel 1, eine ausführliche Diskussion der Resultate in Kapitel 6 zu finden.

Abstract

Holographic models are theories describing higher-dimensional physics through a lower-dimensional theory or vice versa. In string theory there exist two such setups: The Anti-de Sitter/Conformal Field Theory (AdS/CFT correspondence, which will be explained in chapter 2, yields an equivalent description of string theory in asymptotically Anti-de Sitter space-times in terms of gauge theories defined on the conformal boundary of these space-times, in particular four-dimensional Minkowski space. Conversely, the strong coupling regime of these gauge theories can be described in terms of ten-dimensional supergravity. On the other hand, quantum mechanical models of matrix valued degrees of freedom, so-called matrix models, holographically describe $\mathcal{M}$-theory, the proposed quantum theory of eleven-dimensional supergravity. This dissertation deals with three generalisations of these two holographic theories to physical scenarios with a smaller amount of symmetries.

With the goal of future applications to the infrared dynamics of QCD in mind, the AdS/CFT correspondence has been extended to include quark-like degrees of freedom transforming in the fundamental representation of the gauge group. In chapter 3 one of these extensions, the $\mathcal{N} = 2$ supersymmetric D3-D7 model, is analysed [1] in constant external electric and magnetic fields and at finite temperature. These external fields explicitly break Lorentz invariance. I show that the magnetic field induces spontaneous chiral symmetry breaking and stabilises mesonic excitations. The electric field on the other hand destabilises the mesons and induces a metal-insulator phase transition at finite quark mass.

In chapter 4 a new deformation of the D3-D7 model via a Fayet-Iliopoulos term [2] is presented. This deformation breaks supersymmetry, allowing for the description of non-supersymmetric states in the strong coupled gauge theory. I describe an exact map between nonsupersymmetric Coulomb-Higgs states with partially spontaneously broken gauge symmetry of the field theory, and noncommutative instanton configurations on the D7 brane probes in $AdS_5 \times S^5$. This extended AdS/CFT duality is subjected to several consistency checks, in particular of global symmetries and supersymmetry on both sides of the correspondence.

Finally, based on several phenomenological assumptions, in chapter 5 a bosonic $U(N)$ matrix model in the spatially flat Robertson-Walker geometry is derived [3]. The derivation uses the procedure of matrix regularisation of the Nambu-Goto action of a bosonic membrane, and is based on a new kind of gauge fixing procedure replacing light-cone gauge. At the very end of chapter 5 the existence of solutions of the proposed matrix model admitting an interpretation of smooth spatial geometries shortly after the universe expanded to a size of several Planck lengths is shown. Furthermore in section 5.3.1 a self-contained introduction into the connection between matrix models, quantum gravity and $\mathcal{M}$-theory is given.

An extended introduction to this thesis can be found in chapter 1, as well as a detailed discussion of the results in chapter 6.

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Chapter 1

Introduction and Overview

Gleichwohl müßten die Atome zufolge der inneratomischen Elektronenbewegung nicht nur elektromagnetische, sondern auch Gravitationsenergie ausstrahlen, wenn auch in winzigem Betrage. Da dies in Wahrheit in der Natur nicht zutreffen dürfte, so scheint es, daß die Quantentheorie nicht nur die Maxwellsche Elektrodynamik, sondern auch die neue Gravitationstheorie wird modifizieren müssen.

Albert Einstein [4]

In the twentieth century, the description of the fundamental building blocks of the physical world, which are the known particles and their interactions, has seen remarkable progress. The current paradigm behind this success is that both matter particles and the four known fundamental forces between them (electromagnetic, weak, strong and gravitational force) should be described in terms of quantum field theories [5–7]. The currently accepted quantum field theory describing all known particles and interactions except of gravity (which plays a special role explained below) is the standard model of particle physics [8], a gauge theory based on the gauge group $SU(3) \times SU(2) \times U(1)_Y$. The $SU(3)$ sector governs the dynamics of the strong interaction, i.e. of quarks and gluons, while the $SU(2) \times U(1)_Y$ sector gives a unified description of the electromagnetic and weak forces [9–11], where the $U(1)_Y$ factor couples to hypercharge. The electroweak symmetry breaking mechanism [12–15] breaks the $SU(2) \times U(1)_Y$ gauge symmetry to the electromagnetic $U(1)_{EM}$ group, leaving only the photon massless and giving the $W^{\pm}/Z$ bosons their mass. The electroweak symmetry breaking mechanism also predicts the existence of a massive scalar particle, the Higgs boson, but it does not predict its mass. To date, the predictions of the standard model of particle physics are confirmed by experiments with incredible accuracy [16], in particular in the electroweak sector[1] but also in QCD, where e.g. lattice calculations can determine the proton mass with an accuracy of a few percent [18].

[1]The prime example is of course the anomalous magnetic moment of the electron, with an experimental value of $(g-2)/2 = 1159652181.1(7) \times 10^{-12}$ favoured by the particle data group [16], while one of the

a) Quantum Gravity, String Theory and Holography

Albeit all successes, the standard model has several shortcomings of more or less severe nature, some of which lead to the belief that the standard model itself cannot be considered as a complete description of nature but needs to be embedded in a larger theoretical framework. The arguably most important shortcoming is that gravity, i.e. Einstein's theory of general relativity, cannot simply be incorporated into the standard model as a perturbatively quantised sector. The underlying problem is our ignorance about how to quantise gravity, being a perturbatively nonrenormalisable theory, in a consistent fashion. The usual way of coping with this situation is to either incorporate gravity as a classical theory which determines the background space-time for the standard model sector (i.e. to analyse quantum field theories on curved space-times) or to accept the nonrenormalisability [19–24] and simply use the standard model with perturbatively quantised gravity as an effective theory which will break down at the Planck scale. Neither of these approaches are satisfactory: Although the arguments against coupling a classical sector to a quantum theory are not totally conclusive [25–27], the thought experiments by Eppley and Hannah [28] indicate that interactions of a classical gravitational field with a quantum system may violate e.g. the uncertainty principle. Living with nonrenormalisability is neither a good option, since the theory looses its predictability at the Planck scale due to the necessity to experimentally fix infinitely many coupling constants for the higher dimension operators generated by quantum corrections, surely not a wanted feature of a fundamental theory of nature.

The **cosmological constant problem**, i.e. the inability of the standard model of particle physics (or, to date, of any other theory) to explain the small but nonzero measured value of the cosmological constant of (in Planck units)

$$\Lambda = 10^{-123}\,,$$

is probably the most severe indication that treating the standard model together with general relativity as an effective field theory is not consistent: In the absence of an additional mechanism (such as supersymmetry) to cancel loop corrections, loop contributions from both the standard model particles and from gravitons to the vacuum energy density generate [29, 30] a cosmological constant of order of the imposed energy cutoff to the power four. Since from the point of view of the standard model the only natural choice for the cutoff is the Planck scale, the cosmological constant is predicted by this effective field theory treatment to be of order one in Planck units, hence one-hundred and twenty-three magnitudes too large. This is indeed a remarkable failure of our understanding of reality! Unfortunately this observation does not give many hints which way to pursue towards a more satisfactory formulation of quantum gravity.

The conclusion which can be drawn from the perturbative nonrenormalisability of the stan-

best current theoretic predictions coming from an eight-loop calculation including hadronic and weak contributions is $1159652182.79(7.71) \times 10^{-12}$ [17].

dard model coupled to general relativity is that there must exist an ultraviolet completion, i.e. a theory which at high energies may be formulated in terms of fundamentally different degrees of freedom but which reduces at distances greater than the Planck length to the standard model plus gravity. There exists at least one consistent candidate theory for such an ultraviolet completion: String theory (see sec. 2.2 and references therein for more details) replaces the pointlike nature of particles and the local interactions in quantum field theory by one-dimensional extended objects, so-called strings, which interact nonlocally and in this way render scattering amplitudes generically better behaved in the ultraviolet than in perturbative quantum field theory.[2] As will be explained in chapter 2, string theory includes both gauge theories as open string excitations as well a gravity in the form of closed string excitations. It provides a framework for calculating low energy effective actions, in particular the contributions from higher derivative operators. Perturbative string theory thus provides a well-defined framework for quantising gravity, and the same is believed to be true at the nonperturbative level.

Concerning the cosmological constant problem, the obvious conclusion is the necessity to cancel in some way the vacuum fluctuations which generate the huge vacuum energy density (as well as nonperturbative contributions). To date, this is neither achieved by supergravity theories nor by string theory [30,31]. Thus the cosmological constant problem seems not to be tied in an obvious way to the problem of finding a consistent theory of quantum gravity, which presumably makes it much harder to solve. It is however conceivable that imposing a yet unknown symmetry principle could protect the volume term in the Einstein-Hilbert action from quantum corrections [32].

On the other hand, the **holographic principle** put forward by 't Hooft and Susskind, a peculiar property believed to be a necessary feature of any theory of quantum gravity [33,34], might play a role in a solution of the cosmological constant problem [35–40]. At a glance, the holographic principle is based on the holographic entropy bound, which states that the entropy enclosed in a spatial volume cannot scale faster than the enclosing area of this spatial volume. If it would, the energy carried by the enclosed states would be, roughly, enough to collapse and form a black hole. For the actual estimate see section 2.1. On this basis 't Hooft boldly conjectured that the quantum theory governing the degrees of freedom inside the volume must be describable in terms of a theory (maybe even a quantum field theory) living in one dimension less, namely on the bounding surface of the volume. In particular, the entropy scaling of this holographic theory can not be faster than with the area of the bounding surface, i.e. the information density should not exceed one bit per Planck area. This amounts to a huge reduction of the possible number of degrees of freedom in nature which, if a well-defined calculation in some proposed model of quantum gravity could be done, would presumably reduce the contribution to the vacuum energy by quite

[2] Closed string one-loop amplitudes, which includes the gravitational sector, can even be shown to be finite due to modular invariance. Open string ultraviolet divergences can be reinterpreted as infrared divergences in closed string exchange channels due to open-closed string duality and do not pose a problem. See chapter 5 of [31] and references therein for more details.

an amount, thus either solving or at least lessening the cosmological constant problem. So far this is all speculation, since despite several attempts [35–40] no convincing calculation of the cosmological constant in a well-defined holographics setup has been carried out yet.

In string theory there naturally occur at least two intrinsically holographic descriptions of quantum gravitational physics, both of which play a role in this thesis. These descriptions are the **Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence** [41], a holographic duality between string theories on asymptotically Anti-de Sitter space-times and gauge theories defined on the boundary of these space-times, and the **nonperturbative definition of $\mathcal{M}$-theory** (the supposed ultraviolet completion of eleven-dimensional supergravity) **via matrix models** in general, and in particular for the flat eleven-dimensional background via the matrix model proposed by Banks, Fischler, Shenker and Susskind (BFSS) [42]. Both descriptions are well-defined setups for formulating quantum gravitational questions, and in both cases the holographic bound on information density holds [42,43]. In the AdS/CFT setup, which will be explained in great detail in chapter 2, the holographic bound holds after regulating the infinite area of the boundary of AdS space with a ultraviolet cutoff, which translates into an infrared (large volume) cutoff for the interior of AdS space [43]. The BFSS matrix model fulfills the holographic bound in a very particular way [42]: The transverse size of a threshold bound state of N D0 branes (which are the fundamental degrees of freedom described by the matrix model) grows, for large N, as $N^{\frac{1}{9}}$ in eleven-dimensional Planck units, which is an indication of the incompressibility of the D0 branes in the nine-dimensional transverse space. The "holographic surface" in the case of the discrete light-cone quantisation (DLCQ) of $\mathcal{M}$-theory is thus the nine-dimensional space transverse to the light-cone. Although it is surely too bold to conclude from this that either AdS/CFT or the BFSS matrix model is an in all aspects satisfying description of quantised gravity, the fulfilled holographic information bound reassures at least a little bit that we are on the right track.

The main theme of this thesis, as the title suggests, is the application of holography in the presence of external fields and time-dependent backgrounds. In the following I will use the AdS/CFT correspondence as a tool to describe the physics of strongly coupled gauge theories, and in particular extend it in chapters 3 and 4 to situations with a lower amount of symmetry. The rationale of this approach to gauge theories is the converse of what was just explained: Instead of trying to define quantum gravity in a bulk region by a theory on the boundary of this region, which is of course also possible in the AdS/CFT setup, classical gravitational physics in the bulk is used, in a certain limit, to describe the dynamics of strong coupled gauge theories. In chapter 5 I will then come back to the holographic approach to quantum gravity, more precisely to the matrix model approach. There I will present a bottom-up proposal for a matrix model that could describe quantum gravitational physics near the big bang singularity in Robertson-Walker geometries.

b) Confinement and Quantum Chromodynamics

Another fundamental issue which prevails in the standard model even without the gravitational sector is the understanding of the strong coupling dynamics of QCD. This issue stems from the phenomenon of renormalisation group running of coupling constants in quantum field theory, i.e. the energy scale dependence of the actual value of e.g. the fine structure constant or the strong coupling constant. In QED the one-loop corrections of leptons to the photon propagator lead to a screening of the bare electron charge: The formally infinite bare charge can only be seen when probing the electron at very high energies, leading to a divergence of the measured charge at high energies discovered first by Landau [44]. Strictly speaking this renders the theory ill-defined in the absence of gravity. However, since the Landau pole is far above the Planck scale, and since all known types of matter couple to gravity in a universal way, a description of QED physics neglecting gravity will not be valid any longer at the Planck scale. Hence the problem to address is not how to embed QED or the electroweak sector[3] into an asymptotically free theory, but how to quantise gravity.

In nonabelian gauge theories the gauge bosons contribute to the renormalisation of their own propagator in an antiscreening way [47, 48]. For large enough gauge groups or a small enough number of fermions the charges get antiscreened, i.e. the observed charge becomes smaller if probed at higher energies. This is the case for QCD which is, since the strong coupling constant α_s is the expansion parameter in the perturbative quantisation of the theory, weakly coupled in the ultraviolet, a feature dubbed **asymptotic freedom**. On the other hand, the perturbatively calculated α_s grows large in the infrared, and perturbation theory is not valid any longer below the QCD scale $\Lambda_{QCD} \approx 200$ MeV. The theory is thus well-defined in the ultraviolet, but hard to solve in the low energy region region that is of interest to us due to the hardship of solving quantum field theories on the nonperturbative level. Moreover, QCD is believed to undergo a (de)confinement phase transition at energies around Λ_{QCD}. More precisely, at low densities the transition is believed to be a rapid crossover – see e.g. the phase diagram in [49] – which however becomes a first order transition at large N_c due to a discontinuous change in the free energy from an $\mathcal{O}(N_c^2)$ to an $\mathcal{O}(N_c^0)$ behaviour [50–53]. In any case, at low energies colour charges are confined into colourless states like glueballs, mesons and baryons, while at higher energies there is both theoretical as well as experimental evidence from the RHIC experiment [54] that new states of matter, at low densities and high temperatures in particular the quark-gluon plasma, are forming.

[3]Interestingly, the electroweak sector of the standard model, which is renormalisable [45], is not asymptotically free either [46]. The $SU(2)$ part of the gauge group is asymptotically free, but the $U(1)_Y$ factor admits a Landau pole as well. The persistence of the Landau pole problem is another indication that gravity has to be consistently quantised as well.

c) The Anti-de Sitter/Conformal Field Theory Correspondence

There exist several methods to analyse QCD (and also other nonabelian gauge theories) at strong coupling, such as lattice regularisation [55], the large N_c expansion [56], instanton gases and liquids [57,58] and other models of the QCD vacuum such as bag models [59] and dual superconductors [60], effective field theory approaches [61–64], or methods based on truncations of the full set of nonperturbative Schwinger-Dyson equations [65, 66].[4] Each of these approaches led to remarkable insight into QCD dynamics, but each of them is based on additional assumptions. For example, the lattice is a tractable nonperturbative regularisation of QCD, but breaks four-dimensional Lorentz invariance. The large N_c expansion simplifies the theory alot, but not all observables of QCD are insensitive to a change of the gauge group to $N_c > 3$. The approach based on the Schwinger-Dyson equations includes the necessity of a consistent truncation of the infinite set of consistency equations amongst all n-point functions to a finite subset. Last but not least there is the idea of the QCD string [67–69], from which string theory originated. Although the original QCD string theories were abandoned for reasons explained in section 2.2, the idea recently resurfaced, starting with an observation of Alexander Polyakov [70–73] that the emergence of the additional scalar Liouville field in the quantisation of noncritical bosonic strings should properly be interpreted as the emergence of an additional fifth spatial dimension. The actual description of the QCD string would then be in terms of a string theory in a curved five-dimensional space-time with a flat four-dimensional boundary. On its own, this remarkable observation did not lead to a concrete realisation. An analysis of the physics of nonperturbative objects in ten-dimensional string theory, so-called Dirichlet branes, lead Juan Maldacena in 1997 to a remarkable proposal for such a dual realisation of gauge theories in terms of a string in a curved higher-dimensional space-time, called the **Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence** [41].

As will be explained in detail in chapter 2, the AdS/CFT correspondence actually adds six additional curved dimensions to the four-dimensional space-time, instead of only one. One of these six dimensions is an addition radial direction which, together with the four Minkowski space directions forms five-dimensional Anti-de Sitter space, AdS_5, the unique maximally symmetric five-dimensional space-time with constant negative scalar curvature, while the other five additional dimensions form a compact space, the five-dimensional sphere S^5. Both spaces are of the same radius R. At a glance, Maldacena's original proposal states that the physical state space of ten-dimensional type IIB superstring theory on the space-time $\mathrm{AdS}_5 \times \mathrm{S}^5$ is equivalent to the space of physical states of a certain four-dimensional supersymmetric conformal field theory, $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group $U(N_c)$. This superconformal field theory is constructed from the largest supersymmetry multiplet with a maximal spin of one which can be built in four dimensions, the $\mathcal{N} = 4$ vector multiplet, and thus is uniquely defined by the gauge coupling g_{YM} and the rank of the gauge group which is given by N_c.[5] Furthermore there

[4] The author apologises for possible omissions in this list.

[5] There is also a θ-angle in $\mathcal{N} =$ SYM theory, fixed by the vacuum expectation value of the axion C_0

is strong evidence (reviewed in section 2.3.2) that this gauge theory is ultraviolet finite. There is a precise mapping between the quantities defining the string theory, which is the AdS_5 radius measured in string R/ℓ_s and the string coupling g_s, in terms of the quantities defining the $\mathcal{N}=4$ SYM theory. The map explicitly reads

$$\left(\frac{R}{\ell_s}\right)^4 = 2\lambda\,, \quad \lambda = g_{YM}^2 N_c = 2\pi g_s N_c\,. \tag{1.1}$$

Although it involves the 't Hooft coupling λ, which is the natural expansion parameter in the large N_c approach to gauge theories [56], the actual correspondence in its strongest version is expected to hold for all values of N_c, as well as all values of the Yang-Mills coupling g_{YM}^2. However, to date no actual proof of the correspondence on any level of rigor has been found (although much progress in the determination of the spectra on both sides of the correspondence [74–78] has been made), but a huge amount of circumstantial evidence (see e.g. the reviews [79, 80]) in favour of this conjectured duality is known. Together with the beautiful decoupling argument of Maldacena (see section 2.3.1) this circumstantial evidence is the reason for at least assuming its validity, which is sufficient for using the AdS/CFT correspondence as the foundation of chapters 3 and 4 of this thesis.

In the form just stated, the AdS/CFT correspondence is a remarkable and highly nontrivial statement about the dynamics of two seemingly very different theories, namely a string theory and a conformal gauge theory. In particular, the idea of the QCD string, which was connected to the confinement phenomenon, does not apply here. Confining dynamics as in QCD is always bound to asymptotic freedom at high energies and thus a running coupling growing strong at some energy scale. Since $\mathcal{N}=4$ SYM theory is conformal, i.e. the coupling does not run with the energy scale, it does not show a (de)confinement phase transition on flat Minkowski space.[6] Nevertheless the AdS/CFT correspondence implies that the dynamics of $\mathcal{N}=4$ SYM theory has an interpretation in terms of a string theory, not only in some limiting regime but for all values of the 't Hooft coupling and for all ranks of the $U(N_c)$ gauge group. In this respect the AdS/CFT correspondence is a stronger tool than the other approaches to strongly coupled gauge theory dynamics discussed above. However, solving full type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ is arguably as hard as solving QCD dynamics analytically. Fortunately the correspondence simplifies and becomes more tractable in two limits: Taking the **'t Hooft limit**, i.e. considering only the leading behaviour of observables at large N_c while keeping λ fixed, the string coupling g_s becomes small. Since the string coupling governs string loop effects, the analogue of loop effects in quantum field theory, this reduces the dynamics to semiclassical strings on $\mathrm{AdS}_5\times\mathrm{S}^5$. Furthermore taking $\lambda \gg 1$ to be large but finite, according to eq. (1.1) the radius of curvature of $\mathrm{AdS}_5\times\mathrm{S}^5$ grows large compared to the string length, thus

from the Ramond-Ramond sector of type IIB string theory. I however omit it in this discussion, since it will play no role in this thesis.

[6] On spaces with different topologies, such as $\mathrm{Time}\times\mathrm{S}^3$, also $\mathcal{N}=4$ SYM theory shows a (de)confinement phase transition. In this case the curvature scale introduced by the background breaks conformal invariance. The phase transition is dual to a geometric Hawking-Page phase transition on the gravity side.

reducing the string dynamics to its point particle limit given by ten-dimensional type IIB supergravity on the background $AdS_5 \times S^5$. Since classical supergravity calculations on this background are much better understood than semiclassical string computations, this is often the most useful formulation of the AdS/CFT correspondence. In particular, most of the circumstantial evidence for the correspondence was obtained in the large N_c, large 't Hooft coupling limit.

The large N_c, large 't Hooft coupling limit thus is most promising in two ways: Calculations in supergravity are technically feasible, and the dual gauge theory is strongly coupled, albeit with a rather simple, namely conformal, dynamics. In this sense the AdS/CFT correspondence is a strong/weak coupling duality, which makes it promising to search for generalisations describing the strong coupling dynamics in confining theories such as QCD. There is good reason to believe that any ten-dimensional gravity background which asymptotically approaches the metric of $AdS_5 \times S^5$ defines a gravity dual description for some (to be determined) gauge theory. In particular, the background could break many of the symmetries of $AdS_5 \times S^5$, or even supersymmetry. As long as the background solves the type IIB supergravity equations of motion, it defines a **gauge/gravity duality**. This opens up a road to mimic the dynamics of QCD more closely by gradually constructing and exploring less symmetric gauge/gravity duals. Pursuing this road, superconformal theories with less supersymmetry [81, 82], nonconformal theories exhibiting confinement and chiral symmetry breaking [83–88] and theories describing $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the infrared [89, 90] have been found. The correspondence can also be extended to a **finite temperature** version by replacing the AdS_5 space-time by a five-dimensional black hole solution in Anti-de Sitter space [91]. Although no absolutely satisfactory gravity dual of QCD has been found yet, the progress made so far has been remarkable (for recent reviews see [49, 92, 93]).

d) Holographic Flavour at Finite Temperature

A particularly important step towards a gravity dual of QCD was the introduction of quark-like degrees of freedom, i.e. fields which transform in the fundamental representation of the gauge group, into the AdS/CFT correspondence [94,95]. Since the $\mathcal{N} = 4$ SYM theory and its deformations only feature fields in the adjoint representation of the gauge group, in order to approach QCD closer it is necessary to extend the correspondence to fields in the fundamental representation. As explained in more detail in section 2.3.6, string theory yields a natural way to introduce fundamental fields into a gauge theory by introduction of branes of higher dimensionality. Karch, Katz and Randall [94, 95] put forward that for $\mathcal{N} = 4$ SYM theory, which arises as the low energy effective action on a stack of N_c D3 branes, an additional stack of N_f D7 branes parallel to the D3 branes should be added. The low energy excitations of strings connecting the two stacks then introduce N_f hypermultiplets of supersymmetric quarks and antiquarks, i.e. give rise to a **flavour sector** with an $U(N_f)$ flavour symmetry. The gauge theory on the D7 branes then decouples from the 3-3 and 3-7 strings and yields an **open string sector on the gravity side of the correspondence**. This defines an extended AdS/CFT correspondence in which the type IIB supergravity fields are dual to the adjoint sector of the resulting $\mathcal{N} = 2$ supersymmetric field theory, while the fields on a D7 brane embedded into the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background are dual to field theory excitations involving the quark fields. The mass of the quark hypermultiplets, which can take arbitrary values without breaking $\mathcal{N} = 2$ supersymmetry, directly corresponds in a certain coordinate system to the transversal distance (divided by $2\pi\alpha'$) between the original stacks of D3 and D7 branes.

Also in this case there occurs a simplification in the large N_c limit: Keeping the number of flavours N_f fixed and small while the number of colours tends to infinity, loop contributions from the flavour sector are suppressed. This limit is called the **quenched approximation**. On the gravity side this corresponds to neglecting the backreaction of the D7 branes onto the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, which therefore only probe the space-time, hence the name **probe approximation**. In the large N_c, fixed N_f limit, open strings ending on a stack of N_f D7 branes embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ are therefore dual to the flavour sector of a certain $\mathcal{N} = 2$ supersymmetric field theory in the quenched approximation. The flavour sector itself will be introduced in more detail in section 2.3.6. Additionally, the strong coupling $\lambda \gg 1$ can be taken to reduce the open string sector to the Dirac-Born-Infeld-Wess-Zumino action describing the D7 branes in the low energy limit. This extension enriches the dynamics of the theory considerably.

In particular, the D3-D7 model of holographic flavour allows to study vacuum properties such as **spontaneous chiral symmetry breaking** [87] and the dynamics of mesonic excitations [96]. The latter are dual to excitations of the D7 branes on the gravity side. The chiral symmetry breaking mechanism is implemented as follows: The $\mathcal{N} = 2$ theory has a $U(1)_{\mathcal{R}}$ symmetry, which is realised on the gravity side of the correspondence as a rotation symmetry in the directions transverse to the D7 branes. This $U(1)_{\mathcal{R}}$ symmetry acts on the fermions in the quark hypermultiplets as a chiral rotation, and in particular can be broken if the supersymmetric mass operator of the theory (cf. eq. (2.151)) acquires a vacuum expectation value (henceforth called "the condensate"). Holographically this vacuum expectation value corresponds to nontrivially curved embeddings of the D7 branes. In the pure $\mathrm{AdS}_5 \times \mathrm{S}^5$ background however the equations of motion determining the D7 embeddings only have static solutions which correspond to flat embeddings at an arbitrary distance from the D3 brane stack. This is consistent with the $\mathcal{N} = 2$ supersymmetry preserved by the field theory, which forbids the development of a condensate.

In order to generate nonvanishing condensates, one thus has to deform the background away from $\mathrm{AdS}_5 \times \mathrm{S}^5$. This situation occurred in the work [87] where $\mathrm{AdS}_5 \times \mathrm{S}^5$ was replaced by the Constable-Myers background [84]. In this case even at zero mass a condensate was found, indicating that the chiral $U(1)_{\mathcal{R}}$ symmetry is spontaneously broken. This is consistent with the confinement property of the Constable-Myers background [84] – similar to the case of QCD, confinement and chiral symmetry breaking are linked in this case.[7] A

[7]In fact, the author is not aware of the existence of a general classifications of which backgrounds admit

particular simpler situation which does not show spontaneous chiral symmetry breaking but admits condensates at finite quark mass is the finite temperature case, where the D7 branes are embedded into an AdS-Schwarzschild background times a five-sphere. The finite temperature is responsible for the broken supersymmetry. This case shows an interesting first order **meson melting transition** [87, 99], i.e. a transition between a mesonic phase characterised by (in the large N_c limit) stable mesons, and a molten phase characterised by spectral functions showing quasiparticle excitations [100–102] which are given by the quasinormal modes of the AdS black hole. As explained in section 2.3.6, the transition is holographically realised as a transition between D7 brane embeddings which do not fall into the black hole, and embeddings which do fall into the horizon. Since at strong coupling the holographic mesons have large binding energies proportional to the bare quark mass [96], the transition happens at a certain critical quark mass below which the thermal fluctuations destabilise the meson. The meson melting transition manifests itself in a jump of the chiral condensate. Since in the quenched approximation the meson physics at finite temperature can be thought of as happening in the background of the deconfined $\mathcal{N} = 4$ SYM plasma, the destabilised mesons melt in the hot $\mathcal{N} = 4$ SYM plasma.

e) Chapter 3: Holographic Quarks in External Electric and Magnetic Fields

In view of the general idea of constructing gauge/gravity duals with lower degrees of symmetry, it is quite natural to try to break Lorentz invariance in the Minkowski space directions by introducing additional background fields. This approach has for example led to gravity duals of noncommutative $\mathcal{N} = 4$ SYM theory [103, 104]. In chapter 3, I present an analysis [1] of the D3-D7 model of holographic flavour in the presence of constant external electric and magnetic fields. The fields are holographically realised as constant field modes of the Kalb-Ramond field $B_{\mu\nu}$. In both situations the Lorentz symmetry is broken to $SO(1, 1) \times SO(2)$ by the external field, as is supersymmetry. In both cases, an interesting phase structure and behaviour of mesons is found.

In particular I analyse the dynamics of chiral symmetry breaking, as well as the meson spectrum, in a constant magnetic field background at finite temperature. The main result in this case is that the **constant magnetic field catalyses spontaneous chiral symmetry breaking** in the same way as was observed at zero temperature in [105]. This phenomenon is well-known from both QED and QCD in strong magnetic fields [106–109]. In particular I show the existence of a critical magnetic field strength (depending on the temperature) which divides the phase space into two different regions: Above the critical field strength

both confinement and chiral symmetry breaking. In [97] a criterion for chiral symmetry breaking was given: The D7 brane embeddings in properly chosen coordinates should end for all quark masses before reaching the singularity which is typically present in nonsupersymmetric deformations of $\mathrm{AdS}_5 \times \mathrm{S}^5$. In this case there does not exist a coordinate transformation which makes this gap vanishing, and the gap sets the energy scale for chiral symmetry breaking. However, to date no attempt to combine this criterium with e.g. the classification of singularities by Gubser [98] has been made, which thus is an interesting road for further study.

the system is in the mesonic phase for any value of the quark mass, even for massless quarks. The strong magnetic field in this case overcomes the effect of thermal fluctuations and stabilises the mesons completely, such that the meson melting transition disappears. Below the critical field strength, the meson melting transition persists, but it is shifted to a lower critical quark mass due to the stabilising effect of the magnetic field. While the critical field strength is approached, the critical quark mass goes smoothly to zero. The phase below the critical field strength is characterised by a vanishing chiral condensate, as is the case at finite temperature without magnetic field, while above the critical field strength spontaneous chiral symmetry breaking occurs. The spectrum of the corresponding lowest-lying pseudoscalar meson is investigated and is shown, via verification of the Gell-Mann-Oakes-Renner relation, to include the Goldstone boson of the spontaneously broken chiral symmetry. The phase diagram for the system is depicted in figure 3.5.

I then proceed in section 3.3 to the analysis of the D3-D7 system in a constant electric field, both at zero and at finite temperature. The main physical feature in this case will be that, contrary to the magnetic field, the electric field tends to destabilise quark-antiquark bound states. Starting from a high enough quark mass, the system is found to undergo a new kind of first order **meson dissociation phase transition** at a finite critical quark mass (depending on the electric field strength). Similarly to the meson melting transition discussed above, the dissociation transition manifests itself in a jump of the chiral condensate. The electric field alone also does not induce spontaneous chiral symmetry breaking.

Above the critical quark mass an analysis of the pseudoscalar mesons shows that there exist stable mesonic excitations, which are however lighter than in the zero field case. I furthermore analytically show the existence of a second order ($\mathcal{O}(E^2)$) Stark shift of the meson masses. Below the critical mass the situation is not fully understood yet,[8] but the boundary conditions which have to imposed on the D7 brane fluctuations [110] are partially transmittive and thus forbid the existence of stable excitations. Another indication for the dissociating nature of this phase transition is a current which starts to form at the transition point, which can be interpreted as vacuum pair production of quarks and antiquarks by the electric field. In this case the vacuum is however not destabilised and the current even is finite for massless quarks, which is a strong coupling effect still awaiting full understanding. Because of the generation of a current and thus a conductivity at the transition point, this new kind of phase transition is similar to metal-insulator transitions in condensed matter systems.

[8]It was found in [110] that in the presence of a finite quark number density the spectrum of excitations of the system changes from well-defined sharp peaks admitting a quasiparticle interpretation to broad resonances. The first order phase transition is washed out by the finite density to a crossover behaviour, but the change in the spectrum and in thermodynamic quantities is rather quick for small densities. The unclear point is the treatment of a special class of D7 brane embeddings which admit conical singularities – see section 6.1 for a more in-depth discussion of this point.

f) Chapter 4: Holographic Fayet-Iliopoulos Terms from Kalb-Ramond Fields

While chapter 3 is concerned with the breaking of Lorentz symmetry, chapter 4 puts forward a **holographic construction of Fayet-Iliopoulos couplings** in the D3-D7 theory. This construction, which was published in [2], allows a holographic identification of noncommutative instanton configurations on the D7 probes with nontrivial supersymmetry breaking states on the Coulomb-Higgs branch of the D3-D7 theory. The Fayet-Iliopoulos coupling is induced by switching on a constant anti-selfdual Kalb-Ramond B field in the directions of the D7 brane worldvolume transverse to the boundary of AdS space. Based on the knowledge of existence of such a coupling in the low energy effective action of the flat space D3-D7 system in the presence of a anti-selfdual B field, it is argued that the coupling actually survives the decoupling limit and thus is also present in the holographic context. On the gravity side the B field induces a noncommutative worldvolume gauge theory on the D7 stack along the lines of [111].

Without B field, a holographic correspondence between the mixed Coulomb-Higgs vacua of the D3-D7 theory and instanton configurations on the D7 probe branes embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ has been proposed in [112–115]. The mixed Coulomb-Higgs vacua of the $\mathcal{N} = 2$ D3-D7 theory are vacua in which some colour directions (called Higgs directions) of the gauge group are broken completely (including the Cartan generators) by fundamental squark vacuum expectation values, while the gauge group is broken down to its Cartan $U(1)$ generator in other colour directions (called Coulomb directions) by adjoint scalar vacuum expectation values. In terms of the D3-D7 system in flat space the Higgs directions correspond to D3 branes being dissolved in the D7 branes to form an instanton in the additional D7 brane directions, while the Coulomb directions correspond to the usual breaking of gauge symmetry by separating a D3 brane from the D3 stack (see fig. 4.1 for an illustration). The gauge symmetry breaking mechanism is thus that separating a D3 brane from the stack first breaks the corresponding colour direction to a $U(1)$ subgroup, while moving it onto the D7 brane and letting it dissolve corresponds to further even breaking this particular $U(1)$ subgroup completely. The dissolving induces one unit of instanton charge for every dissolved D3 brane.

The conjecture put forward and tested in [112–115] relies on a known property of instantons, which can be rewritten as solitons in a higher-dimensional gauge theory (see [116] for a review). In terms of the low energy effective action of the flat space D3-D7 system [117] this property manifests itself as a equivalence between the part of the D and F term equations describing the Higgs directions of the gauge theory vacua and the ADHM equations [118] which crucially enter the ADHM construction of instantons in four-dimensional Euclidean Yang-Mills theory: The sizes, positions and gauge orientations of instantons in the additional directions of the D7 branes are the vacuum expectation values defining the field theory vacuum. The vacuum expectation values in turn are governed by the D and F term equations of the field theory. It is known that both sides are deformed in the same way by the Fayet-Iliopoulos coupling: The D term equations of the gauge theory of course acquire the Fayet-Iliopoulos term, but it is also known from the construction of instantons in noncommutative Euclidean Yang-Mills theories [119] that the ADHM equations are of the same structure as in the commutative case, plus a constant coupling of Fayet-Iliopoulos form. Thus even with the deformation induced by the Kalb-Ramond field, the noncommutative ADHM equations are still identical to the vacuum equations for the Higgs branch of the field theory. This is not surprising since in terms of D brane physics the only effect of the deformation is to destabilise the Coulomb branch by breaking supersymmetry, i.e. to create an attractive potential which accelerates the D3 branes towards the D7 branes until they dissolve therein. The mechanism of dissolving is still the same – the D3 branes become instantons in the (in this case noncommutative) Euclidean Yang-Mills theory in the transverse D7 directions.

Based on this equivalence of the Higgs branch vacuum equations and the noncommutative ADHM equations, I conjecture in chapter 4 that the noncommutative instanton configurations in the D7 worldvolume theory are, after taking Maldacena's decoupling limit, dual to Coulomb-Higgs states governed by a Fayet-Iliopoulos deformed set of D and F term equations. These states break supersymmetry, since the D and F term equations can not all be simultaneously satisfied. More precisely, the colour directions which belong to the Coulomb branch, i.e. which in the flat space picture correspond to not yet dissolved D3 branes and which in the gravity dual source the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, do not fulfill the D term equation with the Fayet-Iliopoulos term. This is the field theory manifestation of the fact that the whole Coulomb branch of the moduli space is lifted by the deformation, and only the Higgs vacua in which the gauge group would be completely broken is supersymmetric. I argue in chapter 5 that the longlivedness of the setup is ensured in an adiabatic limit in which the Fayet-Iliopoulos parameter is very small. I will give an argument that even in the probe approximation the decay probability of the holographic setup is of order of the Fayet-Iliopoulos parameter, i.e. that the decay is not suppressed in the probe approximation. However, the Fayet-Iliopoulos parameter can be chosen to be arbitrarily small, and thus the setup can be made arbitrarily stable.

To test this extended AdS/CFT duality, I furthermore show that the global symmetries, the scaling dimensions and the supersymmetry breaking pattern on both sides of the correspondence match after introduction of the holographic Fayet-Iliopoulos coupling. Also, the simplest case of one noncommutative $U(1)$ instanton [119] on a single D7 brane embedded flatly (corresponding to vanishing quark mass) in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is considered, and in particular the global symmetries of both the Nekrasov-Schwarz instanton and the corresponding field theory vacuum configuration are analysed.

g) Chapter 5: A Matrix Model Proposal for the Robertson-Walker Universe

In contrast to the approach to strongly coupled gauge theories via classical gravity used in chapters 3 and 4, the work presented in chapter 5, which was published in [3], proposes a bottom-up derivation of a matrix model (eq. (5.71)) in the Robertson-Walker geometry. As will be explained in detail in a self-contained introduction to $\mathcal{M}$-theory and matrix models in section 5.3.1, the dynamics of M2 branes in flat space can be recovered from the formulation of $\mathcal{M}$-theory in flat space via the Banks-Fischler-Shenker-Susskind (BFSS) matrix model [42]. This supersymmetric matrix quantum mechanics describes on a nonperturbatively level the physics of $\mathcal{M}$-theory, the supposed ultraviolet completion of eleven-dimensional supergravity. What the matrix model actually describes is the physics of a collection of D0 branes in flat ten-dimensional space-time. As will be laid out in section 5.3.1, D0 branes have properties which make them good candidates for the "partons" of $\mathcal{M}$-theory, from which a state space can be build. There is by now good evidence [120–124] that the matrix model approach actually gives a nonperturbative definition of $\mathcal{M}$-theory, and hence of eleven-dimensional quantum gravity.

Conversely, the BFSS matrix model can be recovered by a regularisation procedure of the worldvolume theory of a supersymmetric membrane [125]. Behind this connection between the physics of D0 branes and membranes stands the physical requirement that matrix models, being background dependent formulations of quantum gravity, should be able to recover the physics of membranes in their spectrum. This means they should include in their spectrum semiclassical states with an emergent dynamics matching the membrane dynamics in the background under consideration. This logic can now be turned around and used for the bottom-up derivation of matrix models in more complicated backgrounds, such as the Robertson-Walker geometry, in which a top-down derivation from string theory may be unknown. These matrix models then have a chance to describe aspects of quantum gravity on these backgrounds. In the Robertson-Walker geometry this means in particular that they should describe aspects of the big bang singularity.

This is the rationale behind the derivation of the matrix model in chapter 5: Starting from the bosonic membrane in the Robertson-Walker geometry, I use the matrix regularisation prescription of [125] to derive the corresponding matrix model. An additional guiding principle which enters the derivation of the matrix model is that a collection of only a few D0 branes should be described in a freely falling frame, i.e. in particular the motion of a single D0 brane should follow the geodesics in the Robertson-Walker geometry. This approach is a bottom-up one, and a embedding of the matrix model in, or a derivation from string theory, is surely a point which needs to be addressed in future work (see also the discussion in section 6.3). Nevertheless even in this bottom-up approach there is a main technical obstacle which needs to be circumvented before applying matrix regularisation, namely the necessity to gauge-fix the worldvolume diffeomorphisms of the membrane. In backgrounds with a lightlike isometry such as eleven-dimensional flat space or lightlike dilaton backgrounds the worldvolume diffeomorphisms are usually fixed in light-cone gauge by restricting the dynamics to a fixed value of light-cone momentum. In geometries such as the Robertson-Walker geometry which lack this symmetry, another approach is needed. In the course of chapter 5 such an approach is developed which instead fixes an energy density profile on the membrane worldvolume. This fixes equally well the diffeomorphisms up to static area preserving ones, and the matrix regularisation can be applied.

Matrix models derived in a bottom-up approach in general need to be checked a posteriori for the validity of their physical implications. As a consistency check, I give at the end of chapter 5 an argument for the emergence of a well-defined, smooth spatial geometry once the universe described by the Robertson-Walker geometry has expanded to a size well above the Planck scale. An extensive discussion of the interpretation of the proposed model is given in section 6.3 of this thesis.

h) Structure of this Thesis

To conclude the above introductory remarks, the present thesis covers two main topics: Generalisations of the holographic D3-D7 model of AdS/CFT with flavour to situations where external fields are present, and with holographic descriptions of cosmological backgrounds via matrix models. For a quick overview over the rest of this work, the structure of this thesis is summarised below.

- The present chapter 1 includes a general introduction and an overview over this thesis.
- Chapter 2 introduces in detail the idea of holography and the Anti-de Sitter/Conformal Field Theory correspondence.
- Chapter 3 is devoted to the study of holographic quarks in external electric and magnetic fields.
- Chapter 4 presents the construction of holographic Fayet-Iliopoulos terms from Kalb-Ramond fields.
- In chapter 5 the matrix model proposal for the Robertson-Walker universe of [3] is presented.
- Chapter 6 contains a more detailed discussion of the results as well as comments on open questions and possible further developments.

Chapter 2

The Anti-de Sitter/Conformal Field Theory Correspondence

One main theme of this thesis is the so-called Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, which is a new string theoretic duality relation between gravity theories in Anti-de Sitter space and gauge theories on the boundary of this space-time. The purpose of this chapter is to introduce into this correspondence. Since the AdS/CFT correspondence provides a realisation of the holographic principle in quantum gravity, this principle will be introduced in section 2.1. Section 2.2 is devoted to a short introduction to string theory and D brane physics. Finally, in section 2.3 the AdS/CFT correspondence is stated and its generalisations to finite temperature as well as the introduction of additional quark degrees of freedom, i.e. fields in the fundamental representation of the gauge group, are discussed.

2.1 The Holographic Principle

Holography as a technique of storing the full amplitude and phase information of a wave front reflected by an object is known since 1948 [126], when the Hungarian physicist Dennis Gabor, based on Gabriel Lippmann's color photography, conveyed the idea of storing and reconstructing the full information content of the light front through interference with a reference field. He was awarded with the 1971 Nobel Prize, "for his invention and development of the holographic method" [127].

The holographic principle as a property of quantum theories of gravity was first proposed by Gerard 't Hooft [33]. Based on Bekenstein's insights that black hole horizons need to carry entropy [128, 129] in order for the second law of thermodynamics not to be violated, and that the entropy of a semiclassical black hole is a quarter of its horizon surface area in Planck units,[1]

$$S_{BH} = \frac{A_{hor}}{4\ell_p^2}\,, \tag{2.1}$$

't Hooft arrives at the conclusion that the dynamics of every quantum field theory which includes gravity in a spatial volume must be equivalently describable by a model of Boolean degrees of freedom on the surface bounding this spatial volume. 't Hooft assumes the degrees of freedom of this holographic description to be Boolean variables, i.e. variables which can only take two values ("bits"), for the entropy encoded on the surface's area to have an interpretation in terms of microstates of the system.

More precisely, consider a compact region of space with spatial volume V and bounding area A. If the total entropy in V has an interpretation in terms of the dimension $\mathcal{N}$ of the Hilbert space of microstates of quantum fields inside V, then we can as well represent this Hilbert space by n Boolean Variables, where the number of "bits" is given by the relation

$$e^S = \mathcal{N} = 2^n \Rightarrow n = \frac{S}{\log 2}\,. \tag{2.2}$$

For regions which constitute a black hole the relation with the bounding area is known, its simply the area law (2.1). Inserting (2.1) into (2.2) yields the result that the black hole horizon area should be quantised in Planck units,

$$\frac{A}{\ell_p^2} = 4\log(2)n\,. \tag{2.3}$$

Thus any unified quantum theory of matter and gravity which provides discrete black hole microstates needs to be able to reproduce this quantisation condition in a semiclassical approximation, including the factor $4\log(2)$.

't Hooft then generalises this argument to general closed spacelike surfaces. His premise is that gravity sets a natural cutoff for the total energy inside V for the state of matter enclosed in V to be observable: The total energy of the state of matter inside V cannot exceed the energy of a black hole with horizon area A, as otherwise the state would vanish behind the horizon of the black hole it creates by its gravitational backreaction onto space-time. In this way, he argues, the gravitational coupling should, at least qualitatively[2], provide a natural cutoff for physical processes involving the matter only. For simplicity, I will assume that the region of space under consideration is approximately spherically symmetric, and that we work in a four-dimensional space-time. In that case, the known expression for the Schwarzschild radius of a spherically symmetric energy distribution with total energy E,

$$R_S = 2G_N E\,, \tag{2.4}$$

[1] Throughout this thesis, if not useful otherwise, I use natural units $\hbar = c = k_b = 1 = G_N = 1$. For example in this section I keep $G_N = \ell_P^2$ in order to keep track of the powers of the Planck length.

[2] It has to be said that the ideas presented in [33] are of a qualitative, general nature. He also gives a realisation in terms of a cellular automaton model, but in principle other physical models might implement the holographic principle. This has to be, of course, proven model by model.

can be applied. Assuming that the spatial volume V is in contact with a heat bath of temperature T, the total energy of Z different fundamental particle species should scale according to a Stefan-Boltzmann law, i.e.

$$E = C_1 Z V^3 T^4 \,, \tag{2.5}$$

with C_1, as well as the other constants C_i appearing in the following, being of order one. The requirement that the typical linear dimension (assuming three spatial dimensions) of this energy concentration should be larger than its own Schwarzschild radius,

$$2G_N E < \left(\frac{V}{\frac{4}{3}\pi}\right)^{\frac{1}{3}} \,, \tag{2.6}$$

then leads to an upper bound on the temperature applicable to the system,

$$T < \frac{C_2}{G_N^{\frac{1}{4}} Z^{\frac{1}{4}} V^{\frac{1}{6}}} \,. \tag{2.7}$$

The fact that more particle species press down the bound is easily understood, as increasing the species at fixed temperature increases the number of possible excitations in the system, but the volume dependence seems counterintuitive at first: One would expect that increasing the volume would relax the bound on the maximally allowed temperature, as creating a black hole with larger area needs more energy, and thus a higher temperature. This reasoning, however, does not take into account the increase in the number of states in V. This is most easily seen in the case of V being a cubic box of length L with periodic boundary conditions, and with Z species of bosonic matter. In momentum space, the standing waves with periodic boundary conditions form a cubic lattice with unit cell volume $8\pi^3/L^3$. The phase space volume of all states with energy less or equal to E is the volume of a three-dimensional ball with radius E in momentum space, and thus the number of states with energy less or equal to E is

$$N(E) = Z \frac{\frac{4}{3}\pi E^3}{\frac{8\pi^3}{L^3}} = Z\,\frac{(EL)^3}{6\pi^2} \,. \tag{2.8}$$

The total energy of this system at temperature T is given by

$$E_{\text{tot}} = \int\limits_0^\infty \mathrm{d}E N(E) e^{-\frac{E}{T}} = Z \frac{L^3 T^4}{\pi^2} \,. \tag{2.9}$$

As the Schwarzschild radius of the box with this total energy scales cubically with the size of the box itself, but the box should better have a linear dimension larger than its own Schwarzschild radius for not becoming a black hole, the temperature thus must be lowered according to

$$T < \frac{\sqrt{\pi}}{(2Z)^{\frac{1}{4}} G_N^{\frac{1}{4}} L^{\frac{1}{2}}} \,.$$

The bound on the maximal temperature (2.7) then also yields an upper bound on the entropy of the system,

$$S = C_2 Z V T^3 < C_3 \frac{Z^{\frac{1}{4}} V^{\frac{1}{2}}}{G_N^{\frac{3}{4}}} = C_4 Z^{\frac{1}{4}} \left(\frac{A}{\ell_P^2} \right)^{\frac{3}{4}} \tag{2.10}$$

This bound is, in the end, independent of the temperature. Thus, if S is interpreted as the information theoretic entropy of microstates of quantum fields inside the volume V, (2.10) yields a bound on the number of microstates in terms of the enclosing area A,

$$n < C_5 \left(\frac{A}{\ell_P^2} \right)^{\frac{3}{4}} . \tag{2.11}$$

In deriving (2.11), I assumed the number of species Z to be a small natural number and thus of order one and absorbed the factor $Z^{\frac{1}{4}}$ into C_5.[3] As the above reasoning uses results from classical gravity, such as the relation between the Schwarzschild radius and the mass of a black hole (2.4), or semiclassical results such as Hawking's area law (2.1), it is only valid for large spatial volumes V and bounding areas A, and in particular for black holes with large mass. Also the thermodynamic quantities energy and entropy and in particular their scaling behaviors with the volume and temperature is only valid for large thermodynamic systems. We thus have to work with large black holes, which have large horizon area, and also with a large spatial volume which will have a large bounding area. Under these conditions, the maximal entropy of the quantum fields (2.10) will, because of the power of three quarters in (2.10), always be smaller than the entropy of the corresponding black hole with horizon area A, eq. (2.1).[4] This ensures that for large enough spatial volumes the formation of a black hole is always thermodynamically favoured as we reach the entropy bound (2.10) from below. Thus, for the spatial volume we considered, the black hole is, for high enough energy densities inside V, the state with maximal entropy and thus the equilibrium state of the system.[5]

[3]This assumption is of course not valid any more in models with many species such as the ones proposed in [130,131], but the bound on quantum field microstates will in this case be even more stringent due to the number of species entering with a power of one quarter. In fact, the bound gets more and more stringent the more particle species are present.

[4]For a perfectly spherical spatial volume V there is an obvious discrepancy between the maximal number of quantum field states fitting V and the number of states derived from the black hole entropy formula (2.1). One might hope that taking into account interactions, nonperturbative effects or effects on quantum fields on curved space-time might affect the crude estimate (2.7) in a way allowing at the end a scaling linear in the area. What should happen is that free field states which have an energy exceeding the threshold to form a black hole in some region of space are changed by the above-mentioned effects in such a way that the energy is reduced. This can for example happen through effects induced by the curvature of space-time, as it is well-known that even free quantum fields can feature negative energy densities on curved spaces (for example in the vicinity of black hole horizons, see e.g. [132]).

[5]Hawking's area theorem [133], which states that in physical processes like black hole formation or merger of black holes and under the assumption that the weak energy condition is fulfilled the total horizon area of all black holes cannot decrease, ensures that the equilibrium state always is one black hole with total area A, rather than many smaller black holes.

The above result is at first sight counterintuitive: One would expect the number of states of a collection of particles to grow exponentially with the volume rather than with the area. However, gravity imposes a more rigid cutoff in order for the states inside V to be observable, namely that the number of states can only grow with the area as in (2.11). We thus understand that both the number of states of a collection of quantum fields in a spatial volume V and the number of microstates of a black hole scale with the area bounding V or the horizon area, respectively. The difference between both situations is that for large areas the black hole has more microstates than quantum fields enclosed in the corresponding (approximately spherically symmetric) spatial volume. Based on this insight, 't Hooft concluded in a great leap of thought that in both situations the dynamics of the relevant degrees of freedom, inside the horizon and inside the volume V, should be describable entirely by a kind of theory[6] on the bounding surface, may it be a horizon or just some closed surface in three-dimensional space. Furthermore, the theory on the bounding surface has to have the property of being discrete in the sense that the density of degrees of freedom in the information theoretical sense is bounded by roughly one degree of freedom per Planck area. To summarise, the **holographic principle in quantum gravity** is the following conjecture:

> In nature, i.e. in a hypothetical quantum theory of both gravity and matter, there cannot exist more degrees of freedom in a closed spatial volume V than the ones which fit onto the bounding area A, quantised in Planck units. The dynamics of the degrees of freedom inside V are entirely describable by a theory on the bounding surface of V (or on the horizon, if a black hole has formed), and this theory should provide the correct upper bound for the density of states of one state per Planck area.

't Hooft furthermore argues that by considering infinitely large black holes, one could define a limiting situation in which the surface becomes uncompactified, and thus that a two-dimensional hypersurface in three-dimensional space should carry a theory of the degrees of freedom of matter and gravity in either half of the three-dimensional space. In ordinary Einstein-Hilbert gravity in four-dimensional space-time, the horizon topology of a stationary black hole (under the assumption of the dominant energy condition) always is a sphere ([134, 135], see also [136]), which is not the same as ^{2}, but the one-point compactification obtained by adding the point at infinity. Also, taking the infinite mass limit of the Schwarzschild black hole, the horizon will be shifted to $r_S \to \infty$, thus swallowing up all of space and rendering the idea of such a limit rather doubtful. What one would need to conjecture a holographic description of the fields in all of space-time would be extended black objects which also respect the isometries of Minkowski space-time. As will become clear in section 2.2.3 below, string theory in ten dimensions (or better its

[6] 't Hooft calls it a "topological quantum field theory", although it is by no means clear at all that it must be a quantum field theory. In fact, the cellular automaton model he constructs as a realisation of the holographic principle [33] is very different from an ordinary quantum field theory.

low energy effective theory, ten-dimensional supergravity), naturally provides such black objects: They are the Dirichlet branes of string theory.

2.2 Elements of Superstring Theory in Ten Dimensions

Before explaining the AdS/CFT correspondence as a concrete realisation of the holographic principle for four-dimensional, infinitely extended Minkowski space-time, I need to introduce two basic notions of supersymmetric string theory, namely the notion of low energy effective actions and the notion of Dirichlet branes, which are defined as boundary conditions for open strings, but by the peculiar quantum dynamics of string theory actually are dynamical objects by themselves. The purpose of this section is to give an introduction into these concepts. The material given here is mostly taken from the available textbooks on string theory [31,137–147], as well as from the very useful review [148]. In the remainder of this thesis I will always employ natural units $\hbar = c = k_B = 1$, if not stated differently.

2.2.1 Bosonic String Theory

String theory as a theory of one-dimensional extended objects originally arose as dual resonance models invented to describe QCD scattering amplitudes: After the construction of an explicitly crossing symmetric scattering amplitude whose pole structure reproduced Regge behaviour (which was found several years earlier in QCD data [149, 150]) by Gabriele Veneziano in 1968 [151], Yoichiro Nambu [67], Holger Bech Nielsen [68] and Leonard Susskind [69] provided a physical interpretation of Venezianos amplitude in terms of excitations of a one-dimensional extended object connecting the quark-antiquark pair inside a meson. Dual resonance models were an active area of research until 1973/74, but were largely abandoned as a theory of strong interactions after the advent of quantum chromodynamics. Nevertheless they were not totally forgotten, and in 1974 John H. Schwarz together with Joel Scherk [152,153] and independently Tamiaki Yoneya [154,155] discovered that the massless excitations of a bosonic string contains an excitation of spin two, which they interpreted as the graviton. This was the birth of what we nowadays call bosonic string theory. I will not attempt to repeat the full quantisation of the bosonic string in all its details here, as it can be reviewed in the above-mentioned textbooks, but a short review shall be allowed for the sake of completeness.

The Classical Bosonic String

In attempting to write down an action for a relativistic one-dimensionally extended object (a string), Yoichiro Nambu [156] and Tetsuo Goto [157] proposed a very simple action,

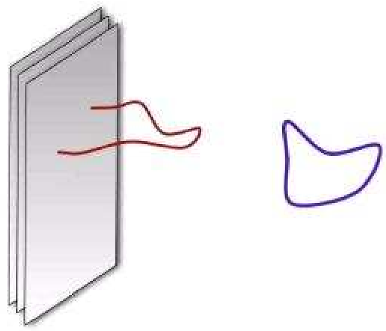


Figure 2.1: Open and closed strings, and Dirichlet branes. (figure taken from [158]).

nowadays named after them. Let $T_F = 1/(2\pi\alpha')$ be the tension of the string, whereas α' is a fundamental constant of dimension length^2, which in the dual resonance models was identified with the slope of the Regge trajectories and had an approximate value of $1GeV^{-2}$. In modern string theory, it is kept arbitrary, but one often its square root to be another (but equivalent) fundamental length scale, the string length, via $\alpha' = \ell_s^2$. Nambu and Goto proposed the following minimal area action principle for the relativistic string in flat D-dimensional Lorentzian space-time,

$$S_{NG} = -T_F \int \mathrm{d}^2\xi \sqrt{-\det_{a,b}\left(\frac{\partial X^\mu(\xi)}{\partial \xi^a}\frac{\partial X^\nu(\xi)}{\partial \xi^b} G_{\mu\nu}(X(\xi))\right)}\,, \quad a,b = 0,1\,. \tag{2.12}$$

This action defines a two-dimensional field theory on the worldsheet of the string, parametrised by coordinates $\xi^a = (\xi^1, \xi^2) = (\tau, \sigma)$, where $\tau \in$ is the world sheet time and σ parametrises the point on the string. As the string has finite length, σ can run for zero to some value, in which case we call the string an **open string** (as depicted in figure 2.1), as it has a beginning and an end. If it is periodically identified, this theory describes a **closed string**. The fields $X^\mu(\xi)$ are scalar fields on the worldsheet and encode the embedding of the string into the D-dimensional space-time manifold which is equipped with a Lorentzian signature metric $G_{\mu\nu}(X)$, as depicted in figure 2.2. The matrix over which the determinant is taken is the metric induced from the space-time onto the worldsheet, and is also called the pull-back of the space-time metric G (via the embedding map $X^\mu(\xi)$), denoted by

$$P[G]_{ab}(\xi) = \frac{\partial X^\mu(\xi)}{\partial \xi^a}\frac{\partial X^\nu(\xi)}{\partial \xi^b} G_{\mu\nu}(X(\xi))\,. \tag{2.13}$$

In fact every space-time totally antisymmetric tensor field of rank $n \leq D$ can be pulled back onto a sub-manifold of dimension m in the case of $n \leq m \leq D$ by a similar formula,

$$P[T]_{a_1 \ldots a_m} = \frac{\partial X^{\mu_1}}{\partial \xi^{a_1}} \cdots \frac{\partial X^{\mu_n}}{\partial \xi^{a_m}} T(X(\xi))_{\mu_1 \ldots \mu_m}\,, \tag{2.14}$$

a fact which will be very useful later on when studying D branes in section 2.2.3. The interpretation of the action (2.12) is that it just computes the total area that the string

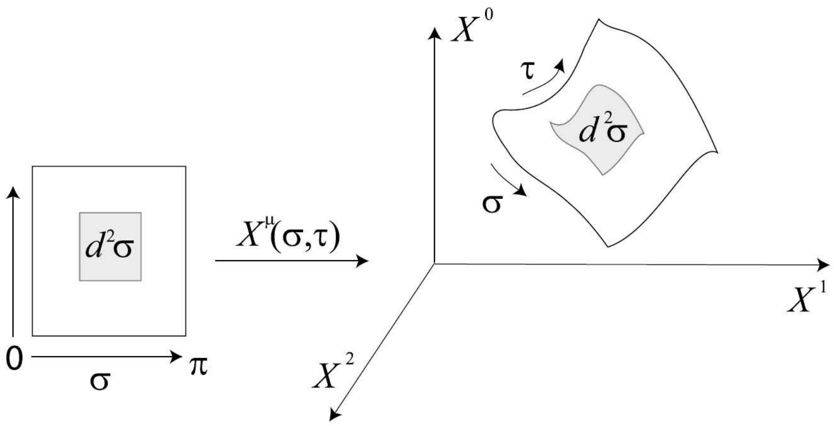


Figure 2.2: The concept of embedding a submanifold into space-time (figure take from [146]).

sweeps out in space-time as it moves through it, multiplied by minus its tension. It is very similar to the action of a relativistic point particle moving on a one-dimensional worldline in space-time,

$$S_{\text{point particle}} = -m \int \mathrm{d}\tau \sqrt{-\dot{X}^\mu(\tau)\dot{X}^\nu(\tau)G_{\mu\nu}(X(\tau))}\,, \tag{2.15}$$

where τ is now the proper time which the particle (or better a clock carried with it) registers. This action just measures the total invariant length of the worldline, multiplied by minus the mass of the point particle. The minus signs in front of both actions, (2.15) and (2.12), are necessary in order for the physical trajectory to minimise the action. For the point particle, this is just due to the well-known fact that a timelike geodesic actually has maximal length rather than minimal length. The Nambu-Goto action needs to have the same sign in order for the string as an extended object to approximately follow timelike geodesics in a curved space too. A similar action principle holds for D branes, as we will see later in section 2.2.3.

The Nambu-Goto action (2.12) is, however, not the one normally used to quantise the bosonic string. The reason is that it is highly nonlinear because of its square root structure, and thus attempts to quantise it will be plagued with operator ordering ambiguities or, in the path integral approach, by the absence of a well-defined distinction between free field propagation and interaction part. The Nambu-Goto model can, however, be transformed into a **two-dimensional nonlinear sigma model** by introducing an auxiliary symmetric two-tensor field h_{ab} on the worldsheet which serves as the two-dimensional world sheet metric. The resulting action was first found by Deser and Zumino [159] and independently by Brink, Di Vecchia and Howe [160] but is commonly denoted after Alexander Polyakov as the **"Polyakov action"** and reads

$$S_P = -\frac{T_F}{2} \int \mathrm{d}^2\xi \sqrt{|h|} h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu\,, \tag{2.16}$$

where the dependence of X on ξ is implicit. The equation of motion for h_{ab}, which is nothing but the condition of vanishing world sheet energy-momentum tensor,

$$0 = T_{ab} = \frac{2}{\sqrt{|h|}}\frac{\delta S_P}{\delta h^{ab}} = -T_F G_{\mu\nu}(X)\left[\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X^\nu\right], \tag{2.17}$$

can be used to classically eliminate the world sheet metric from (2.16) and recover (2.12) by inserting the formula

$$\sqrt{|h|} = \frac{2\sqrt{-\det P[G]}}{h^{cd}G_{\mu\nu}(X)\partial_c X^\mu \partial_d X^\nu}$$

back into (2.16).

Classical and Quantum Scale Invariance and Critical Dimensions

The theory defined by (2.16) can be solved more easily classically and also quantised in a very straightforward way. In this course, its large amount of symmetries are of great help. Eq. (2.16) is invariant both under worldsheet diffeomorphisms

$$\xi^a = \xi^a(\zeta)\,, \quad h_{ab}(\xi(\zeta))\frac{\partial \xi^a}{\partial \zeta^c}\frac{\partial \xi^b}{\partial \zeta^d} = h_{cd}(\zeta)\,, \tag{2.18}$$

local Weyl transformations of the worldsheet metric

$$h_{ab}(\xi) \mapsto e^{2\sigma(\xi)}h_{ab}(\xi)\,, \tag{2.19}$$

as well as diffeomorphisms of the space-time manifold

$$X^\mu = X^\mu(Y)\,, \quad G_{\mu\nu}(X(Y))\frac{\partial X^\mu}{\partial Y^\rho}\frac{\partial X^\nu}{\partial Y^\sigma} = G_{\rho\sigma}(Y)\,. \tag{2.20}$$

Exploiting worldsheet diffeomorphisms and a local Weyl transformation, the metric h_{ab} can locally always be brought into flat form,

$$h_{ab} = \eta_{ab} \text{ or } \delta_{ab}\,, \tag{2.21}$$

depending on the signature of the world sheet manifold. This gauge choice is called conformal gauge. Of course, after fixing a gauge for a gauge symmetry, the corresponding constraints have to be imposed by hand. In the case of worldsheet diffeomorphism symmetry it is the vanishing of the energy-momentum tensor (2.17) which has to be imposed on the classical solutions as well as on the quantum mechanical state space. As the classical equivalence between the actions (2.12) (which is more transparent from the physical point of view because of its interpretation in terms of a minimal worldsheet area principle) and (2.16) (which is less nonlinear) depended on the vanishing of the worldsheet energy-momentum tensor, one actually needs to impose (2.17) also on the quantum level in order to at least have a chance for the quantisation of (2.16) to have any connection to any direct quantisation of (2.12). Now, as we also used a Weyl transformation to fix conformal gauge, we also need to impose the constraint arising from this local symmetry quantum mechanically. The generator for Weyl transformations is the trace of the energy-momentum tensor (2.17), $h^{ab}T_{ab}$. Classically, as the worldsheet is two-dimensional, (2.17) automatically implies the constraint $h^{ab}T_{ab} = 0$. On the quantum level the situation is different. If one understands the action (2.16) for a flat space-time $G_{\mu\nu} = \eta_{\mu\nu}$ as a two-dimensional field theory of D bosonic fields X^μ, $\mu = 0, \ldots, D-1$ coupled to a two-dimensional background metric field h, the conformal anomaly (the calculation is most easily done in Euclidean signature, see e.g. [161]) induces a vacuum expectation value of the trace of the worldsheet energy-momentum tensor proportional to the worldsheet curvature,

$$\langle T_a{}^a \rangle = h^{ab} \langle T_{ab} \rangle = \frac{D-26}{24\pi} R[h] \,. \tag{2.22}$$

Here the contribution proportional to the dimension of space-time comes from the D scalar fields, and the negative contribution comes from integrating out the Faddeev-Popov ghosts arising from gauge fixing. Thus, if we want to carry over the equivalence between (2.12) and (2.16) to the quantum level, the bosonic string needs to live in the critical dimension

$$D_{\text{crit}}^{\text{bos.}} = 26 \,. \tag{2.23}$$

This connection between the conformal anomaly and the critical dimension of space-time was first noticed by Alexander M. Polyakov in his seminal work on the path-integral quantisation of closed bosonic strings [70]. A similar calculation for a $\mathcal{N} = (1,1)$ supersymmetric closed string in a flat (and otherwise free of any additional vacuum expectation values) target space-time yields the critical dimension of superstring theory [162],

$$D_{\text{crit}}^{\text{susy}} = 10 \,. \tag{2.24}$$

First Quantised Closed Strings: Particles and Fields and Field Theories

Of course, for general metric $G_{\mu\nu}(X)$ (2.16) still defines a field theory highly nonlinear in the fields X^μ. As already mentioned above, for a flat ten-dimensional Minkowski background, $G_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, ...1)$, the theory reduces to a collection of ten free bosonic fields X^μ. It is this case for which (2.16) was first quantised and the spectrum of string excitations was found. Probably the easiest way to quantise the string is in light-cone gauge, which means the choice of the conformal gauge $h_{ab} = \eta_{ab}$ (eq. (2.21)) plus light cone coordinates on the world sheet

$$\sigma^\pm = \tau \pm \sigma \,. \tag{2.25}$$

The action (2.16) reduces upon gauge fixing to

$$S_P^{\text{g.f.}} = \frac{T_F}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \left(\dot{X}^\mu \dot{X}_\mu - X'^\mu X'_\mu \right) \,, \tag{2.26}$$

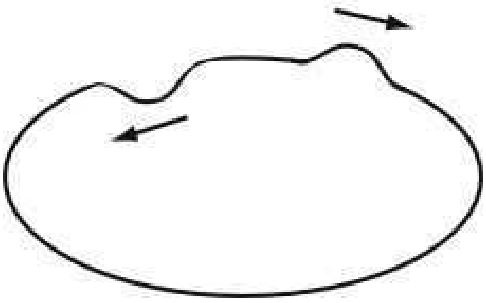


Figure 2.3: Left- and right moving excitations of a closed string (figure taken from [163]).

where the dot denotes a derivative w.r.t τ and the prime denotes the derivative w.r.t. σ. One then proceeds as follows: First, one needs to consider the boundary conditions for the fields X^μ. For a closed string, this is very simple. As a closed string is a closed loop, it has no boundary and thus only a periodic identification

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi n)\,, \quad n \in \tag{2.27}$$

is required. The light-cone gauge fixed equations of motion for the string embedding fields X^μ the simply read

$$0 = \eta^{ab}\partial_a\partial_b X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = \frac{\partial^2}{\partial\sigma^+\sigma^-} X^\mu(\sigma^\pm)\,. \tag{2.28}$$

This wave equation is solved by a superposition of **left-moving** (only depending on σ^+) and **right-moving** excitations (which only depend on σ^-),

$$X^\mu(\sigma,\tau) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma)\,. \tag{2.29}$$

The left and right movers (which are depicted in figure 2.3) can now be expanded in a Fourier series on the circle, s.t. the closed string boundary conditions (2.27) is automatically fulfilled, yielding

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{\ell_s^2}{2}p^\mu\sigma^+ + \frac{i}{2}\ell_s \sum_{n\in\ ,\, n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+}\,, \tag{2.30}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{\ell_s^2}{2}p^\mu\sigma^- + \frac{i}{2}\ell_s \sum_{n\in\ ,\, n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}\,. \tag{2.31}$$

Here the possible constant term x^μ in the solution to (2.28) has been distributed equally amongst the left- and rightmoving part, and the powers of ℓ_s have been introduced such that the integration constant p^μ has dimension of energy, and the oscillation mode coefficients α^μ_n, $\tilde{\alpha}^\mu_n$ are dimensionless. This is natural as (in natural units) ℓ_s is the only dimensionful parameter in the theory defined by (2.16). Introducing different $p^\mu_{L/R}$, which would be possible a priori, is forbidden by the periodicity condition (2.27). The integration constant x^μ serves as the center-of-mass position of the string (i.e. its embedding into space-time is just a constant if the other integration constants $p^\mu = \alpha^\mu_n = \tilde{\alpha}^\mu_n = 0$ all vanish). The second integration constant p^μ is interpreted as the center-of-mass momentum of the string moving through space-time.

The bosonic string can now be canonically quantised just like an ordinary (two-dimensional) quantum field theory by promoting the fields X^μ to operators and postulating canonical commutation relations between X^μ and the canonical momentum

$$P^\mu(\tau,\sigma) = \frac{\delta S_P^{\text{g.f.}}}{\delta \dot{X}_\mu} = T_F \dot{X}^\mu \,, \tag{2.32}$$

at equal world-sheet times,

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \qquad [X^\mu, X^\nu] = [P^\mu, P^\nu] = 0 \,. \tag{2.33}$$

For the closed string, this yields commutators for the center-of-mass modes and the oscillators reading (all other commutators are vanishing)

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \,, \quad [\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \,. \tag{2.34}$$

Note that the center-of-mass modes form the expected Heisenberg algebra. This means that the center-of-mass movement of quantised closed strings in flat space-time and without any additional background fields is equivalent to the quantum movement of a point particle in ordinary, commutative space-time. We will see later in chapter 4 that different background fields can make open strings see space-time rather as a noncommutative one.

The commutation relations of the oscillators (2.34) are, up to a rescaling (similar for the left movers $\tilde{\alpha}$)

$$a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m \,, \quad a^{\mu\,\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m} \quad \text{for } m > 0 \,,$$

just two infinite sets of quantum mechanical harmonic oscillators

$$[a^\mu_m, a^{\nu\dagger}_n] = [\tilde{a}^\mu_m, \tilde{a}^{\nu\dagger}_n] = \eta^{\mu\nu}\delta_{m,n} \quad \text{for } m,n > 0 \,. \tag{2.35}$$

The kinematical Hilbert space of the first quantised closed bosonic string thus simply consists of an infinite direct sum of two copies (one for the left movers and one for the right movers) of the harmonic oscillator Hilbert space. The ground state in each sector is annihilated by the lowering operators, for example for the right movers by a^μ_m for $m > 0$,

$$a^\mu_m|0\rangle = 0 \,, \quad m > 0 \,.$$

One can now construct

The so-constructed state space however has a severe problem: On first sight, as the commutators of the time components of the oscillator modes have the wrong sign,

$$[a^0_m, a^{0\,\dagger}_m] = -1 \,,$$

it seems to contain negative norm states. However, on the quantum level, one now still needs to implement the Virasoro constraint (2.17). Without attempting to prove this here, it turns out that after imposing these constraints on the quantum level, the bosonic string is free of negative norm states provided we choose the dimension to be the critical one, eq. (2.23).[7] For a proof of this statement, see e.g. chapter 2.4 of [146]. The quantum version of (2.17) also provides a mass operator for the states, which for the closed bosonic string reads

$$M^2 = -\hat{p}_\mu \hat{p}^\mu = \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty} n\left(a_n^{\mu\dagger} a_{n,\mu} + \tilde{a}_n^{\mu\dagger}\tilde{a}_{n,\mu}\right) - 2\right). \tag{2.36}$$

Thus the mass squared of a closed string state is basically given by the number of excitations of left- and right movers, weighted with the level n. Note however that the mass squared computed by (2.36) can be negative, which is a sign of an (unwanted) tachyonic excitation. As we will see later in section 2.66, superstring theory has a consistent way of doing away with the tachyonic modes.

To understand the spectrum of the bosonic closed string, we furthermore need two more pieces of information. First, from the two-dimensional field theoretic point of view it is clear that the mass operator (2.36) is nothing but a (normal ordered version of) the generator of worldsheet time translations. For closed strings, it turns out that there is another symmetry, namely translations in the worldsheet space direction (i.e. the σ direction), which needs to be implemented as a symmetry on the quantum mechanical state space. This second constraint yields the so-called "level-matching" condition, i.e. the requirement that the number of left- and right-moving excitations are equivalent,

$$N|\text{phys}\rangle = \tilde{N}|\text{phys}\rangle\,, \quad N = \sum_{n=1}^{\infty}\alpha^\mu_{-n}\alpha_{n,\mu}\,, \quad \tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{n,\mu}\,. \tag{2.37}$$

This condition can be understood to prevent an overall unbalanced energy-momentum distribution on the string being created by the excitations rotating left- and rightwards along the string with the speed of light. It is clear intuitively that for example a single left-moving excitation would break the σ-translation invariance of the string worldsheet theory.

The second piece of information we need is the fact that the operator p^μ actually commutes with all the raising and lowering operators (see (2.34)) and also (without proof) with the quantum version of the Virasoro generators (2.17). Thus, every state in the Hilbert space of the bosonic string is labeled by the excitation numbers of the left- and right movers as well as by a continuous center of mass momentum k^μ. We are now ready to understand the spectrum of the bosonic closed string, whose lowest lying levels are summarised in table 2.1:

[7] It turns out that the Virasoro constraints also allow different critical dimensions, provided that a certain string excitation, the dilaton, acquires a vacuum expectation value which depends linearly on the spatial coordinates. These kinds of vacua are called linear dilaton vacua.

State	$N=\tilde{N}$	**Space-Time Interpretation**	$\alpha' M^2$
$\|0,k\rangle$	0	Tachyon $T(x)$	-4
$[\tilde{\alpha}_{-1}^{(\mu}\alpha_{-1}^{\nu)} - \eta^{\mu\nu}\tilde{\alpha}_{-1}\cdot\alpha_{-1}/26]\|0,k\rangle$	1	Graviton $G_{\mu\nu}$	0
$\tilde{\alpha}_{-1}^{[\mu}\alpha_{-1}^{\nu]}\|0,k\rangle$	1	Antisymmetric Tensor $B_{\mu\nu}$	0
$\eta^{\mu\nu}/26\tilde{\alpha}_{-1}\cdot\alpha_{-1}\|0,k\rangle$	1	Dilaton Φ	0

Table 2.1: Lowest lying modes of the closed bosonic string.

The vacuum, which is defined as the state annihilated by all annihilation operators

$$0 = \alpha_n^\mu|0,k\rangle = \tilde{\alpha}_n^\mu|0,k\rangle\,, \quad n>0\,, \tag{2.38}$$

is the state of lowest mass squared $M^2 = -4/\alpha'$. This negative mass squared indicates a tachyonic mode in the theory, which destabilises the vacuum. At this stage closed bosonic string theory thus looks quite sick but, as mentioned above, superstring theory allows to project out the tachyon from the spectrum in a consistent way.

The next state in the spectrum would be the one with one left- and one right-moving excitation at level one each,

$$\tilde{\alpha}_{-1}^\mu\alpha_{-1}^\nu|0,k\rangle\,, \tag{2.39}$$

which according to (2.36) is a massless state, $M^2 = 0$. Being massless, these states will for sure be important for low energy physics, and the natural question arises how the well-known massless fields like the graviton or the photon arise from these string excitations. The answer lies in the fact that we can build even more general states with nontrivial momentum profiles from (2.38) and (2.39) by including a momentum-dependent profile function $t(k)$ for the tachyon and a polarisation tensor $\zeta_{\mu\nu}(k)$ for the massless states,

$$|T\rangle = \int \mathrm{d}^{10}k\, t(k)|0,k\rangle\,, \quad |\zeta\rangle = \int \mathrm{d}^{10}k\, \zeta_{\mu\nu}(k)\tilde{\alpha}_{-1}^\mu\alpha_{-1}^\nu|0,k\rangle\,. \tag{2.40}$$

Since the center-of-mass position and momentum operator form a Heisenberg algebra (2.34), we could have as well diagonalised the center-of-mass position operator x^μ by introducing a vacuum $|0,x\rangle$. As the center-of-mass movement is that of a quantum particle in all space-time directions, the corresponding transition element between these two bases must be a plane wave,

$$\langle 0,x|0,k\rangle = e^{ik^\mu x_\mu} = e^{ik\cdot x}\,. \tag{2.41}$$

From this it follows that the position space wave functions are given by transition elements between these two bases

$$\langle 0,x|T\rangle = \int \mathrm{d}^{10}k\, t(k)e^{ik\cdot x} =: t(x)\,, \tag{2.42}$$

$$\begin{aligned}\langle 0,x|\alpha_1^\nu\tilde{\alpha}_1^\mu|\zeta\rangle &= \int \mathrm{d}^{10}k\, \zeta_{\rho\sigma}(k)\langle 0,x|\alpha_1^\nu\tilde{\alpha}_1^\mu\tilde{\alpha}_{-1}^\rho\alpha_{-1}^\sigma|0,k\rangle\,,\\ &= \eta^{\mu\rho}\eta^{\nu\sigma}\int \mathrm{d}^{10}k\, \zeta_{\rho\sigma}(k)\langle 0,x|0,k\rangle = \zeta^{\mu\nu}(x)\,.\end{aligned} \tag{2.43}$$

where in the second and third line the definition of the vacuum (2.38) and the commutation relations (2.34) were used. Note that in calculating $\zeta_{\mu\nu}(x)$ it is necessary to take the overlap with the fundamental state $\tilde{\alpha}^{\mu}_{-1}\alpha^{\nu}_{-1}|0,x\rangle$ and not just with the oscillator vacuum $|0,x\rangle$ in order to be able to extract the polarisation tensor. We thus find that the position space wave functions of the string states, which are functions of the eigenvalue of the center-of-mass position operator $\hat{x}^{\mu}|0,x\rangle = x^{\mu}|0,x\rangle$, may be interpreted as fields on space-time. The center-of-mass eigenvalues x^{μ} themselves serve as the coordinates of space-time, and the spin of the corresponding space-time fields is given, through the polarisation tensors which are needed to define general states similar to (2.40), by the index structure of the string of creation operators $\alpha^{\mu_1}_{-n_1}\ldots\alpha^{\mu_k}_{-n_k}\tilde{\alpha}^{\nu_1}_{-m_l}\ldots\tilde{\alpha}^{\nu_k}_{-m_l}$ which acts on the vacuum to create the state under consideration. We now can reduce the polarisation tensor $\zeta_{\mu\nu}$ of the massless closed string state in terms of irreducible two-tensor representations of the 26-dimensional Lorentz group SO(1,25) which are, respectively, the symmetric traceless, antisymmetric and trace part of $\zeta_{\mu\nu}$,

$$
\begin{aligned}
h_{\mu\nu}(x) &= \frac{1}{2}\left(\zeta_{\mu\nu}+\zeta_{\nu\mu}\right)-\frac{1}{26}\eta_{\mu\nu}\eta^{\rho\sigma}\zeta_{\rho\sigma}\,, && (2.44)\\
b_{\mu\nu}(x) &= \frac{1}{2}\left(\zeta_{\mu\nu}-\zeta_{\nu\mu}\right)\,, && (2.45)\\
\varphi(x) &= \frac{1}{26}\eta_{\mu\nu}\eta^{\rho\sigma}\zeta_{\rho\sigma}\,. && (2.46)
\end{aligned}
$$

The interpretation of this decomposition is that $h_{\mu\nu}$, $b_{\mu\nu}$ and φ respectively describe a 26-dimensional **graviton**, an **antisymmetric tensor excitation** called the **Kalb-Ramond field** [164] and a real scalar called the **dilaton**. Note that these string states are interpretated only as linearised excitations of the full nonlinear fields, which can be deduced e.g. from the tracelessness of h. The interpretation as particle excitations also becomes clear if considering a plane wave e^{ipx} as spatial profile: In that case the momentum space profiles just become a delta function, e.g. for the dilaton $t_p(k) = \delta^{10}(k-p)$ and thus the corresponding momentum representation state is just a fundamental string excitation such as (2.38) or (2.39) with momentum p. Also, in this special case the dispersion relations are the correct 26-dimensional relativistic dispersion relations, as the mass operator (2.36) is nothing but the relativistic dispersion relation $M^2 = -k_\mu k^\mu$. Furthermore let us note without proof that also the relativistic wave equations governing the free evolution of these fields can be deduced from the quantum Virasoro constraints (2.17). For completeness, the closed bosonic string states of negative or zero mass squared are summarised in table 2.1.

Clearly, if the strings we consider have a string length comparable to the known four-dimensional Planck Length $\ell_s \approx \ell_P = 1.6\cdot 10^{-35}$m, all modes with higher occupation numbers and thus with positive mass squared will have masses well above the Planck mass. If one is only interested in the physics at low energy scales, well below the Planck scale, these modes thus can be integrated out, giving rise to operators of dimension greater than 26 in the low energy effective field theory. These operators will be suppressed by inverse powers of the Planck mass and thus come with positive powers of α'. Generically, it is thus possible to define a **point particle limit of the low energy effective action** of the massless string modes by taking the limit $\alpha' \to 0$ while keeping all relevant energy scales in the problem fixed in terms of the string length. This limit focuses on the physics of the massless modes, decoupling the massive string excitations. As we will see later, such a scaling limit also lies at the heart of the AdS/CFT correspondence.

Quantum Open Strings: A First Look at Dirichlet Branes

The main difference between open and closed strings is that the open string has a beginning and an end, whereas the closed string is just a closed loop propagating in space and time. For open strings, the world sheet thus has a spacelike boundary, restricting the coordinate σ for example to lie in the interval $[0, \pi]$. The action (2.16) thus will yield boundary terms upon variation w.r.t. X^μ and thus needs to be supplemented by appropriate boundary conditions which make the boundary terms vanish. Variation of the gauge fixed Polyakov action (2.26) yields

$$\begin{aligned} \delta S_P^{\text{g.f.}} &= -T_F \int \mathrm{d}\tau \mathrm{d}\sigma\, \delta X^\mu \left(\ddot{X}_\mu - X''_\mu\right) + T_F \int_0^\pi \mathrm{d}\sigma \left[(\delta X^\mu \dot{X}_\mu)_{\tau=\infty} - (\delta X^\mu \dot{X}_\mu)_{\tau=-\infty}\right] \\ &\quad + T_F \int_{-\infty}^{\infty} \mathrm{d}\tau \left[(\delta X^\mu X'_\mu)_{\sigma=\pi} - (\delta X^\mu X'_\mu)_{\sigma=0}\right] . \end{aligned} \tag{2.47}$$

As always, we require the field variation to vanish for initial and final configurations, $\delta X^\mu|_{\tau=-\infty} = \delta X^\mu_{\tau=\infty} = 0$, but the second boundary term leaves us with two choices for each of the string endpoints, namely

$$\delta X^\mu(\sigma = 0 \text{ or } \pi, \tau) = 0 \quad \Rightarrow X^\mu(\sigma = 0 \text{ or } \pi, \tau) = x_0^\mu(\tau)\,, \quad \textbf{(Dirichlet)} \tag{2.48}$$

$$\partial_\sigma X^\mu(\sigma = 0 \text{ or } \pi, \tau) = 0\,. \quad \textbf{(Neumann)} \tag{2.49}$$

Note that a string can be given different boundary conditions separately at its two ends, $\sigma = 0, \pi$. Starting with a general mode expansion

$$X^\mu = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-)\,, \tag{2.50}$$

$$X_L^\mu = \frac{x_L^\mu}{2} + \frac{\ell_s^2}{2} p_L^\mu(\tau+\sigma) + \frac{i}{2}\ell_s \sum_{n\in\ \ ,n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}\,, \tag{2.51}$$

$$X_R^\mu = \frac{x_R^\mu}{2} + \frac{\ell_s^2}{2} p_R^\mu(\tau-\sigma) + \frac{i}{2}\ell_s \sum_{n\in\ \ ,n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}\,, \tag{2.52}$$

Neumann boundary conditions on both ends imply

$$p_L^\mu = p_R^\mu\,, \quad \tilde{\alpha}_n^\mu = \alpha_n^\mu \quad \forall n \in \quad ,\ n \neq 0\,. \tag{2.53}$$

Thus the left- and right-movers are coupled in a very simple way: For each level n a left-moving excitation gets reflected into a right-moving one once it reaches a boundary and

State	N	Space-Time Interpretation	$\alpha' M^2$
		Spacefilling D25 Brane	
$\lvert 0, k^\mu\rangle$	0	Tachyon	-1
$\alpha^\mu_{-1}\lvert 0, k^\mu\rangle$	1	Vector Boson A_μ	0
		Parallel Dq-Dp Branes ($q < p$) **with Separation** $\lvert\Delta x\rvert = 2\pi\alpha' m$	
		Dq-Dq Strings	
$\alpha^i_{-1}\lvert 0, k^i\rangle$	1	Dq Brane Vector Boson $A_{q,i}$, $i = 0, \ldots, q$	0
$\alpha^a_{-1}\lvert 0, k^i\rangle$	1	Dq "Relatively" Transverse Scalar X^a_q , $I = q+1, \ldots, p$	0
$\alpha^I_{-1}\lvert 0, k^i\rangle$	1	Dq Totally Transverse Scalar X^I_q , $I = p+1, \ldots, 25$	0
		Dq-Dp Strings	
$\lvert 0, k^i\rangle$	0	Massive vacuum state	$\alpha' m^2$
		Dp-Dp Strings	
$\alpha^i_{-1}\lvert 0, k^{i\cup a}\rangle$	1	Dp Brane Vector Boson $A_{p,i\cup a}$, $i \cup a = 0, \ldots, p$	0
$\alpha^I_{-1}\lvert 0, k^{i\cup a}\rangle$	1	Dp Totally Transverse Scalar X^I_p , $I = p+1, \ldots, 25$	0

Table 2.2: Lowest lying modes of the open bosonic string with different boundary conditions. Only modes with finite mass squared in the point particle limit $\alpha' \to 0$ are listed.

vice-versa. The quantisation however proceeds the same way as for closed strings, and the mass formula for a Neumann-Neumann (NN) string is found to be

$$M^2_{NN} = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} n a^{\mu\dagger}_n a_{n,\mu} - 1\right) . \tag{2.54}$$

The **spectrum of the NN string** thus consists again of a tachyonic vacuum $\lvert 0, k\rangle_{NN}$ with $M^2 = -1/\alpha'$ and of a massless vector boson $\alpha^\mu_{-1}\lvert 0, k\rangle_{NN}$. This is a very remarkable observation: Open string excitations naturally give rise to vector bosons such as photons and gauge bosons. The gauge symmetry in that case is also a property of the physical state space which is obtained from the naive state space built up with the creation operators $a^{\mu\dagger}_n$ by imposing the quantum Virasoro constraints: It turns out that some states which can be built up by acting with part of the Virasoro constraints on the vacuum are "spurious" in the sense that they are orthogonal to any physical state. They thus can be added to physical states without changing the transition amplitudes between physical states. It turns out that this addition actually corresponds to a gauge transformation of the space-time field associated with our vector boson, and thus string theory naturally includes gauge symmetries.[8] How the nonabelian nature of gauge bosons arises in string theory will be explained in the following.

The Neumann boundary conditions restricted the derivative of the fields X^μ normal to the worldsheet boundary to vanish. On the other hand **Dirichlet boundary conditions** (2.48) are restricting the positions of the string endpoints themselves. If for example

[8]The same mechanism is at work in implementing the diffeomorphism symmetry in the closed string sector.

the 25-direction of the string is given a Dirichlet-Dirichlet (DD) condition (and the other directions are still NN directions)

$$X^{25}(0,\tau) = 0\,, \quad X^{25}(\pi,\tau) = x^{25}\,,$$

then the string is suspended between these two points on the 25-axis but free to move in all the other directions including time. The string is thus restricted to move between two hypersurfaces in our 26-dimensional space-time, and such hypersurfaces are commonly called **Dirichlet p branes** or just shortly **Dp branes**, where p counts the number of *spatial* dimensions the string is equipped with *Neumann* boundary conditions, i.e. the number of spatial directions the string is allowed to move in. An open bosonic string without any Dirichlet direction would thus move on a **spacefilling D25 brane**, while the above example would correspond to a string suspended between two parallel but separated D24 branes.

> Dp branes are thus by definition hypersurfaces on which open strings end and on which they can move around freely.

Later we will see that Dp branes are also dynamical objects which for example couple to gravity and curve space and time. This dual interpretation of Dp branes, on the one hand as the objects in string theory on which open strings end and which thus carry supersymmetric gauge theories on their worldvolume and on the other hand as sources for the gravitational field, will lead us then to conjecture a duality between a four-dimensional supersymmetric gauge theory called $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, and type IIB string theory on the space-time $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The effect of **Dirichlet-Dirichlet (DD) boundary conditions** is also easily deduced. Let us first consider DD boundary conditions which fix both ends of the string to one and the same particular position in $25-p$ directions, e.g. $X^I(\sigma = 0,\pi,\tau) = x^I\,,\ I = 25-(p-1),\ldots,25$. This imposes on the mode expansion in I-directions (2.50)

$$\frac{x_L^I + x_R^I}{2} = x^I\,, \quad p_L^I = p_R^I = 0\,, \quad \tilde{\alpha}_n^I = -\alpha_n^I\,. \tag{2.55}$$

The center-of-mass coordinate is thus fixed to x^I in each Dirichlet direction, the total center-of-mass momentum $2p^I = p_L^I + p_R^I$ is fixed to zero, and again the left- and right-moving oscillations are coupled, but now with a different sign compared to the NN case. It is exactly this sign which yields a factor $\sin(n\sigma)$ in the sum in (2.50), which is what is needed to fulfill the DD conditions. Because of p^I vanishing, the spectrum for DD directions is now given by a mass operator which represents the momentum in the remaining $p+1$ directions (along the Dp brane),

$$M_{DD}^2 = \sum_{i=0}^{p} p^i p_i = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\left[\sum_{i=0}^{p}\alpha_{-n}^i\alpha_{n,i} + \sum_{I=p+1}^{25}\alpha_{-n}^I\alpha_{n,I}\right] - 1\right)\,. \tag{2.56}$$

Thus we find that if we exchange several NN directions by DD directions, the mass formula basically stays the same, but the oscillator terms split into two groups labeled by the space-time index $\mu = 0, \ldots, 25$ which is split up into two groups $i = 0, \ldots p$ and $I = p+1, \ldots, 25$. However, the oscillators belonging to both groups still contribute to the mass squared in the same way as for the pure NN case, eq. (2.54). The physical difference between the pure NN and mixed NN/DD case is rather that the mass squared is now the mass squared with respect to a lower-dimensional space-time, namely the worldvolume of the Dp brane. The fields we build up out of α^i_{-n} (along the Dp brane) and α^I_{-n} (transversal to it) will thus be localised on the hypersurface the Dp brane is in the 26-dimensional space-time. The **open string spectrum of a single Dp brane** then consists out of the same tachyonic vacuum $|0, k^i\rangle$ as before (but with the momentum now only running in the p+1 worldvolume directions of the brane), of a massless gauge field A_i associated with the state $\alpha^i_{-1}|0, k^i\rangle$ and of massless scalars X^I associated with the state $\alpha^I_{-1}|0, k^i\rangle$, which describe the fluctuations of the strings transverse to the brane. It will become clear later in section 2.3.6 that the corresponding space-time fields also can be interpreted as describing the embedding of the brane into space-time, as well as small fluctuations around these embeddings. Thus Dp branes are not only static, put-in-by-hand hypersurfaces on which open strings have to end, but they will turn out to be dynamical objects which for example curve themselves differently according to the geometry of the surrounding space-time.

We just saw that a Dp brane naturally carries a gauge field, but this was an abelian one, with gauge group U(1). String theory however also provides **nonabelian gauge bosons**: One can attach nondynamical matrix degrees of freedom to the ends of open strings without spoiling space-time Poincaré invariance or world-sheet conformal invariance through the so-called **"Chan-Paton" factors** [165], which were originally an attempt to invent a Veneziano like amplitude including isospin degrees of freedom. The idea is simple: Just add a $U(N)$ matrix sector to the open string Hilbert space via

$$|k; a\rangle = \sum_{i,j=1}^{N} |k, ij\rangle \lambda^a_{ij} \,. \tag{2.57}$$

The indices i and j are transforming in the N and $\bar{N}$ representation of $U(N)$ and are interpreted as belonging, respectively, to the endpoint and the starting point of the oriented open string. Endpoints of oriented open strings are thus transforming in the fundamental of the (by now still global) $U(N)$ symmetry group, while starting points transform in the antifundamental. The space-time interpretation of this construction is that the open string under consideration ends not only on one Dp brane, but on a stack of N Dp branes which lie right on top of each other. The Dp branes are indistinguishable objects, and the $U(N)$ symmetry group effects the indistinguishability by permuting the Dp branes amongst each other. However, as it turns out, also the "spurious states" which were responsible for the gauge transformation of the A_μ field, become equipped with the Chan-Paton factors and through the same mechanism as before the global $U(N)$ symmetry is promoted to a gauged $U(N)$ symmetry. Similarly, the transverse scalars X^I become promoted to fields transforming in the adjoint representation of the $U(N)$ gauge group. We thus arrive at the conclusion that

> Open oriented strings ending on one stack of N Dp branes include nonabelian $U(N)$ gauge fields and massless scalars transforming in the adjoint representation of $U(N)$ in their spectrum.

As we will see later in section 2.3.2, the natural guess then actually is the correct one: For oriented open strings, the worldvolume of a stack of N Dp branes supports a (supersymmetric) nonabelian Yang-Mills theory coupled to the adjoint transverse scalars.

Until now we only considered D branes sitting at the same position in space. But what happens if **strings** are **suspended between different stacks of branes**? If oriented strings stretch between two stacks of Dp branes of the same dimensionality (we will consider directions with mixed Neumann-Dirichlet boundary conditions shortly), e.g. between N_1 and N_2 Dp branes, then the oriented strings running from the former to the latter will have their starting point transforming in the antifundamental of $U(N_1)$ and the endpoint in the antifundamental of $U(N_2)$, and vice-versa for the string running in the other direction. Oriented strings between two stacks of D branes will thus give rise to bifundamental fields, and orientation actually matters here: Each orientation will give rise to its own field or, in the case of supersymmetric strings, to its own multiplet. However, separating the two stacks of branes a distance L away from each other will give the fields a mass, because the string acquires a minimal energy given by its tension times the distance between the branes. The whole spectrum thus will be shifted by that amount. Let for example the first stack of branes sit at x_0^I, and the second stack at x_π^I. The constraints on the mode expansion are now generalised from (2.55) to

$$x_0^I = \frac{x_L^I + x_R^I}{2}\,, \quad p_L^I = -p_R^I = \frac{x_\pi^I - x_0^I}{\pi\alpha'}\,, \quad \tilde{\alpha}_n^I = -\alpha_n^I\,. \tag{2.58}$$

The mode expansion thus becomes

$$X^I(\tau,\sigma) = x_0^I + \frac{x_\pi^I - x_0^I}{\pi}\,\sigma - \frac{\sqrt{\alpha'}}{2}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\alpha_n^I\sin(n\sigma)\,, \tag{2.59}$$

clearly describing a string running from the point x_0^I to the point x_π^I. For a brane configuration with only NN and DD directions (i.e. for a Dq-Dq system) the mass formula reads

$$M^2 = -p^i p_i = m^2 + \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_{n,i} + \alpha_{-n}^I\alpha_{n,I} - 1\right)\,, \quad (\Delta x)^2 = (x_\pi^I - x_0^I)^2 = 4\pi^2\alpha'^2 m^2\,. \tag{2.60}$$

Thus the spectrum now consists of a vacuum which for fixed m^2 and small enough α' always is tachyonic, and gauge bosons and transverse scalars of mass m^2. This mass generating mechanism for gauge bosons is just the **string theoretic version of the Higgs effect**: As noted above, the transverse scalar fields encode informations about the fluctuations of the open strings in the directions transverse to the D branes. Furthermore, these string states are corresponding to small, linearised excitations of the corresponding fields, as we saw from the example of the graviton. Eq. (2.60) tells us that the linearised excitations of the transverse scalars X^I as well as of some gauge bosons A_i have mass m. However, if we would set out to calculate the vacuum expectation value of the transverse scalars, we would find that they are nonzero and proportional to the mass m. We would thus conclude that the separation of the D branes, may it be by a choice of a nontrivial point in the moduli space of the corresponding gauge theory or by an external potential which for example could be generated by a nontrivial gravitational background, is encoded in the vacuum expectation values of the transverse scalar fields, which in turn describe the minimum length a string stretching between these two stacks of D branes can have. We would furthermore identify the massive gauge bosons with the broken generators of the gauge group in the case when both stacks of D branes were coincident. Thus in the example considered here, we would conclude that the gauge group of the theory on the $N_1 + N_2$ Dp branes was broken according to the pattern

$$U(N_1 + N_2) \to U(N_1) \times U(N_2)\,. \tag{2.61}$$

We still have to consider one last choice of boundary conditions to complete our classification of possible open strings: There can exist strings which stretch between D branes of different dimensionality, e.g. *Dp* and *Dq* branes. In this case there are directions (so-called **ND directions**) in which the embedding functions satisfy Dirichlet boundary conditions on the one end and Neumann boundary conditions on the other end. For definiteness, let $q < p$, and let the Dq branes sit at a point $x_q^a\,,\ a = q+1, \ldots, p$ in the space made up by the ND directions. Dirichlet boundary conditions at $\sigma = 0$ and Neumann boundary conditions at $\sigma = \pi$ (i.e. for a string running from the Dq branes to the Dp branes) then imply

$$x_q^a = \frac{x_L^a + X_R^a}{2}\,, \quad p_L^a = p_R^a = 0\,, \quad \tilde{\alpha}_n^a = -\alpha_n^a\,, \quad n \in \quad + \frac{1}{2}\,. \tag{2.62}$$

The constraint for strings running the other way, i.e. from the Dp branes to the Dq branes, are

$$x_q^a = \frac{x_L^a + X_R^a}{2}\,, \quad p_L^a = p_R^a = 0\,, \quad \tilde{\alpha}_n^a = \alpha_n^a\,, \quad n \in \quad + \frac{1}{2}\,. \tag{2.63}$$

The new feature here is that the ND strings need to have half-integer modes in order to fulfill the ND boundary conditions. Intuitively this can be understood as the fact that on a Neumann boundary a standing wave does not experience any phase shift, but on a Dirichlet boundary the phase of the wave will be shifted by π. Thus for a NN or DD wave the phase shift is trivial (zero or $2\pi \sim 0$), but for a ND wave the phase shift will be π. Note furthermore that the modulus x_q^a is the center-of-mass position of the string in the "relatively" transverse directions $a = q + 1, \ldots, p$. The ND directions yield an additional oscillator contribution to the mass squared, but because of the half-integer moding it

	Boson	Fermion
Periodic	$-\frac{1}{24}$	$+\frac{1}{24}$
Antiperiodic	$\frac{1}{48}$	$-\frac{1}{48}$

Table 2.3: Zero point energy contributions from bosons and fermions of different periodicity.

contributes differently to the zero point energy. The zero point energy contributions (see e.g. [143,166]) of one periodic (i.e. integer moded) or antiperiodic (i.e. half-integer moded) boson and fermion are summarised in table 2.3. For the bosonic Dq-Dp system with $p > q$ there are $\nu = p - q$ ND directions, and $26 - \nu$ NN and DD directions. Except of two directions whose contributions are cancelled by the diffeomorphism ghosts, the NN and DD direction contribute all as periodic bosons, since they are integer moded. The ND directions contribute as antiperiodic bosons. The total zero point energy for the bosonic Dq-Dp system in 26 dimensions thus is

$$(24-\nu)\left(-\frac{1}{24}\right)+\frac{\nu}{48}=\frac{3\nu}{48}-1\,.$$

The full mass formula for the p-q strings in the bosonic Dp-Dq system at spatial separation $|\Delta x| = 2\pi\alpha' m$ thus is

$$M^2=-p^ip_i=m^2+\frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha_{n,i}+\alpha^I_{-n}\alpha_{n,I}+\sum_{n=\frac{1}{2}}^{\infty}\alpha^a_{-n}\alpha_{n,a}+\frac{3\nu}{48}-1\right)\,. \tag{2.64}$$

Except for the case of no ND directions ($\nu = 0$), this mass formula does not give rise to any states other than the vacuum with finite mass in the limit $\alpha' \to 0$. This is the point where the analogy of the bosonic with the supersymmetric string ends. As is explained in the next section 2.2.2. The ND strings of bosonic Dq-Dp intersections thus do not yield any additional fields. Interactions between the Dq-Dq strings and the Dp-Dp strings could be induced by integrating out massive ND modes, but these interactions vanish in the $\alpha' \to 0$ limit.

The full massless spectrum of the open bosonic Dq-Dp string is summarised in table 2.2. I omitted the different vacua of the three kinds of strings, as they will be absent for the supersymmetric brane configurations in the superstring. To summarise, the conclusion to draw from this analysis of open bosonic parallel D brane intersections is

1. Each stack of branes carries its own gauge theory, coupled to the corresponding transverse scalars which transform in the adjoint representation of the respective gauge groups. These transverse scalars encode the position of the branes in transverse space and, if acquiring a vacuum expectation value, lead to spontaneous breaking of the corresponding gauge symmetries.

2. The totally transverse adjoint scalars on both brane stacks are not redundant. Simply speaking, they can be redefined into fields encoding the relative position of the

stacks (the difference of X_q^I and X_p^I) and fields encoding the overall "center-of-mass position" of the brane intersection (the sum of the fields). The latter can, of course, always be set to zero by a space-time Poincaré shift.

3. The "relatively" transverse adjoint scalars encode the position of the Dq's inside the Dp's.

4. The spectrum of open bosonic q-p strings does not contain fields surviving the low energy limit $\alpha' \to 0$. This is due to the noncancellation of the zero point energies of NN, DD and ND bosonic strings. In section 2.3 we will see that the superstring in ten dimensions is much better behaved in this respect. It will turn out that for the number of ND directions $p - q$ being a multiple of four, supersymmetry is preserved and the ND directions give rise to additional bifundamental fields which play the roles of supersymmetric quarks in the field theory on the smaller Dq brane.

2.2.2 Ten-Dimensional Type II Superstrings and Supergravities

As the major tools of the AdS/CFT correspondence to describe strongly coupled gauge theories is the use of ten-dimensional type II superstring theories and their low energy effective actions, ten-dimensional $\mathcal{N} = 2$ supergravities, this section will serve as a short summary of the most important facts needed in the course of this thesis. As the reader will understand, the presentation of this material has, due to to the increasing complexity of the material, to be more condensed and less self-contained than in the previous section.

The starting point for the description of ten-dimensional superstrings is a supersymmetrised version of the action (2.16), with the superpartners of the X^μ's being world-sheet fermions $\psi^\mu(\sigma, \tau)$ and with a two-dimensional supergravity multiplet consisting of the worldsheet metric and a corresponding gravitino. I refrain of giving here the rather complicated fully supersymmetrised version of (2.16), just mentioning that it is possible to chose a generalisation of the above-used conformal gauge, the so-called superconformal gauge in which the bosons and fermions each just show up with a kinetic term,

$$S_{II} = \frac{T_F}{2} \int \mathrm{d}\tau \mathrm{d}\sigma \eta_{\mu\nu} \left(\partial_a X^\mu \partial^a X^\nu - i\bar{\psi}^\mu \gamma^a \partial_a \psi^\nu \right) . \tag{2.65}$$

Here γ^a are two-dimensional Dirac "gamma" matrices. In order to have on-shell supersymmetry, the numbers of bosonic and fermionic on-shell degrees of freedom should match. In two space-time dimensions it is in principle possible to impose both a Majorana and a Weyl condition on spinors, reducing the number of real degrees of freedom from four (Dirac spinor) to two (Majorana or Weyl) and then to one (Majorana-Weyl). On-shell the Majorana-Weyl spinor thus only contains one half degree of freedom, which corresponds to either a left- or a right mover, as can be seen by writing the Dirac equation in the light cone frame (using e.g. the gamma matrices (A.7) from appendix A): Naming the left- and right-handed spinor components $\psi^\mu = (\psi^\mu_+, \psi^\mu_-)^T$, one obtains

$$0 = \gamma^a \partial_a \psi^\mu = \gamma^a \partial_a \begin{pmatrix} \psi^\mu_+ \\ \psi^\mu_- \end{pmatrix} = \sqrt{2} \begin{pmatrix} \partial_+ \psi^\mu_- \\ \partial_- \psi^\mu_+ \end{pmatrix} .$$

The left-handed fermions are thus also left movers, $\psi^\mu_+(\tau+\sigma)$, and the right-handed ones are right movers, $\psi^\mu_-(\tau-\sigma)$. For closed strings, where the left- and right moving parts of X^μ are independent of each other, we thus must add two Majorana-Weyl fermions of opposite chirality[9] for each space-time boson. As noted before in eq. (2.24), the requirement of quantum conformal symmetry restricts the number of space-time dimensions to be ten. The theory, if formulated for closed strings, has thus $\mathcal{N} = (1,1)$ world sheet superconformal symmetry in two dimensions and, after a suitable truncation called (after Gliozzi, Scherk and Olive [167]) the GSO Projection, $\mathcal{N} = 2$ target space supersymmetry in ten dimensions.

Working out the spectrum of the theory (2.65), one now finds an ambiguity even for the closed string: The left- and right moving parts of the fermions can separately fulfill either periodic or antiperiodic boundary conditions around the string, giving rise to different mode expansions and thus different sectors. The periodic boundary conditions are called **Ramond** (R) boundary conditions, and the antiperiodic ones are named after **Neveu and Schwarz** (NS) boundary conditions. As the left- and right movers are independent, the closed superstring described by (2.65) has four sectors, summarised in table 2.4. It turns out that in the Ramond sector the fermions have integer-labeled mode expansions, while in the Neveu-Schwarz sector, the modes are half-integers, giving rise to different algebras of creation and annihilation operators, namely

$$\text{NS:} \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0}\,, \quad r,s \in \quad + \frac{1}{2}\,, \tag{2.66}$$

$$\text{R:} \qquad \{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}\,, \quad m,n \in \quad . \tag{2.67}$$

As we are dealing with worldsheet fermions, we need to impose anticommutation relations rather than commutation relations. As the moding in the NS sector is half integer, $r = s$ never yields a nonvanishing right hand side in eq. (2.66), making a unique separation of the operators b^μ_r into pairs of creators and annihilators and thus a definition of a unique nondegenerate ground state possible, which in turn can be identified with a bosonic spin zero state. The Fock space build upon the NS ground state will thus contain space-time bosons. For the R sector however the d^μ_0 operators fulfill a Clifford algebra,

$$\Gamma^\mu = \sqrt{2} d^\mu_0 \quad \Rightarrow \quad \{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\,, \tag{2.68}$$

Furthermore it turns out that the d^μ_0 commute with the fermionic part of the M^2 operator, and thus the vacuum of the R sector and all excited states have to furnish irreducible

[9] One could also try to set one of the fermionic modes (left or right movers) to zero, but then one needs to also set the corresponding scalar mode to zero to preserve supersymmetry. For the closed string this leads to a trivial theory, as the number of left- and right moving excitations have to equal by worldsheet conformal symmetry. Adding more chiral fermions is equivalent to increasing the space-time dimensionality and thus does not solve the problem.

R \ L	NS	R
NS	Space-Time Bosons	Space-Time Fermions
R	Space-Time Fermions	Space-Time Bosons

Table 2.4: Sectors of the closed type II superstring.

representation of the ten-dimensional Clifford algebra (2.68). We thus see that the states in the R sector are space-time fermions. The closed type II superstring Fock space is a tensor product of the two sectors for both the left- and right movers. As the tensor product of two bosonic (tensorial) representations of the Lorentz group $SO(1,9)$ as well as of two fermionic (spinorial) representations both only include tensor representations of $SO(1,9)$, while the tensor product of a tensorial and a spinorial representation decomposes into spinorial ones, the pattern of table 2.4 arises.

The Fock space constructed from eqs. (2.66) and (2.67) however suffers from tachyonic ground states very much in the same way as the bosonic string sector did (which is of course also present in the full type II superstring Fock space). There is however a way to consistently truncate the Fock space, found by Gliozzi, Scherk and Olive [167], by imposing the already mentioned **GSO Projection**: One introduces a fermion number operator F counting the number of creators b/d acting on the respective ground state. To project out the bosonic but tachyonic vacuum in the NS sector but keep the (without proof) massless vector $b^{\mu}_{-\frac{1}{2}}|0\rangle_{NS}$, one has to project out states with even number of fermionic creators, which is achieved by the projection operator

$$P^{GSO}_{NS} = \frac{1}{2}\left(1-(-1)^{F_{NS}}\right) , \quad F_{NS} = \sum_{r>0} b^{\mu}_{-r} b_{r,\mu} . \tag{2.69}$$

The vacuum, which has no b^{μ} acting on it, is projected out, rendering the NS sector tachyon free. The massless vector $b^{\mu}_{-\frac{1}{2}}|0,k\rangle_{NS}$ on the other hand is kept.

In the R sector one uses the same projector, but there are now two choices in the definition of the $(-1)^F$ operator,

$$P^{GSO}_{R,\pm} = \frac{1}{2}\left(1-(-1)^{F^{\pm}_{R}}\right) , \quad (-1)^{F^{\pm}_{R}} = \pm\Gamma^{11} \sum_{m\geq 1} d^{\mu}_{-m} d_{m,\mu} , \tag{2.70}$$

with Γ^{11} being the analogue of the chirality projection matrix Γ^5 in four-dimensional physics. $P^{GSO}_{R,\pm}$ will thus project out respectively the left-handed (upper sign) or right-handed (lower sign) chirality part of the (spinorial) Ramond vacuum[10] $|a\rangle_R$. It turns out that in the Ramond sector the vacuum already is the massless state (i.e. the normal ordering constant in the mass formula vanishes for the Ramond sector). Since in ten dimensions

[10]The index a here is a ten-dimensional spinor index. We define a ten-dimensional left-handed spinor to have positive eigenvalue under Γ^{11}, $\Gamma^{11}\Psi_L = +\Psi_L$, and a right-handed chirality spinor to have negative Γ^{11}-eigenvalue.

the superpartner of a massless vector field (i.e. transforming in the 8_v representation of the massless little group $SO(8)$ of $SO(1,9)$) is a Majorana-Weyl fermion, it should not surprise the reader that the chirality components of the Ramond vacuum exactly correspond to Majorana-Weyl spinors of different chirality. Only in this way, the number of bosonic and fermionic on-shell degrees of freedom match: The massless vector has eight propagating degrees of freedom in ten dimensions, as has the Majorana-Weyl spinor after imposing its equation of motion. We thus find that the massless modes of the supersymmetric string arrange themselves into supersymmetry multiplets in ten dimensions.[11]

The attentive reader will have noticed that the meaning of the freedom of choice to project out either the left- or the right-handed chirality part of the Ramond vacuum still needs to be explained. The interpretation is as follows: Either choice defines a separate theory, as it defines a physically distinct ground state on which the superstring Fock space can be built. As for the closed superstring, we will have two copies of each the Ramond and Neveu-Schwarz sector, one for the left movers and one for the right movers, we can either choose the vacuum in both Ramond sectors to have the same chirality, or to have opposite chiralities. The former choice leads to a string theory called the **type IIB superstring**, while the latter leads to **type IIA string theory**. In this light it will not surprise the reader that the IIB theory is chiral (i.e. all the fermions have the same chirality, while the IIA theory is nonchiral. In particular, the two space-time supercharges of type IIB have the same chirality (and thus the theory strictly speaking is $\mathcal{N} = (2,0)$ space-time supersymmetric), while the supercharges of IIA have opposite chirality (and thus it is $\mathcal{N} = (1,1)$ supersymmetric).

With these pieces of knowledge, the **massless spectrum of both type II string theories** is now easily found. Remember that the massless mode in each NS sector is a vector corresponding to the states $b^\mu_{-\frac{1}{2}}|0_R\rangle_{NS}$ and $\tilde{b}^\mu_{-\frac{1}{2}}|0_L\rangle_{NS}$. Both fields transform in the eight-dimensional vector representation $\mathbf{8_v}$ of the $SO(8)$ little group of $SO(1,9)$. As both the left- and right-moving NS sector are tensored together in the full string Fock space, one can reduce this tensorial representation into irreducible representations of $SO(8)$, yielding the pattern given in table 2.5. The **35** is the representation of a traceless symmetric two-tensor of $SO(8)$, which is identified with the **ten-dimensional graviton** similar as in the bosonic string case before. The **28** yields an antisymmetric two-tensor also called the **Kalb-Ramond field** [164] (as in the bosonic string), and the **1** is the **dilaton**. The NS-R and R-NS sectors each yield the spin-$\frac{3}{2}$ **gravitino** fields $\Psi_\mu\,,\Psi'_\mu$ (of either the same or opposite chirality), and the superpartners of the dilaton, the **dilatini** $\lambda\,,\lambda'$. The corresponding representations can be obtained by reducing the tensor product of the NS sector $\mathbf{8_v}$ and

[11] It might surprise at first that the massless spectrum of the type II superstring comes from the fermionic ψ fields in (2.65) alone. It seems the bosonic sector is not necessary. This would however be a wrong conclusion: On the one hand side, the algebra of generators of the superconformal worldsheet symmetries closes onto the Virasoro constraints, and on the other hand the bosonic fields X^μ contribute to the zero point energies (the normal ordering constants in the mass formulas) in a crucial way. That there are no bosonic modes showing up in the massless spectrum is due to the GSO projection - it projects out all states in which the NS sector is in its vacuum state, no matter how highly excited the bosonic sector is.

the eight-dimensional[12] Majorana-Weyl representation $\mathbf{8_s}$ of $SO(8)$ (coming from the R sector) as given in the table. The Ramond-Ramond sector is a little more interesting, as it is different for type IIA and IIB: Again the massless states in the RR sector lie in the decomposition of the corresponding tensor products of spinor representations, but now the choices are different in IIA and IIB theory. For IIA the product to decompose is that of a left- handed Majorana-Weyl (which is the $\mathbf{8_s}$ in our convention) representation with a right-handed one (which is the conjugate spinor representation to $\mathbf{8_s}$, called 8_c). The decomposition then yields

$$8_s \otimes 8_c = [0] \oplus [2] \oplus [4]\,, \tag{2.71}$$

where $[n]$ denotes the irreducible antisymmetric n-tensor representation of $SO(8)$. The RR sector thus gives rise to fields which are completely antisymmetric ten-dimensional space-time tensors of rank n, or short **form fields**. Eq. (2.72) shows that the **IIA theory includes zero-, two- and four-forms**. Note that these fields are field strengths of corresponding p-form potentials with one rank less, much in the same spirit as the gauge field is the 1-form potential of the two-form field strength, $F = \mathrm{d}A$. In **type IIB** both Ramond sectors have the same chirality, lets say $\mathbf{8_s}$, and thus the decomposition

$$8_s \otimes 8_s = [1] \oplus [3] \oplus [5]_+ \tag{2.72}$$

yields **one-, three and five-form fields**. The five-form field strength is special, as it has to be self-dual with respect to the ten-dimensional Hodge star operator. As will be explained later in section 2.2.3, the existence of these form fields, which again admit a generalisation of the gauge principle for form fields, is closely related to the existence of stable D Branes in the type II theories, as the world-volume of these extended objects naturally couples to the gauge potentials. The D Branes of type II theory will thus turn out to be the objects which carry Ramond-Ramond charges. In conclusion, since there are two independent copies of Ramond and Neveu-Schwarz sectors for the left- and right movers, the massless modes arrange for the closed type II superstring in **ten-dimensional $\mathcal{N} = 2$ supergravity multiplets**, as summarised in table 2.5. For the Ramond-Ramond sector, I listed the corresponding p-form potentials. Note that they are not all independent, but are connected by the Poincaré duality

$$F^{(p+1)} = \mathrm{d}C^{(p)} = *F^{(9-p)} = *\mathrm{d}C^{(8-p)}\,. \tag{2.73}$$

Thus the forms $C^{(p)} \simeq C^{(8-p)}$ are dual to each other. Note that for a p-form electrodynamics, this duality corresponds to exchanging the analogue of Maxwells equation with the analogue of the Bianchi identity, i.e. to electric-magnetic duality.[13]

[12] Note that eight is the dimensionality of the Majorana-Weyl representation after imposing the Dirac equation, i.e. the number of independent dynamically propagating degrees of freedom. As the Dirac equation is one of the supersymmetric Virasoro constraints, the 8_s denotes the representation after imposing the dynamics of the theory. Before doing this, the representation itself is sixteen-dimensional.

[13] Note that, since no "minus-one forms" exist, the notation C_{-1} in table 2.5 is just abstract for the existence of a ten-form field strength F_{10} and a corresponding nine-form potential C_9.

Sector	Statistics	$SO(8)$ Representation	Massless Fields
NS–NS	Bosonic	$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{35} \oplus \mathbf{28} \oplus \mathbf{1}$	$g_{\mu\nu}$, $B_{\mu\nu}$, Φ
NS–R	Fermionic	$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_s \oplus \mathbf{56}_s$	Ψ_μ , λ
R–NS	Fermionic	$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_s \oplus \mathbf{56}_s$	Ψ'_μ , λ'
IIA R–R	Bosonic	$\mathbf{8}_s \otimes \mathbf{8}_c = [0] \oplus [2] \oplus [4]$	p-form potentials $C_{-1} \simeq C_9\,,\ C_1 \simeq C_7\,,\ C_3 \simeq C_5$
IIB R–R		$\mathbf{8}_s \otimes \mathbf{8}_s = [1] \oplus [3] \oplus [5]_+$	p-form potentials $C_0 \simeq C_8\,,\ C_2 \simeq C_6\,,\ C_4 \simeq C_4$

Table 2.5: Spectrum of the closed type II superstring (slightly modified table of [148]).

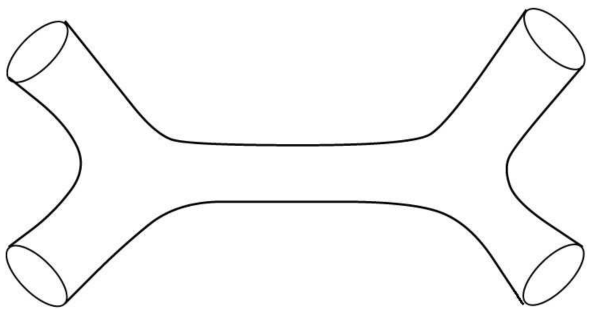


Figure 2.4: A scattering process involving closed strings.

Low Energy Effective Actions: Generalities and Type IIB Supergravity

Since the massless excitations of the type II superstring already come in ten-dimensional $\mathcal{N} = 2$ supergravity multiplets, it is a short way to conjecture that the low energy effective action governing the dynamics of these fields are the two ten-dimensional $\mathcal{N} = 2$ supergravity theories built around these multiplets, which there are type IIA [168–170] and IIB supergravity [171–174].[14] Since the type IIA theory is a nonchiral one, its two Majorana-Weyl supersymmetry generators of opposite chirality can be combined into one Majorana spinor, which can then be interpreted as the supersymmetry generator of the unique supergravity theory in eleven dimensions [176]. It is thus not surprising that type IIA supergravity has first been constructed by dimensional reduction of eleven-dimensional supergravity, while the type IIB theory had to be constructed from scratch by finding supersymmetric equations of motion for the fields in the supergravity multiplet.

In this thesis mostly the IIB theory coupled to low energy effective actions of D branes will play a role, in particular in chapters 3 and 4. But before introducing the features of this theory, it needs to be explained what is meant by a "low energy effective action" of string theory. Generically, strings are interacting objects, which can scatter on each

[14]Type IIA supergravity is the common name of the ten-dimensional supergravity theory which does not include the C_9 potential. Including it leads to a massive version of type IIA supergravity [175], which will, however, not play a role in this thesis.

other as depicted for a closed string in figure 2.4. If the coupling strength between the strings is small, one can define an asymptotic in and out Fock space just by taking two copies of the (super)string Fock space just constructed. The ingoing and outgoing strings now are each in a definite state, lets say a graviton state with certain polarisation, and one can calculate analytically the scattering amplitudes for each string diagram using conformal field theory methods. In the kinematic regions of interest one can now try to find scaling limits $\alpha' \to 0$, in order to find the scattering amplitudes of the corresponding point particles (e.g. graviton-graviton scattering). Physically speaking it is required that for the known fields, e.g. the graviton or the gauge bosons, that this **point particle limit** should reproduce the known interactions of these fields. But for new, intrinsically "stringy" particles, one could in this way at least in principle obtain the scattering amplitudes, and then, in a last step, try to write down an effective field theory action which reproduces these amplitudes. This would be the straigthforward way of obtaining a low energy effective action.

The direct calculation of scattering amplitudes and the finding of an appropriate effective action is, however, a quite complicated procedure. Fortunately, there is a simpler way to obtain the string equations of motion at least for the gravitational sector (the metric, Kalb-Ramond field and dilaton), which can be described by a nonlinear sigma model similar to (2.16),

$$S_{\sigma-\mathrm{Model}} = \frac{T_F}{2} \int \mathrm{d}^2\sigma \sqrt{-h} \left[\left(h^{ab} G_{\mu\nu}(X) + i\varepsilon^{ab} B_{\mu\nu}(X) \right) \partial_a X^\mu \partial_b X^\nu + \alpha' \mathcal{R} \Phi(X) \,. \right] \quad (2.74)$$

Eq. (2.74) can be seen as an attempt to describe bosonic strings moving in a nontrivial background with metric $G_{\mu\nu}(X)$, Kalb-Ramond field $B_{\mu\nu}(X)$ and dilaton vacuum expectation value $\Phi(X)$. Note that $\mathcal{R}$ is the Ricci scalar of the worldsheet, depending on the worldsheet metric h_{ab}. A priori it seems that this action does not know anything about the equations of motion governing these background fields. It just describes an interacting two-dimensional quantum field theory for the fields X^μ, but the interactions are governed by the background fields. One can, however, expand the fields $X^\mu = x_0^\mu + Y^\mu$ around some classical solution to the equations of motion x_0^μ, and, using the well-known background field method, quantise the small fluctuations of the string around this background [177–180]. Since the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ govern the interactions, one can think of them as being the "coupling constants" of the sigma model, and calculate how loop corrections renormalise them. The "beta functionals" for the background fields can then be related to the trace of the world-volume energy momentum tensor via

$$T_a{}^a = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} \partial_a X^\mu \partial^a X^\nu - \frac{i}{2\alpha'} \beta^B_{\mu\nu} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi \mathcal{R} \,. \quad (2.75)$$

If we now want to retain conformal invariance on the quantum level also for strings in these nontrivial backgrounds, $T_a{}^a$ needs to vanish, which means that the beta functions should vanish. They can be calculated on the linearised level (i.e. for a slightly nonflat background for example) and then generalised to the full nonlinear level, yielding to lowest order in α' for the action (2.74)

$$0 = \beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} + \mathcal{O}(\alpha'^2)\,, \tag{2.76}$$

$$0 = \beta^B_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu} + \alpha'\nabla^\omega\Phi H_{\omega\mu\nu} + \mathcal{O}(\alpha'^2)\,, \tag{2.77}$$

$$0 = \beta^\Phi = \frac{D - D_{crit}}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\omega\Phi\nabla^\omega\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\,. \tag{2.78}$$

The first equation resembles Einstein's equation with source terms from the antisymmetric tensor field and the dilaton, while the second equation is a Maxwell-like equation for the field strength

$$H = \mathrm{d}B \tag{2.79}$$

of the Kalb-Ramond field. The last equation is an equation of motion for a scalar field coupled to $H \wedge *H$. Generalisations of Einstein's equations coupled to other fields thus arise from the requirement of the nonlinear sigma model to be conformally invariant.

The term $(D - D_{crit})/6$ in eq. (2.78) is (up to factors) the conformal anomaly of the bosonic string (with $D_{crit} = 26$), see eq. (2.22). The reason for this is that the bosonic nonlinear sigma model (2.74) is only the generalisation of the bosonic part of (2.65). There however exists a generalisation of (2.74) [162,181] for the sector of the IIB string consisting of G, B, Φ plus its superpartners but without the Ramond-Ramond form fields[15], which indeed reproduces the equations (2.76)-(2.78) (with $D_{crit} = 10$ of course) plus the equations of motion for the superpartners.

In attempting to write down a general covariant action for type IIB supergravity, another obstacle occurs: The five-form field strength

$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B \wedge F_5\,, \quad F_5 = \mathrm{d}C_4 \tag{2.80}$$

is Hodge self-dual in ten dimensions,

$$*\,\tilde{F}_5 = \tilde{F}_5\,, \tag{2.81}$$

a constraint whose integration into an action principle for gauge theories is unknown.[16] One can however write down an action which yields the equations of motion and by hand impose the self-duality constraint (2.81). This yields the bosonic part[17] of the **type IIB**

[15]The coupling of the superstring to the form fields cannot be described in terms of a two-dimensional sigma model, since this would require the existence of an antisymmetric tensor analogous to ε^{ab} with more than two indices. They can be incorporated in different descriptions of type II superstrings, namely the Green-Schwarz formalism [182,183] or the pure spinor formulation [184].

[16]An attempt to formulate an action principle incorporating this constraint can be found in [185,186].

[17]I refrain from writing down the fermionic completion of this action, as it will be rather complicated and is not needed in this thesis.

supergravity action in the string frame

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-G}\left[e^{-2\Phi}\left(R+4\nabla_\mu\Phi\nabla^\mu\Phi-\frac{1}{2}|H_3|^2\right)\right. \\ &\quad \left. -\frac{1}{2}\left(|F_1|^2+|\tilde{F}_3|^2+\frac{1}{2}|\tilde{F}_5|^2\right)\right]-\frac{1}{4\kappa^2}\int\limits_{\mathcal{M}_{10}} C_4\wedge H_3\wedge F_3\,. \end{aligned} \tag{2.82}$$

Here

$$2\kappa^2 = 16\pi G_{10} = (2\pi)^7 g_s^2 \ell_s^8 \tag{2.83}$$

is the ten-dimensional gravitational coupling, $\tilde{F}_5$ is defined in eq. (2.80), and the conventions are as in [141], namely

$$\tilde{F}_3 = F_3 - C_0\wedge H_3\,, \tag{2.84}$$

$$F_p = \mathrm{d}C_{p-1} \quad p=1,3,4\,, \quad |F_p|^2=\frac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\ldots\mu_p}\,. \tag{2.85}$$

In conclusion, the action (2.82) is the low energy effective action of type IIB closed string theory.

2.2.3 The Interesting Double Life of Dirichlet Branes

In the last section it was already mentioned that the existence of the p-form fields C_p hints towards the existence of charges which are extended objects. On the other hand we already know such extended objects, namely the D branes from section 2.2.1. And indeed, since a Dp brane has a $p+1$-dimensional worldvolume, one can pull back (see eq. (2.14) for the definition of the pull-back of forms onto submanifolds) a $p+1$-form onto its worldvolume and integrate over it,

$$\int\limits_{\mathcal{M}_{p+1}\,.} P[C_{p+1}] \tag{2.86}$$

A Dp brane thus naturally couples to the Ramond-Ramond $p+1$-form. Since the IIA superstring only admits odd-rank form fields, it only includes Dp branes with p even, while the IIB string only includes Dp branes with p odd.[18] In fact, Joseph Polchinski showed in his seminal paper [189] that the D branes in type II superstring theory are necessary objects for consistency of the interaction between the open and closed superstring sector, and furthermore that only the Dp branes which are able to couple to the respective RR form fields can give rise to consistent open-closed string interactions. The possible Dp branes in both theories are listed in table 2.6. Note that the D(-1) brane is special, since it

[18]The other branes actually can also be described in string theory, but are unstable unless put on an orbifold [187]. The unstable configurations in flat ten-dimensional Minkowski space are believed to undergo tachyon condensation [188].

	Possible Branes
IIA	D0, D2, D4, D6, D8
IIB	D(-1), D1, D3, D5, D7, D9

Table 2.6: Possible Dp branes in type IIA/IIB string theory.

also imposes Dirichlet boundary conditions in the time direction X^0, i.e. it is a pointlike object in space and time, and strings are fixed to that point in all directions. It is the string theoretic analogue of an instanton, and thus often called a **"D-instanton"**. Similar to the relativistic point particle (2.15), the D0 brane of IIA has a one-dimensional worldvolume, and thus is called a **"D-particle"**. The D1 brane has the same dimensionality as a fundamental string, and thus is often called a **"D-string"**. The D8 brane is a **"domain wall"** in ten-dimensional space-time, dividing the nine-dimensional spatial volume into two halfves. The D9 brane in IIB theory on the other hand is a space-time filling brane, but there is no corresponding Ramond-Ramond ten-form potential which it could couple to. Nevertheless it is required to exist as the T-dual partner of the D8 brane in IIA.[19]

The D3 brane of type IIB is also special in the following way: All the other Dp branes couple electrically, i.e. through the coupling (2.86), to C_{p+1}. However, because of the duality (2.73) (which exchanges the Maxwell equation $\mathrm{d} * F = 0$ with the Bianchi identity $\mathrm{d}F = 0$), this coupling can also be thought of as a magnetic coupling to C_{7-p}. Thus, the exchange of electric and magnetic field strengths $F_{p+2} \leftrightarrow F_{8-p}$ will be an electric-magnetic duality symmetry of the theory if, at the same time, the corresponding electric and magnetic charges, i.e. the D branes are interchanged via

$$Dp \;\leftrightarrow\; D(6-p)\,. \tag{2.87}$$

In this sense the D3 brane is self-dual under this electric-magnetic duality: It is both the electric and magnetic source of the five-form field strength (which in turn has to be Hodge self-dual).

Dirichlet Branes as Dynamical Objects

Since we saw in section 2.2.1 that Dirichlet branes are by definition hypersurfaces onto which the movement of open string endpoints are restricted to, it would seem that the branes would be just rigidly defined hypersurfaces in space-time. An absolutely rigid object can however not exist in a theory respecting the laws of special relativity, since perturbations can propagate at most with the speed of light. In the case of D branes the perturbations themselves are open string excitations, and thus D branes should be

[19] It is, however, unprotected by any BPS bound (since the corresponding C_9 form field is nondynamical) and thus unstable in type II string theory. It is however an existing stable brane (and in fact necessary for consistency) in the type I string theory obtained from type IIB by gauging the orientifold projection $\Omega : \sigma \mapsto -\sigma$, i.e. in a theory of unoriented superstrings in ten dimensions [146].

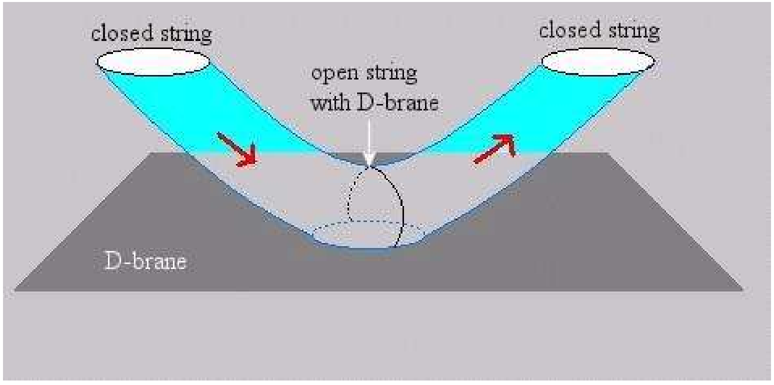


Figure 2.5: Scattering of a closed string on a Dirichlet brane.

dynamical objects whose dynamics is at low energies governed by the massless excitations of the open (super)strings ending on them.

Furthermore, since D branes are charged under closed string fields (e.g. under Ramond-Ramond fields, as argued above), open and closed strings have to be able to interact, which they indeed do: In figure 2.5 the scattering process of a closed string on a D brane is depicted. In this process the closed string opens up through a open-closed string coupling, the open string propagates for some time on the D brane and then closes again through the same process. D branes should thus, if they are dynamical, couple to gravity and thus curve space-time by themselves and also be able to react to the closed string fields in the background space-time, e.g. to the curvature of the background space-time. The low energy effective action describing the dynamics of open strings should include all these couplings between the massless open string excitations (gauge fields and transverse scalars) and the massless closed string excitations. And indeed, the **low energy effective action** at least **for a single Dp brane**[20] is known exactly (see [194] for a discussion of the evidence of its uniqueness), with its bosonic part reading

$$S_{Dp} = S_{DBI} + S_{WZ} \tag{2.88}$$

$$S_{DBI} = -\mu_p \int\limits_{\mathcal{M}_{p+1}} \mathrm{d}^{p+1}\xi\, e^{-\Phi} \sqrt{-\det_{ab}\left(P[G+B]_{ab} + 2\pi\alpha' F_{ab}\right)}\,, \quad a,b = 0,\ldots,p \tag{2.89}$$

$$S_{WZ} = \pm\frac{\mu_p}{g_s} \sum_p \int\limits_{\mathcal{M}_{p+1}} P[C_p] \wedge e^{P[B]+2\pi\alpha' F}\,. \tag{2.90}$$

[20] For stacks of coincident Dp branes the low energy effective action is not known exactly yet, basically because the correct prescription for taking the trace over the Chan-Paton indices is not known to all orders in F_{ab}. Tseytlin gave a prescription for the trace [190] several years ago, which is however known to be incorrect at order F^5 [191]. At this order Tseytlins symmetric trace prescription excludes terms which are present in the open string tree level amplitudes. See also [192, 193] for a discussion of the status of the nonabelian DBI action at that time.

The constant μ_p,

$$\mu_p = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}}}\,, \tag{2.91}$$

renders the action (2.88) dimensionless. I included in eq. (2.88) the dilaton contribution $e^{-\Phi}$ for generality. If the dilaton is constant, the string coupling constant $g_s = e^{\Phi_{const.}}$ can be included in the pre-factor to yield the **Dp brane tension**

$$T_p = \frac{\mu_p}{g_s} = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s}\,. \tag{2.92}$$

If the dilaton is not constant but the space-time under consideration has a well-defined asymptotic region with constant dilaton (for example an otherwise asymptotically flat region), one might shift the dilaton $\Phi \to \Phi + \Phi_\infty$ to extract its asymptotic value and define the string coupling as $g_s^\infty = e^{\Phi_\infty}$.

The action (2.88) consists of two parts, the Dirac-Born-Infeld (DBI) part and the Wess-Zumino (WZ) part. As mentioned earlier, the DBI part is a generalisation of the "minimal area action" which was already implemented in the Nambu-Goto action for strings (2.12). It is also valid for Dp branes in bosonic string theory. It includes the coupling to the NS-NS closed string fields G, B, Φ. The embedding fields $X^\mu(\xi)$, which describe how the D brane is embedded into the target space, are hidden here in the pull-back $P[...]$ (see eq. (2.14)). F_{ab} is the field strength of the U(1) gauge field A_a living on the brane.

For small field strengths $2\pi\alpha' F_{ab} \ll 1$ and constant dilaton, the DBI action (2.89) reduces to a Maxwell type action in $p+1$ dimensions (plus a volume term), as can be seen most easily by assuming the Kalb-Ramond field $B_{\mu\nu}$ to vanish:

$$\begin{aligned}
S_{DBI} &= -\frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}} g_s}\int \mathrm{d}^{p+1}\xi \sqrt{-\det\left(g_{ab} + 2\pi\alpha' F_{ab}\right)} \qquad (g_{ab} = P[G]_{ab}) \\
&= -\frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}} g_s}\int \mathrm{d}^{p+1}\xi \sqrt{-\det(g)\det\left(\delta^a_b + 2\pi\alpha' g^{ac}F_{cb}\right)} \\
&= -\frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}} g_s}\int \mathrm{d}^{p+1}\xi \sqrt{-\det(g)}\sqrt{1 + 2\pi^2\alpha'^2 F_{ab}F^{ab} + \mathcal{O}(F^4)} \\
&= -\frac{1}{(2\pi)^p(\alpha')^{\frac{p+1}{2}} g_s}\int \mathrm{d}^{p+1}\xi \sqrt{-\det(g)} - \frac{1}{2^p\pi^{p-2}(\alpha')^{\frac{p-3}{2}} g_s}\int \mathrm{d}^{p+1}\xi\sqrt{-g}F_{ab}F^{ab} + \mathcal{O}(F^4)\,.
\end{aligned}$$

In the third step, the expansion formula (C.2) for the determinant was used. Comparing to the canonically normalised Maxwell term in curved space,

$$-\frac{1}{4g^2_{YM,p}}\sqrt{-g}F_{ab}F^{ab}\,,$$

we find a relation between the parameters of our string theory and the Yang-Mills coupling in $p+1$ dimensions,

$$g^2_{YM,p} = (2\pi)^{p-2}(\alpha')^{\frac{p-3}{2}} g_s\,. \tag{2.93}$$

As expected, this has the correct dimensions of a Yang-Mills coupling in $p+1$ dimensions, namely $(length)^{p-3}$. In particular, it is dimensionless for a D3 brane. If the Kalb-Ramond field does not vanish, odd powers of the field strength will be retained. Such a situation will be encountered in chapters 3 and 4.

The Wess-Zumino part describes the coupling to the Ramond-Ramond sector, as well as to the Kalb-Ramond field. The notation is such that the sum over p runs over the allowed values for type IIA/IIB string theory. The exponentials are meant to be exponentials with respect to the wedge product,

$$e^{P[B]+2\pi\alpha' F} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(P[B]+2\pi\alpha' F) \wedge \cdots \wedge (P[B]+2\pi\alpha' F)}_{\text{n times}} . \tag{2.94}$$

The integral is then to be taken over all terms in this expansion which are a $p+1$-form. In this way the expansion collapses to a finite number of terms.

Obviously, S_{WZ} includes the already-mentioned coupling (2.86), but it also shows that Dp branes can couple to RR form fields with a rank lower than $p+1$ if either the Kalb-Ramond field or the brane-intrinsic gauge field acquires a nontrivial vacuum expectation value. In this case, when the Dp brane carries lower-dimensional brane charge as well, one speaks of **"branes dissolved within branes"**. We will encounter such a situation in chapter 4. Note that there is a sign choice for the Wess-Zumino action which distinguishes a **Dp brane** (one with C_{p+1}-charge $\propto \mu_p$, corresponding to the upper sign in (2.90)) from an **anti Dp brane** (with negative charge, corresponding to the lower sign in (2.90)). The fact that the Dp branes of table 2.6 are **BPS objects** of type IIA/IIB superstring theory, i.e. that their mass is related to their charge, could also be guessed already from eq. (2.88): The tension of the Dp brane (2.92), which is the equivalent of "mass" for extended objects, is for a brane related to its charge via (2.92). The proportionality factor, the string coupling constant g_s, is understood to be due to the fact that the coupling of the Dp brane as an open string object to the closed string Ramond-Ramond sector is mediated by an open-closed string interaction (see fig. 2.5). Note however that the relation (2.92) only includes the absolute value of the Dp brane charge, but is insensitive of its sign. Thus it is not possible to guess from eq. (2.92) alone that the corresponding antibranes are not BPS objects. It is also not possible from this observation to conclude that Dp branes preserve one half of the supersymmetries of the background, i.e. that they are in fact $\frac{1}{2}$BPS objects of type II string theory. In conclusion,

> Dirichlet (anti-)branes are dynamical objects which react through the shape of their embedding in the ten-dimensional background and through the value of the brane gauge field. Both the embeddings and the gauge field are governed by the equations of motion derived from eq. (2.88). For gauge field strengths small compared to the string length ($2\pi\alpha' F \ll 1$), the low energy effective action (2.88) reproduces the well-known U(1) gauge theories. Furthermore, Dirichlet branes are the charge carriers for the Ramond-Ramond form fields.

Branes with positive charge are $\frac{1}{2}$BPS objects of type II string theory, i.e. break one half of the supersymmetries. Antibranes, which have negative charge, are nonsupersymmetric objects, i.e. break all of the supersymmetries.

Dirichlet Branes as Solitonic Solutions to Type II Supergravity

Since Dirichlet branes couple through open-closed string interactions to the type II supergravity theories, one might expect that they curve space and time in a certain way. Furthermore, since they are $\frac{1}{2}$BPS objects in the corresponding string theories, it is natural to search for half-supersymmetric solutions of type II supergravity theories which have the same charges as the Dp branes and obey a similar relation between (ADM) mass density and charge as (2.92). Horowitz and Strominger [195] did so for Dp branes in flat space. In fact they set out not to search for the extremal $\frac{1}{2}$BPS solutions directly but, in order to be able to trust the supergravity approximation to string theory, they searched for nonextremal solutions, i.e. black holes charged under the p-form fields which are asymptotically flat and obey an isometry $\ ^{1,9} \times SO(9-p)$ ($p+1$-dimensional translations along the black brane, and rotations in the transverse space, i.e. the black brane is a point in transverse space and surrounded by a spherically symmetric horizon),[21] but whose masses are larger than the corresponding charges. The extremal limit of their solutions (in which always only one form field is excited, i.e. these branes do not carry lower-dimensional brane charge) can be written in the form

$$\begin{aligned} \mathrm{d}s^2 &= H_p^{-\frac{1}{2}} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + H_p^{\frac{1}{2}} \delta_{ij} \mathrm{d}y^i \mathrm{d}y^j \,, \quad \mu,\nu = 0,\dots,p\,, \quad i,j = p+1,\dots,9 && (2.95) \\ e^{2\Phi} &= g_s^2 H_p^{\frac{3-p}{2}} \,, \quad C_{p+1} = \left(1 - \frac{1}{H_p}\right) \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^p \,, && (2.96) \\ H_p &= 1 + \left(\frac{R_p}{r}\right)^{7-p} \,, \quad r^2 = \sum_{i=p+1}^{9} y^{i2} \,, \quad R_p = (4\pi)^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right) g_s N (\alpha')^{\frac{7-p}{2}} && (2.97) \end{aligned}$$

Here N denotes the total $p+1$-form charge (i.e. Dp brane charge, i.e. the number of Dp branes which sourced this solution) measured at infinity via the formula[22]

$$\int_{\mathrm{S}^{8-p}} *F_{p+2} = N \,, \tag{2.98}$$

with S^{8-p} being the 8-p-sphere surrounding the pointlike brane charge in the $9-p$-dimensional transverse space. Note that the supergravity equations of motion force the function $H_p(y^i)$ to be a harmonic function with respect to the Laplacian in transverse

[21] A flat extremal Dp brane breaks the Poincaré group $\ ^{1,9} \times SO(1,9)$ down to $\ ^{1,p} \times SO(1,p) \times SO(9-p)$, but a nonextremal black brane does not admit the $SO(1,p)$ part.

[22] In the case of the D3 brane, in order to fulfill the self-duality constraint, one needs to define the five-form field strength as $F_5 = \frac{1}{2}\left(\mathrm{d}C_4 + *\mathrm{d}C_4\right)$.

$9-p$-dimensional Euclidean space, and that the integration constant in H_p is chosen such that the metric is asymptotically flat far away from the brane, i.e. at $r \to \infty$.

Since the solution (2.95) to the type II supergravity equations preserves the same amount of supersymmetries (namely 32 real supercharges, corresponding to $\mathcal{N}=1$ in ten dimensions) as a stack of N coinciding Dp branes in type II string theory, has the same charge (2.98) and since its isometry group is identical with the remainder of ten-dimensional Poincaré symmetry in the presence of a Dp brane (namely $\quad ^{1,p} \times SO(1,p) \times SO(9-p)$), one might inquire about the exact nature of the relationship between Dp branes in flat space which carry the supersymmetrised version of the gauge theory (2.88) and the solitonic solution (2.95). In view of the fact that various string dualities, in particular T and S dualities, require the existence of Dp branes both in the regimes of weak and strong string coupling, the most natural interpretation (put forward by Polchinski in [189]) is that Dp branes are intrinsic objects of type II string theories if one includes open superstrings as well.They can be thought of as nonperturbative (in the string coupling g_s) states of these string theories. The fact that the Dp brane tension (2.92) scales with the inverse of the string coupling supports this idea. The solitonic solutions (2.95) then should be identified with the Dp branes in the full type II string theory, which also explains the quantisation of the charge in (2.98). All this is similar to the nature of field theoretic solitons (as for example in the Sine-Gordon model [196]), but one question remains: If Dp branes are nonperturbative excitations, and if they are the same as the gravitational solitons (2.95), then what are the fundamental particle-like excitations they are made of? Are they open strings, as suggested by the fact that they carry gauge theories, or are the fundamental pieces they are made of closed strings, as suggested by the solitonic picture? Note that the solitonic solutions (2.95) are solutions of type II supergravity, i.e. the low energy effective action of closed strings only. A priori, this gravity theory thus does not have any coupling to a gauge sector or to some extended object. Nevertheless it includes these nonperturbative states which are charged under the Ramond-Ramond gauge fields. If they are the equivalent realisations of the type II string theory D branes in the low energy effective actions, and since D branes in type II string theory require the existence of an open string sector, then where are the open strings and their field theory in this purely gravitational picture?[23]

It turns out that this situation receives a partial resolution by decoupling the physics of open and closed strings in the presence of D branes, which leads to a (conjectured) new duality between the gauge theory living on the D brane (the open string sector) and the gravitational (i.e. closed string) excitations propagating in the near-horizon limiting geometry of the gravitational background (2.95). This new duality, which is known under the name of the **Anti-de Sitter/Conformal Field Theory Correspondence**, is the

[23]Note that this question is similar to the question which kind of energy-momentum tensor sources the vacuum Schwarzschild (or Kerr, or Reissner-Nordstroem) solution. Of course, these are vacuum solutions of general relativity, but since Einstein's equations do break down at the singularity, one might ask what kind of energy-momentum tensor, which would have to be of distributional nature at the singularity and zero away from it, could generate this singularity and the surrounding vacuum solution (see e.g. [197–199]). It could be interesting to investigate similar aspects also for the black branes (2.95).

main tool of this thesis and will be introduced in the next section.

2.3 The Anti-de Sitter/Conformal Field Theory Correspondence

In this section the basics of the Anti-de Sitter/Conformal Field Theory Correspondence are introduced. The material presented here can be found in more detail in the review articles [79, 80, 200, 201] or the cited original works.

2.3.1 Maldacena's Conjecture

A partial resolution[24] of the confusing situation described in the last section was proposed by Juan Maldacena in the seminal paper [41]. It relies on an argument suggesting a full equivalence on the dynamical level between the low energy effective field theory of a stack of N_c D3 branes in type IIB string theory, **four-dimensional** $\mathbf{U(N_c)}$ $\mathcal{N} = \mathbf{4}$ **supersymmetric Yang-Mills theory**,[25] and type IIB string theory formulated on the near-horizon geometry of the solitonic solution (2.95) for N_c D3 branes. This near-horizon geometry is the space-time $\mathbf{AdS_5 \times S^5}$ **with** $\mathbf{N_c}$ **units of five-form flux through the** $\mathbf{S^5}$. "Full equivalence on the dynamical level" means roughly that a one-to-one map between the full spectrum of gauge invariant operators of the supersymmetric Yang-Mills theory and the spectrum of type IIB closed superstrings in this particular geometry has to be given.[26] But before describing the field theory and this very particular space-time in more detail, I want to sketch the very general argument leading to this conjecture, which is remarkable since the supersymmetric Yang-Mills theory, which a priori is a low energy effective action of the open string sector of type IIB in the presence of D3 branes, is conjectured to include the full information of a string theory, namely the theory of closed strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The argument given by Maldacena proceeds in two steps, corresponding to the two different perspectives one can take, namely the open string perspective of D branes in a otherwise flat spatial background, and the closed string (or supergravity) perspective of the solitonic solution. First, one considers the Newton constant dependence of the full low energy effective action of type IIB open and closed string theory in the presence of the D3 branes. The ten-dimensional Newton constant in this case is related to the string theory parameters

[24] Since the AdS/CFT correspondence is still an unproven conjecture, it can only be seen as a partial explanation of how the apparently different descriptions of D branes fit together.

[25] This rather long name will often be abbreviated by $\mathcal{N} = 4$ Super-Yang-Mills theory or even just $\mathcal{N} = 4$ SYM theory in this thesis.

[26] There exists a whole army of physicists trying to exploit integrable structures both in the string theory as well as in the $\mathcal{N} = 4$ Super-Yang-Mills theory to establish this map in all details, a very interesting topic which will however not be touched in this thesis. For a good overview see e.g. [74–78].

by the relation (2.83). The general low energy effective action of open strings on D3 branes coupled to closed strings in a flat space consists of three parts: First, there is the type IIB supergravity action (2.82) describing the closed strings in flat space and their interactions. Secondly, the supersymmetrised version of the D3 brane action for N_c coincident branes, which is similar to the action (2.88), describes the interactions of open superstrings on the D3 branes. For slowly varying fields the D3 brane action yields the $\mathcal{N} = 4$ SYM theory. Thirdly, there are additional interactions between the open and closed string sectors which differ from the usual minimal coupling of gravity to matter fields. The type IIB supergravity action (2.82) scales like κ^{-2}, i.e. with a negative power of the gravitational coupling κ. The D brane effective action also scales with a negative power, namely $\kappa^{-1} = (g_s \ell_s^4)^{-1}$. Since the minimal interaction between the D brane theory and type IIB supergravity is already given by the dependence of eq. (2.88) on the closed string background fields, one might think that no other interaction terms are present. This is, however, not true if the string scale ℓ_s is finite since in this case one has to integrate out all the higher string excitations, which will lead to additional interactions suppressed by positive powers of the gravitational coupling (corresponding to negative powers of the Planck mass). Only in the strict limit low energy limit $\ell_s \to 0$, in which $\kappa \to 0$ (if g_s and N_c are fixed), these interactions will vanish. We can thus summarise the κ-dependence of the full low energy effective action as

$$S_{LEEA} = \underbrace{S_{\text{IIB SUGRA}}}_{\frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}R+\ldots} + \underbrace{S_{D3}}_{\sim\frac{1}{\kappa}} + \underbrace{S_{int}}_{\sim\kappa^{\#>0}} . \tag{2.99}$$

Since gravity in dimensions greater than three is an a priori nonrenormalisable interaction ($[\kappa^2] = (mass)^{-8}$), processes involving the gravitational coupling must have an energy dependence $\omega^4\kappa$ and thus vanish in the infrared limit $\omega \to 0$. Thus gravity is said to be "infrared free", which can be reexpressed in a running of the gravitational coupling to zero in the far infrared. We should thus take an infrared limit, focusing exclusively on processes with very low energies compared to the string scale which in this setting sets the strength of the gravitational interaction (assuming g_s to be fixed). This limit is obviously equivalent to the point particle limit $\ell_s \to 0$ with all other energy scales in the problem fixed. Note that in this limit also the backreaction of the otherwise infinitely heavy brane, since it is $\sim \kappa$, is negligible. In the low energy limit considered here the space-time away from the D3 branes thus stays flat. In this way the open-closed string interactions summarised in S_{int} and even interactions between the IIB fields themselves (which are $\sim \kappa$) vanish, and thus the full low energy effective theory (2.99) should reduce to two independent sectors, namely **free type IIB supergravity on flat space and four-dimensional $\mathcal{N} = 4$ $U(N_c)$ supersymmetric Yang-Mills theory**. That the actual low energy effective action on the D3 branes in this limit is the $\mathcal{N} = 4$ SYM theory and no higher derivative corrections is due to the fact that the low energy limit can be taken as the limit $\alpha' \to 0$ with all other dimensionless parameters kept fixed. In this limit also the higher derivative corrections in the effective field theory on the D3 branes vanish.

On the other hand, we can consider type IIB string theory (or supergravity as its low energy effective action) on the solitonic background (2.95). Since these gravitational solitons are asymptotically flat, it is natural to consider the observer with respect to this asymptotically flat region. The typical experiment of such an observer (if such an "observer" exists at all in the absence of matter and gauge theories) is the classical scattering of gravitational waves and other type IIB supergravity excitations on the D3 brane stack. These processes were analysed by Igor Klebanov in [202] for massless scalars (and also in later works for other supergravity fields), where the typical behaviour of the cross section was found to be

$$\sigma_{D3} \sim \omega^3 R^8 \,, \quad R_3^4 = 4\pi g_s N_c \alpha'^2 \,. \tag{2.100}$$

Thus in the low energy limit $\omega^2\alpha' \to 0$ with g_s and N_c fixed, the D3 branes do not interact with the supergravity excitations thrown onto them from the asymptotic region. This means that in the infrared, the asymptotic region decouples from the "bulk" of the solitonic solution (2.95), and by the same "infrared freedom" argument as above, type IIB supergravity in the asymptotically flat region becomes free. These free gravitational excitations are however not the only excitations seen in the asymptotic region. Since the D3 brane soliton

$$\begin{aligned}
\mathrm{d}s^2 &= H_3^{-\frac{1}{2}}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + H_3^{\frac{1}{2}}\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j \,, \quad \mu,\nu = 0,\ldots,3 \,, \quad i,j = 4,\ldots,9 && (2.101)\\
e^{2\Phi} &= g_s^2 = \mathbf{const.} \,, \quad C_4 = \left(1 - \frac{1}{H_3}\right)\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3 \,, && (2.102)\\
H_3 &= 1 + \left(\frac{R}{r}\right)^4 \,, \quad r^2 = \sum_{i=4}^{9} y^{i2} \,, \quad R^4 = 4\pi g_s N_c \alpha'^2 && (2.103)
\end{aligned}$$

admits a Killing horizon for the timelike Killing vector $\frac{\partial}{\partial x^0}$ at $r = 0$, excitations starting very close from the horizon and traveling to the asymptotic observer will be strongly redshifted. Infinitesimally close to the horizon all excitations, even the massive string modes, will be redshifted to very small energies when traveling towards the flat region. The asymptotic observer will thus be able to see the full type IIB closed string spectrum coming from the near-horizon region, redshifted to very small energies! He (or she of course) thus notices in low energy experiments ($\omega \ll \ell_s^{-1}$) two sectors of excitations: The **free type IIB supergravity excitations** surrounding him **and** the redshifted **type IIB closed string spectrum from the near horizon region**. The near horizon region of the geometry (2.101) naively is obtained by taking the limit $r \to 0$. We however need to take into account the redshift of modes, which gets stronger and stronger near the horizon. Thus the correct near horizon limit is the limit $\alpha' \to 0$ while keeping the energy scale $u = r/\alpha'$ (and g_s as well as N_c) fixed. Introducing spherical coordinates $\mathrm{d}y^{i2} = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2$, this limit yields[27]

$$\begin{aligned}
\frac{\mathrm{d}s^2}{\alpha'} &= \frac{u^2}{\ell^2}\mathrm{d}x^{\mu 2} + \frac{\ell^2}{u^2}\mathrm{d}u^2 + \ell^2\mathrm{d}\Omega_5^2 \,, && (2.104)\\
\alpha'^2 C_4 &= \frac{\ell^4}{u^4}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, && (2.105)
\end{aligned}$$

[27] Also in this equation the interpretation is that of a proper length and a four-form being measured in string units.

	Bulk/Asymptotic Region	Brane/Near Horizon Region
Flat Space D3 Branes	Free IIB Supergravity	$\mathcal{N} = 4$ SYM
Solitonic D3 Branes	Free IIB Supergravity	IIB String on $\mathrm{AdS}_5 \times \mathrm{S}^5$

Table 2.7: Maldacena's conjecture: Identifying the different sectors.

which is the direct product of five-dimensional Anti-de Sitter space (AdS_5) of radius $\ell = (4\pi g_s N_c)^{\frac{1}{4}}$ with a five-dimensional sphere of the same radius. Solutions of this type, $AdS_p \times \mathrm{S}^q$, to type II supergravity and eleven-dimensional supergravity have been found long ago in [203]. Anti-de Sitter space in any dimension is the unique maximally symmetric space of constant negative curvature. $d+1$-dimensional Anti-de Sitter space of radius ℓ and with Minkowskian signature can be defined as a hyperboloid in a flat space with one additional dimension and with signature diag$(-, +, \dots, +, -)$ via the equation

$$- x^{0^2} + \vec{x}^2 - x^{(d+1)^2} = -\ell^2 \,, \tag{2.106}$$

where $\vec{x}$ is a d-dimensional vector. In particular it is clear from this embedding that AdS_{d+1} admits the $SO(2,d)$ isometry group of the embedding space. AdS_{d+1} with Euclidean signature is defined in a similar way via

$$x^{0^2} + \vec{x}^2 - x^{(d+1)^2} = -\ell^2 \,, \tag{2.107}$$

admitting an $SO(1, d+1)$ isometry symmetry group. For a nice and very clear account of the geometrical properties of Anti-de Sitter spaces (e.g. the geodesics in Minkowskian AdS_{d+1}) see e.g. [204]. Note also that the C_4 potential induces N_c units of five-form flux on the S^5, which are calculated again by the formula (2.98).

If we now assume that the D3 branes in flat space-time are the solitonic three branes of type IIB supergravity, and we identify the two sectors of free type IIB supergravity excitations (the yellow ones in table 2.7) with each other, then it is natural to identify the other two sectors (the orange ones in table 2.7) as well. This leads to the conjecture of the

Anti-de Sitter/Conformal Field Theory (AdS/CFT) Correspondence

$U(N_c)$ $\mathcal{N} = 4$ four-dimensional supersymmetric Yang-Mills theory should be fully equivalent on the level of dynamics to ten-dimensional type IIB supergravity on the maximally symmetric space $\mathrm{AdS}_5 \times \mathrm{S}^5$ with N_c units of F_5 flux on the five-sphere.

	Field	$\mathcal{N}=1$ Multiplet	$\mathcal{N}=2$ Multiplet
Gauge Boson	A_μ	Vector V	Vector
Gaugino (Weyl)	λ_4		
-Scalar	$\phi_3 = X^8 + iX^9$	Chiral Φ_3	
Weyl-Fermion	λ_3		
-Scalar	$\phi_1 = X^4 + iX^5$	Chiral Φ_1	Hyper
Weyl-Fermion	λ_1		
-Scalar	$\phi_2 = X^6 + iX^7$	Chiral Φ_2	
Weyl-Fermion	λ_2		

Table 2.8: Field content and symmetry representations of the $\mathcal{N}=4$ vector multiplet in four dimensions.

2.3.2 Four-Dimensional Maximally Supersymmetric Yang-Mills Theory

Before proceeding with the discussion of this remarkable conjecture, it is necessary to discuss some aspects of the field theory involved, $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions with gauge group $U(N_c)$. As usual in the construction of supersymmetric multiplets[28], the higher the wanted amount of supersymmetry is, the larger the spanned range of helicities in the multiplet is. If one wants to construct supersymmetric theories of gauge and matter fields but without including gravity, one should not exceed the helicity of the gauge fields. This sets a restriction for the maximal amount of supersymmetry a supersymmetric gauge theory without gravity can have, which in four dimensions turns out to be $\mathcal{N}=4$ (16 real Poincaré supercharges).

The $\mathcal{N}=4$ vector multiplet in four dimensions has the following field content, summarised in table 2.8: A vector field A_μ in the adjoint of the $U(N_c)$ gauge group, four left-handed Weyl fermions[29] $\lambda_{\alpha A}$ transforming in the **4** representation of the $SU(4)_\mathcal{R}$ symmetry and six real scalars X^i transforming in the **6** representation[30] of $SU(4)_\mathcal{R}$. Since the gauge field transforms in the adjoint representation of the gauge group and all other fields in the multiplet are related to it by supersymmetry transformations, all fields do have to transform in the adjoint representation. The $\mathcal{N}=4$ vector multiplet furthermore decomposes into an $\mathcal{N}=2$ vector multiplet containing the gauge fields, one gaugino, one complex scalar which we choose to be the one corresponding to the transverse 8-9-plane and his superpartner, and an $\mathcal{N}=2$ hypermultiplet containing the complex scalars corresponding to the movements

[28] For reviews about supersymmetric gauge and gravity theories, see e.g. the book by Wess and Bagger [205].

[29] $\alpha = 1, 2$ is the spinor index and $A = 1, \ldots, 4$ is the $SU(4)_\mathcal{R}$ index.

[30] The six real scalars can be combined to three complex scalars plus their complex conjugates as in table 2.8. The **6** of $SU(4)$ is a rank two antisymmetric tensor representation, which is equivalent to the real vector representation of the geometrically realised $SO(6)$ acting on the X^i's directly under the isomorphism $SO(6) \simeq SU(4)$.

of D3 branes in the space transverse to the 8-9-plane (plus their superpartners). Later in section 2.3.6, when we will introduce additional quark-like degrees of freedom transforming in the fundamental representation of the gauge group, these decompositions into smaller supersymmetry multiplets will be useful. The choice of this multiplet thus fixes the field content of the maximally supersymmetric Yang-Mills theory completely, leaving only the value of some continuous parameters like the Yang-Mills coupling and the theta angle or the choice of vacuum expectation values (i.e. the choice of the vacuum) unfixed.

It turns out [206] that the requirement of maximal supersymmetry, and in particular the implied large $\mathcal{R}$-symmetry, are extremely restrictive and constrain the action of $\mathcal{N} = 4$ SYM theory to be

$$\begin{aligned} S_{\mathcal{N}=4} &= \int \mathrm{d}^4x \,\mathrm{tr} \left\{ -\frac{1}{2g_{YM}^2} F_{\mu\nu}^2 + \frac{\Theta_I}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - i \sum_{A=1}^{4} \bar{\lambda}^A \bar{\sigma}^\mu D_\mu \lambda_A - \sum_{i=1}^{6} D_\mu X^i D^\mu X^i \right. \\ &\qquad \left. + \frac{g_{YM}^2}{2} \sum_{i,j} [X^i, X^j]^2 + g_{YM} \sum_{A,B,i} \left(C_i^{AB} \lambda_A [X^i, \lambda_B] + h.c. \right) \right\} \\ &= \int \mathrm{d}^4x \, \Im \left[\tau \int \mathrm{d}^4\theta \,\mathrm{tr} \left(\bar{\Phi}_I e^V \Phi_I e^{-V} \right) + \right. \\ &\qquad \left. + \tau \left(\int \mathrm{d}^2\theta \,\mathrm{tr} \left(W_\alpha W^\alpha + \epsilon_{IJK} \Phi_I \Phi_J \Phi_K \right) + \mathrm{h.c.} \right) \right] , \qquad (2.108) \\ \text{with} \qquad & \tau = \frac{\Theta_I}{2\pi} + i \frac{4\pi}{g_{YM}^2} , \quad W_\alpha = -\frac{1}{4} (\bar{D})^2 \left(e^{-V} D_\alpha e^V \right) . \end{aligned}$$

Here τ is the usual complex coupling known in four-dimensional gauge theories, and W_α is the chiral spinor superfield constructed out of the gauge superfield V. The constants C_i^{AB} are the Dirac matrices for $SO(6)_\mathcal{R}$ spinors, since the $\mathbf{4}$ of $SU(4)$ is the Weyl representation of spinors in six-dimensional Euclidean space. The second form of the action (2.108) is written in $\mathcal{N} = 1$ superspace, with θ_α being the Grassmanian superspace coordinates. Apart from the usual kinetic terms, one recognises in the component field version of the action a quartic (commutator squared) potential as well as Yukawa couplings. In the superfield version all these interactions are hidden in the $\epsilon_{IJK}\Phi_I\Phi_J\Phi_K$ superpotential, and turn up again after integrating over superspace and solving for the D and F term constraints.

The fact that eq. (2.108) is the unique theory with this amount of supersymmetries in four dimensions actually makes its identification with the low energy effective action of N_c D3 branes straightforward: Recalling that the D3 branes break the ten-dimensional $\mathcal{N} = (2, 0)$ supersymmetry of type IIB supergravity (corresponding to 32 real Poincaré supercharges) to $\mathcal{N} = (1, 0)$ (corresponding to 16 real supercharges) and that one Majorana-Weyl spinor in ten dimensions reduces to four Weyl spinors in four dimensions, one readily sees that the low energy effective action of a stack of D3 branes must have four-dimensional $\mathcal{N} = 4$ supersymmetry. Invoking the uniqueness argument of $\mathcal{N} = 4$ SYM, the searched-for low energy effective action must be the theory defined by (2.108).

Before proceeding with the discussion of the AdS/CFT correspondence, let us first collect several important facts about $\mathcal{N} = 4$ SYM which will turn out important later on:

- The theory is classically scale invariant, since there are no dimensionful couplings involved. Poincaré invariance and scale invariance thus combine to the larger conformal symmetry group $SO(2,4) \simeq SU(2,2)$. Note that this is just the isometry group of AdS_5, as shown in the previous section. Furthermore, the $\mathcal{N} = 4$ Poincaré supersymmetry and this conformal invariance combine to the classical **superconformal symmetry** $SU(2,2|4)$.

- The theory admits an $SU(4)_{\mathcal{R}}$ symmetry,[31] which rotates the fields X^i into each other. On the gravity side, there also exists an $SO(6)$ symmetry, namely the isometry group of the S^5. It is thus natural to identify the two groups with each other. We thus find that the global symmetries of $\mathcal{N} = 4$ SYM match with the isometries of the $\text{AdS}_5 \times \text{S}^5$ background. This matching of symmetries is one of the most powerful checks of the correspondence, and will play an important role in the later chapters.

- The scalar fields X^i are the transverse scalars in the open string spectrum[32] and describe the transverse position of the brane in flat space. Since the transverse space is ^{6}, it is clear that the $SO(6)$ rotating in this space is nothing but the $SU(4)_{\mathcal{R}}$ symmetry just described. Due to the gauge group $U(N_c)$ being compact, it's Cartan-Killing form is positive semi-definite and thus each term in the sum

$$-\sum_{i,j} \text{tr}\left[X^i, X^j\right]^2 \tag{2.109}$$

  is either positive or zero. The $\mathcal{N} = 4$ supersymmetric ground states of the zero have vanishing potential and thus are described by the requirement of vanishing commutators,

$$\left[X^i, X^j\right] = 0\,, \quad i,j = 1,\ldots,6\,. \tag{2.110}$$

  There are two classes of solutions to (2.110), namely

  1. The **superconformal phase**, in which the vacuum expectation values $\langle X^i \rangle = 0$ vanish for all $i = 1,\ldots,6$. In this phase the gauge group $U(N_c)$ is unbroken, as is the superconformal symmetry $SU(2,2|4)$. This phase corresponds to all the N_c D3 branes sitting on top of each other.

[31] The $\mathcal{N} = 4$ supersymmetry algebra a priori also has a $U(1)_{\mathcal{R}}$ automorphism, multiplying the supercharges with a phase. The action (2.108) however does not allow a consistent assignment of $U(1)_{\mathcal{R}}$ charges such that the action is invariant, and thus the $U(1)_{\mathcal{R}}$ part is not present in the $\mathcal{N} = 4$ SYM theory.

[32] For simplicity I omit the distinction between a scalar field in field theory and a transverse scalar in string theory. They are connected by factors of $2\pi\alpha'$, such that the scalar field in four dimensions has mass dimension one and the transverse scalar has length dimension one, i.e. $X^i_{\text{string}} = 2\pi\alpha' X^i_{\text{field}}$.

2. The **Coulomb phases**, for which $\langle X^i \rangle \neq 0$ for at least one i. The unbroken part of the gauge group depends on the detailed structure of the vacuum expectation value, but generically the gauge group is broken completely, $U(N_c) \to U(1)^{\mathbf{Rank}(U(N_c))=N_c}$. Superconformal symmetry is also broken by the scale set by the vacuum expectation value. These vacua correspond to separated D3 branes. In the generic case, every D3 brane is separated from all the others in the transverse directions (see e.g. figure 4.1(a)).

- Upon perturbative quantisation, $\mathcal{N} = 4$ SYM theory exhibits no ultraviolet divergences in the correlation functions of its canonical fields. Since instanton corrections only lead to finite contributions [80], the theory is believed to be ultraviolet finite. Perturbative corrections are forbidden by $\mathcal{N} = 4$ supersymmetry: For the complex gauge coupling to run perturbatively, one needs to generate logarithmic corrections of the form $\log\left(\frac{\mu}{\langle \mathcal{O} \rangle}\right)$, where the vacuum expectation value $\langle \mathcal{O} \rangle$ of some $\mathcal{N} = 4$ operator sets the energy scale for the renormalisation group running. Since the operator spectrum of $\mathcal{N} = 4$ SYM theory does not include nontrivial operators preserving the $SU(4)_{\mathcal{R}}$ symmetry [80], giving a vacuum expectation value to any operator will necessarily break $\mathcal{N} = 4$ supersymmetry.[33] Thus there is no need to renormalise the theory perturbatively (up to possible finite shifts of the gauge coupling) and the beta function for the Yang-Mills coupling vanishes exactly and to all orders in perturbation theory. The theory is thus exactly scale invariant even on the quantum level (unless scale invariance is broken by some other mechanism, as for example in the Coulomb phases), and in particular the superconformal invariance group $SU(2, 2|4)$ is an exact quantum symmetry. The property of UV finiteness does match the behaviour of superstring theories, which are (at least on flat ten-dimensional Minkowski space) also UV finite.

- As an aside, $\mathcal{N} = 4$ was conjectured by Montonen, Olive and others [207–209] to be invariant under the S-duality group $SL(2,\ )$, which acts on the complex coupling τ as
$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \quad ad - bc = 1\,, \quad a, b, c, d \in\ . \tag{2.111}$$
In particular for $\Theta_I = 0$ this includes a strong-weak coupling duality, sending $g_{YM} \mapsto g_{YM}^{-1}$. This field theoretic S-duality is believed to be the equivalent of the type IIB S-duality symmetry [210, 211].

- The theta angle term has a dual supergravity interpretation as the charge of smeared D-Instantons (D(-1) branes) carried by the field C_0. It arises from the Wess-Zumino coupling
$$\int\limits_{\mathcal{M}_4} P[C_0] \mathrm{tr}\,(F \wedge F) = P[C_0] \int\limits_{\mathcal{M}_4} \mathrm{tr}\,(F \wedge F)\ . \tag{2.112}$$

[33] The author thanks Xi Yin for this argument.

The D-Instantons are smeared here over the world-volume of the D3 branes in the sense that C_0 is a constant in the D3 brane worldvolume directions. The above observation adds more evidence to the interpretation of D(-1) branes as field theory instantons in the low energy effective theory. In chapter 4 it will be explained why this interpretation actually holds.

2.3.3 AdS/CFT: Forte, Mezzo & Piano

The AdS/CFT conjecture is of course a bold statement, since it states that an "ordinary" field theory can describe the full string theoretic dynamics of the type IIB superstring on the space (2.104), and one would like to test this conjecture by explicit calculations, if not even prove it. In its **strongest form** the AdS/CFT correspondence should hold for arbitrary numbers of D3 branes N_c,[34] and arbitrary string coupling g_s. However, since even for the maximally symmetric space $\mathrm{AdS}_5 \times \mathrm{S}^5$ our knowledge of how to quantise type IIB superstrings on this curved background is rather restricted[35], it is useful to take additional limits in which the type IIB superstring theory simplifies.

The first limit which one can take is the limit of many D3 branes, $N_c \to \infty$, while holding $\lambda = g_{YM}^2 N_c$ fixed. This limit was first proposed by Gerard 't Hooft [56] as a limit in which QCD might simplify and admit an expansion in terms of the new coupling parameter λ, which is named after the inventor as the "**'t Hooft coupling**". The limit is called "**'t Hooft limit**". From the field-theoretic point of view, in this limit all Feynman diagrams which are nonplanar (in the sense that they cannot be drawn on a sphere because some of their internal virtual particle lines cross each other without an interaction vertex) are suppressed by inverse powers of N_c, and gauge theories actually do simplify very much. In the context of the AdS/CFT correspondence, since $g_{YM}^2 = 2\pi g_s$ for D3 branes, the string coupling has to vanish in this limit as $g_s \propto N_c^{-1}$. In the strict $N_c \to \infty$ limit the type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ is thus reduced to a theory of free (so-called semiclassical) strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$, since string splitting/merging interactions and thus string loop corrections are suppressed. As long as α' is finite, the theory will however still know about "stringy effects", i.e. effects coming from the finite extension of strings. This large-N_c limit is thus not a point particle limit yet.

The point particle limit, which makes the supergravity approximation to type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ valid by suppressing stringy (i.e. α') corrections is now equivalent to taking the limit of large 't Hooft coupling. This can be seen from the relation

$$\left(\frac{R}{\ell_s}\right)^4 = 2\lambda \gg 1\,, \tag{2.113}$$

[34]For a discussion of the $U(1)$ factor of the gauge group, which is of course particularly important in the case $N_c = 1$, see chapter 4.

[35]For recent progress in this direction see e.g. [74–78].

which states that the typical curvature radius of both AdS_5 and S^5 is much larger than one in string units. This is exactly what is needed for a point particle limit: The background has no features which are so strongly curved that stringy effects like higher string modes or winding modes can be excited. Thus if the 't Hooft coupling is very large but fixed, $\lambda \gg 1$, the supergravity approximation to the semiclassical string theory is valid and stringy effects are small. There is no need to take a strict $\lambda \to \infty$ limit.

In conclusion, the **weakest version of the AdS/CFT correspondence** reached by taking both the large-N_c and the strong coupling limit, can be stated as the

> Full dynamical equivalence of classical type IIB supergravity on the ten-dimensional background $AdS_5 \times S^5$ and the four-dimensional maximally supersymmetric Yang-Mills theory, which is $U(N_c)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, at large N_c and in the strong 't Hooft coupling region $\lambda \gg 1$.

This version of the AdS/CFT conjecture is the one which is easiest to use and to check, since the type IIB supergravity approximation to type IIB string theory is known. It is also the version of the correspondence exclusively used in this thesis. The most interesting aspect of the large 't Hooft coupling limit is that this weakest version of the correspondence yields a way to describe **strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of a weakly coupled gravity theory**. This opens up a new window to strongly coupled gauge theories in general, as long as one is able to find a corresponding dual gravity background. These methods are complementary to e.g. lattice gauge theory calculations at strong coupling, which have their own restrictions in dealing e.g. with chemical potentials or real-time processes. The gravity side of the correspondence is weakly coupled in the following sense: In the next section it will become clear that it is necessary to reduce the full type IIB supergravity theory of $AdS_5 \times S^5$ through a Kaluza-Klein reduction on the five-sphere.[36] Since we are reducing on the S^5, the effective five-dimensional gravitational coupling is given by

$$G_5 = \frac{G_{10}}{\mathrm{Vol}(S^5_R)} = \frac{8\pi^6 \alpha'^4 g_s^2}{\mathrm{Vol}(S^5_R)} = \frac{\pi R^3}{2N_c^2}\,, \tag{2.114}$$

where in the last step the volume of a five-sphere of radius R, $\pi^3 R^5$, was inserted. The gravity theory thus is already weakly coupled in the large-N_c limit, even before taking the strong 't Hooft coupling limit. In this case however the type IIB action will receive stringy corrections, and thus one better takes the $\lambda \gg 1$ limit to have a treatable setup at hand.

[36]This reduction is necessary since the higher spherical harmonic modes of the ten-dimensional fields do couple separately to operators with larger scaling dimensions in the $\mathcal{N} = 4$ theory.

2.3.4 Operator-Field Dictionary, Witten's Generating Functional Formula and Correlation Functions

We have already seen in section 2.3.2 that the global symmetries of the $\mathcal{N}=4$ SYM theory match the isometries of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. This is an important check for the correspondence. It also gives a hint how to further concretise the statement of the correspondence: In the $\mathcal{N}=4$ SYM theory, operators fall into multiplets of the full $SU(2,2|4)$ superconformal group. These multiplets can be short multiplets or long multiplets, depending on whether they preserve some supersymmetry of the theory or not. Of most interest for checking the AdS/CFT correspondence are short multiplets, in particular the $\frac{1}{2}$BPS ones, as it turns out that they are protected by supersymmetry and their correlators do not renormalise. In particular, their correlators are independent of the 't Hooft coupling λ [212],[37] which makes an extrapolation possible from weak 't Hooft coupling (where the correlator can be calculated in the field theory) to strong 't Hooft coupling, where we can trust the supergravity calculations on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. But how are they identified on the gravity side of the correspondence? There must exist a map between local[38] gauge invariant operators on the field theory side and the excitations of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$ on the field theory side. Since the latter also falls into multiplets of the corresponding global symmetries, in particular of the $SU(4)_{\mathcal{R}}$ symmetry if one Kaluza-Klein reduces the IIB supergravity on the S^5 [217, 218], one needs to show that there is a one-to-one mapping between the representations showing up in the field theory and the representations showing up in the Kaluza-Klein tower of IIB supergravity states. This would then establish an **operator-field dictionary**. And indeed this is the case, as described nicely e.g. in chapter 5.6 of [80], see also the references cited there. I will not try to reproduce the full spectrum here, it should suffice to say that one needs to expand the ten-dimensional supergravity fields of different spin in spherical harmonics on the five-sphere which are labeled by (amongst other quantum numbers) the scaling dimension Δ, which describes the scaling of the corresponding field theory operator under dilatations $x \mapsto \lambda x$. The resulting five-dimensional fields will eventually become massive (just as when reducing a theory on a circle), and these masses are in correspondence with the scaling dimensions via the following mapping (for AdS_{d+1}):

$$\begin{aligned} \text{scalars and massive spin two fields} \quad & R^2 m^2 = \Delta(\Delta - d)\,, \\ \text{massless spin two fields} \quad & R^2 m^2 = 0\,, \quad \Delta = d\,, \\ \text{spin 1/2 and 3/2} \quad & R|m| = \Delta - \frac{d}{2}\,, \\ \text{p-form fields} \quad & R^2 m^2 = (\Delta - p)(\Delta + p - d)\,. \end{aligned} \tag{2.115}$$

Note that it is the linearised fields (i.e. gravitons, dilatons etc.) which are reduced on the five-sphere to yield these representations. Upon reduces the full nonlinear theory, also the

[37] Nonrenormalisation properties of $\frac{1}{4}$BPS operators are also known [213–215].

[38] Nonlocal operators can of course also be described holographically, see e.g. [216] and references therein.

interactions between these modes are obtained. We will see later in this section that these interactions are important in holographic calculations of correlation functions.

The above mapping on the basis of symmetry arguments is partly a kinematical feature of the correspondence, since the symmetries are strong enough in this case to protect the field theory from most or probably all quantum corrections. To establish a full correspondence it is however also necessary to explain how each field mode exactly encodes the information about the corresponding operator and also a possible vacuum expectation value. This explanation was given by Edward Witten in another seminal paper [219] which laid the foundations of the AdS/CFT correspondence. It is remarkable simple: For every field mode φ obtained after Kaluza-Klein reduction, consider the Laplace-Beltrami equation on the Euclidean version of AdS_5 space (in Poincaré coordinates)

$$\mathrm{d}s^2 = R^2 \,\frac{\mathrm{d}x^{i^2} + \mathrm{d}z^2}{z^2}\,, \quad i = 0,1,2,3\,, \quad z = \frac{R^2}{r}\,, \tag{2.116}$$

where a new radial coordinate z was introduced. The reason for doing this calculation in Euclidean AdS is two-fold: Historically this route was taken in e.g. [220, 221], and one does not need to worry about boundary conditions too much, since the metric (2.116) has a boundary of topology $\ ^4$ at $z = 0$ (henceforth called "the boundary"), plus a single point in the "interior" of Anti-de Sitter space at $z = \infty$. In Minkowskian signature the boundary would be four-dimensional Minkowski space, while $z = \infty$ will become a Killing horizon. In the Euclidean case one can just impose regularity as the boundary condition at $z = \infty$, but in the Minkowskian case different boundary conditions (e.g. regularity for spacelike momenta and infalling wave boundary conditions for timelike momenta) have to be imposed, complicating the situation. It is however enough to consider the Euclidean case for extracting the wanted information.

What is important for establishing the exact dictionary between the behaviour of the supergravity modes and the field theory operators is the behaviour of the solutions to the Laplace-Beltrami equation at the boundary of AdS_5, which is at $z = 0$. We thus need to solve

$$0 = \Box_{\mathrm{AdS}_5} \varphi(x^\mu, z) \tag{2.117}$$

and extract the behaviour of the solution as $z \to 0$. It turns out [219] that any solution of eq. (2.117) in an asymptotically Anti-de Sitter space-time (i.e. a space-time which is of the form (2.116) for $z \to 0$) has a leading and a subleading behaviour. For e.g. a scalar excitation φ on AdS_5 the asymptotic behaviour is

$$\varphi(x^i, z)_{m^2 = \Delta(\Delta-4)} \simeq z^{4-\Delta} J_{\mathcal{O}}(x) + z^{\Delta} \langle \mathcal{O}(x) \rangle\,, \tag{2.118}$$

where the leading term is interpreted as a **source for the operator** $\mathcal{O}$ corresponding to this field (which has mass $m^2 = \Delta(\Delta - 4)$), and the subleading term is identified with the **expectation value** (the condensate) of this operator. Note that this is only the asymptotic behaviour of the eq. (2.117) and no boundary conditions in the interior have been imposed so far, so the source and the expectation value are actually the two integration constants of eq. (2.117). Imposing a boundary condition in the interior would then relate the expectation value with the source, enabling one to calculate the vacuum expectation value $\langle \mathcal{O} \rangle|_{J_{\mathcal{O}}=0}$.

This interpretation of classical gravity solutions also yields a correspondence between the **ultraviolet and infrared physics** of the field theory, **and the different regions of Anti-de Sitter space-time**. In terms of renormalisation group flows, the procedure is normally such that one defines a theory through an action in the ultraviolet, and studies its behaviour under the renormalisation group flow, i.e. when integrating out high-energetic degrees of freedom in the Wilsonian sense [222]. The theory then can flow to an infrared fixed point (this is believed to happen for QCD), or it can be already at a fixed point (this is what happens e.g. for conformal theories like $\mathcal{N} = 4$ SYM). For the latter case of theories, it is then the interesting to perturb the theory by adding additional operators with corresponding coefficients (the sources which can for example be mass matrices or additional couplings), and investigate whether this creates a new renormalisation group flow [223]. If the system is driven by this deformation to a new fixed point, the deformed Lagrangian serves as the ultraviolet definition of a new theory with a nontrivial RG flow. This theory can then flow to another fixed point in the infrared, but the important point here is that the deformation with its nontrivial sources for certain operators was added in the ultraviolet description of the theory. Since we know that sources in the AdS/CFT context are encoded in the nonnormalisable modes of supergravity fields, i.e. that they die off slowest towards $z = 0$, it is natural to identify the $z = 0$ region of the Anti-de Sitter space as the region encoding the ultraviolet physics of the dual field theory. On the other hand, the interior part of Anti-de Sitter space, in particular the point $z = \infty$, should correspond to the infrared physics in the field theory since for example condensates, which in the field theory are often governed by infrared effects, are calculated after giving boundary conditions at $z = \infty$. Later on in section 2.3.6 we will see an example of this mechanism. In conclusion, we thus identify the region $z \to 0$ with ultraviolet effects in the dual field theory, and the region $z \to \infty$ with infrared effects. A renormalisation group flow of some observable from the ultraviolet to the infrared is encoded in the full z dependence of the corresponding dual supergravity field.

One might wonder why the mass squares in eq. (2.115) are not bounded by zero from below. The reason for this is a peculiar property of fields in Anti-de Sitter space-time found by Peter Breitenlohner and Daniel Z. Freedman [224, 225] (see also [226]): Since the metric (2.116) of AdS_{d+1} diverges as one approaches the boundary at $z = 0$, less restrictive fall-off conditions at spatial infinity are necessary for fields to have positive energy, and thus tachyonic (i.e. negative) mass squares are allowed if they are not too negative. For scalars[39] one finds

$$R^2 m^2 \geq -\frac{d^2}{4}\,. \tag{2.119}$$

[39]The Breitenlohner-Freedman bound for fields other than scalars can readily be deduced from the unitarity bound $\Delta \geq 0$ by using eqs. (2.115).

This restriction is called the **Breitenlohner-Freedman bound**. There is actually a nice correspondence between the (ir)relevance of operators and the masses of the corresponding supergravity fields (see. ch. 9.1 of [80] or [219]), which I will describe for scalars: In general, deforming the field theory by a certain operator corresponds to searching for a new supergravity background with the corresponding field excited on the nonlinear level, i.e. solving the coupled equations of the new field together with the equations of motion of the old theory. In particular, scalar fields can have quite complicated potentials in the gauged supergravity theories which arise from the reduction on the five-sphere (or other internal spaces). Scalar operators with $m^2 < 0$ then correspond to **relevant deformations** of the field theory, creating a RG flow towards a different infrared fixed point. Massless scalars correspond to **marginal deformations**, which might or might not create a RG flow, depending on whether they are marginally relevant, irrelevant or exactly marginal. Finally, scalar operators with positive mass squared (i.e. $\Delta > 4$) correspond to **irrelevant deformations** of the dual CFT, describing a renormalisation group flow to $\mathcal{N} = 4$ SYM in the infrared from some other unknown theory in the UV. The way to describe these RG flows holographically is via domain wall solutions of truncated gauged supergravities interpolating between different asymptotically Anti-de Sitter spaces (see e.g. ch. 9 of [80]).

We have seen that free supergravity fields encode in their boundary behaviour both ultraviolet physics (the sources) and infrared physics (the vacuum expectation values). For giving a detailed description of the correspondence we also need to give a recipe of how to calculate correlation functions. This was given again by Witten in [219] (and independently in [220]), by stating the following **equivalence between the supergravity on-shell action and the generating functional of the dual field theory**. Let $\mathcal{O}_i$ be a set of operators of $\mathcal{N} = 4$ SYM theory, then, according to [219], the generating functional of Euclidean field theory correlators is identified with the on-shell type IIB supergravity action, where the solutions for the linearised fields φ^i dual to the correlators $\mathcal{O}_i$ are subject to the boundary behaviour as in (2.118). In a formula,

$$\left\langle \exp\left(- \int_4 J^i \mathcal{O}_i \right) \right\rangle_{\mathrm{CM}} = \exp\left(-S_{IIB}\left[\varphi^i(J^i, \langle \mathcal{O}_i(J^i) \rangle) \right] \right) \ . \tag{2.120}$$

Here the chosen vacuum expectation values $\langle \mathcal{O}_i \rangle|_{J^i=0}$ on the right-hand side define the vacuum in which the generating functional on the left-hand side is taken, while the non-normaliseable mode of the linearised fields (2.118) provides the dependence on the source J. Possible additional boundary conditions imposed on the linearised field equations relate the normaliseable behaviour, i.e. the expectation value, with the source for the operator. The action on the right-hand side is the type IIB supergravity action after Kaluza-Klein reduction on the S^5. The correct definition of the on-shell action still needs the addition of surface terms, such that the theory admits a well-defined semi-classical approximation, a process called "holographic renormalisation". I will not dwell on this at this point, but come back to it in the appropriate places in the next chapters. See however [227] for a good review of this extensive topic and [228, 229] for a systematic treatment of the simpler

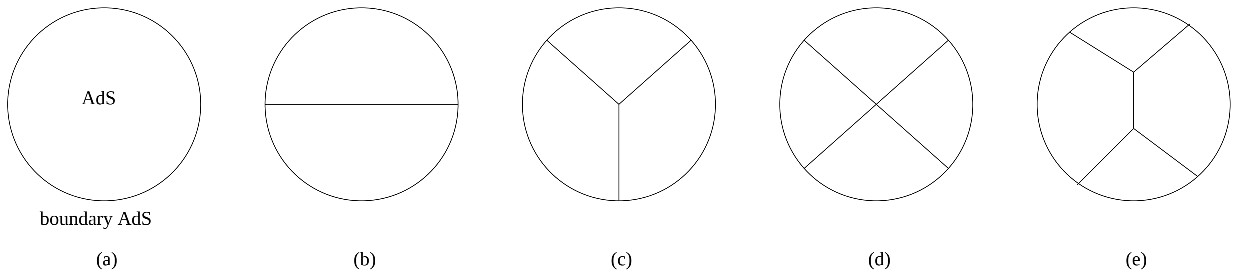


Figure 2.6: Several Witten diagrams describing a) the vacuum, b,c) two- and three-point functions and d,e) contributions to a four-point function (figure taken from [80]).

and thus more transparent situation in 1+1 dimensions. In any case the formula (2.120) now allows to calculate Euclidean correlators by differentiating with respect to the sources as usual. Note that the on-shell type IIB supergravity action on the right-hand side is the full action, including interaction terms between the linearised fields. These interactions are important, since they encode the nontrivial information about the field theory interactions between the operators. Since, as explained in section 2.3.3, the five-dimensional gravitational coupling (i.e. the relevant gravitational coupling after reduction on the S^5) $\kappa_5^2 \sim N_c^{-2}$ is small in the large N_c limit, it is possible to set up a perturbation theory scheme on the gravity side by expanding the supergravity action in κ_5. The on-shell supergravity action can then be evaluated in perturbation theory to the required order by the following prescription: Connect the sources J^i located at the boundary of Euclidean AdS with the interior by "bulk-to-boundary propagators", and connect the end points of the bulk-to-boundary propagators with "bulk-to-bulk" propagators. If necessary, loops in the interior of Euclidean AdS can be build up by bulk-to-bulk propagators. Include integrations over Euclidean AdS space for each internal interaction point. At the end, do a summation over all diagrams to the wanted order in κ_5. These diagrams are called "Witten diagrams", depicted in figure 2.6. I will however not dwell longer on the actual calculation of correlators, since in this thesis we will mostly work with one-point functions. The interested reader is referred to the reviews [79,80,200,201] or the classics [212,219–221,230–237]. The classical checks of the AdS/CFT correspondence via the calculation of correlators of $\frac{1}{2}$BPS operators which are protected by supersymmetry and thus do not depend on the 't Hooft coupling can also be found in these references.

Some additional words have to be said about the relation between Minkowskian and Euclidean signature in the AdS/CFT correspondence. It has been observed by Dam T. Son and Andrei Starinets in [238] that Witten's formula for the generating functional (2.120) does not apply to the case of Minkowskian signature. In particular, it would yield real boundary propagators for timelike momenta, while e.g. the retarded/advanced propagators are complex. Son and Starinets proposed a recipe which correctly provides the right boundary propagators by imposing corresponding boundary conditions at the $z = \infty$ horizon of the Poincaré patch of AdS_5. However, since their approach is ad hoc and does not generalise eq. (2.120), it cannot be applied to higher n-point functions. A step forward to such a generalisation was made in [239], where it was shown that holographically implementing the Schwinger-Keldysh formalism with its doubling of the degrees of freedom on the contour needs the full global Anti-de Sitter manifold rather than just the Poincaré patch (which covers only half of the manifold, see e.g. [204]). Physically speaking, the second copy of $\mathcal{N} = 4$ in the Schwinger-Keldysh formalism lives on the other boundary of Anti-de Sitter space which bounds the second Poincaré patch. In this way [239] it was possible to obtain the Schwinger-Keldysh propagator by functional variation of a corresponding boundary action. The upshot is that the relevant on-shell gravity action is a sum over action functionals, one with Minkowskian signature for each real-time segment of the contour, and a Wick-rotated one for each imaginary-time segment. Recently Kostas Skenderis and Balt C. van Rees gave an in-depth analysis of this procedure for general contours [240, 241], including holographic renormalisation and several examples.

In conclusion, we have seen that and how the AdS/CFT correspondence can be formulated on the levels of operators and fields, correlation functions and generating functionals. It is possible to incorporate vacua with nontrivial expectation values and to describe different field theory perturbations and holographic renormalisation group flows. This is already a rather concrete framework which can be used to analyse the dynamics of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at strong coupling. In the remaining two sections of this chapter I will describe two generalisations of the AdS/CFT correspondence to include finite temperature as well as matter fields in the fundamental representation of the gauge group (recall that all fields in $\mathcal{N} = 4$ SYM where adjoints). These two generalisations will be most important for the understanding of the research work of the author presented in chapters 3 and 4.

2.3.5 Finite Temperature AdS/CFT and Black Holes

The generalisation of the AdS/CFT correspondence to put the $\mathcal{N} = 4$ SYM theory at finite temperature is remarkably simple and again was given by Witten in [91,219] (see also [242]): Instead of taking the near-horizon limit of the extremal D3 soliton (2.101) to arrive at the AdS geometry (2.104), one should take the decoupling limit in the nonextremal black D3 brane geometry [195], which reads

$$
\begin{aligned}
\mathrm{d}s^2 &= H_3^{-\frac{1}{2}}\left(-f(r)\mathrm{d}t^2 + \mathrm{d}x^{i^2}\right) + H_3^{\frac{1}{2}}\left(\frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_5^2\right)\,,\\
H_3(r) &= 1 + \frac{R^4}{r^4}\left[\sqrt{1+\frac{r_h^8}{4r^8}} - \frac{r_h^4}{2r^4}\right]\,, \quad f(r) = 1 - \frac{r_h^4}{r^4}\,,\\
C_4 &= \left(1 - \frac{1}{H_3}\right)\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\,, \quad e^{2\Phi} = g_s^2\,, \quad R^4 = 4\pi g_s N_c \alpha'^2\,.
\end{aligned}
\tag{2.121}
$$

Maldacena's decoupling argument is still valid, since the only difference is now the nonextremality of the D3 brane[40] on the field theory side, and the presence of the Killing horizon at finite $r = r_H$ on the gravity side. In particular, the modes infinitesimally close to the horizon still receive an infinite redshift, and an asymptotic observer sees the full type IIB string theory spectrum on the near horizon geometry. The near horizon limit is now taken holding fixed the horizon temperature

$$T = \frac{r_H}{\pi R^2} = \frac{u_h}{\pi \ell^2}\,, \tag{2.122}$$

i.e. holding fixed both $u = \frac{r}{\alpha'}$ and $u_H = \frac{r_H}{\alpha'}$. Holding the temperature fixed in this case corresponds to holding the energy density above extremality fixed, since the extremal D3 brane solution has zero horizon temperature. This scaling limit yields the metric of a Schwarzschild black hole in the Poincaré patch of Anti-de Sitter space [242],

$$\begin{aligned} \frac{\mathrm{d}s^2}{\alpha'} &= \frac{u^2}{\ell^2}\left(-f(u)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \frac{\ell^2}{u^2}\frac{\mathrm{d}u^2}{f(u)} + \ell^2 \mathrm{d}\Omega_5^2\,, \\ f(u) &= 1 - \frac{u_H^4}{u^4}\,,\ u_H = \frac{r_H}{\alpha'}\,, \quad \ell^4 = 4\pi g_s N_c\,. \end{aligned} \tag{2.123}$$

The four-form potential C_4 is unchanged compared to the zero temperature case. This kind of black hole is flat, i.e. the horizon topology is instead of S^3, times the usual S^5 of radius ℓ. Its ADM mass [247][41] is, using eq. (2.114), readily computed to be

$$M = \frac{3\pi r_H^4}{8 G_5 R^2} = \frac{3}{4}\frac{N_c^2}{\ell^5} u_H^4 \ell_s^3\,. \tag{2.124}$$

Though the appearance of the factor ℓ_s^3 may surprise here, it just means that what is kept fixed in the decoupling limit is actually the energy density rather than the mass itself. From dimensional analysis of eq. (2.124) one finds that the energy density $\varepsilon = \frac{M}{\ell_s^3}$ must be a density with respect to three dimensions, i.e. it can be interpreted as the three-dimensional spatial energy density on the boundary of AdS_5. One can now combine eqs. (2.122) and (2.124) to the following Stephan-Boltzmann law,

$$\varepsilon = \frac{3\pi^4 \ell^3 N_c^2}{4} T^4\,. \tag{2.125}$$

The appearance of this T^4 behaviour already yields a hint that the boundary field theory is in a finite temperature state. Since the metric (2.123) asymptotically returns to AdS space, the AdS/CFT correspondence as described in the previous sections is affected by this replacement of AdS by AdS-Schwarzschild only by the choice of boundary conditions at the horizon. In particular, the operator-field mapping is the same as before.

[40] The string theoretic description of certain near-extremal black holes in four-dimensional $\mathcal{N} = 8$ supergravity has been studied in [243, 244], and the Bekenstein-Hawking entropy was reproduced. For work on near-extremal five branes see [245, 246].

[41] The mass definition by Arnowitt, Deser and Misner is the most applicable definition of masses in asymptotically flat black hole space-times. It in particular reproduces the mass of the Schwarzschild black hole correctly.

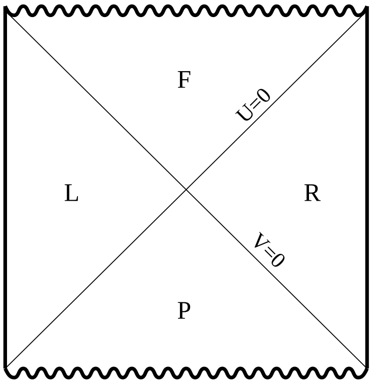


Figure 2.7: Penrose diagram of the AdS-Schwarzschild space-time (figure taken from [239]).

The metric (2.123) is written in Poincaré coordinates and thus only describes half of the full manifold, which is depicted as a Penrose diagram in figure 2.7. The global metric of AdS-Schwarzschild (omitting the five-sphere) is

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_3^2\,, \quad f(r) = 1 + \frac{r^2}{R^2} - \frac{r_0^4}{R^2 r^2}\,. \tag{2.126}$$

Here r_0 sets the mass of the space-time, but the horizon will not be any longer exactly at r_0, but the horizon is at

$$r_H^2 = \sqrt{\frac{R^4}{4} + r_0^4} - \frac{R^2}{2}\,, \tag{2.127}$$

with a Hawking temperature of

$$T = \frac{2r_H^2 + R^2}{2\pi R^2 r_H}\,. \tag{2.128}$$

Note that the boundary of this space-time at $r = \infty$ is $\times \mathrm{S}_R^3$ rather than ^{4}. Calculating the holographic vacuum expectation value in this space-time yields the result (see e.g. ch. 18 of [143] for the details on holographic renormalisation etc.)

$$\begin{aligned} T_{tt} &= \frac{3}{8\pi G_5}\left(\frac{1}{8R} + \frac{1r_0^4}{2R^5}\right) + \mathcal{O}(r^{-1})\,, & (2.129)\\ T_{ij} &= \frac{1}{8\pi G_5}\left(\frac{1}{8R} + \frac{r_0^4}{2R^5}\right) g_{ij} + \mathcal{O}(r^{-1})\,, & (2.130)\end{aligned}$$

where g_{ij} is the metric in the angular directions of (2.126). In the $r \to \infty$ limit, the first term in the brackets describes the Casimir energy of $\mathcal{N} = 4$ SYM theory on S_R^3, and vanishes in the $R \to \infty$ limit. The second term is the contribution to the energy density and pressure above extremality. The second term (and actually also the Casimir energy term) fulfill the equation of state $\varepsilon = 3p$, which is the equation of state for radiation. We thus find that the dual field theory is in a thermal state with temperature T, and the equation of state is that of a gas of massless particles, as expected for $\mathcal{N} = 4$ SYM.

Note that the energy density (2.125) scales with N_c^2, which is consistent with the interpretation of the $\mathcal{N}=4$ SYM theory being in a **deconfined phase**, i.e. a phase with gluons and their superpartners as the fundamental degrees of freedom. Only at zero temperature $T=0$, there is no such scaling (since $\varepsilon=0$ there). Thus the theory can possibly confine only at zero temperature, meaning that the Poincaré patch by itself is not sufficient to show the confinement-deconfinement phase transition known from $\mathcal{N}=4$ SYM theory. The reason for this is the lack of a second dimensionful parameter in flat Minkowski space which could together with the temperature yield a dimensionless ration governing the phase transition. It is however possible to describe the **(de)confinement phase transition** also in the Poincaré patch by introducing an infrared cutoff for the coordinate $u \geq \Lambda_{IR}$ [248]. In this case Λ_{IR} serves as the dimensionful parameter. There is however a much more elegant description of the (de)confinement phase transition [91, 249] in terms of the global coordinate system (2.126): The thermodynamics of black holes in AdS space [250] admits a **phase transition found by** Stephen **Hawking** and Don **Page** between the Euclideanised version of (2.126) and "**thermal Anti-de Sitter space**". The latter space is Euclidean Anti-de Sitter space in global coordinates (i.e. eq. (2.126) with $r_0=0$) with the Euclidean time being periodically identified, $\tau \sim \tau + \frac{1}{\beta}$. Here $\beta = T^{-1}$ is the inverse Hawking temperature in units in which the Boltzmann constant $k_B=1$. The transition between both phases is of purely thermodynamic nature: Comparing the free energies of both metrics, which is identified with the on-shell five-dimensional Euclidean Einstein-Hilbert action plus a negative cosmological constant (plus appropriate boundary terms), yields a difference

$$F_{\mathrm{BH}} - F_{\mathrm{thermal}} = \frac{\pi^2 r_H^3 (R^2 - r_H^2)}{4G_5(2r_H^2 + R^2)}\,. \tag{2.131}$$

The phase transition thus occurs at $r_H = R$, i.e. when the Schwarzschild radius of the global AdS black hole exceeds the curvature radius R of the background AdS space. For $r_H > R$ the black hole has smaller free energy, while for $r_H < R$ the thermal AdS is thermodynamically preferred. Comparing with the Hawking temperature (2.128) we find that the transition happens at

$$T_{\mathrm{deconf}} = \frac{3}{2\pi R}\,. \tag{2.132}$$

This is expected since $\mathcal{N}=4$ SYM as a conformal theory has no intrinsic scale and thus the transition temperature must be inversely proportional of the radius of the spatial S^3. Note that this phase transition is a first order transition. It also has a topological interpretation: Since the boundary of both thermal AdS space and the AdS black hole in global coordinates has topology $\mathrm{S}^1_\beta \times \mathrm{S}^3_R$, but the full spaces are topologically distinct ($B^2 \times \mathrm{S}^3$ for the AdS black hole and $\mathrm{S}^1 \times B^4$ for thermal AdS), the transition is between two topologically distinct saddle points of the type IIB supergravity path integral. Another, though a bit different, phase transition of a topological nature will show up in section 2.3.6 when introducing quarks into the theory via probe branes.

The limit which reduces the global AdS-Schwarzschild black hole to the flat Poincaré patch form must send the transition temperature to zero, which is achieved by sending $R \to \infty$, i.e. for small AdS curvature. It is not obvious from the forms of the metrics (2.123) and (2.126) that this is the correct limit to take, since both metrics are written in different coordinates. However, since the horizon topology of the global AdS black hole is S^3 while the topology of the Poincaré patch black hole is $\ ^3$ a suitable limit which decompactifies the horizon might be the right choice.[42] By separating one polar angle and expanding for small angle around one pole (e.g. the north pole) one can write the standard metric on S^3 as

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\Omega_2^2 \simeq \mathrm{d}\theta^2 + \theta^2 \mathrm{d}\Omega_2^2 \,. \tag{2.133}$$

This expansion needs $\theta \ll 1$. If θ would be related to some radial variable, then (2.133) would be a multiple of the three-dimensional metric $\mathrm{d}\vec{x}^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2$ in spherical coordinates. This can be achieved while keeping θ small via

$$\theta = \frac{\rho}{R} \,. \tag{2.134}$$

In this way, if sending $R \to \infty$, one can allow for larger and larger values of the radial coordinate ρ without spoiling $\theta \ll 1$, which achieves the decompactification of the horizon. This brings the metric into the form

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{R^2}\left(1 - \frac{r_0^4}{r^4}\right)\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 + \frac{r^2}{R^2}\left(1 - \frac{r_0^4}{r^4}\right)\right)} + \frac{r^2}{R^2}\left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2\right) \,.$$

Now, since $u = \frac{r}{\alpha'}$ is a coordinate with dimension of mass rather than length, we must rescale

$$r = R^2 u \,, \quad r_0 = R^2 u_0 \,, \tag{2.135}$$

which finally allows us to drop the one in $f(r)$ and brings the metric into the form

$$\mathrm{d}s^2 = \ell^2 \alpha' \left[u^2 \left(-\left(1 - \frac{u_0^4}{u^4}\right)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 \right) + u^2 \frac{\mathrm{d}u^2}{\left(1 - \frac{u_0^4}{u^4}\right)} \right] \,,$$

which up to a rescaling of the coordinate $u \mapsto \frac{u}{\ell^2}$ (which does not change the dimension of u) is the Poincaré black hole (2.123). Since in this limit the confinement temperature (2.132), the dual gauge theory "gluons" as always deconfined in the Poincaré patch, except at zero temperature. The Poincaré patch AdS black hole thus describes the $\mathcal{N} = 4$ SYM theory at high-temperatures, i.e. when the ratio $T/T_{\mathrm{deconf}} \gg 1$. We thus cannot expect to describe exactly phenomena associated with the phase transition itself just by looking at the Poincaré patch.

To summarise, the AdS/CFT correspondence at finite temperature can be stated as follows:

> In the large-N_c, large 't Hooft coupling limit, four-dimensional $U(N_c)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a space-time $\ \times S_R^3$ and a temperature

[42] The author thanks Andrew O'Bannon for suggesting this argument.

> T given by eqs. (2.128) and (2.127) is holographically dual to the S^5-reduction of type IIB supergravity on the space-time AdS-BH$\times S^5_R$, where the Anti-de Sitter black hole geometry is given by eq. (2.126). In the limit in which the spherical black hole horizon decompactifies, the $\mathcal{N}=4$ SYM theory lives on four-dimensional Minkowski space ^{1,3}, while the gravity dual is given by the S^5 reduction of type IIB supergravity on eq. (2.123).

Note that in this thesis we will exclusively work with the Poincaré patch version of the finite temperature correspondence.

2.3.6 Quarks with Flavour and Probe Branes in Anti-de Sitter Space

So far we have extended the gauge-gravity duality to finite temperature, but for a theory of adjoint fields only, namely $\mathcal{N}=4$ SYM theory. In chapters 3 and 4 we however will make use of an even extended gauge-gravity duality, which includes fields in the fundamental representation of the gauge group, i.e. supersymmetric versions of **quarks** and antiquarks. They can be added in the quenched approximation[43] by considering probe branes in the corresponding gravity dual background, an idea which was proposed by Andreas Karch, Emanuel Katz and Lisa Randall in [94,95] (see also [251,252]). This construction basically adds an open string sector to the gravity dual through the additional fields in the low energy effective action on the probe branes (or excited open strings for higher spin fields). For the purposes needed in the following two chapters it will be sufficient to describe the gravity dual of $\mathcal{N}=4$ SYM coupled to a number N_f of $\mathcal{N}=2$ quark hypermultiplets, which is constructed by embedding D7 probe branes into the geometries already described. This model is known as the **D3-D7 model of AdS/CFT with flavour**, and my description will mostly follow the review [253]. Alternative models with fundamental flavour degrees of freedom were proposed in [88,254,255].

The field theory under consideration is a $\mathcal{N}=2$ supersymmetric D3-D7 intersection in type IIB string theory, depicted in figure 2.8. It consists of a stack of $N_c = N$ D3 branes and another stack of N_f D7 branes, oriented parallel to each other in the 0123 directions (henceforth called x^μ) of flat Minkowski space ^{1,9}. The D7 branes also fill the 4567 directions (henceforth called y_m), and there are two totally transverse directions z^i, $i=8,9$. The embedding profile of the D7 branes will be described by the embedding functions $z^i(x^\mu, y_m)$. To describe fundamental quarks with mass m_q, the D7 branes have to be

[43] The 't Hooft limit for gauge theories with fundamental fields organises in a series expansion in powers of N_c^{-1}. In a deconfined phase, the leading order $N_c^{-}2$ is given by planar gluon diagrams, while the fundamental degrees of freedom show up at order $\frac{N_f}{N_c}$. Truncating the expansion to this order (which requires $N_f/N_c \to 0$, i.e. N_f fixed in the large-N_c limit) corresponds to truncating the theory to planar diagrams with only gluons running in loops, while fundamental fields are allowed as external particles. This is called the "quenched approximation".

Coordinates		
0 1 2 3	4 5 6 7	8 9
N_c D3 branes		
N_f D7 branes		
x^μ	y_m ρ S^3	z^i
$^{1,3} \times SO(1,3)$	$SO(4)_{4567} \equiv SU(2)_\Phi \times SU(2)_\mathcal{R}$	$SO(2)_{89} \equiv U(1)_\mathcal{R}$

Table 2.9: Embedding of a stack of N_f flavour D7 branes.

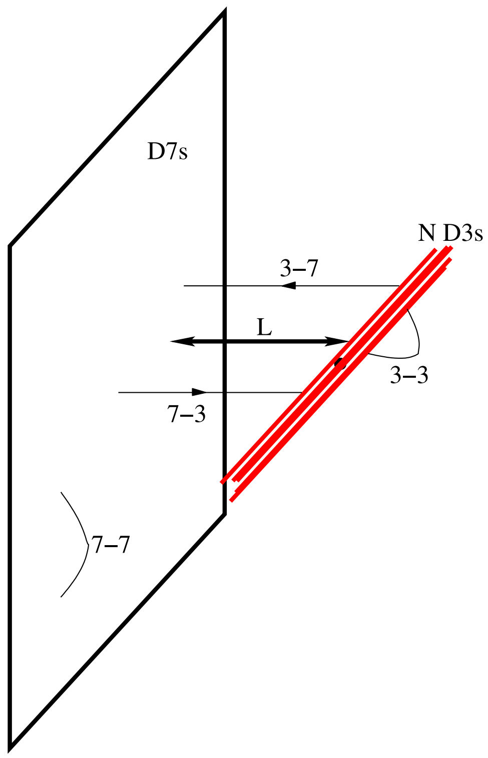


Figure 2.8: The D3-D7 model of AdS/CFT with flavour.

a distance apart from the D3 branes, specified by $\sqrt{(z^8)^2+(z^9)^2} = 2\pi\alpha' m_q$. Since the setup is invariant under a $SO(2)$ symmetry rotating the 8-9 plane, one can always choose $z^8 = 2\pi\alpha' m_q$ and $z^9 = 0$. The fundamental fields then arise from the excitations of the 3-7 and 7-3 superstrings[44], and group themselves into $\mathcal{N} = 2$ hypermultiplets. The full geometric symmetry of the setup is

$$SO(1,3)_{0123} \times SO(4)_{4567} \times SO(2)_{89} \,. \tag{2.136}$$

In addition to the fundamentals, there is of course a full $\mathcal{N} = 4$ SYM sector coming from the 3-3 strings, as well as an eight-dimensional DBI theory on the D7 branes with gauge group $U(N_f)$. However, since the different gauge couplings (2.93) scale differently with α', the D7 brane gauge coupling vanishes if the D3 brane gauge coupling is kept fixed in the $\alpha' \to 0$ limit,

$$g^2_{YM,7} = (2\pi)^4 \alpha'^2 g^2_{YM} \,, \tag{2.137}$$

and thus the $U(N_f)$ gauge symmetry becomes a global (ungauged symmetry) in this limit. Furthermore, since the interactions of the 3-7 strings with the 7-7 modes are all $\propto g^2_{YM,7}$, the whole section of 3-3 and 3-7 fields decouples from the 7-7 sector. The formerly bifundamental 3-7/7-3 modes become fundamental fields in the $U(N_c)$ gauge group, while the $U(N_f)$ index just becomes a global index.

It is now possible to set up a decoupling limit similar to the one presented in section 2.3.1. In the open string picture (i.e. branes in flat space), the decoupling is again achieved because of the κ-dependence of the effective action which now consists of (2.99) plus the D7 brane action (2.88) and a piece for the 3-7/7-3 interactions. From the point of view of gravity, the D7 brane tension scales like

$$T_7 \sim \frac{1}{g_s \ell_s^8} \sim \frac{g_s}{\kappa^2} \,, \tag{2.138}$$

and thus the whole D3-D7 system will decouple from the bulk IIB supergravity (which again becomes free) in the limit $\alpha' \to 0$. We need, however, also to decouple the D7 brane from the 3-3 and 3-7 parts of the action. Since both of these parts of the low energy effective action scale like g^{-2}_{YM}, a fact which will become clear in a minute when writing down the corresponding action, the $\alpha' \to 0$ limit would automatically also decouple the D7 fields from the rest of the D brane construction. However, we still need to make sure that the D7 brane does not backreact onto the flat space background, i.e. that its coupling to gravity vanishes. This is called the **probe limit**, i.e. the D7 brane acts as a probe which gets acted upon by the background, but which itself is a negligible source of gravitational fields. This is achieved by the large-N_c, fixed N_f limit which in the field theory corresponds to the quenched approximation, as becomes clear when comparing the N_c behaviour of the

[44] In fact only their ND directions yield massless modes.

different low energy effective actions,[45]

$$S = \underbrace{\frac{1}{2g_s^2\alpha'^4}}_{\sim\frac{N_c^2}{\lambda^2\alpha'^4}} \int \mathcal{L}_{IIB} + \underbrace{\frac{N_f}{(2\pi)^5 g_s\alpha'^4}}_{\sim\frac{N_f N_c}{\lambda\alpha'^4}} \int \mathcal{L}_{D7}\,. \tag{2.139}$$

The D7 contribution to the energy-momentum tensor is thus suppressed by a factor $\lambda\frac{N_f}{N_c}$ compared to the leading order behaviour of the IIB supergravity action. In the strict $N_c \to \infty$, $\frac{N_f}{N_c}$ fixed limit one can thus naively neglect the D7 brane contribution to the energy-momentum tensor, and solve for the D7 equations of motion separately from solving for the D3 brane solitonic solution. This yields, on the gravity side of the correspondence, the usual type IIB supergravity sector, plus an additional sector of open string fields summarised in the D7 brane effective action (2.88). The open string fields will then couple holographically to operators built out of the 3-7/7-3 fields.

The new field content introduced by the 3-7/7-3 sector is a hypermultiplet of (anti)fundamentals, which consists of one chiral multiplet Q^i, $i = 1, \dots, N_f$, which comes from the 3-7 strings and transforms in the $(\bar{N}_c, N_f)$ representation of the gauge and flavour groups, and another chiral multiplet $\tilde{Q}_i$ coming from the 7-3 strings and transforming in the $(N_c, \bar{N}_f)$. Since both chiral multiplets form a hypermultiplet, there will be an $U(2)_{\mathcal{R}}$ symmetry rotating them into each other. The Lagrangian describing this theory in $\mathcal{N} = 1$ superspace language is given by the $\mathcal{N} = 4$ SYM piece, usual additional kinetic terms for the two chiral multiplets plus an additional superpotential term. Altogether it reads[46]

$$\begin{aligned} S_{D3D7} \;=\; & \int \mathrm{d}^4x\, \Im\Big[\tau \int \mathrm{d}^4\theta\, \Big(\mathrm{tr}\,\big(\bar{\Phi}_I e^V \Phi_I e^{-V}\big) + Q_i^\dagger e^V Q^i + \tilde{Q}_i e^{-V} \tilde{Q}^{\dagger i}\Big) \\ & + \tau\Big(\int \mathrm{d}^2\theta\, \mathrm{tr}\,(W_\alpha W^\alpha + \epsilon_{IJK}\Phi_I\Phi_J\Phi_K) + \tilde{Q}_i(m_q + \Phi_3)Q^i + \text{h.c.}\Big)\Big] \end{aligned} \tag{2.140}$$

with all the other definitions as in eq. (2.108). The interesting and new part is the coupling between the transverse scalar Φ_3 and the chiral multiplets, and the mass deformation given by the "quark masses" m_q, which corresponds to separating the D7 brane stack from the D3 branes in the transverse 8-9 direction. The above action not only has $\mathcal{N} = 2$ supersymmetry, but is also classically conformal for $m_q = 0$, as there are no other dimensionful parameters, so in the massless case the $\mathcal{N} = 2$ supersymmetry enhances to

[45]This argument only requires the contribution of the nonabelian D7 brane action to the energy-momentum tensor to be finite, i.e. not to diverge. This is always fulfilled, ensured by correct holographic renormalisation. I however assumed here additionally that the contribution of N_f D7 branes to the energy-momentum tensor scales like N_f, which is surely obvious for the decoupled $U(1) \subset U(N_f)$ part of the DBI action.

[46]In this thesis the convention is employed that a fundamental gauge index is a column index, while an antifundamental gauge index is a row index. Contrary to this, the flavour indices are defined upstairs for the fundamental of $U(N_f)$, and downstairs for the antifundamental representation. In the literature the conventions often differ.

$\mathcal{N}=2$	components	spin	$SU(2)_\Phi \times SU(2)_\mathcal{R}$	$U(1)_\mathcal{R}$	Δ	$U(N_f)$	$U(1)_q$
(Φ_1, Φ_2)	X^4, X^5, X^6, X^7	0	$(\frac{1}{2}, \frac{1}{2})$	0	1	1	0
hyper	λ_1, λ_2	$\frac{1}{2}$	$(\frac{1}{2}, 0)$	-1	$\frac{3}{2}$	1	0
(Φ_3, W_α)	$X_V^A = (X^8, X^9)$	0	$(0,0)$	$+2$	1	1	0
vector	λ_3, λ_4	$\frac{1}{2}$	$(0, \frac{1}{2})$	$+1$	$\frac{3}{2}$	1	0
	v_μ	1	$(0,0)$	0	1	1	0
$(Q, \tilde{Q}^\dagger)$	$q^m = (q, \bar{\tilde{q}})$	0	$(0, \frac{1}{2})$	0	1	N_f	$+1$
fund. hyper	$\psi_i = (\psi, \tilde{\psi}^\dagger)$	$\frac{1}{2}$	$(0,0)$	± 1	$\frac{3}{2}$	N_f	$+1$

Table 2.10: Fields of the D3/D7 low energy effective field theory and their quantum numbers under the global symmetries (table taken from [253]). Note that $U(1)_q \subset U(N_f)$.

a superconformal symmetry.[47] In fact, requiring $\mathcal{N}=2$ supersymmetry by requiring an unbroken $SU(2)_\mathcal{R}$ symmetry and unbroken $U(N_f)$ flavour symmetry uniquely determines eq. (2.140). Note that the $U(1)_\mathcal{R}$ symmetry is broken by a finite quark mass. As we will see below, this has a geometric interpretation in the embeddings of D7 probes in $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The action (2.140) has several global symmetries which match the geometric symmetries of the D brane constructions: There is of course Lorentz symmetry $SO(1,3)$. The rotation symmetry $SO(4)_{4567}$ in the "relatively transverse" directions shows up again as the $SU(2)_\Phi \times SU(2)_\mathcal{R}$ symmetry, with the first part of which being a global symmetry rotating Φ_1 and Φ_2 into each other, while the second part is the nonabelian $\mathcal{R}$-symmetry of the theory. The $SO(2)_{89}$ rotations translate into the $U(1)_\mathcal{R}$ symmetry. The quantum numbers of the fields are summarised in table 2.10. From the flat space D brane construction we now understand that a finite quark mass, which breaks $U(1)_\mathcal{R}$, breaks the rotational symmetry in the totally transverse 8-9 plane.

In order to holographically describe this theory, we need to find an embedding of the D7 brane which preserves these geometric symmetries as well. I will only describe the embedding in the Poincaré coordinate system here, as it suffices for the purpose of this thesis. For the situation in global AdS space see [257]. Once can rewrite the Poincaré patch of $\mathrm{AdS}_5 \times \mathrm{S}^5$ (2.104) as a warped product of ^{1,3} and ^{6},

$$\mathrm{d}s^2 = \frac{\vec{y}^2 + \vec{z}^2}{R^2} \mathrm{d}x^{\mu^2} + \frac{R^2}{\vec{y}^2 + \vec{z}^2} \left(\mathrm{d}\vec{y}^2 + \mathrm{d}\vec{z}^2 \right) . \tag{2.141}$$

In this form the several geometric symmetries are already evident. Since we do not want to break Lorentz symmetry, the embedding functions should not depend on x^μ, and since we do not want to break the rotations in 4567 direction, the embedding can only depend

[47] The conformal symmetry is also preserved at the quantum level in 't Hooft limit since the β function for the 't Hooft coupling $\beta(\lambda) = \frac{1}{2\pi} \left(\frac{\lambda}{4\pi}\right)^2 \frac{N_f}{N_c}$ vanishes in the strict $N_c \to \infty$ limit with N_f fixed [256]. Note however that there is a restriction on N_f coming from global issues with a deficit angle in the fully backreacted D3-D7 solution (see sec. 4.1.2 of [253] for a discussion).

on the radial coordinate ρ in that space. Introducing polar coordinates in transverse space via

$$z^8 = L\cos\phi\,, \quad z^9 = L\sin\phi\,, \tag{2.142}$$

one finds that the $U(1)_{\mathcal{R}}$ symmetry acts by shifts of the angle Φ. Introducing a mass corresponds to breaking this shift symmetry. But even if the shift symmetry is broken, one can always set $\Phi = 0$ by a $U(1)_{\mathcal{R}}$ rotation. Thus the relevant embedding coordinate is just $L(\rho)$, the radial distance in totally transverse space. The induced metric on the D7 worldvolume $P[G]$ then reads

$$\mathrm{d}s^2 = \frac{\rho^2 + L(\rho)^2}{R^2}\mathrm{d}x^{\mu^2} + \frac{R^2}{\rho^2 + L(\rho)^2}\left((1 + L'(\rho)^2)\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_3^2\right)\,. \tag{2.143}$$

Employing Hopf coordinates for the S^3,

$$d\Omega_3^2 = d\psi^2 + \cos^2\psi d\beta^2 + \sin^2\psi d\gamma^2\,, \tag{2.144}$$

the DBI action for the center-of-mass embedding of N_f D7 branes, which is described by the $U(1) \subset U(N_f)$ part of the gauge group, reduces to[48]

$$S_{D7} = -\frac{2\pi^2 \mathrm{Vol}(\quad^4) N_f}{(2\pi)^7(\alpha')^4 g_s}\int\limits_0^\infty \mathrm{d}\rho\rho^3\sqrt{1 + L'(\rho)^2}\,, \tag{2.145}$$

where the $2\pi^2$ is just the volume of the unit radius three sphere coming from the angular integrations,

$$\mathrm{Vol}(\mathrm{S}^3) = \int\limits_0^{2\pi}\mathrm{d}\beta\int\limits_0^{2\pi}\mathrm{d}\gamma\int\limits_0^{\pi/2}\mathrm{d}\psi\sin\psi\cos\psi = 2\pi^2\,. \tag{2.146}$$

The action is formally divergent because of the volume of Minkowski space appearing, so this infinite factor should be divided off and one should work with the action density only. This procedure is understood to be applied in the rest of this thesis whenever applicable. Note that there is no contribution from the Wess-Zumino term, since there is neither a $B_{\mu\nu}$ field in the background nor a $F_{\mu\nu}$ on the brane excited. Excitation of either of them would break some of the global symmetries. We will see examples of this symmetry breaking mechanism in chapters 3 an 4.

The Euler-Lagrange equation for $L(\rho)$ derived from eq. (2.145),

$$0 = \left(\frac{\rho^3 L'(\rho)}{\sqrt{1 + L'}}\right)'\,, \tag{2.147}$$

[48]The $U(1) \subset U(N_f)$ sector of the DBI dynamics describes the center-of-mass fluctuations of a stack of D7 branes, i.e. the collective fluctuations of all D7 branes together. Embeddings which excite different branes differently are also useful, for example in the study of isospin chemical potentials.

which is readily integrated once to yield

$$0 = \frac{\rho^3 L'(\rho)}{\sqrt{1+L'}} = c \quad \Rightarrow \quad L'(\rho)^2 = \frac{c^2}{\rho^2 - c^2}\,. \tag{2.148}$$

L' has a pole at some finite value of ρ, unless $c = 0$. Since L' diverges as ρ^{-3} there, L will diverge like ρ^{-2}. There are several reasons why $c \neq 0$ is unphysical: First, this pole in L would correspond to a D7 brane generating a throat with finite "mouth" diameter[49], a situation which is surely not wanted. Since the D7 branes in the flat space D brane construction were parallel to the D3 branes. In particular the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background space cannot create this effect, since it has no special features like curvature singularities at $\rho^6 = c^2$, and it has no additional free parameter which could correspond to the integration constant c. There is also a holographic reason for such divergences to be unphysical: Since we want to identify $r^2 = \rho^2 + L^2(\rho)$ as an energy scale in the dual field theory, $r^2(\rho)$ should monotonically increase with ρ. This is not the case if $L(\rho)$ diverges for small ρ, and thus such embeddings do not admit a valid renormalisation group flow interpretation. We thus set $c = 0$, and conclude that the valid embeddings must be flat,

$$L = 2\pi\alpha' m_q\,, \tag{2.149}$$

where the second constant of integration was interpreted as the quark mass times the inverse string tension. This interpretation can be understood as coming from the operator-field dictionary (similar to eq. (2.118)): From (2.148) one finds that for large ρ the embedding behaves as a constant plus a ρ^{-2} fall-off,

$$L = 2\pi\alpha' m_q + \frac{c}{\rho^2}\,. \tag{2.150}$$

In the light of the operator-field mapping (2.118) we interpret the constant as the quark mass, and c as the corresponding operator vacuum expectation value.[50] Since the quark mass should be the source for the operator, it is easily identified from (2.140) [260, 261]. It is the supersymmetric completion of the quark mass term[51] [52]

$$-\frac{N_f N_c}{\lambda}\frac{c}{(2\pi\alpha')^3} = \mathcal{O}_{m_q} = \frac{\delta S_{D3D7}}{\delta m_q} = \tilde{\psi}\psi + \tilde{q}\left(m_q + \sqrt{2}\phi_3\right)\tilde{q}^\dagger + q\left(m_q + \sqrt{2}\phi_3\right)q^\dagger + h.c.\,. \tag{2.151}$$

[49] Such solutions correspond to a D7 brane and an Anti-D7 brane joining up to form a throat [258]. Without a baryon chemical potential they are indeed thermodynamically disfavoured w.r.t. the constant embeddings, but at finite chemical potential they become important in the region $\mu < m_q$ [259].

[50] The full spectrum of operators and their mapping to supergravity fields was worked out in [96]. Not surprisingly, the operators fall into $\mathcal{N} = 2$ supersymmetric multiplets.

[51] Eliminating the factor of α' via $\alpha'^{-2} = 2\lambda$ (i.e. $R^4 = 1$) sets $\langle \mathcal{O}_{m_q} \rangle = -\frac{\sqrt{2\lambda}}{4\pi^3} N_f N_c c$. This is a convention often used in the literature, meaning that lengths are measured in terms of the AdS radius R instead of the string scale $\sqrt{\alpha'}$.

[52] We use the superspace conventions of [205], which are also used in the review [80].

The factor $(2\pi\alpha')^{-3}$ in the identification of c with $\mathcal{O}_{m_q}$ is necessary for dimensional reasons. The other factors and the minus sign come from careful holographic renormalisation, see e.g. the appendix of [261] for an example of the calculation. According to the quantum numbers of table 2.10, $\mathcal{O}_{m_q}$ has $U(1)_{\mathcal{R}}$ charge +2, and a vacuum expectation value will thus break the $U(1)_{\mathcal{R}} \equiv SO(2)_{89}$ symmetry. This is consistent with the picture of D7 branes being separated from the D3s in flat space.

The $U(1)_{\mathcal{R}}$ symmetry is acting with the same quantum numbers on the left-handed Weyl fermion ψ and the right-handed Weyl fermion $\tilde{\psi}$, and thus has the properties of a **chiral U(1) symmetry**. The spontaneous breaking of this symmetry by a vacuum expectation value $\langle\mathcal{O}_{m_q}\rangle$ is thus the analogue of the field-theoretic **chiral symmetry breaking mechanism**[53] in this supersymmetric toy model. As we have seen above by holographic means, the condensate c vanishes in the theory (2.140), i.e. such a spontaneous chiral symmetry breaking cannot occur in that theory. In fact, unbroken supersymmetry forbids the generation of such a condensate: Since it originates from an F term, giving it a vacuum expectation value would break supersymmetry. Accordingly, this phenomenon was first found in [87] by studying D7 probe branes in the nonsupersymmetric Constable-Myers [84] background. In chapter 3 I will show that switching on an external magnetic background field produces such a condensate and thus induces chiral symmetry breaking. It turns out that it is also possible to holographically calculate the spectrum mesons in the theory (2.140) by solving the linearised field equations for the embedding functions L and ϕ (and for the D7 brane gauge field A_a) after imposing appropriate boundary conditions. For the supersymmetric embedding here this was first done in [96]. The technicalities of this calculation will be presented in chapter 3, but it will turn out that the natural candidate for the **Goldstone boson of chiral symmetry breaking**, i.e. the analogue of the pions, is given by the operator dual to the angular coordinate ϕ. This is not surprising since it is exactly the shift in the ϕ angle, i.e. rotations in the 8-9 plane, which is broken by a spontaneous generation of a nontrivial profile for the radial coordinate $L(\rho)$.

At finite temperature, the quarks undergo another first order phase transition, the **meson melting transition** [87,99]. As the Hawking-Page transition encountered in section 2.3.5, this transition is also of geometrical nature: It is the transition between branes which do not touch the horizon of the black hole, and those which have an induced horizon on the world volume. The details of this transition will be explained in section 3.2.2, so I will just summarise the results here: Working in the Poincaré patch and with a slightly different coordinate system than (2.123), one can integrate the analogue of eq. (2.147) only analytically.[54] The result is shown in figure 2.10. It turns out that the physics is governed in this case by the dimensionless ration m_q/T, and thus in this figure the black hole horizon radius has been scaled to one. Technically, one fixes the quark mass at a large value of ρ together with the requirement that the first derivative of L vanishes at that value,

[53] Chiral symmetry breaking in field theories is reviewed in sec. 6.1 of [253].

[54] Analytic expressions for the embeddings can be found in the limits of very small or very large temperature.

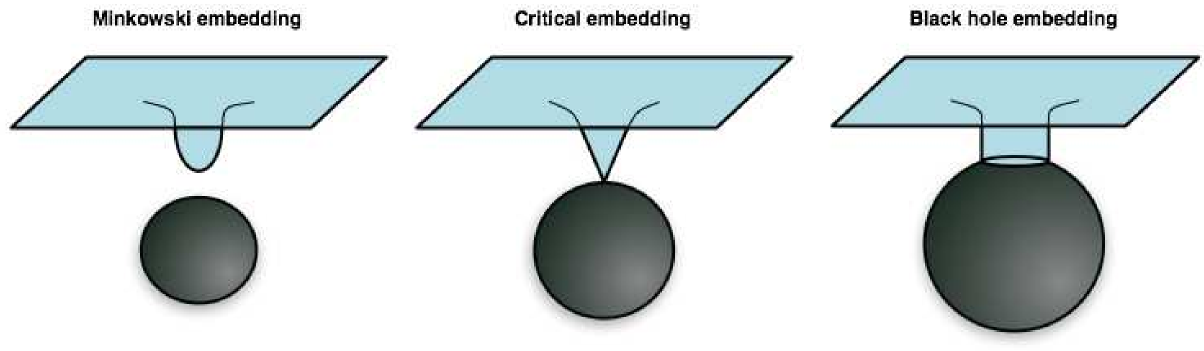


Figure 2.9: A sketch of the possible embeddings of a probe D7 brane into Poincaré AdS-Schwarzschild (figure taken from [262]).

and integrates the embedding equation numerically towards $\rho = 0$. One finds the two classes of embeddings: The "**Minkowski embeddings**" (depicted in blue) which reach $\rho = 0$ smoothly,[55], and "**black hole embeddings**" (depicted in green) which reach the horizon.Although not obvious from figure 2.10, there is no mass gap between the Minkowski and the black hole embeddings. Note that figure 2.10 only shows the physical embeddings for each value of m_q/T, i.e. the one with smallest free energy $F = -TS_{D7}|_{\text{onshell}}$. There is also a critical embedding which just touches the horizon, shown in the sketch figure 2.9. Once can now fit these curves with the formula (2.150) to obtain the value of the "quark condensate" $\langle \mathcal{O}_{m_q} \rangle (m_q/T)$. It turns out that this condensate is not a continuous function of m_q for any fixed temperature, but jumps exactly at the point where the physical Minkowski embeddings cease to exist and the black hole embeddings start, namely

$$\left(\frac{m_q}{T} \right)_{\text{crit.}} \approx 1.332 \,. \tag{2.152}$$

The condensate is thus an order parameter for the phase transition happening at this point. Note that it is not an order parameter in the usual sense that it is zero in one phase and nonzero in another, and the phase transition is connected with a change in symmetry of the system. The condensate is nonvanishing on both sides of the transition, it just has a jump. Also, in this case it is rather a change in topology than in symmetry: The Euclideanised embeddings would wrap a cycle $\mathrm{S}^1 \times \mathrm{S}^3$, where the S^1 is the Euclidean time circle, while the S^3 is the sphere wrapped by the D7 embedding. It turns out that for the Minkowski embeddings the S^3 shrinks to zero size while the Euclidean time circle stays finite, while for the black hole embeddings the (induced) Euclidean time circle shrinks to zero size since the brane reaches the horizon, but the S^3 stays finite. For the critical embedding, both cycles shrink to zero size. All these statements will be proven in chapter 3. That the transition is of first order is obvious from the relation of the free energy and the condensate: Since

[55]They have to reach this point with zero derivative $L'(0) = 0$ in order to prevent a conical singularity in the induced metric. This will be shown explicitly later in chapter 3.

the free energy is proportional to minus the properly holographically renormalised on-shell D7 brane action, the condensate is

$$\langle \mathcal{O}_{m_q} \rangle = -\frac{1}{T}\frac{\partial F(m_q, T)}{\partial m_q}\,, \tag{2.153}$$

i.e., up to a factor involving the temperature, the first derivative of the free energy. If the condensate has a discontinuity, the transition is thus of first order. That it is a meson melting transition can be seen by the following qualitative argument: For the Minkowski embeddings a small fluctuation propagating towards $\rho = 0$ has to be reflected there, since the coordinate ρ is a radial coordinate. This is one boundary condition. At infinity, the fluctuation has to fulfill another boundary condition, namely it has to be normaliseable in order to be dual to a field theory state (which in this case is a meson) [263,264]. These two conditions for a second order ODE on a half with oscillatory solutions line yield a discrete spectrum for the oscillation frequencies, which are the meson masses $M^2 = -k_\mu k^\mu$.[56] For the black hole embeddings one still searches for normaliseable modes, but now the fluctuation can fall into the black hole and excite it. The allowed spectrum will thus be given by the oscillation frequencies of the black hole, which are quasinormal modes (see e.g. the nice book [267]), which in general have an imaginary part and thus correspond to unstable excitations. The mesonic excitations thus will decay in the high temperature or black hole phase. Since the gluons of $\mathcal{N} = 4$ SYM theory are already in a deconfined phase in the Poincaré patch, the transition is thus a transition between stable mesons in a deconfined $\mathcal{N} = 4$ plasma and unstable quasiparticle excitations in the same plasma. At the phase transition the mesons thus "melt" in the hot $\mathcal{N} = 4$ plasma.

So far we have already extended the AdS/CFT correspondence quite far, to include temperature, quark like degrees of freedom and to describe phase transitions for example, but we have not broken Lorentz invariance explicitly (except at finite temperature). In the next chapter I will show how to introduce background electric and magnetic fields which break Lorentz invariance explicitly, and investigate their effects on the supersymmetric quarks.

[56]More exactly, in the 't Hooft limit the decay width of these particles is suppressed width N_c^{-1} and thus zero in the strict $N_c \to \infty$ limit. It can however receive contributions nonanalytically in $\sqrt{\lambda}$ via worldsheet instantons [265, 266].

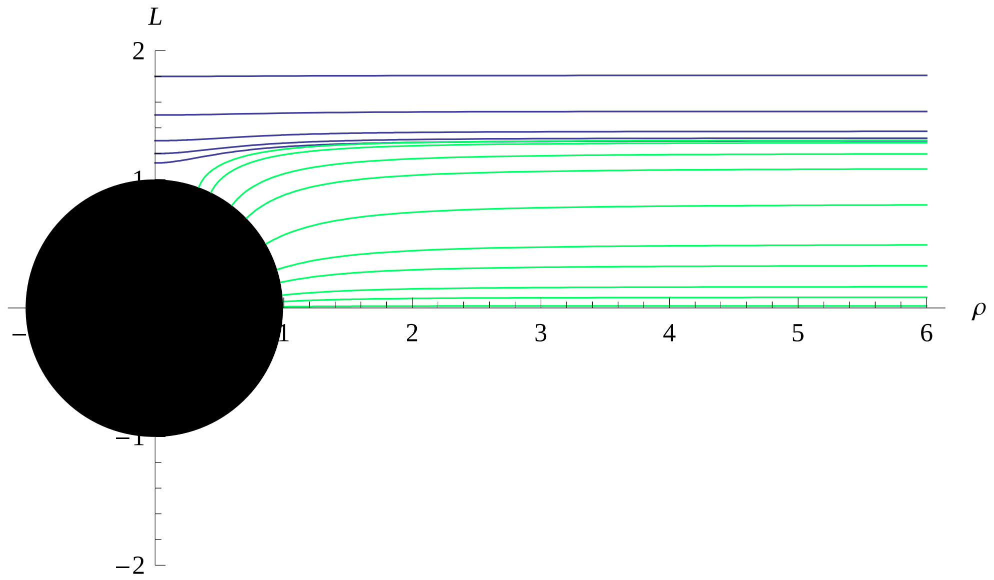


Figure 2.10: D7 probe embeddings into the Poincaré AdS black hole.

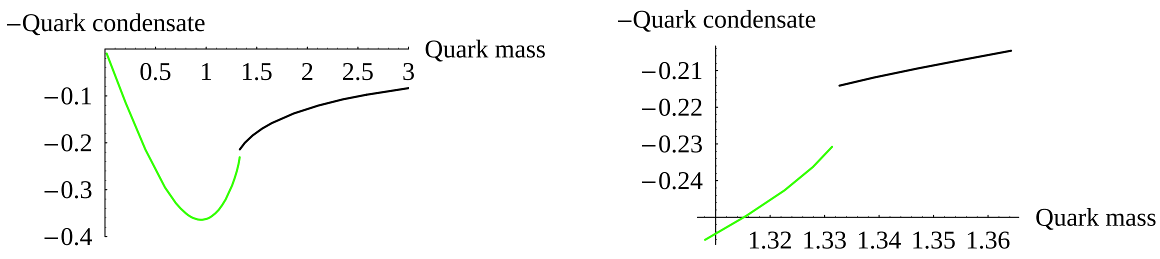


Figure 2.11: Condensate $\langle \mathcal{O}_{m_q} \rangle$ versus quark mass at finite temperature, both in the full mass range and near the meson melting transition.

Chapter 3

Holographic Quarks in External Electric and Magnetic Fields

3.1 Introduction and Summary

In the last chapter, two extensions of the original AdS/CFT correspondence were presented: Considering type IIB supergravity on an AdS-Schwarzschild black hole space-time rather than $\mathrm{AdS}_5 \times \mathrm{S}^5$ corresponded to a dual field theory at finite temperature; and the addition of fundamental matter using probe branes and the breaking of supersymmetry due to finite temperature has allowed a study of QCD-like theories and many measurable quantities and interesting phenomena such as an analogue of chiral symmetry breaking.[1] As an example, I explicitly discussed the embedding of probe D7 branes [95] into both the zero and finite temperature background. In particular, the gravity dual constructions describing spontaneous chiral symmetry breaking have been found in [87, 88, 254, 268]. In principle these results may be tested against results from the light-quark sector of QCD, and qualitative agreement has been reached e.g. in the details of the chiral symmetry breaking dynamics: The Goldstone modes found holographically fulfill the Gell-Mann-Oakes-Renner relation [87] and the chiral Lagrangian can be computed [268]. Furthermore, meson mass ratios were found to match, the order of magnitude of the energy densities of the $\mathcal{N} = 4$ plasma and matches values from lattice QCD, and the magnitude of the shear viscosity to entropy density ratio of the quark gluon plasma measured at RHIC was predicted from a gravity calculation [269] even before it was measured – see the reviews [49, 158, 253, 270, 271] for a guide to the literature. In particular the evidence suggests that the quark-gluon plasma of QCD is in many respects very similar to the $\mathcal{N} = 4$ plasma, which is believed to be due to universal features of strongly coupled field theories. The exploration of the phase structure of this sector is ongoing [87, 99, 101, 257, 259, 262, 272–279].

[1] In this chapter the term "chiral symmetry breaking" is referring to the breaking of the $U(1)_{\mathcal{R}}$ symmetry by developing a vacuum expectation value for the operator $\mathcal{O}_{m_q}$, see eq. (2.151).

In this chapter I consider the effect of an external Kalb-Ramond $B_{\mu\nu}$ field for D7 branes embedded in the dual gravity background. My motivation to look at such configurations originates from the investigations in [280, 281], where D7 brane probes were embedded into the Polchinski-Strassler background [85]. In this background, a Kalb-Ramond field of quite complicated structure, depending on the radial coordinate of the dual space, is turned on in all six directions perpendicular to the boundary. In the original work of Polchinski and Strassler [85] this has been shown to correspond to mass terms for the adjoint chiral multiplets in the dual gauge theory. For the theory with added flavour, this implies a repulsion of the D7 branes by the shell forming in the background due to the Myers effect [282]. This leads to a shift of the meson spectrum induced by the adjoint masses, which was worked out to second order in the adjoint masses in [280].

It is also quite natural to try to break Lorentz invariance in a further generalisation of the AdS/CFT correspondence by introducing additional background fields. In contrast to the work of Polchinski and Strassler, in [105] such an attempt was made by turning on a pure-gauge magnetic Kalb-Ramond field in two spatial directions of the D3 brane worldvolume, parallel to the boundary. This was found to induce chiral symmetry breaking and a Zeeman splitting of the meson states. This setup was investigated further in [283]. A related approach using both external magnetic and electric fields was used in [284–286] to calculate conductivities and to study the holographic Hall effect (for related work in 2+1 dimensions see also [287]).

In the present chapter, I consider both external magnetic and electric fields separately. In the electric case, the Kalb-Ramond field extends into the temporal direction and one spatial direction parallel to the boundary. I begin by considering the magnetic field of [105] in the AdS-Schwarzschild background dual to a finite-temperature field theory, and study the phase structure of chiral symmetry breaking for one fundamental hypermultiplet in this background by embedding a probe D7 brane in it. I find two competing mechanisms at work: As first discussed in [87], the black hole attracts the D7 brane which bends towards it. For very large values of the quark mass in units of the temperature, as determined by the UV boundary value of the D7 embedding, this leads to a very small value of the quark condensate. Moreover, for brane probes ending on the black hole, there is a phase in which mesons are unstable and melt in the $\mathcal{N} = 4$ plasma. As argued in the last section, the melting transition for the mesons occurs when the D7 brane probe reaches the black hole horizon. On the other hand, as discussed in [105], the magnetic Kalb-Ramond field leads to spontaneous chiral symmetry breaking, since the quark condensate is large even at zero quark mass. This is essentially due to the fact that the magnetic field has the effect of repelling the D7 probe from the origin. I will argue below that for sufficiently large magnetic field, the second mechanism is stronger than the first one and find spontaneous chiral symmetry breaking even in the black hole background. A critical line in the temperature-field phase diagram illustrates the interplay between the two effects: The magnetic field acts by repelling the embeddings from the horizon. Above a critical value for the field strength (at a fixed temperature) this repulsion is so strong that no embedding, not even the embedding corresponding to massless quarks, can flow into the black hole any more - all embeddings are Minkowski embeddings for which the meson spectrum is discrete. For fixed mass and varying temperature, this is equivalent to stating that for increasing magnetic field the critical temperature for the melting phase transition increases. I also investigate the meson spectrum for this scenario and find that a Goldstone mode occurs above the critical magnetic field value, in agreement with spontaneous chiral symmetry breaking.

In section 3.3, I consider the case where the Kalb-Ramond field is turned on in spatio-temporal directions, corresponding to an **external electric field** in the gauge theory. In this case, there is a singular region with topology S^5 (henceforth often denoted "singular shell") where the Dirac-Born-Infeld action for the D7 brane has a zero and becomes complex inside this region. It is necessary to turn on a gauge field on the brane in order to ensure a regular action beyond the shell where the brane action vanishes, similar to [284]. The singular shell of vanishing brane action has an attracting effect on the D7 brane probes, as opposed to the repulsion observed in the magnetic case. This may be interpreted as the holographic incarnation of an ionisation effect. I provide evidence that the singular shell acts similarly to a black hole horizon: Between Minkowski embeddings which do not reach the singular shell, and embeddings which flow into it, a phase transition characterised by the jump of the vacuum expectation value $\langle \mathcal{O}_{m_q} \rangle$ occurs. I will also give evidence that the branes flowing into the singular shell can not support stable excitations, and interpret this as the dissociation of mesonic bound states in the electric field. The phase transition is thus a **meson dissociation transition**.

Far away from this region, i.e. in the weak field limit at zero temperature, I investigate a particular branch of the pseudoscalar meson spectrum both analytically and numerically and find a mass shift for the pseudoscalar mesons $\delta M \sim B^2$. This is very similar to the second-order Stark effect for atoms in electric fields, where the energy levels of s-orbitals are shifted by an amount proportional to the square of the applied field strength. For the analytical calculation, I perform a perturbative analysis in the external field to lowest order.

However, I also consider the case of a general, not necessarily small, electric field strength. After obtaining a regular action as in [284] by introducing gauge fields on the brane, I find that both at zero and finite temperature, regular embeddings exist which pass through the shell of vanishing action. The employed ansatz for the world volume gauge field corresponds to quark number densities and currents in the dual gauge theory, and I am considering the canonical ensemble in which the quark number density is fixed. Minkowski embeddings can only exist if the quark density vanishes [262], and this is the situation I will be mostly interested in. The physics at finite quark number densities is largely unknown, but I will comment on it in chapter 6. Besides Minkowski embeddings, I find two different classes of embeddings at zero temperature: Embeddings that reach all the way to the extremal horizon of AdS space, and, at quark masses between the Minkowski embeddings and the ones ending at the extremal AdS horizon, embeddings which end in a conical singularity, the latter of which were first noticed in [288]. The meaning and fate of these conical singularities is discussed in chapter 6.

At finite temperature and for vanishing quark densities we find, in order of decreasing asymptotic quark mass, Minkowski embeddings, conically singular embeddings and those that fall into the black hole. The physical meaning of the conically singular embeddings, in particular their stability and role in this phase transition, remains to be investigated. Nevertheless they are important since they cover a finite range of quark masses, and also, as can be seen from figure 3 of [288], the first order phase transition is a transition between a Minkowski embedding and a conically singular one.

At finite quark number density and temperature, black hole embeddings exist covering the whole range of asymptotic quark mass. As Minkowski embeddings are inconsistent in that case, the dissociation phase transition should then occur between different black hole embeddings, similar to the situation in the canonical ensemble without electric field [262].

This chapter is organised as follows: Section 3.2 is devoted to the magnetic field and some review of well-known facts from the literature. I begin by considering the general ansatz for both magnetic and electric fields in section 3.2.1, and comment on some aspects of the magnetic case at zero temperature in particular. In section 3.2.2 I consider the magnetic case at finite temperature and show that spontaneous chiral symmetry breaking occurs even at finite temperature above a critical value of the magnetic field strength. In section 3.3 I consider the electric case. In particular, I derive the meson mass shift which corresponds to the Stark effect. Moreover, I discuss the behaviour of the embeddings for general, not necessarily small, values of the Kalb-Ramond field. I conclude briefly with a discussion of the instability which potentially occurs for the electric Kalb-Ramond field, and differences between the canonical and grand canonical ensemble in the electric case. Some calculational details are relegated to the appendices B.1 and B.2.

The material presented in this chapter was obtained in collaboration with and published in a paper together with Johanna Erdmenger and Jonathan P. Shock [1]. In particular the numerical calculations were carried out by Jonathan P. Shock.

3.2 Fundamental Matter in Constant External Magnetic Fields

3.2.1 Constant External Magnetic Fields and Zero Temperature

In [105] a pure gauge magnetic Kalb-Ramond field was added to the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry and quenched fundamental matter was included using a D7 brane probe. The D7 brane embeddings were calculated and it was found that chiral symmetry is spontaneously broken, as indicated by a nonzero quark condensate at vanishing quark mass. The spectrum of mesons was also calculated both for small and large magnetic field strength and the Goldstone mode was found to satisfy the Gell-Mann-Oakes-Renner relation [289].

In this section I first review the main results of [105]. Moreover, I simultaneously consider a new ansatz for the Kalb-Ramond field, corresponding to an electric field.

General ansatz

In this section I deal in parallel with the magnetic and electric ansätze for the Kalb-Ramond field.

The background of interest is the pure AdS geometry in the Poincaré patch, given by

$$ds^2 = \frac{\omega^2}{R^2}\left(dx_0^2 + d\vec{x}^2\right) + \frac{R^2}{\omega^2}\left(d\rho^2 + \rho^2 d\Omega_3^2 + dL^2 + L^2 d\Phi^2\right). \tag{3.1}$$

Here the AdS radial coordinate is given by $\omega^2 = \rho^2 + L^2$. As before, for the S^3 I use Hopf coordinates

$$d\Omega_3^2 = d\psi^2 + \cos^2\psi d\beta^2 + \sin^2\psi d\gamma^2 . \tag{3.2}$$

In addition, the background involves the usual four-form and constant dilaton

$$C_{(4)} = \frac{\omega^4}{R^4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 , \quad e^\phi = g_s , \quad R^4 = 4\pi g_s N \alpha'^2 . \tag{3.3}$$

There are two obvious choices of a pure gauge ansatz for the Kalb-Ramond fields[2],

$$B_{\mathrm{mag}} = B dx_2 \wedge dx_3, \qquad B_{\mathrm{el}} = B dx_0 \wedge dx_1. \tag{3.4}$$

I will call these Ansätze the magnetic and electric ansatz respectively, since, after embedding a D7 brane in the background and trading the constant B field for the $U(1)_F$ gauge field on the brane via a Ramond-Ramond gauge transformation to obtain $F_{\mu\nu} = -\frac{B_{\mu\nu}}{2\pi\alpha'}$, the two choices correspond to constant electric and magnetic field strengths. Provided that $C_2 = 0$, which ensures the absence of a boundary term $\mathrm{d}(C_2 \wedge B)$, both these pure-gauge ansätze are solutions to the IIB supergravity equation of motion (see eg. [85]), since B only enters the supergravity equations of motion through the three-form field strength $H = \mathrm{d}B = 0$, which is vanishing. The B field thus does not deform the $\mathrm{AdS}_5 \times \mathrm{S}^5$.

The additional fields introduced in this way a priori couple to the $U(1)_q \subset U(N_f)$ quark number symmetry, see table 2.10. But is their effect equivalent to an actual constant electric or magnetic background field? To answer this question one first needs to identify a

[2]A third choice, as pointed out to the author by P. Aschieri, would be a lightlike B field. It has the advantage that for spacelike (i.e. magnetic) or lightlike B, there is a decoupling limit leading to a noncommutative field theory [111, 290], while in the electric or timelike case the best one can do is to decouple the closed string modes [291–293]. This then leads to so-called noncommutative open string theory.

proper $U(1)$ factor in the gauge group of the $\mathcal{N}=2$ theory. The obvious choice is the trivial $U(1) \subset U(N_c)$ factor which is not described by supergravity fields in $\mathrm{AdS}_5 \times \mathrm{S}^5$ but by singleton degrees of freedom living directly at the boundary and which does decouple from the nonabelian sector in the $\mathcal{N}=4$ theory [294, 295]. However, from table 2.10 one can see that the chiral multiplets Q and $\tilde{Q}$ have respective $U(1)_q$ charge ± 1. From eq. (2.140) one finds that Q and $\tilde{Q}$ have respective $U(1) \subset U(N_c)$ charge ± 1, too. The so-introduced background fields thus couple to quark number as real electric or magnetic background fields would couple to the electromagnetic $U(1)$ charge and one can thus expect the effects to be equivalent. This will be confirmed by the results described in the following.

In this chapter, I will always choose static gauge $\xi^a = (x^\mu, \rho, \psi, \beta, \gamma)$, for the D7 brane, and parametrise its embedding by an ansatz $L = L(\rho)$, $\Phi = 0$, thus preserving the rotational symmetry of the S^3 wrapped by the D7 brane. The rotational symmetry perpendicular to the D7 brane will be broken for nonzero embeddings L. In the background (3.1) and for the two B fields (3.4), the DBI action is given by

$$\mathcal{L}_{m,e} = -2\pi^2 \frac{\mu_7}{g_s} \rho^3 \sqrt{1+L'^2} \sqrt{1 \pm \frac{R^4 B^2}{(\rho^2+L^2)^2}}\,, \tag{3.5}$$

where the positive (negative) sign corresponds to the magnetic (electric) ansatz. Note that the angle ψ from the Hopf fibration (3.2) disappeared after integration over the wrapped S^3, which yields the three-sphere volume $2\pi^2$. The Wess-Zumino part of the D7 action does not contribute, as $P[C_4] \wedge B_{(2)m,e} = 0$. The magnetic dual of C_4 [105]

$$\tilde{C}_4 = R^4 \frac{2\rho^2 + L^2}{(\rho^2+L^2)^2} L^2 \sin\psi \cos\psi \mathrm{d}\psi \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma \wedge \mathrm{d}\phi \tag{3.6}$$

gives rise to a pull-back

$$P[\tilde{C}_4] = R^4 \frac{2\rho^2 + L(\rho)^2}{(\rho^2+L(\rho)^2)^2} L(\rho)^2 \sin\psi \cos\psi (\partial_\mu \phi) \mathrm{d}\psi \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma \wedge \mathrm{d}x^\mu\,, \tag{3.7}$$

which thus vanishes for embeddings with constant 8-9 plane angle ϕ. It is however exactly this contribution which gives rise to couplings of ϕ-fluctuations and brane gauge fields in the calculation of the meson spectra, as will become clear later on.

The D7 brane embedding is found by solving the Euler-Lagrange equation for $L(\rho)$,

$$0 = \partial_\rho \left(\frac{\rho^3 L' \sqrt{1 \pm \frac{B^2 R^4}{(\rho^2+L^2)^2}}}{\sqrt{1+L'^2}} \right) \pm \frac{2B^2 R^4 \rho^3 L \sqrt{1+L'^2}}{(\rho^2+L^2)^3 \sqrt{1 \pm \frac{B^2 R^4}{(\rho^2+L^2)^2}}}\,, \tag{3.8}$$

which is a scalar from the point of view of the world volume field theory. In both the magnetic and electric cases, the UV (i.e. large ρ) behaviour of the embeddings is given by

$$L(\rho) \sim m + \frac{c}{\rho^2}\,, \tag{3.9}$$

i.e. the embeddings asymptote to the pure $AdS_5 \times S^5$ solution $L = m$ for $\rho \to \infty$. The quantity m is proportional to the quark mass, $m_q = m/2\pi\alpha'$, while c is related to the chiral condensate via $\langle \mathcal{O}_{m_q} \rangle = c/(2\pi\alpha')^3$. As discussed in [105], supersymmetry is broken on the brane, though, by virtue of the probe limit, not in the background: On the brane, the magnetic B field breaks the Lorentz symmetry from $SO(1,3)$ to $SO(1,1) \times SO(2)$ and thus also breaks supersymmetry. *A priori* there might be a possibility that some supersymmetry is preserved though.[3] We will see in the next section that there is no room for this: The embedding of a D7 brane at zero temperature in the presence of the magnetic field shows spontaneous chiral symmetry breaking and thus has to be nonsupersymmetric. At finite temperature there can exist a flat embedding for $m_q = 0$ if the magnetic field is sufficiently small (see below), but in this case already the temperature breaks the supersymmetry. For the electric field, which does not induce spontaneous chiral symmetry breaking, i.e. the embedding is flat for zero quark mass. In this case a calculation of the κ-symmetry along the lines of [297] shows the absence of supersymmetry [298]. The nonflatness of the embeddings we find numerically at finite quark mass, i.e. the nonvanishing condensate, shows the supersymmetry breaking for $m_q > 0$. Since through the Wess-Zumino coupling

$$\mu_7 \int B \wedge C_6 \,,$$

D1 brane charge is sourced by the D7 brane, and the C_6 field could break the supersymmetry of the $AdS_5 \times S^5$ background. This is however a backreaction effect of the brane sourcing background fields, and thus is suppressed in the large-N_c, fixed N_f limit. However, since we are mostly interested in the brane physics, the breaking of supersymmetry on the brane is of relevance here. The induced C_6 shows this in both the electric and magnetic case, since the D1-D3 system is known to be nonsupersymmetric.

Magnetic Kalb-Ramond field at zero temperature

I now review the results of [105] for the magnetic case at zero temperature. Some examples for the embedding in the magnetic case with varying IR boundary are shown in figure 3.1. For completeness I have also shown solutions with negative quark mass as fixed by the UV asymptotic behaviour. For small quark mass the embeddings intersect the ρ axis, in some cases several times. I will show that these solutions describe a well-behaved renormalisation group flow, in contrast to the argument in [105]. Nevertheless these solutions are ruled out by an energy argument which shows that they are not the lowest-energy configurations. Although the D7 brane does cross the ρ axis multiple times for small quark masses, it does not indicate a multiple intersection with the D3 brane stack. The distance of the D7 brane from the origin of the (ρ, L) plane, which is given by $\sqrt{\rho^2 + L(\rho)^2}$, is monotonically decreasing as the solution flows towards the IR. This can be seen in the right hand graph of figure 3.1.

[3]Usual no-go theorems against partial breaking of global supersymmetry [296] may not apply here, since Lorentz invariance is broken.

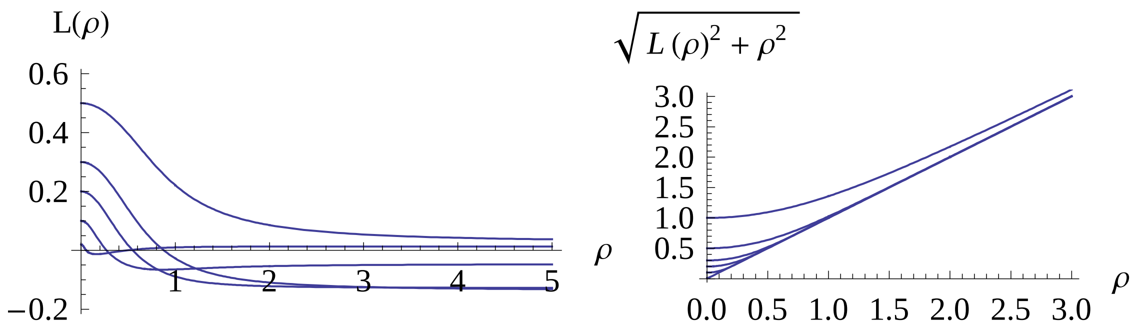


Figure 3.1: D7 brane embeddings in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with external magnetic field.

In the normalisation of [105], for which I perform a rescaling $\rho \to R\sqrt{B}\rho$, $L \to R\sqrt{B}L$, the equation of motion is given by

$$0 = \partial_\rho \left(\frac{\rho^3 L' \sqrt{1 + \frac{1}{(\rho^2+L^2)^2}}}{\sqrt{1+L'^2}} \right) + \frac{2\rho^3 L \sqrt{1+L'^2}}{(\rho^2+L^2)^3 \sqrt{1+\frac{1}{(\rho^2+L^2)^2}}} \,. \tag{3.10}$$

The UV asymptotics of the solutions is

$$\tilde{L}(\tilde{\rho}) = \tilde{m} + \frac{\tilde{c}}{\tilde{\rho}^2} \,, \quad \tilde{m} = \frac{2\pi\alpha' m_q}{R\sqrt{B}} \,, \quad -\tilde{c} = \left\langle \mathcal{O}_{m_q} \right\rangle \frac{\lambda}{N_f N_c} \frac{(2\pi\alpha')^3}{R^3 B^{\frac{3}{2}}} \,. \tag{3.11}$$

The extra factor of B in the normalisation is convenient in the zero temperature case. Note that B is, as the metric G, dimensionless. Using (3.11), one can extract the rescaled mass and condensate from the numerically determined D7 embeddings. It was found in [105] that $\tilde{c}(\tilde{m})$ shows a discrete self-similar spiral behaviour near the zero mass embedding, which was further investigated in [283, 299]. This indicates that for a given quark mass, there may be more than one solution. It is thus important to find the physical embeddings, which will be done by an energy argument in the following.

It was stated in [105] that the inner arms of the spiral intersect with the D3 branes multiple times, and thus only the outer portion of the spiral in the lower right quadrant of figure 3.2 is physical. I now show that this conclusion may be found through simple energy considerations. In order to determine which of the degenerate solutions is the physical one, I calculate the energy of each solution. The energy of each one has a UV divergence which must be removed by an appropriate normalisation. Since I am interested in only comparing the energies and not, for example, in thermodynamics of these branes, it is sufficient for this argument to impose a cutoff Λ, which is sent to infinity at the end, and subtract a

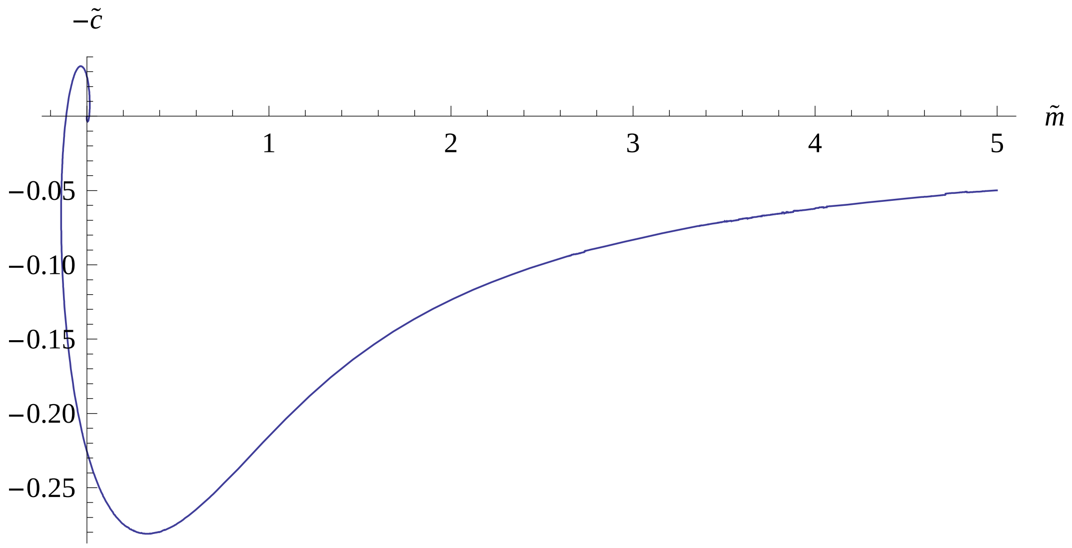


Figure 3.2: Reduced condensate versus reduced mass $-\tilde{c}(\tilde{m})$ for the magnetic ansatz at zero temperature.

reference solution [4],

$$E_{norm} = \int_{\rho_0}^{\Lambda} d\rho \rho^3 \sqrt{1 + L_0'(\rho)^2} \sqrt{1 \pm \frac{B^2 R^4}{\left(\rho^2 + L_0(\rho)^2\right)^2}} - E_{ref} \ , \tag{3.12}$$

where

$$E_{ref} = \int_{\rho_0}^{\Lambda} d\rho \rho^3 \sqrt{1 + L_{ref}'(\rho)^2} \sqrt{1 \pm \frac{B^2 R^4}{\left(\rho^2 + L_{ref}(\rho)^2\right)^2}} \ . \tag{3.13}$$

Here L_0 is the classical solution to the embedding equation of motion. The lower integration limit ρ_0 is zero for the magnetic case at zero temperature, as in that case the D7 brane fills the whole range $0 \leq \rho < \infty$. We are using the physical $m = 0$ solution as a reference solution. This is just a convenient choice, as a shift of the normalised energy does not change the relative energies between physical and unphysical embeddings.

The $-\tilde{c}(\tilde{m})$ spiral in figure 3.2 cuts the $\tilde{m} = 0$ axis an infinite number of times, dividing the spiral in different branches in each quadrant of the $(\tilde{m}, \tilde{c})$-plane. The physical embeddings are those which have the lowest energy according to (3.12). Figure 3.3 shows the $-\tilde{c}(\tilde{m})$-spiral (green line) along with the corresponding normalised energy $E_{norm}(\tilde{m})$ (blue line). The dashed lines link $\tilde{m} = 0$ points on the condensate curve which correspond to massless

[4] As the energy for our static D7 brane configuration is just the negative of the action, I drop here all volume factors arising from integration over $(x^\mu, \psi, \beta, \gamma)$. Note that here I am not considering a holographic renormalisation and regularisation which is necessary for the correct calculation of the free energy and thermodynamic quantities [300]. In the coordinate system used in this work, no counterterms including m or c are necessary to cancel the large volume divergence $\propto \Lambda^4$, which is achieved by our subtraction method as well.

embeddings, with the corresponding points on the normalised energy curve. The numbers next to the dashed lines are the values of the normalised energies for these massless embeddings. We find that the lowest energy configuration for zero quark mass is the one where the lowest branch of the condensate curve in the bottom right quadrant of figure 3.3 intersects the $-\tilde{c}$-axis. The corresponding normalised energy is $E_{norm} = -0.007773$. It is also clear from figure 3.3 that the energy for the embeddings on this branch is indeed smaller than for all the other branches.[5] The lowest lying branch in the bottom right quadrant of figure 3.3 thus corresponds to the physical embeddings, while the other branches have to be considered unphysical. As this physical branch admits a nonzero quark condensate $-\tilde{c}(0)$ at zero quark mass, spontaneous breaking of the $U(1)$ chiral symmetry is possible in the presence of the magnetic Kalb-Ramond field.

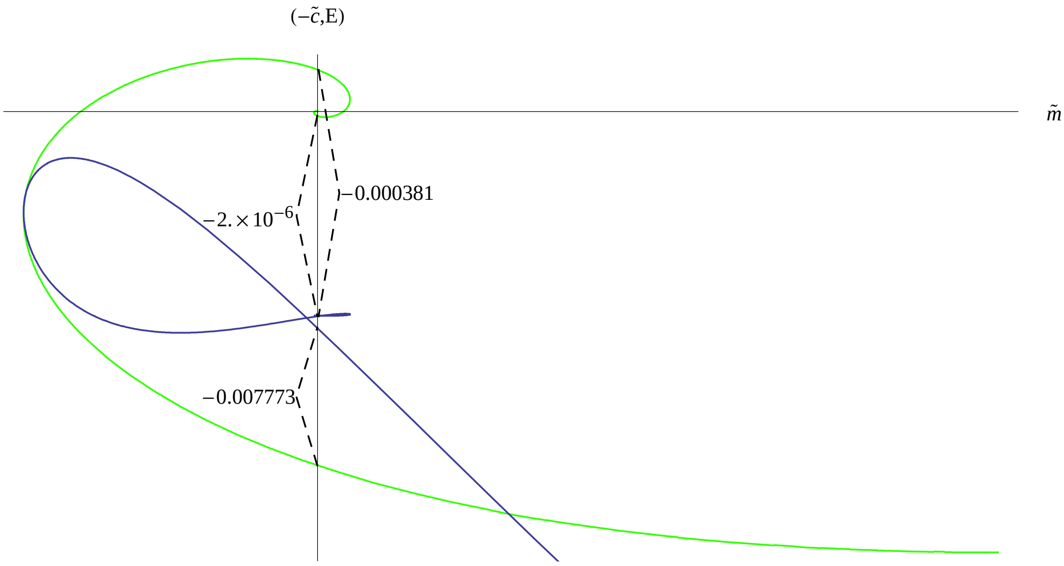


Figure 3.3: Reduced condensate versus reduced mass $-\tilde{c}(\tilde{m})$ (green line) for the magnetic ansatz and normalised energy $E_{norm}(\tilde{m})$ (blue line). The numbers correspond to energies, and the dashed lines link corresponding points on the two curves (see text). Note that the origin of the $\tilde{c}$ axis does not coincide with the origin of the E_{norm} axis.

[5]From the figure 3.3 it might appear that the big "loop" the blue curve is making has lower energy at its bottom than $E_{norm} = -0.007773$. As we checked, this is not the case. In any case, this loop would correspond to negative masses, which would correspond to D7 branes sitting on the other side of the D3 branes and thus could not be observed in the Poincaré patch.

3.2.2 Constant External Magnetic Fields and Finite Temperature

The embedding of D7 brane probes into the AdS-Schwarzschild black hole background, dual to a finite-temperature field theory, was first studied in [87]. It was found that in the black hole background no spontaneous chiral symmetry breaking by a quark condensate occurs, i.e. the vacuum expectation value $\langle \mathcal{O}_{m_q} \rangle$ vanishes for zero quark mass. On the other hand, there is an interesting first order phase transition [99] when the embedded D7 brane reaches the horizon. This has been studied in further detail by many authors [257,272–274], including the effect of a quark chemical potential and of a finite quark number density [259,276–278]. The embeddings which terminate before reaching the horizon do so because the S^3 which they wrap shrinks to zero size, as discussed in [95]. The phase transition corresponds to the transition from the mesonic to the molten phase [101]. As discussed in section 2.3.6 in detail, in the former phase there is a discrete meson spectrum with a mass gap, whereas in the latter the spectrum becomes continuous. The mesons melt at this phase transition due to the interaction with the hot $\mathcal{N} = 4$ plasma. Subsequently, I will refer to D7 embeddings reaching the black hole horizon as black hole embeddings or as “in a molten phase”, while those which do not are named “mesonic phase” or Minkowski ones.

I now consider the effects of the Kalb-Ramond field (3.4) in the Poincaré patch of the AdS-Schwarzschild background that is dual to a finite temperature field theory. The metric in Minkowski signature is given by ($\omega^2 = \rho^2 + L^2$)

$$ds^2 = \frac{\omega^2}{2R^2}\left(\frac{\mathrm{d}\vec{x}^2\,(b^4+\omega^4)}{\omega^4} - \frac{\mathrm{d}t^2\,(\omega^4-b^4)^2}{\omega^4\,(b^4+\omega^4)}\right) + \frac{R^2}{\omega^2}\left(\mathrm{d}L^2 + \mathrm{d}\rho^2 + \mathrm{d}\Phi^2 L^2 + \rho^2 \mathrm{d}\Omega_3^2\right) . \quad (3.14)$$

Here a slightly different radial coordinate

$$|\vec{y}| = \omega = r\left(1 + \sqrt{1 - \frac{r_H^4}{r^4}}\right)^{\frac{1}{2}} , \quad b^4 = r_H^4 \quad (3.15)$$

was used compared to (2.123). The dilaton is constant for the black (i.e. nonextremal) D3 brane solution, whose near-horizon geometry is just (3.14) [91]. The horizon of the black hole (3.14) is now located at

$$\omega_H^2 = \rho_H^2 + L_H^2 = b^2 .$$

The temperature of the dual field theory is given by the Hawking temperature of the black hole horizon

$$T = \frac{b}{\pi R^2} . \quad (3.16)$$

With the same static gauge as before, the DBI Lagrangian for a D7 brane probe in this background reads for the magnetic (upper sign) and electric (lower sign) ansatz

$$\mathcal{L} = -\,2\pi^2 \frac{\mu_7}{g_s} \frac{\tilde{\rho}^3 \left((\tilde{L}^2 + \tilde{\rho}^2)^2 \pm 1 \right)}{4(\tilde{L}^2 + \tilde{\rho}^2)^4} \times \\ \times \sqrt{\left(\left(1 \mp (\tilde{L}^2 + \tilde{\rho}^2)^2\right)^2 \mp \tilde{B}^2 (\tilde{L}^2 + \tilde{\rho}^2)^2 \right) (\tilde{L}'^2 + 1)} \,. \tag{3.17}$$

Here I have introduced rescaled dimensionless quantities

$$(L, \rho, B) = (b\tilde{L}, b\tilde{\rho}, \frac{\tilde{B}b^2}{2R^2}) \,, \tag{3.18}$$

such that the horizon is at $\tilde{\rho}^2 + \tilde{L}^2 = 1$.

Let us now turn to the **magnetic field case**. We numerically solve the Euler-Lagrange equation for $\tilde{L}(\tilde{\rho})$ obtained from the Lagrangian (3.17). For this purpose it is convenient to introduce another rescaling for the B field (the physical magnetic field strength is $|\vec{B}| = B/(2\pi\alpha')$ – it has dimension $mass^2$)

$$\hat{B} = \frac{B\lambda}{2\pi^2 R^2 m_q^2} = \frac{|\vec{B}|\sqrt{\lambda}}{\sqrt{2}\pi m_q^2} \,, \tag{3.19}$$

as well as an appropriate dimensionless quark mass and condensate

$$m_q = \frac{1}{2}\sqrt{\lambda} T \tilde{m} \,, \qquad \tilde{T} = \frac{1}{\tilde{m}} \,, \qquad \langle \mathcal{O}_{m_q} \rangle = -\frac{1}{8}\sqrt{\lambda} N_c T^3 \tilde{c} \,. \tag{3.20}$$

Here $\lambda = g_s N_c$ is the 't Hooft coupling. Note that the factor of T^3 comes from the rescaling (3.18).

The numerical results are plotted in figure 3.4 for increasing values of the magnetic field at a fixed temperature (or equivalently fixed Schwarzschild radius). We see that the increasing external magnetic field repels the branes from the horizon further and further, until there are no black hole solutions any more. This is exactly the point where the molten phase disappears, at a critical value

$$\tilde{B}_{crit} = \frac{2^{\frac{3}{2}}}{\sqrt{\lambda}} \frac{|\vec{B}|_{crit}}{T^2} \approx 16 \,, \tag{3.21}$$

where I introduced the physical value of the magnetic field $|\vec{B}| = B/(2\pi\alpha')$. Above this critical value, there is spontaneous chiral symmetry breaking, since the lowest-energy solution at quark mass $\tilde{m} = 0$ is a mesonic one and has a condensate $\tilde{c} > 0$. On the other hand, in the case where the zero quark mass solution reaches the horizon and therefore corresponds to molten mesons, this solution is given by $L(\rho) = 0$ and thus no condensate develops and no spontaneous chiral symmetry breaking occurs.

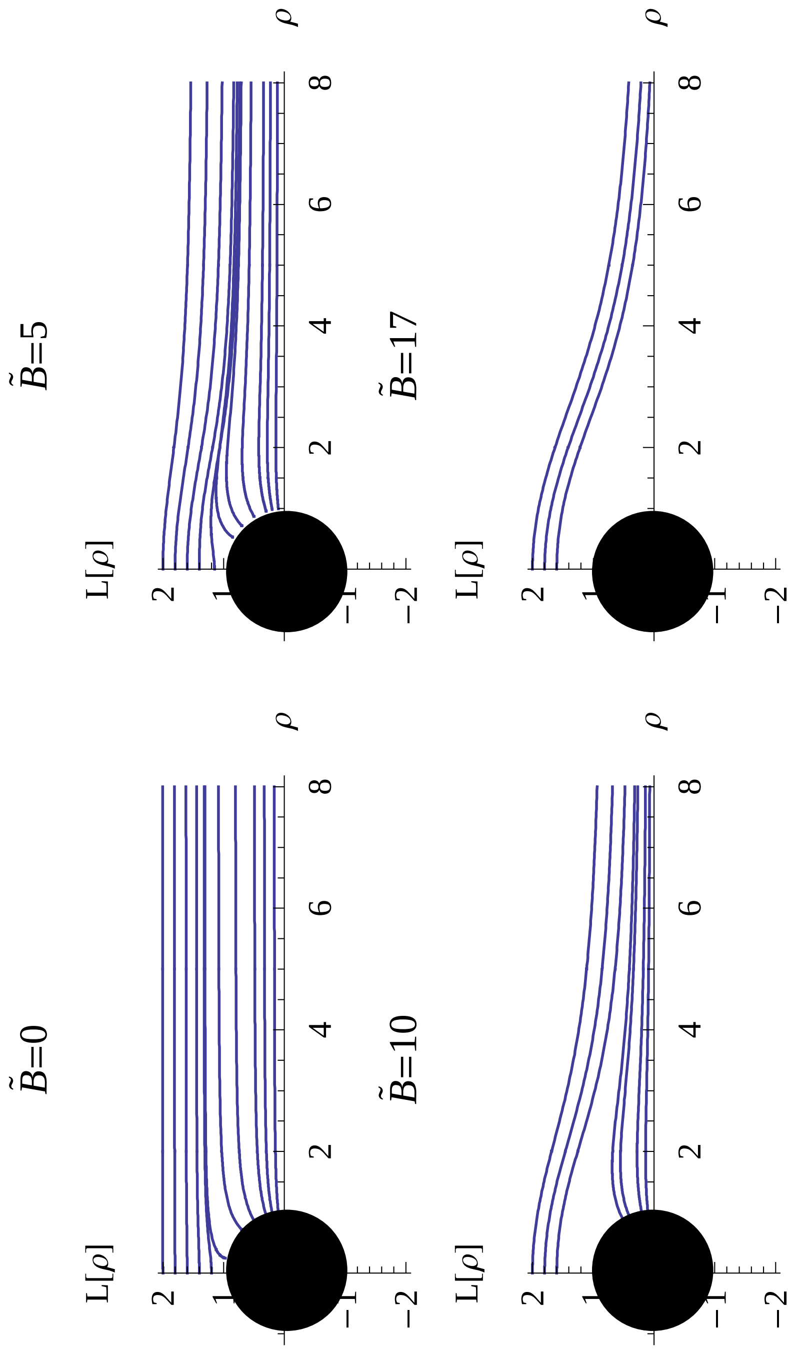


Figure 3.4: The D7 brane embeddings for increasing values of $\tilde{B}$ show the repulsive nature of the magnetic Kalb-Ramond field. The reduced temperature has been fixed to one. For large enough $\tilde{B}$, the molten phase is never reached. In this regime the chiral symmetry is spontaneously broken.

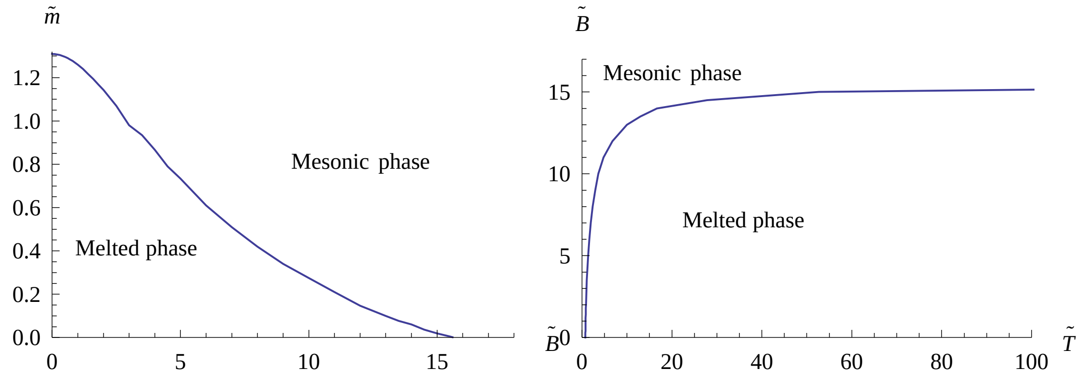


Figure 3.5: Phase diagram in the $(\tilde{m}, \tilde{B})$ and $(\tilde{T}, \tilde{B})$ planes for the magnetic field case. The left graph shows the largest quark mass for a given $\tilde{B}$ for which the embedding still reaches the black hole. In the right graph, the critical line reaching a constant $\tilde{B}$ at large temperature corresponds to a quadratic dependence $|\vec{B}|_{crit} \sim \sqrt{\lambda} T^2$ at large temperature.

The phase diagram is depicted in figure 3.5. On the left hand side I plot the largest quark mass $\tilde{m}$ for given $\tilde{B}$ for which the embedding still reaches the horizon. The temperature is fixed. Above the critical value $\tilde{B}_{crit} \approx 16$, there are no more black hole embeddings reaching the black hole horizon. On the right hand side of figure 3.5, I consider the same phase diagram for fixed m_q while varying T. We see that for large $\tilde{T}$, the critical value $\tilde{B}_{crit}$ tends to a constant value around 16. Because $\tilde{B}$ is explicitly a function of T, the appropriate dimensionless quantities to plot on the (T, B) phase diagram are $(\tilde{T}, \hat{B})$. Because $\tilde{B}_{crit}$ becomes constant for large $\tilde{T}$, we see that $\hat{B}_{crit}$ behaves as $\tilde{T}^2$ for large $\tilde{T}$, corresponding to large temperature at fixed quark mass.

The condensate as a function of the mass for different $\tilde{B}$ is shown in figure 3.6: The blue part of the curve corresponds to the mesonic phase, while the green part at small quark masses corresponds to the molten phase. Increasing the magnetic field strength lowers the critical quark mass at which the first order phase transition occurs, until zero quark mass is reached at the critical value $\tilde{B}_{crit}$. At this point there is no molten phase any more, but a nonzero condensate at zero quark mass indicates spontaneous chiral symmetry breaking in the dual gauge theory. It should be noted that for a given quark mass there may be several possible embeddings. Just as in the case of pure finite temperature or pure magnetic field, the physical solutions can be found by energy considerations or, as done explicitly in [105], by careful holographic renormalisation and calculation of the free energy $F = -TS_{\text{ren,onshell}}$. Although not clearly recognisable in figure 3.6, the joining point of the two different embedding branches there occurs a multivaluedness of the condensate versus the quark mass for all values of the magnetic field, indicating that below the critical magnetic field strength the meson melting transition is present as in the pure finite temperature case.

Yet another graph showing clearly the onset of spontaneous chiral symmetry breaking at

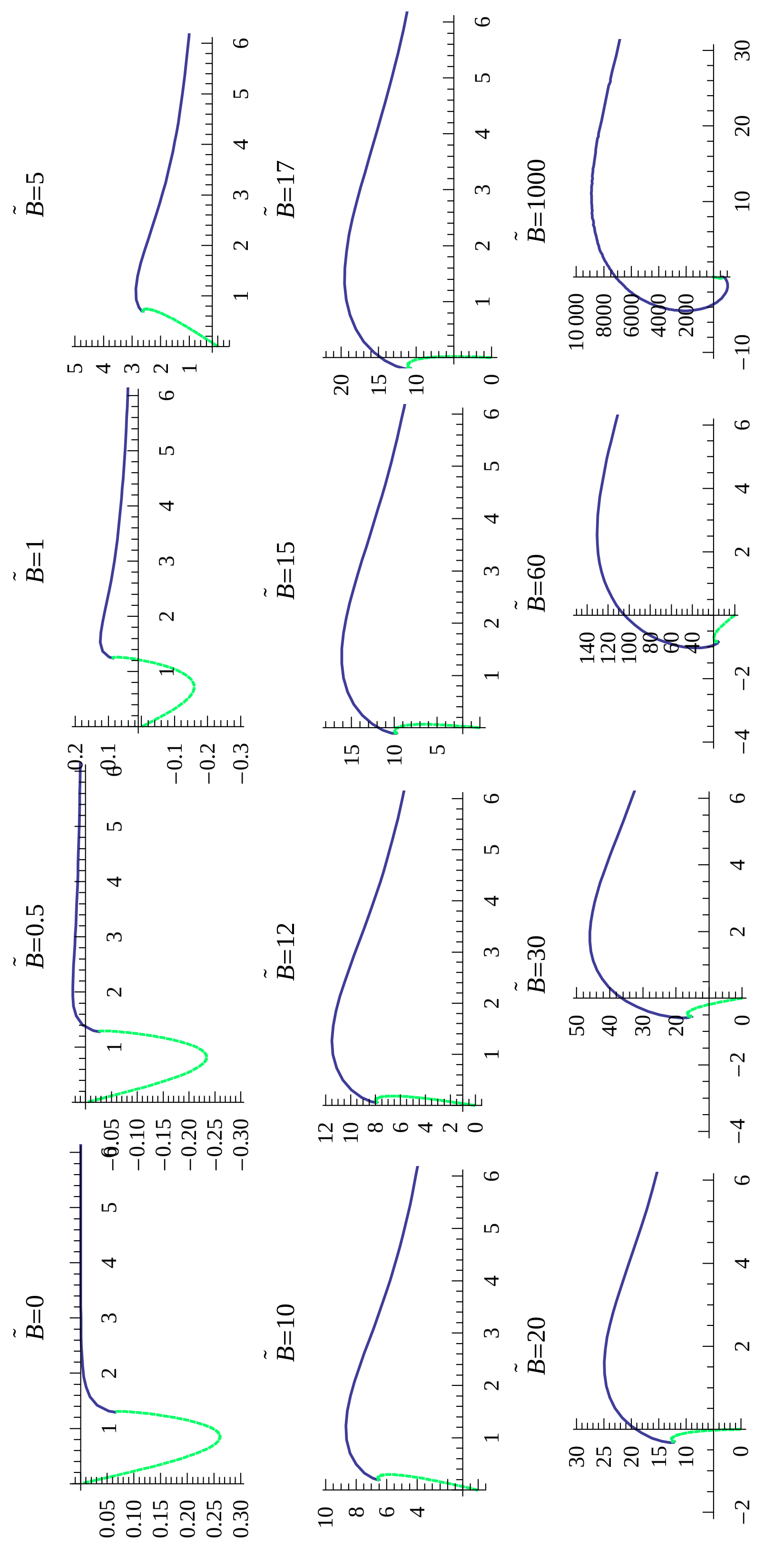


Figure 3.6: Condensate $\frac{8\langle\mathcal{O}_{m_q}\rangle}{\sqrt{\lambda}N_cT^3}$ against quark mass $\frac{2m_q}{\sqrt{\lambda}T}$ for increasing magnetic field $\tilde{B}$ from 0 to 1000. There is a critical value of $\tilde{B} \approx 16$, above which there is no molten phase.

the critical value of the magnetic Kalb-Ramond field is figure 3.7, in which the condensate at zero quark mass is plotted versus the external magnetic field. We observe that the phase transition is first order, as there is a jump in the order parameter $\tilde{c}$.

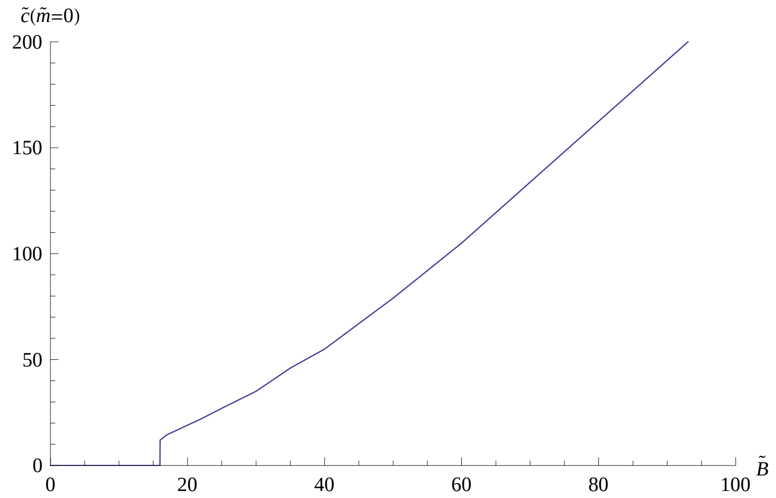


Figure 3.7: Condensate for the lowest-energy embedding at zero quark mass $\tilde{c}(0)$ as a function of the magnetic field $\tilde{B}$. There is a first-order phase transition at which spontaneous chiral symmetry breaking occurs.

Pseudoscalar Meson Spectrum

We also calculated the masses of some of the mesons for the Minkowski embeddings in the magnetic finite temperature background by solving the eigenvalue problem for the Φ-fluctuations. Since the angle Φ is the angle of the polar coordinate system introduced in the X^8-X^9 directions, it is odd under a parity transformation in that plane. The fluctuation of the field Φ thus corresponds to pseudoscalar mesons, and I restrict myself to states with zero S^5 angular momentum for simplicity. These mesons are particularly interesting as the field Φ is the Goldstone mode of the chiral symmetry breaking in our model, i.e. it is the shift symmetry in Φ (the rotation symmetry in the 8-9 plane) which is spontaneously broken by a vacuum expectation value $\langle \mathcal{O}_{m_q} \rangle$. This calculation was performed at zero temperature in [105], where it was found that it is possible to decouple the gauge field fluctuations on the brane and from the fluctuations of the embedding coordinate ϕ (i.e. to consistently set the gauge field fluctuations to zero), if the latter depends only on the coordinates (x^2, x^3). The problematic coupling is the Wess-Zumino coupling to the magnetic dual $\tilde{C}_4$ (which is unchanged in the finite temperature case and still given by eq. (3.7)) of C_4 (after integrating over the S^3 coordinates $\{\psi, \beta, \gamma\}$)

$$\frac{\mu_7}{g_s} \int P[\tilde{C}_4] \wedge B \wedge (2\pi\alpha' F) = \frac{4\pi^3 \alpha' B R^4}{(2\pi)^7 \alpha'^4 g_s} \int \frac{2\rho^2 + L^2}{(\rho^2 + L^2)^2} L^2 (\partial_\mu \phi) \mathrm{d}x^\mu \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge F \,.$$

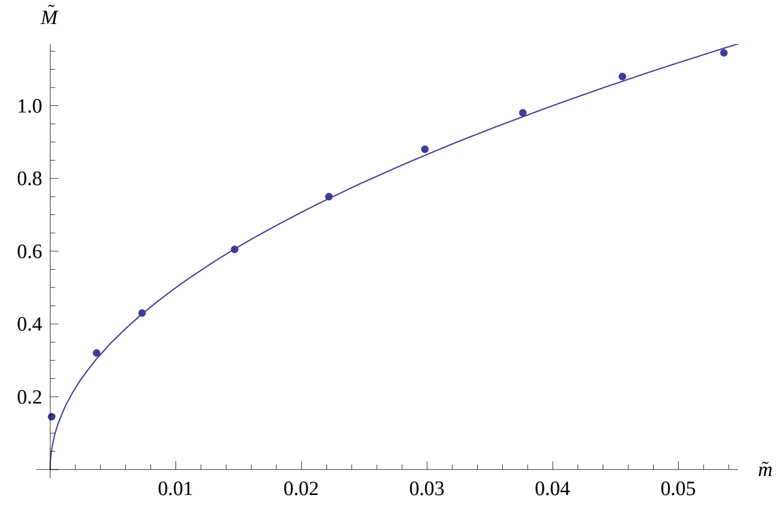


Figure 3.8: Mass of the Goldstone boson for $\tilde{B} = 16$. The solid line is the fitted curve $\tilde{M} = 5\sqrt{\tilde{m}}$, showing the typical Gell-Mann-Oakes-Renner behaviour.

This term thus couples $A_{0,1}$ with ϕ (and also L), unless $\partial_0\phi = \partial_1\phi = 0$. The modes considered here can thus be thought of as a meson moving in the Euclidean (x^2, x^3)-plane [105]. This will suffice for showing the main features of the meson dynamics of our interest in this chapter, namely spontaneous chiral symmetry breaking and the existence of the corresponding Nambu-Goldstone mode. We thus followed the same strategy as [105] and used the ansatz

$$\phi(\rho, x^2, x^3) = h(\rho)e^{-ik_2x_2 - ik_3x_3}\,. \tag{3.22}$$

The effective Lagrangian for the fluctuations obtained after integration over the S^3 reads

$$\begin{aligned}\mathcal{L} = -\pi^2\frac{\mu_7}{g_s}\rho^3\sqrt{g_{\rho\rho}^4 g_{x^1x^1}^3 g_{tt}\left(\frac{B^2}{g_{x^1x^1}^2}+1\right)(L'^2+1)}\,\times \\ \times g_{\phi\phi}\left(\frac{g_{x^1x^1}\left((\partial_2\phi(\rho,x^2,x^3))^2+(\partial_3\phi(\rho,x^2,x^3))^2\right)}{\left(B^2+g_{x^1x^1}^2\right)}+\frac{(\partial_\rho\phi(\rho,x^2,x^3))^2}{g_{\rho\rho}(L'^2+1)}\right)\,,\end{aligned} \tag{3.23}$$

where the metric is given by (3.14).

The meson masses are obtained from

$$M^2 = -k_2^2 - k_3^2\,. \tag{3.24}$$

As first noticed in [105], the magnetic field actually induces a Zeeman-like splitting of the pseudoscalar spectrum. The half of the pseudoscalar meson spectrum for fixed temperature and different B field values which gets heavier is plotted in figure 3.9, where the dashed lines are the pure AdS result [96],

$$\tilde{M} = 2\tilde{m}\sqrt{(n+1)(n+2)}\,, \quad n = 0, 1, 2, \ldots$$

i.e. the meson masses for this excitation at zero temperature and no additional fields calculated in the Poincaré patch of $AdS_5 \times S^5$. n is a radial quantum number. There is another branch of the spectrum depicted in figure 16 of [300] which can only be seen if considering the full system of coupled $A - \phi$ fluctuations. With the ansatz $k_0 = k_1 = 0$ considered here these modes decouple, but the features of the upper branch of the pseudoscalar states suffices for my discussion of the induced spontaneous chiral symmetry breaking and its effects. The mesons in this upper branch are heavier than their pure AdS counterparts. The situation is intuitively comparable to the hydrogen atom: Put into a magnetic field along e.g. the z-axis, the degeneracy between modes of different spin orientation $m_s = -s, \dots, s$ split up in a way such that positive m states get shifted to higher energies due to the coupling[6] $\mu_B \vec{B} g_S \vec{S} = \mu_B B g_S S_z$, while modes with negative S_z-eigenvalue m_S get shifted to lower energies. This analogy might not carry too far, since a string spinning in a magnetic field is not exactly a hydrogen atom. For example the quark-antiquark pair potential in the simplest case of a single flavour is not purely coulombic, but is linear at small separations and coulombic at large distances (see fig. 1 of [301] and fig. 3 of [96] for the meson spectrum). We can however at least conclude that the magnetic B field induces a Zeeman-like splitting of the pseudoscalar meson spectrum while it catalyses spontaneous chiral symmetry breaking.[7]

Furthermore we found that above the critical magnetic field strength $\tilde{B}_{crit} \approx 16$, the lowest-lying pseudoscalar state becomes massless for zero quark mass and thus is the **Goldstone mode of chiral symmetry breaking**. Figure 3.8 zooms into the zero mass region for this state, calculated for $\tilde{B} = 16$. The solid curve, $\tilde{M} = 5\sqrt{\tilde{m}}$, is obtained by fitting to the data points, and shows that our identification of this state with the Goldstone mode of chiral symmetry breaking is correct, as it satisfies the **Gell-Mann-Oakes-Renner relation** [289]. The deviation of the data point for the lowest reduced quark mass $\tilde{m}$ is due to numerical artefacts. Note however that a full independent test of the Gell-Mann-Oakes-Renner relation

$$M_\pi^2 = -\frac{2\langle \mathcal{O}_{m_q} \rangle}{f_\pi^2} m_q \tag{3.25}$$

would need an independent calculation of the pion decay constant. This was recently achieved in [299] by calculating the effective chiral action from the probe brane dynamics.

[6] $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton, $g_L = 1$ the gyromagnetic ratio for angular momentum and $g_S \approx 2.0023192$ the anomalous gyromagnetic ratio for spin.

[7] The situation was more closely analysed in [301] for mesons with large total angular momentum J. In the region $1 \ll J \ll \sqrt{\lambda}$ it was found for strings with equally opposite charged endpoints that the lowest-order mass splitting is $\propto B^2$. However, since we are considering the DBI theory here, the angular momentum is small in this case and thus the results of [301] do not apply. The Zeeman effect found in [105] is linear in the magnetic field.

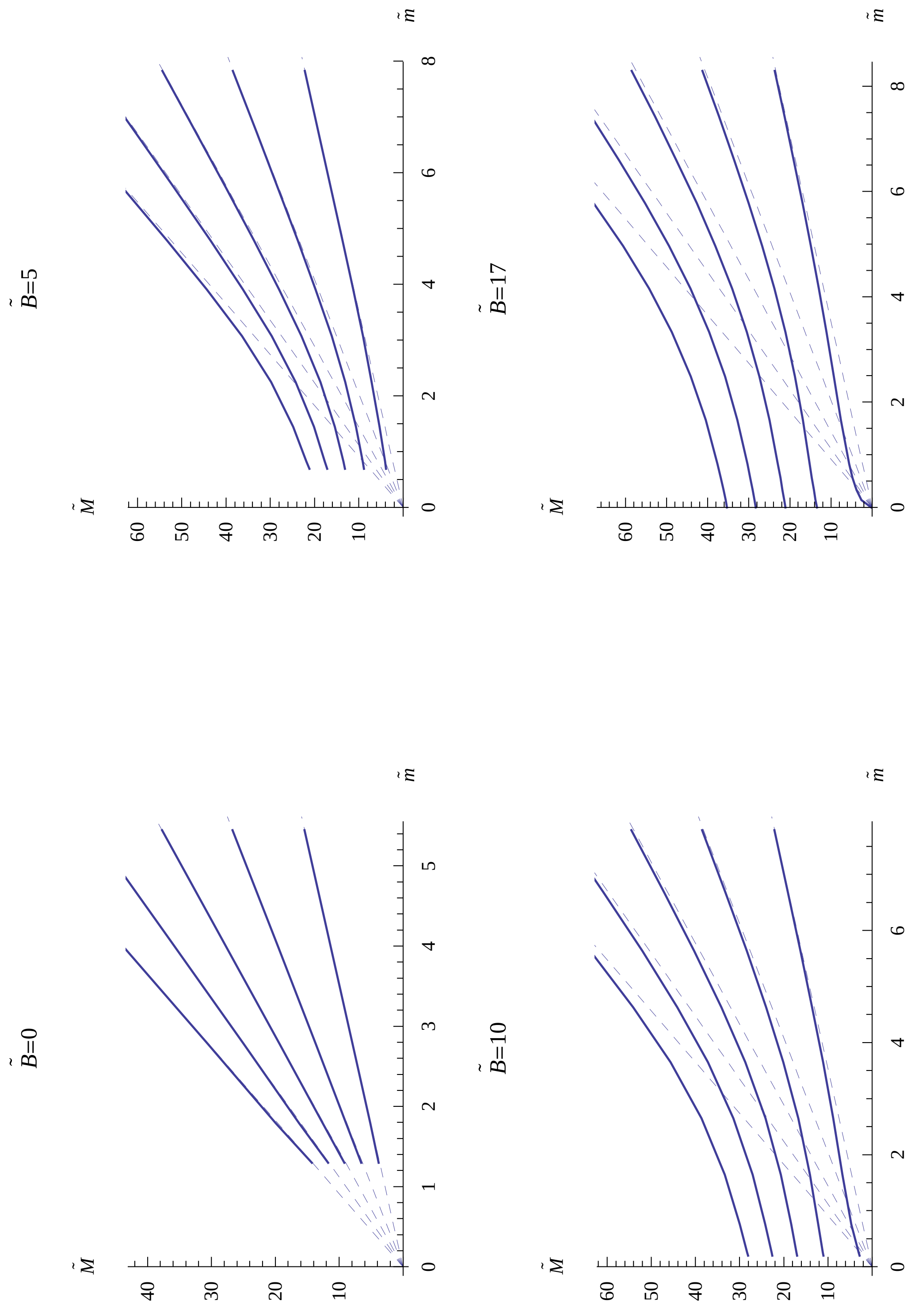


Figure 3.9: Pseudoscalar meson masses for the AdS-Schwarzschild black hole background for different magnetic field strengths. The dashed lines are the pure AdS result $\tilde{M} = 2\tilde{m}\sqrt{(n+1)(n+2)}$, cf. [96]. Above the critical field strength $\tilde{B}_{crit} \approx 16$ the lowest meson state becomes massless at zero quark mass and thus is identified with the Goldstone boson of chiral symmetry breaking.

3.3 Fundamental Matter in Constant External Electric Fields

3.3.1 Constant External Electric Fields and Zero Temperature

I now turn to the case where the Kalb-Ramond B field is switched on in the spatio-temporal directions, corresponding to an external electric field. There is a subtle but important difference between the Lagrangians for the magnetic and electric ansatz (3.5): In the case of the electric field, there is a zero of the action at

$$\omega^2 = \rho^2 + L^2 = BR^2 . \tag{3.26}$$

I will call the five-sphere defined by this equation the **singular shell** in the following. Even worse, inside the singular shell the action (3.5) becomes imaginary in the electric case. Without further modifications this setup thus seems to be inconsistent, since the embedding field L has to be real, but the equation of motion derived from (3.5) for L becomes complex inside the singular shell. Qualitatively we will see that in contrast to the magnetic case where the external field has a repulsive effect, now the singular shell is attracting the D7 solutions, in a sense similar to the black hole metric.

For values of the radius ω inside the singular shell, the DBI Lagrangian (3.5) becomes imaginary, which indicates a tachyonic instability [302–305]. We stabilise the D7 configuration by switching on a compensating gauge field on the D7 brane near and inside the singular shell, along the lines of [284, 285]: We switch on components of the gauge field's Faraday tensor which correspond holographically to a finite quark number density n_q, i.e. switching on $F_{t\rho}$, and a quark number current expectation value $\langle J_x \rangle$, i.e. through switching on $F_{x\rho}$. Demanding the DBI action to be real throughout ten-dimensional space-time then yields a regularity constraint relating number density and current.[8] Intuitively this should cure the sickness in our setup, since we were trying to put the system into a constant electric field without allowing for a current flow to develop.

Using the ansatz of [284],

$$A_x = -\frac{f(\rho)}{2\pi\alpha'} , \quad A_t = -\frac{g(\rho)}{2\pi\alpha'} , \tag{3.27}$$

the action for the D7 brane in the external electric B field takes the form

$$S_{D7} \;=\; -\mathcal{N} \int_0^\infty \mathrm{d}\rho \rho^3 \sqrt{f'(\rho)^2 - g'(\rho)^2 + \left(1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}\right)(1 + L'^2)} . \tag{3.28}$$

[8]In fact, for the reality of the DBI action it is enough to introduce the magnetic component $F_{\rho x}$. We however also consider the electric component $F_{\rho t}$, which corresponds to introducing a chemical potential and a finite quark number density on the brane. This seems to be complicating the situation at first, but it might be interesting in future studies at finite chemical potential.

The factor $\mathcal{N}$ is defined by $\mathcal{N} = 2\pi^2 \frac{\mu_7}{g_s}$, where the factor of $2\pi^2$ is again the volume of a unit radius S^3. For the Ramond-Ramond four-form potential (3.3), the Chern-Simons part of the D7 brane action only contributes through the gauge field strength components $F_{\hat{a}\hat{b}}$, $\{\hat{a}, \hat{b}\} \in \{\rho, \psi, \beta, \gamma\}$. The coupling to the magnetic dual four-form $\tilde{C}_4$ vanishes because $P[\tilde{C}_4] = 0$ for our choice of the embedding, as seen before in the magnetic case. Furthermore, I am suppressing here the customary factor of $2\pi\alpha'$ in front of the Faraday tensor which can easily be restored. As the action depends on the ρ-derivatives of A_x and A_t only through the field strength [284], the Euler-Lagrange equations for A_t and A_x demand the existence of two conserved quantities,

$$\delta A_t : D = \frac{\mathcal{N}\rho^3 g'(\rho)}{\sqrt{f'(\rho)^2 - g'(\rho)^2 + \left(1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}\right)(1 + L'^2)}}, \tag{3.29}$$

$$\delta A_x : \mathcal{B} = -\frac{\mathcal{N}\rho^3 f'(\rho)}{\sqrt{f'(\rho)^2 - g'(\rho)^2 + \left(1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}\right)(1 + L'^2)}}. \tag{3.30}$$

The field theory interpretation of these quantities is given in [284]: D corresponds to a finite quark number density $\langle J_t \rangle = n_q$, and $\mathcal{B}$ to a current in x-direction $\langle J_x \rangle$, both of which are defined through the asymptotic behaviour for large ρ [284]

$$A_t \simeq \mu - \frac{D}{2\mathcal{N}(2\pi\alpha')^2\rho^2} = \mu - \frac{(2\pi\alpha')^2\lambda}{N_f N_c}\frac{n_q}{\rho^2}, \tag{3.31}$$

$$A_x \simeq b + \frac{\mathcal{B}}{2\mathcal{N}(2\pi\alpha')^2\rho^2} = b + \frac{(2\pi\alpha')^2\lambda}{N_f N_c}\frac{\langle J_x \rangle}{\rho^2}. \tag{3.32}$$

We thus find that both integration constants are directly identified with the operator vacuum expectation values,

$$D = \langle J_t \rangle = n_q, \quad \mathcal{B} = \langle J_x \rangle. \tag{3.33}$$

Inverting (3.29) and (3.30) yields

$$f'(\rho) = -\mathcal{B}\frac{\sqrt{1 + L'^2(\rho)}\sqrt{1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}}}{\sqrt{\mathcal{N}^2\rho^6 + D^2 - \mathcal{B}^2}}, \tag{3.34}$$

$$g'(\rho) = +D\frac{\sqrt{1 + L'^2(\rho)}\sqrt{1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}}}{\sqrt{\mathcal{N}^2\rho^6 + D^2 - \mathcal{B}^2}}. \tag{3.35}$$

Using these solutions, as in [284] we now perform a Legendre transform eliminating f and g completely from the action and replacing them by the conserved quantities $\mathcal{B}$ and D,

$$\begin{aligned}\bar{S}_{D7} &= S_{D7} - \int \mathrm{d}\rho \left(g'(\rho)\frac{\delta S_{D7}}{\delta g'(\rho)} + f'(\rho)\frac{\delta S_{D7}}{\delta f'(\rho)} \right) \\ &= -\int \mathrm{d}\rho \sqrt{(\mathcal{N}^2\rho^6 + D^2 - \mathcal{B}^2)(1 + L'^2)\left(1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}\right)}.\end{aligned} \tag{3.36}$$

It is thus possible to obtain an action which is real everywhere by demanding that at the point $\rho_{IR}^2 + L(\rho_{IR})^2 = BR^2$, where the last bracket in the square root of (3.36) changes sign, the first bracket therein should also change sign. This implies a relation between the two conserved quantities and the position ρ_{IR} where the brane hits the singular shell,

$$\mathcal{B}^2 = \mathcal{N}^2 \rho_{IR}^6 + D^2 \,. \tag{3.37}$$

Reinserting this relation into (3.36), one finds

$$\bar{S}'_{D7} = -\mathcal{N} \int \mathrm{d}\rho \sqrt{(\rho^6 - \rho_{IR}^6)(1 + L'^2)\left(1 - \frac{B^2 R^4}{(\rho^2 + L^2(\rho))^2}\right)} \,. \tag{3.38}$$

In the finite temperature AdS-Schwarzschild background (3.14) the corresponding expressions read ($L_{IR} = L(\rho_{IR})$)

$$\begin{aligned}
\omega_{IR}^2 &= \rho_{IR}^2 + L_{IR}^2 = \frac{BR^2}{2} + \frac{1}{2}\sqrt{4b^4 + B^2 R^4}\,, && (3.39)\\
\mathcal{B}^2 &= \frac{(\omega_{IR}^4 - b^4)^2}{(\omega_{IR}^4 + b^4)^2} D^2 + \frac{\mathcal{N}^2 B^2 R^4 (\omega_{IR}^4 + b^4)\rho_{IR}^6}{\omega_{IR}^8}\,, && (3.40)\\
\bar{S}'_{D7} &= -\int d\rho \sqrt{a\left(\mathcal{N}^2\left(\rho^6 - \frac{b_{IR}c_{IR}}{bc}\rho_{IR}^6\right) + D^2 \frac{b - b_{IR}}{bc}\right)}\,, &&\\
\text{with} & &&\\
a &= (1 + L'^2)\left(1 + \frac{b^4}{\omega^4}\right)^2 \left(\left(1 - \frac{b^4}{\omega^4}\right)^2 - \frac{B^2 R^4}{\omega^4}\right)\,, &&\\
b &= \left(1 - \frac{b^4}{\omega^4}\right)^2 \left(1 + \frac{b^4}{\omega^4}\right)\,, \quad c = \left(1 + \frac{b^4}{\omega^4}\right)^3 \,. && (3.41)
\end{aligned}$$

At finite temperature the singular shell always resides outside the horizon at $\omega_{\mathrm{hor}}^2 = b^2$. There is an important difference between the cases of finite and zero temperature: While (3.41) does, after Legendre transformation, explicitly depend on D, the zero temperature action (3.38) does not. Both are, however, real, over the whole range $\rho \in [0, \infty)$ *per constructionem*. The embeddings in the zero temperature case thus will not depend on the quark number density, but the finite temperature embeddings will: The choice of D at finite temperature influences the asymptotic values of the quark masses for the D7 branes falling into the horizon, i.e. sets an energy scale.

Embeddings

Let us begin with a study of the D7 brane embeddings at zero temperature and with $D = 0$, which are depicted in figure 3.10. For Minkowski embeddings (depicted in blue), which do not reach the singular shell but flow all the way to $\rho_{IR} = 0$, the gauge field can be turned off consistently ($A_x = A_t = 0$) and one can work with the original DBI action. There are also embeddings which do intersect the singular shell (depicted in green): Some of these flow towards the origin $L = \rho = 0$, i.e. the location of the AdS horizon, while others end in a conical singularity at $\rho = 0$ and finite $L(0)$. The presumable nature of this conical singularity is discussed in section 6.1. For the singular shell embeddings, the infrared value ρ_{IR} is given by $\rho_{IR}^2 + L_{IR}^2 = BR^2$, where L_{IR} is the value of L at which they intersect the singular shell. Similar to the black hole case, there is one critical embedding which is just touching the singular shell.

The action (3.38) is real everywhere, and reduces to (3.5) with the lower sign for $\rho_{IR} = 0$, i.e. for the Minkowski embeddings and the critical embedding. The Euler-Lagrange equation for $L(\rho)$ as derived from (3.38) reads

$$\partial_\rho \left(\frac{\sqrt{\rho^6 - \rho_{IR}^6} L' \sqrt{1 - \frac{B^2R^4}{(\rho^2+L^2)^2}}}{\sqrt{1+L'^2}} \right) - \frac{2B^2R^4 L \sqrt{\rho^6 - \rho_{IR}^6}\sqrt{1+L'^2}}{(\rho^2+L^2)^3 \sqrt{1 - \frac{B^2R^4}{(\rho^2+L^2)^2}}} = 0\,. \tag{3.42}$$

In practice we obtain the Minkowski embeddings numerically by shooting out from the L-axis to infinity, while the singular shell embeddings are obtained by shooting inwards and outwards starting from the shell. The boundary condition can be obtained by an expansion of the embedding equation (3.42) near the singular shell as follows:[9] Introducing rescaled dimensionless coordinates $\rho = \sqrt{B}R\tilde{\rho}$, $L = \sqrt{B}R\tilde{L}$, one first needs to expand the embedding near $\tilde{\rho}_{IR}$,

$$\tilde{L} = \sqrt{1 - \tilde{\rho}_{IR}} + \tilde{L}'_{IR}(\tilde{\rho} - \tilde{\rho}_{IR})\,. \tag{3.43}$$

The value of $\tilde{L}$ is restricted by the circle equation to $\tilde{L}_{IR} = \sqrt{1 - \tilde{\rho}_{IR}}$. The embedding equation (3.42) then reads (using $\tilde{\rho}^2 - \tilde{\rho}_{IR}^2 = -2\tilde{\rho}_{IR}^2 + 2\tilde{\rho}\tilde{\rho}_{IR} + \mathcal{O}((\tilde{\rho} - \tilde{\rho}_{IR})^2)$)

$$\begin{aligned} &\left(\frac{\sqrt{\tilde{\rho}^6 - \tilde{\rho}_{IR}^6}\tilde{L}'_{IR}\sqrt{1 - (1 - 2\tilde{\rho}_{IR}^2 + 2\tilde{\rho}\tilde{\rho}_{IR} + 2\tilde{L}'_{IR}\sqrt{1 - \tilde{\rho}_{IR}^2}(\tilde{\rho} - \tilde{\rho}_{IR}))^{-2}}}{\sqrt{1 + \tilde{L}'^2_{IR}}} \right)' = \\ &\qquad 2\frac{(\sqrt{1 - \tilde{\rho}_{IR}^2}) + \tilde{L}'_{IR}(\tilde{\rho} - \tilde{\rho}_{IR})\sqrt{\tilde{\rho}^6 - \tilde{\rho}_{IR}^6}\sqrt{1 + \tilde{L}'^2_{IR}}}{(1 - 2\tilde{\rho}_{IR}^2 + 2\tilde{\rho}\tilde{\rho}_{IR} + 2\tilde{L}'_{IR}\sqrt{1 - \tilde{\rho}_{IR}^2}(\tilde{\rho} - \tilde{\rho}_{IR}))^3} \times \\ &\qquad \left(1 - (1 - 2\tilde{\rho}_{IR}^2 + 2\tilde{\rho}\tilde{\rho}_{IR} + 2\tilde{L}'_{IR}\sqrt{1 - \tilde{\rho}_{IR}^2}(\tilde{\rho} - \tilde{\rho}_{IR}))^{-2}\right)^{-\frac{1}{2}} \end{aligned} \tag{3.44}$$

Taylor expanding near $\tilde{\rho}_{IR} \in [0, 1]$ and picking the constant piece near $\tilde{\rho}_{IR} \in [0, 1]$ yields

$$0 = \frac{\sqrt{6\tilde{\rho}_{IR}^5}\left(2\tilde{L}'_{IR}\tilde{\rho}_{IR} - \sqrt{1 - \tilde{\rho}_{IR}^2}(1 - \tilde{L}'^2_{IR})\right)}{\sqrt{1 + \tilde{L}'^2_{IR}}\sqrt{\tilde{\rho}_{IR} + \tilde{L}'_{IR}\sqrt{1 - \tilde{\rho}_{IR}^2}}}\,,$$

[9]In fact we checked numerically that different crossing angles affect the UV and IR behaviour only minimally.

which has two solutions

$$\tilde{L}'_{IR,1} = \frac{\sqrt{1-\tilde{\rho}_{IR}}}{\sqrt{1+\tilde{\rho}_{IR}}}\,, \quad \tilde{L}'_{IR,2} = -\frac{\sqrt{1+\tilde{\rho}_{IR}}}{\sqrt{1-\tilde{\rho}_{IR}^2}}\,. \tag{3.45}$$

The second solution can be discarded as unphysically, as the zero mass solution to (3.42) is $L = 0$, but $L'_{IR,2}$ diverges at $\tilde{\rho}_{IR} = 1$. We thus conclude that $L'_{IR,1}$ is the correct boundary condition.

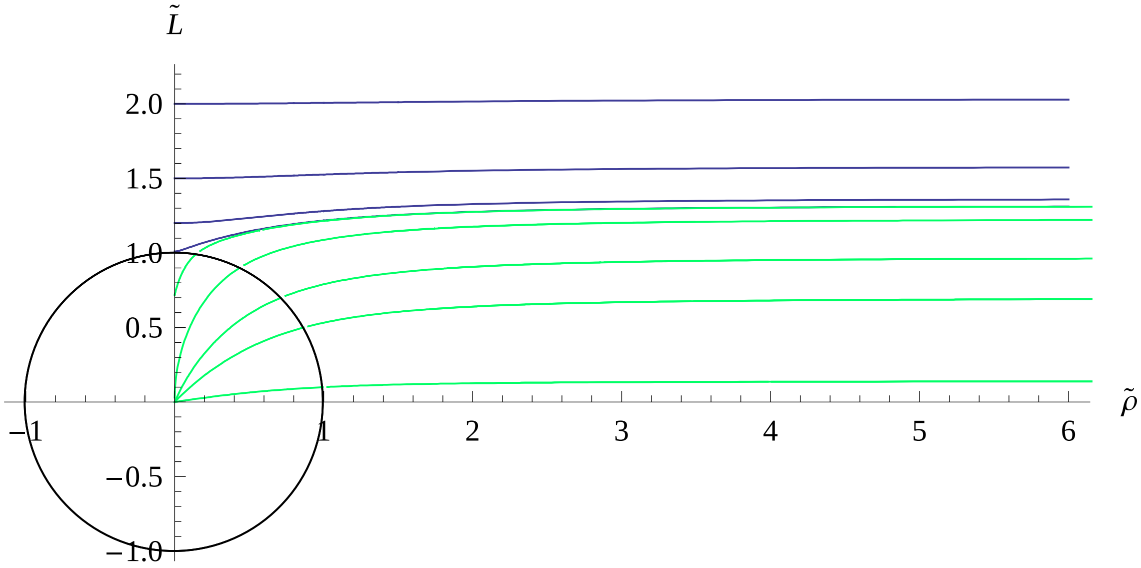


Figure 3.10: D7 brane embeddings for the electric field at zero temperature. The blue curves are Minkowski embeddings, while the green ones reach the singular shell. The singular shell attracts the D7 brane probes similar to the attractive force exerted on the D7 brane by a black hole horizon, and bends them towards the origin.

We extract the UV asymptotic values of $\tilde{m}$ for each flow according to (3.11). In figure 3.11 I plot the condensate $\tilde{c}$ as a function of the reduced quark mass $\tilde{m}$. The blue curve again corresponds to the Minkowski embeddings, while the green curve is for the embeddings flowing into the shell of vanishing action. There are three different kinds of the latter ones: There is one unique critical embedding which just touches the singular shell, then (for decreasing quark mass) there follow the already-mentioned conically singular embeddings and furthermore, down to zero quark mass, embeddings reaching to the AdS horizon at $\rho^2 + L^2 = 0$. There is a **first order phase transition** at the point where both curves join. This region is shown in detail in figure 3.12, where we find a three-fold degeneracy of the function $\tilde{c}(\tilde{m})$ for a range of masses between $\tilde{m} = 1.316$ and $\tilde{m} = 1.319$, in exact analogy to the finite temperature case [273], where $\tilde{m}$ is defined as in (3.11). The exact point of the phase transition can be found by the equal area method:[10] As $\tilde{c}(\tilde{m})$ in the holographic

[10]Strictly speaking the energy and momentum of the D7 branes are not conserved in this situation due

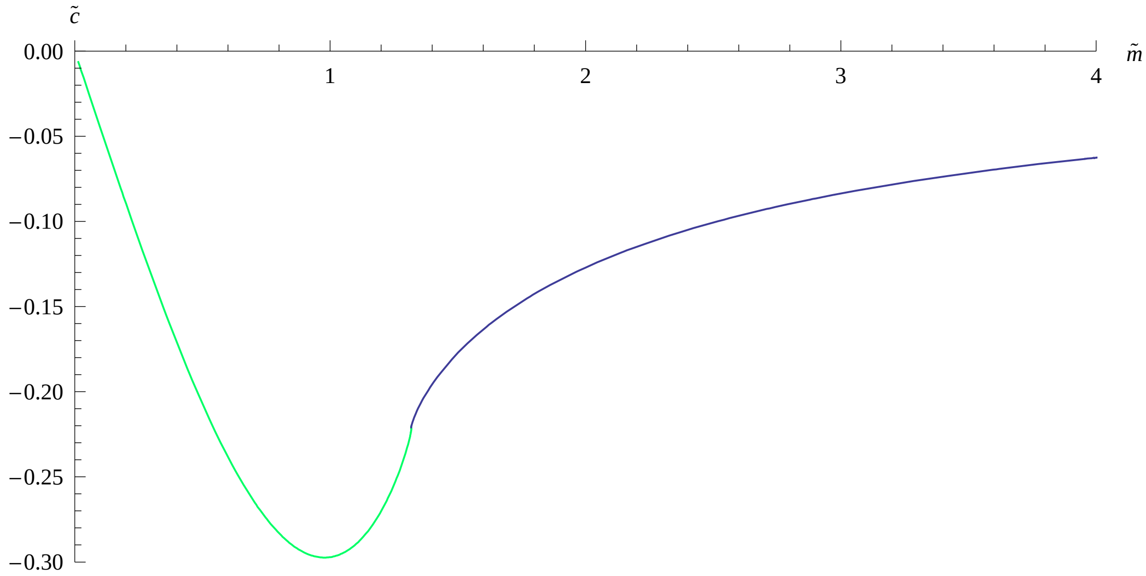


Figure 3.11: Reduced condensate versus reduced mass $\tilde{c}(\tilde{m})$ for the embeddings in the electric zero temperature case. The blue line corresponds to the Minkowski solutions, and the green line to those reaching the singular shell.

context is proportional to the first derivative of the free energy, the area below this curve is proportional to the free energy itself. In a region with several possible embeddings, the phase transition then occurs where the difference of the free energies of the two phases vanishes, i.e. where the areas below the $\tilde{c}(\tilde{m})$ curves are equal below both phases. We find that in the region below a reduced quark mass of ($|\vec{E}| = B/(2\pi\alpha')$ is the physical, dimensionful electric field strength)

$$\tilde{m}_{\rm crit} = \frac{2^{\frac{1}{4}}\pi^{\frac{1}{2}}}{\lambda^{\frac{1}{4}}} \left(\frac{m_q}{|\vec{E}|^{\frac{1}{2}}} \right)_{\rm crit} = 1.31775\,, \tag{3.46}$$

the solutions flowing into the shell of vanishing action become energetically favoured. The electric field however does not induce spontaneous chiral symmetry breaking, since $\tilde{c}(0) = 0$.

to the presence of the constant electric field [306]. The work done by the electric field on the flavour charge carriers pumps energy into the system at a constant rate, which is then dissipated into the $\mathcal{N} = 4$ plasma. The dissipation mechanism works, once the mesons are dissociated into quarks and antiquarks, via the formation of a diffusion wake [307, 308]. The drag force at finite quark baryon number densities in the regime in which the probe limit holds is independent of the number density [284], and hence the energy dissipation must be, although the notions of particles might not apply at strong coupling and conformality, due to interactions of the quarks with the $\mathcal{N} = 4$ gluons. The D7 brane alone is thus out of equilibrium and the usual laws of equilibrium thermodynamics and also the equal area law may not apply in this situation. Nevertheless the application of the equal area law localises the transition in exactly the small region with multivalued condensate, where it is expected from previous experience (see also references [254, 300] for the application of the equal area law to such situations). This treatment can thus not be totally wrong, but it should be justified by a correct treatment of the thermodynamics in future work. The author thanks Andy O'Bannon for guidance on the energy loss mechanism.

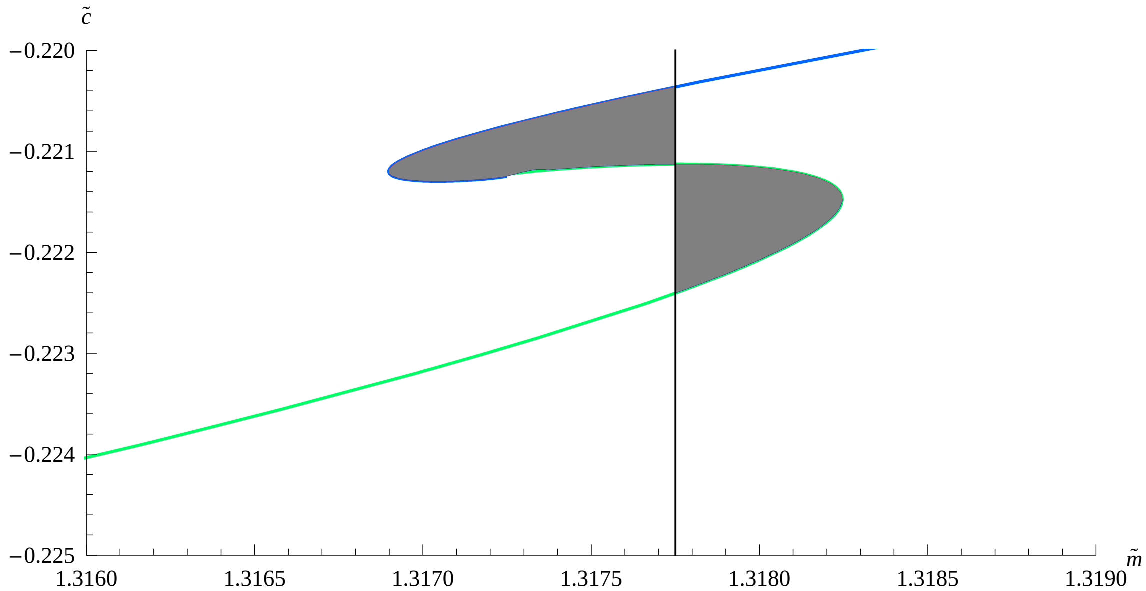


Figure 3.12: Reduced condensate versus reduced mass $\tilde{c}(\tilde{m})$ for embeddings in the region $1.317 \le \tilde{m} \le 1.322$. The "S"-shaped bend indicates a first order phase transition at $\tilde{m} \approx 1.31775$, found by the "equal area law" (see text).

This is to be contrasted with the magnetic field case, where spontaneous chiral symmetry breaking is observed. Nevertheless the condensate as a function of the quark mass shows an interesting structure near the (unphysical) critical embedding: There is a double-logarithmic spiral behaviour in $c(m)$ at that point. A similar structure also appears in the AdS-Schwarzschild case [275], as well as in the case of R-charged black holes [309]. The scaling exponents characterising the logarithmic spiral turn out in both cases to be universal in the sense that they only depend on the dimension of the internal S^n wrapped in a more general Dp-Dq intersection, but neither on the magnitude of temperature, chemical potential or electric field. However, it was found in [310] that the scaling exponents are the same for the electric field and R-charged black hole case (and also universal in the above sense), but are different from the finite temperature AdS-Schwarzschild case. It was thus conjectured in [310] that the two phase transitions are in different universality classes. Note that it is not surprising that the scaling behaviour of the electric field and R-charged (i.e. rotating) black hole case are the same, since both backgrounds are connected by T duality. It is more surprising that the thermal phase transition is, albeit with similarities, different in some respects from the dissociation transition.[11]

[11]The recent work [110] offers a hint towards explaining this difference: The boundary conditions for fluctuations at the singular shell were found there to be different from simple incoming wave boundary conditions, although they reduce to the incoming wave conditions in the limit of vanishing electric field. However the connection to the spiral behaviour for the critical embedding, which is after all thermodynamically unstable, is still to be understood. Furthermore, the spiral structure for the case with magnetic field has been analysed in [299]. The critical exponents are different both from the electric case as well as the

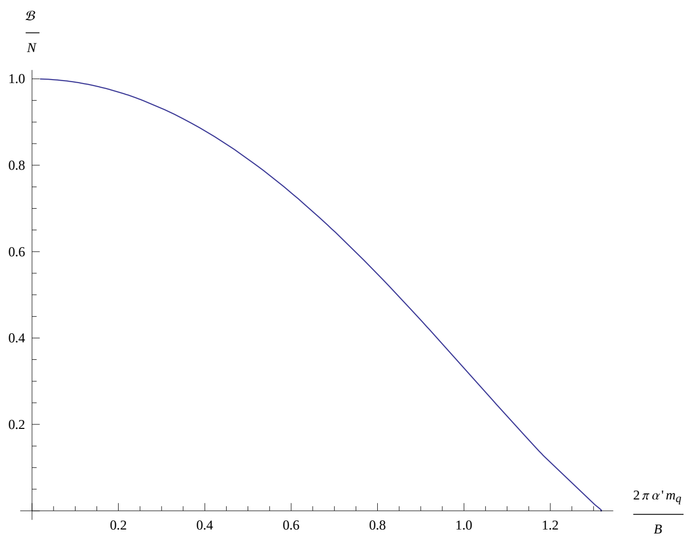


Figure 3.13: Induced quark number current over reduced quark mass for zero temperature and quark density.

In figure 3.10 we assumed a vanishing quark number density D. The induced quark number current $\langle J_x \rangle = \mathcal{B}$, which is shown in figure 3.13, shows an interesting behaviour too: For a fixed value of the electric field it decreases with growing quark mass until the phase transition point is reached, where the Minkowski embeddings, which have zero induced current, take over. Note that this plot also includes the singular shell embeddings, whose physical fate still needs to be decided, cf. section 6.1. This is consistent with an interpretation of the phase transition as **dissociation phase transition of mesons**: If an external electric field is applied to a bound state of charge carries (e.g. a hydrogen atom), the additional linear part of the electric potential destabilises the system, but a threshold for the induced current at zero temperature and quark density is expected, since it simply takes a certain electric field to bend the potential sufficiently in order to destabilise the ground state. Once the potential is sufficiently bent such that not even a ground state can exist, the charge carries are unbounded and free to move with the electric field, and thus a current (of quarks in this case) starts to flow. The transition is thus between the insulating mesonic phase to the conducting phase described by the singular shell embeddings, and can thus be characterised as an **insulator-metal transition** [288].

Pseudoscalar Meson Spectrum

Let us now turn to the calculation of the meson spectrum in the presence of the electric Kalb-Ramond field at zero temperature. Up to small modifications, the calculation is very

finite temperature case. This is to be expected since the chiral symmetry breaking mechanism is present already at zero temperature and electric field, and also in no obvious way connected to those situations by dualities like T-duality.

similar to the one performed for the magnetic case in [105]. For the fluctuations I use the ansatz

$$L(\rho) = L_0(\rho)\,, \quad \Phi = \phi = h(\rho)e^{ik_0t-ik_1x_1}Y_l(\mathrm{S}^3)\,, \tag{3.47}$$

for which I show the decoupling from the L- and gauge field fluctuations on Minkowski embeddings in appendix B.2. Here L_0 is the embedding obtained from the equation of motion (3.42) by numerical integration. Note that in (3.47) we now consider a meson in the (t,x_1) plane, as opposed to the magnetic case (3.22) where the meson was in the (x_2,x_3) plane, which decouples the ϕ fluctuations from the gauge field fluctuations consistently, much in the same way as explained before eq. (3.22). Linearising the D7 brane action in an analogous way to the calculation performed in [105], we obtain the equation of motion for the excitations ϕ,

$$\frac{1}{g}\partial_\rho\left(\frac{gL_0^2\partial_\rho h(\rho)}{1+{L_0'}^2}\right) - \frac{L_0^2 l(l+2)}{\rho^2}h(\rho) + \frac{R^4L_0^2M_{01}^2}{(\rho^2+L_0^2)^2 - R^4B^2}h(\rho) = 0\,. \tag{3.48}$$

Here $M_{01}^2 = k_0^2 - k_1^2$ is the meson mass and

$$g = \rho^3\sqrt{1+{L_0'}^2}\sqrt{1-\frac{B^2R^4}{(\rho^2+L_0^2)^2}}\,.$$

We consider only s-wave fluctuations, corresponding to pseudoscalar mesons ($l=0$), and calculate the spectrum for Minkowski embeddings. The results for $BR^2=1$ are displayed in Figure 3.14. The dashed lines in figure 3.14 again show the analytic AdS solutions,

$$\tilde{M} = 2\tilde{m}\sqrt{(n+1)(n+2)}\,.$$

The behaviour shown in figure 3.14 is consistent with intuitive expectations: For large reduced quark mass $\tilde{m}$ the D branes do not feel the forces exerted by the singular shell (see figure 3.10) and thus are approximately flat, as dictated by supersymmetry of the pure AdS setting [95].

Note that in contrast to the effect of a magnetic Kalb-Ramond field, the mesons in the presence of the electric field are lighter than without applied field. The critical value of the electric field being proportional to the bare quark mass (see eq. (3.46)) is consistent with the fact that the binding energy of these holographic mesons itself is proportional to the bare quark mass [96]. Since the electric field needs to be sufficiently large to overcome the binding energy, the critical electric field must then be proportional to the bare quark mass m_q, too.

In the light of the above interpretation of this phase transition as a dissociation transition one expects the meson excitations to destabilise for embeddings ending inside the shell via a mechanism similar to ionisation, in analogy to the finite temperature meson melting transition. It would be highly interesting to investigate whether the singular shell embeddings support quasiparticle excitations similar to the black hole embeddings in the pure

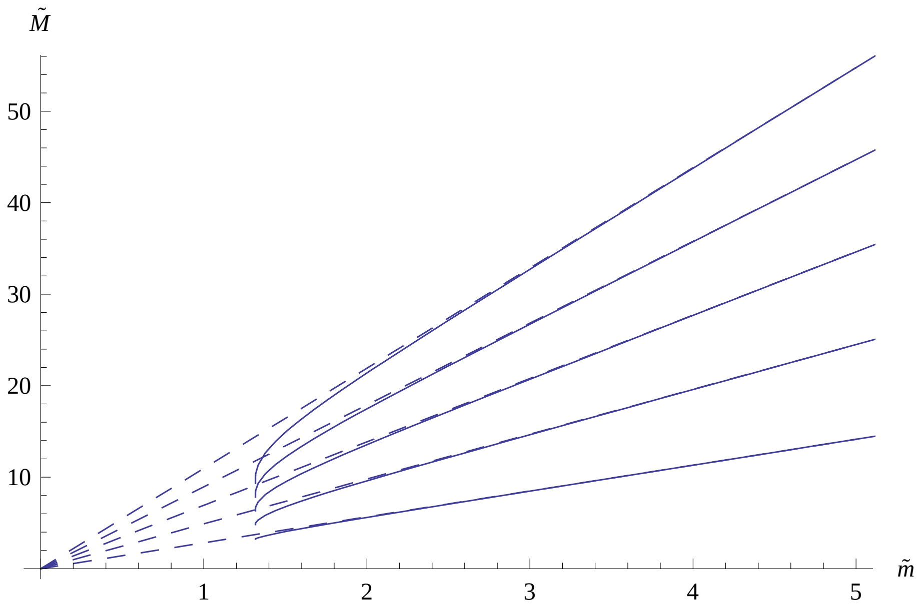


Figure 3.14: First five pseudoscalar meson states in the electric field for the Minkowski embeddings. The reduced meson and quark masses are defined as $\tilde{M} = \frac{R}{\sqrt{B}} M_{01}$, $\tilde{m} = \frac{m_q 2\pi\alpha'}{\sqrt{B}R}$.

finite temperature case along the lines of [311, 312]. In the recent work [110] this analysis has been partially carried out for the D7 brane gauge field fluctuations transverse to the electric field, which comprise a subset of fluctuations decoupling from the others. Their spectral function at small quark number density was shown to have well-defined peaks admitting a quasi-particle interpretation, while near the phase transition, which was found to be a crossover rather than a first order transition for finite quark density, the spectrum rapidly changes to having wide oscillations which do not admit such an interpretation any longer (see figure 11 of [110]). However, this analysis is not complete yet, and comes with several caveats discussed in section 6.1.

Stark effect

For weak external electric fields at zero temperature, I now analytically calculate the meson mass shift of the $n = l = 0$ state, which in the corresponding region of large mass is the lightest meson in figure 3.14, and thus the ground state.[12] For this purpose, I use a technique similar to first order perturbation theory familiar from quantum mechanics, which in this context was suggested to us by Derek Teaney and which was subsequently

[12]Strictly speaking it is unclear whether the meson masses of the higher levels cross the ground state for $\tilde{m} \to \tilde{m}_{crit}$, and also whether the meson masses stay positive in that limit at all. For large $\tilde{m}$ the ordering of the levels is however fixed and well-defined.

used in [313, 314].

I start by looking at the perturbation exerted by the electric field on a Minkowski embedding, i.e. an embedding that has high enough quark mass such that it does not reach the singular shell. Its fluctuation spectrum is then discrete, and the meson masses are well-defined. To expand the equation for the Φ fluctuations (3.48) to lowest nontrivial order in the electric field, which in fact is $\mathcal{O}(B^2)$, I use the following perturbative solution for the brane embedding to $\mathcal{O}(B^2)$,

$$L = m - \frac{B^2 R^4}{4m(\rho^2 + m^2)} \,. \tag{3.49}$$

This result is easily obtained by expanding the embedding equation (3.42) up to second order in B. The $\mathcal{O}(B)$ term would not satisfy the Minkowski embedding boundary condition $L'(0) = 0$ and thus has to vanish. By comparing (3.49) with (3.9) we find that the condensate at small field strength or equivalently large quark mass is

$$c(m) = -\frac{B^2 R^4}{4m} \,. \tag{3.50}$$

Figure 3.15 shows the quark condensate as a function of $\frac{B^2 R^4}{m}$. The dark curve corresponds

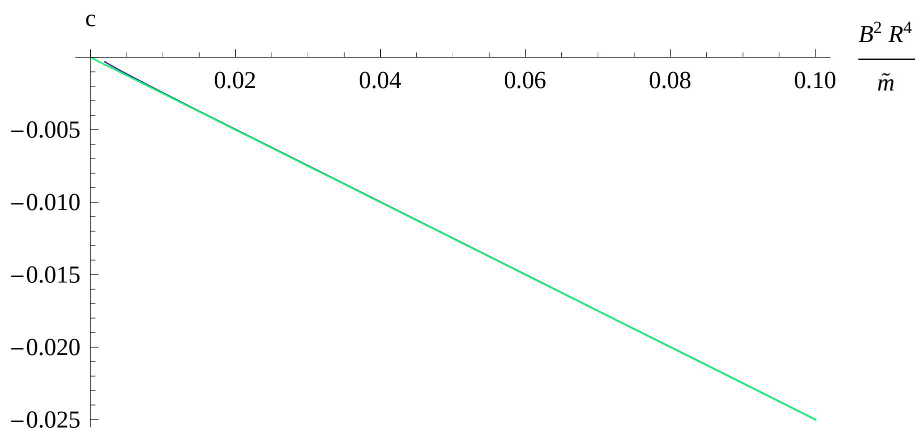


Figure 3.15: The condensate as a function of $B^2 R^4/m$, both numerically (dark curve) and in the weak field approximation (green curve).

to numerical data, while the light green curve is the weak field result (3.50). Both curves coincide to high precision, and both have a slope of minus four, thus validating (3.50). The slight mismatch near the origin, i.e. in the small B or large m region where the approximation should hold very well, is due to numerical instabilities.

Using (3.49) as well as the identities (B.1)-(B.3) in appendix B.1, the equation for the

Φ-fluctuations (3.48) up to second order in B reads

$$\rho^{-3}\partial_\rho\left(\rho^3 h'(\rho)\right)+\frac{(M_0^2+\delta(M^2))R^4 h(\rho)}{(\rho^2+m^2)^2}=$$
$$\frac{B^2R^4}{2m^2}\left[-\frac{M_0^2R^4(3m^2-\rho^2)h(\rho)}{(\rho^2+m^2)^4}-\frac{4\rho m^2 h'(\rho)}{(\rho^2+m^2)^3}+\rho^{-3}\partial_\rho\left(\frac{\rho^3 h'(\rho)}{\rho^2+m^2}\right)\right]. \quad (3.51)$$

Here I split the exact mass of the ground state $M^2 = M_0^2+\delta(M^2)$ into the unperturbed AdS piece $M_0^2 = 8m^2/R^4$ and a lowest order correction. It is convenient to introduce dimensionless quantities[13] $\tilde\rho = \rho/m$, $\tilde M^2 = R^4M^2/m^2$ and $\tilde B^2 = B^2R^4/(2m^4)$. In these units and after multiplying by $\tilde\rho^3 m^2$, eq. (3.51) becomes

$$\partial_{\tilde\rho}\left(\tilde\rho^3 h'(\tilde\rho)\right)+W(\tilde\rho)(\tilde M_0^2+\delta(\tilde M^2))h(\tilde\rho)=$$
$$\tilde B^2\left[-\frac{\tilde M_0^2 W(\tilde\rho)(3-\tilde\rho^2)h(\tilde\rho)}{(\tilde\rho^2+1)^2}-\frac{4\tilde\rho W(\tilde\rho)h'(\tilde\rho)}{\tilde\rho^2+1}+\partial_{\tilde\rho}\left(W(\tilde\rho)h'(\tilde\rho)(\tilde\rho^2+1)\right)\right], \quad (3.52)$$

where I defined the weight function $W(\tilde\rho)=\frac{\tilde\rho^3}{(1+\tilde\rho^2)^2}$. One finds that the lowest order shift in masses will be proportional to $\tilde B^2$. I now use a strategy similar to first order perturbation theory in quantum mechanics: The fluctuation equation for the pure $\mathrm{AdS}_5\times\mathrm{S}^5$ case

$$\partial_{\tilde\rho}\left(\tilde\rho^3 h'(\tilde\rho)\right)+W(\tilde\rho)\tilde M_n^2 h(\tilde\rho)=0\,,$$

is a Sturm-Liouville problem. This implies [315] that its normalisable eigenfunctions

$$h_n(\tilde\rho)=c_n(1+\tilde\rho^2)^{-(n+1)}{}_2F_1(-(n+1),-n,2,-\tilde\rho^2)\,,\ \tilde M_n^2=4(n+1)(n+2)\,, \quad (3.53)$$

which have been found in [96], can be used to define an orthonormal basis of functions $f_n(\tilde\rho)$ on the interval $\tilde\rho\in[0,\infty)$ w.r.t to the inner product

$$(f,g)=\int_0^\infty \mathrm{d}\tilde\rho\, W(\tilde\rho)f(\tilde\rho)g(\tilde\rho)\,. \quad (3.54)$$

Here ${}_2F_1(a,b,c,z)$ is the Gauss hypergeometric function. Let $\{f_n, n=0,1,...\}$ be such a set satisfying the orthonormality relation

$$(f_n,f_m)=\delta_{nm}\,, \quad (3.55)$$

and let us normalise the ground state wave function $h_0(\tilde\rho)$ by choosing the coefficient $c_0=\sqrt{12}$ such that $f_0=h_0$. In what follows I will only need the explicit form of this ground state wave function[14]

$$f_0(\tilde\rho)=\frac{\sqrt{12}}{1+\tilde\rho^2}\,, \quad (3.56)$$

[13] Note $m = m_q/(2\pi\alpha')$ has dimension length.
[14] Note that ${}_2F_1(-1,0,2,-\tilde\rho^2)=1$.

and the fact that the above orthonormal basis exists [315]. The main point is that the normalised ground state wave function f_0 only gets perturbed by a small amount, which is encoded in the ansatz $h(\tilde\rho) = a_0 f_0 + \sum_{n>0} a_n f_n \rho$, where $a_0 = 1 + \mathcal{O}(\tilde B^4)$ and $a_n = \mathcal{O}(\tilde B^2)$ for all $n > 0$. Plugging this ansatz into (3.52), keeping only terms $\mathcal{O}(\tilde B^2)$ and using that the f_n satisfy the equation[15]

$$\partial_{\tilde\rho}\left(\tilde\rho^3 f_n\right) = -\tilde M_n'^2 f_n W\,,$$

we find

$$\begin{aligned}\sum_{n>0} a_n f_n(\tilde M_n'^2 - \tilde M_0^2)W + \delta(\tilde M^2) W f_0 = \\ \tilde B^2\left[-\frac{\tilde M_0^2 W f_0(3-\tilde\rho^2)}{(1+\tilde\rho^2)^2} - \frac{4W\tilde\rho f_0'}{1+\tilde\rho^2} + \partial_{\tilde\rho}\left(W(\tilde\rho)f_0'(\tilde\rho)(\tilde\rho^2+1)\right)\right]\,. \end{aligned}\tag{3.57}$$

Multiplying with f_0 and integrating over $\tilde\rho$ yields then, after making use of the orthonormality relation (3.55), an expression for the mass shift

$$\begin{aligned}\delta(\tilde M^2) = \tilde B^2\Big[&\underbrace{-\tilde M_0^2\left(f_0, f_0\frac{3-\tilde\rho^2}{(1+\tilde\rho^2)^2}\right)}_{I_1}\underbrace{-4\left(f_0, \frac{\tilde\rho f_0'}{1+\tilde\rho^2}\right)}_{I_2} \\ &+\underbrace{\left(\frac{f_0}{W}, \partial_\rho\left(W(\tilde\rho)f_0'(\tilde\rho)(\tilde\rho^2+1)\right)\right)}_{I_3}\Big]\,.\end{aligned}\tag{3.58}$$

The individual contributions are ($\tilde M_0^2 = 8$)

$$\begin{aligned}
I_1 &= -96\int_0^\infty \mathrm{d}\rho\frac{\rho^3(3-\rho^2)}{(1+\rho^2)^6} = -\frac{28}{5}\,,\\
I_2 &= -48\int_0^\infty \mathrm{d}\rho\frac{\rho^4}{(1+\rho^2)^4}\left(\frac{1}{(1+\rho^2)}\right)' = 96\int_0^\infty \mathrm{d}\rho\frac{\rho^5}{(1+\rho^2)^6} = \frac{8}{5}\,,\\
I_3 &= -24\int_0^\infty \frac{\mathrm{d}\rho}{(1+\rho^2)}\left(\frac{\rho^4}{(1+\rho^2)^3}\right)' = -48\int_0^\infty \mathrm{d}\rho\frac{\rho^3(2-\rho^2)}{(1+\rho^2)^5} = -2\,.
\end{aligned}$$

In dimensionless units, the mass of the ground state is thus shifted by

$$\delta(\tilde M^2) = -6\tilde B^2 = -3\frac{B^2R^4}{m^4}\,.\tag{3.59}$$

[15] Note that $\tilde M_n'^2 \neq \tilde M_n^2$ in general, but $\tilde M_0'^2 = \tilde M_0^2$ because of our choice $f_0 = h_0$.

Reinstating physical units yields

$$\delta M = -\frac{3}{4\sqrt{2}}\frac{B^2R^2}{m^3} = -\frac{3}{16\sqrt{2}\pi^{5/2}}\frac{B^2\sqrt{\lambda}}{\alpha'^2 m_q^3} \approx -0.00758\frac{B^2\sqrt{\lambda}}{\alpha'^2 m_q^3}\,. \tag{3.60}$$

There are several points to mention for this result: First, it has the expected B^2 behaviour of the second order Stark effect. Secondly, it has the correct sign, i.e. the mass of the ground state is lowered compared to the pure AdS case (c.f. figure 3.14). Furthermore, by introducing physical units for the B field $\bar{B} = B/2\pi\alpha'$, which is of dimension mass squared, the result becomes independent of α' and is thus finite in the $\alpha' \to 0$ limit,

$$\delta M \approx -0.00758\frac{\bar{B}^2}{{m_q}^3}\,4\pi^2\sqrt{\lambda}\,. \tag{3.61}$$

Level n	c_n	α_n
0	0.54	3.0
1	1.80	3.1
2	3.62	3.2
3	7.85	3.4
4	8.85	3.4

Table 3.1: Coefficients of the Stark shift, fitted using the formula $\delta M_n = -\frac{c_n B^2 R^2}{m^{\alpha_n}}$.

The dependence on the 't Hooft coupling λ is also easily understood: As the meson masses themselves are proportional to $1/\sqrt{\lambda}$ [96] (c.f. e.g. (3.53) and note $R^2 = \sqrt{4\pi\lambda}\alpha'$), the combination $\bar{B}^2/m_q^3$ must scale like λ^{-1}, i.e. must be small in the 't Hooft limit $N \to \infty$ with $\lambda = g_s N = \text{const.} \gg 1$. This means just that the small $\tilde{B}$ expansion is valid if either the physical B field $\bar{B}$ is small or the quark mass m_q is large.

Let us now compare this analytical calculation of the Stark effect with the numerical data displayed in figure 3.14. In the limit of large quark mass $\tilde{m}$, we studied the difference between the AdS meson spectrum and the value of $\tilde{M}$. Numerically we find[16]

$$\delta\tilde{M} = \tilde{M} - \tilde{M}_{AdS} = -\frac{0.54R^2}{\tilde{m}^3}\,. \tag{3.62}$$

Performing the appropriate rescaling of $\tilde{M}$ and $\tilde{m}$ to reintroduce B we find

$$\delta M = -\frac{0.54B^2R^2}{m^3}\,. \tag{3.63}$$

[16]Note that in the units of figure 3.14, B is scaled to $B = 1$.

From equation (3.60) we find

$$\delta M = -\frac{0.53B^2R^2}{m^3}\,, \tag{3.64}$$

which is in very good agreement with the numerical calculation. The discrepancy is likely to come from both numerical errors and higher order corrections in the expansion around small B (large m). Table 3.1 shows the coefficients c_n and α_n in the mass shift

$$\delta M = -c_n\frac{B^2R^2}{m^{\alpha_n}} \tag{3.65}$$

as obtained by fitting (3.65) to the numerical result for the meson masses displayed in figure 3.14, also for the higher modes $n \geq 1$. Keeping in mind that according to our experience, the numerical errors get larger for higher states (although I can not specify the error quantitatively), there is a chance that α_n is n-independent with a value of three, as can be expected on dimensional grounds by requiring the mass shift to be proportional to the dimensionful Kalb-Ramond field $\bar{B}^2$ and simultaneously independent of α'.

3.3.2 Constant External Electric Fields and Finite Temperature

I conclude this chapter by commenting on the embeddings in the finite temperature AdS-Schwarzschild background with electric field, which we obtain numerically from the action (3.41). Since we are introducing quark number density through the D7 brane gauge field, we have to distinguish between embeddings for zero and finite density, as smooth Minkowski embeddings are only consistent for vanishing density [262]. The reason for this is the following: A nonvanishing quark number density, i.e. a nonvanishing brane gauge field configuration, is a sign of strings ending on the D7 probe and stretching towards the black hole horizon. These strings will try to minimise their length, i.e. in cases where they are parallel to the L-axis they tend to attach to the probe brane at $\rho = 0$.[17] The string tension will have to be balanced by the D7 brane tension, which creates either a cusp at $\rho = 0$ or, if force balance is not achievable, the brane is dragged into the black hole. Since the Minkowski embeddings do not show such a behaviour but close off smoothly with $L'(0) = 0$ they can not be physical in the presence of a finite quark number density. Note that considering finite densities is the more general case: For ensuring a regular DBI action at the singular shell, $D = 0$ would suffice to satisfy equation (3.40).

Figure 3.16 shows embeddings calculated for different strengths of the electric field, and vanishing quark number density $D = 0$. At zero quark number density both Minkowski and singular shell embeddings are consistent (cf. figure 3.10). A well-defined zero temperature limit in the canonical ensemble reproducing figure 3.10, i.e. reproducing the Minkowski embeddings, is possible for $D = 0$ only. Note however that in the grand canonical ensemble

[17] With a finite electric field a more plausible configuration might be a string stretching downwards in the $L - \rho$ diagram while trailing in the x-axis too, its other endpoint being dragged by the electric field.

at zero temperature considered in [259], new kinds of throat-like embeddings appear for chemical potentials smaller than the quark mass. These embeddings, which correspond to $D7-\overline{D7}$ configurations, have already been discussed in section 2.3.6, cf. footnote 49 there.

As can be seen from figure 3.16, in contrast to finite quark number density, the black hole embeddings cover only a finite range of asymptotic quark mass up to a maximal value. Above this mass, whose B field dependence we fitted numerically to be

$$\tilde{m} = (1.27 \pm 0.03)\sqrt{BR^2 + \sqrt{4b^4 + B^2R^4}}/\sqrt{2}\,,$$

Minkowski embeddings take over. This now allows for a well-defined zero temperature limit. Lowering the temperature, the black hole horizon shrinks to the extremal AdS horizon at $\rho = L = 0$, where the black hole embeddings then end. The corresponding chiral condensate, in particular its dependence on the field strength, on the temperature and on the quark mass, was calculated in [288].

Figures 3.17 and 3.18 show black hole embeddings calculated for a quark number density of $D = 10$ and $D = 20$, respectively. For comparison we also included Minkowski embeddings which, however, can only coexist with black hole embeddings at finite quark number density in a grand canonical ensemble [275], where the quark chemical potential is fixed instead of the number density. The reason for this is that in the grandcanonical ensemble the number density can vary and actually drop to zero for a finite quark mass, above which Minkowski embeddings can then take over without having strings end on them. In the grandcanonical ensemble a meson melting phase transition between black hole and Minkowski embeddings is thus possible [316], in contrast to the canonical ensemble [262] where the transition happens between two black hole embeddings. Recent work published in [110] indicates that in the presence of the electric field the phase transition is washed out to a crossover, at least at not too small densities. The authors of [110] however verified that the phase transition is still present at small but nonzero densities, i.e. the line of phase transitions of [262] is shortened, but not immediately destabilised by the electric field [317].

The black hole embeddings on the other hand pass through the shell of vanishing action smoothly and reach the black hole horizon. We observe that black hole embeddings at finite quark number density can cover the whole range of asymptotic quark masses, a fact already noted without external electric fields in [262]. Comparing the two figures 3.17 and 3.18, we find the effect of changing the number density D as given by (3.29): In both figures, the Minkowski and black hole embeddings were calculated for the same values of the infrared boundary condition $L(\rho_{IR}) = L_{IR}$. The black hole embeddings (drawn in green colour) are sensitive to the value of D: For a given infrared boundary condition, they reach larger asymptotic quark masses $L(\infty)$ for larger D. Thus the scaling of D corresponds to a dilation of the energy scale. Note that this scaling effect is also present for black hole embeddings in the canonical ensemble at zero external electric field, c.f. e.g. figure 4 in [262].

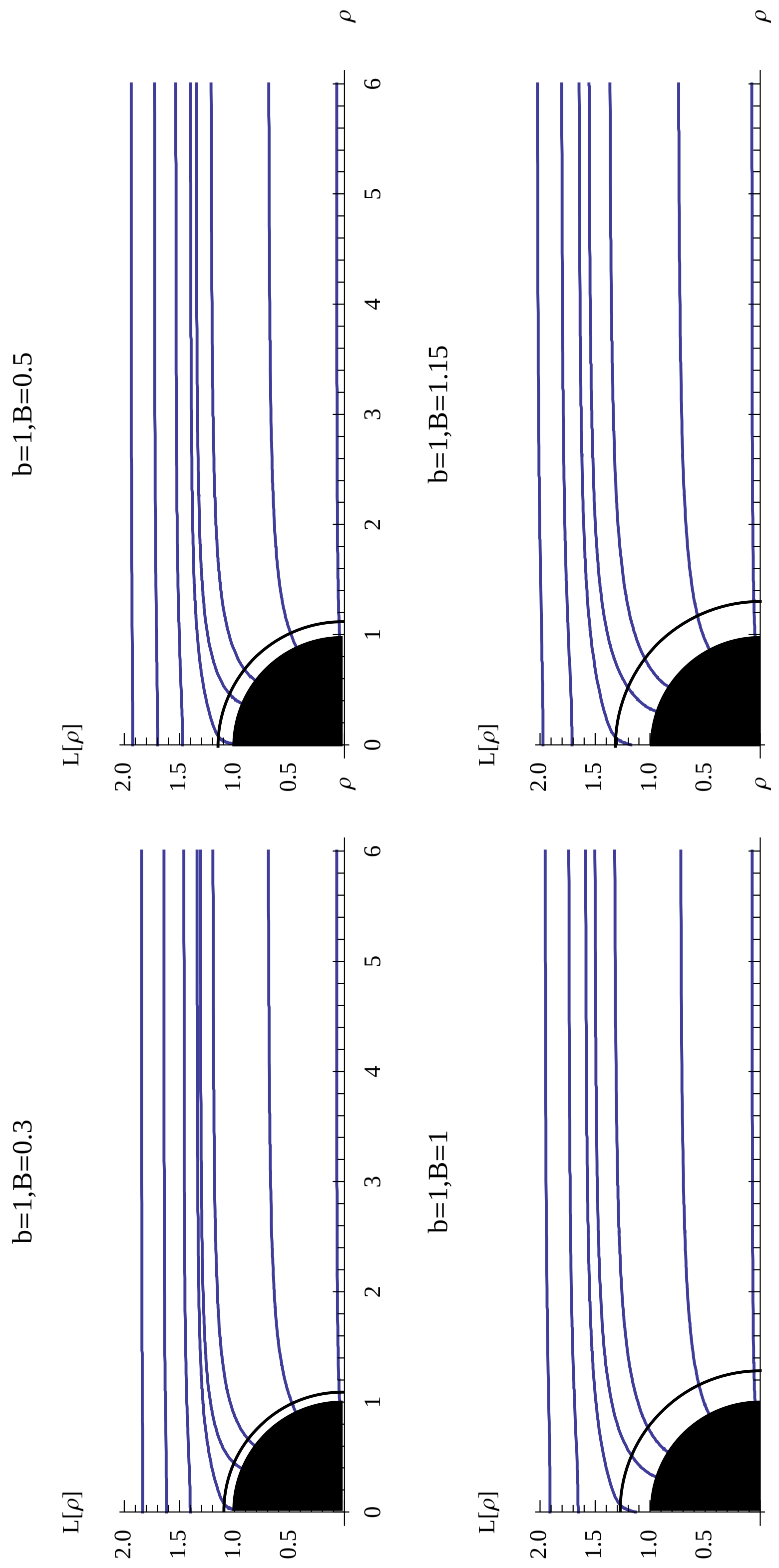


Figure 3.16: Minkowski and singular shell embeddings for the electric case at finite temperature and zero quark number density $D = 0$.

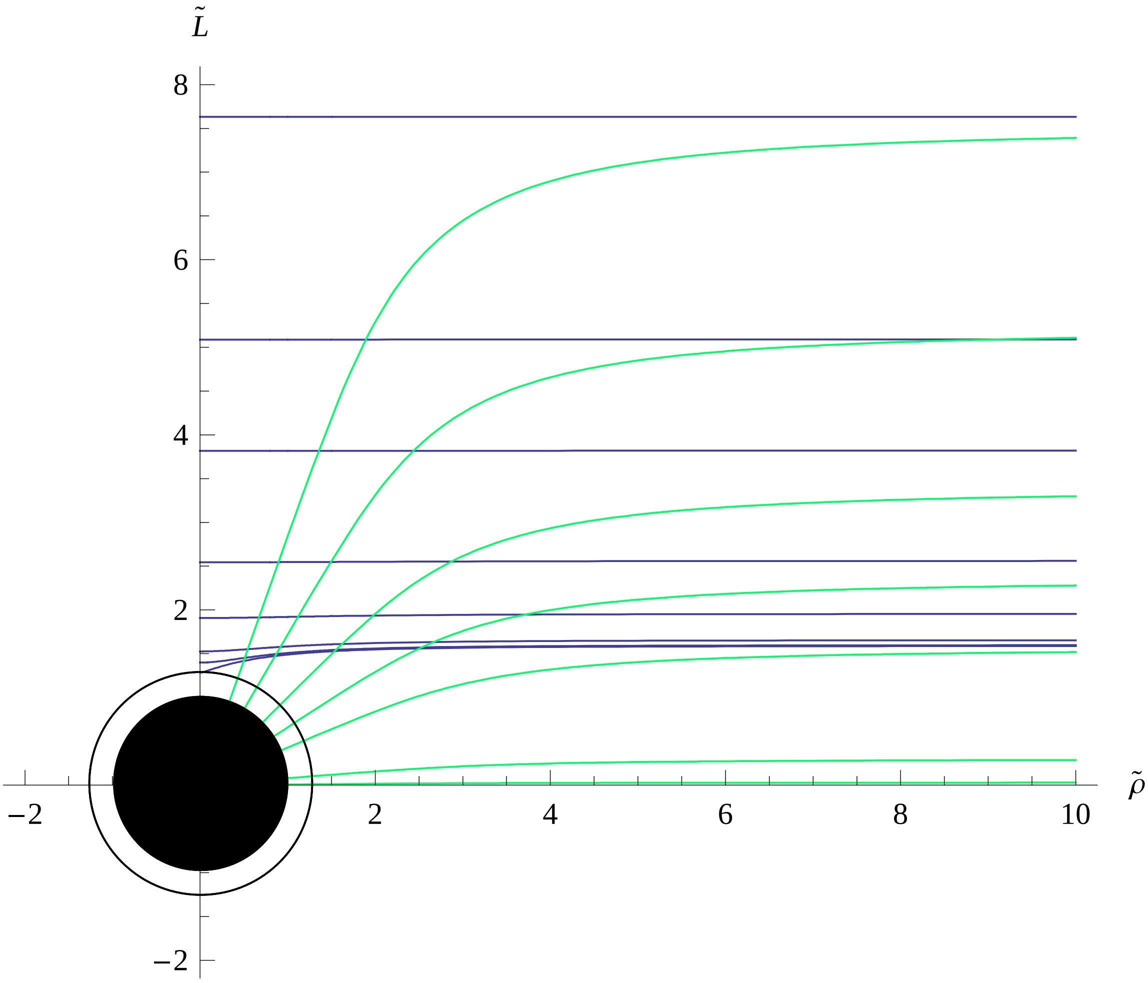


Figure 3.17: Minkowski ($b = B = R = 1$, $D = 0$) and black hole embeddings for the electric case at finite temperature ($b = B = R = 1$, $D = 10$). Note that both types os embeddings can only exist simultaneously in the grand canonical ensemble [275].

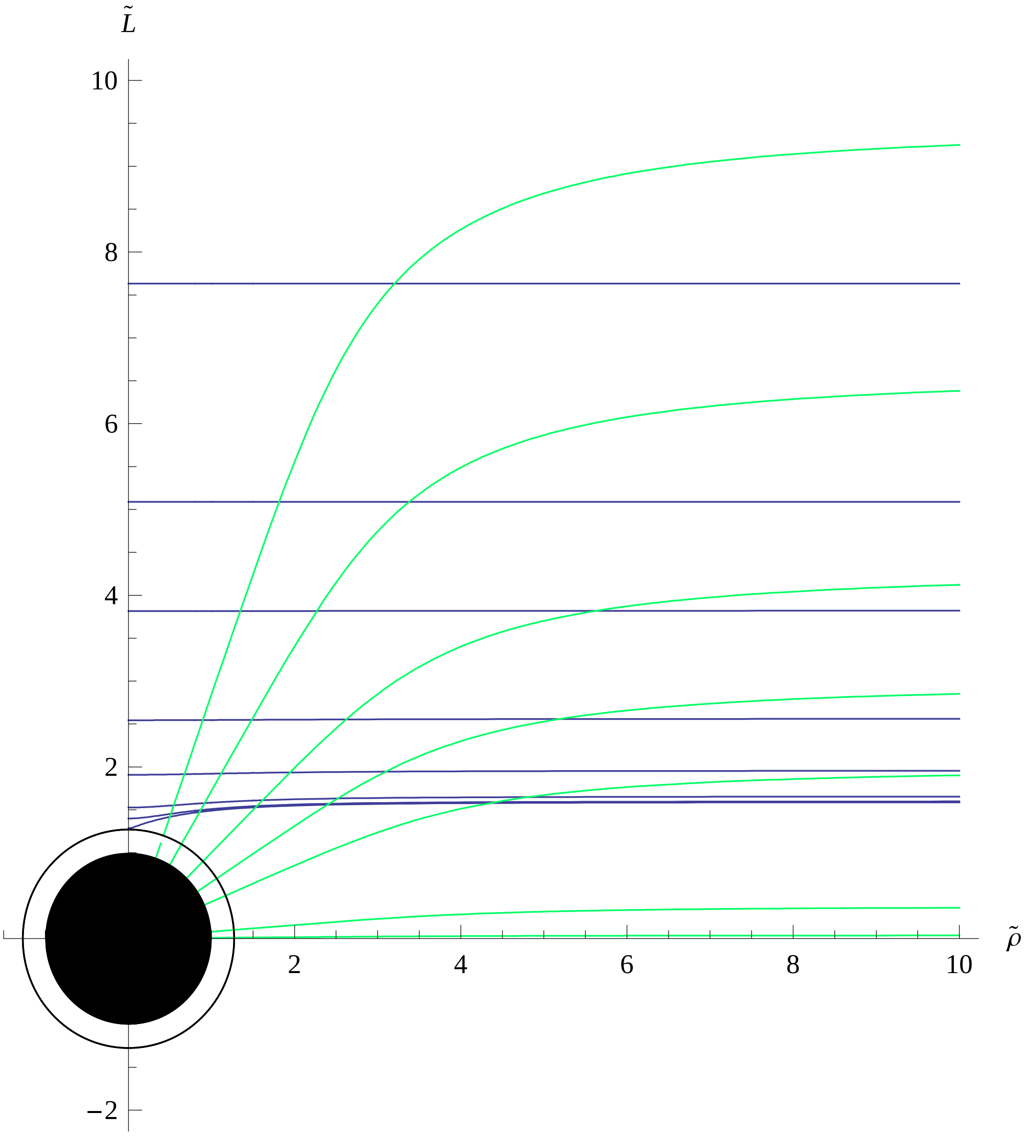


Figure 3.18: Minkowski ($b = B = R = 1,\ D = 0$) and black hole embeddings for the electric case at finite temperature ($b = B = R = 1,\ D = 20$). Note that both types of embeddings can only exist simultaneously in the grand canonical ensemble [275].

Chapter 4

Holographic Fayet-Iliopoulos Terms from Kalb-Ramond Fields

4.1 Introduction and Summary

In view of making further progress towards generalizing the AdS/CFT correspondence [41, 219, 220] to physically relevant phenomena, it is useful to study the gravity duals of quantum field theories with nontrivial moduli spaces. This applies in particular to theories with flavour added by virtue of probe D7 branes wrapping a subspace of $\mathrm{AdS}_5 \times \mathrm{S}^5$ which is asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^3$ [95]. It has been shown in [112–115] that the mixed Coulomb-Higgs branch of the $\mathcal{N} = 2$ theory with two flavours is dual to instanton configurations in the supergravity theory. These are instanton solutions for the $SU(2)$ gauge field in the four directions of a probe of two D7 branes perpendicular to the AdS boundary.[1] These solutions arise since the Dirac-Born-Infeld and Wess-Zumino contributions to the D7 probe brane action combine in such a way as to give an action containing only the anti-selfdual part of the field strength tensor. The instantons are selfdual with respect to the flat four-dimensional metric. This is due to the fact that the metric dependence of the D7 brane action is limited to an overall factor which only depends on the AdS radial coordinate. Moreover, it was shown that the Higgs vacuum squark expectation value is dual to the size of the instanton in the supergravity theory. In [115] the vector meson spectrum of fluctuations about this instanton background was computed as a function of the instanton size, and it was shown that the spectra at zero and infinite instanton size are related by a singular gauge transformation. In [320] the instanton analysis was extended to all orders in α', with special emphasis on the small instanton behaviour.

In this chapter I present the results of the work [2], in which the construction of a gravity

[1] This is based on the fact that Yang-Mills instantons can be described by Dp branes dissolved inside the worldvolume of D(p+4) branes [318, 319] – for a review see [116].

dual of a quantum field theory in which Fayet-Iliopoulos terms are present was proposed. For this purpose I consider a constant Kalb-Ramond B field in the four directions perpendicular to the AdS boundary, but parallel to the D7 brane probe. The B field generates noncommutativity of the coordinates in these directions [111]. We found in [2] that for a D7 brane probe in a selfdual B field background, a no-force condition is satisfied, and there is no Fayet-Iliopoulos term present. On the other hand, for a D7 brane probe in an anti-selfdual B field background a no-force condition does not exist and the supersymmetry breaking is parametrized by a Fayet-Iliopoulos term. In this case, the B field is dual to an $\mathcal{N} = 2$ Fayet-Iliopoulos term, coupling to the triplet (D, F_1, F_2) of real fields in the dual field theory. We showed that the quantum numbers of the B field and the Fayet-Iliopoulos term coincide. The analysis of [117] for the D3-D(-1)-system in flat space also supports this claim: The anti-selfdual part of the B field is shown there to induce the Fayet-Iliopoulos term in the effective action via string disk diagrams.

In the case of an anti-selfdual B field background, the D3/D7 brane intersection in flat space-time, which consists of N_c D3 branes generating the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space-time in the near-horizon limit and N_f (coincident) D7 branes, is not static. A no-force condition does not exist and the D3 and D7 branes attract each other. The attractive force is parametrically small when the Fayet-Iliopoulos term is small. If the distance of the D3 and D7 branes is below a critical value, tachyons appear in the spectrum of 3-7 strings. Consider one of the D3 branes at a distance below the critical value: After tachyon condensation, it will be dissolved in the D7 brane and can be described by instanton configurations on the D7 brane. The low energy effective field theory description is that the dissolved D3 brane manifests itself through a squark vacuum expectation value for the direction in colour space associated to it. The colour direction described by this particular D3 brane will then be broken, and the field theory will be in a vacuum state on the mixed Coulomb-Higgs branch of the moduli space of vacua. If all D3 branes are dissolved in the N_f D7 branes, the configuration will be static. I will show that this configuration is supersymmetric, which can also be understood from the gauge theory point of view. However, the probe approximation in AdS/CFT is not applicable for the D7 any longer, if infinitely many D3 branes are dissolved in the D7 brane.

Therefore I assume in this chapter that k D3 branes are dissolved in the D7 brane with $k \ll N_c$, giving rise to an instanton configuration with charge k, and that the remaining $N_c - k$ D3 branes are separated from the D7 branes and generate the $\mathrm{AdS}_5 \times \mathrm{S}^5$ space. If k and N_f are small, the probe limit applies but is not sufficient for ensuring the stability of the setup: The configuration is not supersymmetric and thus in general will decay into the supersymmetric ground state, which is the pure Higgs vacuum with N_c instantons on the D7 stack. It however can be rendered arbitrarily longlived if the Fayet-Iliopoulos parameter ζ is tuned to a very small value: Although the decay process, which involves a quantum of five-form flux to be transferred from the S^5 onto the D7 stack and to manifest itself on the D7 branes as an instanton (while at the same time the radius of AdS_5 and S^5 shrinks by one unit) is a backreaction effect and hence suppressed by $\frac{N_f}{N_c}$, there are N_c

identical quanta of five-form flux which could jump over. This means that the total decay probability is of order $\frac{N_f}{N_c}N_c = N_f$, and thus in general not negligible. However, since for vanishing Fayet-Iliopoulos parameter the setup is supersymmetric and therefore stable, the decay probability must be, at least for small ζ, proportional to ζ itself,

$$P_{decay} \propto \zeta \frac{N_f}{N_c} N_c = \zeta^\alpha N_f \, ,$$

and hence can be tuned to an arbitrarily small number. Therefore the setup can be rendered arbitrarily longlived by tuning ζ to be very small. It is therefore safe to let the $N_c - k$ D3 branes for large N_c generate the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric and the selfdual five-form flux, and embed the D7 stack with instantons in the usual way. This approach can be viewed as an adiabatic approximation, since the decay time will be proportional to $(N_f\zeta)^{-1}$, i.e. the system will not decay except at very late times if ζ is very small.

Another indication for the decay of this setup being suppressed is that the $\overline{\mathrm{D3}}$ brane charge induced on the D7 by the B field can not leave the D7 through the on-shell process of condensing first to a zero-size anti-instanton which then detaches from the D7 as a $\overline{D3}$. The reason is that the small instanton limit is no longer available if a Fayet-Iliopoulos term is present in the ADHM equations, such that the $\overline{D3}$ would need to go off-shell to condense outside the D7. This process might only be possible through a tunneling process, being exponentially suppressed.[2]

I show that although from the geometric point of view supersymmetry is broken in the presence of an anti-selfdual B field, the modified ADHM equations for the noncommutative instantons on the D7 probes can still be identified with the D and F term equations for the Higgs part of mixed Coulomb-Higgs states in the dual gauge theory on the boundary of AdS. In these D and F term equations a Fayet-Iliopoulos term is present. By Higgs part of a Coulomb-Higgs state I mean the colour directions for which the squark fields acquire a vacuum expectation value. The other colour directions, in which the adjoint scalars acquire a vacuum expectation value, comprise the Coulomb part of the state. In the scenario presented, supersymmetry is then broken on the Coulomb part of the Coulomb-Higgs branch, but the D and F term equations (i.e. the ADHM equations) for the Higgs part of the Coulomb-Higgs branch can still be satisfied simultaneously.

According to Nekrasov and Schwarz [119], there is a noncommutative $U(1)$ instanton[3] solution to the modified ADHM equations. In analogy to the case without B field [112,115], I argue that the flat space Nekrasov-Schwarz instanton solution remains a solution of the equations of motion for a D7 probe brane embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Note that the instanton

[2] The author thanks Luca Martucci for this argument.

[3] In this chapter instantons are selfdual. Originally, Nekrasov and Schwarz constructed anti-selfdual instantons in flat space-time with selfdual noncommutativity, but call them instantons (see footnote 2 in [321] to avoid future confusion). In this chapter I consider selfdual instantons in backgrounds with anti-selfdual B field (and therefore anti-selfdual noncommutativity) which can be obtained from the original Nekrasov-Schwarz solution by a parity transformation. This instanton is also called Nekrasov-Schwarz instanton in this chapter.

is selfdual with respect to the flat metric in the respective directions. Still, it is a solution of the gauge theory on the D7 brane.

To summarise, the picture which emerges is the following: In the presence of an anti-selfdual B field, a Fayet-Iliopoulos term is generated for the gauge theory on the boundary. This is supported by the matching quantum numbers for the B field in the bulk and the auxiliary fields in the boundary field theory in section 4.4.1. The Fayet-Iliopoulos term is associated with the $U(1)$ factor of the $U(N_c)$ gauge group. In fact, the presence of the D7 brane probe is essential for our construction, since it ensures that the constant background B field may no longer be gauged away by means of a Ramond-Ramond gauge transformation. This implies that the dual field theory at the boundary has $U(N)$ gauge symmetry instead of $SU(N)$, as is necessary for a Fayet-Iliopoulos term to be present. The $U(1)$ factor corresponds to singleton degrees of freedom [294, 295]. It will be broken by instanton solutions in the dual gravity theory.

On the gravity side, there are noncommutative $U(1)_f$-instantons on a single D7 brane, which we conjectured in [2] to be dual to a particular mixed Coulomb-Higgs state in the dual gauge theory determined by the instanton charge. These states do not correspond to supersymmetric vacua of the theory, as the D and F term equations for the Coulomb part cannot be satisfied in the presence of the Fayet-Iliopoulos term. Rather, they are excited states with an excitation energy set by the Fayet-Iliopoulos parameter. Hence, throughout this chapter I refer to them as "states" rather than "vacua", except in section 4.2, where the Coulomb-Higgs vacua are actual true vacua of the theory. The size moduli of the instanton configurations are identified with the squark Higgs vacuum expectation values on the gauge theory side. For the selfdual Nekrasov-Schwarz instanton on a single D7 brane we obtained as a nontrivial prediction the existence of this new Higgs state in the gauge theory with a single flavour. The Higgs vacuum squark expectation value q is shown to be given by the square root of the Fayet-Iliopoulos parameter ζ, $q = \sqrt{\zeta}$ whereas the anti-squark vacuum expectation value $\tilde{q}$ vanishes, $\tilde{q} = 0$.

In order to test the conjectured equivalence between Coulomb-Higgs states of the field theory and noncommutative instantons on the D7 probe brane, we investigated the breaking of global symmetries by the anti-selfdual B field which generates the Fayet-Iliopoulos term and then also in the special case of one noncommutative $U(1)_f$ instanton. We found that an anti-selfdual B field breaks the $SU(2)_L \times SU(2)_R \times U(1)_{89}$ of the D7 brane configuration to $SU(2)_L \times U(1)_R \times U(1)_{89}$. This corresponds to the symmetries which remain on the field theory side. We also conjectured that switching on the selfdual noncommutative $U(1)_f$ instanton on the D7 brane further breaks these symmetries down to $SU(2)_L \times \mathrm{diag}(U(1)_R \times U(1)_f) \times U(1)_{89}$. I will argue in favour of this conjecture in section 4.5, and also discuss the possible reasons for the missing proof of it. I will also show that the squark vacuum expectation values $q = \sqrt{\zeta}$, $\tilde{q} = 0$ in this case break the flavor and $U(1)_R$ symmetries to the diagonal subgroup $\mathrm{diag}(U(1)_R \times U(1)_f)$. Furthermore, these relations break the $U(1)$ factor of the gauge group $U(N_c)$.

Our construction is also similar to models studied within cosmology, in the context of inflation. D3/D7 systems with B field [322] or on the resolved conifold [323–325] have been investigated, and D term generation has been studied in [325, 326]. Our analysis of D3 branes dissolving into D7 branes bears similarities with the inflationary models of [327–329]. It will be interesting to generalize the gauge/gravity construction presented here, in particular by using a suitable stabilisation mechanism, to investigate these relations further. Also, in AdS/CFT with flavour the impact of internal B fields on the meson spectrum has been investigated for the Polchinski-Strassler [85] and Maldacena-Lunin [330] backgrounds [280, 331], for a review and further references see [253].

This chapter is organized as follows: In section 4.2 I review the description of the Higgs branch of the dual theories using instanton configurations on these probe branes. In section 4.3 some prerequisites concerning noncommutative field theories, the noncommutative ADHM equations and their string theoretic origin are reviewed. In section 4.4 I show how the noncommutativity induced by the B field in the internal directions of the probe brane translates into the Fayet-Iliopoulos term on the dual gauge theory side. In section 4.5 I investigate the effect of switching on a noncommutative instanton in this internal noncommutative field theory. A conclusion with discussions of the results and possible further developments can be found in chapter 6, section 6.2. The material presented in this chapter was obtained in collaboration together with Martin Ammon, Johanna Erdmenger, Stephan Höhne and Dieter Lüst, and published in the paper [2]. In particular the κ-symmetry calculation in section 4.4.3 was worked out by Martin Ammon.

4.2 Commutative Instantons on Flavour Branes

This section is intended to review the work of [115, 320], in which it was shown that instantons on D7 brane probes describe the mixed Coulomb-Higgs branch of $\mathcal{N}=4$ $U(N_c)$ Super-Yang-Mills theory coupled to N_f $\mathcal{N}=2$ fundamental quark hypermultiplets. For selfcontainedness of this chapter I will first review some basic facts about D7 brane probes in AdS/CFT, which were introduced in detail in section 2.3.6.

In the standard AdS/CFT correspondence, the $\mathcal{N}=4$ $U(N_c)$ Super-Yang-Mills theory is realized as the near horizon limit of N_c D3 branes. The type IIB supergravity background is given by

$$\begin{aligned} ds^2 &= H_3^{-1/2}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + H_3^{1/2}\left(dy^m dy^m + dz^i dz^i\right), && (4.1)\\ C_{(4)} &= H_3^{-1}\,dx^0\wedge\cdots\wedge dx^3, && (4.2)\\ e^\phi &= g_s = \text{const.}, && (4.3)\end{aligned}$$

where $r^2 = y^m y^m + z^i z^i$ and

$$H_3(r) = \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_S N_c \alpha'^2. \tag{4.4}$$

Here the Minkowski coordinates are $x^\mu\,,\ \mu = 0,\dots,3$, while the internal coordinates are split into two sets, $y^m\,,\ m = 4,\dots,7$ and $z^i\,,\ i = 8,9$. To couple the dual $U(N_c)$ $\mathcal{N}=4$ Super-Yang-Mills theory to fundamental matter fields, I follow [95] and embed N_f D7 branes into this background in the way given by table 2.9. The D7 branes fill the x^μ- and y^m-directions, while their profile is parametrized by the two transversal directions $z^i(x^\mu, y^m)$. To describe fundamental matter with mass m, the embedding of the D7 brane is specified by $z^8 = 2\pi\alpha' m$ and $z^9 = 0$. In the stringy picture, i.e. before replacing the D3 branes by their near-horizon geometry, the matter hypermultiplets arise as the massless excitations of strings stretching between the D3 and D7 branes. Since I consider N_f to be small, in the limit of large N_c at large (but fixed) 't Hooft coupling λ we can ignore the backreaction of the D7 branes to the background. This corresponds to the quenched approximation.

The dual field theory is an $\mathcal{N}=2$ supersymmetric $U(N_c)$ gauge theory, which has N_f hypermultiplets in the fundamental representation of the gauge group, coupled to the $\mathcal{N}=4$ vector multiplet in the adjoint representation. The scalar components $\Phi_{1,2,3}$ of the latter encode the positions $X^{4,5,6,7,8}$ of the D3 branes in the transverse six directions,

$$2\pi\alpha'\Phi_1 = X^4 + iX^5\,,\ 2\pi\alpha'\Phi_2 = X^6 + iX^7\,,\ 2\pi\alpha'\Phi_3 = X^8 + iX^9\,. \tag{4.5}$$

Note that in this chapter the scalar components of these chiral superfields are denoted with the same letter as the superfields themself. The action, written in $\mathcal{N}=1$ superspace formalism, is given by eq. (2.140),

$$\begin{aligned} S_{D3D7} \;=\; & \int \mathrm{d}^4x\, \Im \Bigg[\tau \int \mathrm{d}^4\theta \left(\mathrm{tr}\left(\bar\Phi_I e^V \Phi_I e^{-V}\right) + Q_i^\dagger e^V Q^i + \tilde Q_i e^{-V} \tilde Q^{\dagger i} \right) \\ & + \tau \left(\int \mathrm{d}^2\theta\, \mathrm{tr}\left(W_\alpha W^\alpha + \epsilon_{IJK}\Phi_I\Phi_J\Phi_K\right) + \tilde Q_i (m+\Phi_3) Q^i + \text{h.c.} \right) \Bigg]\,, \end{aligned} \tag{4.6}$$

where the last two terms comprise the superpotential W,

$$W = \mathrm{tr}\left(\epsilon_{IJK}\Phi_I\Phi_J\Phi_K\right) + \tilde Q_i\,(m+\Phi_3) Q^i\,. \tag{4.7}$$

Here m denotes the mass of the quarks, which I choose to be equal for all flavours. This theory has a number of global symmetries: For massless quarks ($m=0$), it has a $SO(4,2) \times SU(2)_\Phi \times SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ global symmetry, of which the first factor is the conformal group in four dimensions.[4] In the massive case, the $SO(4,2)$ gets broken to the Lorentz group $SO(3,1)$, and the $U(1)_\mathcal{R}$ factor is broken by the mass term in the superpotential. In both cases, the second $SU(2)$ factor is a $\mathcal{N}=2$ $\mathcal{R}$-symmetry, $SU(2)_\mathcal{R}$, while the first $SU(2)$ is an additional global $SU(2)_\Phi$ symmetry, rotating the chiral superfields Φ_1 and Φ_2. Additionally, there is a $U(N_f)$ flavour symmetry, the obvious $U(1)_f$ factor of which is a quark number symmetry [253]. The different component fields and their quantum

[4]The β function vanishes for the 't Hooft coupling vanishes in the strict $N_c \to \infty$ limit with N_f fixed [256].

numbers are given in table 4.1, where I also listed the quantum numbers of the auxiliary fields (D, F_1, F_2) of the $\mathcal{N} = 2$ $U(N_c)$ vector multiplet (W_α, Φ_3), as their $U(1)$ part can couple to a Fayet-Iliopoulos term, which will become important later on.

The identification of symmetries between the gravity description and the gauge theory is now straightforward: The Lorentz-group $SO(3,1)$ (or the conformal group $SO(4,2)$ if only massless fundamental hypermultiplets are considered) corresponds to isometries of the induced metric on the embedded D7 brane derived from the AdS$_5$ part of AdS$_5 \times$ S^5. The internal $SO(4)_{4567} \simeq SU(2)_L \times SU(2)_R$, which rotates the coordinates y^m into each other, is identified with $SU(2)_\Phi \times SU(2)_\mathcal{R}$ for a D7 brane. The rotations acting on $\vec{z}$, $U(1)_{89}$, are identified with the $U(1)_\mathcal{R}$-symmetry on the field theory side.

$(\mathcal{N} = 2)$	components	spin	$SU(2)_\Phi \times SU(2)_\mathcal{R}$	$U(1)_\mathcal{R}$	Δ	$U(N_f)$	$U(1)_f$
Φ_1, Φ_2	X^4, X^5, X^6, X^7	0	$(\frac{1}{2}, \frac{1}{2})$	0	1	1	0
	λ_1, λ_2	$\frac{1}{2}$	$(\frac{1}{2}, 0)$	-1	$\frac{3}{2}$	1	0
Φ_3, W_α	$X_V^A = (X^8, X^9)$	0	$(0, 0)$	$+2$	1	1	0
	λ_3, λ_4	$\frac{1}{2}$	$(0, \frac{1}{2})$	$+1$	$\frac{3}{2}$	1	0
	v_μ	1	$(0, 0)$	0	1	1	0
	(D, F_1, F_2)	0	$(0, 1)$	0	2	1	0
Q, $\tilde{Q}$	$q^m = (q, \bar{\tilde{q}})$	0	$(0, \frac{1}{2})$	0	1	N_f	$+1$
	$\psi_i = (\psi, \tilde{\psi}^\dagger)$	$\frac{1}{2}$	$(0, 0)$	± 1	$\frac{3}{2}$	N_f	$+1$

Table 4.1: Field content and quantum numbers of the $\mathcal{N} = 2$ theory including the auxiliary fields from the $\mathcal{N} = 2$ vector multiplet

The supersymmetric field theory (4.6) with superpotential (4.7) has **Coulomb- and Higgs vacua**, i.e. vacua with nonvanishing expectation value for the adjoint scalars Φ_i, or for the fundamental scalars q^i and $\tilde{q}_i$, respectively. For a mixed choice of color space components of these fields, i.e. for nonzero vacuum expectation values for the adjoints in some color directions and the fundamentals in other colour directions, the corresponding vacua are called **mixed Coulomb-Higgs vacua**, meaning that some generators of the gauge group are broken down to its respective Cartan subalgebra generators, yielding $U(1)$ factors on the Coulomb branch, while other parts of the gauge group are broken completely - this is the Higgs part of the mixed vacuum.

The vacua are solutions of the F and D term equations

$$
\begin{aligned}
0 &= (m + \Phi_3) q^i = \tilde{q}_i (m + \Phi_3) && (4.8)\\
0 &= [\Phi_1, \Phi_3] = [\Phi_2, \Phi_3] && (4.9)\\
0 &= q^i \tilde{q}_i + [\Phi_1, \Phi_2] && (4.10)\\
0 &= |q^i|^2 - |\tilde{q}_i|^2 + [\Phi_1, \Phi_1^\dagger] + [\Phi_2, \Phi_2^\dagger]\,. && (4.11)
\end{aligned}
$$

Here $q, \tilde{q}$ are the squark fields, while Φ_I are the scalar components of the adjoint transverse scalar superfields. The mixed Coulomb-Higgs branch is accessed by solutions of (4.8) with

$$\Phi_3 = \begin{pmatrix} \tilde{m}_1 & & & & & \\ & \ddots & & & & \\ & & \tilde{m}_{N-k} & & & \\ & & & -m & & \\ & & & & \ddots & \\ & & & & & -m \end{pmatrix}, \quad q^i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ q_1^i \\ \vdots \\ q_k^i \end{pmatrix}, \tag{4.12}$$

$$\tilde{q}_i = \left(0 \cdots 0, \tilde{q}_i^1 \cdots, \tilde{q}_I^k\right) . \tag{4.13}$$

The $\tilde{m}$ are the masses of the first $N_c - k$ colour components of the fundamental hypermultiplet. The remaining equations (4.9), (4.10) and (4.11) then reduce to

$$0 = (\tilde{m}_b - \tilde{m}_a)(\Phi_1)_{ab} = (\tilde{m}_b - \tilde{m}_a)(\Phi_2)_{ab}, \quad a, b = 1, \ldots, N-k \tag{4.14}$$

$$0 = q_a^i \tilde{q}_{bi} + [\Phi_1, \Phi_2]_{ab}, \quad a, b = 1, \ldots, k \tag{4.15}$$

$$0 = |q^i|_{ab}^2 - |\tilde{q}_i|_{ab}^2 + [\Phi_1, \Phi_1^\dagger]_{ab} + [\Phi_2, \Phi_2^\dagger]_{ab} . \tag{4.16}$$

Equation (4.14) can be solved either by setting the masses $\tilde{m}$ equal, or by switching off some off-diagonal elements of $\Phi_{1,2}$ in the corresponding gauge directions where $\tilde{m}_b - \tilde{m}_a \neq 0$. In any case, the first $N-k$ diagonal elements of $\Phi_{1,2}$ may be chosen freely. The off-diagonal (lower-left and upper-right) $k \times (N-k)$ blocks of $\Phi_{1,2}$ are not constrained by (4.14). The lower $k \times k$ block is constrained by the last two equations (4.15) and (4.16).

With the identification (the factor $2\pi\alpha'$ must be included for dimensional reasons)

$$\{(I)_a{}^i, (J)_j{}^b, (B_0)_{ab}, (B_1)_{ab}\} \equiv 2\pi\alpha' \{q_a^i, \tilde{q}_J^b, (\Phi_1)_{ab}, (\Phi_2)_{ab}\}, \ a, b = 1, \ldots, k, \ i, j = 1, \ldots, N_f, \tag{4.17}$$

equations (4.15) and (4.16) are just the ADHM equations [118] for the moduli of k gauge instantons in a $SU(N_f)$ Yang-Mills theory,

$$0 = \mu_r = [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J, \tag{4.18}$$

$$0 = \mu_c = [B_0, B_1] + IJ, \tag{4.19}$$

from which all instanton solutions can, in principle, be constructed in the following way [119]: Once the $k \times k$ matrices B_0 and B_1 and the $k \times N_f$ and $N_f \times k$ matrices I and J are known from equations (4.18)-(4.19), one complexifies the $\ ^4 \simeq \ ^2$ via $(z_0, z_1) = (y^4 + iy^5, y^6 + iy^7)$ and constructs an operator $\mathcal{D}_z^\dagger : \ ^k \oplus \ ^k \oplus \ ^{N_f} \to \ ^k \oplus \ ^k$ via[5]

$$\mathcal{D}_z^\dagger = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix}, \tag{4.20}$$

$$\tau_z = (B_0 - \bar{z}_0 \ _{k\times k}, z_1 \ _{k\times k} - B_1, I), \tag{4.21}$$

$$\sigma_z = \begin{pmatrix} B_1 - z_1 \ _{k\times k} \\ B_0 - \bar{z}_0 \ _{k\times k} \\ J \end{pmatrix}. \tag{4.22}$$

[5]This is the definition for a selfdual configuration. It differs from the definitions in [119] by the exchange $z_0 \leftrightarrow \bar{z}_0$.

The zero modes of the operator $\mathcal{D}_z^\dagger$ are matrices $\psi : {}^{N_f} \to {}^{k} \oplus {}^{k} \oplus {}^{N_f}$ fulfilling the zero mode condition and a normalisation condition

$$\mathcal{D}_z^\dagger \psi = 0 \,, \quad \psi^\dagger \psi = {}_{N_f \times N_f} \,. \tag{4.23}$$

They can now be used to construct the instanton gauge potential via

$$A = \psi^\dagger \mathrm{d}\psi \,. \tag{4.24}$$

The claim of [118] is now that this construction yields all instanton configurations in four-dimensional Euclidean space, where the field strength is given by

$$F = \psi^\dagger \left(\mathrm{d}\mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathrm{d}\mathcal{D}_z^\dagger \right) \psi \,. \tag{4.25}$$

For the proofs of uniqueness and self-duality of the so-constructed gauge field configurations, see e.g. the extensive review [116] and references therein or the shorter explanations in [119].

The moduli space of k $U(N_f)$ instantons is thus given by the solutions of the ADHM equations (4.18)-(4.19), i.e. it is the space

$$\mathcal{M}_k = (\mu_r^{-1}(0) \cap \mu_c^{-1}(0))/U(k) \,. \tag{4.26}$$

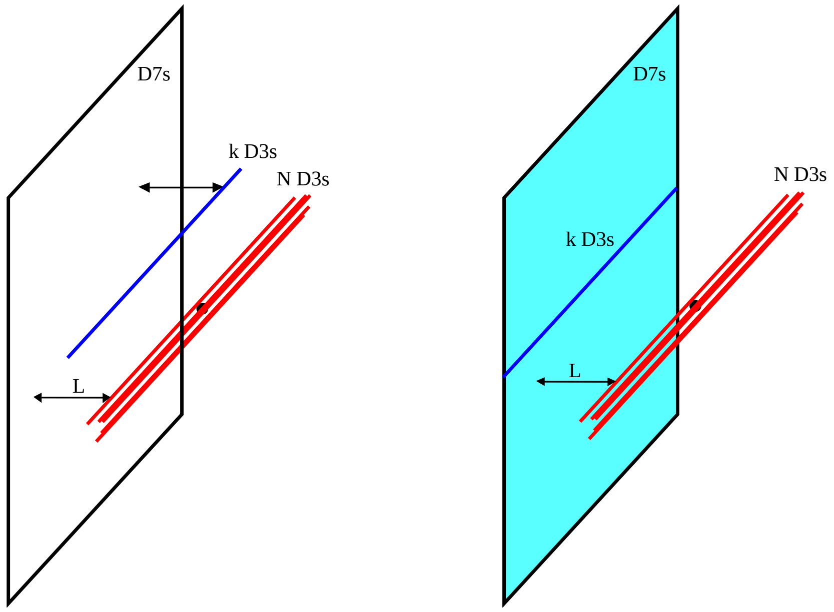
Here $U(k)$ are unitary rotations acting on the indices a, b of eqs. (4.18)-(4.19). Both ADHM equations are invariant under such rotations, and thus solutions differing by such a rotation have to be identified. This moduli space turns out to have dimension

$$\dim \ (\mathcal{M}_k) = 4N_f k \,, \tag{4.27}$$

and it is a singular space. The singularities are fixed points of the $U(k)$ action, i.e. special solutions to the ADHM equations for which subgroups of $U(k)$ act trivially on the $k \times k$ matrices B_0, B_1, $II^\dagger$, $J^\dagger J$. They exactly occur if some of the generically nonzero eigenvalues of the matrices $II^\dagger$ and $J^\dagger J$ vanish. Since the matrices I, J encode the "sizes" of the instantons while B_0, B_1 encode the positions of the instanton centers, the singularities correspond to the zero size limits of instanton solutions. As will be seen later, this is exactly the limit in which the instantons on higher-dimensional D branes have interpretations as dissolved lower-dimensional branes which detach from the higher-dimensional one.

We thus just rediscovered the well-known fact [116, 318, 319, 332] that in a system of N_c Dp and N_f D(p+4) branes the ADHM equations of k $SU(N_f)$ (anti)instantons in the four transverse directions of the D(p+4) brane field theory are exactly the D and F term equations of the intersection $SU(N_c)$ theory.[6] The D and F term equations parametrise the mixed Coulomb-Higgs vacua, namely the vacuum in which k generators of the Cartan

[6] Without noncommutativity, the $U(1)$ factor of the gauge group decouples and is of no importance for this construction.



(a) Pure Coulomb branch

(b) Mixed Coulomb-Higgs branch

Figure 4.1: Different supersymmetric vacua in the D3–D7 model of AdS/CFT with flavour.

subalgebra of $SU(N_c)$ are broken (cf. equation (4.16)). Through the identification (4.17) and the fact that there are no $U(1)$ instanton solutions to the ADHM equations [333], we find that for only one quark flavour the theory (4.6) does not posses Higgs vacua, as there are no nontrivial $U(1)$ instantons in commutative space-time. This changes, as we will see in section 4.4 once the directions transversal to the Dp branes become noncommutative through the introduction of a constant internal Kalb-Ramond field.

Flat Space D Brane Picture

From the geometric point of view of strings and D branes in flat space, this appearance of instantons as supersymmetric Higgs vacua can be understood as either lower dimensional branes dissolving into the higher dimensional ones [332], or equivalently higher dimensional branes acquiring an induced charge associated with lower Ramond-Ramond p-forms. Both Coulomb and Higgs type breaking of gauge symmetries have D brane analogues. In figure 4.1 the N_c colour D3 branes are located at the origin of the six-dimensional transverse space, while the N_f flavour D7 branes are put parallel to the D3 branes, but at a perpendicular separation $|\vec{z}| = \sqrt{X^{8^2} + X^{9^2}} = L = 2\pi\alpha' m$. **Coulomb vacua** – cf. figure 4.1(a) – are configurations with some of the lower-dimensional branes separated from the stack of N_c Dp branes. The transverse scalars (4.5) acquire vacuum expectation values, as they encode the positions of the D3 branes in transverse space. The point in moduli space at which the k separated colour branes coincide with the N_f flavour branes thus also lies on the Coulomb branch. This is only possible as coinciding D3 and D7 branes form a marginal bound state [334], since the Dp-D(p+4) intersection preserves one quarter (i.e. eight real supercharges) of the full thirty-two supercharges of ten-dimensional type IIB supergravity.

If some of the D3 branes coincide with the D7 branes, the D3 can dissolve into the D7. This is understood in terms of the low energy effective action on the D7 brane, which includes a coupling to the Ramond-Ramond four form potential C_4 via

$$-\frac{\mu_7}{g_s}(2\pi\alpha')^2 \int_{\mathcal{M}} P[C_4] \wedge \mathrm{Tr}(F \wedge F)\,.$$

The Pontryagin density of the flavour gauge field on the D7 brane

$$\mathrm{Tr}(F \wedge F)$$

is a source term for C_4.Thus field configurations on the D7 stack with nontrivial Pontryagin number in the $\vec{y}$-directions, i.e. instantons, behave as D3 branes [332], at least concerning their charges. Reversely, starting with a dissolved D3 brane (an instanton) on the D7 brane, one can reach the pure Coulomb branch by sending the size moduli (the vacuum expectation values for the squarks) of the instanton to zero. In this way, the D3 branes sitting on top of the D7 which formerly had a Φ_3 vacuum expectation value $-m$ (cf. equation (4.12)), can move away from the D7 brane (changing Φ_3 to some other value). Thus in the **zero size limit of the instantons on the D7 brane**, the dissolved D3 branes separate themselves from the D7 and can move away from them. The point in moduli space where some or all of the D3 branes lie on top of the D7 branes thus connects the Coulomb- and Higgs parts of the mixed Coulomb-Higgs branch.

Commutative Instantons on D7 Branes in $\mathbf{AdS_5 \times S^5}$

In [115] instantonic configurations on the internal directions of D7 brane probes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ were considered.[7] It was shown that the Dirac-Born-Infeld and Wess-Zumino parts of the action for N_f D7 branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ (cf. equation (4.2)) combine[8] to give to order $(2\pi\alpha')^2$

$$\begin{aligned} S_{D7} &= \frac{(2\pi\alpha')^2\mu_7}{2g_s}\int \mathrm{d}^8\xi\sqrt{g}\mathrm{Tr}(F_{\alpha\beta}F^{\alpha\beta}) - \frac{(2\pi\alpha')^2\mu_7}{2g_s}\int_{\mathcal{M}_7} P[C_4]\wedge F \wedge F \\ &= \frac{(2\pi\alpha')^2\mu_7}{2g_s}\int \mathrm{d}^4x\,\mathrm{d}^4y\frac{(\vec{y}^2+m^2)^2}{R^4}F_-^2\,. \end{aligned} \tag{4.28}$$

[7] See [335] for a more detailed account of the work on instantonic configurations on D7 branes.

[8] This restriction is necessary since the full nonabelian DBI action is not known exactly. The existence of instanton solutions when including higher derivative terms however puts constraints on the unknown higher order terms [112, 114].

In the Yang-Mills term in the first line indices are raised with the pull-back of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ metric (4.1) using the embedding $\xi^\alpha = (x^\mu, y^m)$, $|\vec{z^i}| = 2\pi\alpha' m = \text{const.}$. The (anti)selfdual part of the field strength used in the second line is taken with respect to the flat metric, and is defined as

$$(F_\pm)_{\alpha\beta} = \frac{1}{2}\left(F_{\alpha\beta} \pm \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F_{\gamma\delta}\right) , \tag{4.29}$$

with $\epsilon_{\alpha\beta\gamma\delta}$ being the Euclidean signature four-dimensional totally antisymmetric tensor defined by $\epsilon_{4567} = +1$. The idea of deriving (4.29) is to first solve the leading order in α' part of the D7 brane action to find the embedding $|\vec{z}| = 2\pi\alpha' m$ at vanishing field strengths. Equation (4.28) then holds for these general constant embeddings $|\vec{z}| = 2\pi\alpha' m$, which are the only solutions to the embedding equations of a D7 brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$ consistent with supersymmetry [96]. Every instanton configuration

$$F_- = 0 \tag{4.30}$$

then minimizes the D7 action to order α'^2 and thus solves the equations of motion for the gauge field on the probe brane. As the instanton-probe brane system has an on-shell energy (which is just minus the action since the configuration is static) independent of the quark mass m (i.e. the separation from the N_c D3 branes which sourced the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background), it was concluded in [115] that the instanton-probe brane system in $\mathrm{AdS}_5 \times \mathrm{S}^5$ preserves supersymmetry, since the dual field theory (4.6) also preserves $\mathcal{N} = 2$ supersymmetry independently of the choice of m. [9]

In [320], this argument was pushed further to all orders in α', which is believed to be possible for instantonic configurations, as the radicant of the square root of the DBI action turns out to be a complete square, hence the series expansion in α' terminates. This belief is supported by the work[10] [336]: There it was shown that the Born-Infeld electrodynamics solution in flat space and with otherwise vanishing dilaton and Kalb-Ramond field describing a fundamental string ending perpendicular on a Dp brane is actually an exact solution to the theory of open strings on a Dp brane. In particular all higher-order derivative terms to the gauge fields beta function β^μ_A vanish. Considering the case $p = 3$ this D3-F1 system is connected to the D0-D4 system which describes instantonic configurations in $\mathcal{N} = 4$ SYM theory by first S-dualising to the D3-D1 configuration and then T-dualising along the D1 brane world volume direction which, by construction, is perpendicular to the D3 brane. In this way one ends up with the well-known D4-D0 system in flat space. However, this line of argument and the calculations in [336] are only valid for the abelian theory. However, much less is known in the nonabelian case. What is known is that instantons solve the low energy effective action for N D3 branes to order α'^3 [337]. A full proof along the lines of [336] is however still missing. In any case, a similar situation will occur in section 4.4.3 below when I consider the system in the constant Kalb-Ramond field.

[9] In general a static and supersymmetric state of two or more BPS objects (such as D branes) is again a BPS object, as its mass is determined only by some charge. The latter is, by charge conservation, determined by the charges of the two constituent BPS objects (see e.g. ch. 13.3 of [141]).

[10] The author thanks Paul Koerber for bringing this reference to his attention.

The authors of [115] went on in analyzing the symmetries of the instanton-probe brane configuration for two flavours, i.e. $N_f = 2$, and the Belavin-Polyakov-Shvarts-Tyupkin (BPST) instanton [338],

$$A_\mu = 0\,, \;\; A_m = \frac{2\Lambda^2 \bar{\sigma}_{nm} y_n}{\vec{y}^2(\vec{y}^2 + \Lambda^2)} \text{ with } \bar{\sigma}_{mn} = \frac{1}{2}\bar{\sigma}_{[m}\sigma_{n]}\,, \;\; \sigma_m = (i\vec{\tau}, 1_{2\times 2})\,, \;\; \bar{\sigma}_m = \sigma_m^\dagger\,, \tag{4.31}$$

where Λ is the instanton size modulus. In the case of the BPST instanton ($N_f = 2$, $k = 1$) the $4N_f k = 8$ moduli consist of four position moduli for the instanton core (which are set to zero in (4.31) and correspond to the vacuum expectation value of $\Phi_{1,2}$ via the identification (4.17)), three global $SU(2)$ gauge rotations (which generate an orbit of solutions when applied to (4.31) and correspond to a choice of the $SU(2)_{\mathcal{R}}$ orientation of the vacuum expectation values on the field theory side), and the size of the instanton core Λ. The authors of [115] found that the instanton (in singular gauge) breaks the geometric symmetries of the D7 brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$ down to

$$SU(2)_L \times SU(2)_R \times SU(2)_f \rightarrow SU(2)_L \times \mathrm{diag}(SU(2)_R \times SU(2)_f)$$

by the necessary identification of the space-time $SO(3)$ at infinity with the internal gauge symmetry $SU(2)_f$ which is induced by the form of the instanton field configuration (4.31).[11] This lead the authors of [115] to conclude that the right AdS/CFT identification for the size modulus of this one-instanton solution with the squark vacuum expectation values of the field theory vacuum must be

$$q_{i\alpha} = \frac{\Lambda}{2\pi\alpha'}\varepsilon_{i\alpha}\,. \tag{4.32}$$

The factor of $2\pi\alpha'$ is necessary for dimensional reasons, since Λ has dimension *length*, while the scalar field q has dimension *mass*. The $SU(2)_f$ index is $i = 1, 2$, while the $SU(2)_{\mathcal{R}}$ index is denoted by $\alpha = 1, 2$. This is exactly the identification (4.17) for $k = 1$ and $N_f = 2$, if one keeps in mind that the squarks transform as a doublet under the $SU(2)_{\mathcal{R}}$ symmetry (see table 4.1).

4.3 Noncommutativity and Instantons in Flat Space

This section is devoted to collect the necessary prerequisites for setting up the duality conjecture between noncommutative instantons on noncommutative D7 probes and the

[11] The instanton configurations in an $SU(2)$ gauge theory are classified by continuous maps from the sphere at infinity S^3_∞ to the group manifold $SU(2) \simeq \mathrm{S}^3$. The instanton number $\pi_3(SU(2)) =$, which is the winding number of the $\mathrm{S}^3_\infty \mapsto SU(2)$ maps, is measured by the Chern-Pontryagin index $\propto \int \mathrm{d}^4x \mathrm{Tr} F \wedge F$. Once a representative such as the BPST instanton (4.31) with fixed winding number is chosen and the position and size moduli are fixed, the instanton solution eq. (4.31) is only invariant under a combined $SU(2)_f$ and $SU(2)_R \subset SO(4)$ rotation, due to the intertwining effect of the $\bar{\sigma}_{mn} = -\eta_{amn}\tau^a$ on the two groups. The η^a_{mn} here are the selfdual 't Hooft symbols, which intertwine the selfdual representation $(1, 0)$ of $SU(2)_L \times SU(2)_R \simeq SO(4)$ with the adjoint representation of $SU(2)$.

Fayet-Iliopoulos term deformed vacuum moduli space of the field theory in section 4.4. Section 4.3.1 reviews how noncommutative field theories arise as the low energy effective actions of D branes in constant Kalb-Ramond fields. Section 4.3.2 briefly reviews the noncommutative ADHM equations and how the Fayet-Iliopoulos term enters them, as well as the work [117], in which the low energy effective action of the D3-D(-1) intersection was calculated and shown to give rise to the noncommutative ADHM equations, including the supersymmetry breaking Fayet-Iliopoulos term.

4.3.1 Kalb-Ramond Field Induced Noncommutativity in String Theory

Let us review how noncommutative field theories arise when D branes are embedded into a flat space with constant Kalb-Ramond B field background. Seiberg and Witten showed in [111] that a constant B field B_{ij} in the Euclidean directions[12] of a D brane embedded flatly (i.e. with constant embedding functions in a Cartesian coordinate system) in flat spacetime with constant metric g_{ij} induces a noncommutative behaviour for the endpoints of open strings ending on the brane. It turns out that with these restrictions the string theory with Dirichlet boundary conditions can still be quantised exactly, and the two-dimensional propagator can be calculated (see eq. (2.4) of [111]). Restricted to the boundary of the string world sheet, the propagator for one brane (i.e. without Chan-Paton factors) reads

$$\langle X^i(\tau,\sigma)X^j(\tau',\sigma)\rangle\big|_{\sigma=0,\pi} = -\alpha'\mathcal{G}^{ij}\log(\tau-\tau') + \frac{i}{2}\theta^{ij}\epsilon(\tau-\tau')\,, \tag{4.33}$$

with $\mathcal{G}^{ij}$ being the **open string metric** (defined in eq. (4.36)), θ^{ij} the **noncommutativity parameter** (defined in eq. (4.35)) and

$$\epsilon(x) = \begin{cases} +1 & x>0\,, \\ 0 & x=0\,, \\ -1 & x<0 \end{cases}$$

the sign function. Calculating the expectation value of the anticommutator of X^i and X^j, the open string metric drops out. Thus open strings in the Dirichlet directions in which the Kalb-Ramond field is switched on have noncommuting endpoint coordinates,

$$\langle [X^i(\tau,\sigma), X^j(\tau,\sigma=)]\rangle\big|_{\sigma=0,\pi} = i\theta^{ij}\,. \tag{4.34}$$

The noncommutativity parameter is related, in an obvious matrix notation, with the closed string metric g and the B field through (in an obvious matrix notation)

$$\theta = -(2\pi\alpha')^2\frac{1}{g+2\pi\alpha' B}B\frac{1}{g-2\pi\alpha' B}\,. \tag{4.35}$$

[12] This restriction is necessary since switching on the B field in spatio-temporal directions leads to instabilities due to string pair production [302, 304]. When attempting to write down unitary noncommutative field theories these instabilities translate into problems of defining a time ordering which preserves unitarity, as well as when defining the notion of a Wick rotation.

Note that because of the additional factor $2\pi\alpha'$, B has dimension of $length^{-2}$. Although the X^i are scalar fields from the point of view of the two-dimensional sigma model, i.e. they are dimensionless, they have to have dimension of *length* from a space-time point of view. Thus when interpreting eq. (4.34) as a relation which holds in a low energy effective field theory, i.e. in a field theory on noncommutative space-time, the noncommutativity parameter must have dimension $length^2$ (as is obvious from (4.35)) and thus the dimensions in (4.34) work out.

The result (4.34) can be obtained without a double scaling limit. However, to show that correlators of vertex operators only differ from the zero B field correlators by the well known noncommutative phase factors [339] $e^{-\frac{i}{2}p_i\theta^{ij}k_j}$ and do not receive contributions from anomalous dimensions (which would depend on $\mathcal{G}$), the **zero slope limit** $\alpha' \to 0$ has to be taken while θ and the open string metric $\mathcal{G}$

$$\mathcal{G}^{-1} = \frac{1}{g + 2\pi\alpha' B} g \frac{1}{g - 2\pi\alpha' B}\,, \tag{4.36}$$

are kept fixed. As shown in [111] this implies scaling $\alpha' \sim \sqrt{\varepsilon}$ and $g \sim \varepsilon$ (i.e. the metric has to scale twice as fast as the string length) while keeping B fixed. In this limit the noncommutativity parameter and open string metric simplify to

$$\theta = B^{-1}\,, \quad \mathcal{G}^{-1} = -\frac{1}{(2\pi\alpha')^2} B^{-1} g B^{-1}\,, \tag{4.37}$$

Clearly θ and $\mathcal{G}$ are fixed in this scaling limit.

For slowly varying fields (compared to the string scale α') the low energy effective theory on the Dp brane in the directions in which the B field is nonvanishing then reduces to the well-known noncommutative Yang-Mills theory[13] [111]

$$S_{\text{NCYM}} = \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2} G_s} \int \sqrt{\det \mathcal{G}}\, \mathcal{G}^{ij} \mathcal{G}^{kl} \text{Tr}(\hat{F}_{ik} * \hat{F}_{jl})\,. \tag{4.38}$$

Here the **open string coupling constant** is

$$G_s = g_s \det{}^{\frac{1}{2}} \left((g + 2\pi\alpha' B) g^{-1} \right)\,,$$

and the star product is defined as

$$(f * g)(x) = f(x) e^{\frac{i}{2} \overleftarrow{\partial}_i^x \theta^{ij} \overrightarrow{\partial}_j^y} g(y) \Big|_{x=y}\,. \tag{4.39}$$

[13]See [340, 341] for two concise reviews of the vast topic of noncommutative field theory. Note that for nonabelian theories, the trace runs over the Hilbert space the operators are realized on, as well as over colour space, and that this formula can only be true for the B field extending in all directions of the Dp brane for the dimensions to match ($\int$ Tr has dimension $length^{p+1}$). If the B field extends only in some directions, there is a commutative gauge theory sector as well as a noncommutative one.

With this definition, the star commutator of two noncommutative coordinates

$$[x^i, x^j]_* = x^i * x^j - x^j * x^i = i\theta^{ij} \tag{4.40}$$

reproduces the algebra (4.34). The noncommutative gauge field strength for both the abelian and nonabelian case reads

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + i\left(\hat{A}_i * \hat{A}_j - \hat{A}_j * \hat{A}_i\,,\right) \tag{4.41}$$

and thus also includes a star product. In the nonabelian case the Hilbert space on which the operators are acting is a tensor product of a representation of the algebra (4.40) times the usual gauge group representation by matrices. Gauge transformations act on the gauge field as

$$\delta_\varepsilon \hat{A}_i = \partial_i \varepsilon + i[\hat{A}_i, \varepsilon] \tag{4.42}$$

In the $\alpha' \to 0$ limit, (4.38) becomes exact in describing the dynamics of open strings [111].

4.3.2 Noncommutative Instantons and the Modified ADHM Construction

In this subsection I briefly review the noncommutative ADHM construction and the underlying noncommutative ADHM equations constraining the instanton moduli, as explained in e.g. [342, 343]. As shown by Nekrasov and Schwarz [119], noncommutative instantons on $\mathbb{R}^4$ can be constructed from the moduli obtained by solving the **noncommutative ADHM equations**

$$\begin{aligned} 2(\theta^{45} - \theta^{67}) = \zeta &= [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J\,, && (4.43)\\ 0 &= [B_0, B_1] + IJ\,. && (4.44)\end{aligned}$$

These equations are very similar in form to the commutative ADHM equations (4.18)-(4.19), except of the constant term on the left-hand side of (4.43). It is however exactly this term which actually forces the underlying space-time to be noncommutative: Using the operators τ_z, σ_z as defined in (4.21)-(4.22), the ADHM equations can be reexpressed as

$$\begin{aligned} 0 &= \tau_z\tau_z^\dagger - \sigma_z^\dagger\sigma_z = [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J - [z_0, \bar{z}_0] + [z_1, \bar{z}_1]\,,\\ 0 &= \tau_z\sigma_z = [B_0, B_1] + IJ + [\bar{z}_0, z_1]\,.\end{aligned}$$

It is clear that one wants to keep this structure of the ADHM equations, since it lies at the heart of the whole construction. The constant term on the left-hand side of (4.43) can then only be reproduced if space-time is noncommutative, i.e. if the coordinates fulfill commutation relations of the form

$$[\hat{y}^m, \hat{y}^n] = i\theta^{mn} = i\begin{pmatrix} 0 & \theta^{45} & 0 & 0\\ -\theta^{45} & 0 & 0 & 0\\ 0 & 0 & 0 & \theta^{67}\\ 0 & 0 & -\theta^{67} & 0\end{pmatrix}\,,\ m, n = 4, 5, 6, 7\,, \tag{4.45}$$

On purpose I denote here the transversal coordinates in the D brane setup (4.1) with the same symbol as the coordinate operators. Since the noncommutativity parameter θ^{mn} is a antisymmetric matrix in the four-dimensional Euclidean space spanned by the y^m, it can always be brought into block diagonal form by an appropriate $SO(4)_{4567}$ rotation. From now on, I omit the hats on operators and leave it to the reader to distinguish between operators and c-numbers where necessary.

The noncommutative ADHM construction then exactly proceeds the same way as described in section 4.2. As can be seen from (4.43), the Fayet-Iliopoulos term ζ appearing in the ADHM construction for instantons vanishes for selfdual noncommutativity parameter, i.e. for the upper sign of the definition

$$\theta^{mn} = \pm\frac{1}{2}\epsilon^{mnop}\theta^{op} \Leftrightarrow \theta^{45} = \pm\theta^{67}\,.$$

In order to arrive at anti-instantons (which were actually considered by the authors of [119]) one needs to exchange $z_0 \leftrightarrow \bar{z}_0$. This generates a right-hand side of (4.43) of $-2(\theta^{45}+\theta^{67})$, which vanishes for an anti-selfdual noncommutativity parameter. It is in these two special cases, i.e. (anti)-instantons on an (anti)-selfdual noncommutative Euclidean space, in which the small instanton singularities in the moduli space are not regulated, i.e. in which the equation (4.43) has solutions with some components of the size moduli I, J vanishing. The noncommutative instantons thus have a zero size limit exactly when the Fayet-Iliopoulos parameter ζ vanishes. These are exactly the cases in which $\mathcal{N} = 2$ supersymmetry is preserved in the flat space D brane configuration [322, 344]. We will see in section 4.4 that these are exactly the cases where a) a κ-symmetry calculation for an (anti)-D7 brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with (anti)-selfdual B field confirms the preservation of supersymmetry, and b) a no-force condition for the D7 brane in the background of D3 branes holds, which is another indication of preserved supersymmetry.

As elaborated in section 4.2, without the B field, the D term and F term equations reduced to the Higgs part of a mixed Coulomb-Higgs vacuum of the field theory on the D3 brane are just the ADHM equations for k $U(N_f)$ instantons on the D7 worldvolume transversal to the D3 branes. This statement was checked by direct calculation of the effective action for D3-D(-1) in [345], which included the derivation of the correct ADHM measure, i.e. the low energy effective action was shown to reproduce the supersymmetric version of the ADHM equations which show up in the analysis of instantons in Euclidean $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In the D3-D(-1) system the $\mathcal{N} = 4$ theory lives on the stack of N D3 branes, while the instantons are represented by the k D(-1) branes. The effective action then contains three pieces: First there is the Euclideanised $\mathcal{N} = 4$ piece (eq. (3.11) in [345]), encoding the interactions of the 3-3 strings. Then there is the low energy effective action for the (-1)-(-1) strings, which are part of the ADHM moduli fields and their superpartners. That action can be obtained by reducing ten-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory to zero dimensions. Since the D(-1) brane has no world volume, the "fields" in that case are just matrices and Grassmann numbers. The interaction terms coming from the 3-(-1) and (-1)-3 strings then yields the remaining ADHM moduli. This piece is also a zero-dimensional matrix action, similar to fact that the interaction terms coming from the 3-7 strings in (2.140) are four-dimensional. The (-1)-(-1) and 3-(-1) pieces of the effective action then can be rewritten to include a triplet of bosonic auxiliary fields D_c (with c being a $SU(2)_{\mathcal{R}}$ symmetry index) which multiply the bosonic ADHM constraints, and other auxiliary fields generating the rest of the supersymmetric ADHM constraints. The effective action just described could actually be obtained by dimensional reduction of the low energy effective action of N D7 and k D3 branes. From this point of view it is not surprising that the triplet D_c turns out to be the triplet of auxiliary fields of the $\mathcal{N}=2$ vector multiplet in the intersection theory, and that it is exactly the D term equation for the $\mathcal{N}=1$ vector field (4.11) and the F term equation (4.10) which on the Higgs branch reduces to the ADHM equations (4.18)-(4.19).

In [117] this was extended to the case with constant B field on the D3 brane. It turned out that the relevant couplings generating the ADHM measure exist also in this case. Furthermore, a disk diagram with $D(-1)$ boundary conditions containing an insertion of the auxiliary D_c fields of the $\mathcal{N}=2$ vector multiplet and a closed string vertex operator for the Kalb-Ramond B field generates the correct Fayet-Iliopoulos term (cf. eq. (6.3) in [117]),

$$\langle V_D V_B \rangle \propto \frac{1}{g^2_{(-1)}} D_c \bar{\eta}^c_{mn} \theta^{mn} \,. \tag{4.46}$$

As noted above, the Lagrange multiplier D_c also multiplies the bosonic ADHM constraint. It is consistent with the vanishing of ζ in (4.43) for a self-dual noncommutativity parameter that for a configuration of a D instanton (and not an anti-D instanton) bound to a D3 brane, the Fayet-Iliopoulos term only depends on the anti-selfdual part of B. The selfdual part gets projected out by the anti-selfdual 't Hooft symbol $\bar{\eta}$. If we now think about the effective theory on the D3-D7 intersection with B field as connected to the effective action of the D3-D(-1)-System via dimensional reduction, D_c corresponds to the triplet (D, F_1, F_2) of auxiliary fields in the $\mathcal{N}=2$ vector multiplet, with quantum numbers listed in table 4.1. The index c thus transforms in the fundamental representation $SU(2)_R$-symmetry, and the triplet of fields is in the adjoint representation of the gauge group $U(N_c)$. Furthermore, the gauge coupling on the D instanton $g_{(-1)}$ after dimensional oxidation becomes the gauge coupling on the D3 branes of the D3-D7 system. Note that the B field in [117] is dimensionless, while in this work it has dimension of $energy^2$. After an appropriate $SU(2)_{\mathcal{R}}$ transformation which maps $(D, F_1, F_2) \mapsto (D, 0, 0)$, where D is considered the "z component" in the three-dimensional field space on which $SU(2)_{\mathcal{R}} \simeq SO(3)$ acts, I thus conclude that the low energy effective action on a D3-D7 intersection in flat space-time with a constant B field in the directions on the D7 brane which are transversal to the D3 brane contains an additional Fayet-Iliopoulos term

$$\frac{\bar{\eta}^3_{mn}\theta^{mn}}{4g^2_{YM}} \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \mathrm{tr} V \,. \tag{4.47}$$

Note that the normalisation is the correct one to generate the right hand side of equation (4.43), namely

$$\zeta = \bar{\eta}^3_{mn}\theta^{mn} = 2(\theta^{45} - \theta^{67}) \,,$$

where I used the standard form of the third anti-selfdual 't Hooft symbol (A.8). Of course also the field theory in the D7 directions transverse to the D3 brane becomes noncommutative via the mechanism explained in section 4.3.1, but the $\mathcal{N}=4$ theory on the D3 branes stays a commutative field theory.

The authors of [117] also discussed the different possible point particle limits in the $D3-D(-1)$ system. In particular, there is a limit in which $g_{(-1)}$ is held fixed while $\alpha' \to 0$, which necessarily decouples the D3 brane degrees of freedom by sending the gauge coupling g_3 to zero. Before dimensional reduction, i.e. in the D3-D7 setup, this corresponds to the limit in which the gauge coupling on the N_c D3 branes, g_{YM}, is held fixed, which then implies that 7-7 string degrees of freedom, i.e. the ones on the flavour brane, decouple from the 3-3 and 3-7 strings. This is exactly the open string sector decoupling limit described in section 2.3.6.

To summarise, I conclude that from the point of view of the flat space D3-D7 intersection theory with a constant anti-selfdual B field along the 4567 directions of the D7 brane induces a Fayet-Iliopoulos term in the $\mathcal{N}=2$ theory (2.140). This term will survive the decoupling limit necessary to set up the AdS/CFT correspondence with the open string sector, and thus the field theory side of the correspondence will include this Fayet-Iliopoulos term. On the gravity side of the correspondence the decoupling limit also works as explained in section 2.3, with the exception that now also the energy scale $B=\tilde{B}/(2\pi\alpha')$ has to be held fixed in addition to the radial energy scale $u=r/(\alpha')$. These observations motivate an extended AdS/CFT conjecture which includes the pure gauge B field in $\mathrm{AdS}_5 \times \mathrm{S}^5$ plus noncommutative instantons on the embedded D7 probe branes. This setup is conjectured to be dual to the $\mathcal{N}=2$ flavour theory (2.140) with the additional Fayet-Iliopoulos term (4.47). The remaining two sections of this chapter elaborate on the precise statement and several tests of this conjecture.

4.4 Holographic Fayet-Iliopoulos Term from Internal Noncommutativity

In the last section we saw evidence for an extended AdS/CFT correspondence including a Fayet-Iliopoulos term in the $\mathcal{N}=2$ field theory with flavour and a constant Kalb-Ramond field on the D7 probes. In this section I thus turn to the analysis of N_f D7 brane probes in the near horizon limit of N_c D3 branes with a B field switched on, in order to find evidence for this conjecture. The N_f D7 probe branes are embedded according to table 2.9. Replacing the N_c D3 branes by the near horizon limit yields the background (4.1)-(4.3). Additionally, a constant Kalb-Ramond B field is switched on in the $\vec{y}$-directions. Due to $H=\mathrm{d}B=0$ and a vanishing C_2 field, the supergravity solution (4.1)-(4.3) is not perturbed. In the following I use the skew-diagonalized form

$$B=b_1 \mathrm{d}y^4 \wedge \mathrm{d}y^5 + b_2 \mathrm{d}y^6 \wedge \mathrm{d}y^7 \,. \tag{4.48}$$

This form can always be reached by an $SO(4)$ rotation on $\vec{y}$, and may be specialized further to the **selfdual case** $\mathbf{b_1} = \mathbf{b_2} = \mathbf{b}$ or to the **anti-selfdual case** $\mathbf{b_1} = -\mathbf{b_2} = \mathbf{b}$. The former (latter) case yields upon usage of the relation (4.37) a selfdual (anti-selfdual) noncommutativity parameter θ. In this background probe D7 branes are embedded by specifying $z^8(y^m)$ and $z^9(y^m)$, such that the equations of motion of the D7 brane action S_{D7},

$$\begin{aligned} S_{D7} &= S_{DBI} + S_{WZ}\,, && (4.49)\\ S_{DBI} &= -\frac{\mu_7}{g_s}\int\limits_{D7} d^8\xi\,\mathrm{STr}\sqrt{-\det\left(\mathcal{P}[G+B]+2\pi\alpha' F\right)}\,, && (4.50)\\ S_{WZ} &= \frac{\mu_7}{g_s}\int\limits_{D7}\mathrm{STr}\,\mathcal{P}[C_{(4)}]\wedge e^{\mathcal{P}[B]+2\pi\alpha' F}\,, && (4.51) \end{aligned}$$

are fulfilled. The pull-back of the background metric $g_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ is denoted by $\mathcal{P}$. STr is the symmetrised trace prescription of Tseytlin [190]. This prescription is known in general to be correct only for the lowest several orders of α', but for special BPS configurations the series will terminate and is believed to be correct to all orders in α'. Also, since both C_4 and B are in the directions of the D7 world volume, there is no Myers polarisation effect [282] to be expected in this configuration.

For one D7 brane ($N_f = 1$) one can readily show that in the presence of a B field the usual embedding $z^8 = 2\pi\alpha' m$, $z^9 = 0$ is no longer a solution of the equations of motion of the D7 brane action, unless $m = 0$. Therefore I will only study massless embeddings of the D7 probe branes, i.e.

$$z^8 = z^9 = 0\,. \qquad (4.52)$$

The case of one D7 brane will be of major interest in the following, since noncommutative instantons do not have commutative counterparts for a $U(1)$ gauge group. Also, the noncommutative instanton solutions are best understood and simplest for gauge group $U(1)$. Since there is no polarisation effect happening in the setup considered here, once can also expect the flat embedding to be a valid solution of the nonabelian embedding equations by diagonalising the hermitian $N_f \times N_f$ matrices $Z^{8,9}$ and solving the equations of motion for each eigenvalue.

Due to the analysis of the flat space D3-D7 system of section 3.2, the Fayet-Iliopoulos term (4.47) is expected to survive the decoupling limit of AdS/CFT, and thus a description of the mixed Coulomb-Higgs branch of the dual field theory in terms of noncommutative instantons on probe branes is possible along the lines of [112–115, 320, 346–348]. This is supported by the fact that the deformation induced on the instanton moduli space by the Fayet-Iliopoulos term is rather mild, as only the small instanton limit is affected. The map between instantons on the probe branes and the Coulomb-Higgs branch on the field theory is therefore expected to carry over to the holographic setting, in analogy to the case without B field [112, 115].

4.4.1 Global Symmetries

Let us first analyse the geometric symmetries of one D7 brane embedded in the background with the B field (4.48). As can be seen from (4.1), the $(\vec{y}, \vec{z})$-directions perpendicular to the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ are flat up to the warp factor. The components of the B field in the $(\vec{y}, \vec{z})$-directions can then be written as an antisymmetric 6×6-matrix $B_{ij} = -B_{ji}$, transforming in the antisymmetric tensor representation $[1, 0, 1] = \mathbf{15}$ of the six-dimensional rotation group $SO(6)$. To find the possible representations compatible with the symmetry breaking induced by the probe brane, I decompose this representation of $SO(6)$ into irreducible representations of the symmetry group which is preserved by the flavour D7 brane, $SU(2)_L \times SU(2)_R \times U(1)_{89}$. The branching rule for $\mathbf{15}$ of $SO(6)$ into $SU(2)_L \times SU(2)_R \times U(1)_{89}$ is, according to table 58 of [349],

$$\mathbf{15} = (0, 0)_0 \oplus (1, 0)_0 \oplus (0, 1)_0 \oplus (\frac{1}{2}, \frac{1}{2})_2 \oplus (\frac{1}{2}, \frac{1}{2})_{-2}\,. \tag{4.53}$$

Since I switch on a B field in the $\vec{y}$-directions only, the $U(1)_{89}$ charge is zero, which leaves only the first three terms on the right hand side of (4.53). The first term corresponds to a B field in the $\vec{z}$-directions, which I do not want to consider here. The second and third terms in the decomposition are, respectively, the selfdual and anti-selfdual parts of the B field in the $\vec{y}$ directions (cf. e.g. [333]). These are the field configurations we are interested in. These B field contributions are inert under the $SO(1, 3)$ Lorentz group and thus dual to scalar operators in the dual field theory. They are however not inert under the conformal group $SO(2, 4)$, and so the operators will have definite conformal scaling properties.

A general B field in the $\vec{y}$-directions transforms both under $SU(2)_L \times SU(2)_R$ in the respective vector representation and thus breaks the $SU(2)_L \times SU(2)_R \simeq SO(4)$ rotation invariance down to $U(1)_L \times U(1)_R$. The selfdual case $b_1 = b_2 = b$ preserves the $SU(2)_R$, while the anti-selfdual case preserves the $SU(2)_L$, which is obvious from the representations $(1, 0)_0$ and $(0, 1)_0$. As we have seen from the analysis of the D3-D(-1) system, there is no Fayet-Iliopoulos term in the dual gauge theory in the case of self-dual B field since the self-dual part of the noncommutativity parameter gets projected out by the $\bar{\eta}^3$ in eq. (4.47). However, for an anti-selfdual B field transforming in the $(0, 1)_0$ of the $SU(2)_L \times SU(2)_R \times U(1)_{89}$, there is a Fayet-Iliopoulos term present: The anti-selfdual B field in the representation $(0, 1)_0$ of $SU(2)_L \times SU(2)_R \times U(1)_{89}$ has the right quantum numbers to couple to the auxiliary field triplet (D, F_1, F_2), which transforms under $(0, 1)_0$ of $SU(2)_L \times SU(2)_R \times U(1)$ (see section 4.2). This is consistent both with the brane picture in which the Fayet-Iliopoulos term (4.47) in the D3-D(-1) system only depends on the anti-selfdual part of the B field, as well as with the analysis of the noncommutative ADHM construction [342]. For instantons, the small instanton singularity is only resolved if the noncommutativity parameter is not purely selfdual, and vice-versa for anti-instantons. The global symmetries on the gravity side are thus consistent with the existence of a holographic coupling of the form (4.47), in the standard holographic sense, of the B field to the auxiliary field triplet (D, F_1, F_2).

4.4.2 Scaling Dimensions

Since the operator dual to the $(0,1)_0$ part of the internal B field is not inert under the $SO(4,2)$ symmetry, it has to transform nontrivially under conformal transformations. Since the representations of the conformal group are labeled by the usual Lorentz quantum numbers (s_+, s_-) plus the scaling dimension Δ and we already found out that the operator dual to $(0,1)_0$ must be a Lorentz scalar, we only need to know the scaling dimension to characterise the operator completely. The other global symmetry quantum numbers $SU(2)_\Phi \times SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ are of course just given by the representation $(0,1)_0$ of $SU(2)_L \times SU(2)_R \times U(1)_{89}$.

The scaling dimensions for such a holographic coupling work out as follows: The operator dual to the Fayet-Iliopoulos term, i.e. the triplet (D, F_1, F_2), has scaling dimension $\Delta = 2$. This can also be deduced as follows. The chiral primary operator of the $\mathcal{N} = 2$ gauge multiplet is given by the scalar component of Φ_3 and hence has scaling dimension one. By applying the supersymmetry generators on the chiral primary twice, we obtain the Fayet-Iliopoulos triplet of auxiliary fields of the $\mathcal{N} = 2$ vector multiplet (D, F_1, F_2).[14] Therefore, the scaling dimension of the operator dual to the Fayet-Iliopoulos term is

$$\Delta = 1 + 2 \cdot \frac{1}{2} = 2\,, \tag{4.54}$$

which is consistent with the coupling (4.47).

The quantum numbers of the $\mathcal{N} = 2$ vector multiplet imply that the auxiliary fields (D, F_1, F_2) transform in the $(0,1)_0$ representation of $SU(2)_L \times SU(2)_R \times U(1)_{89}$, as can be seen from the second line in table 4.1. Their scaling dimension is $\Delta = 2$ and they are uncharged under $U(1)_f$. Thus the coupling (4.47) has the right scaling dimensions, as well as the correct quantum numbers under the global symmetries, to be interpreted as the holographic version of the coupling (4.47) of the noncommutativity parameter to the triplet (D, F_1, F_2).

4.4.3 Supersymmetry and No-Force Conditions

Another consistency check of the proposed AdS/CFT duality between the anti-selfdual B field in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the Fayet-Iliopoulos term (4.47) is matching of the supersymmetry breaking pattern derived from the Dp-D(p+4) system in flat space [111, 117] with the pattern derived from no-force conditions of probe branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$, as well as with the κ-symmetry of the D7 brane embedding.

In flat space, a Dp-D(p+4) system is $\mathcal{N} = 2$ supersymmetric if and only if B is selfdual. The lower-dimensional brane can then be viewed as an instanton in the four additional directions

[14] The $\mathcal{N} = 2$ vector multiplet in four dimensions splits up into the $\mathcal{N} = 1$ vector multiplet V with auxiliary field D and into an $\mathcal{N} = 1$ chiral multiplet Φ_3 with complex auxiliary field $F = F_1 + iF_2$.

of the D(p+4) brane. Equivalently, $\overline{\text{Dp}}$-D(p+4) or Dp $-$ $\overline{\text{D(p+4)}}$ are supersymmetric if B is anti-selfdual. This configuration corresponds to an anti-instanton. In the cases where B has the selfduality properties which do not lead to a no-force condition, a Fayet-Iliopoulos term is expected to be generated which parametrises the supersymmetry breaking.

To calculate the unbroken supersymmetry of the D3-D7 system, I embed a D7 probe brane into the background given by (4.1)-(4.3) and (4.48). Following [322], I present a κ-symmetry calculation for one probe D7 brane. I choose (x^μ, y^m) as the set of worldvolume coordinates and consider only massless embeddings, i.e. $z^8 = z^9 = 0$. To describe dissolved D3 branes in the D7 probe brane, a $U(1)$ field strength F_{ab} is switched on the worldvolume of the D7 brane in the directions y^m. This D7 probe brane preserves some supersymmetries if there are nontrivial spinor solutions to the equation [350]

$$\Gamma_\kappa \epsilon = \epsilon \,, \tag{4.55}$$

where ϵ is a Killing spinor of the $\text{AdS}_5 \times \text{S}^5$ background and the κ-symmetry projector Γ_κ for a D7 probe brane in this background is given by [351]

$$\Gamma_\kappa = e^{-a} \,(i\sigma_2) \otimes \Gamma_{01234567} \,, \tag{4.56}$$

where a is a function of Y_{ik}, which depends on $\mathcal{F}_{ij} = \mathcal{P}[B]_{ij} + 2\pi\alpha' F_{ij}$ in a nonlinear way,

$$a = \frac{1}{2\sqrt{H_3}} Y_{jk} \sigma_3 \otimes \Gamma^{jk} \,. \tag{4.57}$$

The gamma matrices used here fulfill the flat space Clifford algebra

$$\Gamma^\mu, \Gamma^\nu = 2\eta^{\mu\nu} \,.$$

Using the identity

$$\Gamma^{mn}\Gamma_{4567} = -\frac{1}{2}\epsilon^{mnop}\Gamma_{op}$$

for m, n, o, p running from 4 to 7, one can rewrite $\Gamma_\kappa \epsilon = \epsilon$ in the form

$$\exp\left(-\frac{1}{4\sqrt{H_3}}\,\sigma_3 \otimes \Gamma^{mn}\left[Y^+_{mn}\,(1-\Gamma_{4567}) + Y^-_{mn}\,(1+\Gamma_{4567})\right]\right)(i\sigma_2)\otimes\Gamma_{01234567}\epsilon = \epsilon \tag{4.58}$$

with $Y^\pm_{mn} = \frac{1}{2}\left(Y_{mn} \pm (\star Y)_{mn}\right)$. The dual two-form $\star Y$ is calculated with respect to the flat metric δ_{mn} in the 4567-directions.

Let us first consider **selfdual field strengths** on the D7 brane, i.e. **instantons**. This implies that only F^+_{mn} will depend on the worldvolume coordinates of the D7 brane, and $F^-_{mn} = 0$. For **selfdual B field** $B^-_{mn} = 0$ then also $Y^-_{mn} = 0$, but Y^+_{mn} is unconstrained and can depend on the worldvolume coordinates y^m. Since the spinor ϵ in (4.58) is a constant spinor fulfilling the kappa projection condition of a stack of D3 branes along the 0123 directions of flat space,

$$(i\sigma_2) \otimes \Gamma_{0123})\epsilon = \epsilon \,, \tag{4.59}$$

equation (4.58) can be fulfilled only if

$$(1-\Gamma_{4567})\epsilon = 0\,.$$

Since the matrices $[(i\sigma_2)\otimes\Gamma_{0123}]^2 = \Gamma_{4567}^2 =$ both square to unity, these two conditions are projection conditions on the spinor ϵ, each projecting out half of its components. They are compatible with each other since

$$[\Gamma_{0123},\Gamma_{4567}] = 0\,,$$

and thus the setup with an instanton and a self-dual B field preserves one quarter of the 32 real supercharges of ten-dimensional space-time, i.e. $\mathcal{N}=2$ supersymmetry in four dimensions. Note that these two conditions are exactly the same conditions for a probe D7 brane in $\mathrm{AdS}_5\times\mathrm{S}^5$ with zero B field.

In the case of an **anti-selfdual B field** $Y^-_{mn} = B_{mn}$ is no longer zero. Due to the factor $H_3^{-1/2}$ in the exponent of (4.58), which also depends on the worldvolume coordinates of the D7 brane, we additionally have to satisfy

$$(1+\Gamma_{4567})\,\epsilon = 0\,.$$

Thus, the spinor ϵ has to satisfy both conditions $(1\pm\Gamma_{4567})\,\epsilon = 0$. Since Γ_{4567} squares to one it can only have eigenvalues plus or minus one, and thus imposing both conditions sets the full spinor $\epsilon=0$. Therefore supersymmetry is completely broken, which is in agreement with the above field-theoretical considerations. Note that in globally supersymmetric theories a partial breaking of supersymmetry (i.e. from $\mathcal{N}=2$ to $\mathcal{N}=1$) is not possible [296]. For an anti-D7 brane the sign of the right-hand side of (4.58) changes to a minus sign, and thus one needs to impose (note $\Gamma_{01234567}=\Gamma_{0123}\Gamma_{4567}$) the background projection condition (4.59), $\Gamma_{4567}\epsilon=-\epsilon$ and thus $Y^+_{mn}=0$ for preserving $\mathcal{N}=2$ supersymmetry. Anti-D7 branes in $\mathrm{AdS}_5\times\mathrm{S}^5$ will preserve $\mathcal{N}=2$ supersymmetry if and only if the B field is anti-selfdual and they carry anti-selfdual gauge field flux, i.e. if they carry anti-instantons.

This supersymmetry pattern for instantons can be confirmed by **no-force conditions** for N_f D7 probe branes. An expansion of (4.50) up to $\mathcal{O}(\alpha'^2)$ yields for selfdual B ($b_1=b_2=b$)

$$S_{DBI}+S_{CS} = -\frac{\mu_7}{g_s}\int\! \mathrm{d}^4x\int\! \mathrm{d}^4y\;\left(1+\frac{1}{2}\frac{(2\pi\alpha')^2}{H_3+b^2}\mathrm{tr}F_-F_-\right)\,, \tag{4.60}$$

where F_- is defined in (4.29). The trace runs over $SU(N_f)$ indices. For calculational details see Appendix C.1. The crucial feature in the calculation is that contributions of $\mathcal{O}(\alpha')$ coming from mixed terms including B and F cancel between the DBI-part and the Wess-Zumino part of the brane action. Thus, despite the presence of the B field, the conclusion from this calculation is as in the case without B field, described in section 4.2: There is no force exerted on probe branes if both the B field and the field strength on the branes are selfdual, and instantonic solutions still solve the gauge theory living on the D7 branes, at least to $\mathcal{O}(\alpha'^2)$. For an anti-D7 brane in an anti-selfdual B field one would find that anti-instantons are force-free.

Assuming that the full DBI nonabelian action (4.50) holds for instantonic configurations and that the diagonal embedding $Z^i = z^i \ {}_{N_f \times N_f} = 0$ solves the nonabelian embedding equations (and not specifying the actual trace prescription), the no-force condition (4.60) can also be obtained in a different way, following [320]: Rewrite the DBI action as

$$S_{DBI} = -\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \sqrt{-\det\left(P[G] + \mathcal{F}\right)} \tag{4.61}$$

$$= -\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \sqrt{-\det\left(\eta_{\mu\nu}/\sqrt{H_3}\right)} \sqrt{\det\left(\sqrt{H_3}\delta_{ij} + \mathcal{F}_{ij}\right)} \tag{4.62}$$

$$= -\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \sqrt{\det\left(\delta_{ij} + M_{ij}\right)}\,. \tag{4.63}$$

Here I defined the Levi-Civita symbol $\tilde{\epsilon}_{01234567} = +1$ and the volume element through $\mathrm{d}^4x\mathrm{d}^4y = \frac{1}{8!}\tilde{\epsilon}_{\mu_1 \dots \mu_8} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_8}$. Now one can use the following identity for antisymmetric four by four matrices M with definite duality properties $(*M)_{ij} = \frac{1}{2}\epsilon_{ijkl}M_{kl} = \pm M_{ij}$,

$$\det(\mathbf{1} + M) = 1 + \frac{1}{2}M^2 + \frac{1}{16}(*MM)^2 = \left(1 + \frac{M^2}{4}\right)^2 . \tag{4.64}$$

Here $M^2 = M_{ij}M_{ij}$. Thus, for (anti)selfdual $\mathcal{F} = B + 2\pi\alpha' F$,

$$*\left(B + 2\pi\alpha' F\right) = \pm(B + 2\pi\alpha' F)\,, \tag{4.65}$$

the DBI action acquires a simpler square-root free form,

$$S_{DBI} = -\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \left(1 + \frac{M^2}{4}\right) . \tag{4.66}$$

The Chern-Simons action can be evaluated in a similar way, yielding

$$S_{CS} = \pm\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \frac{M^2}{4}\,, \tag{4.67}$$

where the sign depends on the duality properties of M. Thus the D7 brane action reads

$$S_{D7} = S_{DBI} + S_{CS} = -\frac{\mu_7}{g_s} \int \mathrm{d}^4x \mathrm{d}^4y \left(1 + \frac{M^2}{4}\left(1 \mp 1\right)\right)\,, \tag{4.68}$$

Therefore, as found in the κ-symmetry calculation before, $B + 2\pi\alpha' F$ has to be selfdual for a D7 brane in order to have a force-free situation. For an anti D7 brane, the opposite holds. If B and F have different selfduality properties, supersymmetry will be broken and a Fayet-Iliopoulos-term will be created. Table 4.2 lists the different possibilities for (non)supersymmetric configurations.

	$*B = +B$	$*B = -B$
D3-D7	$\mathcal{N} = 2$ SUSY if $*F = +F$	no SUSY, FI
D3-$\overline{D7}$	no SUSY, FI	$\mathcal{N} = 2$ SUSY if $*F = -F$

Table 4.2: Supersymmetry conditions

4.4.4 Duality Conjecture: Fayet-Iliopoulos Terms from Kalb-Ramond Fields

To summarise, the above elaborations showed the following:

1. The global symmetries of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background with the flatly ($z^i = 0$) embedded D7 probe brane and an anti-selfdual B field, which are $SU(2)_L \times U(1)_R \times U(1)_{89}$, match the field theory symmetries of the $\mathcal{N} = 2$ theory (2.140) with the additional Fayet-Iliopoulos term coupling (4.47), namely $SU(2)_\Phi \times U(1) \times U(1)_\mathcal{R}$, where the first $U(1)_\mathcal{R}$ factor is the Cartan subgroup of $SU(2)_\mathcal{R}$. The presence of the Fayet-Iliopoulos coupling (4.47) will thus break the $SU(2)_\mathcal{R}$ symmetry.
2. The scaling dimension of the operator dual to the Fayet-Iliopoulos term, $\Delta = 2$, matches the scaling dimensions of the triplet (D, F_1, F_2) of auxiliary fields of the $\mathcal{N} = 2$ vector multiplet.
3. The supersymmetry breaking pattern summarised in table 4.2 obtained from calculating the preserved supersymmetries of a D7 brane with (anti)-selfdual gauge field flux embedded in either selfdual or anti-selfdual B field backgrounds matches the appearance of a Fayet-Iliopoulos term in the noncommutative ADHM equations (4.43)-(4.44).

Based on this evidence, I thus propose the following extended AdS/CFT conjecture:

> The degrees of freedom of N_f D7 brane probes with k units of instanton charge on its worldvolume, embedded flatly into $\mathrm{AdS}_5 \times \mathrm{S}^5$ with a constant anti-selfdual B field in the $\vec{y}$-directions, are dual, in the standard holographic sense, to the $\mathcal{N} = 2$ field theory (2.140) for massless quarks at strong coupling, with the additional Fayet-Iliopoulos coupling (4.47).

In particular, the F term equations are given by (4.8)-(4.10) with $m = 0$, and the D term equation is modified by the Fayet-Iliopoulos term to

$$\zeta \ _{N_c \times N_c} = |q^i|^2 - |\tilde{q}_i|^2 + [\Phi_1, \Phi_1^\dagger] + [\Phi_2, \Phi_2^\dagger] \,. \qquad (4.69)$$

Note that in the AdS/CFT correspondence, a gravity dual does not only describe a dual field theory, but also fixes a particular state (which does not necessarily need to be a vacuum state), in which relevant quantities are computed [240]. I thus propose that the D7 probe with instanton charge k describes nonsupersymmetric Coulomb-Higgs states of the dual field theory, which are however not vacua of the theory. This can be seen from equation (4.69), which cannot be strictly fulfilled by a Coulomb branch ansatz with nonvanishing $\Phi_{1,2}$ expectation value but vanishing squark vacuum expectation values. The supersymmetry breaking via the Fayet-Iliopoulos term thus strictly lifts the Coulomb branch, which is the holographic manifestation of the fact that D3 branes sitting outside of the D7 branes in the flat space construction will be attracted towards it and finally dissolve in the D7 brane. In the holographic setup this instability is somewhat hidden, since we neglect the backreaction of the D7 probe onto the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. I will comment in section 6.2 on how this instability could show up again in the holographic setup.

The field theory on the D7 brane in the 4567 direction is expected to become equivalent to a kind of a noncommutative $U(N_f)$ Yang-Mills theory, in the sense that there is a Seiberg-Witten map [111], a field redefinition of the commutative DBI fields to a noncommutative Yang-Mills theory (4.38). However, since the background space $\mathrm{AdS}_5 \times \mathrm{S}^5$ is curved, the induced geometry of the 4567 directions will only be flat Euclidean space up to a warp factor depending on $|\vec{y}|^2$. Not much is known about noncommutative gauge theories on curved space-times and even less is known about the instanton solutions of such theories. However I have argued that there is good evidence that many aspects of the physics of flat space D3-D7 intersections with B field carry over to the holographic setting. I will thus assume in the following that the noncommutative ADHM construction for instantons is applicable also in the case of the unknown noncommutative gauge theory on the D7 brane. Since in flat space the noncommutative ADHM equations coincide with the Fayet-Iliopoulos deformed D and F term equations, and since the operator $|\vec{\hat{y}}|$ is also well-defined in the flat space noncommutative setup, there is a chance to gain more evidence for the above conjecture by considering a noncommutative gauge theory in flat space. In any case, finding the actual form of the induced noncommutative field theory on the D7 brane would be an extremely nontrivial research project of its own, on which I will comment in section 6.2.

In the next section I will thus study noncommutative instantons in flat space satisfying the noncommutative ADHM equations on the supergravity side. An explicit example will be the Nekrasov-Schwarz instanton for a $U(1)$ gauge theory [119], for which the geometric symmetry breaking will be calculated and compared to the field theory symmetries of the nontrivial Coulomb-Higgs state.

4.5 Field Theory Implications of Noncommutative Instantons

In this section I study some predictions of the proposed correspondence in the explicit example of a noncommutative $U(1)$ instanton. I consider $U(1)$ instantons since they do not have a commutative counterpart and thus should give rise to new phenomena in the dual gauge theory which may be easily identified. Furthermore, the technical complications of dealing with solutions to noncommutative field theories are smallest in the case of the simplest charge one noncommutative $U(1)$ instanton.

4.5.1 The $U(1)$ Nekrasov-Schwarz Instanton

In this section the instanton constructed by Nekrasov and Schwarz is reviewed. Following [119], I introduce complex coordinates $z_0 = y^4 + iy^5$, $z_1 = y^6 + iy^7$, with commutators

$$[z_0, \bar{z}_0] = 2\theta^{45}\,,\ [z_1, \bar{z}_1] = 2\theta^{67}\,,\ [z_0, z_1] = 0\,. \tag{4.70}$$

Hence θ^{mn} is given in a skew-diagonalized form (4.45), which can be achieved by an appropriate $SO(4)$ rotation. This basis has the advantage that the complex coordinate operators can be mapped onto two copies of standard bosonic Fock space generators. For θ^{45} and θ^{67} both negative,[15] the map is

$$a = \bar{z}_0/\sqrt{2|\theta^{45}|} = (a^\dagger)^\dagger\,, \quad b = \bar{z}_1/\sqrt{2|\theta^{67}|} = (b^\dagger)^\dagger\,.$$

If one or both of the components of θ change sign, the roles of creation and annihilation operators are interchanged. Again, $\theta^{45} = \pm\theta^{67}$ respectively correspond to a purely selfdual or anti-selfdual noncommutative space. In this section I consider an anti-selfdual θ.

The solution constructed in [119] is an anti-selfdual one, i.e. an anti-instanton. Within the framework of the D3-D7 system with a Fayet-Iliopoulos term, we need to consider an instanton instead. By a parity transformation $y^5 \mapsto -y^5$,[16] amounting to the exchange $z_0 \leftrightarrow \bar{z}_0$ already encountered in the ADHM construction in section 4.2, the solution of [119] is straightforwardly changed to be selfdual,

$$A = \frac{1}{d\left(d+\frac{\zeta}{2}\right)}\left[z_0 \mathrm{d}\bar{z}_0 + \bar{z}_1 \mathrm{d}z_1\right]\,, \tag{4.71}$$

$$F = \frac{\zeta}{(d-\zeta/2)d(d+\zeta/2)}\left[f_3(\mathrm{d}z_0\mathrm{d}\bar{z}_0 + \mathrm{d}z_1\mathrm{d}\bar{z}_1) + f_+\mathrm{d}z_0\mathrm{d}z_1 + f_-\mathrm{d}\bar{z}_1\mathrm{d}\bar{z}_0\right]\,, \tag{4.72}$$

with

$$f_3 = z_1\bar{z}_1 - \bar{z}_0 z_0\,, \quad f_+ = 2\bar{z}_0\bar{z}_1\,, \quad f_- = 2z_1 z_0\,.$$

Although the radial distance operator

$$d = \sum_{i=4}^{7}(y^i)^2$$

[15] By this choice, I follow the conventions of [119].

[16] Since the Pontryagin density is a pseudoscalar, parity changes the sign of the instanton number $\int \mathrm{Tr} F \wedge F$.

is invariant under parity, its form in complex coordinates changes, compared to [119], to

$$d = \bar{z}_0 z_0 + \bar{z}_1 z_1 \,. \tag{4.73}$$

Note that the parameter $\zeta = \theta^{45}/4 = -\theta^{67}/4$ is negative in our conventions (which are those of [119]). Clearly equation (4.72) fulfills the complexified selfduality equations

$$F_{z_0\bar{z}_0} = F_{z_1\bar{z}_1}\,, \quad F_{\bar{z}_0 z_1} = F_{z_0\bar{z}_1} = 0\,, \quad F_{z_0 z_1}\mathrm{d}z_0 \wedge \mathrm{d}z_1 = (F_{\bar{z}_0\bar{z}_1}\mathrm{d}\bar{z}_0 \wedge \mathrm{d}\bar{z}_1)^\dagger \,. \tag{4.74}$$

It is easy to show that this solution has instanton number plus one by first noticing that anti-selfduality $F_+ = 0$ changes into selfduality $F_- = 0$ under parity, and then realizing that the Lagrange density given in [119],

$$\mathcal{L} = -\frac{1}{8\pi^2} F_{mn} F_{mn} = \frac{\zeta^2}{4\pi^2} \frac{1}{d^2\,(d - \frac{\zeta}{2})\,(d + \frac{\zeta}{2})}\,\Pi\,, \tag{4.75}$$

is invariant under $z_0 \mapsto \bar{z}_0$. The reason is that this only amounts to exchanging the annihilators and creators in the z_0 Fock space. The projector $\Pi = \ - |0,0\rangle\langle 0,0| = \ - :e^{-a^\dagger a - b^\dagger b}:$ is normal ordered [352] and thus invariant. The integral-trace over the Lagrange density stays positive under parity, while the relation between the latter and the instanton number changes sign, and thus the solution (4.72) needs to have Pontryagin number $+1$, since the anti-instanton of [119] has -1.

The Nekrasov-Schwarz solution (4.72) does not have a freely selectable size modulus, in contrary to the BPST instanton [338]. This is expected since the dimensionality of the instanton moduli space is $4N_f k$. The $N_f = 1$ one-instanton solution thus has a four-dimensional moduli space, which is a copy of $\mathbb{R}^4$ encoding the instanton position, but it has no size modulus. Note however that a size of the instanton can still be defined by the squark expectation value (cf. equation (4.87)) $q = \sqrt{\zeta} = \sqrt{2|\theta^{45} - \theta^{67}|}$. It is fixed by the Fayet-Iliopoulos term. This also is exactly the minimal separation above which a Dp-D(p+4) system in flat space does not have a tachyon [322]. Exactly at this separation one of the 3-7 string modes becomes tachyonic, and the system ends up in the state with an instanton on D(p+4) after tachyon condensation [321]. The squark expectation value sets the instanton size as expected, as was argued in section 4.3 for the equivalence between D and F term equations on the Higgs branch and ADHM equations also in the noncommutative setup. The Fayet-Iliopoulos deformation of the D and F term equations thus removes the connecting point between the Coulomb- and Higgs branch of the moduli space of field theory vacua, which amounts to regularising the small instanton singularity.

Now let us assume the Nekrasov-Schwarz instanton solves the equations of motion of the yet to be known noncommutative field theory on the 4567-directions of the D7 brane. The instanton background further breaks the **remaining symmetries**. Using as a basis of rotations in $\mathbb{R}^4$ the rotations in the plane of two directions, e.g. the $4-5$-plane, one can parametrize an infinitesimal $SU(2)_L \times U(1)_R$ rotation of the coordinate operators by

$$\begin{aligned} z_0' &= (1+i(c+d))z_0 + (a+ib)\bar{z}_1 \,, && (4.76)\\ \bar{z}_0' &= (1-i(c+d))\bar{z}_0 + (a-ib)z_1 \,, && (4.77)\\ z_1' &= (1+i(c-d))z_1 - (a+ib)\bar{z}_0 \,, && (4.78)\\ \bar{z}_1' &= (1-i(c-d))\bar{z}_1 - (a-ib)z_0 \,. && (4.79) \end{aligned}$$

Here a, b, c generate $SU(2)_L$-transformations (c generates $U(1)_L$), while d generates $U(1)_R$ rotations. By evaluating the transformation law for (operator-valued) one-forms

$$A_i'(z')\mathrm{d}z'^i = A_i(z)\mathrm{d}z^i \tag{4.80}$$

to first order in the rotation parameters, one can show that the U(1) one-instanton solution (4.71) leaves the full

$$SU(2)_L \times U(1)_R \tag{4.81}$$

symmetry invariant.

4.5.2 Implications for the Dual Gauge Theory

The dual field theory of a probe D7 brane embedded in $\mathrm{AdS}_5 \times \mathrm{S}^5$ as in (4.52) is a $\mathcal{N}=2$ supersymmetric $U(N_c)$ gauge theory, which has one massless hypermultiplet in the fundamental representation of the gauge group, coupled to the $\mathcal{N}=4$ vector multiplet in the adjoint representation. As was argued in section 4.4, switching on a constant anti-selfdual B field on the gravity side is dual to Fayet-Iliopoulos terms for the auxiliary fields (D, F_1, F_2) of the $\mathcal{N}=2$ $U(1) \subset U(N_c)$ vector multiplet.

Due to the special form (4.48) of the B field (with $b_1 = -b_2$), only the Fayet-Iliopoulos term for the auxiliary D field is present. The global symmetries of the gauge theory (which were discussed in section 4.2) which coincide with the symmetries of gravity side setup are

$$SO(1,3) \times SU(2)_\Phi \times U(1) \times U(1)_\mathcal{R} \times U(1)_f, \tag{4.82}$$

with $U(1)$ being the remaining unbroken part of the $SU(2)_\mathcal{R}$, unbroken $U(1)_\mathcal{R}$ symmetry (corresponding to rotations of the z^i), and global flavour symmetry $U(1)_f$. Note that the conformal symmetry $SO(2,4)$ is broken by the B field (which is dimensionful) to the Lorentz group.

Consider the equations of the proposed dual gauge theory which determine the supersymmetric vacua. These are the F and D term equations

$$\begin{aligned} 0 &= \Phi_3 q = \tilde{q}\Phi_3 \,, && (4.83)\\ 0 &= [\Phi_1, \Phi_3] = [\Phi_2, \Phi_3] \,, && (4.84)\\ 0 &= q\tilde{q} + [\Phi_1, \Phi_2] \,, && (4.85)\\ \zeta_{\ N_c \times N_c} &= |q|^2 - |\tilde{q}|^2 + [\Phi_1, \Phi_1^\dagger] + [\Phi_2, \Phi_2^\dagger] \,. && (4.86) \end{aligned}$$

First let us consider the case in which all squark vacuum expectation values $\tilde{q}$ and q vanish. Since $U(1)$ factors drop out of the commutator terms in (4.86), and since the $\zeta\ _{N_c\times N_c}$ term is in the $U(1)\subset U(N_c)$, it is impossible to solve (4.86). Therefore, a pure Coulomb state is not supersymmetric, as expected from the flat space brane picture. The Fayet-Iliopoulos term lifts the whole Coulomb branch of vacua.

Now we are interested in the mixed Coulomb-Higgs state which is dual to the Nekrasov-Schwarz solution with instanton charge $k=1$. Since on the gravity side the gauge group was chosen to be the $U(1)_f$ residing on the single D7 brane, there is only one flavour present in the field theory. The number of nonvanishing squark components q_a is related to the instanton charge, and thus only one colour component of the squark fields q and $\tilde{q}$ in the dual gauge theory is nonvanishing. I choose the colour direction to be the first one, and the respective component will be called q_1 and $\tilde{q}_1$. Due to the ansatz (4.12) and (4.13) for the mixed Coulomb-Higgs branch equation (4.83) is trivially satisfied. Using this ansatz the F term equation (4.84) gives the constraint (4.14). Furthermore the equations (4.85) and (4.86) reduce to

$$0=q_1\tilde{q}_1=|q_1|^2-|\tilde{q}_1|^2-\zeta\,. \tag{4.87}$$

As already discussed, the other D term equations involving q_a and $\tilde{q}_a$ for $a=2,\ldots,N_c$ cannot be satisfied, which reflects the instability of the Coulomb part of the Coulomb-Higgs branch.

Solving (4.87), we find that the gauge theory with only one flavour has a unique state with squark vacuum expectation value at strong 't Hooft coupling, with the expectation value given by

$$q_1=\sqrt{\zeta}\,,\quad \tilde{q}_1=0\,. \tag{4.88}$$

We recognize that the squark vacuum expectation value is in one-to-one correspondence with the size of the $U(1)$ noncommutative instanton on the gravity side, as elaborated in section 4.5.1. This is in agreement with the fact that except of the position moduli, noncommutative $U(1)$ instantons do not have additional moduli and their size modulus is given by the noncommutativity parameter of the underlying space-time [321].

Due to the squark vacuum expectation values $q_1=\sqrt{\zeta}$ and $\tilde{q}_1=0$, the flavour symmetry $U(1)_f$, the $\mathcal{R}$-symmetry $U(1)\subset SU(2)_{\mathcal{R}}$ and the $U(1)$ part of the $U(N_c)$ gauge group are broken. Since the squark vacuum expectation values $(q,\tilde{q})$ transform under $U(1)_f$ as

$$q\to e^{i\alpha_f}q,\quad \tilde{q}\to e^{-i\alpha_f}\tilde{q} \tag{4.89}$$

the $U(1)_f$ rotation can be undone by an appropriate $U(1)\subset SU(2)_{\mathcal{R}}$ transformation of the form

$$q\to e^{i\alpha_R}q,\quad \tilde{q}\to e^{i\alpha_R}\tilde{q}. \tag{4.90}$$

Therefore, the diagonal subgroup $\mathrm{diag}(U(1)\times U(1)_f)$ defined by

$$\alpha_f=-\alpha_R$$

Object	Coordinates			
	x^μ	ρ	y_m S^3	z^i
$\mathrm{AdS}_5 \times \mathrm{S}^5$	$SO(2,4)$		$SO(6) \simeq SU(4)$	
D7	$SO(2,4)$		$SU(2)_L \times SU(2)_R$	$U(1)_{89}$
$*B = -B$	$SO(1,3)$	$SU(2)_L \times U(1)_R$		$U(1)_{89}$
Eq. (4.71)	$SO(1,3)$	$SU(2)_L \times \mathrm{diag}(U(1)_R \times U(1)_f)$		$U(1)_{89}$

Table 4.3: The symmetry breaking pattern (see text for an explanation of the colour coding).

is preserved by the squark vacuum expectation value. Note that this further breaking of the symmetries is not seen in the symmetry calculation of the previous section 4.5.1. Possible reasons for resolving this mismatch are discussed in section 6.2. In short, it might be that the rather simple symmetry calculation of the previous section neglects the fact that space-time and gauge symmetries do mix in noncommutative field theories, i.e. that the $U(1) \subset SU(2)_R \subset SO(4)_{4567}$ rotations actually change the gauge choice underlying the Nekrasov-Schwarz solution and thus might not all be independently preserved. The **symmetry breaking pattern** is summarised in table 4.3. The colour coding is as follows: Red, yellow and blue indicate the range of coordinates which are acted upon by the respective groups given there. The mixed-color orange and green fields indicate coordinates which are acted upon by both of the two adjacent symmetry transformations. In the last line I included the field theory symmetries rather than the geometric symmetries of the AdS side since, although the situation is not fully understood yet, I find this result more trustworthy.

Finally, for completeness let us consider the case of the Higgs vacuum, for which all D3 branes are dissolved in the D7, and for which an AdS/CFT dual is not describable in the probe limit. The Higgs vacuum is given by $\Phi_3 = 0$ and $|q_a| = \sqrt{\zeta}, \tilde{q}_a = 0$ for $a = 1, \ldots, N_c$. Furthermore, all commutators involving Φ^1, Φ^2 and Φ^3 vanish. It is easy to verify that the F and D term equations are simultaneously satisfied. The Higgs vacuum is therefore supersymmetric. This vacuum corresponds to a charge N_c noncommutative $U(1)$ instanton on the D7 brane which, since the on-shell action of instantons is proportional to their charge, violates the probe approximation.

Chapter 5

A Matrix Model Proposal for the Robertson-Walker Universe

5.1 Introduction and Summary

In the last two chapters, different generalisations of the AdS/CFT correspondence, a realisation of the holographic idea arising from string theory, were presented. In this chapter I turn to another form of holographic description of gravitational theories in terms of nongravitational ones, namely the description of quantum gravity (and in particular in eleven dimensions of $\mathcal{M}$-theory) by quantum mechanical models of hermitian matrices, so-called matrix models. The main result of this chapter will be the proposal of a matrix model which ought to describe the physics in a well-known cosmological background, the Robertson-Walker geometry. This chapter thus is concerned with holographic descriptions of cosmology and in particular of the big bang singularity.

Recent astronomical data, in particular the WMAP three-year data of the cosmic microwave background fluctuations [353], shows that the current universe is very well described by the spatially flat Robertson-Walker geometry. This indicates that the universe has evolved from a big bang singularity, near which quantum effects of matter and gravity are expected to have played an equally important role. This expectation is based on one hand on the phenomenon of coupling constant unification observed in the standard model of particle physics: Due to the renormalisation group running of the coupling constants in the standard model, the coupling strength of all four fundamental interactions (gravity, electroweak and strong interaction) grows large (of order one) at energies of around 10^{15}GeV. On the other hand, per definition the Planck mass is the mass scale at which the Compton wavelength of a particle is roughly equal to the Schwarzschild radius of a Schwarzschild black hole of the same mass. Since the Compton wavelength $\lambda = h/(Mc)$ is a purely quantum mechanical quantity while the Schwarzschild radius (2.4) is a purely gravitational quantity, the Planck mass is the energy scale at which gravitational effects such as black hole formation and quantum effects such as the energy-time uncertainty are equally strong. In particular, quantum fluctuations in the local energy density are strong enough to create "virtual black holes"[1]. Thus the physics at the Planck scale cannot be described any more as quantum field theory on a curved but otherwise classical background, but must be described by a fully quantised description of both matter and gravity.

While a completely satisfactory quantum gravitational description of the big bang singularity has not been achieved yet, string theory and $\mathcal{M}$-theory, its eleven-dimensional origin, is believed to be a candidate for a consistent theory of quantum gravity. In particular $\mathcal{M}$-theory, since it is the strong-coupling completion of ten-dimensional type IIA string theory, is believed to be applicable to realistic cosmological singularites at which the coupling strengths in both the matter and gravity sectors, as set by the string coupling g_s in string theory, equally grow large. Although a fully satisfactory formulation of $\mathcal{M}$-theory has not been found yet either, there is evidence based on the work of Banks, Fischler, Shenker and Susskind [42] that certain models of supersymmetric matrix quantum mechanics capture the full nonperturbative physics of $\mathcal{M}$-theory. As will be explained in section 5.3.1, this remarkable conjecture relies on properties of D0 branes, which in a specific sense behave similar to free particles and thus can be used to build a state space for $\mathcal{M}$-theory similar to the Fock space in ordinary quantum field theory.

Although originally formulated in eleven-dimensional flat space, matrix model and 1+1-dimensional supersymmetric Yang-Mills descriptions[2] of $\mathcal{M}$-theory on time dependent backgrounds have lead to a number of insights [361, 362] (see [360] for a review). The main focus of these studies so far has been on supersymmetry-preserving orbifold, plane wave or linear dilaton backgrounds, but not on the physically more relevant (supersymmetry breaking) Robertson-Walker geometry. Nevertheless it has been for example possible to show in the 1+1-dimensional supersymmetric Yang-Mills description of $\mathcal{M}$-theory on the linear dilaton background that interactions give rise to a one-loop effective potential [363] which falls off sufficiently fast at late times and for large brane separations. This implies that the D1 branes behave as distinct free test particles probing a smooth nine-dimensional transverse space. This phenomenon is interpreted as the emergence of a classical spacetime[3] out of the quantum evolution around the singularity, since if separated branes (in

[1]Though it is barely understood how to describe virtual black holes in higher-dimensional proposals for quantum gravity, it is possible to give a precise meaning to this term in two-dimensional dilaton gravity theories coupled to bosonic and fermionic matter [354–359]. In these theories effective vertices for gravitational interactions between matter fields can be calculated by integrating out the gravitational sector of the theory. Those effective vertices can be given an interpretation as a continuous sum (i.e. integration) over virtual black hole geometries.

[2]These two-dimensional supersymmetric Yang-Mills theories are related to the IIA BFSS matrix model, which is the low energy effective descriptions of D0 branes in type IIA theory in the infinite momentum frame as follows: Discrete Light-Cone Quantisation of $\mathcal{M}$-Theory on $^{1,8} \times \mathrm{S}^1_{R_9} \times \mathrm{S}^1_{R_s}$ leads to IIA string theory on $^{1,8} \times \mathrm{S}^1_{R_9}$. In this procedure, as will be explained in detail later, the string length $\ell_s \to \infty$ has to be taken to infinity, and thus the physics is best described after a T-duality along the $\mathrm{S}^1_{R_9}$, which decompactifies in this way. This yields 1+1-dimensional SYM theory on flat ^{1,9} [360].

[3]Strictly speaking time does not so easily emerge in matrix models since it is used as the evolution pa-

a vacuum in which the transverse scalar matrix fields commute with each other) do not interact their coordinates can be interpreted as the "points" of the underlying manifold. In other, more symmetric models, the resolution of the singularity has been made precise too. In general, singularity resolution in string theory is connected to the breakdown of the effective field theory description in terms of gravity due to the appearance of some light states near the singularity. These states can be perturbative or nonperturbative, i.e. strings or branes. They become massless as the singularity is approached and thus should be included in a good effective description at the singularity rather than being integrated out. These additional degrees of freedom, whose emergence may be interpreted in specific circumstances in terms of an enhancement of symmetries, resolve the singularity via an essentially nonsingular dynamics, in the sense that by including these new light modes in calculations it is possible to obtain well-defined and most importantly finite answers for observables which were ill-defined in the effective theory [360]. In the context of matrix models this happens very naturally, since off-diagonal modes of the matrices become light if the D0 branes come close together at the singularity. The matrix model itself is then the valid effective description including the light modes, and meaningful observables have to be identified and investigated in order to learn about how certain singularities are resolved.

However, since the concordance model of cosmology is based on the Robertson-Walker geometry and describes our universe very accurately, one of the most important tasks for string/$\mathcal{M}$-theory as a theory of quantum gravity thus is to provide a resolution of the big bang singularity in the same controlled fashion as just described. In particular, it would be highly interesting to find matrix models, either from a bottom-up approach or derived from string theory, which reproduce certain aspects of the physics near the big bang singularity. One technical obstacle posed by the Robertson-Walker geometry is, aside from its nonsupersymmetric nature, the lack of a null isometry. Hence the conventional light-cone quantisation of the M2 brane is not applicable and a new approach is required.

In this chapter I present work obtained in collaboration with Johanna Erdmenger and Jeong-Hyuck Park and published in [3], which provides such an approach. The characteristic feature of this formalism is the presence of a Hamiltonian constraint, which replaces the gauge-fixing constraints in the light-cone approach to membrane quantisation. This will lead to a proposal for a matrix model which includes in its spectrum semiclassical states describing bosonic membranes in the Robertson-Walker geometry. The existence of such states is ensured *per constructionem* through deriving the matrix model from the Nambu-Goto action for a bosonic membrane, which is the bosonic part of the world-volume action of an M2 brane. The underlying rationale is, as explained in detail later, that a quantisation of the spatial world volume of a membrane leads directly to a description of its dynamics in terms of "bits", which are, via our experience with the flat eleven-dimensional

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rameter for the quantum evolution of the model. Nevertheless I will often speak of "space-time emergence" in this chapter, since this is the commonly used term in the literature. In light-like dilaton backgrounds, the coupling decreases from infinity to zero as time passes from the past to the future, in this sense time is emergent in these models.

situation, identified with D0 branes in type IIA string theory (or at least the bosonic part of their effective action). The quantisation of the spatial membrane world volume is a deformation quantisation which introduces noncommutativity for the spatial world volume coordinates. In this approach to derive matrix models it is important to fix the membrane world volume diffeomorphisms up to static area preserving ones, since these are exactly the automorphisms of the two-dimensional noncommutative algebra $[\sigma^i, \sigma^j] = i\epsilon^{ij}$. In the approach presented here, this gauge fixing is not achieved by fixing the light-cone momentum, but by fixing the Hamiltonian density to a particular world volume profile. In this way it becomes possible to construct a tree-level matrix model for the realistic Robertson-Walker geometry and demonstrate the emergence of classical space-time from an originally fuzzy geometry.

This chapter is organised as follows: I begin with an analysis of the geodesic motion of a single point particle, i.e. by analysing the dynamics of geodesics, in the Robertson-Walker geometry in section 5.2. In particular, I propose a classical mechanical model with a particular global $\mathbf{so}(1,2)$ symmetry and I show that for two different parameter regimes and different choices of one-dimensional diffeomorphism gauge, this mechanical system describes either the geodesic motion of a point particle in the spatially flat Robertson-Walker background, *or* homogeneous metric fluctuations around the Robertson-Walker geometry in Einstein-Hilbert gravity with a positive cosmological constant. More precisely, in each case a conserved quantity can be found such that any sector with fixed value of the quantity is described by the conformal mechanics. In section 5.3, I first review in some detail the connection between string theory, $\mathcal{M}$-theory and matrix models in section 5.3.1. In section 5.3.2 I present the main result of this chapter, namely the derivation of a matrix model from the action of the bosonic membrane in the Robertson-Walker background. I furthermore show that imposing the Hamiltonian constraint in the matrix model ensures the emergence of space-time. Emergence here means that the Hermitian matrices in the matrix model, whose eigenvalues encode the positions of D-particles probing the underlying space-time geometry, become simultaneously diagonalisable once the size of the universe grows significantly larger than the Planck length. Once the matrices are simultaneously diagonalisable, the D0 brane positions can be simultaneously measured and thus the underlying space-time geometry they probe is a smooth manifold structure, and geometric quantities become classically well-defined [362]. A further summary and discussion of the results as well as possible further developments can be found in section 6.3.

5.2 A Classical Mechanical Model with Dynamical Conformal Symmetry

In this section I first introduce a general classical mechanical model which depends on two parameters c_1 and c_2 and has a dynamical $\mathbf{so}(1,2)$ symmetry, i.e. a symmetry with algebra $\mathbf{so}(1,2)$ but acting on the degrees of freedom in a special way different from conformal transformations of the time direction. I then proceed to show that two different aspects of the physics of the Friedmann-Robertson-Walker (FRW) universe are described by this model for two different choices of c_1 and c_2. The first aspect is the geodesic motion of a point particle in the FRW background after fixing certain constants of motion, while the second aspect is the dynamics of homogeneous (i.e. space-independent) metric fluctuations around that background, again in a subsector with fixed constants of motion. The geodesic motion dynamics will be the basis for the derivation of the matrix model in section 5.3.

5.2.1 The Conformal Mechanical Model

The conformal mechanics (henceforth abbreviated "CM") which I show in the following to be closely related to the spatially flat FRW universe, is of the general form

$$\mathcal{S}_{\mathrm{CM}} = \int \mathrm{d}t \left[\tfrac{1}{2}\eta\dot{\varphi}^2 + \tfrac{1}{2}\eta^{-1}\left(\frac{c_1}{\varphi^2} + c_2\right)\right]. \tag{5.1}$$

Here c_1, c_2 are dimensionless constants, $\varphi(t)$ is the only dynamical variable and $\eta(t)$ is the inverse of an einbein density, the analogue of $\sqrt{-g}$. $\varphi(t)$ is dimensionless, but the einbein has the same dimension as the time, i.e. dimension of *length*.

Integrating out the auxiliary variable η by use of its nondynamical equation of motion reduces the action to

$$S = \int \mathrm{d}t \; |\dot{\varphi}| \sqrt{c_1\varphi^{-2} + c_2}\,,$$

where the sign of the square root was chosen to be positive in order to have a positive semi-definite action. The action (5.1) is thus the "Polyakov form" of this square root action, in the sense of the usual treatment of the Nambu-Goto action in string theory.

Both φ and η transform under one-dimensional diffeomorphisms $t \to s(t)$ as $(\varphi(t), \eta(t)) \to (\varphi(s), \eta(s)/\dot{s})$, i.e. η transforms as a scalar density with weight -1. Fixing the gauge symmetry with an arbitrary function of time, $\eta \equiv \hat{\eta}(t)$, the mechanical system (5.1) essentially corresponds to a conformal mechanics proposed in [364]. The gauge fixed action is invariant under the transformation

$$\delta\varphi = \hat{\eta}(f\dot{\varphi} - \tfrac{1}{2}\dot{f}\varphi)\,, \tag{5.2}$$

where $f(t)$ is given by a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\eta}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\eta}\dot{f})\right] = 0\,. \tag{5.3}$$

This third order differential equation has three solutions forming the symmetry algebra $\mathbf{so}(1,2)$: With the definitions

$$\beta(t) := \int^t \frac{\mathrm{d}\tilde{t}}{\hat{\eta}(\tilde{t})}\,, \quad \gamma(t) = \int^t \frac{\mathrm{d}\tilde{t}}{\hat{\eta}(\tilde{t})}\beta(\tilde{t})\,,$$

and

$$p_\varphi = \hat{\eta}\dot{\varphi}\,, \tag{5.4}$$

the three solutions are

$$f_0 = \tfrac{1}{\sqrt{2}}\left(1 + \tfrac{1}{2}\beta(t)^2\right)\,, \quad f_1 = \beta(t)\,, \quad f_2 = \tfrac{1}{\sqrt{2}}\left(1 - \tfrac{1}{2}\beta(t)^2\right)\,.$$

The corresponding Noether charges

$$Q_f = \tfrac{1}{2}f\left(p_\varphi^2 - c_1\varphi^{-2}\right) - \tfrac{1}{2}\hat{\eta}\dot{f}\varphi p_\varphi + \frac{1}{4}\hat{\eta}\varphi^2\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\eta}\dot{f}) \tag{5.5}$$

generate the transformation (5.2) via the usual Poisson bracket,

$$\{\varphi, Q_f\}_{\mathrm{PB}} = \delta_f\varphi\,. \tag{5.6}$$

They form an $\mathbf{so}(1,2)$ Lie algebra,

$$\{Q_1, Q_2\}_{\mathrm{PB}} = -Q_0\,, \quad \{Q_2, Q_0\}_{\mathrm{PB}} = Q_1\,, \quad \{Q_0, Q_1\}_{\mathrm{PB}} = Q_2\,. \tag{5.7}$$

After the redefinition

$$D = -Q_2\,, \quad P = Q_0 - Q_2\,, \quad K = Q_0 + Q_2\,, \tag{5.8}$$

this is the same Lie algebra as the conformal symmetry algebra in $0+1$ dimensions,

$$\{D, P\}_{\mathrm{PB}} = -P\,, \quad \{D, K\}_{\mathrm{PB}} = K\,, \quad \{P, K\}_{\mathrm{PB}} = 2D\,. \tag{5.9}$$

The Hamiltonian corresponding to the gauge fixed Lagrangian reads, with the canonical momentum (5.4),

$$\mathcal{H}_{\mathrm{CM}} = \frac{1}{2\hat{\eta}(t)}\left(p_\varphi^2 - \frac{c_1}{\varphi^2} - c_2\right)\,. \tag{5.10}$$

The physical states must lie on the surface of vanishing energy

$$\mathcal{H}_{\mathrm{CM}} \equiv 0$$

in phase space, as implied by the gauge fixing of diffeomorphism invariance.[4]

Note that this $\mathbf{so}(1,2)$ symmetry is an additional symmetry of the system, not connected in an obvious way to the conformal symmetry in 0+1 dimensions (time shifts, dilatations and special conformal transformations). This can be understood from the fact that since our model is invariant under diffeomorphisms of t, the conformal transformations of the time are actually embedded in the diffeomorphism group of the real line as a subgroup.

[4]Throughout this chapter, '$\equiv$' denotes gauge fixings or on-shell relations, depending on the context in which this symbol is used. The double meaning appearing here comes from the fact that in this chapter certain diffeomorphism symmetries are gauged by demanding certain diffeomorphism noninvariant objects such as Lagrangians to take fixed on-shell values in sectors defined by conserved quantities.

Gauge fixing with a general function $\hat{\eta}(t)$ breaks all the diffeomorphisms including time shifts. Also no gauging with time shifts $t'(t) = t + \epsilon$, dilatations $t'(t) = \lambda t$ and special conformal transformations $t'(t) = t - \epsilon t^2$ all together as residual gauge freedom can be chosen: time shifts would restrict $\hat{\eta}$ to a constant, dilatations to the form const $\times\, t$, and special transformations to const $\times\, e^{t^2}$. The symmetry must thus be an accidental additional global symmetry of the gauge-fixed action which is realised on the dynamical variable $\varphi(t)$ by the Noether charges (5.5). It may however be very useful in solving the system (5.1) and, as explained in section 6.3, may also play a role in the dynamics of the matrix model presented in this chapter.

5.2.2 Conformal Mechanics from Point Particle Dynamics

In D-dimensional space-time, requiring both homogeneity and isotropy of the $D-1$ spatial dimensions, the metric is constrained to the Robertson-Walker (RW) geometry [365]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\mathbf{x}^2 + \kappa(\mathbf{x}\cdot\mathrm{d}\mathbf{x})^2/(1-\kappa\mathbf{x}^2)\right] , \tag{5.11}$$

where $\kappa = +1, 0, -1$ is a constant, and $a(t)$ is the only undetermined function, called the scale factor, which depends on the cosmic time t. The spatial part of this metric is constrained by isotropy and homogeneity to maximally symmetric three-dimensional Euclidean spaces, which there are the three-sphere S^3, flat Euclidean plane E^3 or the hyperbolic plane H^3. They have constant positive, zero and negative curvature, respectively. Detailed measurement [353] of the fluctuation spectrum in the cosmic microwave background are consistent with a universe that is spatially flat ($\kappa = 0$) and otherwise described by the RW metric (5.11). These are the assumptions which enter the so called ΛCDM model which assumes that the structure formation of the universe is governed by quantum fluctuations in an inflationary model based on a simple cosmological constant Λ, as well as by the dynamics of cold dark matter, i.e. dark matter with velocities in the nonrelativistic regime $v \ll c$. It is the simplest model which fits with all observations of the cosmic microwave background and with the large-scale distribution of matter and galaxies. Fitting the input parameters of the ΛCDM model to the data yields values for the equation of state parameter w and the spatial curvature

$$w = \frac{p}{\rho} = -0.926^{+0.051}_{-0.075}\,, \quad \kappa = -0.010^{+0.014}_{-0.012}\,, \tag{5.12}$$

with $p = w\rho$ being the equation of state for the energy and matter in the universe on large scales. These numbers are consistent with a pure cosmological constant $w = -1$ and a spatially flat universe $\kappa = 0$. Note however that when fitting to models with a time varying equation of state parameter $w(t)$ (e.g. Quintessence models), the data seems to slightly favour a universe with negative spatial curvature (see figure 17 in [353]).

Before analysing the motion of point particles along geodesics in the RW metric, let us analyse a generic mechanical system satisfying the following two conditions:

1. The Hamiltonian is given by the inverse of the Lagrangian

$$\mathcal{H}\mathcal{L} = -m^2 \,, \tag{5.13}$$

where m is a constant parameter with unit of mass.

This always holds for a relativistic point particle Lagrangian of the form

$$\mathcal{L} = -m\sqrt{1 - g_{ij}(t,x)\dot{x}^i\dot{x}^j} \,,$$

after choice of the gauge $\tau = t$ for the world line time and furthermore $g_{00} = g_{0i} = 0$.

2. There exists a conserved quantity with on-shell value ν such that for a sector with fixed ν the Lagrangian is completely fixed on-shell as a time- and ν-dependent function,

$$\mathcal{L} \equiv e_\nu(t) \,. \tag{5.14}$$

The requirement here is thus the existence of at least one conserved quantity, since the on-shell Lagrangian always is a function of the on-shell values of conserved quantities and of time.

Then **the square-root free Lagrangian**

$$\mathcal{L}_\nu := \frac{\mathcal{L}^2}{2e_\nu(t)} - \frac{m^2}{e_\nu(t)} + \frac{e_\nu(t)}{2} \tag{5.15}$$

together with the Hamiltonian constraint which has to be imposed after fixing the world-line reparametrisation invariance equivalently describes the sector of fixed ν. Notice the similarity between (5.15) and the "Polyakov action" $\mathcal{L}_{\text{Polyakov}} = \frac{1}{2}\left(e^{-1}\mathcal{L}^2 + e\right)$. The trick at work here is thus similar to the Deser-Zumino-Brink-Di Vecchia-Howe method [159,160] for rewriting the Nambu-Goto to obtain the Polyakov action. The truth of the above statement can be seen by observing that all the canonical momenta of (5.15) take the same values after imposing the constraint (5.14) (which is equivalent to fixing time diffeomorphisms) as those of the original Lagrangian $\mathcal{L}$. Explicitly, for every pair of conjugate coordinates q and momenta p, p_ν, the following identity holds:

$$p_\nu = \frac{\partial \mathcal{L}_\nu}{\partial \dot{q}} \overset{(5.15)}{=} \frac{\mathcal{L}}{e_\nu}\frac{\partial \mathcal{L}}{\partial \dot{q}} \overset{(5.14)}{\equiv} \frac{\partial \mathcal{L}}{\partial \dot{q}} = p \,.$$

The Hamiltonian corresponding to $\mathcal{L}_\nu$ can be derived from (5.13) and (5.15), reading

$$\mathcal{H}_\nu = \left(\mathcal{L}^2 - e_\nu^2\right)/(2e_\nu). \tag{5.16}$$

Imposing the Hamiltonian constraint $\mathcal{H}_\nu \equiv 0$ then is equivalent to (5.14) (up to an irrelevant choice of sign $\mathcal{L} = \pm e_\nu(t)$). This procedure has been applied in [366] to the relativistic point particle dynamics of noncritical type 0A string theory black holes and linear dilaton backgrounds in order to derive matrix model descriptions of these backgrounds.

I now turn to the dynamics of a relativistic particle in the four-dimensional Robertson-Walker background (5.11) and apply the method described above to this system. In spherical coordinates the metric (5.11) reads

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big[\mathrm{d}\rho^2 + r^2(\rho)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\Big]\,,$$

where $r(\rho) = \sqrt{\mathrm{x}^2}$ is given by

$$r(\rho) = \begin{cases} \sin(\sqrt{\kappa}\rho)\,/\sqrt{\kappa} & \kappa = +1 \\ \rho & \kappa = 0 \\ \sinh\big(\sqrt{-\kappa}\rho\big)\,/\sqrt{-\kappa} & \kappa = -1\,. \end{cases} \tag{5.17}$$

Note that at this stage the scale factor $a(t)$ is left undetermined, since we do not want to restrict to any particular matter content of the underlying theory.

After identifying the world line affine parameter with cosmic time, $\tau = t$, the point particle Lagrangian in the Robertson-Walker background becomes

$$\mathcal{L} = -m\sqrt{1 - a^2(t)\left(\dot{\rho}^2 + r^2(\rho)\dot{\theta}^2 + r^2(\rho)\sin^2\theta\,\dot{\phi}^2\right)}\,. \tag{5.18}$$

Note that since the DBI action is also of square-root form, this action also governs the motion of a D0 brane or D-particle in the background (5.11), if the mass $m = T_0$ is identified with the tension of the D0 brane and if no further background fields are present.[5] A possible homogeneous dilaton factor can be absorbed into the Robertson-Walker metric via a redefinition of 'frame', i.e. by a conformal transformation and a redefinition of time, through which of course the functional form of $a(t)$ will change.

The canonical momenta for ρ, θ and ϕ derived from (5.18) are

$$p_\rho = M(t)\dot{\rho}\,,\;\; p_\theta = M(t)r^2(\rho)\dot{\theta}\,,\;\; p_\phi = M(t)r^2(\rho)\sin^2\theta\,\dot{\phi}\,, \tag{5.19}$$

where I introduced the compact notation

$$M(t) := -m^2a^2(t)/\mathcal{L}\,.$$

The corresponding Hamiltonian

$$\mathcal{H} = \sqrt{m^2 + a^{-2}(t)\left[p_\rho^2 + r^{-2}(\rho)\left(p_\theta^2 + p_\phi^2/\sin^2\theta\right)\right]}$$

[5]In superstring theories there will arise a problem with setting the fermionic excitations of the D0 brane consistently to zero, since the background (5.11) does not preserve supersymmetry. However, since gravity is always attractive in classical general relativity, the presence of additional matter will only change the detailed functional form of $a(t)$, and not the existence of a singularity.

satisfies the requirement (5.13).

In spite of the arbitrariness of the scale factor $a(t)$, the dynamics of (5.18) is *integrable*, as there exist three mutually Poisson-bracket commuting conserved quantities

$$p_\phi \equiv \text{constant}\,, \tag{5.20}$$
$$J^2 := p_\theta^2 + p_\phi^2/\sin^2\theta \equiv j(j+1)\,, \tag{5.21}$$
$$p_\rho^2 + J^2/r^2(\rho) \equiv (\nu m)^2\,. \tag{5.22}$$

The constant j plays the role of a classical $\mathbf{so}(3)$ angular momentum. Note that ν itself is not a measure of the energy of the system, as the energy is not conserved due to the time dependence introduced by $a(t)$, but it is a measure for the initial energy the system starts with at a time t_0. Indeed, at this time, the Hamiltonian has a value $\sqrt{m^2 + (\nu m)^2/a^2(t_0)}$. Interestingly, this set of conserved quantities is time-independent, although the Hamiltonian itself is time-dependent. Introducing a time-dependent mass as the on-shell value of $M(t)$,

$$m_\nu(t) := m a(t)\sqrt{a^2(t) + \nu^2} \equiv M(t)\,, \tag{5.23}$$

both the Hamiltonian and the Lagrangian assume the on-shell values, $\mathcal{H} = -m^2/\mathcal{L} \equiv m_\nu(t)/a^2(t)$.

The square-root free Lagrangian (5.15) now explicitly reads

$$\mathcal{L}_\nu = \frac{m_\nu(t)}{2}\Big(\dot\rho^2 + r^2(\rho)\dot\theta^2 + r^2(\rho)\sin^2\theta\,\dot\phi^2\Big) + \frac{(\nu m)^2}{2m_\nu(t)}\,. \tag{5.24}$$

Again, all the canonical momenta of (5.24) match with the on-shell ones of (5.18). Furthermore, the corresponding Hamiltonian

$$\mathcal{H}_\nu = \frac{1}{2m_\nu(t)}\left(p_\rho^2 + \frac{J^2}{r^2(\rho)} - (\nu m)^2\right) \tag{5.25}$$

exhibits the same mutually commuting conserved quantities as (5.20)-(5.22). Thus the surface of the vanishing energy $\mathcal{H}_\nu \equiv 0$ in the phase space of the dynamical system (5.25) describes precisely the relativistic point particle in the Robertson-Walker background for a sector of fixed ν.

Since there are two other conserved quantities, total angular momentum and momentum in the ϕ direction, the dynamics can further be reduced. The subsector of fixed angular momentum is reached by setting

$$J^2 \equiv j(j+1)\,,$$

which reduces (5.25) to the mechanical system (5.10). In this way, the following conclusion is reached:

> The conformal mechanics (5.10) with the choice
>
> $$\varphi = \rho\,, \quad \hat\eta = m_\nu(t)\,, \quad c_1 = -j(j+1) \le 0\,, \quad c_2 = (\nu m)^2 \ge 0$$

describes the geodesic motion of a relativistic particle with respect to cosmic time in the spatially flat Robertson-Walker universe, as well as with fixed conserved quantities ν, j.

Note that, since $c_1 \leq 0$, $c_2 \geq 0$, the classical energy (5.25) of the system is positive for large enough j or small enough ν.

5.2.3 Conformal Mechanics from Homogeneous Gravity

In the last section I showed that in a sector of fixed initial energy and fixed total angular momentum, the dynamical system governing the geodesic motion in the spatially flat Robertson-Walker geometry reduces to a dynamical system described by the conformal mechanics (5.1). In this section I present a similar reduction for the nonperturbative but homogeneous modes in Einstein-Hilbert gravity with positive cosmological constant, which also is the theory giving rise to the solution (5.11).

Pioline and Waldron observed in [367] that for a solely time-dependent, generic D-dimensional metric in longitudinal gauge ($g_{0i} = 0$),

$$\mathrm{d}s^2 = -e^2\varrho^{-2}\mathrm{d}t^2 + \varrho^{2/(D-1)}\hat{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \quad \det\hat{g} = 1\,, \tag{5.26}$$

the Einstein-Hilbert action with cosmological constant reduces to a mechanical system for a relativistic "fictitious" point particle,

$$-\int \mathrm{d}^D x\sqrt{-g}\,(R-2\Lambda) = V\int \mathrm{d}t\,\left[\frac{1}{e}\left(\frac{D-2}{D-1}\right)\dot{\varrho}^2 - 2e\left(\frac{\hat{C}}{\varrho^2} - \Lambda\right)\right] \tag{5.27}$$

Here V is the $D-1$-dimensional spatial volume which needs to be dropped in the reduction, and

$$\hat{C} := \frac{1}{8}e^{-2}\varrho^4\mathrm{tr}(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}})\,,$$

which contains a nonlinear σ-model metric for the coset

$$\frac{\mathrm{SL}(D-1)}{\mathrm{SO}(D-1)}$$

via

$$\mathrm{tr}(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}}) = h_{ab}(\theta)\dot{\theta}^a\dot{\theta}^b\,.$$

In terms of the momenta

$$p_\varrho = \omega\dot{\varrho}/(4e)\,, \quad p_a = -\tfrac{1}{2}\varrho^2 e^{-1}h_{ab}\dot{\theta}^b\,,$$

the conserved quantity $\hat{C}$ can be identified as

$$\hat{C} = \tfrac{1}{2}h^{ab}(\theta)p_a p_b\,,$$

the kinetic energy on the coset space. The Hamiltonian of the system reads

$$\mathcal{H}_{\text{E.H.}} = \frac{2e}{\omega}\left(p_\varrho^2 - \frac{\omega\hat{C}}{\varrho^2} - \omega\Lambda\right), \quad \omega := 8\left(\frac{D-2}{D-1}\right). \tag{5.28}$$

$\hat{C}$ is conserved [367] and positive semi-definite[6].

The original motivation of [367] to consider the homogeneous modes only was based on the observation in [368] that near cosmological singularities inhomogeneous modes generically decouple and thus the physics of cosmological singularities should be describable by the homogeneous modes alone. On the basis of the above reduction I make the further observation that since the spatially flat Robertson-Walker geometry (5.11) is a special case of (5.26) with $e = \varrho = a^{D-1}$, $\hat{g}_{ij} = \delta_{ij}$, and hence (5.26) is the most general homogeneous and nonperturbative fluctuation around the Robertson-Walker metric. In the cosmic time gauge $e = \varrho$, comparing eq. (5.28) and eq. (5.10) the following conclusion is reached:

> The mechanical system (5.10) with the choice (for $D > 1$)
>
> $$\varphi = \varrho\,, \quad \hat{\eta} = \frac{1}{4}\omega a^{1-D}\,, \quad c_1 \equiv \omega\hat{C} \geq 0\,, \quad c_2 = \omega\Lambda \geq 0$$
>
> describes the homogeneous metric fluctuations of the spatially flat Robertson-Walker universe with respect to cosmic time and in a sector with fixed value of kinetic energy on the coset space.

Note that in the spatial flat case $\kappa = 0$ the cosmological constant has to be positive (yielding an expanding de Sitter universe) or zero (yielding a steady state universe) [365], in order for the spatially flat Robertson-Walker geometry to solve eq. (5.27). The new observation thus is the $\mathbf{so}(1,2)$ symmetry which is also present in this system. Since we found in this and the last section that both the dynamics of matter (point particles) and of gravitational fluctuations (modes of the metric) are described by the same mechanical system (5.1) in different parameter ranges, the situation is reminiscent of the AdS/CFT correspondence between open and closed string excitations. I comment in section 6.3 on whether this reminiscence could have a deeper meaning as a new form of matter/gravity duality.

5.3 A Matrix Model for the Robertson-Walker Singularity

In section 5.2.2, I presented a reduction of the dynamics of geodesic motion in the Robertson-

[6] With the diagonalisation $\hat{g} = o\lambda o^t$, $oo^t = 1$, $\lambda > 0$, $\mathrm{tr}\left(\hat{g}^{-1}\dot{\hat{g}}\hat{g}^{-1}\dot{\hat{g}}\right) = 2\sum_{i>j}\left[(\sqrt{\lambda_i/\lambda_j} - \sqrt{\lambda_j/\lambda_i})(o^t\dot{o})_{ij}\right]^2 + \sum_i(\dot{\lambda}_i/\lambda_i)^2 \geq 0$.

Walker background to a subsector that can be described by the conformal mechanical system (5.1). Based on this I now turn to the description of many point particles in the spatially flat Robertson-Walker universe. Since the relativistic point particle action is, in the absence of NS-NS and R-R background fields, just the DBI action for a D0 brane (or D-particle), this study will be useful in describing the physics of many D0 branes in the presence of the big bang singularity. Using several general arguments I propose a matrix model which encodes the bosonic part of the open string dynamics in the presence of many D-particles.

5.3.1 A Short Review of the $\mathcal{M}$-Theory – Matrix Model Connection

Before presenting the actual proposal, it is however necessary to review the way of arguments that lead to the conjecture that the dynamics of D0 branes, encoded in certain matrix models, actually could capture the full dynamics of $\mathcal{M}$-theory, which is defined as the ultraviolet completion of eleven-dimensional supergravity, and thus is theory of quantum gravity.

The Strong Coupling Limit of Type IIA String Theory

The description of the bosonic sector of open strings in the presence of many D0 branes is generically given by a Yang-Mills quantum mechanics [334], i.e. by a dynamical system of hermitean matrices with a $U(N)$ gauge symmetry. This idea stems from the observation that a D0 brane only carries a nondynamical 0+1-dimensional gauge field A_0, and all the other components of the gauge field in the ten-dimensional $\mathcal{N} = 1$ vector multiplet A_I, $I = 1, \ldots, 9$, become transverse scalars X^I upon dimensional reduction to 0+1 dimensions. Since the resulting low energy theory has no spatial but only a time direction, it is a quantum mechanical model describing the dynamics of D0 branes. Furthermore, since one D0 brane carries a $U(1)$ gauge symmetry, and for N coincident D0 branes the gauge symmetry gets promoted to $U(N)$, the transverse scalars X^I become matrices in the adjoint representation of $U(N)$. Hence the quantum mechanics we are searching for is a Yang-Mills type of quantum mechanics with a $U(N)$ gauge symmetry.

In flat ten-dimensional space, a background of type IIA closed string theory which preserves the maximal amount of thirty-two supercharges, the dynamics of N D0 branes can be obtained from the dimensional reduction of the action for ten-dimensional $U(N)$ $\mathcal{N} = 1$ supersymmetric Yang-Mills theory [146],

$$S_{10D} = \frac{1}{g_{10}^2} \int \mathrm{d}^{10}x \mathrm{Tr} \left(\frac{1}{2} F_{\mu\nu}^2 - i\bar{\Psi}\Gamma^\mu D_\mu \Psi \right) , \tag{5.29}$$

with Ψ being a ten-dimensional Majorana-Weyl spinor of either chirality in the adjoint representation of the gauge group, and g_{10} the ten-dimensional Yang-Mills coupling. The Γ^μ are ten-dimensional Dirac matrices, and I follow here the definitions of [146]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \quad D_\mu \cdots = \partial_\mu \cdots + [A_\mu, \ldots] \ .$$

The generators of the adjoint representation T_A are normalised such that $\mathrm{Tr} T_A T_B = -\delta_{AB}/2$. The dimensional reduction to 0+1 dimensions is now effected by requesting all fields to solely depend on time t, relabeling the nine spatial components of the gauge field $A^i = X^i$ and using the following identities

$$\begin{aligned} F_{0i} &= \partial_t X_i + [A_0, X_i] \ , \quad F_{ij} = [X_i, X_j] \ , & (5.30)\\ D_t\Psi &= \partial_t \Psi + [A_0, \Psi] \ , \quad D_i\Psi = [X_i, \Psi] \, . & (5.31) \end{aligned}$$

With these relations, eq. (5.29) reduces to the matrix model

$$S = \frac{1}{g_1^2} \int \mathrm{d}t \mathrm{Tr} \left(-(D_t X^i)^2 + \frac{1}{2}[X^i, X^j]^2 - \frac{i}{2}\bar{\Psi}\Gamma^t D_t \Psi - \frac{i}{2}\bar{\Psi}\Gamma^i [X_i, \Psi] \right) \ . \qquad (5.32)$$

This action describes the low energy physics of the strings stretched between N D0 branes on flat ten-dimensional space-time in the limit $\alpha' \to 0$. It is an $\mathcal{N} = 16$ supersymmetric Yang-Mills quantum mechanics. The 0+1-dimensional Yang-Mills coupling is related to the ten-dimensional one via $g_1^2 = g_{10}^2 \ell_s^{-9}$, where the volume of nine-dimensional Euclidean space makes up for the difference in the dimensions of the two coupling constants.[7] Possible higher-derivative corrections to the ten-dimensional theory (5.29) will of course also be present after dimensional reduction.

So far nothing remarkable happened. We simply derived the low energy effective action for N D0 branes. The remarkable connection to quantum gravity as described by $\mathcal{M}$-theory comes via the works of Banks, Fischler, Shenker and Susskind [42] and Witten [210, 334]. In [210] Witten analysed the spectrum of supersymmetric states in type IIA supergravity which are charged under the Ramond-Ramond one-form field C_1, i.e. the spectrum of states with D0 brane charge.[8] Ten-dimensional type IIA theory has two Majorana-Weyl supercharges of opposite chirality, Q_α and $\tilde{Q}_{\dot\alpha}$. Witten argued that since the charge Q_{D0} appears as a central charge in the ten-dimensional supersymmetry algebra, which schematically reads[9]

$$\{Q_\alpha, Q_\beta\} \sim \sum_{M=0}^{9} \sigma^M_{\alpha\beta} P_M \ , \quad \left\{\tilde{Q}_{\dot\alpha}, \tilde{Q}_{\dot\beta}\right\} \sim \sum_{M=0}^{9} \sigma^M_{\dot\alpha\dot\beta} P_M \ , \quad \left\{Q_\alpha, \tilde{Q}_{\dot\alpha}\right\} \sim \delta_{\alpha\dot\alpha} Q_{D0} \, ,$$

[7]More precisely I redefined the volume of infinite space $\int \mathrm{d}^9 x = \ell_s^9 \int \mathrm{d}^9 \tilde{x}$ in a dimensionless way and then rescaled the action S by this dimensionless infinity. In this way the action is kept dimensionless, and the Yang-Mills coupling constants acquire their correct dimensions.

[8]At the time of publication of [210], the identification of D0 branes with the corresponding solitonic solutions of IIA supergravity [189] was still unclear, so Witten argued from the point of view of the solitonic zero branes. In [334] he the used the insight of [189] to recognise that the bound states of D0 branes follow the same pattern.

[9]σ^M are the ten-dimensional analogues to the four-dimensional matrices $\sigma^\mu = (\ , \tau^i)$ which generate the Clifford algebra acting on spinors of definite chirality.

supersymmetric states charged under Q_{D0} must combine into short ($\frac{1}{2}$BPS) multiplets of the ten-dimensional $\mathcal{N} = 2$ algebra, and their mass (in the string frame) is thus bound from below by their charge via a BPS relation

$$M \geq \frac{c}{\ell_s g_s} |Q_{D0}| \,. \tag{5.33}$$

Here c is a mere constant, not depending on any additional parameters. The string coupling is given by the expectation value of the dilaton, $g_s = \langle e^{\phi} \rangle$, and the charge Q_{D0} must be quantised in integer units (a fact which only becomes clear after the identification of Dp branes with solitonic p branes) and also independent of the string coupling g_s in order to guarantee the validity of charge conservation for Q_{D0}. At small coupling these states are thus heavy, while they become light at large g_s. On this basis, Witten argued that these states correspond to bound states of the solitonic zero branes in type IIA supergravity, and thus, since these arguments based on supersymmetry are valid both in the point particle limit and when including α' corrections, to supersymmetric bound states of D0 branes in the full type IIA string theory. He observed that the spectrum (5.33) resembles that of a **Kaluza-Klein spectrum** of eleven-dimensional $\mathcal{N} = 1$ supergravity (which is the unique supergravity theory in eleven dimensions), compactified on a spacelike circle of radius

$$R_{11} = \ell_s g_s \,, \tag{5.34}$$

henceforth called the "$\mathcal{M}$-theory circle". From this observation he concluded that the strong coupling dynamics of type IIA supergravity is given by the dynamics of eleven-dimensional supergravity compactified on a circle with large radius. The Ramond-Ramond field C_M is part of the eleven-dimensional metric, as is obvious from the decomposition

$$\mathrm{d}s_{11}^2 = g_{MN}^{10} \mathrm{d}x^M \mathrm{d}x^N + e^{\frac{4}{3}\phi} (\mathrm{d}x^{11} - C_M \mathrm{d}x^M)^2 \,.$$

Since the eleven-dimensional excitations are massless, the mass of the ten-dimensional excitation then just becomes the momentum along the $\mathcal{M}$-theory circle,

$$M_{11}^2 = -p_M p^M - p_{11}^2 = 0 \Rightarrow M_{10}^2 = -p_M p^M = p_{11}^2 = \left(\frac{N}{R_{11}} \right)^2 \,, \quad N \in \quad ,$$

where the momentum p_{11} is quantised due to the compactness of the $\mathcal{M}$-theory circle. Thus, the perturbative regime of IIA supergravity with small g_S corresponds to the limit $R_{11} \to 0$, while in the strong coupling region the radius R_{11} goes to infinity, and thus the $\mathcal{M}$-theory circle decompactifies.

This correspondence between strongly coupled type IIA supergravity and eleven-dimensional supergravity is a new form of duality, which seems to be forced upon us by the uniqueness of eleven-dimensional supergravity and by supersymmetry. Since the low energy limit of type IIA superstring theory in ten dimensions is IIA supergravity, and eleven-dimensional supergravity as a perturbatively nonrenormaliseable theory is also only valid at low energies, the question arises whether there is a dual theory in eleven dimensions which, upon Kaluza-Klein reduction, describes full type IIA string theory at strong coupling. This theory would be the ultraviolet completion of eleven-dimensional supergravity and was given the name $\mathcal{M}$-**Theory**.

The BFSS Matrix Model: $\mathcal{M}$-Theory on Flat Space-Time

A proposal for a concrete formulation of $\mathcal{M}$-theory in terms of a matrix model was made by **Banks, Fischler, Shenker and Susskind (BFSS)** in [42]. It is based on the fact that the type IIA D0 brane bound states (i.e. the above-described states with fixed D0 brane charge) are marginal bound states [334], i.e. have vanishing binding energy. This means that for example a bound state with N D0 branes can be separated into two bound states with N_1 and N_2 D-particles which are infinitely far from each other at no cost of energy, as long as the charge conservation law $N = N_1 + N_2$ is fulfilled. The D0 brane bound states thus behave very similar to the in-states and out-states of free particles in ordinary quantum field theory, and may be used to build up a space of fundamental states for $\mathcal{M}$-theory. For this reason the D0 bound states are called "supergravitons". Interactions between supergravitons are then generated by loop corrections arising from integrating out massive modes in the BFSS matrix model.

The central idea of [42] is to use the idea of **discrete light-cone quantisation (DLCQ)**, which has been successfully applied to QCD before [369, 370]. DLCQ is a quantisation method where time is taken to be one light cone direction, while restricting the dynamics to a subsector of fixed light-cone momentum p^+. This is clearly possible in a theory on a possibly curved background which admits a lightlike isometry generated by a Killing vector $\propto \frac{\partial}{\partial x^-}$. Taking the lightlike direction to be a circle of radius R, the light-cone momentum

$$p^+ = \frac{N}{R} \tag{5.35}$$

is quantised in integer units, as in the case of the Kaluza-Klein reduction on a spacelike circle discussed above. The advantage of taking the light-cone direction to be time is that this generically yields nonrelativistic dynamics. This can be seen already in the simple example of a massless real scalar field ϕ on flat $d+2$-dimensional space-time, which recently resurfaced in an attempt to find holographic duals with Schrödinger symmetry [371,372]. Call the coordinates of the $d+2$-dimensional space-time (t, x^i, x^{d+1}), $i = 1, \ldots, d$. Introducing light-cone coordinates $x^\pm = (t \pm x^{d+1})/\sqrt{2}$, the Klein-Gordon equation reads

$$\left(-2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} + \sum_{i=1}^{d} \partial_i^2 \right) \phi = 0\,.$$

Since the system is Poincaré invariant, $P_- = P^+$ is a constant of motion. Restricting to a fixed light-cone momentum $P^+ = P_- = \frac{1}{i}\frac{\partial}{\partial x^-} = -m < 0$, the dynamics in this subsector is given by the free Schrödinger equation (with $\hbar = 1$) in d spatial dimensions,

$$i\partial_+\phi = -\frac{1}{2m}\sum_{i=1}^{d} \partial_i^2 \phi\,,$$

with x^+ playing the role of time. This relation of nonrelativistic dynamics in the transverse directions and fully relativistic dynamics is generally true in DLCQ. Obviously, DLCQ is a noncovariant quantisation method, i.e. if carried out in flat space it breaks Lorentz invariance. Galileian invariance in transverse space is however preserved.

Returning to the physics of $\mathcal{M}$-theory in eleven dimensions and of D0 branes in type IIA theory, I now argue that D0 brane dynamics in DLCQ is described by the gauged matrix model (5.32). I follow the discussion in [42] and in chapter 14.1 of [31]. The starting point is $\mathcal{M}$-theory compactified on a spacelike circle x^{10} with radius R_s, i.e. with the identification

$$\left(x^{10}, t\right) \sim \left(x^{10} - R_s, t\right) . \tag{5.36}$$

We know that this is per definition equivalent to type IIA string theory with the identifications

$$R_s = g_s \ell_s \,, \quad \ell_{11}^3 = g_s \ell_s^3 \,, \tag{5.37}$$

where ℓ_{11} is the eleven-dimensional Planck length. Now introduce a length scale R (which becomes the radius of the lightlike circle at the end), and boost the system with a velocity

$$v = \frac{1}{\sqrt{1 + 2\frac{R_s^2}{R^2}}} \overset{\frac{R_s}{R} \ll 1}{\approx} 1 - \frac{R_s^2}{R^2} + \mathcal{O}(R_s^4/R^4)$$

close to the speed of light $c = 1$ in the x^{10} direction. After the boost, the identification (5.38) becomes

$$\left(x^{10}, t\right) \sim \left(x^{10} - \frac{R}{\sqrt{2}}, t + \frac{R}{\sqrt{2}}\right) + \mathcal{O}(R_s^2/R^2) \,. \tag{5.38}$$

Thus $x^- \sim x^- + R$ becomes a lightlike circle of radius R in the strict $R_s \to 0$ limit (with R held fixed), and the associated momentum p^+ is quantised as in (5.35). This boosted frame is known as the **infinite momentum frame**. What happens to the energy and momentum of a collection of N D0 branes in this limit? Let the energy of this collection and the momentum in the $\mathcal{M}$-theory circle direction be

$$E = \frac{N}{R_s} + \Delta E \,, \quad P = \frac{N}{R_s} \Leftrightarrow P^- = \frac{E - P}{\sqrt{2}} = \frac{\Delta E}{\sqrt{2}} \,, \quad P^+ = \frac{E + P}{\sqrt{2}} = \sqrt{2}\frac{N}{R_s} \,.$$

ΔE includes possible excitations of the D0 branes above their ground state energy N/R_s (which is dictated by supersymmetry). P^- thus encodes excitations and therefore plays the role of a "light-cone Hamiltonian", while P^+ is still quantised. The Lorentz boost acts as

$$\tilde{P}^- = \sqrt{\frac{1+v}{1-v}} P^- \approx \frac{R}{R_s} \Delta E \,, \quad \tilde{P}^+ = \sqrt{\frac{1-v}{1+v}} P^+ \approx \frac{N}{R} \,,$$

We thus observe that in the lightlike limit

$$R_s \to 0 \,, \quad \ell_{11} \text{ fixed} \Rightarrow g_s \to 0 \,, \quad \ell_s \to \infty \tag{5.39}$$

the light-cone momentum of N D0 branes P^+ is consistent with a compactification of $\mathcal{M}$-theory on a lightlike circle with radius R. The spacelike compactification, which through the connection with type IIA string theory gave the original definition of of $\mathcal{M}$ theory, and the lightlike compactification of DLCQ are thus smoothly related by the infinite boost, hinting towards a possible definition of $\mathcal{M}$-theory via the effective action of D0 branes in the infinite momentum frame. At first the lightlike limit (5.39), which is the tensionless limit of type IIA string theory, seems very complicated, since the whole tower of string states seems to become massless in this limit. However, any fluctuation energy measured in terms of the string scale

$$\frac{\Delta E}{M_s} \sim \frac{R_s}{R} P^- \ell_s = \frac{\sqrt{R_s \ell_{11}^3}}{R} P^- \to 0$$

vanishes in the lightlike limit. Here $P^- = -i\frac{\partial}{\partial x^+}$, the light-cone Hamiltonian in the effective nonrelativistic Schrödinger equation we are aiming at, is kept fixed in the limit, since otherwise we would focus only on the sector of zero energy states of $\mathcal{M}$-theory. In particular, any fluctuation of the D0 brane bound state which could be strong enough to excite higher derivative (i.e. α') corrections in the effective D0 brane action is suppressed, which is more than welcome since our knowledge of higher-order corrections to the nonabelian DBI action is not complete. It thus suffices to consider the lowest order in derivatives, which is the Yang-Mills quantum mechanics (5.32).

The lightlike limit (5.39) has to be taken with some care in order to not get confused about dimensions. From the form of the D0 brane coupling (2.92) we know that the Yang-Mills coupling in the low energy effective action is given by (2.93), i.e. $g_1^2 \sim g_s \ell_s^{-3}$, which does not seem to have the right ℓ_s dependence to be kept fixed in the lightlike limit.[10] In string units however, i.e. after rescaling $t \mapsto \ell_s t$, $A_0 \mapsto \ell_s^{-1} A_0$, $X^i \mapsto \ell_s^{-1} X^i$ and $\Psi \mapsto \ell_s^{-\frac{3}{2}} \Psi$, the action is simply multiplied by the inverse string coupling which, in string units, just becomes the eleven-dimensional Planck length $\ell_{11}^3 = g_s \ell_s^3$ which we kept fixed in the lightlike limit. In the next section we will see that the ℓ_{11} dependence is just right to reproduce the BFSS matrix model (5.32) from a quantisation of the M2 brane in eleven-dimensional supergravity. To summarise, based on the arguments presented above, the authors of [42] proposed that

> The discrete light-cone quantisation of $\mathcal{M}$-theory on eleven-dimensional flat Minkowski space-time in each sector with fixed but not necessarily large light-cone momentum $P^+ = N/R$ is described by the $U(N)$ matrix model (5.32), which is the low energy effective action of N D0 branes, in the limit (5.39).

This proposal provides a nonperturbative formulation of $\mathcal{M}$-theory in terms of the physics of D0 branes, which has passed many nontrivial tests such as the calculation of the correct

[10] Note the change in notation compared to chapter 2: The D0 brane gauge coupling there was denoted $g_{YM,0}$, while here it is g_1.

eleven-dimensional graviton scattering amplitude from the matrix model or the existence of membrane-like solutions of (5.32) [42].[11] It is nonperturbative in the sense that in the above arguments no assumption on the coupling strength in eleven dimensions was made. The only eleven-dimensional scale which enters is the Planck scale ℓ_{11}, which sets the tension of the M2 brane in $\mathcal{M}$-theory

$$T_2 = \frac{1}{(2\pi)^2 g_s \ell_s^3} = \frac{1}{(2\pi)^2 \ell_{11}^3} \,. \tag{5.40}$$

The appearance of the M2 brane tension in this context already gives a hint that another set of fundamental degrees of freedom of $\mathcal{M}$-theory might be M2 branes (often also simply called "membranes"). Recent progress in formulating a world-volume theory of coincident M2 branes at $\ ^4/\ _k$ singularities [373–379] and setting up a corresponding holographic (AdS_4/CFT_3) duality [380] between certain three-dimensional superconformal Chern-Simons-Matter theories and eleven-dimensional supergravity on $AdS_3 \times S^7/\ _k$ supports this claim. In the following we will see that there is also an inverted connection between membranes and D0 branes: A quantisation of the world volume theory of a single membrane will yield the matrix model (5.32). This will be the guideline for the proposal of a matrix model on the Robertson-Walker background, presented in section 5.3.2.

The Membrane – Matrix Model Connection

Interestingly, the matrix model (5.32) can also be obtained by a regularisation and subsequent quantisation of the world volume theory of a supersymmetric M2 brane. For simplicity I will consider the bosonic part of the M2 brane world volume theory only. The full supersymmetric case is treatable in a similar way [125]. In this presentation I follow [31, 124].

Consider the Nambu-Goto action for the bosonic membrane in flat space-time $g_{\mu\nu} = \eta_{\mu\nu}$ and with eleven-dimensional three-form field vanishing,

$$S_{M2} = -T_2 \int \mathrm{d}^3\xi \sqrt{-\det_{\alpha,\beta}(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu})} \,. \tag{5.41}$$

The world volume coordinates are denoted $\xi^\alpha = (\tau, \sigma^1, \sigma^2)$. The membrane tension is given by (5.40). Introducing world volume gravity with a metric $\gamma_{\alpha\beta}$, it can be recast in "Polyakov form",

$$\tilde{S}_{M2} = -\frac{T_2}{2} \int \mathrm{d}^3\xi \sqrt{-\gamma} \left(\gamma^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - 1 \right) \,. \tag{5.42}$$

In the gauge

$$\gamma_{0i} = 0 \,, \quad \gamma_{00} = -\det_{\alpha,\beta}(\partial_\alpha X^\mu \partial_\beta X_\mu) \,, \tag{5.43}$$

[11] For an overview of the vast topic of $\mathcal{M}$-theory and matrix models see also the reviews [120–124].

the action (5.42) can be rewritten as

$$\tilde{S}_{M2} = -\frac{T_2}{2}\int \mathrm{d}^3\xi \left[-\dot{X}^\mu \dot{X}_\mu - \frac{1}{2}\{X^\mu, X^\nu\}_{\mathrm{PB}}\{X_\mu, X_\nu\}_{\mathrm{PB}}\right], \tag{5.44}$$

where the Poisson bracket (often also called "Nambu bracket", although this term originally refers to a totally antisymmetric three-bracket structure) if defined in terms of the spatial world-sheet derivatives as

$$\{F, G\}_{\mathrm{PB}} \equiv \varepsilon^{ij}\partial_i F \partial_j G = \partial_1 F \partial_2 G - \partial_2 F \partial_1 G\,. \tag{5.45}$$

For obtaining (5.44) from (5.42) the simple identity

$$\det_{i,j}(\partial_i X^\mu \partial_j X_\mu) = \frac{1}{2}\{X^\mu, X^\nu\}_{\mathrm{PB}}\{X_\mu, X_\nu\}_{\mathrm{PB}} \tag{5.46}$$

is useful. Since we already fixed a world volume diffeomorphism gauge, this action has to be supplemented by the constraints

$$\dot{X}^\mu \dot{X}_\mu + \frac{1}{2}\{X^\mu, X^\nu\}_{\mathrm{PB}}\{X_\mu, X_\nu\}_{\mathrm{PB}} = 0\,, \quad \dot{X}^\mu \partial_i X_\mu = 0\,. \tag{5.47}$$

The second constraint implies another one,

$$0 = \{\dot{X}^\mu, X_\mu\}_{\mathrm{PB}} = \partial_1(\dot{X}^\mu \partial_2 X_\mu) - \partial_2(\dot{X}^\mu \partial_1 X_\mu)\,, \tag{5.48}$$

which will turn out to be the Gauß constraint after regularising the matrix. So far no target space diffeomorphism gauge has been chosen yet. Doing so by choosing light-cone coordinates $X^\pm = (X^0 \pm X^{10})/\sqrt{2}$ and restricting to light-cone gauge

$$X^+(\tau, \sigma^1, \sigma^2) = \tau\,,$$

the constraints can be solved for X^- in a way analogous to string theory in light-cone gauge, yielding

$$\dot{X}^- = \frac{1}{2}\dot{X}^i \dot{X}_i + \frac{1}{4}\{X^i, X^j\}_{\mathrm{PB}}\{X_i, X_j\}_{\mathrm{PB}}\,, \quad \partial_i X^- = \dot{X}^j \partial_i X_j\,, \tag{5.49}$$

with X^j being the transverse directions to the light cone, i.e. $j = 1, \ldots, 9$. Since the underlying flat eleven-dimensional space-time has a lightlike isometry, the conserved total momentum conjugate to X^- reads

$$P^+ = \int \mathrm{d}^2\sigma \frac{\delta \tilde{L}_{M2}}{\delta \dot{X}^-} = T_2 \int \mathrm{d}^2\sigma \dot{X}^+ = V_2 T_2\,, \tag{5.50}$$

where V_2 denotes the volume of the spatial part of the membrane's world volume, which is ^{2} in this case. In particular, one deduces immediately from the conservedness of P^+ that in the gauge (5.43) the world volume diffeomorphisms are broken to static area preserving ones, i.e. diffeomorphisms with $\tau' = \tau$ and spatial coordinate redefinitions $\sigma'^i(\sigma)$ with unit Jacobian $|\det(\partial_i \sigma'^j)| = 1$. The corresponding light-cone Hamiltonian of the membrane is then given by

$$H_{M2} = \frac{T_2}{2} \int \mathrm{d}^2\sigma \left(\dot{X}^i \dot{X}^i + \frac{1}{2} \{X^i, X^j\}_{\mathrm{PB}} \{X_i, X_j\}_{\mathrm{PB}} \right) . \tag{5.51}$$

We thus arrive in light-cone gauge at a nonrelativistic system for the nine transverse coordinates X^i.

So far, every step has been a manipulation of the classical field theory. To quantise this system a regularisation has to be used. The proposal of [125] is to discretise the two-dimensional spatial world-volume by deforming the classical Riemannian manifold through a kind of deformation quantisation which is effected by replacing the spatial coordinates σ^i by hermitian $N \times N$ matrices $\hat{\sigma}^i$ satisfying in the case of $\sigma^i \in {}^2$ an algebra

$$[\hat{\sigma}^i, \hat{\sigma}^j] = i\varepsilon^{ij} , \tag{5.52}$$

i.e. which parametrise a noncommutative space. The transverse coordinates $X^i(\tau, \sigma^i)$ become time-dependent hermitian matrices $X^i(\tau)$, and the Poisson bracket as well as the integral are replaced as

$$\{.,.\}_{\mathrm{PB}} \to -i[.,.] , \quad \frac{1}{V_2} \int \mathrm{d}^2\sigma \to \frac{1}{N} \mathrm{Tr} , \quad V_2 \to N . \tag{5.53}$$

This procedure is called **matrix regularisation** and, after dropping the overall factor of N, gives rise to the light-cone Hamiltonian

$$H = \frac{T_2}{2} \mathrm{Tr} \left[\dot{X}^i \dot{X}^i - \frac{1}{2} [X^i, X^j]^2 \right] . \tag{5.54}$$

Note that this Hamiltonian still is a classical one, no quantisation of the fields X^i has been carried out yet. The deformation quantisation of the membrane world volume has to be understood as a mere regularisation of the infinitely many degrees of freedom. The Hamiltonian (5.54) still has to be supplemented by the matrix version of the constraint (5.48),

$$[\dot{X}^i, X^i] = 0 , \tag{5.55}$$

which is the Gauß constraint of the BFSS matrix model (5.32) in the gauge $A_0 = 0$. This constraint has to be imposed by hand for the following reason: After imposing light-cone gauge the membrane action (5.44) does not depend on $\partial_j X^-$, and thus the solution $\partial_j X^- = \dot{X}^i \partial_j X_i$ (5.49) to the second constraint in (5.47) gets lost during the procedure of solving the first equation in (5.47). It can however be recovered from the identity (5.48), which is an integrability condition for the one-form $a = (\dot{X}^j \partial_i X^j) \mathrm{d}\sigma^i$, i.e.

$$\mathrm{d}a = 0 \Rightarrow a = \mathrm{d}f = (\partial_i f) \mathrm{d}\sigma^i ,$$

$$
\begin{tikzcd}
\text{IIA String} \arrow[r, "\text{KK-Decompactification}"] \arrow[d, "\text{Quanti-}"', "\text{sation}"] & \text{M2 Brane} \arrow[d, "\text{Quanti-}"', "\text{sation}"] \\
\text{Type IIA String Theory} \arrow[d, "\text{Low Energy}"', "\text{Limit}"] & \mathcal{M}\text{-Theory} \arrow[d, "\text{Low Energy}"', "\text{Limit}"] \\
\text{IIA Supergravity} \arrow[r, "\text{KK-Decompactification}"'] & \text{11D Supergravity}
\end{tikzcd}
$$

Table 5.1: Strings, Membranes and $\mathcal{M}$-Theory.

which is possibly globally on ^{2}. If $\partial_i f$ is now identified with $\partial_i X^-$ in (5.49), the "lost constraint" (the second equation in (5.49)) is recovered.[12]

The matrix regularised bosonic membrane in light-cone gauge is thus equivalent to the bosonic part of the BFSS matrix model in temporal ($A_0 = 0$) gauge. For the supermembrane one has to additionally project out half of the fermionic degrees of freedom by imposing $\Gamma^+\Psi = 0$, such that the remaining degrees of freedom form a Majorana spinor of $SO(9)$. In that case the BFSS model describes the light-cone gauge dynamics of the fully supersymmetric M2 brane [125]. Conversely it was shown in [42] that the BFSS matrix model includes supermembranes as semiclassical states for large N. This is additional evidence for the underlying conjecture that D0 branes comprise a space of fundamental states for $\mathcal{M}$-theory.

This connection between the proposed description of $\mathcal{M}$-theory via the matrix model and membrane physics is not totally unexpected. Consider an M2 brane wrapping the $\mathcal{M}$-theory circle. The effective tension of the two-dimensional (i.e. string-like) object in ten dimensions is

$$2\pi R_s\, T_2 = \frac{2\pi g_s \ell_s}{(2\pi)^2 g_s \ell_s^3} = \frac{1}{2\pi\alpha'} = T_{F1}\,, \tag{5.56}$$

the tension of the fundamental string. The Kaluza-Klein compactification of the M2 brane is thus the fundamental type IIA string. Since we defined $\mathcal{M}$-theory as the ultraviolet completion of the strong coupling limit (i.e. the Kaluza-Klein decompactification) of type IIA supergravity, and since type IIA supergravity arises from the quantisation of the type IIA string via a low energy limit, it is natural to expect that $\mathcal{M}$-theory should be connected to a quantisation of the M2 brane, yielding an effective theory which in its low energy limit gives eleven-dimensional supergravity. The circle of relations between $\mathcal{M}$-theory, M2 branes and the matrix model (5.32) thus closes via the above derivation of the matrix model from the M2 brane world volume theory. The situation is summarised in table 5.1.

5.3.2 D0 Brane Dynamics in the Robertson-Walker Universe

In the last section it was shown that the BFSS matrix model can be recovered from the M2 brane action in light-cone gauge. The guiding principle of this section will be

[12]The author thanks Elias Kiritsis for explaining this subtle point.

to turn this logic around and, since any matrix model description of $\mathcal{M}$-theory should reproduce large semiclassical membranes, to derive the bosonic part of a matrix model for the spatially flat Robertson-Walker background from the dynamics of a bosonic membrane in this background. In this course I will use the ideas of section 5.2.2 and reduce the dynamics of the membrane by fixing the world volume diffeomorphisms through a gauge fixing constraint which involves the Hamiltonian density of the theory. The dynamics in this subsector will then be described by a square root free Lagrangian whose potential can be expressed in terms of the Poisson bracket and thus admits a matrix regularisation. This step then leads to the proposed matrix model. From the time dependence of the coefficient of the commutator-squared potential it is then possible to deduce statements about the emergence of the classical geometry close to the big bang singularity.

In a flat background, the coupling of the Yang-Mills potential $[X^i, X^j]^2$ in the matrix model can be freely scaled by a rescaling of the fields and of time, and its actual magnitude therefore is irrelevant for the dynamics of the system. However, in the Robertson-Walker background the coupling coefficient should encode the time dependence of the scale factor $a(t)$ and cannot be simply deduced from the one-particle action (5.24) due to the missing commutator-squared potential in the one-particle action. I determine this time-dependent coupling by deriving the tree-level matrix model from the bosonic membrane action in the spatially flat Robertson-Walker background. The dynamics of a bosonic membrane with tension T_2 embedded in a D-dimensional target space-time is governed by the Nambu-Goto action (5.41) with the flat metric $\eta_{\mu\nu}$ replaced by the target space metric $g_{\mu\nu}$,

$$S_{\mathrm{M2}} = -T_2 \int \mathrm{d}^3\xi \sqrt{-\det\Big(\partial_{\hat{\alpha}} x^\mu \partial_{\hat{\beta}} x^\nu g_{\mu\nu}(x)\Big)}\,. \tag{5.57}$$

The three-dimensional world volume of the membrane is parameterised by coordinates $\xi^{\hat{\alpha}}$, $\hat{\alpha} = 0, 1, 2$, and the embedding functions are given by $x^\mu(\xi)$, $\mu = 0, 1, \cdots, D-1$.

As in the flat space case described before, a certain partial gauge fixing is needed in order to derive the matrix model. Since the Robertson-Walker background (5.11) does not admit a lightlike isometry, light-cone gauge is not useful in this setup. We thus have to proceed in a different way to fix all the world volume diffeomorphisms until only static area preserving ones are left over. The static area preserving diffeomorphisms are the ones compatible with the deformation (5.52), since they keep the symplectic structure ϵ^{ij} invariant and thus, after fixing all but these diffeomorphisms, the matrix regularisation procedure can be applied.[13] Demanding cosmic gauge

$$t = x^0 = \xi^0 \tag{5.58}$$

as well as the longitudinal gauge

$$\partial_t x^\mu \partial_\alpha x^\nu g_{\mu\nu} = 0\,, \quad \alpha = 1, 2\,, \tag{5.59}$$

[13]Since the spatial part of the world volume becomes a two-dimensional phase space with symplectic structure after this deformation, the static area preserving diffeomorphisms are the only ones allowed by Liouville's theorem of classical mechanics. The author thanks Corneliu Sochichiu for pointing out this connection.

the remaining world volume diffeomorphisms are, at this stage, the static ones $t' = t$ and $\sigma^{i'} = \sigma^{i'}(\sigma)$.[14] A proof that longitudinal gauge can always be chosen is presented e.g. in [381].

The Nambu-Goto Lagrangian in the spatially flat Robertson-Walker background then reduces to

$$\mathcal{L}_{\mathrm{M2}} = -T_2\, a^2(t)\sqrt{(1 - a^2(t)\dot{x}^2)\det\mathcal{G}}\,, \tag{5.60}$$

where the determinant

$$\det\mathcal{G} = \det(\partial_\alpha x^i \partial_\beta x_i)\,, \quad \alpha, \beta = 1, 2 \tag{5.61}$$

is taken over spatial membrane coordinates only. Since we work with the spatially flat Robertson-Walker background, spatial indices i, j are contracted with the flat metric δ_{ij}, i.e. e.g. $\dot{x}^2 := \dot{x}^i \dot{x}^j \delta_{ij}$. With the momenta

$$p_i = T_2\, a^4 \dot{x}_i \sqrt{\det\mathcal{G}/(1 - a^2\dot{x}^2)}\,, \tag{5.62}$$

the Hamiltonian equations of motion derived from (5.60) read

$$\dot{p}_i \equiv \partial_\alpha \left[T_2\, a^2 \sqrt{(1 - a^2\dot{x}^2)\det\mathcal{G}}\, \mathcal{G}^{-1\alpha\beta}\partial_\beta x_i \right]\,, \tag{5.63}$$

and the longitudinal gauge condition becomes

$$\dot{x}_i \partial_\alpha x^i = 0 \quad \Longleftrightarrow \quad p_i \partial_\alpha x^i = 0\,. \tag{5.64}$$

In terms of the Poisson bracket $\{x, y\}_{\mathrm{PB}} := \epsilon^{\alpha\beta}\partial_\alpha x \partial_\beta y$ (with $\epsilon^{12} = 1$, $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$), the determinant can again be expressed as

$$\det\mathcal{G} = \tfrac{1}{2}\left\{x^i, x^j\right\}_{\mathrm{PB}} \{x_i, x_j\}_{\mathrm{PB}}\,.$$

Furthermore the longitudinal gauge condition (5.64) implies the identities

$$\partial_t(p^2) + T_2^2 a^6\, \partial_t \det\mathcal{G} \equiv 0\,, \quad \{p_i\,,\, x^i\}_{\mathrm{PB}} = 0\,, \tag{5.65}$$

the second of which becomes the Gauß law constraint in the matrix model.

Consider the sector of solution space with fixed on-shell value of the Hamiltonian density

$$\mathcal{H}_{\mathrm{M2}} = a^{-1}\sqrt{p^2 + T_2^2 a^6 \det\mathcal{G}} \equiv \Omega(\xi)\,. \tag{5.66}$$

Since Ω transforms as a scalar density, imposing (5.66) finally breaks the remaining static diffeomorphisms down to the static area preserving ones.

[14] Note the slight change in notation compared to the previous section: Here spatial world-volume indices are denoted by α, β in order to distinguish them from the spatial indices x^i of the target space coordinates.

By the same argument as presented in section 5.2.2, the dynamics after gauge-fixing is then equally described by a square root free Lagrangian

$$\mathcal{L}_\Omega := \tfrac{1}{2}\left(\Omega a^2\, \dot{x}^i \dot{x}_i - \Omega^{-1} T_2^2 a^4 \det \mathcal{G} + \Omega\right) . \tag{5.67}$$

Again the canonical momenta as well as the equations of motion are, after imposing the gauge condition (5.66), identical to those of (5.60), namely eq. (5.62) and eq. (5.63), respectively. Similarly to the one-particle case in section 5.2.2, the Hamiltonian constraint for $\mathcal{L}_\Omega$ matches with (5.66) as

$$\mathcal{H}_\Omega = \left(p^2 + T_2^2 a^6 \det \mathcal{G} - a^2\Omega^2\right) / \left(2a^2\Omega\right) \stackrel{(5.66)}{\equiv} 0 . \tag{5.68}$$

Having fixed all the world volume diffeomorphisms and derived the nonrelativistic action (5.67), the matrix regularisation procedure can now be applied. The prescription is, as before, to replace the dynamical fields $x^i(t, \xi^\alpha)$ by time-dependent $N \times N$ Hermitian matrices $X^i(t)$, the Nambu bracket $\{x^i, x^j\}_{\mathrm{PB}}$ by a matrix commutator $-i[X^i, X^j]$ [125], and the world volume coordinates σ^α by nondynamical matrices $\hat{\sigma}^\alpha$ satisfying the noncommutative relation $[\hat{\sigma}^\alpha, \hat{\sigma}^\beta] = i\epsilon^{\alpha\beta}$. The determinant $\det \mathcal{G}$ then yields the Yang-Mills potential and the resulting matrix model reads

$$\hat{\mathcal{L}}_{\mathrm{M2}} = \mathrm{Tr}\left[\frac{\hat{\Omega} a^2}{2}\left(D_t X^i\right)^2 + \frac{T_2^2}{4}\frac{a^4}{\hat{\Omega}}\left[X^i, X^j\right]^2 + \frac{\hat{\Omega}}{2}\right] , \tag{5.69}$$

where $\hat{\Omega}(t) := \Omega(t, \hat{\xi}^\alpha)$. The covariant time derivative $D_t X^i = \dot{X}^i - i[A_0, X^i]$ involves a nondynamical gauge field A_0, such that the matrix model admits a $\mathrm{U}(N)$ gauge symmetry. With the canonical momenta

$$P^i = \tfrac{1}{2} a^2 (\hat{\Omega} D_t X^i + D_t X^i \hat{\Omega}) ,$$

the equation of motion for A_0 gives the Gauß constraint

$$[P^i, X_i] = 0 ,$$

which is the quantum analogue of the second equation in (5.65) and which has to be implemented on physical states.

The matrix model (5.69) is the first main result of this chapter. It is still, however, far too general, since it contains the arbitrary matrix function $\hat{\Omega}(t)$ which was used for gauge fixing. For example it is hard to say anything about the static ($D_t X^i = 0$) solutions (the vacua) of the theory, since the arbitrary matrix function $\hat{\Omega}$ shows up in the static equation of motion

$$0 = [X^j, [X^i, X^j]\hat{\Omega}^{-1}(t)] .$$

What is needed is a physical guiding principle to fix a convenient gauge in which the function $\hat{\Omega}$ simplifies considerably. Choosing a time-independent $\hat{\Omega}$ would allow to learn about the time-dependent effects induced by the expansion or contraction of the universe encoded in $a(t)$, without the additional time-dependent effects from the gauge choice. However this restriction is too strong, since it does not reproduce the correct motion of a single D0 brane (the case $N = 1$): Note that we already derived part of the coefficient of the commutator-squared potential, namely the $a^4(t)$ dependence, by matrix regularising the bosonic membrane in the RW background. We thus already ensured the existence of semiclassical (i.e. large N) states that describe bosonic membrane dynamics in the RW background, at least away from any singularity $a(t_*) = 0$. The matrix model should however also give the correct description of the RW background in the case of only a few D0 branes or, in the limiting case, only one D0 brane, probing the space-time. It is thus reasonable to fix a gauge such that the $U(1)$ part of $\hat{\Omega}$, which describes the center-of-mass motion of the D-particles, coincides with the motion of one D-particle (or relativistic point particle) in the case $N = 1$. Comparing the result for one D-particle from section 5.2.2 (with $\kappa = 0$) with the matrix model (5.69) lead then to the identifications

$$\hat{\Omega}(t) = \frac{m_\nu(t)}{a^2(t)\ell_{11}^2} \quad , \quad \hat{\eta} = m_\nu(t) \quad , \tag{5.70}$$

with $m_\nu(t)$ defined in (5.23). The eleven-dimensional Planck length is necessary for dimensional reasons, since $\hat{\Omega}$ has dimension $mass^3$. Note that this is the simplest possible choice, setting the $SU(N)$ part of $\hat{\Omega}$ to zero. It is surely not the unique choice which reproduces the D-particle dynamics for $N = 1$, but it suffices for the purposes in the rest of this chapter, and I thus leave the analysis of more complicated choices for future work. The matrix model (5.69) for N D-particles in the flat Robertson-Walker background with this choice of $\hat{\Omega}$ is thus described (in temporal gauge $A_0 = 0$) by the Hamiltonian

$$\hat{\mathcal{H}}_{M2} = \frac{1}{2\hat{\eta}(t)} \mathrm{Tr}\left[P_i^2 - \frac{T_2^2}{2} a^6(t) \big[X^i, X^j\big]^2 - \frac{m^2(a^2+\nu^2)}{\ell_{11}^4}\right] . \tag{5.71}$$

This matrix model Hamiltonian (5.71), and in particular the above derivation of the coefficients of the different terms, is the second main result of this chapter.

Emergent Classical Geometry

The Hamiltonian (5.71), if imposed as the constraint $\hat{\mathcal{H}}_{M2} \equiv 0$ gives rise to an explicit realisation of **space-time emergence**: Since the Yang-Mills potential

$$-\mathrm{Tr}[X^i, X^j]^2 = \mathrm{Tr}[X^i, X^j][X_i, X_j]^\dagger \geq 0 \tag{5.72}$$

is multiplied by the high power of the scale factor $a^6(t)$, it must decrease on-shell as the scale factor $a(t)$ increases in order for the Hamiltonian constraint to be satisfied: First note that from (5.65) $\partial_t(a^4/\Omega^2) \equiv 6a\dot{a}p^2/\Omega^4 > 0$ and thus for all $\hat{\Omega}$ the coefficient of the Yang-Mills potential term is monotonically increasing with time. One might worry about the time-dependency of the last term in (5.71). Two facts ensure that the second term dominates the third for fixed values of m and ν: Since $a(t)$ measures the spatial volume in the metric (5.11) in Planck units, once the scale factor $a(t)$ exceeds the Planck scale, i.e. the value of one, a^6 grows much faster than a^2. Furthermore, the second term comes with a factor ℓ_{11}^{-6}, while the last term only has ℓ_{11}^{-4}. For small eleven-dimensional Planck lengths this yields a further enhancement of the second term compared to the third one. The first term in (5.71) is always positive and thus leads to an even faster decrease of the squared commutator. We thus find that very shortly after the universe described by (5.11) started to expand, once it reached the size of several Planck lengths, the commutator between the matrices

$$[X^i(t), X^j(t)] \approx 0$$

must have vanished up to small corrections. This means that the matrices are approximately simultaneously diagonalisable, and thus the positions of the D0 branes probing the space-time are sharp and well-defined. This is interpreted as the emergence of the smooth geometry of the underlying space-time, while the system evolves from the "quantum geometrical phase" at the big bang into a well-defined smooth manifold structure.

On the other hand, near the singularity there is no obvious reason why the X^i should commute, since the Yang-Mills coupling vanishes and thus the potential for the off-diagonal modes of the matrices which normally accounts for their masses vanishes, too. Physically this means that the D0 branes come closer together near the singularity, and thus their off-diagonal modes become light. Therefore, generically a magnitude for all components of the matrices X^i of order one in Planck units, in particular also for the off-diagonal modes, can be expected. Explicit solutions to the equations of motion will be needed in order to check under which special circumstances (e.g. for which choices of $a(t)$) space-time does not become fuzzy near the big bang, since also the P^2 term can balance the last term in (5.71), with the Yang-Mills potential still vanishing. If the commutator of the X^i is of order one, the particle positions are fuzzy of order one (in Planck units) and hence the space-time geometry they probe cannot be described any more in geometrical terms. This is the deeply quantum gravitational regime at the big bang singularity. The above argument showing that the Yang-Mills potential term has to become considerably small once the universe reaches a size of several Planck lengths together with the maximal fuzzyness of space-time geometry at distances below a Planck length leads to the expectation that the transition from the quantum to the semiclassical regime predicted from this matrix model is actually very fast. It will be interesting to study this aspect further in future work. I will also comment on additional directions of generalisation and application of the matrix model (5.71) in the discussion in section 6.3.

Chapter 6

Conclusions and Possible Further Developments

The common theme of the work presented in chapters 3, 4 and 5 of this thesis is the generalisation of holographic theories to situations in which additional external fields and nontrivial backgrounds are present, i.e. to situations with a lower degree of symmetry. In chapter 3, by employing the AdS/CFT correspondence the physics of a strongly coupled $\mathcal{N} = 2$ supersymmetric gauge theory with quark-like degrees of freedom in external electric and magnetic fields, i.e. in a setup with broken Lorentz symmetry, was analysed. In chapter 4, the AdS/CFT description of the same $\mathcal{N} = 2$ supersymmetric gauge theory was deformed by a Fayet-Iliopoulos coupling in order to describe nontrivial supersymmetry-breaking vacua, which correspond to instanton configurations on D7 probe branes on the gravity side of the correspondence. Chapter 5 dealt with another holographic description[1] of a gravitational system, namely a proposal for a matrix model which should capture part of the bosonic dynamics of $\mathcal{M}$-theory (the conjectured ultraviolet completion of eleven-dimensional supergravity) in the Robertson-Walker metric. The final chapter of this thesis is devoted to a summary and discussion of the results, and a description of possible further directions of research. The sections 6.1, 6.2 and 6.3 are respectively devoted to chapters 3, 4 and 5.

[1] The BFSS matrix model satisfies the holographic bound on the number of states (2.11) in a very particular way [42]: The transverse size of a threshold bound state of N D0 branes grows, for large N in a mean-field approximation, as $N^{\frac{1}{9}}$ in eleven-dimensional Planck units, which is interpreted as the incompressibility of the D0 branes in the nine-dimensional transverse space. The "holographic screen" in the case of the discrete light-cone quantisation (DLCQ) of $\mathcal{M}$-theory is in this case the nine-dimensional space transverse to the light-cone.

6.1 Holographic Quarks in External Electric and Magnetic Fields

The first generalisation of the AdS/CFT correspondence presented in chapter 3 of this thesis was concerned with the introduction of external electric and magnetic fields [1] into the $\mathcal{N} = 2$ supersymmetric D3-D7 model of AdS/CFT with additional flavour degrees of freedom. In this model, which was explained in detail in section 2.3.6, the $\mathcal{N} = 2$ quark hypermultiplets originate in the flat space picture from strings stretching between the D3 and D7 branes. These hypermultiplets each consist of two $d = 4$ $\mathcal{N} = 1$ chiral multiplets, with one chiral multiplet originating from strings running from the D3 to the D7 branes, and the other from the strings oriented in the opposite way. Assigning appropriate quantum numbers, the two chiral multiplets are respectively interpreted as quarks (i.e. degrees of freedom in the fundamental representation of the gauge group), and the corresponding antiquarks. The (anti)quarks are the fermions in each chiral multiplet, the corresponding scalar superpartners are interpreted as (anti)squarks. Furthermore, quarks in different, say N_f flavours, are constructed by considering a stack of N_f coincident D7 branes. In this case the system exhibits a global $U(N_f)$ flavour symmetry, which arises from the $U(N_f)$ gauge symmetry on the D7 stack. The overall $U(1)_q \subset U(N_f)$ factor is a "quark number symmetry", i.e. a symmetry under which the (s)quarks have charge plus one, while the anti(s)quarks have charge minus one. For a single flavour[2] ($N_f = 1$) the quark number assignment thus coincides with the assignment of electric charges between quarks and antiquarks under an electromagnetic $U(1)$ gauge symmetry. As usual in the AdS/CFT correspondence, global symmetries on the field theory side correspond to gauge symmetries on the gravity side. In this case the $U(1)_q$ number symmetry is identified with the overall $U(1)$ gauge symmetry of a stack of probe D7 branes embedded into $\mathrm{AdS}_5 \times \mathrm{S}^5$. Introducing constant background $U(1)_q$ gauge fields on the worldvolume of the D7 brane will therefore generate effects equivalent to actual constant background fields in the actual gauge symmetries, which are more difficult to model in the AdS/CFT correspondence.[3]

The starting point of chapter 3 was thus to consider D7 probe branes in constant electric and magnetic Kalb-Ramond background fields (see eq. (3.4)).[4] The analysis of their prop-

[2]Assigning different charges to different flavours is also possible by introducing constant nonabelian brane gauge fields. For example, for two flavours the combined effect from $F^3_{\mu\nu}\tau_3$ of the $SU(2)_f$ field strength together with the $U(1)_q$ field strength will add up for one flavour, and subtract for the other flavour.

[3]The difficulty at first sight is that the gravity dual description of gauge theories at strong couplings can only describe gauge-invariant objects. There is however an exception for the $U(1) \subset U(N_c)$ factor of the gauge group [294,295], which is not described by the local type IIB supergravity excitations but by a sector of boundary degrees of freedoms, so-called "singletons". Simply speaking the boundary term $\int \mathrm{d}(B \wedge C_2)$ of the supergravity action directly induces the corresponding $U(1)$ gauge sector on the boundary. It will be interesting to study this phenomenon in the setup used here in future work, which may involve a calculation of the backreaction of the D7 branes with B field onto the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, inducing a $C_2 \simeq C_6$ potential in the background.

[4]These closed string fields do not couple directly to the $U(1)_q$ number, but can be exchanged with a

erties was carried out both at zero and finite temperature, i.e. in the $AdS_5 \times S^5$ and AdS black hole backgrounds. The magnetic (spatial-spatial) and electric (temporal-spatial) external B fields are switched on in the directions parallel to the boundary, which has quite different effects on D7 probe brane embeddings and on the physics of quarks in the dual gauge theory.

In the magnetic case considered in section 3.2, it was found that the external field induces a repulsion of the D7 brane from the black hole. This repulsive effect competes with the attraction of the D7 brane probe by the black hole, such that for strong enough magnetic fields spontaneous chiral symmetry breaking occurs in the dual field theory even at finite temperature. The exact expression for the critical magnetic field strength as a function of temperature and 't Hooft coupling is given in eq. (3.21). Above the critical field strength spontaneous chiral symmetry breaking occurs, i.e. the operator (2.151) dual to the quark mass acquires a vacuum expectation value. An analysis of the spectrum of lowest-lying (i.e. with vanishing orbital angular momentum) pseudoscalar mesons, which are dual to fluctuations of the polar angle in the plane transverse to the D7 brane, confirmed this picture of induced chiral symmetry breaking: The lightest pseudoscalar meson was identified as the Goldstone boson of the broken symmetry by showing that its mass satisfies the Gell-Mann-Oakes-Renner relation (3.25). From the form of the D7 probe brane embeddings at finite temperature (fig. 3.4) it is immediately clear what happens when the magnetic field exceeds its critical value at finite temperature: Even the embedding corresponding to zero quark mass cannot fall into the black hole horizon any more. The boundary conditions which have to be imposed on the fluctuations around this brane embedding then force the meson spectra to be discrete, such that the mesons are stable at leading order in N_c.[5] Furthermore, the first order meson melting transition present for subcritical magnetic fields disappears above the critical field strength (see the phase diagram in fig. 3.5). Therefore the mesons cannot melt any more in the hot $\mathcal{N} = 4$ plasma for strong enough magnetic fields. The magnetic field thus has a confining effect on the quarks, and the mesonic phase is characterised by a nonzero chiral condensate, similar to the situation in QCD. Note however that in QCD the chiral symmetry restoration transition and the deconfinement phase transition occur at about the same energy scale, while in the D3-D7 model the two transitions are independent of each other [257]. In particular, in the Poincaré coordinates of AdS space used in chapter 3 the $\mathcal{N} = 4$ glue is deconfined, but the quarks can still form stable bound states. The underlying reason is the quenched approximation, i.e. the large N_c limit with fixed numbers of flavours, which decouples the thermodynamic behaviour of the $\mathcal{N} = 4$ glue from the physics of the $\mathcal{N} = 2$ matter fields, while at the same time the conformal glue still determines the physics of the flavour fields themselves.

The catalysation of spontaneous chiral symmetry breaking in strong magnetic fields, i.e. a dynamical mass generation for fermions, is known both from QED and QCD [106–109], as

D7 brane gauge field by means of a Ramond-Ramond gauge transformation.

[5] String worldsheet instanton effects [265, 266] yield nonperturbative contributions to the meson. Note also that the mesons will not be stable in a vacuum with finite quark number densities.

well as in effective models of QCD dynamics such as the Nambu-Jona-Lasinio model [62,63]. The underlying physical mechanism is the effective reduction of fermionic dynamics from 1+3 to 1+1 dimensions, since a strong magnetic field restricts the dynamics of the fermions to the lowest Landau level. For this reason, in QCD this mechanism was conjectured to be universal in the sense that it occurs for any number of flavours and also in a wide class of effective models of quark dynamics. The results obtained here using holographic techniques are consistent with these general expectations of the magnetic field acting as a catalyst of spontaneous chiral symmetry breaking. A more quantitative comparison between holographic models resembling QCD more closely and the known field theory results in future work would in my eyes be an extremely interesting enterprise. A good starting point would for example be the calculation of the dynamically generated masses which can then be compared to QCD results. However, some effects known from field theory such as anisotropic confinement are beyond the quenched approximation and thus will not be visible in the holographic setup.[6]

While the magnetic field tends to confine the quarks into mesons, a constant electric field, as considered in section 3.3, tends to destabilise quark-antiquark bound states. It also induces a new kind of phase transition similar to an insulator-metal transition, during which the mesons dissociate in the external electric field. This happens at an energy scale set by the temperature of the system and the mass of the quarks. In particular, lowering the quark mass at fixed temperature leads from the mesonic phase to the dissociated one. Although the mesons considered in these theories here are highly relativistic (the binding energy makes up nearly one hundred percent of the meson mass [96]) and therefore cannot be described effectively by a nonrelativistic effective potential picture, it is known [96] that the binding energy of the holographic mesons is proportional to the quark mass itself. In this way it becomes intuitively clear why lowering the quark mass leads to a dissociation of the bound states at fixed electric field: The binding energy linearly decreases with the quark mass, until it can be overcome by the electric field.

Holographically this phase transition manifests itself in the following way: In the six directions of $\mathrm{AdS}_5 \times \mathrm{S}^5$ transverse to the boundary, the electric field induces a five-dimensional spherical locus at which the DBI action vanishes. This locus was dubbed “singular shell” or “vanishing locus” in the literature. Inside the shell, the DBI action becomes imaginary, indicating an instability of the open strings in the electric background field. This instability can be cured by switching on gauge fields on the D7 brane which correspond to vacuum expectation values for a current in the direction of the electric field, as well as for a nonvanishing charge density (which in this case is the quark number density). The requirement of the DBI action with these additional fields to be real inside the vanishing locus then yields a relation between the electric field, the charge density and the induced current. Physically

[6] An attempt to compare the induced condensates in the Constable-Myers background [84] with Nambu-Jona-Lasinio and chiral perturbation theory results has been recently made in [382]. However, chiral perturbation theory captures only effects of order N_f/N_c, while the leading order matter effect in the quenched approximation is of order $N_f N_c$. It is therefore not surprising that the holographic results differ from chiral perturbation theory.

speaking the instability corresponds to an electric field trying to accelerate charge carriers while forbidding a current to form, while exciting the additional brane gauge fields renders the situation consistent by introducing the operators corresponding to the induced current in the field theory. In this consistent setup, both the embedding of the D7 brane as well as the meson spectrum can be calculated.

The study of the embeddings at zero temperature (and zero quark number density) then already shows the holographic manifestation of the dissociation transition: The embeddings, depicted in fig. 3.10, come into two distinct classes, namely "Minkowski embeddings" which do not reach the singular shell, and "singular embeddings" which do. The shell has an overall attractive effect on the embeddings. The transition between these two classes of embeddings occurs at a certain critical quark mass, at which the chiral condensate (fig. 3.12) jumps by a finite amount, indicating that the phase transition is of first order. The similarity to the finite temperature case reviewed in section 2.3.6 then led to the conjecture that this geometric transition is to be interpreted as a meson dissociation transition. Note that in the electric field no chiral condensate is created at zero quark mass, and that the dissociation transition happens at a finite quark mass. An indication for the interpretation of the transition as a metal-insulator transition is that the current induced by the electric field (fig. 3.13) drops to zero exactly at the point where the quark mass is high enough to allow for stable quark-antiquark bound states in the presence of the electric field. This is interpreted as the impossibility of quark-antiquark pair creation out of the vacuum for high enough quark masses. This latter interpretation coincides with the result of [284] that the quark number conductivity is finite even in the limit of vanishing temperature and density.The phase transition is thus between an insulating mesonic phase and a conducting quarkonic phase.

A calculation of the masses of the lowest-lying pseudoscalar mesons (fig. 3.14) confirms that the mesons are lighter in the electric field than without, i.e. the lowering of the binding energy. Furthermore an analytic calculation in section 3.3.1 showed that a second order Stark shift of the meson masses occurs for the pseudoscalar sector in weak electric fields. The calculation of the meson spectrum presented in [1] was, due to our ignorance about the correct boundary conditions at the singular shell, restricted to the mesonic phase, i.e. to embeddings which do not touch the vanishing locus. This restriction was recently overcome in [110], where the correct boundary conditions at the singular shell for gauge field fluctuations were derived. Only one set of boundary conditions consistent with the zero field limit, i.e. with infalling wave boundary conditions at the finite temperature black hole horizon, was found. Building on this the spectral functions for current-current correlators could be calculated numerically, clearly showing a fast change of the quasiparticle spectrum encoded in the spectral function from well-defined, sharp peaks to resonances with broad widths at the dissociation transition. It was also found in that work that at finite quark number density the first order dissociation transition gets smoothed out to a crossover. In summary, the evidence presented in the works [1, 110] is in favour for the interpretation of the geometric brane embedding transition in the electric field as a dissociation or metal-insulator transition.

In order to interpret the results of chapter 3 correctly, a comment on the question of the thermodynamical ensemble considered in the electric case is in order. As shown in [262], switching on a gauge field A_t corresponds to including quark number density and a quark chemical potential into the thermodynamics of the dual gauge theory. The canonical ensemble is then characterized by a fixed number density and a varying chemical potential, while in the grand canonical ensemble the chemical potential is held fixed [316]. Minkowski embeddings which close off smoothly at $\rho = 0$ are only consistent for vanishing quark number density: A nonvanishing quark number density would correspond to strings creating a cusp at $\rho = 0$ on the brane while stretching down to the black hole horizon [262], or to the AdS horizon in the zero temperature limit. Thus, as we choose to fix the number density D as in [284] while varying the quark mass and thus the induced current, we are naturally considering the system in the canonical ensemble. Fixing the density to zero as in section 3.3 then ensures the existence of the Minkowski embeddings in figure 3.10. As mentioned above, finite densities in the canonical ensemble were considered in [110].

From the above, I conclude that an interesting future task is to explore the phase diagram for finite external electric field in the grand canonical ensemble. I expect that Minkowski embeddings will play a role for the dissociation transition at finite temperature similar to the situation described in [316], and that the dissociation of the mesons at high temperatures, or equivalently at small quark masses, will lead to nonzero quark number densities due to the fixed chemical potential. Note however that for finite electric field the application of equilibrium thermodynamics is justified only in the strict large N_c-limit with $N_f/N_c \to 0$, for the following reason: The charge carriers accelerated by the electric field do lose energy to the $\mathcal{N} = 4$ plasma by a diffusion wake [307, 308] created due to their movement. This is the strong coupling analogue of heavy quark energy loss in the quark-gluon plasma at weak coupling, where the dominant processes are two-body collisions and gluon bremsstrahlung [383]. If N_c is large but finite, the energy pumped into the glue, which grows linearly in time, will not be negligible any longer for time scales of the order N_c/N_f, in which case the glue will react to the movement of the quarks and start to develop a flow by itself. At large but finite N_c the situation is thus at most quasistatic. In the strict $N_c \to \infty$ limit the relevant time scale however diverges, and in this case the $\mathcal{N} = 4$ plasma can absorb an infinite amount of energy from the $\mathcal{N} = 2$ quark number charge carriers without being accelerated itself (see the conclusions of [284] for details). The situation is thus static only in the strict $N_c \to \infty$ limit.

The results for the electric Kalb-Ramond field presented in chapter 3 come with an open question which needs to be answered in future work: It remains to analyse whether the conically singular embeddings, whose presence was noticed first in [288] and which are also seen in our results are physical or not. These embeddings occur both at zero and finite temperature. At zero temperature they are easily spotted (see fig. 3.10): The exist in the intermediate quark mass range between the embeddings which reach the origin of the diagram and the Minkowski embeddings which do not reach the singular shell. At finite temperature they exist between the black hole and Minkowski embeddings. Since a conical singularity has diverging curvature, what is the status of these solutions to the DBI action? It is clear that they cannot be just discarded, since they exist in a finite interval of quark masses, which is outside the phase transition region [288]. Discarding them would mean to discard a finite interval of quark masses, and it is not even clear what is the physical meaning of excluding a region in the space of external parameters for a thermodynamical system. It is thus reasonable to assume that they play a part in the physics of this system. They could for instance describe a new intermediary phase of matter in the presence of an external $U(1)$ field. In this case the presence of the conical singularity needs to be explained, presumably by some object pulling the brane at this point, such that a force-balanced (i.e. static) state is possible. What could this object be – a string, another brane? For vanishing electric field but with finite density and temperature a similar situation was analysed in [262] (around eq. (2.40) there), with the result that static configurations of a string pulling the D7 brane to create a cusp do not exist. In that case the D7 branes must always end on the horizon, being pulled into by the tension of a bundle of fundamental strings attached to it. The calculation in [262] shows that in the region $\rho \approx 0$ the D7 brane action reduces to an action for a density n_q of Nambu-Goto strings smeared in the Minkowski directions and stretching along the L-axis from the black hole horizon to the cusp. With electric field, an analogous calculation approximates the D7 brane action eq. (3.41) as[7]

$$\bar{L}'_{D7} \simeq n_q \sqrt{(g_{tt}(r) - g_{tt}(r_{IR}))\left(1 - \frac{B^2}{g_{tt}(r) g_{xx}(r)}\right)(g_{\rho\rho}(r) + g_{\theta\theta}(r)(\partial_r \theta)^2)}\,. \tag{6.1}$$

Without the B^2 term and the $g_{tt}(r_{IR})$ contribution in the square root, this would reduce to a Nambu-Goto type action for strings smeared in the Minkowski directions with density n_q. It is thus unclear what is the actual object creating this conical singularity.[8] It is however conceivable that the strings get dragged along in the x-direction by the electric field, and thus a trailing string dragged through the plasma (i.e. a more complicated string embedding) reproducing eq. (6.1) could be attached at the cusp. Also, stringy α'-corrections to the DBI action [384] could resolve the conical singularity in the embedding. In any case, this question should be addressed in future work.

Generically, also from the field theory perspective, an instability is expected in the electric case, since turning on an external electric Kalb-Ramond field requires the presence of a

[7] Note the change of coordinates $L = r\cos\theta$, $\rho = r\sin\theta$.

[8] Ref. [110] used a workaround to this problem by numerically integrating the embeddings not to $\rho = 0$ but to $\rho = \epsilon$. In this way the otherwise conically singular solutions extend down to the horizon by being "reflected" at the L axis everytime they come close to it. Extrapolating this treatment to $\epsilon = 0$ yields selfintersecting D7 branes. Nevertheless, this procedure seems to give physically reasonable results. However, if this would be the correct way to embed the D7 branes at these particular quark masses, α'-corrections could remove the conical singularities, and the embedding could relax into a black hole embedding reaching smoothly from the singular shell to the horizon. I thus doubt that this is the final answer to this problem.

non-zero gauge field background corresponding to a non-vanishing quark number density and quark chemical potential μ. It is known for supersymmetric theories, which involve scalars, that this leads to an upside-down potential of the form [385]

$$V(\phi) = -\mu^2|\phi|^2\,, \tag{6.2}$$

which will, in the end, lead to Bose-Einstein condensation of the scalars. For an isospin chemical potential, this instability was reproduced in the holographic context in [346]. Also in the study of quark chemical potentials, thermodynamical instabilities of D7 brane embeddings at which tachyonic meson modes appear have been found [262, 275]. These latter instabilities are of a more intricate structure, since they are confined to particular regions of the phase diagram. The detailed structure of the phase diagram for the electric external field is thus very subtle. It will be interesting to study it further, in particular in the light of the work [259, 316].

Several additional directions of future research work are:

- The ansatz for a constant electric field (4.48) (as well as the ansatz of a constant quark chemical potential) is plagued by the fact that constant spatio-temporal differential forms on Lorentzian manifolds are generically ill-defined at Killing horizons unless they vanish there [386]. A possible way out would be to allow for a radial dependence in (4.48), and attempt to solve the IIB supergravity equations of motion with nontrivial H flux to find a generalisation of the D3 soliton with electric field in the wanted directions. To the author's knowledge no such attempt has been undertaken so far.[9]

- In the electric case at finite temperature it is conceivable that besides the dissociation phase transition there is a second phase transition (or a crossover) when the conically singular solutions reach the black hole horizon. In the work [110] this seems to be explicitly excluded by the workaround described in footnote 8 of this chapter. Thus as long as the status of the conically singular embeddings and their exact treatment is not settled, such details of the dissociation phase transition remain unclear and require further study.

- As noted already in chapter 1, the quark-antiquark pair production rate in the electric field is finite even for vanishing bare quark mass. This fact is even more puzzling since the chiral condensate also vanishes in the limit of zero bare quark mass, i.e. there is no dynamical mass generated at zero bare quark mass. The finiteness of the pair production rate at zero quark mass thus seems to be due to a genuine strong coupling effect which deserves further investigation.

[9] The recent paper [387] used a null Melvin twist in order to derive D3 soliton solutions with C_1 switched on, i.e. with D0 brane charge. This form field is however part of the Ramond-Ramond sector and not of the NS-NS sector, and furthermore the solutions presented do not even admit a limit in which the two-form field strength $\mathrm{d}C_1$ becomes constant far away from the D3 branes.

- Since the Lorentz symmetry is broken in the presence of the electromagnetic background fields, a detailed analysis of the holographic dictionary in this situation of reduced symmetry along the lines of [297] will be interesting.
- As was noted in footnote 10 of chapter 3, the *ad hoc* application of the equal area law to the dissociation phase transition seems to yield correct results, but the thermodynamical treatment in the presence of persisting currents and in particular a proof of the equal area law still needs to be worked out. The underlying problem is that a component of a current is not a conserved quantity, and thus no chemical potential conjugate to it can be defined in the usual way. Nevertheless it is a static situation, and therefore not too different from thermal equilibrium.
- It might be instructive to embed D7 probes into global AdS-Schwarzschild spacetimes in the presence of electric and in particular magnetic fields. In this way new insights into the interplay of the (de)confinement and meson melting transitions in this toy model of strongly interacting quantum field theories may be gained.

6.2 Holographic Fayet-Iliopoulos Terms from Kalb-Ramond Fields

In the last section, the work presented in chapter 3 on the breaking of the Lorentz symmetry of the dual field theory by external electric and magnetic fields was summarised. As a further step of generalising the AdS/CFT correspondence to situations with less symmetry, chapter 4 presented a holographic construction [2] of nontrivial supersymmetry breaking Coulomb-Higgs states in a deformation of the $\mathcal{N} = 2$ supersymmetric gauge theory with flavour. The original theory, which was described in section 2.3.6, is deformed by the Fayet-Iliopoulos coupling (4.47). Fayet-Iliopoulos couplings in supersymmetric gauge theories (see e.g. chapter VIII of [205]) generically break either just supersymmetry (if the matter fields are sufficiently massive), or both supersymmetry and the $U(1)$ gauge symmetry to which the Fayet-Iliopoulos term couples (if the mass squares are light compared to the Fayet-Iliopoulos parameter). Due to technical restrictions that will be explained below, only the massless case was considered in chapter 4, and hence both the $U(1) \subset U(N_c)$ gauge symmetry as well as supersymmetry is broken.

The holographic construction of chapter 4 was to a large extend inspired by known facts about the D3-D7 brane intersection in flat space. The Fayet-Iliopoulos deformation is induced in the D3-D7 system by switching on the constant Kalb-Ramond field given in eq. (4.48) in the directions of the D7 brane world volume perpendicular to the D3 brane. The Fayet-Iliopoulos coupling (4.47) only depends on the anti-selfdual part of the B field, i.e. on $b_1 - b_2$ in the notation of eq. (4.48). The self-dual part cannot be gauged away by Ramond-Ramond gauge transformation due to the presence of the D7 brane, but it does not influence the physics of open strings ending on the D7 probe. It can be shown from direct calculations of string disk amplitudes [117] that the low-energy effective action includes the correct Fayet-Iliopoulos coupling (4.47). As argued in section 4.3, the derivation of the AdS/CFT duality with fundamental flavour fields via the decoupling argument explained in section 2.3.6 is also valid in the Fayet-Iliopoulos deformed theory. Thus the Fayet-Iliopoulos deformed $\mathcal{N} = 2$ theory is conjectured to be holographically dual to type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$ with N_f probe D7 branes and with the constant Kalb-Ramond field in the directions of the D7 branes perpendicular to the AdS boundary. This is the first main result of chapter 4.

Several nontrivial checks described in section 4.4 confirm this conjecture. In section 4.4.1 it was shown that the anti-selfdual part of the B field (4.48) has the correct transformation behaviour under the $SU(2)_\Phi \times SU(2)_\mathcal{R} \times U(1)_\mathcal{R}$ global symmetry of the undeformed $\mathcal{N} = 2$ field theory (see section 2.3.6 for more details) in order to couple to the $SU(2)_\mathcal{R}$ triplet of auxiliary fields in the $\mathcal{N} = 2$ vector multiplet. Also, if only the auxiliary D field from the $\mathcal{N} = 1$ vector multiplet couples to the B field (which corresponds to a choice of $SU(2)_\mathcal{R}$ frame), the global $SU(2)_\mathcal{R}$ symmetry is broken to its $U(1)$ subgroup, both in the field theory as well as on the gravity side. For an argument that the operator corresponding holographically to the anti-selfdual B field has the scaling dimension $\Delta = 2$ of auxiliary D and F fields, see section 4.4.2. Finally, in section 4.4.3 the supersymmetry breaking pattern of the embedding of one probe D7 brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$ was shown to match the expectations from the theory of noncommutative instantons. In particular both a κ-symmetry calculation as well as the existence of no-force conditions confirmed that supersymmetry is preserved for a D7 brane if the B field is self-dual, while it is broken if the B field has an anti-selfdual part. In the latter case the supersymmetry breaking is parametrised by the Fayet-Iliopoulos term in the dual field theory. The supersymmetry breaking pattern is summarised in table 4.2.

The second main result of chapter 4 is the conjecture of a holographic duality between mixed Coulomb-Higgs states of the Fayet-Iliopoulos deformed $\mathcal{N} = 2$ theory, and noncommutative instanton solutions in the D7 brane worldvolume directions perpendicular to the AdS boundary. This conjecture relies on two properties of string theory in the presence of D branes: First, the presence of a constant Kalb-Ramond field induces noncommutativity on the worldvolume of a D brane. The open string endpoints on the D brane behave like coordinates of a noncommutative space [111], and hence the low-energy effective gauge theory in the worldvolume directions with B field is a noncommutative gauge theory. Secondly, in the supersymmetric Dp-D(p+4) intersections in type II string theory, the D and F term equations whose solutions describe the different vacua of the (p+1)-dimensional gauge theory can be mapped onto the Atiyah-Drinfeld-Hitchin-Manin (ADHM) equations which classify all possible instanton solutions in the four directions of the D(p+4) brane transverse to the Dp brane. The underlying physical picture is that instantons in these transverse directions of the D(p+4) brane can be thought of as “dissolved” Dp branes in the D(p+4) brane, or equivalently Dp-D(p+4) bound states. In terms of the classification of vacua in the (p+1)-dimensional field theory, Dp branes which dissolve into the D(p+4) branes correspond to colour directions of the $U(N_c)$ gauge group which are broken completely (i.e. including the corresponding $U(1)$ generator of the Cartan subalgebra) by the Higgs mechanism. In this case scalar fields in the fundamental representation (squarks) acquire a vacuum expectation value in the corresponding colour direction. Such vacua are called “Higgs vacua”. On the other hand, Dp branes which are not bound to the higher-dimensional branes but separated from the main stack of Dp branes correspond to “Coulomb vacua” of the gauge theory, i.e. directions in colour space which are broken by the Higgs mechanism down to the corresponding $U(1)$ Cartan generator. In this case a scalar field transforming in the adjoint representation of the gauge group acquires a vacuum expectation value. The general case is that of mixed Coulomb-Higgs vacua, in which several colour directions are broken completely and others unbroken or only broken down to the respective $U(1)$ subgroups. The part of the D and F term equations which describe the Higgs branch, i.e. which describe the vacuum expectation values of fundamental and adjoint squark fields in the colour directions with broken Cartan generators, are then exactly the ADHM equations for instantons in the transverse D(p+4) directions. In particular the vacuum expectation values of the adjoint and fundamental squark fields are in one-to-one correspondence with, respectively, the position and size moduli of the instantons.

In the case of the Fayet-Iliopoulos deformed theory the situation is similar: The D and F term equations of the field theory are deformed by the Fayet-Iliopoulos term (4.47), and the ADHM equations of noncommutative instantons are as well deformed by a term of this type [119]. The full low-energy effective action of the Dp-D(p+4) intersection in flat space in the presence of a constant Kalb-Ramond field was calculated in [117], where it was found explicitly that the mapping between vacuum equations in the field theory and transverse instantons persists also in the deformed case. I conjectured in chapter 4 of this thesis that this mapping between the D and F term equations for the Higgs part of the Coulomb-Higgs branch and the noncommutative ADHM equations (4.43)-(4.44) also persists in the holographic setting, i.e. after taking Maldacena’s decoupling limit. As a consequence, nontrivial states on the Coulomb-Higgs branch of the field theory at strong coupling should be described by a collection of N_f D7 probe branes embedded in the standard way into $\mathrm{AdS}_5 \times \mathrm{S}^5$, and with the constant Kalb-Ramond field (4.48) switched on. This is the second main result of the work [2] presented in chapter 4. More precisely, the mixed Coulomb-Higgs branch of the field theory with k colour directions broken completely, i.e. with k fundamental squark fields acquiring a vacuum expectation value, is conjectured to be dual to a stack of N_f D7 branes in $\mathrm{AdS}_5 \times \mathrm{S}^5$ with a charge k noncommutative instanton in the $U(N_f)$ noncommutative gauge theory present in the transverse directions of the D7 brane.

For the analysis presented in chapter 4 it was necessary to work in an adiabatic approximation, in which the Fayet-Iliopoulos parameter is very small, $\zeta \ll 1$, corresponding to a large B field. In this limit, as explained below, the setup is sufficiently stable in order to make the following construction: Let $N_c - k$ D3 branes generate the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry, and embed a D7 stack into this space, with k D3 branes dissolved as instantons on the brane probes. Evidently, k must be small for the usual probe approximation to be valid, i.e. for the backreaction of the D7 probe onto the background to be negligible. The probe approximation alone however is not sufficient for ensuring the stability of this setup: Although the decay process, which involves a quantum of five-form flux to be transferred from the S^5 onto the D7 stack and to manifest itself on the D7 branes as an instanton (while at the same time the radius of AdS_5 shrinks by one unit) is a backreaction effect and hence suppressed by $\frac{N_f}{N_c}$, there are N_c identical quanta of five-form flux which could jump over. This means that the total decay probability is of order $\frac{N_f}{N_c}N_c = N_f$, and thus in general not negligible. However, since for vanishing Fayet-Iliopoulos parameter the setup is supersymmetric and therefore stable, the decay probability must be, for small ζ, proportional to ζ itself,

$$P_{decay} \propto \zeta \frac{N_f}{N_c} N_c = \zeta^\alpha N_f \,, \tag{6.3}$$

and hence can be tuned to an arbitrarily small number. Therefore the setup can be made arbitrarily longlived by tuning ζ to be very small. In this adiabatic (or quasistatic) approximation the setup of chapter 4 describes metastable states than stable ones (and hence I refer to them as "states" and not as "vacua"). However, it is not a potential barrier that protects the setup from decaying, but only the smallness of the Fayet-Iliopoulos parameter: Already the flat space D3-D7 intersection is not stable in the anti-selfdual B field, but it is known from models of D3-D7 inflation [322,327–329] that the supersymmetry breaking generates a potential between the D3 and D7 branes, pulling the D3 branes towards the D7 branes until they dissolve to form instantons. The actual stable and $\mathcal{N} = 2$ supersymmetric state is the one where all N_c D3 branes are dissolved into an instantonic configuration on the D7 brane. This is consistent with the field theory since the D term equation with Fayet-Iliopoulos term actually cannot be solved without giving all the squark fields a vacuum expectation value, which generates a potential for the squark fields on the Coulomb part of the Coulomb-Higgs branch. The D term contribution to the scalar potential will thus be nonvanishing, but will directly drive the system to the pure Higgs vacuum. On the gravity side of the correspondence the flat space process of D3 branes approaching and finally dissolving into the D7 branes will correspond to a backreaction effect of the D7 brane onto the background in which five-form flux is be transferred from the internal S^5 onto the D7 brane, manifesting itself as an additional instanton. It is an interesting question for future work to investigate the dynamics of the actual decay process involved.

In order to test the proposed duality between Coulomb-Higgs states and noncommutative instantons, in section 4.5 the symmetry properties of both sides were compared explicitly in the simplest situation, namely that of a single ($k = 1$) noncommutative instanton on a single ($N_f = 1$) flavour brane embedded flatly into AdS space. In the case of a single instanton, the squark fields acquire a vacuum expectation value in a single colour direction. The situation describes one D3 brane dissolved into the D7 brane. Since this calculation involves finding an analytic expression for the D7 brane embedding in the presence of the B field, which is only possible for flat embeddings $L = 0$, the analysis was restricted to the case of massless flavour fields. On the gravity side, the Nekrasov-Schwarz noncommutative $U(1)$ instanton was shown to preserve the full global symmetry left invariant by the B field ansatz, i.e. $SU(2)_\Phi \times (U(1) \subset SU(2)_\mathcal{R}) \times U(1)_f \times U(1)_\mathcal{R}$, while the squark vacuum expectation value[10] calculated from the D and F term equations of the field theory preserves only $SU(2)_\Phi \times \mathrm{diag}(U(1) \times U(1)_f) \times U(1)_\mathcal{R}$. Here $U(1)_f$ is the global flavour symmetry. There is thus an obvious symmetry mismatch which needs explanation. I believes that the field theory result is the correct one, since in the work [115] a similar breaking of the $\mathcal{R}$-symmetry and the flavour symmetry has been observed for a single Belavin-Polyakov-Shvarts-Tyupkin (BPST) instanton [338] on two flavour branes but without B field. The backdraw of comparing with the commutative $SU(2)$ instanton is of course that no nontrivial commutative instanton solutions exist for gauge group $U(1)$ [333], the Nekrasov-Schwarz instanton does not admit a commutative limit. An analysis of the symmetry properties of the noncommutative version of the $SU(2)$ BPST instanton in future work could thus resolve the situation. Assuming that the field theory result is the correct one, there are still several assumptions entering the analysis of section 4.5 which might be the origin of this mismatch. One basic assumption is that the instanton gauge potential (4.71) transforms under rotations in noncommutative space-time as an ordinary vector. Since space-time symmetries mix with gauge symmetries in noncommutative gauge theories [340], it might be that some of the rotations allowed by the choice of B field actually are broken by the choice of singular gauge in the Nekrasov-Schwarz solution. This is possible since even infinitesimal rotations in noncommutative space-time correspond to large gauge transformations. As in the AdS/CFT context global symmetries in the field theory do correspond to local symmetries on the gravity side, it is necessary to work out the noncommutative diffeomorphisms and $U(1)_f$ gauge transformations which leave invariant the asymptotic behaviour $A_i \simeq |\vec{y}|^{-3}$ of the gauge potential (4.71). This is a rather complicated calculation since it involves complicated nonpolynomial operators in the Fock space representation of the noncommutative space-time. Preliminary attempts did not lead to any conclusive results, but I nevertheless assume this to be the true reason of the symmetry mismatch, and the problem of finding the correct symmetries of the Nekrasov-Schwarz instanton should surely be addressed again in future work.

Another, more basic assumption entering the analysis of section 4.5 is that although the induced metric on the D7 brane is not flat, the Nekrasov-Schwarz instanton is a solution of the actual unknown noncommutative gauge on the D7 brane. I assume this to be true, since the induced metric on the D7 brane worldvolume for the flat embedding is actually flat in the internal directions up to a function of the radial coordinate $\vec{y}^2$, which is rotationally invariant and can easily be represented in terms of Fock space operators

[10] As can be seen explicitly in this case (see e.g. (4.88)), an important conclusion which can be drawn from the proposed duality conjecture is that the $U(1)$ factor of the $U(N_c)$ gauge group in the boundary field theory plays an important role, namely it is broken generically by the solutions to the F and D term equations since the fundamental squark fields (which are charged under the $U(1)$) acquire a vacuum expectation value. It will be interesting to study the singleton mechanism [294, 295] involved further.

(see eq. (4.73)). Although the actual noncommutative deformation of this curved space is unknown, a modification involving the radial distance will not break any additional rotation symmetries (which correspond to $\mathcal{R}$ symmetries in the field theory). This argument does of course not exclude exotic modifications of the noncommutative algebra due to the curvature of this space-time. It may thus be interesting to investigate possible noncommutative deformations of conformally flat four-dimensional Euclidean space when the conformal factor depends only on the radius. Since the induced geometry on the D7 brane is $\mathrm{AdS}_5 \times \mathrm{S}^3$, which is a quotient manifold, an application of the results of [388] to this case could also help to deduce the actual form of the noncommutative gauge theory on the D7 brane.

Finally, let me note several possible applications and generalisations of the work presented. It will be interesting to investigate similar dualities for the more involved case where the scenario is stabilized, for instance by considering D7 brane probes in a singular geometry such as the conifold. This may allow for constructing gravity dual descriptions of metastable vacua, which have been discussed for theories without fundamental flavour degrees of freedom in [389–394]. These models are stabilised by fluxes which lead to configurations of branes at special points, such as the resolved conifold. It may also be interesting to study supersymmetry breaking for D7 branes in the warped throat geometry, for instance based on the results of [323, 324]. The stabilisation issue in the presence of instantons may also be addressed by considering D7 brane probes in the recently discussed deformed Sasaki-Einstein geometries of [395]. Finally, as both supersymmetry and gauge symmetry is broken in this setup, an interesting application of the proposal made in this work would be to analyse the corresponding Higgs mechanism and find e.g. the Goldstino in the spectrum of the D7 brane fluctuations. In order for such a project to be feasible it is necessary to first translate the Nekrasov-Schwarz instanton into an instantonic solution of the corresponding commutative DBI field theory via the Seiberg-Witten map [111], which is possible at least in a series expansion if the noncommutativity parameter θ^{ij} is small.

6.3 A Matrix Model Proposal for the Robertson-Walker Universe

In the last two sections of this chapter I discussed the work presented in chapters 3 and 4 of this thesis, which generalised the D3-D7 model of holographic flavour physics to situations in different external fields, namely to constant electric and magnetic field backgrounds in chapter 3, and to a Fayet-Iliopoulos deformation in chapter 4. In this section, I will discuss the results of chapter 5, which was concerned with a different kind of holographic models, namely with matrix models in the Robertson-Walker geometry.

The main result of the work presented in chapter 5 and published in [3] is the matrix quantum mechanical model (5.71) and its more general form (5.69). These two models were derived from the dynamics of a bosonic membrane in the spatially flat Robertson-

Walker geometry (5.11) (with $\kappa = 0$), with so-far unspecified scale factor, by use of a special regularisation procedure for the worldvolume theory of the membrane known as matrix regularisation [125]. This regularisation method replaces the embedding functions of the membrane by $N \times N$ matrices transforming in the adjoint representation of the $U(N)$ gauge group of the matrix model. The underlying logic is to derive the matrix model such that it captures special features of the background geometry, rather than from string theoretic considerations in the first place. In this sense it is a bottom-up rather than a top-down approach, which may be useful in its own right. In a second step it should be checked whether this matrix model actually captures part of the physics of a collection of D0 branes near the big bang singularity. In the same step one should also try to derive the model from string theoretic considerations, which is a very hard problem for nonsupersymmetric backgrounds such as the Robertson-Walker geometry. Alternatively, it can be fruitful to perform nontrivial crosschecks with $\mathcal{M}$-theory matrix models on backgrounds including a Robertson-Walker type singularity by comparing dynamical features of both kinds of matrix models. I will comment on both approaches below. It should also be noted that the procedure of matrix regularisation is a purely classical procedure, replacing the infinitely many degrees of freedom of the membrane field theory by finitely many ones encoded in the matrix fields. The so-derived matrix models (5.71) and (5.69) are still classical mechanical models which need to be quantised in a second step. Nevertheless I will refer to them as "matrix quantum mechanics", since their physical content is to a large part encoded in quantum corrections arising from integrating out heavy off-diagonal modes encoded in the matrix fields.

Technically, the actual derivation of the matrix models went along the following steps: Starting from the Nambu-Goto action for a membrane in the Robertson-Walker geometry, the worldvolume diffeomorphism gauge was fixed by choosing cosmic gauge (5.58) for the worldvolume time, as well as longitudinal gauge (5.59) for the induced metric on the membrane, which breaks the worldvolume diffeomorphisms to static ones. The fixing of a certain energy density profile on the membrane via eq. (5.66) then breaks the static diffeomorphisms down to static area preserving ones. This last gauge fixing condition replaces the fixing of the light-cone momentum known from light-cone quantisation of the membrane, and is thus applicable to the Robertson-Walker background which does not admit a lightlike isometry. The gauge-fixed Nambu-Goto action is then replaced by a classically equivalent square-root free mechanical model using a procedure explained in section 5.2.2. The crucial point here is that the vanishing of the Hamiltonian of the new square-root free system has to be imposed as a constraint in order to ensure classical equivalence of the two models. The square-root free action (5.67) is then subjected to the matrix regularisation procedure. The group of static area preserving two-dimensional diffeomorphisms leaves a symplectic form on the two-dimensional spatial worldvolume of the membrane invariant. It is thus the largest group of spatial diffeomorphisms compatible with the matrix regularisation of the spatial worldvolume (5.52), which essentially amounts to a "deformation quantisation" turning the membrane worldvolume into a noncommutative space, and the mechanical system (5.67) into the matrix quantum mechanics (5.69). The so-obtained matrix model admits a $U(N)$ gauge symmetry, reflecting the indistinguishability of the "partons" (i.e. D0 branes) which probe the underlying space-time.

The matrix model (5.69) derived in this way depends explicitly on the gauge-fixing function $\hat{\Omega}$, which encodes the chosen energy density profile of the original membrane. There is a huge freedom in choosing this function, and this choice influences the structure of solutions and in particular of the static solutions of the theory. This is not surprising since the whole procedure of deriving matrix models from membrane dynamics is not only background dependent but also gauge dependent. On the other hand, the *a priori* choice of $\hat{\Omega}$ fixes the total energy of the membrane configuration considered. This raises the question of the physical meaning of the derived matrix model: Does its spectrum include only membranes of fixed energy, or is there a way to *a posteriori* find a continuum of membrane states with different energies? Even if the former is the case, there might be a mathematically well-defined mapping between matrix models with different $\hat{\Omega}$, transforming matrix models with different spectra into each other and corresponding to a change of gauge choice for the worldvolume diffeomorphisms. In order to answer these questions and thus gain a better understanding of which sector of $\mathcal{M}$-theory physics the matrix model actually describes, it will be helpful to investigate in future work the space of static (i.e. $D_t X^i = 0$) solutions of the matrix model with more general choices of the gauge function $\hat{\Omega}(t)$.

In order to derive the second model (5.71), a guiding principle for choosing a suitable gauge $\hat{\Omega}$ was needed. The principle used in [3] was the requirement that a single D0 brane (the case $N = 1$) should behave as a point particle in the Robertson-Walker geometry, i.e. it should follow the geodesics in that space-time. As explained in section 5.2.2 it is actually possible to derive a classical mechanical model (see eq. (5.25)) admitting a certain global $\mathbf{so}(1,2)$ symmetry and describing a subsector of the geodesic motion in the Robertson-Walker background with fixed initial energy and angular momentum. This model is a square-root free rewriting of the relativistic point particle action, and it has striking similarities with the matrix model Lagrangian (5.69), which made it possible to fix the gauge fixing function $\hat{\Omega}$ by direct comparison of eqs. (5.25) and (5.69) to the values given in eq. (5.70) and hence arrive at the matrix model (5.71). This choice of gauge physically corresponds to the requirement that the center-of-mass motion of all the N D0 branes follows certain geodesics in the Robertson-Walker geometry, corresponding to a membrane with "freely falling" center-of-mass motion. It seems to be the simplest possible and at the same time physically interesting gauge choice, but surely not the unique one. In particular switching on $SU(N)$ components of $\hat{\Omega}$ do not change the center-of-mass motion of the D0 branes, so there is a large degeneracy of choices which satisfy our physical guiding principle, and their significance should be investigated further in future work. Looking at eq. (5.72), it is very plausible that gauge choices with nontrivial $SU(N)$ part give rise to static solutions exhibiting certain polarisation features similar to Myer's effect [282], i.e. that the collective behaviour of the D0 branes is such that they form a higher-dimensional object in target space.[11]

[11]Note however that in the case of a 1+1-dimensional universe, i.e. with only one $X^1 = X$ present,

Within this second, more specialised matrix model (5.71) I gave a general argument at the very end of chapter 5 for the emergence of a smooth geometric structure from the matrix model dynamics once the universe has expanded to a size few times larger than the Planck scale. One may argue that the circle of argumentation closes here and that one simply recovers the Robertson-Walker geometry which was put into the calculation via the membrane action at the beginning. Though it is true that matrix models derived from membrane regularisation generally know about the background (and hence are explicitly background dependent), it is in my eyes a highly nontrivial consistency check for the procedure of matrix regularisation as well as for the special choice of $\hat{\Omega}$ that the coefficient of the Yang-Mills-like commutator-square potential in (5.71) goes with a high enough power of the scale factor such that the emergence of space is obvious at all in (5.71). Other, more complicated choices of $\hat{\Omega}$ could have made this behaviour more complicated to observe or not present at all. Another reassuring observation is that similar to the model considered in [362], with our choice of gauge fixing (5.71) the Yang-Mills coupling constant in front of the commutator-squared potential becomes small near a singularity and vanishes at the singularity. The observation of space-time emergence thus gives confidence in the sometimes rather *ad hoc* assumptions entering the derivation of (5.71), but on the other hand demands even stronger for a more in-depth analysis of the role of the choice of $\hat{\Omega}$ in this setting. In any case, the proposal laid out in chapter 5 yields the opportunity for several directions of future investigations on which I want to briefly comment before closing this thesis. I also comment on some wilder speculations.

- Presumably the most interesting investigation would be to embed Robertson-Walker cosmologies in string theory backgrounds with spacelike singularities such as the ones classified in [399–402], try to derive matrix models from this starting point both with the gauge fixing method used here as well as with other more conventional methods such as light-cone gauge fixing, and compare the approaches. If this proves feasible, nontrivial crosschecks with eleven-dimensional supergravity on backgrounds with a Robertson-Walker singularity such as the calculation of scattering amplitudes directly from the matrix model as well as from the supergravity point of view would be possible. The application of the laid-out method for gauge fixing and subsequent matrix regularisation to backgrounds with already known matrix model descriptions such as flat eleven-dimensional space or linear dilaton backgrounds would also be a good consistency check, and could in particular clarify the physical significance of the gauge choice encoded in $\hat{\Omega}$.

- Even without embedding the model into string theory, in the bottom-up approach itself, a case worth considering would be Einstein-Hilbert gravity with a positive cosmological constant. In this case the de Sitter universe with scale factor $a(t) =$

eq. (5.72) forces the D0 brane configuration to be diagonal independently of the choice of $\hat{\Omega}$. In this extremely low-dimensional case the question is rather how much dynamics one would at all expect from a theory of quantum gravity. There are interesting examples with nontrivial dynamics known in the literature, see e.g. [396–398] for an overview on two-dimensional toy models of quantum gravity.

const $\times$ e^{Ht} is a solution to Einstein's equations, and induced interactions from a one-loop calculation in the matrix model could be compared to tree-level scattering amplitudes in the gravity theory. In any case the matrix regularisation procedure only yields a classical matrix mechanical model, which needs to be subjected to quantisation in order to learn about stability properties of the model, in particular of the ground state. Quantum corrections to our scenario can be calculated in analogy to [362]. Such a calculation may also shed light on the question whether the holographic entropy bound holds in the models presented here - the strategy would be to calculate the one-loop effective action for initial conditions of two well-separated groups of D0 branes and then analyse the scaling behaviour of the typical length scale of a large cluster of D0 branes along the lines of section VII of [42].

- Also an *ad hoc* supersymmetrisation of the matrix models (5.69) and (5.71) could be feasible. The result could be compared to matrix regularisations of supersymmetric membrane theories on Robertson-Walker like supersymmetric geometries, or to the supersymmetric matrix model which presumably can be derived by application of the gauge fixing procedure and matrix regularisation procedure proposed in chapter 5 to the full supersymmetric M2 brane action [403].

- Already in the work of Banks, Fischler, Shenker and Susskind [42] it has been observed that in order to ensure boost invariance of the eleven-dimensional theory, a kind of scale invariance must be present in the dynamics of the BFSS matrix model. It would be interesting to investigate whether the $\mathbf{so}(1,2)$ invariance found in the analysis of point particle dynamics in section 5.2.2 carries over to the matrix models (5.69) and (5.71), and whether there is a connection with the mysterious scale invariance of BFSS.

- Another generalisation would be to use spatially nonflat geometries, i.e. Robertson-Walker geometries with $\kappa = \pm 1$. The main obstacle in this case is that the spatial geometry is not flat anymore, and thus the determinant of the pulled-back metric is not straightforwardly expressed in terms of the Poisson bracket. Most probably either a modification of the Poisson bracket or the deformation quantisation procedure of the spatial membrane worldvolume is needed.

- In section 5.2 it was shown that the general conformal mechanical system (5.1) covers through different choices of the constants c_1 and c_2 as well as of the gauge fixing function $\hat{\eta}$, not only the point particle dynamics in the Robertson-Walker universe, but also the dynamics of a subsector of homogeneous gravity modes around the spatially flat Robertson-Walker background with positive cosmological constant (i.e. the de Sitter universe). In particular, near the big bang singularity $a \simeq 0$, the choice of small $\hat{\eta} = ma\sqrt{a^2+\nu^2}$ and negative c_1 describes the point particle dynamics, while large $\hat{\eta} = \frac{1}{4}\omega a^{1-D}$ and positive c_1 describe the gravity fluctuations. It is a wild speculation to think about a possibly deeper meaning of this apparent coincidence, but an identification of the two pictures would for example map the volume element

$\varrho = \sqrt{\det g_{ij}}$ in the gravity picture to the radial coordinate ρ of the point particle trajectory. Singularities in the homogeneous space-times occur when the metric degenerates ($\varrho = 0$), whereas nothing particular happens to the point particle at the point $\rho = 0$ in the spatially flat Robertson-Walker geometry. A similarly wild speculation is that since the conformal mechanics model (5.1) describes both one-particle motion as well as homogeneous gravity excitations, and since the matrix model (5.71) is the generalisation of (5.1) to many D-particles and hence should describe inhomogeneous quantum fluctuations around the homogeneous Robertson-Walker geometry, there could either exist a connection between the matrix models derived here and the quantum mechanics of inhomogeneous modes of Einstein-Hilbert gravity with a cosmological constant, or a physical picture from which a matrix model description of Einstein-Hilbert gravity with a cosmological constant can be derived. In either I dont know how to give a more precise meaning to these interesting speculations.

- Finally, I would like to draw the attention onto another observation concerning the matrix models (5.69) and (5.71): Assuming that the point particles we are considering are actually D0 branes, the D0 brane tension $m = T_0 \propto (g_s \ell_s)^{-1} = 1/R_s$ explicitly appears in the matrix model. It seems that a successful embedding of this matrix model into $\mathcal{M}$-theory would need a very particular background which connects the scale factor $a(t)$ in such a way with the $\mathcal{M}$-theory circle, in order to give rise to a constant energy contribution $\propto a^2/(R_s^2 \ell_{11}^4)$ as in (5.71). This might be used as a guiding principle for the search for the correct string/$\mathcal{M}$-theory embedding of these models.

To conclude this thesis, the overall common theme of the work presented in chapters 3, 4 and 5 was the generalisation of two holographic models arising in string/$\mathcal{M}$-theory to physical situations with a lower degree of symmetry. In chapters 3 and 4 the D3-D7 model of AdS/CFT with flavour has been extended to include constant electric and magnetic background fields which break Lorentz invariance, as well as to a supersymmetry-breaking Fayet-Iliopoulos deformation. Chapter 5 presented a derivation of a matrix model for a particular nonsupersymmetric background, the Friedmann-Robertson-Walker geometry. In all three cases, interesting new physical effects with the potential for many further developments discussed in this chapter were found. The application of holographic methods to both the description of strongly coupled gauge theories as well as to quantum gravity thus seems to be a field with a high potential for interesting future applications. Several additional research directions which emerged during my PhD studies, such as the study of boundary effects in two-dimensional quantum dilaton gravity [404] and in the BFSS matrix model [405], a holographic analysis of the $\mathcal{N} = 2$ gauge theory with flavour in the presence of a four-dimensional boundary cosmological constant [406] and of black hole formation as a holographic dual for quark-gluon plasma equilibration [407], as well as a systematic classification of possible extensions of the ABJM theory [380] to include flavour degrees of freedom [408] confirm my expectation that the field of gauge/gravity duality has not reached its limits yet.

Appendix A

Notational Conventions

Throughout this thesis, if not mentioned otherwise, natural units

$$\hbar = c = G_N = 1$$

are employed. Space-time is assumed to be a Lorentzian manifold with signature $(-,+,,\dots,+)$. Latin indices a, b are local Lorentz indices, while Greek indices μ, ν refer to the manifold. However be aware of additional index conventions in the different chapters – it may have been necessary at some points to introduce for example capital latin letters form ten-dimensional space-time indices. The totally antisymmetric symbol both in tangent space and on the manifold is defined by $\epsilon_{01\dots d} = \tilde{\epsilon}_{01\dots d} = +1$. The symbols with upper indices are defined as usual via raising all indices with the corresponding metric. Light cone coordinates are defined as

$$u^{\pm} = \frac{1}{\sqrt{2}}(u^0 \pm u^1)\,. \tag{A.1}$$

The components of a p-form are defined via

$$\Omega_p = \frac{1}{p!}\Omega_{\mu_1\dots\mu_p}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_1}\,, \tag{A.2}$$

and the Hodge dual of a p-form (in D dimensions) is defined with the coefficients

$$*\,\Omega_p = \Omega'_{D-p} = \frac{1}{p!(D-p)!}\epsilon_{\mu_1\dots\mu_{D-p}}{}^{\nu_1\dots\nu_p}\Omega_{\nu_1\dots\nu_p}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{D-p}}\,, \tag{A.3}$$

with the Levi-Civita tensor $\epsilon_{\mu_1\dots\mu_D} = |\det e_\mu^a|\tilde{\epsilon}_{\mu_1\dots\mu_D}$. For even dimension D and Lorentzian signature this yields

$$*\,{*\Omega_p} = (-1)^{p+1}\Omega_p \tag{A.4}$$

and thus

$$*\,\epsilon = 1\,, \quad *1 = -\epsilon \tag{A.5}$$

for the Hodge star acting on the volume form ϵ. With these conventions the hermitian conjugate of the exterior derivative [409] reads

$$\mathrm{d}^\dagger = *\mathrm{d} * \ . \tag{A.6}$$

A Dirac fermion in two dimensions has two complex components, $\chi = (\chi_0, \chi_1)^\mathrm{T}$. Dirac matrices in two-dimensional Minkowski space are chosen as

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad & \gamma^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \gamma^+ &= \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \qquad & \gamma^- &= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} \ . \end{aligned} \tag{A.7}$$

The analogue of the γ^5 matrix is $\gamma_* = \gamma_0\gamma_1 = \mathrm{diag}(+-)$. They satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\{\gamma_*, \gamma^a\} = 0$. For calculations in Euclidean space γ^0 is defined as above, but $\gamma^1 = \mathrm{diag}(+-)$ and $\gamma_* = \gamma_0\gamma_1$, thus satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. The Dirac matrices in Euclidean space are thus hermitian, $\gamma^a = \gamma^{a\dagger}$, whereas γ_* becomes anti-hermitian.

The standard form of the third anti-selfdual 't Hooft symbol, which is used in chapter 4, is

$$\bar{\eta}^3_{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ . \tag{A.8}$$

Appendix B

Appendices of Chapter 3

B.1 Some useful Expansion Formulae

To expand eq. (3.48) while calculating the Stark shift, the following identities which hold up to $\mathcal{O}(B^2)$ are useful:

$$\begin{aligned} g &= \rho^3\sqrt{1+L'^2}\sqrt{1-\frac{B^2R^4}{(\rho^2+L^2)^2}} \\ &\simeq \rho^3\left(1-\frac{B^2R^4}{2(\rho^2+m^2)^2})\right)\,, \qquad \text{(B.1)} \\ g^{-1} &\simeq \rho^{-3}\left(1+\frac{B^2R^4}{2(\rho^2+m^2)^2}\right)\,, \qquad \text{(B.2)} \\ \left[(\rho^2+L^2)^2-B^2R^4\right]^{-1} &\simeq (\rho^2+m^2)^{-2}\left(1+\frac{B^2R^4}{(\rho^2+m^2)^2}\right)\,. \qquad \text{(B.3)} \end{aligned}$$

B.2 Decoupling of Pseudoscalar Fluctuations

In this appendix we show that for the magnetic and electric Kalb-Ramond field (3.4), both at finite and zero temperature, the L- and gauge field fluctuations can be decoupled from the Φ-fluctuations and consistently set to zero by the respective Ansätze (3.22) and (3.47). For the electric case, as we will show, this holds only for the Minkowski embeddings, which have both zero baryon number density and current, i.e. for a trivial gauge field background. For a nontrivial gauge field background, like the one which renders the singular shell embeddings consistent, this decoupling will only be possible if the Φ-fluctuations do not depend on the Minkowski coordinates at all, $\phi = \phi(\rho)$. The calculation follows [105].

To show the decoupling, we have to verify that in the part of the action quadratic in the fluctuations no couplings of χ and A to ϕ appear. We take the embedding fluctuations to be $L = L_0(\rho) + \chi$, $\Phi = \phi$, where $L_0(\rho)$ is the embedding of the D7 brane into the appropriate background. For now, let us assume a trivial gauge background $F = 0$ on the brane, such that the gauge field A is a pure fluctuation. For a diagonal background metric $G_{MN}(\rho, L) = \mathrm{diag}(-G_{tt}, G_{xx}, G_{yy}, G_{zz}, G_{\rho\rho}, G_{\eta\eta}, G_{\xi_1\xi_1}, G_{\xi_2\xi_2}, G_{LL}, G_{\Phi\Phi})$ not depending on Φ and the magnetic ansatz (3.4), the pull-back of metric and B field can be split into an embedding part, a part linear and a part quadratic in the fluctuations:

$$
\begin{aligned}
E_{ab} &= P[G+B]_{ab} = E^{(0)}_{ab} + E^{(1)}_{ab} + E^{(2)}_{ab}\,, && \text{(B.4)}\\
E^{(0)}_{ab} &= (\mathcal{G}^{-1}+\theta)^{-1}_{ab}\,, && \text{(B.5)}\\
(\mathcal{G}^{-1})^{ab} &= \mathrm{diag}\Bigg(-G_{tt}^{-1}, G_{xx}^{-1}, \frac{G_{zz}}{G_{yy}G_{zz}+B^2}, \frac{G_{yy}}{G_{yy}G_{zz}+B^2}, (G_{\rho\rho}+{L_0'}^2 G_L L)^{-1}, \\
&\qquad G_{\eta\eta}^{-1}, G_{\xi_1\xi_1}^{-1}, G_{\xi_2\xi_2}^{-1}\Bigg)\,, && \text{(B.6)}\\
\theta^{23} &= -\theta^{32} = -\frac{B}{G_{yy}G_{zz}+B^2}\,, \quad \text{others zero}\,, && \text{(B.7)}\\
E^{(1)}_{ab} &= E^{(1)}_{Sab} + E^{(1)}_{Aab}\,, && \text{(B.8)}\\
E^{(1)}_{Sab} &= 2L_0' G_{LL}(L_0)\delta^\rho_{(a}\partial_{b)}\chi + \chi\partial_L G_{ab}(L_0) + {L_0'}^2\chi\delta^\rho_a\delta^\rho_b\partial_L G_{LL}(L_0)\,, && \text{(B.9)}\\
E^{(1)}_{Aab} &= 0\,, && \text{(B.10)}\\
E^{(2)}_{ab} &= E^{(2)}_{Sab} + E^{(2)}_{Aab}\,, && \text{(B.11)}\\
E^{(2)}_{Sab} &= G_{\Phi\Phi}(\partial_a\phi)(\partial_b\phi) + G_{LL}(\partial_a\chi)(\partial_b\chi) + 2\delta^\rho_{(a}(\partial_{b)}\chi)\chi L_0'\partial_L G_{LL}(L_0) && \text{(B.12)}\\
&\quad + \frac{1}{2}\delta^\rho_a\delta^\rho_b {L_0'}^2\chi^2\partial_L^2 G_{LL}(L_0) + \frac{1}{2}\chi^2\partial_L^2 G_{ab}\,, && \text{(B.13)}\\
E^{(2)}_{Aab} &= 0\,. && \text{(B.14)}
\end{aligned}
$$

Here, we split the inverse of $E^{(0)}$ further into its symmetric (the "open string metric") $\mathcal{G}^{-1}$ and its antisymmetric part (the "noncommutativity parameter") θ, as well as the linear and quadratic fluctuation parts. Note that the antisymmetric parts at linear and quadratic order vanish.

We now pull out the $E^{(0)}$ from the square root in the DBI action and use the usual determinant expansion to obtain the fluctuation part of the DBI action, dropping a factor

$-T_7/g_s$,

$$\mathcal{L}^{(2)}_{DBI} = \frac{1}{2}\sqrt{-\det E^{(0)}}\Big[\mathrm{Tr}(E^{(0)^{-1}}E^{(2)}) + \frac{1}{4}(\mathrm{Tr}(E^{(0)^{-1}}E^{(1)}))^2 \tag{B.15}$$

$$+\frac{1}{4}(\mathrm{Tr}(E^{(0)^{-1}}F))^2 + \frac{1}{2}\mathrm{Tr}(E^{(0)^{-1}}E^{(1)})\mathrm{Tr}(E^{(0)^{-1}}F) \tag{B.16}$$

$$-\frac{1}{2}\mathrm{Tr}(E^{(0)^{-1}}E^{(1)})^2 - \frac{1}{2}\mathrm{Tr}(E^{(0)^{-1}}F)^2 - \mathrm{Tr}(E^{(0)^{-1}}E^{(1)}E^{(0)^{-1}}F)\Big], \tag{B.17}$$

$$\sqrt{-|E^{(0)}|} = \sqrt{\left(\prod_{M\in\{t,x,\eta,\xi_1,\xi_2\}} G_{MM}\right)(G_{yy}G_{zz}+B^2)(G_{\rho\rho}+L_0'^2G_LL)}\,. \tag{B.18}$$

Because the antisymmetric parts (B.10) and (B.14) vanish and the Φ-fluctuations only show up at second order in (B.12), it is seen by checking term by term that (B.15) includes couplings between F and χ only, but not between F or χ with ϕ, and thus $A = \chi = 0$ decouples the angular fluctuations. The ϕ-part of the DBI action thus reads

$$\mathcal{L}^{(2)}_{DBI,\phi} = \frac{1}{2}\sqrt{-\det E^{(0)}}G_{\Phi\Phi}(L_0)(\mathcal{G}^{-1})^{ab}\partial_a\phi\partial_b\phi\,,$$

which reduces to (3.23) for the AdS-Schwarzschild background. This reasoning, as is easily checked, also holds for the gauge field background (3.27), which is needed for the electric case.

The dangerous part, which may couple gauge field and angular fluctuations, is the Wess-Zumino part of the action to second order in the fluctuations,

$$\int\limits_{\mathcal{M}_8} \frac{1}{2}P[C_4]\wedge F\wedge F + P[\tilde{C}_4]\wedge P[B]\wedge F\,,$$

which could in principle introduce couplings of F to $\partial_t\phi$, $\partial_x\phi$ and $\partial_t\partial_\rho\phi$ or $\partial_x\partial_\rho\phi$, as the magnetic dual of the Ramond-Ramond four-form potential includes Φ-fluctuations, $P[\tilde{C}_4]\propto\partial_a\phi$ (see eq. (45) in [105]). With the ansatz (3.22), which does not depend on t or x, these couplings vanish. Thus $\chi = 0$, $A = 0$ and the ansatz (3.22) yields (3.23).

For the electric case, the Wess-Zumino action does lead to additional couplings for a nontrivial gauge field background F_0. As for the Minkowski embeddings no gauge background is needed for consistency, the ansatz (3.27) used to calculate the spectrum at zero temperature is consistent. The new couplings, which spoil the above reasoning for the singular shell embeddings, are again due to the magnetic dual of C_4, namely $\int P[\tilde{C}_4]\wedge F_0\wedge F$. As $P[\tilde{C}_4]$ containts $\partial_a\phi\mathrm{d}\xi^a$, this induces the following couplings (schematically): $(\partial_y\phi)(F_{zx}+F_{zt})+(\partial_z\phi)(F_{xy}+F_{ty})$. These couplings vanish only if $\phi(\rho)$ is a function independent of all Minkowski coordinates, i.e. a zero mode in the Minkowski directions, $(\omega, k_1, k_2, k_3) = 0$.

Appendix C

Appendices of Chapter 4

C.1 Expansion of the Determinant in the DBI-Action

For completeness, we write down the most important steps of the expansion of the D7 brane action (4.60) to second order $2\pi\alpha'$ in more detail. In the following discussion we abbreviate[1]

$$E_{ab} = \mathcal{P}_{ab}[g_{\mu\nu} + B_{\mu\nu}]. \tag{C.1}$$

Using the expansion

$$\begin{aligned} &\sqrt{-\det(E_{ab} + 2\pi\alpha' F_{ab})} = \sqrt{-\det E_{ab}\det(\delta_{ab} + 2\pi\alpha'(E^{-1}F)_{ab})} \\ &= \sqrt{-\det E_{ab}} \cdot \Bigg(1 + \frac{2\pi\alpha'}{2}\mathrm{Tr}\, E^{-1}F + \frac{(2\pi\alpha')^2}{8}\left(\mathrm{Tr}\, E^{-1}F\right)^2 \\ &\qquad - \frac{(2\pi\alpha')^2}{4}\mathrm{Tr}\left(E^{-1}F\right)^2 + \dots\Bigg), \end{aligned} \tag{C.2}$$

the action of the D7 brane up to order α'^2 is given by

$$S_{D7} = S^{(0)} + 2\pi\alpha' S^{(1)} + (2\pi\alpha')^2 S^{(2)} + \mathcal{O}(\alpha'^3)\,, \tag{C.3}$$

$$S^{(0)} = -T_7 \int_{D7} dx^\mu dy^m \sqrt{-\det E_{ab}} \pm \frac{T_7}{2}\int_{D7} C_{(4)} \wedge B \wedge B\,, \tag{C.4}$$

[1] In this appendix we suppress the colour traces to avoid confusion between them and traces over euclidean space indices.

$$S^{(1)} = -\frac{T_7}{2}\int\limits_{D7} dx^\mu dy^m \sqrt{-\det E_{ab}} E^{ab} F_{ba} \pm T_7 \int\limits_{D7} C_{(4)} \wedge B \wedge F\,, \tag{C.5}$$

$$S^{(2)} = -T_7 \int\limits_{D7} dx^\mu dy^m \sqrt{-\det E_{ab}} \left[\frac{1}{8}\left(E^{ab}F_{ba}\right)^2 - \frac{1}{4} E^{ab} F_{bc} E^{cd} F_{da}\right] \tag{C.6}$$

$$\pm \frac{T_7}{2} \int\limits_{D7} C_{(4)} \wedge F \wedge F\,. \tag{C.7}$$

The inverse matrix of E_{ab} will be denoted by upper indices, i.e. E^{ab}, with

$$E^{ab}E_{bc} = \delta^a_c. \tag{C.8}$$

Switching on the selfdual, constant B field (4.48), a non-vanishing $U(1)$ field strength in the directions transversal to the D3 brane, and using the massless embedding of the D7 brane, we obtain

$$\begin{aligned}
\sqrt{-\det(E_{ab})} &= \frac{H_3(r) + b^2}{H_3(r)} = 1 + \frac{b^2}{H_3(r)}\,, \\
E^{ab}F_{ba} &= 2\,b\,A\,(F_{45} + F_{67}) = \frac{2b}{H_3(r) + b^2}\left(F_{45} + F_{67}\right)\,, \\
\frac{1}{8}\left(E^{ab}F_{ba}\right)^2 - \frac{1}{4}E^{ab}F_{bc}E^{cd}F_{da} &= \frac{1}{4(H_3(r)+b^2)^2}\left(H_3(r) F_{ab}F_{ab} + \frac{b^2}{2}\epsilon_{klmn}F_{kl}F_{mn}\right)\,, \\
C_{(4)} \wedge B \wedge B &= \frac{2b^2}{H_3(r)} d^4x\, d^4y\,, \\
C_{(4)} \wedge B \wedge F &= \frac{b}{H_3(r)}\left(F_{45} + F_{67}\right) d^4x d^4y\,, \\
C_{(4)} \wedge F \wedge F &= \frac{1}{4\,H_3(r)} \epsilon_{klmn} F_{kl} F_{mn} d^4x d^4y\,,
\end{aligned}$$

where $d^4x\, d^4y = dx^0 \wedge \cdots \wedge dx^3 \wedge dy^4 \wedge \cdots \wedge dy^7$. Using these results, we have for the D7 brane action $S_{D7} = S_{DBI} + S_{WZ}$ up to order α'^2

$$S_{D7} = S^{(0)} + 2\pi\alpha' S^{(1)} + \left(2\pi\alpha'\right)^2 S^{(2)} + \mathcal{O}(\alpha'^3)\,, \tag{C.9}$$

$$S^{(0)} = -T_7 \int dx^\mu\, dy^m\, 1\,, \tag{C.10}$$

$$S^{(1)} = 0\,, \tag{C.11}$$

$$S^{(2)} = -\frac{T_7}{4} \int dx^\mu\, dy^m\, \frac{1}{H_3(y) + b^2}\left(F_{ab}F_{ab} - \frac{1}{2}\epsilon_{klmn}F_{kl}F_{mn}\right) \tag{C.12}$$

$$= -\frac{T_7}{2} \int dx^\mu\, dy^m\, \frac{1}{H_3(y) + b^2} F_- F_-\,, \tag{C.13}$$

which is precisely the action (4.60).

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