Lifting of Nichols Algebras

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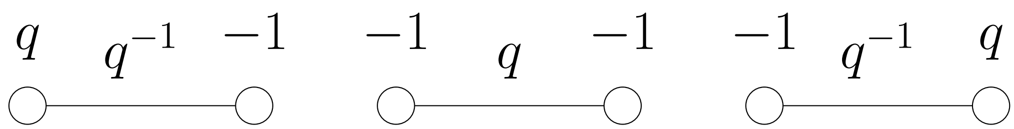

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Abstract

Nichols algebras are a fundamental building block of pointed Hopf algebras. Part of the classification program of finite-dimensional pointed Hopf algebras with the lifting method of Andruskiewitsch and Schneider [6] is the determination of the liftings, i.e., all possible deformations of a given Nichols algebra. The classification was carried out in this way in [11] when the group of group-like elements is abelian and the prime divisors of the order of the group are > 7. In this case the appearing Nichols algebras are of Cartan type.

Based on recent work of Heckenberger about diagonal Nichols algebras [29, 28, 27] we compute explicitly the liftings of some Nichols algebras not treated in [11]; namely we lift

- all Nichols algebras with Cartan matrix of type A_2 (Theorem 6.3.3),
- some Nichols algebras with Cartan matrix of type B_2 (Theorem 6.4.3), and
- some Nichols algebras of two Weyl equivalence classes of non-standard type (Theorem 6.5.3),

giving new classes of finite-dimensional pointed Hopf algebras.

Crucial is the knowledge of a "good" presentation of the Nichols algebra and its liftings: We want to have an explicit description in terms of generators and (non-redundant) relations, and a basis; this requires new ideas and methods that generalize those in [11].

In this spirit, we describe Hopf algebras generated by skew-primitive elements and an abelian group with action given via characters (including Nichols algebras and their liftings) in Theorem 5.4.1. The relations form a Gröbner basis and are given by a combinatorial property involving the theory of Lyndon words.

Furthermore, in Theorem 7.3.1 we give a necessary and sufficient criterion to check whether a given set of iterated q-commutators establishes a restricted PBW basis for a given realization of the relations. Also with the help of this criterion we determine the redundant relations in the examined Nichols algebras and their liftings.

Zusammenfassung

Nicholsalgebren sind ein fundamentaler Baustein punktierter Hopfalgebren. Teil des Klassifizierungprogramms endlich-dimensionaler punktierter Hopfalgebren mit der Lifting Methode von Andruskiewitsch und Schneider [6] ist die Bestimmung der Liftings, d.h. aller möglichen Deformationen einer gegebenen Nicholsalgebra. Die Klassifizierung wurde mit dieser Methode in [11] durchgeführt, falls die Gruppenelemente eine abelsche Gruppe bilden und die Primteiler der Gruppenordnung > 7 sind. Die dort auftretenden Nicholsalgebren sind vom Cartan-Typ.

Basierend auf neueren Arbeiten von Heckenberger über diagonale Nicholsalgebren [29, 28, 27] bestimmen wir explizit die Liftings einiger Nicholsalgebren, welche nicht in [11] behandelt wurden: Wir liften

- alle Nicholsalgebren mit Cartan Matrix vom Typ A_2 (Theorem 6.3.3),
- einige Nicholsalgebren mit Cartan Matrix vom Typ B_2 (Theorem 6.4.3) und
- einige Nicholsalgebren aus zwei Weyl-Äquivalenzklassen vom nicht-standard-Typ (Theorem 6.5.3).

Dies liefert neue Klassen von endlich-dimensionalen punktierten Hopfalgebren.

Es ist entscheidend eine "gute" Beschreibung der Nicholsalgebra und ihrer Liftings zu besitzen: wir wollen eine explizite Angabe von Erzeugern und (nicht redundanten) Relationen, desweiteren eine Basis; dazu braucht man neue Ideen und Methoden, die jene in [11] verallgemeinern.

In diesem Sinne beschreiben wir Hopfalgebren, die von schief-primitiven Elementen und einer abelschen Gruppe mit einer durch Charaktere gegebenen Wirkung erzeugt sind (diese Klasse beinhaltet Nicholsalgebren und ihre Liftings), in Theorem 5.4.1. Die Relationen bilden eine Gröbnerbasis und sind durch eine kombinatorische Eigenschaft gegeben, für deren Formulierung die Theorie der Lyndonwörter eingeht.

Desweiteren liefern wir mit Theorem 7.3.1 ein notwendiges und hinreichendes Kriterium, ob eine gegebene Menge von iterierten q-Kommutatoren eine PBW-Basis für eine gegebene Realisierung der Relationen bildet. Ebenfalls bestimmen wir mit Hilfe dieses Kriteriums die nicht benötigten Relationen in den untersuchten Nicholsalgebren und deren Liftings.

Introduction

Hopf algebras and quantum groups. *Hopf algebras* are named in honor of Heinz Hopf, who used this algebraic structure in 1941 [33] to solve a problem in the cohomology theory of group manifolds; see also [5]. The first book on Hopf algebras [51] was published in 1969 and in spite of many interesting results, there were only few people studying this field. The interest in Hopf algebras grew dramatically when Drinfel'd [20, 21] and Jimbo [35] introduced the so-called *quantum groups* $U_q(\mathfrak{g})$ in the 80s. These were a totally new class of non-commutative and non-cocommutative Hopf algebras coming from q-deformations of universal enveloping algebras $U(\mathfrak{g})$ of semi-simple complex Lie algebras $\mathfrak{g}$. Later Lusztig [40, 41] found another important class of finite-dimensional Hopf algebras, the so-called *Frobenius-Lusztig kernels* $u_q(\mathfrak{g})$, also called *small quantum groups*. Further quantum groups showed to have connections to knot theory, quantum field theory, non-commutative geometry and representation theory of algebraic groups in characteristic $p > 0$, only to name a few.

Classification of Hopf algebras. Finite-dimensional Hopf algebras give rise to finite tensor categories in the sense of [22] and thus classification results of these should have applications in conformal field theory [23]. Not only for this reason it is of great interest to classify Hopf algebras. Although there are some results (see [1] for a discussion of what is known on classification of finite-dimensional Hopf algebras), an answer to this question in general may be impossible. Therefore one needs to restrict to a subclass of finite-dimensional Hopf algebras: At the moment the most promising general method is the *lifting method* developed by Andruskiewitsch and Schneider [6] for the classification of *pointed Hopf algebras*.

Pointed Hopf algebras. A Hopf algebra is called *pointed*, if all its simple subcoalgebras are one-dimensional, or equivalently the coradical equals the group algebra of the group of group-like elements; see Section 1.3.

Any Hopf algebra generated as an algebra by group-like and skew-primitive elements is pointed. In particular the above mentioned quantum groups: The cocommutative universal enveloping algebras $U(\mathfrak{g})$ and their non-cocommutative deformations $U_q(\mathfrak{g})$ and $u_q(\mathfrak{g})$ are all pointed [34, 42].

The converse statement is the following conjecture of Andruskiewitsch and Schneider, which is proven for a large class in [11], see also [7, 8, 9, 10]:

Conjecture 0.0.1. [8] *Any finite-dimensional pointed Hopf algebra over the complex numbers is generated by group-like and skew-primitive elements.*

We want to mention that this is long known to be true for cocommutative Hopf algebras, which are pointed if the ground field is algebraically closed: The Cartier-Kostant-Milnor-Moore theorem of around 1963 states that any cocommutative Hopf algebra over the complex numbers is a semi-direct product of a universal enveloping algebra and a group algebra. Further we want to mention the classification results on pointed Hopf algebras of rank one by Krop and Radford [37] in characteristic zero and by Scherotzke [50] in positive characteristic. Finally, the conjecture is false if the ground field has positive characterisitc or the Hopf algebra is infinite-dimensional; see [10, Examples 2.5, 2.6].

The lifting method. Given a finite-dimensional pointed Hopf algebra A with coradical $A_0 = \Bbbk[\Gamma]$ and (abelian) group of group-like elements $\Gamma = G(A)$. Then we can decompose its associated graded Hopf algebra into a smash product $\mathrm{gr}(A) \cong B\#\Bbbk[\Gamma]$ where B is a braided Hopf algebra; see Section 2.6. The subalgebra of B generated by its primitive elements $V := P(B)$ is a *Nichols algebra* $\mathfrak{B}(V)$, see Section 2.7. Now the classification is carried out in three steps:

(1) Show that $B = \mathfrak{B}(V)$ (this is equivalent to Conjecture 0.0.1).

(2) Determine the structure of $\mathfrak{B}(V)$.

(3) *Lifting*: Determine the *liftings* of $\mathfrak{B}(V)$, i.e., all Hopf algebras A such that $\mathrm{gr}(A) \cong \mathfrak{B}(V)\#\Bbbk[\Gamma]$.

Let us mention briefly some classification results for pointed Hopf algebras of dimension p^n with an odd prime p and $1 \leq n \leq 5$, obtained in this way: If the dimension is p or p^2, then the Hopf algebra is a group algebra or a Taft Hopf algebra. The cases of dimension p^3 and p^4 were treated in [6] and [8], and the classification of dimension p^5 follows from [7] and [24]. Also, the lifting method was used in [25] to classify pointed Hopf algebras of dimension $2^5 = 32$.

The most impressive result obtained by this method by Andruskiewitsch und Schneider [11] is the classification of all finite-dimensional pointed Hopf algebras where the prime divisors of the order of the abelian group Γ are > 7. In this case the diagonal braiding of V is of Cartan type and the Hopf algebras are generalized versions of small quantum groups. The classification when the braiding is not of Cartan type or the divisors of the order of Γ are ≤ 7 is still an open problem. Also the case where Γ is not abelian is widely open and of different nature, e.g., the defining relations have another form [32, 4].

Concerning (2), Heckenberger recently showed that Nichols algebras of diagonal type have a close connection to semi-simple Lie algebras, namely he introduces a *Weyl groupoid* [28], *Weyl equivalence* [27] and an *arithmetic root system* [30, 26] for Nichols algebras. With the help of these concepts he classifies the diagonal braidings of V such that the Nichols algebra $\mathfrak{B}(V)$ has a finite set of PBW generators [31]. Moreover, he determines the structure of all rank two Nichols algebras in terms of generators and relations [29].

This is the starting point of our work which addresses to step (3) of the program, namely the *lifting* in the cases not treated in [11].

The main results and organization of this thesis

In Chapters 1 and 2 the basic notions of Hopf algebrs and Nichols algebras are recalled, taking into account the recent developement of Nichols algebras. Then in Chapter 3 we develop a general calculus for q-commutators in an arbitrary algebra, which is needed throughout the thesis; new formulas for q-commutators are found in Proposition 3.2.3.

We recall in Chapter 4 the theory of Lyndon words, super letters and super words; super letters are iterated q-commutators and super words are products of super letters. We show that the set of all super words can be seen indeed as a set of words, i.e., as a free monoid. This is a consequence of Proposition 4.3.2.

In Chapter 5 we give in Theorem 5.4.1 a structural description of Hopf algebras generated by skew-primitive elements and an abelian group with action given via characters, in terms of generators and relations, with the help of a result by Kharchenko [36]; he calls these Hopf algebras *character Hopf algebras*. As we will see, these relations build up a Gröbner basis for such Hopf algebras.

Based on the previous chapters we then formulate the two main results of this thesis:

Lifting of Nichols algebras, Chapter 6. We generalize the methods of Andruskiewitsch and Schneider to compute explicitly the liftings of all Nichols algebras with Cartan matrix of type A_2, some with Cartan matrix of type B_2 and some with Cartan matrix of non-standard type; see Theorems 6.3.3, 6.4.3 and 6.5.3. These are a new class of finite-dimensional pointed Hopf algebras. We explain our method in Section 6.1.

When lifting arbitrary diagonal Nichols algebras, new phenomena occur: In the setting of [11] there are only three types of defining relations, namely the Serre relations, the linking relations and the root vector relations. The algebraic structure in the general setting is more complicated: Firstly, the Serre relations do not play the outstanding role. Other relations are needed and sometimes the Serre relations are redundant; we give a complete answer for the Serre relations in Lemma 6.1.3. Secondly, in general the lifted relations from the Nichols algebra do not remain in the group algebra.

By Theorem 5.4.1 we know the structure of the defining relations. As it turned out, it is enough to find a counterterm such that the relation is a skew-primitive element, see Section 6.1. In order to show this, one needs to calculate certain coproducts; for this, new methods are found in Section 5.5.

Part of the lifting is the knowledge of the dimension resp. a basis and to find the redundant relations. The here obtained liftings could not be treated by existing basis criterions like [11, Sect. 4]. For this reason we develop in Chapter 7 a PBW basis criterion which is applicable for all character Hopf algebras, i.e., generated by skew-primitive elements and an abelian group with action given via characters (in particular liftings of Nichols algebras), see Theorem 7.3.1.

A PBW basis criterion for a class of pointed Hopf algebras, Chapter 7. In the famous Poincaré-Birkhoff-Witt theorem for universal enveloping algebras of finite-dimensional Lie algebras a class of new bases appeared. Since then many PBW theorems for more general situations were discovered. We want to name those for quantum groups:

Lusztig's axiomatic approach [39, 42] and Ringel's approach via Hall algebras [48]. Let us also mention the work of Berger [15], Rosso [49], and Yamane [54].

The very general and for us important work is [36], where a PBW theorem for all of the above mentioned quantum groups and also Nichols algebras and their liftings is formulated: Kharchenko shows in [36, Thm. 2] that character Hopf algebras have a PBW basis in special q-commutators, namely the *hard super letters* coming from the theory of Lyndon words, see Chapter 5. Thereby we use the term *PBW basis* in the sense of Definition 5.1.1. However, the definition of *hard* is not constructive (see also [18, 17] for the word problem for Lie algebras) and in view of treating concrete examples there is a lack of deciding whether a given set of iterated q-commutators establishes a PBW basis resp. is the set of hard super letters in the language of [36].

On the other hand the diamond lemma [16] (see also Section 7.6, Theorem 7.6.1) is a very general method to check whether an associative algebra given in terms of generators and relations has a certain basis, or equivalently the relations form a Gröbner basis. As mentioned before, we construct such a Gröbner basis for a character Hopf algebra in Theorem 5.4.1 and give a necessary and sufficient criterion for a set of super letters being a PBW basis, see Theorem 7.3.1. The PBW Criterion 7.3.1 is formulated in the languague of q-commutators. This seems to be the natural setting, since the criterion involves only q-commutator identities of Proposition 3.2.3; as a side effect we find redundant relations.

The main idea is to combine the diamond lemma with the combinatorial theory of Lyndon words resp. super letters and the q-commutator calculus of Chapter 3. In order to apply the diamond lemma we give a general construction to identify a smash product with a quotient of a free algebra, see Proposition 7.4.5 in Section 7.4 (this presentation fits perfectly for the implementation in computer algebra programs, see Appendix A).

Further the PBW Criterion 7.3.1 is a generalization of [15] and [11, Sect. 4] in the following sense: In [15] a condition involving the q-Jacoby identity for the generators x_i occurs (it is called "q-Jacobi sum"). However, this condition can be formulated more generally for iterated q-commutators (not only for x_i), so also higher than quadratic relations can be considered. The intention of [15] was a q-generalization of the classical PBW theorem, so powers of q-commutators are not covered at all and also his algebras do not contain a group algebra.

On the other hand, [11, Sect. 4] deals with powers of q-commutators (root vector relations) and also involves the group algebra. But here it is assumed that the powers of the commutators lie in the group algebra and fulfill a certain centrality condition. As mentioned above these assumptions are in general not preserved; in the PBW Criterion 7.3.1 the centrality condition is replaced by a more general condition involving the restricted q-Leibniz formula of Proposition 3.2.3.

Finally in Chapters 8 and 9 we apply the PBW Criterion 7.3.1 to classical examples and the obtained liftings. In this way we find PBW bases and the redundant relations. In the Appendix we give an example of the program code for the computer algebra system FELIX [13].

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Chapter 1

Hopf algebras

In this chapter we recall the definitions of the structures which we study in our work. It is meant for fixing the notations we use. For an introduction see for example [43, 51].

Throughout the thesis let $\Bbbk$ be a field of $\operatorname{char}\Bbbk = p \geq 0$, although much of what we do is valid over any commutative ring. We denote the multiplicative order of any $q \in \Bbbk^\times$ by $\operatorname{ord} q$. All tensor products are assumed to be over $\Bbbk$.

1.1 Coalgebras

A *coalgebra* is the dual version of an associative and unital algebra, namely a vector space C together with two $\Bbbk$-linear maps $\Delta : C \to C \otimes C$ (*comultiplication*) and $\varepsilon : C \to \Bbbk$ (*counit*) that satisfy

$$\begin{aligned} (\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta && (\textit{coassociativity}),\\ (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} &= (\mathrm{id} \otimes \varepsilon)\Delta && (\textit{counitality}). \end{aligned}$$

A morphism $\phi : C \to D$ of coalgebras is a $\Bbbk$-linear map such that

$$\Delta_D \phi = (\phi \otimes \phi)\Delta_C \quad \text{and} \quad \varepsilon_D \phi = \varepsilon_C.$$

For calculations we use the following version of the Heyneman-Sweedler notation: For $c \in C$ we write

$$\Delta(c) = c_{(1)} \otimes c_{(2)},$$

keeping in mind that the right-hand side is in general a sum of simple tensors. Thus the coassociativity and counitality read

$$\begin{aligned} c_{(1)} \otimes c_{(2)} \otimes c_{(3)} : &= (c_{(1)(1)} \otimes c_{(1)(2)}) \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)(1)} \otimes c_{(2)(2)}),\\ \varepsilon(c_{(1)})c_{(2)} &= c = c_{(1)}\varepsilon(c_{(2)}). \end{aligned}$$

A morphism ϕ then has to fulfill

$$\Delta(\phi(c)) = \phi(c_{(1)}) \otimes \phi(c_{(2)}) \quad \text{and} \quad \varepsilon(\phi(c)) = \varepsilon(c).$$

The set

$$G(C) := \{g \in C \mid \Delta(g) = g \otimes g,\ \varepsilon(g) = 1\}$$

is called the set of *group-like* elements; this is a linearly independent set. For two $g, h \in G(C)$ the set

$$P_{g,h}(C) := \{x \in C \mid \Delta(x) = x \otimes g + h \otimes x\}$$

is called the space of g, h-*skew primitive* elements; this is a subspace.

A coalgebra is called *simple*, if it has no nontrivial subcoalgebras. It is said to be *pointed*, if every simple subcoalgebra is one-dimensional, i.e., spanned by some $g \in G(C)$. The *coradical* is the sum of all simple subcoalgebras, and it is denoted by C_0.

1.2 Comodules

We also want to give the dual version of a module over an algebra, namely M is called a (left) *comodule* over the coalgebra C, if there is a $\Bbbk$-linear map $\delta : M \to C \otimes M$ (*coaction*) that satisfies

$$(\Delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \delta)\delta \qquad (\textit{coassociativity}),$$
$$(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id} \qquad (\textit{counitality}).$$

A morphism $f : M \to N$ of comodules is a $\Bbbk$-linear map such that

$$\delta_N f = (\mathrm{id} \otimes f)\delta_M \qquad (\textit{colinearity}).$$

For the coaction we use a version of the Heyneman-Sweedler notation

$$\delta(m) = m_{(-1)} \otimes m_{(0)}.$$

The coassociativity and counitality then read

$$m_{(-2)} \otimes m_{(-1)} \otimes m_{(0)} := m_{(-1)(1)} \otimes m_{(-1)(2)} \otimes m_{(0)} = m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)},$$
$$\varepsilon(m_{(-1)})m_{(0)} = m,$$

and a colinear map fulfills

$$\delta(f(m)) = m_{(-1)} \otimes f(m_{(0)}).$$

1.3 Bialgebras and Hopf algebras

Let (C, Δ, ε) be a coalgebra and (A, μ, η) be an algebra, where $\mu : A \otimes A \to A$ is the multiplication map and $\eta : \Bbbk \to A$ the unit map. The space $\mathrm{Hom}_\Bbbk(C, A)$ becomes an algebra with the *convolution product*

$$f \star g := \mu(f \otimes g)\Delta$$

for all $f, g \in \mathrm{Hom}_\Bbbk(C, A)$, and unit $\eta\varepsilon$. The tensor product gives a monoidal structure for algebras and coalgebras: $A \otimes A$ resp. $C \otimes C$ is again an algebra resp. a coalgebra by

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb', \quad \Delta(c \otimes d) := \big(c_{(1)} \otimes d_{(1)}\big) \otimes \big(c_{(2)} \otimes d_{(2)}\big),$$

for all $a, a', b, b' \in A$ and $c, d \in C$. Thus we can define the following: A *bialgebra* is a collection $(H, \mu, \eta, \Delta, \varepsilon)$, where

- (H, μ, η) is an algebra,
- (H, Δ, ε) is a coalgebra,
- $\Delta : H \to H \otimes H$ and ε are algebra maps.

A bialgebra H is called a *Hopf algebra*, if $\mathrm{id} \in \mathrm{End}_{\Bbbk}(H)$ is convolution invertible, i.e., there is a $S \in \mathrm{End}_{\Bbbk}(H)$ (*antipode*) with $\mathrm{id} \star S = \eta\varepsilon = S \star \mathrm{id}$, in Heyneman-Sweedler notation

$$h_{(1)}S(h_{(2)}) = \varepsilon(h)1 = S(h_{(1)})h_{(2)}.$$

A morphism $\phi : H \to K$ of bialgebras is a morphism of algebras and coalgebras. If the antipodes exist then $S_K\phi = \phi S_H$.

In a bialgebra H, we have $1 \in G(H)$. The elements of $P(H) := P_{1,1}(H)$ are called *primitive* elements. We call a bialgebra *pointed*, if it has the property as a coalgebra.

1.4 The smash product

Let A be an algebra, H a bialgebra and $\cdot : H \otimes A \to H$, $h \otimes a \mapsto h \cdot a$ a $\Bbbk$-linear map. One says that A is a *(left) H-module algebra* if

- $(A, \cdot)$ is a left H-module,
- $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,
- $h \cdot 1_A = \varepsilon(h)1_A$,

for all $h \in H$, $a, b \in A$.

Let A be a left H-module algebra. We define the *smash product algebra*

$$A\#H := A \otimes H$$

as $\Bbbk$-spaces with multiplication defined for $a, b \in A, g, h \in H$ by

$$(a\#g)(b\#h) := a(g_{(1)} \cdot b)\#g_{(2)}h.$$

$A\#H$ is indeed an associative algebra with identity element $1_A\#1_H$. Further $H \cong 1_A\#H$ and $A \cong A\#1_H$, so we may just write ah instead of $a\#h$. In this notation

$$ha = (h_{(1)} \cdot a)h_{(2)}. \tag{1.1}$$

1.5 Yetter-Drinfel'd modules

Let H be a Hopf algebra. A (left-left) *Yetter-Drinfel'd module* V over H is a left H-module and H-comodule with action $\cdot$ and coaction δ fulfilling the *compatibility condition* for all $h \in H$ and $v \in V$

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v_{(0)}.$$

We denote the category of Yetter-Drinfel'd modules with linear and colinear maps as morphisms by ${}^H_H\mathcal{YD}$. There is again a monoidal structure: $V \otimes W \in {}^H_H\mathcal{YD}$ for $V, W \in {}^H_H\mathcal{YD}$ by

$$h \cdot (v \otimes w) := (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w), \quad \delta(v \otimes w) := v_{(-1)} w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}$$

for all $h \in H$, $v \in V$ and $w \in W$. For any $V, W \in {}^H_H\mathcal{YD}$ we define the *braiding*

$$c_{V,W} : V \otimes W \to W \otimes V, \quad c(v \otimes w) := (v_{(-1)} \cdot w) \otimes v_{(0)}$$

which turns (V, c) with $c := c_{V,V}$ into a braided vector space, i.e., V is a vector space and $c \in \mathrm{Aut}_{\Bbbk}(V \otimes V)$ satisfies the *braid equation*

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

We are mainly concerned with the case when $H = \Bbbk[\Gamma]$ is the group algebra with abelian Γ, see Section 2.1.

Chapter 2

Nichols algebras

Braided Hopf algebras, especially Nichols algebras play an important role in the structure theory of pointed Hopf algebras, as we mentioned in the introduction. See also Section 2.7 and [7, 10]. Nichols algebras were introduced in [44]. They can be seen as generalizations of the symmetric algebra of a vector space, where the flip map of the tensor product is replaced by a braiding.

We want to define braided Hopf algebras and Nichols algebras in the context of a braided category, namely in the category ${}^H_H\mathcal{YD}$ in the special case $H = \Bbbk[\Gamma]$. One can give also a definition in a non-categorical way, which sometimes provides additional information [3]. For general results about braided Hopf algebras we want to refer to [52]. However, there are many open problems, especially in the theory of Nichols algebras. Recent results of Heckenberger connecting Nichols algebras with the theory of semi-simple Lie algebras are found in [28, 31].

Our main reference is the survey article [10, Sect. 1,2]. In this chapter let Γ be again an abelian group, but not necessarily finite.

2.1 Yetter-Drinfel'd modules of diagonal type

For a group Γ we denote by $\widehat{\Gamma}$ the character group of all group homomorphisms from Γ to the multiplicative group $\Bbbk^\times$. At first we want to recall the notion of a Yetter-Drinfel'd module over an abelian group Γ, the special case of ${}^H_H\mathcal{YD}$ in Section 1.5 with $H = \Bbbk[\Gamma]$:

The category ${}^\Gamma_\Gamma\mathcal{YD}$ of (left-left) Yetter-Drinfel'd modules over the Hopf algebra $\Bbbk[\Gamma]$ is the category of left $\Bbbk[\Gamma]$-modules which are Γ-graded vector spaces $V = \bigoplus_{g\in\Gamma} V_g$ such that each V_g is stable under the action of Γ, i.e.,

$$h \cdot v \in V_g \quad \text{for all} \quad h \in \Gamma, v \in V_g.$$

The Γ-grading is equivalent to a left $\Bbbk[\Gamma]$-comodule structure $\delta : V \to \Bbbk[\Gamma] \otimes V$: One can define δ or the other way round V_g by the equivalence $\delta(v) = g \otimes v \iff v \in V_g$ for all $g \in \Gamma$. The morphisms of ${}^\Gamma_\Gamma\mathcal{YD}$ are the Γ-linear maps $f : V \to W$ with $f(V_g) \subset W_g$ for all $g \in \Gamma$.

We consider the following monoidal structure on ${}^{\Gamma}_{\Gamma}\mathcal{YD}$: If $V, W \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$, then also $V \otimes W \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ by

$$g \cdot (v \otimes w) := (g \cdot v) \otimes (g \cdot w) \quad \text{and} \quad (V \otimes W)_g := \bigoplus_{hk=g} V_h \otimes W_k$$

for $v \in V, w \in W$ and $g \in \Gamma$. The *braiding* in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is the isomorphism

$$c = c_{V,W} : V \otimes W \to W \otimes V, \quad c(v \otimes w) := (g \cdot w) \otimes v$$

for all $v \in V_g$, $g \in \Gamma$, $w \in W$. Thus every $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a braided vector space $(V, c_{V,V})$.

We have the following important example:

Definition 2.1.1. Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. If there is a basis x_i, $i \in I$, of V and $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ for all $i \in I$ such that

$$g \cdot x_i = \chi_i(g) x_i \quad \text{and} \quad x_i \in V_{g_i},$$

then we say V is of *diagonal type*.

Remark 2.1.2.

1. Let V be a vector space with basis $x_1, \dots, x_\theta$, and let $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ for all $1 \le i \le \theta$. Then $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ (of diagonal type) by setting

$$g \cdot x_i := \chi_i(g) x_i \quad \text{and} \quad x_i \in V_{g_i}.$$

2. If $\Bbbk$ is algebraically closed of characteristic 0 and Γ is finite, then all finite-dimensional $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ are of diagonal type.

3. For the braiding we have $c(x_i \otimes x_j) = \chi_j(g_i) x_j \otimes x_i$ for $1 \le i, j \le \theta$. Hence the braiding is determined by the matrix

$$(q_{ij})_{1 \le i,j \le \theta} := (\chi_j(g_i))_{1 \le i,j \le \theta}$$

called the *braiding matrix* of V.

2.2 Braided Hopf algebras

A collection (B, μ, η) is called an *algebra in* ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, if

- (B, μ, η) is an algebra,
- $B \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$,
- μ and η are morphisms of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, i.e., Γ-linear and Γ-colinear.

The tensor product in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ further allows to define the following: If B is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, then also $B \underline{\otimes} B := B \otimes B \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ by defining $\Bbbk$-linearly

$$(a \otimes b)(a' \otimes b') := a(g \cdot a') \otimes bb', \quad \text{for all } a, a', b, b' \in B, b \in B_g, g \in \Gamma.$$

Exactly in the same manner a collection (B, Δ, ε) is called a *coalgebra in* ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, if

- (B, Δ, ε) is a coalgebra,
- $B \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$,
- Δ and ε are morphisms of ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

A *braided bialgebra in* ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a collection $(B, \mu, \eta, \Delta, \varepsilon, S)$, where

- (B, μ, η) is an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$,
- (B, Δ, ε) is a coalgebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$,
- $\Delta : B \to B\underline{\otimes}B$ and ε are algebra maps.

If further there is an $S \in \mathrm{End}_{\Bbbk}(B)$ with $\mu(\mathrm{id} \otimes S)\Delta = \eta\varepsilon = \mu(S \otimes \mathrm{id})\Delta$, then B is called a *braided Hopf algebra in* ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

If the *antipode* S exists then it is a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ [52]. A morphism $\phi : B \to B'$ of braided bialgebras in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a morphism of algebras and coalgebras and also a morphism in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ (Γ-linear and Γ-colinear). A braided Hopf algebra B in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is called *graded*, if there is a grading $B = \oplus_{n\geq 0}B(n)$ of Yetter-Drinfel'd modules which is a grading of algebras and coalgebras.

Note that braided bialgebras are generalizations of bialgebras: the basic idea is to replace the usual flip map $\tau : V \otimes V \to V \otimes V$, $\tau(v \otimes w) = w \otimes v$ with the braiding c in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Example 2.2.1. Let V be a vector space with basis X. Then the tensor algebra $T(V) \cong \Bbbk\langle X\rangle$ is a graded braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ with structure determined by

$$\begin{aligned} g \cdot u &:= \chi_u(g)u, \quad u \in V_{g_u}, && \text{for all } g \in \Gamma, u \in \langle X\rangle, \\ \Delta(x_i) &:= x_i \otimes 1 + 1 \otimes x_i && \text{for all } 1 \leq i \leq \theta. \end{aligned}$$

It is $\mathbb{N}$-graded by the length of a word $u \in \langle X\rangle$.

2.3 Nichols algebras

Let $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$. B is called a *Nichols algebra* of V, if

- $B = \oplus_{n\geq 0}B(n)$ is a graded braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$,
- $B(0) \cong \Bbbk$,
- $P(B) = B(1) \cong V$,
- B is generated as an algebra by $B(1)$.

Any two Nichols algebras of V are isomorphic, thus we write $\mathfrak{B}(V)$ for "the" Nichols algebra of V. One can construct the Nichols algebra in the following way: Let I denote the sum of all ideals of $T(V)$ that are generated by homogeneous elements of degree ≥ 2 and that are also coideals. Then $\mathfrak{B}(V) \cong T(V)/I$.

2.4 Cartan matrices

A matrix $(a_{ij})_{1\le i,j\le\theta} \in \mathbb{Z}^{\theta\times\theta}$ is called a *generalized Cartan matrix* if for all $1 \le i, j \le \theta$

- $a_{ii} = 2$,
- $a_{ij} \le 0$ if $i \ne j$,
- $a_{ij} = 0 \Rightarrow a_{ji} = 0$.

Let $\mathfrak{B}(V)$ be a Nichols algebra of *diagonal type*, i.e., V is of diagonal type. Recall that V resp. $\mathfrak{B}(V)$ with braiding matrix (q_{ij}) is called of *Cartan type*, if there is a generalized Cartan matrix (a_{ij}) such that

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}.$$

Not every Nichols algebra is of Cartan type (see Sections 6.3, 6.4, 6.5), but still we have the following:

Lemma and Definition 2.4.1. If $\mathfrak{B}(V)$ is finite-dimensional, then the matrix (a_{ij}) defined for all $1 \le i \ne j \le \theta$ by

$$a_{ii} := 2 \quad \text{and} \quad a_{ij} := -\min\{r \in \mathbb{N} \mid q_{ij}q_{ji}q_{ii}^r = 1 \text{ or } (r+1)_{q_{ii}} = 0\}$$

is a generalized Cartan matrix fulfilling

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}} \quad \text{or} \quad \mathrm{ord} q_{ii} = 1 - a_{ij}.$$

We call (a_{ij}) the *Cartan matrix associated to* $\mathfrak{B}(V)$.

Proof. See [28, Sect. 3]: We prove this more generally in the situation when the set $\{r \in \mathbb{N} \mid [x_i^r x_j] \ne 0 \text{ in } \mathfrak{B}(V)\}$ is finite for all $1 \le i \ne j \le \theta$. It is well-known that if $1 \le i \ne j \le \theta$ and $r \ge 1$, then in $\mathfrak{B}(V)$

$$[x_i^r x_j] = 0 \quad \Longleftrightarrow \quad (r)!_{q_{ii}} \prod_{0\le k\le r-1} (1 - q_{ij}q_{ji}q_{ii}^k) = 0.$$

Thus the matrix (a_{ij}) is well-defined and it is indeed a generalized Cartan matrix. $\square$

2.5 Weyl equivalence

Heckenberger introduced in [27, 28, Sect. 2] the notion of the *Weyl groupoid* and *Weyl equivalence* of Nichols algebras of diagonal type. With the help of these concepts Heckenberger classified in a series of articles [30, 26, 31] all braiding matrices (q_{ij}) of diagonal Nichols algebras with a finite set of PBW generators. We are mainly concerned with the list of rank 2 Nichols algebras given in Table 2.1 from [27, 30, Figure 1], see below.

We want to recall the following: For diagonal $\mathfrak{B}(V)$ with braiding matrix (q_{ij}) we associate a *generalized Dynkin diagram*: this is a graph with θ vertices, where the i-th vertex is labeled with q_{ii} for all $1 \le i \le \theta$; further, if $q_{ij}q_{ji} \ne 1$, then there is an edge between the i-th and j-th vertex labeled with $q_{ij}q_{ji}$: Thus, if $q_{ij}q_{ji} = 1$ resp. $q_{ij}q_{ji} \ne 1$, then we have

$$\cdots \overset{q_{ii}}{\circ} \qquad \overset{q_{jj}}{\circ} \cdots \qquad \text{resp.} \qquad \cdots \overset{q_{ii}}{\circ}\!\!\overset{q_{ij}q_{ji}}{\frac{\qquad\qquad}{}}\!\!\overset{q_{jj}}{\circ} \cdots$$

So two Nichols algebras of the same rank θ with braiding matrix (q_{ij}) resp. (q'_{ij}) have the same generalized Dynkin diagram if and only if they are *twist equivalent* [10, Def. 3.8], i.e., for all $1 \le i, j \le \theta$

$$q_{ii} = q'_{ii} \quad \text{and} \quad q_{ij}q_{ji} = q'_{ij}q'_{ji}.$$

Definition 2.5.1. Let $1 \le k \le \theta$ be fixed and $\mathfrak{B}(V)$ finite-dimensional with braiding matrix (q_{ij}) and Cartan matrix (a_{ij}). We call $(q_{ij}^{(k)})$ defined by

$$q_{ij}^{(k)} := q_{ij}q_{ik}^{-a_{kj}}q_{kj}^{-a_{ki}}q_{kk}^{a_{ki}a_{kj}}$$

the *at the vertex k reflected braiding matrix.*

We introduce for $i \neq j$

$$p_{ij} := \begin{cases} 1, & \text{if } q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \\ q_{ii}^{-1}q_{ij}q_{ji}, & \text{if } \mathrm{ord} q_{ii} = 1 - a_{ij}. \end{cases}$$

Then by the definition of (a_{ij}) in Remark 2.4.1 for $1 \le i, j \le \theta$

$$\begin{aligned} q_{kk}^{(k)} &= q_{kk}, \quad q_{kj}^{(k)} = q_{kj}^{-1}q_{kk}^{a_{kj}}, \quad q_{ik}^{(k)} = q_{ik}^{-1}q_{kk}^{a_{ki}}, \\ q_{ii}^{(k)} &= p_{ki}^{-a_{ki}}q_{ii} = \begin{cases} q_{ii}, & \text{if } q_{ki}q_{ik} = q_{kk}^{a_{ki}}, \\ q_{ii}(q_{ik}q_{ki})^{-a_{ki}}q_{kk}, & \text{if } \mathrm{ord} q_{kk} = 1 - a_{ki}. \end{cases} \end{aligned}$$

Concerning Dynkin diagrams it is usefull to know the following products for all $1 \le i, j \le \theta$, $i, j \neq k$:

$$q_{ki}^{(k)}q_{ik}^{(k)} = p_{ki}^{-2}q_{ki}q_{ik}, \quad q_{ij}^{(k)}q_{ji}^{(k)} = p_{ki}^{-a_{kj}}p_{kj}^{-a_{ki}}q_{ij}q_{ji}.$$

Definition 2.5.2. Two Nichols algebras with braiding matrix (q_{ij}) resp. (q'_{ij}) are called *Weyl equivalent*, if there are $m \ge 1$, $1 \le k_1, \ldots, k_m \le \theta$ such that the generalized Dynkin diagrams w.r.t. the matrices $\big((\ldots(q_{ij}^{(k_1)})^{(k_2)}\ldots)^{(k_m)}\big)$ and (q'_{ij}) coincide, i.e., one gets the Dynkin diagram of (q'_{ij}) by successive reflections of (q_{ij}).

Example 2.5.3. The braiding matrix $(q_{ij}) := \begin{pmatrix} q & 1 \\ q^{-1} & -1 \end{pmatrix}$ with $q \neq 1$ has the generalized Dynkin diagram $\overset{q}{\circ}\!\!\overset{q^{-1}}{\frac{\qquad}{}}\!\!\overset{-1}{\circ}$ and associated Cartan matrix $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ of type A_2, since $q_{12}q_{21} = q_{11}^{-1}$ and $\mathrm{ord} q_{22} = 1 - (-1)$. Then the at the vertex 2 reflected braiding matrix is $(q'_{ij}) := (q_{ij}^{(2)}) = \begin{pmatrix} -1 & -1 \\ -q & -1 \end{pmatrix}$, since

$$\begin{aligned} q_{11}^{(2)} &= q_{11}(q_{12}q_{21})^{-a_{21}}q_{22} = -1, & q_{12}^{(2)} &= q_{12}^{-1}q_{22}^{a_{21}} = -1, \\ q_{21}^{(2)} &= q_{21}^{-1}q_{22}^{a_{21}} = -q, & q_{22}^{(2)} &= q_{22} = -1. \end{aligned}$$

Its Dynkin diagram is $\overset{-1}{\circ}\!\!\overset{q}{\frac{\qquad}{}}\!\!\overset{-1}{\circ}$ and the associated Cartan matrix is also of type A_2. The two braiding matrices (q_{ij}) and (q'_{ij}) are by definition Weyl equivalent; they are twist equivalent if and only if $q = -1$. See also Table 2.1 row 3 if $q \neq \pm 1$ and row 2 if $q = -1$.

Remark 2.5.4.

1. Both twist equivalence and Weyl equivalence are equivalence relations, and twist equivalent Nichols algebras are Weyl equivalent.

2. Weyl equivalent Nichols algebras have the same dimension and Gel'fand-Kirillov dimension [27, Prop. 1], but can have different associated Cartan matrices. If the whole Weyl equivalence class has the same Cartan matrix, then the Nichols algebras of this class are called of *standard* type [2, 12].

Examples 2.5.5. Let $\mathfrak{B}(V)$ be of rank 2. Then two Nichols algebras are Weyl equivalent if and only if their generalized Dynkin diagrams appear in the same row of Table 2.1 and can be presented with the same set of fixed parameters [27].

1. $\mathfrak{B}(V)$ is of standard type, if and only if it appears in the rows 1–7, 11 or 12 of Table 2.1. The Cartan matrices are

   - $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ of type $A_1 \times A_1$ of row 1,
   - $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ of type A_2 of rows 2 and 3,
   - $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ of type B_2 of rows 4–7, and
   - $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ of type G_2 of rows 11 and 12.

   All Nichols algebras of type $A_1 \times A_1$, A_2 and some of B_2 are lifted in Sections 6.2, 6.3, 6.4.

2. In the non-standard Weyl equivalence class of row 8 of Table 2.1 the Cartan matrices

   - $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ of $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$,
   - $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ of type B_2 of $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$, $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$, and
   - $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$ of type G_2 of $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$

   appear. These Nichols algebras are lifted in Section 6.5. The same Cartan matrices appear in row 9, where we lift the Nichols algebras corresponding to the last two Dynkin diagrams.

2.6 Bosonization

Let B be a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. We will use the notation $\Delta_B(x) = x^{(1)} \otimes x^{(2)}$ for $x \in B$ to distinguish the comultiplication in the braided Hopf algebra B from the comultiplication in a usual Hopf algebra. The smash product $H = B\#\Bbbk[\Gamma]$ is a (usual) Hopf algebra, the *bosonization* of B, with structure given by

$$(x\#g)(y\#h) := x(g \cdot y)\#gh, \quad \Delta(x\#g) := x^{(1)}\#x^{(2)}{}_{(-1)}g \otimes x^{(2)}{}_{(0)}\#g,$$

for all $x, y \in B$, $g, h \in \Gamma$. We then have a Hopf algebra projection

$$\pi : B\#\Bbbk[\Gamma] \to \Bbbk[\Gamma], \quad \pi(x\#g) := \varepsilon(x)g$$

on the Hopf subalgebra $\Bbbk[\Gamma] \overset{\iota}{\hookrightarrow} B\#\Bbbk[\Gamma]$, $\iota(g) := 1\#g$; it is $\pi\iota = \mathrm{id}$.

Also the converse is true by a theorem of Radford [45]: Let H be a Hopf algebra with a Hopf subalgebra $\Bbbk[\Gamma] \overset{\iota}{\hookrightarrow} H$ (more exactly a Hopf algebra injection) and a Hopf algebra projection $\pi : H \to \Bbbk[\Gamma]$ such that $\pi\iota = \mathrm{id}$, then the subalgebra of H of right coinvariants with respect to π,

$$B := H^{\mathrm{co}\,\pi} := \{h \in H \mid (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\},$$

is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ in the following way: For any $x \in B$, $g \in \Gamma$ set

$$\begin{aligned} \delta(x) &:= \pi(x_{(1)}) \otimes x_{(2)}, \quad g \cdot x := \iota(g)x\iota(g^{-1}), \\ \Delta_B(x) &:= x_{(1)}\iota S_H \pi(x_{(2)}) \otimes x_{(3)}. \end{aligned}$$

Then the following map is a Hopf algebra isomorphism

$$B\#\Bbbk[\Gamma] \to H, \quad x\#g \mapsto x\iota(g), \qquad \text{for all } x \in B, g \in \Gamma.$$

2.7 Nichols algebras of pointed Hopf algebras

To determine the structure of a given pointed Hopf algebra it is useful to study its associated Nichols algebra, which is easier (e.g. it is graded):

Let A be a pointed Hopf algebra with abelian group of group-like elements $G(A) = \Gamma$ and

$$\Bbbk[\Gamma] = A_0 \subset A_1 \subset \ldots \subset A \quad \text{with} \quad A = \cup_{n\geq 0} A_n$$

be its *coradical filtration*, i.e.,

$$A_n := \Delta^{-1}(A \otimes A_{n-1} + A_0 \otimes A)$$

for $n \geq 1$; see [43, Sect. 5.2]. Recall that the *associated graded algebra*

$$\mathrm{gr}(A) := \oplus_{n\geq 0} A_n/A_{n-1} \quad \text{with} \quad A_{-1} := 0$$

is a pointed Hopf algebra [43, Lem. 5.2.8] of same dimension $\dim_{\Bbbk} A = \dim_{\Bbbk} \mathrm{gr}(A)$. By Section 2.6 we can write

$$\mathrm{gr}(A) \cong B\#\Bbbk[\Gamma],$$

with $B := \mathrm{gr}(A)^{\mathrm{co}\,\pi}$, π the projection of $\mathrm{gr}(A)$ on $A_0 = \Bbbk[\Gamma]$. The subalgebra of B generated by $V := P(B) \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ is the Nichols algebra $\mathfrak{B}(V)$ of V [7].

In general one hopes that $B = \mathfrak{B}(V)$, because then all finite-dimensional pointed Hopf algebras A are just the *liftings* of $\mathfrak{B}(V)$, see Chapter 6 and the introduction. Note that $B = \mathfrak{B}(V)$ is equivalent to the Conjecture 0.0.1.

	generalized Dynkin diagrams	fixed parameters
1	$\overset{q}{\circ}\qquad\overset{r}{\circ}$	$q, r \in \Bbbk^{\times}$
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \in \Bbbk^{\times}\setminus\{1\}$
3	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^{\times}\setminus\{-1, 1\}$
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^{2}}{\circ}$	$q \in \Bbbk^{\times}\setminus\{-1, 1\}$
5	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ}\quad\overset{-q^{-1}}{\circ}\overset{q^{2}}{\text{——}}\overset{-1}{\circ}$	$q \in \Bbbk^{\times}\setminus\{-1, 1\}$, $\mathrm{ord} q \neq 4$
6	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{——}}\overset{\zeta q^{-1}}{\circ}$	$\mathrm{ord}\zeta = 3$, $q \in \Bbbk^{\times}\setminus\{1, \zeta, \zeta^{2}\}$
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 3$
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-\zeta^{2}}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 12$
9	$\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-\zeta^{2}}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 12$
10	$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{\zeta^{3}}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 9$
11	$\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^{3}}{\circ}$	$q \in \Bbbk^{\times}\setminus\{-1, 1\}$, $\mathrm{ord} q \neq 3$
12	$\overset{\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}\quad\overset{\zeta^{2}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 8$
13	$\overset{\zeta^{6}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^{-4}}{\circ}\quad\overset{\zeta^{6}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}\quad\overset{-\zeta^{-4}}{\circ}\overset{\zeta^{5}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 24$
14	$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 5$
15	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 20$
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{\zeta^{5}}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{-\zeta^{4}}{\text{——}}\overset{-\zeta^{-4}}{\circ}\quad\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-2}}{\text{——}}\overset{-1}{\circ}\quad\overset{\zeta^{3}}{\circ}\overset{-\zeta^{2}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 15$
17	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}\quad\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\mathrm{ord}\zeta = 7$

Table 2.1: Weyl equivalence for rank 2 Nichols algebras [27, 30, Figure 1]

Chapter 3

q-commutator calculus

In this section let A denote an arbitrary algebra over a field $\Bbbk$ of characteristic $\operatorname{char}\Bbbk = p \geq 0$. The main result of this chapter is Proposition 3.2.3, which states important q-commutator formulas in an arbitrary algebra.

3.1 q-calculus

For every $q \in \Bbbk$ we define for $n \in \mathbb{N}$ and $0 \leq i \leq n$ the *q-numbers* and *q-factorials*

$$(n)_q := 1 + q + q^2 + \ldots + q^{n-1} = \begin{cases} n, & \text{if } q = 1 \\ \frac{q^n-1}{q-1}, & \text{if } q \neq 1 \end{cases} \quad \text{and} \quad (n)_q! := (1)_q(2)_q \ldots (n)_q,$$

and the *q-binomial coefficients*

$$\binom{n}{i}_q := \frac{(n)_q!}{(n-i)_q!(i)_q!}.$$

Note that the right-handside is well-defined since it is a polynomial over $\mathbb{Z}$ evaluated in q. We denote the *multiplicative order* of any $q \in \Bbbk^\times$ by $\operatorname{ord}q$. If $q \in \Bbbk^\times$ and $n > 1$, then

$$\binom{n}{i}_q = 0 \text{ for all } 1 \leq i \leq n-1 \Longleftrightarrow \begin{cases} \operatorname{ord}q = n, & \text{if } \operatorname{char}\Bbbk = 0 \\ p^k \operatorname{ord}q = n \text{ with } k \geq 0, & \text{if } \operatorname{char}\Bbbk = p > 0, \end{cases} \tag{3.1}$$

see [46, Cor. 2]. Moreover for $1 \leq i \leq n$ there are the *q-Pascal identities*

$$q^i \binom{n}{i}_q + \binom{n}{i-1}_q = \binom{n}{i}_q + q^{n+1-i}\binom{n}{i-1}_q = \binom{n+1}{i}_q, \tag{3.2}$$

and the *q-binomial theorem*: For $x, y \in A$ and $q \in \Bbbk^\times$ with $yx = qxy$ we have

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i}_q x^i y^{n-i}. \tag{3.3}$$

Note that for $q = 1$ these are the usual notions.

3.2 q-commutators

Definition 3.2.1. For all $a, b \in A$ and $q \in \Bbbk$ we define the *q-commutator*

$$[a, b]_q := ab - qba.$$

The q-commutator is bilinear. If $q = 1$ we get the classical commutator of an algebra. If A is graded and a, b are homogeneous elements, then there is a natural choice for the q. We are interested in the following special case:

Example 3.2.2. Let $\theta \geq 1$, $X = \{x_1, \ldots, x_\theta\}$, $\langle X\rangle$ the free monoid and $A = \Bbbk\langle X\rangle$ the free $\Bbbk$-algebra. For an abelian group Γ let $\widehat{\Gamma}$ be the character group, $g_1, \ldots, g_\theta \in \Gamma$ and $\chi_1, \ldots, \chi_\theta \in \widehat{\Gamma}$. If we define the two monoid maps

$$\deg_\Gamma : \langle X\rangle \to \Gamma,\ \deg_\Gamma(x_i) := g_i \quad \text{and} \quad \deg_{\widehat{\Gamma}} : \langle X\rangle \to \widehat{\Gamma},\ \deg_{\widehat{\Gamma}}(x_i) := \chi_i,$$

for all $1 \leq i \leq \theta$, then $\Bbbk\langle X\rangle$ is Γ- and $\widehat{\Gamma}$-graded.

Let $a \in \Bbbk\langle X\rangle$ be Γ-homogeneous and $b \in \Bbbk\langle X\rangle$ be $\widehat{\Gamma}$-homogeneous. We set

$$g_a := \deg_\Gamma(a), \quad \chi_b := \deg_{\widehat{\Gamma}}(b), \quad \text{and} \quad q_{a,b} := \chi_b(g_a).$$

Further we define $\Bbbk$-linearly on $\Bbbk\langle X\rangle$ the q-commutator

$$[a, b] := [a, b]_{q_{a,b}}. \tag{3.4}$$

Note that $q_{a,b}$ is a bicharacter on the homogeneous elements and depends only on the values

$$q_{ij} := \chi_j(g_i) \text{ with } 1 \leq i, j \leq \theta.$$

For example $[x_1, x_2] = x_1x_2 - \chi_2(g_1)x_2x_1 = x_1x_2 - q_{12}x_2x_1$. Further if a, b are $\mathbb{Z}^\theta$-homogeneous they are both Γ- and $\widehat{\Gamma}$-homogeneous. In this case we can build iterated q-commutators, like $\big[x_1, [x_1, x_2]\big] = x_1[x_1, x_2] - \chi_1\chi_2(g_1)[x_1, x_2]x_1 = x_1[x_1, x_2] - q_{11}q_{12}[x_1, x_2]x_1$.

Later we will deal with algebras which still are $\widehat{\Gamma}$-graded, but not Γ-graded such that Eq. (3.4) is not well-defined. However, the q-commutator calculus, which we next want to develop, will be a major tool for our calculations such that we need the general definition with the q as an index.

Proposition 3.2.3. *For all $a, b, c, a_i, b_i \in A$, $q, q', q'', q_i, \zeta \in \Bbbk$, $1 \leq i \leq n$ and $r \geq 1$ we have:*

(1) q-derivation properties:

$$[a, bc]_{qq'} = [a, b]_q c + qb[a, c]_{q'}, \qquad [ab, c]_{qq'} = a[b, c]_{q'} + q'[a, c]_q b,$$

$$[a, b_1 \ldots b_n]_{q_1 \ldots q_n} = \sum_{i=1}^{n} q_1 \ldots q_{i-1} b_1 \ldots b_{i-1} [a, b_i]_{q_i} b_{i+1} \ldots b_n,$$

$$[a_1 \ldots a_n, b]_{q_1 \ldots q_n} = \sum_{i=1}^{n} q_{i+1} \ldots q_n a_1 \ldots a_{i-1} [a_i, b]_{q_i} a_{i+1} \ldots a_n.$$

(2) q-Jacobi identity:

$$\big[[a,b]_{q'},c\big]_{q''q} = \big[a,[b,c]_q\big]_{q'q''} - q'b[a,c]_{q''} + q[a,c]_{q''}b.$$

(3) q-Leibniz formulas:

$$[a,b^r]_{q^r} = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta b^i \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r-i}\big]_{q\zeta^{r-i-1}},$$

$$[a^r,b]_{q^r} = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta \big[\underbrace{a,\dots\big[a,[a}_{r-i},b]_q\big]_{q\zeta}\dots\big]_{q\zeta^{r-i-1}} a^i.$$

(4) restricted q-Leibniz formulas: *If* $\operatorname{char}\Bbbk = 0$ *and* $\operatorname{ord}\zeta = r$, *or* $\operatorname{char}\Bbbk = p > 0$ *and* $p^k\operatorname{ord}\zeta = r$ *, then*

$$[a,b^r]_{q^r} = \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r}\big]_{q\zeta^{r-1}},$$

$$[a^r,b]_{q^r} = \big[\underbrace{a,\dots\big[a,[a}_{r},b]_q\big]_{q\zeta}\dots\big]_{q\zeta^{r-1}}.$$

Proof. (1) The first part is a direct calculation, e.g.

$$[a,bc]_{qq'} = abc - qq'bca = abc - qbac + qbac - qq'bca = [a,b]_q c + qb[a,c]_{q'}.$$

The second part follows by induction.

(2) Using the $\Bbbk$-linearity and (1) we get

$$\begin{aligned}\big[[a,b]_{q'},c\big]_{q''q} &= [ab,c]_{q''q} - q'[ba,c]_{q''q} = a[b,c]_q + q[a,c]_{q''}b - q'\big(b[a,c]_{q''} + q''[b,c]_q a\big)\\ &= \big[a,[b,c]_q\big]_{q'q''} - q'b[a,c]_{q''} + q[a,c]_{q''}b.\end{aligned}$$

(3) By induction on r: $r=1$ is obvious, so let $r \geq 1$. Using (1) we get

$$[a,b^{r+1}]_{q^{r+1}} = [a,b^r b]_{q^r q} = [a,b^r]_{q^r} b + q^r b^r [a,b]_q.$$

By induction assumption $[a,b^r]_{q^r} b = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta b^i \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r-i}\big]_{q\zeta^{r-i-1}} b$, where

$$b^i \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r-i}\big]_{q\zeta^{r-i-1}} b =$$

$$b^i \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r+1-i}\big]_{q\zeta^{r-i}} + q\zeta^{r-i} b^{i+1} \big[\dots \underbrace{\big[[a,b]_q,b\big]_{q\zeta}\dots,b}_{r-i}\big]_{q\zeta^{r-i-1}}.$$

In total we get

$$[a, b^{r+1}]_{q^{r+1}} = \sum_{i=0}^{r} q^i \binom{r}{i}_\zeta b^i \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r+1-i}\big]_{q\zeta^{r-i}}$$
$$+ \sum_{i=0}^{r-1} q^{i+1} \binom{r}{i}_\zeta \zeta^{r-i} b^{i+1} \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r-i}\big]_{q\zeta^{r-i-1}}.$$

Shifting the index of the second sum and using Eq. (3.2) for ζ we get the formula. The second formula is proven in the same way.

(4) Follows from (3) and Eq. (3.1). □

Remark 3.2.4.

1. If we are in the situation of Example 3.2.2 and assume that the elements are homogeneous, we can replace the arbitrary commutators by Eq. (3.4) and also replace the general q's above in the obvious way; e.g., in the first one of (1) set $q = q_{a,b}$, $q' = q_{a,c}$ and in (3), (4) $\zeta = q_{b,b}$ resp. $\zeta = q_{a,a}$.

2. If all q's are equal to one, we obtain the classical formulas. The name *restricted* is chosen, because of the analogous formula in the theory of restricted Lie algebras (also p-Lie algebras).

Chapter 4

Lyndon words and q-commutators

In this chapter we recall the theory of Lyndon words [38, 47] as far as we are concerned and then introduce the notion of super letters and super words [36].

We want to emphasize that the set of all super words can be seen indeed as a set of words (more exactly as a free monoid, see Section 4.5), which is a consequence of Proposition 4.3.2. Moreover, we introduce a well-founded ordering of the super words which makes way for inductive proofs along this ordering.

4.1 Words and the lexicographical order

Let $\theta \geq 1$, $X = \{x_1, x_2, \ldots, x_\theta\}$ be a finite totally ordered set by $x_1 < x_2 < \ldots < x_\theta$, and $\langle X \rangle$ the free monoid; we think of X as an alphabet and of $\langle X \rangle$ as the words in that alphabet including the empty word 1. For a word $u = x_{i_1} \ldots x_{i_n} \in \langle X \rangle$ we define $\ell(u) := n$ and call it the *length* of u.

The *lexicographical order* $\leq$ on $\langle X \rangle$ is defined for $u, v \in \langle X \rangle$ by $u < v$ if and only if either v begins with u, i.e., $v = uv'$ for some $v' \in \langle X \rangle \backslash \{1\}$, or if there are $w, u', v' \in \langle X \rangle$, $x_i, x_j \in X$ such that $u = wx_iu'$, $v = wx_jv'$ and $i < j$. E.g., $x_1 < x_1x_2 < x_2$. This order $<$ is stable by left, but in general not stable by right multiplication: $x_1 < x_1x_2$ but $x_1x_3 > x_1x_2x_3$. Still we have:

Lemma 4.1.1. *Let $v, w \in \langle X \rangle$ with $v < w$. Then:*

(1) *$uv < uw$ for all $u \in \langle X \rangle$.*

(2) *If w does not begin with v, then $vu < wu'$ for all $u, u' \in \langle X \rangle$.*

4.2 Lyndon words and the Shirshov decomposition

A word $u \in \langle X \rangle$ is called a *Lyndon word* if $u \neq 1$ and u is smaller than any of its proper endings, i.e., for all $v, w \in \langle X \rangle \backslash \{1\}$ such that $u = vw$ we have $u < w$. We denote by

$$\mathcal{L} := \{u \in \langle X \rangle \mid u \text{ is a Lyndon word}\}$$

the set of all Lyndon words. For example $X \subset \mathcal{L}$, but $x_i^n \notin \mathcal{L}$ for all $1 \leq i \leq \theta$ and $n \geq 2$. Moreover, if $i < j$ then $x_i^n x_j^m \in \mathcal{L}$ for $n, m \geq 1$, e.g. x_1x_2, $x_1x_1x_2$, $x_1x_2x_2$, $x_1x_1x_2x_2$; also $x_i(x_ix_j)^n \in \mathcal{L}$ for any $n \in \mathbb{N}$, e.g. $x_1x_1x_2$, $x_1x_1x_2x_1x_2$.

For any $u \in \langle X\rangle \backslash X$ we call the decomposition $u = vw$ with $v, w \in \langle X\rangle\backslash\{1\}$ such that w is the minimal (with respect to the lexicographical order) ending the *Shirshov decomposition* of the word u. We will write in this case

$$\mathrm{Sh}(u) = (v|w).$$

E.g., $\mathrm{Sh}(x_1x_2) = (x_1|x_2)$, $\mathrm{Sh}(x_1x_1x_2x_1x_2) = (x_1x_1x_2|x_1x_2)$, $\mathrm{Sh}(x_1x_1x_2) \neq (x_1x_1|x_2)$.

If $u \in \mathcal{L}\backslash X$, this is equivalent to w is the longest proper ending of u such that $w \in \mathcal{L}$. Moreover we have another characterization of the Shirshov decomposition of Lyndon words:

Theorem 4.2.1. *Let $u \in \langle X\rangle\backslash X$ and $u = vw$ with $v, w \in \langle X\rangle$. Then the following are equivalent:*

(1) *$u \in \mathcal{L}$ and $\mathrm{Sh}(u) = (v|w)$.*

(2) *$v, w \in \mathcal{L}$ with $v < u < w$ and either $v \in X$ or else if $\mathrm{Sh}(v) = (v_1|v_2)$ then $v_2 \geq w$.*

Proof. This is equivalent to [38, Prop. 5.1.3, 5.1.4] □

With this property we see that any Lyndon word is a product of two other Lyndon words of smaller length. Hence we get every Lyndon word by starting with X and concatenating inductively each pair of Lyndon words v, w with $v < w$.

Definition 4.2.2. We call a subset $L \subset \mathcal{L}$ *Shirshov closed* if

- $X \subset L$,
- for all $u \in L$ with $\mathrm{Sh}(u) = (v|w)$ also $v, w \in L$.

For example $\mathcal{L}$ is Shirshov closed, and if $X = \{x_1, x_2\}$, then $\{x_1, x_1x_1x_2, x_2\}$ is not Shirshov closed, whereas $\{x_1, x_1x_2, x_1x_1x_2, x_2\}$ is. Later we will need the following:

Lemma 4.2.3. [36, Lem. 4] *Let $u, v \in \mathcal{L}$ and $u_1, u_2 \in \langle X\rangle\backslash\{1\}$ such that $u = u_1u_2$ and $u_2 < v$. Then we have*

$$uv < u_1v < v \text{ and } uv < u_2v < v.$$

4.3 Super letters and super words

Let the free algebra $\Bbbk\langle X\rangle$ be graded as in Example 3.2.2. For any $u \in \mathcal{L}$ we define recursively on $\ell(u)$ the map

$$[\,.\,] : \mathcal{L} \to \Bbbk\langle X\rangle, \quad u \mapsto [u]. \tag{4.1}$$

If $\ell(u) = 1$, then set $[x_i] := x_i$ for all $1 \leq i \leq \theta$. Else if $\ell(u) > 1$ and $\mathrm{Sh}(u) = (v|w)$ we define $[u] := \big[[v], [w]\big]$. This map is well-defined since inductively all $[u]$ are $\mathbb{Z}^\theta$-homogeneous such that we can build iterated q-commutators; see Example 3.2.2. The elements $[u] \in$

$\Bbbk\langle X\rangle$ with $u \in \mathcal{L}$ are called *super letters*. E.g. $[x_1x_1x_2x_1x_2] = \big[[x_1x_1x_2],[x_1x_2]\big] = \big[[x_1,[x_1,x_2]],[x_1,x_2]\big]$.

If $L \subset \mathcal{L}$ is Shirshov closed then the subset of $\Bbbk\langle X\rangle$

$$[L] := \big\{[u]\,\big|\, u \in L\big\}$$

is a set of iterated q-commutators. Further $[\mathcal{L}] = \big\{[u]\,\big|\, u \in \mathcal{L}\big\}$ is the set of all super letters and the map $[\,.\,] : \mathcal{L} \to [\mathcal{L}]$ is a bijection, which follows from the Lemma 4.3.1 below. Hence we can define an order $\le$ of the super letters $[\mathcal{L}]$ by

$$[u] < [v] :\Leftrightarrow u < v,$$

thus $[\mathcal{L}]$ is a new alphabet containing the original alphabet X; so the name "letter" makes sense. Consequently, products of super letters are called *super words*. We denote

$$[\mathcal{L}]^{(\mathbb{N})} := \big\{[u_1]\dots[u_n]\,\big|\, n \in \mathbb{N},\, u_i \in \mathcal{L}\big\}$$

the subset of $\Bbbk\langle X\rangle$ of all super words. In order to define a lexicographical order on $[\mathcal{L}]^{(\mathbb{N})}$, we need to show that an arbitrary super word has a unique factorization in super letters. This is not shown in [36].

For any word $u = x_{i_1}x_{i_2}\dots x_{i_n} \in \langle X\rangle$ we define the *reversed word*

$$\overleftarrow{u} := x_{i_n}\dots x_{i_2}x_{i_1}.$$

Clearly, $\overleftarrow{\overleftarrow{u}} = u$ and $\overleftarrow{uv} = \overleftarrow{v}\,\overleftarrow{u}$. Further for any $a = \sum \alpha_i u_i \in \Bbbk\langle X\rangle$ we call the lexicographically smallest word of the u_i with $\alpha_i \neq 0$ the *leading word* of a and further define $\overleftarrow{a} := \sum \alpha_i \overleftarrow{u_i}$.

Lemma 4.3.1. *Let $u \in \mathcal{L}\backslash X$. Then there exist $n \in \mathbb{N}$, $u_i \in \langle X\rangle$, $\alpha_i \in \Bbbk$ for all $1 \le i \le n$ and $q \in \Bbbk^\times$ such that*

$$[u] = u + \sum_{i=0}^{n} \alpha_i u_i + q\overleftarrow{u} \quad \textit{and} \quad \overleftarrow{[u]} = \overleftarrow{u} + \sum_{i=0}^{n} \alpha_i \overleftarrow{u_i} + qu.$$

Moreover, u is the leading word of both $[u]$ and $\overleftarrow{[u]}$.

Proof. We proceed by induction on $\ell(u)$. If $\ell(u) = 2$, then $u = x_ix_j$ for some $1 \le i < j \le \theta$ and $[u] = [x_ix_j] = x_ix_j - q_{ij}x_jx_i = u - q_{ij}\overleftarrow{u}$. Let $\ell(u) > 2$, $\mathrm{Sh}(u) = (v|w)$ and $[u] = [v][w] - q_{vw}[w][v]$. By induction

$$[v] = v + \sum_i \beta_i v_i + q\overleftarrow{v} \quad \text{and} \quad \overleftarrow{[v]} = \overleftarrow{v} + \sum_i \beta_i \overleftarrow{v_i} + qv, \quad \text{resp.}$$

$$[w] = w + \sum_j \gamma_i w_i + q'\overleftarrow{w} \quad \text{and} \quad \overleftarrow{[w]} = \overleftarrow{w} + \sum_i \gamma_i \overleftarrow{w_i} + q'w$$

with $q, q' \neq 0$ and leading word v resp. w. Hence $[v][w]$ and $\overleftarrow{[v]}\overleftarrow{[w]}$ resp. $[w][v]$ and $\overleftarrow{[w]}\overleftarrow{[v]}$ have the leading words vw resp. wv. Since u is Lyndon we get $u = vw < wv$, thus the leading word of $[u]$ and $\overleftarrow{[u]}$ is u and further they are of the claimed form. □

Proposition 4.3.2. *Let $u_1, \dots, u_n, v_1, \dots, v_m \in \mathcal{L}$. If $[u_1][u_2]\dots[u_n] = [v_1][v_2]\dots[v_m]$, then $m = n$ and $u_i = v_i$ for all $1 \le i \le n$.*

Proof. Induction on $\max\{m, n\}$, we may suppose $m \le n$. If $n = 1$ then also $m = 1$, hence $[u_1] = [v_1]$ and both have the same leading word $u_1 = v_1$.

Let $n > 1$: By Lemma 4.3.1 $[u_1]\dots[u_n] = [v_1]\dots[v_m]$ has the leading word $u_1 \dots u_n = v_1 \dots v_m$ and

$$\overleftarrow{[u_n]}\dots\overleftarrow{[u_1]} = \overleftarrow{[u_1]\dots[u_n]} = \overleftarrow{[v_1]\dots[v_m]} = \overleftarrow{[v_m]}\dots\overleftarrow{[v_1]}$$

has the leading word $u_n \dots u_1 = v_m \dots v_1$.

If $\ell(u_1) \ge \ell(v_1)$, then $u_1 = v_1 u$ and $u_1 = u' v_1$ for some $u, u' \in \langle X \rangle$. If $u, u' \ne 1$, we get the contradiction $v_1 < v_1 u = u' v_1 < v_1$, since u_1 is Lyndon. Else if $\ell(u_1) < \ell(v_1)$, it is the same argument using that v_1 is Lyndon. Hence $u_1 = v_1$ and by induction the statement follows. $\square$

Now the lexicographical order on all super words $[\mathcal{L}]^{(\mathbb{N})}$, as defined above on regular words, is well-defined. We denote it also by $\le$.

4.4 A well-founded ordering of super words

The *length* of a super word $U = [u_1][u_2]\dots[u_n] \in [L]^{(\mathbb{N})}$ is defined as $\ell(U) := \ell(u_1 u_2 \dots u_n)$.

Definition 4.4.1. For $U, V \in [\mathcal{L}]^{(\mathbb{N})}$ we define $U \prec V$ by

- $\ell(U) < \ell(V)$, or
- $\ell(U) = \ell(V)$ and $U > V$ lexicographically in $[\mathcal{L}]^{(\mathbb{N})}$.

This defines a total ordering of $[\mathcal{L}]^{(\mathbb{N})}$ with minimal element 1. As X is assumed to be finite, there are only finitely many super letters of a given length. Hence every nonempty subset of $[\mathcal{L}]^{(\mathbb{N})}$ has a minimal element, or equivalently, $\preceq$ fulfills the descending chain condition: $\preceq$ is *well-founded*.

4.5 The free monoid $\langle X_L \rangle$

Let $L \subset \mathcal{L}$. We want to stress the two different aspects of a super letter $[u] \in [L]$:

- On the one hand it is by definition a polynomial $[u] \in \Bbbk\langle X \rangle$.
- On the other hand, as we have seen, it is a letter in the alphabet $[L]$.

To distinguish between these two point of views we define for the latter aspect a new alphabet corresponding to the set of super letters $[L]$:

To be technically correct we regard the free monoid $\langle 1, \dots, \theta \rangle$ of the ciphers $\{1, \dots, \theta\}$ ("telephone numbers" in ciphers $1, \dots, \theta$), together with the trivial bijective monoid map

$$\nu : \langle x_1, \dots, x_\theta \rangle \to \langle 1, \dots, \theta \rangle, \quad x_i \mapsto i \ \text{ for all } 1 \le i \le \theta.$$

Hence we can transfer the lexicographical order to $\langle 1, \ldots, \theta \rangle$. The image $\nu(\mathcal{L}) \subset \langle 1, \ldots, \theta \rangle$ can be seen as the set of "Lyndon telephone numbers". We define the set

$$X_L := \{x_u \mid u \in \nu(L)\}.$$

Note that if $X \subset L$ (e.g. $L \subset \mathcal{L}$ is Shirshov closed), then $X \subset X_L$. E.g., if $X = \{x_1, x_2\} \subset L = \{x_1, x_1x_2, x_2\}$ then $\nu(L) = \{1, 12, 2\}$ and $X \subset X_L = \{x_1, x_{12}, x_2\}$.

Notation 4.5.1. From now on we will not distinguish between L and $\nu(L)$ and write for example x_u instead of $x_{\nu(u)}$ for $u \in L$. In this manner we will also write $g_{\nu(u)}, \chi_{\nu(u)}$ equivalently for g_u, χ_u if $u \in L$, as defined in Example 3.2.2. E.g. $g_{112} = g_{x_1x_1x_2} = g_{x_1}g_{x_1}g_{x_2} = g_1g_1g_2$, $\chi_{112} = \chi_{x_1x_1x_2} = \chi_{x_1}\chi_{x_1}\chi_{x_2} = \chi_1\chi_1\chi_2$.

Notabene, the notation of the x_u, like x_{112}, fits perfectly for the implementation in computer algebra systems like FELIX, see Appendix A.

By Proposition 4.3.2 we have the bijection

$$\rho : [L]^{(\mathbb{N})} \to \langle X_L \rangle, \quad \rho\big([u_1] \ldots [u_n]\big) := x_{u_1} \ldots x_{u_n}. \tag{4.2}$$

E.g., $[x_1x_2x_2][x_1x_2] \overset{\rho}{\mapsto} x_{122}x_{12}$. Hence we can transfer all orderings to $\langle X_L \rangle$: For all $U, V \in \langle X_L \rangle$ we set

$$\begin{aligned} \ell(U) &:= \ell(\rho^{-1}(U)), \\ U < V &:\Leftrightarrow \rho^{-1}(U) < \rho^{-1}(V), \\ U \prec V &:\Leftrightarrow \rho^{-1}(U) \prec \rho^{-1}(V). \end{aligned}$$

Chapter 5

A class of pointed Hopf algebras

In this chapter we deal with a special class of pointed Hopf algebras. Let us recall the notions and results of [36, Sect. 3]: A Hopf algebra A is called a *character Hopf algebra* if it is generated as an algebra by elements $a_1, \dots, a_\theta$ and an abelian group $G(A) = \Gamma$ of all group-like elements such that for all $1 \leq i \leq \theta$ there are $g_i \in \Gamma$ and $\chi_i \in \widehat{\Gamma}$ with

$$\Delta(a_i) = a_i \otimes 1 + g_i \otimes a_i \quad \text{and} \quad g a_i = \chi_i(g) a_i g.$$

As mentioned in the introduction this covers a wide class of examples of Hopf algebras.

The aim of this chapter is to construct for any *character Hopf algebra* A a smash product $\Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ together with an ideal I such that

$$A \cong (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I.$$

Note that any character Hopf algebra is $\widehat{\Gamma}$-graded by

$$A = \bigoplus_{\chi \in \widehat{\Gamma}} A^\chi \quad \text{with} \quad A^\chi := \{a \in A \mid ga = \chi(g) ag\},$$

since A is genereated by $\widehat{\Gamma}$-homogeneous elements, and elements of different A^χ are linearly independent.

5.1 PBW basis in hard super letters

At first we want to give a formal definition of the term PBW basis of an arbitrary algebra.

Definition 5.1.1. Let A be an algebra, $P, S \subset A$ subsets and let $N_s \in \{1, 2, \dots, \infty\}$ for all $s \in S$. Assume that $(S, \leq)$ is totally ordered. If the set of all products

$$s_1^{r_1} s_2^{r_2} \dots s_t^{r_t} g$$

with $t \in \mathbb{N}$, $s_i \in S$, $s_1 > \dots > s_t$, $0 < r_i < N_{s_i}$ and $g \in P$, is a basis of A, then we call it a *PBW basis*. More simple, we also say S is a *PBW basis*.

Let from now on A be again a character Hopf algebra. The algebra map

$$\Bbbk\langle X\rangle \to A, \quad x_i \mapsto a_i$$

allows to identify elements of $\Bbbk\langle X\rangle$ with elements of A: By abuse of language we will write for the image of $a \in \Bbbk\langle X\rangle$ also a. Further let $\Bbbk\langle X\rangle$ be Γ-, $\widehat{\Gamma}$-graded and $q_{u,v}$ as in Example 3.2.2 with the g_i and χ_i above. Then a super letter $[u] \in A$ is called *hard* if it is not a linear combination of

- $U = [u_1]\dots[u_n] \in [\mathcal{L}]^{(\mathbb{N})}$ with $n \geq 1$, $\ell(U) = \ell(u)$, $u_i > u$ for all $1 \leq i \leq n$, and
- Vg with $V \in [\mathcal{L}]^{(\mathbb{N})}$, $\ell(V) < \ell(u)$ and $g \in \Gamma$.

Note that if $[u]$ is hard and $\mathrm{Sh}(u) = (v|w)$, then also $[v]$ and $[w]$ are hard; this follows from [36, Cor. 2]. We may assume that $a_1, \dots, a_\theta$ are hard, otherwise A would be generated by Γ and a proper subset of $a_1, \dots, a_\theta$. But this says that the set of all hard super letters is Shirshov closed.

For any hard $[u]$ we define $N'_u \in \{2, 3, \dots, \infty\}$ as the minimal $r \in \mathbb{N}$ such that $[u]^r$ is not a linear combination of

- $U = [u_1]\dots[u_n] \in [\mathcal{L}]^{(\mathbb{N})}$ with $n \geq 1$, $\ell(U) = r\ell(u)$, $u_i > u$ for all $1 \leq i \leq n$, and
- Vg with $V \in [\mathcal{L}]^{(\mathbb{N})}$, $\ell(V) < r\ell(u)$ and $g \in \Gamma$.

Theorem 5.1.2. [36, Thm. 2, Lem. 13] *Let A be a character Hopf algebra. Then the set of all*

$$[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$$

with $t \in \mathbb{N}$, $[u_i]$ is hard, $u_1 > \dots > u_t$, $0 < r_i < N'_{u_i}$, $g \in \Gamma$, forms a $\Bbbk$-basis of A.

Further, for every hard super letter $[u]$ with $N'_u < \infty$ we have $\mathrm{ord} q_{u,u} = N'_u$ if $\mathrm{char}\,\Bbbk = 0$ resp. $p^k \mathrm{ord} q_{u,u} = N'_u$ for some $k \geq 0$ if $\mathrm{char}\,\Bbbk = p > 0$.

We now generally construct a smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ with an ideal I.

5.2 The smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$

Let $\Bbbk\langle X\rangle$ be Γ- and $\widehat{\Gamma}$-graded as in Example 3.2.2, and $\Bbbk[\Gamma]$ be endowed with the usual bialgebra structure $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in \Gamma$. Then we define

$$g \cdot x_i := \chi_i(g)x_i, \quad \text{for all } 1 \leq i \leq \theta.$$

In this case, $\Bbbk\langle X\rangle$ is a $\Bbbk[\Gamma]$-module algebra and we calculate $gx_i = \chi_i(g)x_i g$, $gh = hg = \varepsilon(g)hg$ in $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$. Thus $x_i \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_i}$ and $\Bbbk[\Gamma] \subset (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\varepsilon}$ and in this way

$$\Bbbk\langle X\rangle\#\Bbbk[\Gamma] = \bigoplus_{\chi \in \widehat{\Gamma}} (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi}.$$

This $\widehat{\Gamma}$-grading extends the $\widehat{\Gamma}$-grading of $\Bbbk\langle X\rangle$ in Example 3.2.2 to $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$.

Further $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ is a Hopf algebra with structure determined by

$$\Delta(x_i) := x_i \otimes 1 + g_i \otimes x_i \quad \text{and} \quad \Delta(g) := g \otimes g,$$

for all $1 \leq i \leq \theta$ and $g \in \Gamma$.

5.3 Ideals associated to Shirshov closed sets

In this subsection we fix a Shirshov closed $L \subset \mathcal{L}$. We want to introduce the following notation for an $a \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ and $W \in [\mathcal{L}]^{(\mathbb{N})}$: We will write $a \prec_L W$ (resp. $a \preceq_L W$), if a is a linear combination of

- $U \in [L]^{(\mathbb{N})}$ with $\ell(U) = \ell(W)$, $U > W$ (resp. $U \geq W$), and
- Vg with $V \in [L]^{(\mathbb{N})}$, $g \in \Gamma$, $\ell(V) < \ell(W)$.

Furthermore, we want to distinguish the set of Lyndon words $w = uv$ with $u, v \in L$ such that

$$u < v, \quad \mathrm{Sh}(uv) = (u|v), \quad \text{and} \quad uv \notin L. \tag{5.1}$$

For example, if $L = \{x_1, x_1x_1x_2, x_1x_2, x_2\}$, then all uv with $u, v \in L$ as in Eq. (5.1) are $x_1x_1x_1x_2$, $x_1x_1x_2x_1x_2$ and $x_1x_2x_2$, see also Section 9.3.

We set $N_u := \infty$ or $N_u := \mathrm{ord} q_{u,u}$ for all $u \in L$ (resp. $N_u := p^k \mathrm{ord} q_{u,u}$ with $k \geq 0$ if $\mathrm{char}\, \Bbbk = p > 0$). Moreover, let $c_{uv} \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_{uv}}$ for all $u, v \in L$ with Eq. (5.1) such that $c_{uv} \prec_L [uv]$; and let $d_u \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_u^{N_u}}$ for all $u \in L$ with $N_u < \infty$ such that $d_u \prec_L [u]^{N_u}$. Then let I be the ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ generated by the following elements:

$$[uv] - c_{uv} \qquad \text{for all } u, v \in L \text{ with Eq. (5.1)}, \tag{5.2}$$

$$[u]^{N_u} - d_u \qquad \text{for all } u \in L \text{ with } N_u < \infty. \tag{5.3}$$

Note that the ideal I is $\widehat{\Gamma}$-homogeneous. Examples of the ideal I for certain L are found in Chapters 6, 8, and 9.

In the next Lemma we want to define $c_{(u|v)} \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ for all $u, v \in L$ with $u < v$, such that $\big[[u],[v]\big] = c_{(u|v)}$ modulo I. In this way we show that the relations of type Eq. (5.2) with $\mathrm{Sh}(uv) \neq (u|v)$ or $uv \in L$ are redundant.

Lemma 5.3.1. *Let $I' \subset \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ be the ideal generated by the elements Eq.* (5.2)*. Then there are $c_{(u|v)} \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_{uv}}$ for all $u, v \in L$ with $u < v$ such that*

(1) $\big[[u],[v]\big] - c_{(u|v)} \in I'$,

(2) $c_{(u|v)} \preceq_L [uv]$.

The residue classes of

$$[u_1]^{r_1}[u_2]^{r_2}\ldots[u_t]^{r_t}g$$

with $t \in \mathbb{N}$, $u_i \in L$, $u_1 > \ldots > u_t$, $0 < r_i < N_{u_i}$, $g \in \Gamma$, $\Bbbk$-generate $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I$.

Proof. For all $u, v \in L$ with $u < v$ and $\mathrm{Sh}(uv) = (u|v)$ we set

$$c_{(u|v)} := \begin{cases} [uv], & \text{if } uv \in L, \\ c_{uv}, & \text{if } uv \notin L. \end{cases}$$

We then proceed by induction on $\ell(u)$: If $u \in X$ then $\mathrm{Sh}(uv) = (u|v)$ by Theorem 4.2.1 and by definition the claim is fulfilled. So let $\ell(u) > 1$. Again if $\mathrm{Sh}(uv) = (u|v)$ then we argue as in the induction basis. Conversely, let $\mathrm{Sh}(uv) \neq (u|v)$, and further $\mathrm{Sh}(u) = (u_1|u_2)$; then $u_2 < v$ by Theorem 4.2.1 and by Lemma 4.2.3

$$u_1 < u_1u_2 = u < uv < u_2v, \text{ and } uv < u_1v. \tag{5.4}$$

By induction hypothesis there is a $c_{(u_2|v)} = \sum \alpha U + \sum \beta V g$ (we omit the indices to avoid double indices) of $\widehat{\Gamma}$-degree χ_{u_2v} with $U = [l_1]\dots[l_n] \in [L]^{(\mathbb{N})}$, $\ell(U) = \ell(u_2v)$, $l_1 \geq u_2v$, $V \in [L]^{(\mathbb{N})}$, $\ell(V) < \ell(u_2v)$, $g \in \Gamma$ and $\big[[u_2],[v]\big] - c_{(u_2|v)} \in I'$. Then

$$\big[[u_1], c_{(u_2|v)}\big] = \sum \alpha\big[[u_1], U\big] + \sum \beta\big[[u_1], Vg\big].$$

Since U is χ_{u_2v}-homogeneous we can use the q-derivation property of Proposition 3.2.3 for the term

$$\big[[u_1], U\big] = \sum_{i=1}^{n} q_{u_1,l_1\dots l_{i-1}}[l_1]\dots[l_{i-1}]\big[[u_1],[l_i]\big][l_{i+1}]\dots[l_n].$$

By assumption $u_2v \leq l_1$, hence we deduce $uv < l_1$ and $u_1 < l_1$ from Eq. (5.4); because of the latter inequality, by the induction hypothesis there is a $\chi_{u_1l_1}$-homogeneous $c_{(u_1|l_1)} = \sum \alpha' U' + \sum \beta' V' g'$ with $U' \in [L]^{(\mathbb{N})}$, $\ell(U') = \ell(u_1l_1)$, $U' \geq [u_1l_1]$, $V' \in [L]^{(\mathbb{N})}$, $\ell(V') < \ell(u_1l_1)$, $g' \in \Gamma$ and $\big[[u_1],[l_1]\big] - c_{(u_1|l_1)} \in I'$. Since $u_2v \leq l_1$ we have $[uv] = [u_1u_2v] \leq [u_1l_1] \leq U'$. We now define $\partial_{u_1}(c_{(u_2|v)})$ $\Bbbk$-linearly by

$$\begin{aligned}\partial_{u_1}(U) &:= c_{(u_1|l_1)}[l_2]\dots[l_n] + \sum_{i=2}^{n} q_{u_1,l_1\dots l_{i-1}}[l_1]\dots[l_{i-1}]\big[[u_1],[l_i]\big][l_{i+1}]\dots[l_n],\\ \partial_{u_1}(Vg) &:= \big[[u_1],V\big]_{q_{u_1,u_2v}\chi_{u_1}(g)}g.\end{aligned}$$

Then $\partial_{u_1}(c_{(u_2|v)}) \preceq_L [uv]$ with $\widehat{\Gamma}$-degree χ_{uv}. Moreover $\big[[u_1],\big[[u_2],[v]\big]\big] - \partial_{u_1}(c_{(u_2|v)}) \in I'$, since $\big[[u_1],U\big] - \partial_{u_1}(U) \in I'$ and $\partial_{u_1}(Vg) = \big[[u_1],Vg\big]_{q_{u_1,u_2v}}$.

Finally, because of $u_1 < u < v$ there is again by induction assumption a $c_{(u_1|v)} \preceq_L [u_1v]$, which is χ_{u_1v}-homogeneous and $c_{(u_1|v)} - \big[[u_1][v]\big] \in I'$ (moreover, $u_1v > uv$ by Eq. (5.4)). We then define for $\mathrm{Sh}(uv) \neq (u|v)$

$$c_{(u|v)} := \partial_{u_1}(c_{(u_2|v)}) + q_{u_2,v}c_{(u_1|v)}[u_2] - q_{u_1,u_2}[u_2]c_{(u_1|v)}. \tag{5.5}$$

We have $u_2 > u$ since u is Lyndon and u cannot begin with u_2, hence $u_2 > uv$ by Lemma 4.1.1. Thus $c_{(u|v)} \prec_L [uv]$. Also $\deg_{\widehat{\Gamma}}(c_{(u|v)}) = \chi_{uv}$ and by the q-Jacobi identity of Proposition 3.2.3 we have $\big[[u],[v]\big] - c_{(u|v)} \in I'$.

For the last assertion it suffices to show that the residue classes of $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$ $\Bbbk$-generate the residue classes of $\Bbbk\langle X\rangle$ in $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I'$: this can be done as in the proof of [36, Lem. 10] by induction on $\preceq$ using (1),(2). □

5.4 Structure of character Hopf algebras

Theorem 5.4.1. *If A is a character Hopf algebra, then there is a Shirshov closed $L \subset \mathcal{L}$ and an ideal $I \subset \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ as in Section 5.3 such that*

$$A \cong (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I.$$

Proof. Let $[L]$ be the set of hard super letters in A; then $L \subset \mathcal{L}$ is Shirshov closed as mentioned above. By Theorem 5.1.2 the elements $[u_1]^{r_1}[u_2]^{r_2} \dots [u_t]^{r_t} g$ with $t \in \mathbb{N}$, $u_i \in L$, $u_1 > \dots > u_t$, $0 < r_i < N'_{u_i}$, $g \in \Gamma$, form a $\Bbbk$-basis. We consider the $\Bbbk$-linear map

$$\phi : A \to \Bbbk\langle X\rangle \# \Bbbk[\Gamma], \quad [u_1]^{r_1} \dots [u_t]^{r_t} g \mapsto [u_1]^{r_1} \dots [u_t]^{r_t} g,$$

and define $c_{uv} := \phi\big([uv]\big)$ for all $u, v \in L$ with $u < v, \mathrm{Sh}(uv) = (u|v), uv \notin L$, $d_u := \phi\big([u]^{N_u}\big)$ for all $u \in L$ with $N_u := N'_u < \infty$. Note that these elements are as stated in Lemma 5.3.1 since $[uv]$ is not hard. Then there is the surjective Hopf algebra map

$$(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I \to A, \quad x_i \mapsto a_i, \ g \mapsto g.$$

By Lemma 5.3.1 the residue classes of $[u_1]^{r_1} \dots [u_t]^{r_t} g$ $\Bbbk$-generate $(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I$; they are linearly independent because so are their images. Hence the map is an isomorphism. $\square$

5.5 Calculation of coproducts

Let in this section $\operatorname{char} \Bbbk = 0$. For any $g \in \Gamma, \chi \in \widehat{\Gamma}$ we set

$$P_g^\chi := P_g^\chi(A) := P_{1,g}(A) \cap A^\chi = \{a \in A \mid \Delta(a) = a \otimes 1 + g \otimes a, \ ga = \chi(g) ag\}.$$

Although the following calculations are for $\Bbbk\langle X\rangle \# \Bbbk[\Gamma]$, we can use the results in any character Hopf algebra A by the canonical Hopf algebra map $\Bbbk\langle X\rangle \# \Bbbk[\Gamma] \to A$. Assume again the situation of Example 3.2.2.

Lemma 5.5.1. *Let $1 \le i < j \le \theta$ and $r \ge 1$.*

(1) *If $\mathrm{ord} q_{ii} = N$, then $x_i^N \in P_{g_i^N}^{\chi_i^N}$.*

(2) *If $q_{ij} q_{ji} = q_{ii}^{-(r-1)}$ and $r \le \mathrm{ord} q_{ii}$, then $[x_i^r x_j] \in P_{g_i^r g_j}^{\chi_i^r \chi_j}$.*

(3) *If $q_{ij} q_{ji} = q_{jj}^{-(r-1)}$ and $r \le \mathrm{ord} q_{jj}$, then $[x_i x_j^r] \in P_{g_i g_j^r}^{\chi_i \chi_j^r}$.*

Proof. (1) We have $(g_i \otimes x_i)(x_i \otimes 1) = q_{ii}(x_i \otimes 1)(g_i \otimes x_i)$ hence by Eq. (3.3) we obtain the claim. For (2) and (3) see [7, Lem. A.1]. $\square$

Next we want to examine certain coproducts in the special case when $q_{ii} = -1$ for a $1 \le i \le \theta$. Note that in the following two Lemmata we could write more generally i and j with $1 \le i < j \le \theta$ instead of 1 and 2:

Lemma 5.5.2. *Let $\mathrm{ord} q_{12,12} = N$.*

(1) *If $q_{22} = -1$, we have for the quotient $(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/(x_2^2)$*

$$\Delta\big([x_1 x_2]^N\big) = [x_1 x_2]^N \otimes 1 + g_{12}^N \otimes [x_1 x_2]^N + q_{2,12}^{N-1}(1 - q_{12} q_{21})[x_1 (x_1 x_2)^{N-1}] g_2 \otimes x_2.$$

(2) *If* $q_{11} = -1$*, we have for the quotient* $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/(x_1^2)$

$$\begin{aligned}\Delta\big([x_1x_2]^N\big) &= [x_1x_2]^N \otimes 1 + g_{12}^N \otimes [x_1x_2]^N \\ &\qquad + q_{1,12}^{N-1}(1-q_{12}q_{21})x_1 g_{12}^{N-1} g_2 \otimes [(x_1x_2)^{N-1}x_2].\end{aligned}$$

Proof. We calculate directly in $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$

$$\Delta([x_1x_2]) = [x_1x_2]\otimes 1 + (1-q_{12}q_{21})x_1g_2\otimes x_2 + g_{12}\otimes [x_1x_2].$$

For $\alpha := (1-q_{12}q_{21})$, $q := q_{12,12}$, $U := [x_1x_2]\otimes 1$, $V := \alpha x_1 g_2 \otimes x_2$ and $W := g_{12}\otimes [x_1x_2]$ we have $WU = qUW$ and

$$\begin{aligned}VU - qUV &= \alpha q_{2,12}[x_1x_1x_2]g_2\otimes x_2,\\ WV - qVW &= \alpha q_{12,1} x_1 g_{12} g_2 \otimes [x_1x_2x_2].\end{aligned}$$

We further set for $r\geq 1$

$$\begin{aligned}[V] &:= V, & [VU^r] &:= \alpha q_{2,12}^r [x_1(x_1x_2)^r]g_2\otimes x_2,\\ [W] &:= W, & [W^rV] &:= \alpha q_{1,12}^r x_1 g_{12}^r g_2 \otimes [(x_1x_2)^r x_2].\end{aligned}$$

(1) We have $[x_1x_2x_2] = [x_1, x_2^2] = 0$ by the restricted q-Leibniz formula and $x_2^2 = 0$. Hence $WU = qUW$ and $WV = qVW$. By Eq. (3.3) we have

$$\Delta([x_1x_2]^r) = (U+V+W)^r = (U+V)^r + W^r.$$

We state for $r\geq 1$

$$(U+V)^r = U^r + \sum_{i=0}^{r-1}\binom{r}{i}_q U^i[VU^{(r-1)-i}],$$

from where the claim follows. This we prove by induction on r: For $r=1$ the claim is true. By induction assumption

$$\begin{aligned}(U+V)^{r+1} &= (U+V)^r(U+V)\\ &= U^{r+1} + \sum_{i=0}^{r-1}\binom{r}{i}_q U^i[VU^{(r-1)-i}]U + U^rV + \sum_{i=0}^{r-1}\binom{r}{i}_q U^i[VU^{(r-1)-i}]V,\end{aligned}$$

where the last sum is zero since $[VU^{(r-1)-i}]V = \ldots\otimes x_2^2 = 0$ for all $0\leq i\leq r-1$. Further

$$\begin{aligned}[VU^{(r-1)-i}]U &= \alpha q_{2,12}^{(r-1)-i}[x_1(x_1x_2)^{(r-1)-i}]g_2[x_1x_2]\otimes x_2\\ &= \alpha q_{2,12}^{r-i}\big([x_1(x_1x_2)^{r-i}][x_1x_2]\\ &\qquad + q_{1,12}q^{(r-1)-i}[x_1x_2][x_1(x_1x_2)^{(r-1)-i}]\big)g_2\otimes x_2\\ &= [VU^{r-i}] + q^{r-i}U[VU^{(r-1)-i}].\end{aligned}$$

Thus $(U+V)^{r+1} =$

$$\begin{aligned}
&= U^{r+1} + \sum_{i=0}^{r-1} \binom{r}{i}_q U^i[VU^{r-i}] + U^r V + \sum_{i=0}^{r-1} \binom{r}{i}_q q^{r-i} U^{i+1}[VU^{(r-1)-i}] \\
&= U^{r+1} + \sum_{i=0}^{r} \Big(\binom{r}{i}_q + \binom{r}{i-1}_q q^{r+1-i}\Big) U^i [VU^{r-i}],
\end{aligned}$$

by shifting the index of the second sum. By Eq. (3.2) this is the desired formula.

(2) is proven analogously with the formula $(V+W)^r = W^r + \sum_{i=0}^{r-1} \binom{r}{i}_q [W^{(r-1)-i}V]V^i$. $\square$

A direct computation in $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ shows that

$$\begin{aligned}
\Delta([x_1x_1x_2x_1x_2]) = {} & [x_1x_1x_2x_1x_2] \otimes 1 + g_1^3 g_2^2 \otimes [x_1x_1x_2x_1x_2] \\
& + \alpha [x_1x_1x_2] g_1 g_2 \otimes [x_1x_2] \\
& + (1-q_{12}q_{21})\big(q_{21}q_{22}\beta[x_1x_1x_1x_2] + \alpha[x_1x_1x_2]x_1\big) g_2 \otimes x_2 \\
& + (1-q_{12}q_{21})(1-q_{11}q_{12}q_{21}) x_1^2 g_1 g_2^2 \\
& \qquad \otimes \big(q_{11}q_{21}(1+q_{11}-q_{11}^3q_{12}^2q_{21}^2q_{22})[x_1x_2x_2] + \alpha x_2[x_1x_2]\big) \\
& + q_{21}(1-q_{12}q_{21})^2(1-q_{11}q_{12}q_{21})(1-q_{11}^2q_{12}^2q_{21}^2q_{22}) x_1^3 g_2^2 \otimes x_2^2 \\
& + x_1 g_1^2 g_2^2 \otimes \big(\gamma[x_1x_2]^2 + q_{11}^2 q_{21}(1-q_{12}q_{21})[x_1x_1x_2x_2]\big),
\end{aligned}$$

with

$$\begin{aligned}
\alpha &:= (2)_{q_{11}} q_{11}q_{12}q_{21}q_{22}(1-q_{11}q_{12}q_{21}) + 1 - q_{11}^4 q_{12}^3 q_{21}^3 q_{22}^2, \\
\beta &:= 1 - q_{11}q_{12}q_{21} - q_{11}^2 q_{12}^2 q_{21}^2 q_{22}, \\
\gamma &:= q_{11}^2 q_{21} q_{12}(1-q_{12}q_{21})(q_{22}-q_{11}) \\
&\qquad + (2)_{q_{11}}(1-q_{11}q_{12}q_{21})(1-q_{11}^3q_{12}^2q_{21}^2q_{22}).
\end{aligned}$$

Lemma 5.5.3. *Let* $q_{22} = -1$. *Then*

$$\begin{aligned}
\alpha &= (3)_{q_{12,12}}(1-q_{11}^2q_{12}q_{21}), \\
\beta &= (3)_{q_{12,12}}, \\
\gamma &= (2)_{q_{11}}(3)_{q_{12,12}}(1-q_{11}^2q_{12}q_{21}).
\end{aligned}$$

As a consequence we have the following:

(1) *If* $\mathrm{ord} q_{12,12} = 3$, *then*

$$\begin{aligned}
\Delta([x_1x_1x_2x_1x_2]) = {} & [x_1x_1x_2x_1x_2] \otimes 1 + g_1^3 g_2^2 \otimes [x_1x_1x_2x_1x_2] \\
& + (1-q_{12}q_{21})(1-q_{11}q_{12}q_{21}) x_1^2 g_1 g_2^2 \\
& \qquad \otimes q_{11}q_{21}(1+q_{11}-q_{11}^3q_{12}^2q_{21}^2q_{22})[x_1x_2x_2] \\
& + q_{21}(1-q_{12}q_{21})^2(1-q_{11}q_{12}q_{21})(1-q_{11}^2q_{12}^2q_{21}^2q_{22}) x_1^3 g_2^2 \otimes x_2^2 \\
& + x_1 g_1^2 g_2^2 \otimes q_{11}^2 q_{21}(1-q_{12}q_{21})[x_1x_1x_2x_2].
\end{aligned}$$

Hence $[x_1x_1x_2x_1x_2] \in P_{g_1^3g_2^2}^{\chi_1^3\chi_2^2}$ *in the quotient* $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/(x_2^2)$.

(2) *If* $q_{12}q_{21} = q_{11}^{-2}$ *and* $\mathrm{ord} q_{11} = 3$, *then*

$$\begin{aligned}\Delta([x_1x_1x_2x_1x_2]) &= [x_1x_1x_2x_1x_2] \otimes 1 + g_1^3g_2^2 \otimes [x_1x_1x_2x_1x_2]\\ &+ (1-q_{12}q_{21})q_{21}q_{22}\beta[x_1x_1x_1x_2]g_2 \otimes x_2\\ &+ q_{21}(1-q_{12}q_{21})^2(1-q_{11}q_{12}q_{21})(1-q_{11}^2q_{12}^2q_{21}^2q_{22})x_1^3g_2^2 \otimes x_2^2\\ &+ x_1g_1^2g_2^2 \otimes q_{11}^2q_{21}(1-q_{12}q_{21})[x_1x_1x_2x_2].\end{aligned}$$

Hence $[x_1x_1x_2x_1x_2] \in P_{g_1^3g_2^2}^{\chi_1^3\chi_2^2}$ *in the quotients* $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/(x_2^2,\, [x_1x_1x_1x_2])$ *or* $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/(x_1^3,\, [x_1x_1x_2x_2])$.

Proof. This is also a straightforward calculation using the following identities: Since $q_{22} = -1$, we have $[x_1x_2x_2] = [x_1, x_2^2]$ by the restricted q-Leibniz formula of Proposition 3.2.3, thus $[x_1x_1x_2x_2] = \big[x_1, [x_1x_2x_2]\big] = \big[x_1, [x_1, x_2^2]\big]$. So we see that both are zero if $x_2^2 = 0$. If $\mathrm{ord} q_{11} = 3$ analogously $[x_1x_1x_1x_2] = [x_1^3, x_2] = 0$ for $x_1^3 = 0$. □

We want to state some basic combinatorics on the g_i's and χ_i's for later reference:

Lemma 5.5.4. *Let* $1 \le i \ne j \le \theta$, $1 < N := \mathrm{ord} q_{ii} < \infty$, *and* $r \in \mathbb{Z}$. *Then:*

(1) $\chi_i^N \ne \chi_i$.

(2) *If* $q_{jj} \ne 1$, *then* $\chi_i^N \ne \chi_j$ *or* $g_i^N \ne g_j$.

(3) *If* $\chi_i^N = \varepsilon$, *then* $q_{ji}^N = 1$. *Especially, if* $\chi_i^2 = \varepsilon$, *then* $q_{ji} = \pm 1$.

(4) *If* $q_{ij}q_{ji} = q_{ii}^{-(r-1)}$ *and* $q_{jj} \ne 1$, *then* $\chi_i^r\chi_j \ne \chi_i$.

(5) *If* $q_{ii}^r \ne 1$, *then* $\chi_i^r\chi_j \ne \chi_j$.

(6) *If* $q_{ij}q_{ji} = q_{ii}^{-(r-1)}$ *and* $\chi_i^r\chi_j = \varepsilon$, *then*

$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix} = \begin{pmatrix} q_{ii} & q_{ii}^{-r} \\ q_{ii} & q_{ii}^{-r} \end{pmatrix}.$$

Especially, if $q_{jj} = -1$, *then* $q_{ii}^r = -1$ *and* N *is even.*

Proof. (1) Assume $\chi_i^N = \chi_i$. Hence $q_{ii}^{N-1} = 1$, a contradiction.
(2) If $\chi_i^N = \chi_j$ and $g_i^N = g_j$, then $1 = q_{ii}^N = q_{ij}$, $q_{ji}^N = q_{jj}$, $1 = q_{ii}^N = q_{ji}$ and $q_{ij}^N = q_{jj}$. Hence $q_{jj} = q_{ij} = q_{ji} = 1$.
(3) is clear.
(4) If $\chi_i^r\chi_j = \chi_i$, then $q_{ii}^{r-1}q_{ij} = 1$, $q_{ji}^{r-1}q_{jj} = 1$. We deduce $q_{ji} = q_{jj} = 1$.
(5) If $\chi_i^r\chi_j = \chi_j$, then $q_{ii}^r = 1$.
(6) We have $q_{ii}^rq_{ij} = 1$, $q_{ji}^rq_{jj} = 1$. Now the assumption implies the claim. □

Chapter 6

Lifting

We proceed as in [6, 8]: In this chapter let $\operatorname{char}\Bbbk = 0$ and A be a finite-dimensional pointed Hopf algebra with abelian group of group-like elements $G(A) = \Gamma$ and assume that the associated graded Hopf algebra with respect to the coradical filtration (see Section 2.7) is

$$\operatorname{gr}(A) \cong \mathfrak{B}(V)\#\Bbbk[\Gamma],$$

where V is of diagonal type of dimension $\dim_{\Bbbk} V = \theta$ with basis $x_1, x_2, \ldots, x_\theta$. It is $\dim_{\Bbbk} A = \dim_{\Bbbk} \operatorname{gr}(A) = \dim_{\Bbbk} \mathfrak{B}(V) \cdot |\Gamma|$. In particular $\mathfrak{B}(V)$ is finite-dimensional and we can associate a Cartan matrix as in Definition 2.4.1.

Definition 6.0.5. In this situation we say that A is a *lifting* of the Hopf algebra $\mathfrak{B}(V)\#\Bbbk[\Gamma]$, or simply of the Nichols algebra $\mathfrak{B}(V)$.

By [6, Lem. 5.4], we have that

$$\begin{aligned} &P_g^{\varepsilon} = \Bbbk(1-g) \text{ for all } g \in \Gamma, \text{ and if } \chi \neq \varepsilon, \text{ then} \\ &P_g^{\chi} \neq 0 \iff g = g_i,\ \chi = \chi_i \text{ for some } 1 \leq i \leq \theta. \end{aligned} \tag{6.1}$$

Thus we can choose $a_i \in P_{g_i}^{\chi_i}$ with residue class $x_i \in V\#\Bbbk[\Gamma] \cong A_1/A_0$ for $1 \leq i \leq \theta$.

Lemma 6.0.6. *Let $u, v \in L \subset \mathcal{L}$.*

(1) (a) *If $q_{i,uv} \neq 1$ for some $1 \leq i \leq \theta$, then $\chi_{uv} \neq \varepsilon$.*
(b) *If $\chi_{uv} \neq \varepsilon$ and for all $1 \leq i \leq \theta$ there are $1 \leq j \leq \theta$ such that $q_{j,uv} \neq q_{ji}$ or $q_{uv,j} \neq q_{ij}$, then*

$$P_{g_{uv}}^{\chi_{uv}} = 0.$$

(2) *Let $\operatorname{ord} q_{u,u} = N_u < \infty$.*
(a) *If $q_{i,u}^{N_u} \neq 1$ for some $1 \leq i \leq \theta$, then $\chi_u^{N_u} \neq \varepsilon$.*
(b) *If $\chi_u^{N_u} \neq \varepsilon$ and for all $1 \leq i \leq \theta$ there are $1 \leq j \leq \theta$ such that $q_{j,u}^{N_u} \neq q_{ji}$ or $q_{u,j}^{N_u} \neq q_{ij}$, then*

$$P_{g_u^{N_u}}^{\chi_u^{N_u}} = 0.$$

Proof. (1a) If $\chi_{uv} = \varepsilon$, then $q_{i,uv} = 1$ for all $1 \leq i \leq \theta$.
(1b) Let $\chi_{uv} \neq \varepsilon$ and $P^{\chi_{uv}}_{g_{uv}} \neq 0$, then $\chi_{uv} = \chi_i$ and $g_{uv} = g_i$ for some i by Eq. (6.1). Hence $q_{j,uv} = q_{ji}$ and $q_{uv,j} = q_{ij}$ for all $1 \leq j \leq \theta$.
(2a) If $\chi_u^{N_u} = \varepsilon$, then $q_{i,u}^{N_u} = 1$ for all $1 \leq i \leq \theta$.
(2b) Let $\chi_u^{N_u} \neq \varepsilon$ and $P^{\chi_u^{N_u}}_{g_u^{N_u}} \neq 0$, then $\chi_u^{N_u} = \chi_i$ and $g_u^{N_u} = g_i$ for some i. Thus $q_{j,u}^{N_u} = q_{ji}$ and $q_{u,j}^{N_u} = q_{ij}$ for all $1 \leq j \leq \theta$. □

This and Eq. (6.1) motivate the following:

Definition 6.0.7. Let $L \subset \mathcal{L}$. Then we define coefficients $\mu_u \in \Bbbk$ for all $u \in L$ with $N_u < \infty$, and $\lambda_{uv} \in \Bbbk$ for all $u, v \in L$ with Eq. (5.1) by

$$\begin{aligned} \mu_u &= 0, \text{ if } g_u^{N_u} = 1 \text{ or } \chi_u^{N_u} \neq \varepsilon, \\ \lambda_{uv} &= 0, \text{ if } g_{uv} = 1 \text{ or } \chi_{uv} \neq \varepsilon, \end{aligned}$$

and otherwise they can be chosen arbitrarily.

6.1 General lifting procedure

Suppose we know the PBW basis $[L]$ of $\mathfrak{B}(V)$, then a lifting A has the same PBW basis $[L]$; see [53, Prop. 47]. Hence we know by Theorem 5.4.1 the structure of the ideal I such that

$$A \cong (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I.$$

Let us order the relations Eqs. (5.2) and (5.3) of I, namely the two types

$$[uv] - c_{uv} \text{ for } u, v \in L \text{ with Eq. (5.1)} \quad \text{and} \quad [u]^{N_u} - d_u \text{ for } u \in L \text{ with } N_u < \infty,$$

with respect to $\prec$ by the leading super word $[uv]$ resp. $[u]^{N_u}$. Yet we don't know the $c_{uv}, d_u \in \Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ explicitly; our general procedure to compute these elements is the following, stated inductively on $\prec$:

- Suppose we know all relations $\prec$-smaller than $[uv]$ resp. $[u]^{N_u}$.
- Then we determine a counterterm r_{uv} resp. $s_u \in \Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ such that

  $$[uv] - r_{uv} \in P^{\chi_{uv}}_{g_{uv}} \quad \text{resp.} \quad [u]^{N_u} - s_u \in P^{\chi_u^{N_u}}_{g_u^{N_u}}$$

  modulo the relations $\prec$-smaller than $[uv]$ resp. $[u]^{N_u}$; we conjecture that we can do this in general (see below).
  Further if $\chi_{uv} \neq \chi_i$ or $g_{uv} \neq g_i$ resp. $\chi_u^{N_u} \neq \chi_i$ or $g_u^{N_u} \neq g_i$ for all $1 \leq i \leq \theta$, then by Eq. (6.1) we get

$$\boxed{c_{uv} = r_{uv} + \lambda_{uv}(1 - g_{uv}) \quad \text{resp.} \quad d_u = s_u + \mu_u(1 - g_u^{N_u}).} \tag{6.2}$$

In order to formulate our conjecture, we define the following ideal: For any super word $U \in [\mathcal{L}]^{(\mathbb{N})}$ let I_U denote the ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ generated by the elements

$$\begin{aligned} &[uv]-c_{uv} && \text{for all } u,v\in L \text{ with Eq. (5.1) and } [uv]\prec U,\\ &[u]^{N_u}-d_u && \text{for all } u\in L \text{ with } N_u<\infty \text{ and } [u]^{N_u}\prec U. \end{aligned}$$

Note that $I_U \subset I$. See Appendix A for an example of I_U.

Conjecture 6.1.1. *For all $u,v\in L$ with Eq. (5.1) resp. for all $u\in L$ with $N_u<\infty$ there are $r_{uv}\in(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_{uv}}$ resp. $s_u\in(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_u^{N_u}}$ with $r_{uv}\prec_L [u]$ resp. $s_u\prec_L [u]^{N_u}$ such that $[uv]-r_{uv}$ resp. $[u]^{N_u}-s_u$ is skew-primitive modulo the relations $\prec$-smaller than $[uv]$ resp. $[u]^{N_u}$, i.e.,*

$$\begin{aligned} \Delta([uv]-r_{uv})-([uv]-r_{uv})\otimes 1-&g_{uv}\otimes([uv]-r_{uv})\\ &\in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]\otimes I_{[uv]}+I_{[uv]}\otimes\Bbbk\langle X\rangle\#\Bbbk[\Gamma],\\ \Delta([u]^{N_u}-s_u)-([u]^{N_u}-s_u)\otimes 1-&g_u^{N_u}\otimes([u]^{N_u}-s_u)\\ &\in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]\otimes I_{[u]^{N_u}}+I_{[u]^{N_u}}\otimes\Bbbk\langle X\rangle\#\Bbbk[\Gamma]. \end{aligned}$$

Remark 6.1.2.

1. If the conjecture is true, then one could investigate from the list of braidings in [31] where a free paramter λ_{uv} resp. μ_u occurs in the lifting, without knowing r_{uv} resp. s_u explicitly.

2. To determine the generators of the ideal I explicitly, i.e., to find r_{uv} resp. s_u, it is crucial to know which relations of I are redundant. We will detect the redundant relations with Theorem 7.3.1 in Chapter 9.

3. In general r_{uv} resp. s_u is not necessarily in $\Bbbk[\Gamma]$, like it was the case in [11]; see Lemma 6.1.3 (2b),(3b) below or the liftings in the following sections.

At first we lift the root vector relations of $x_1,\dots,x_\theta$ and the Serre relations in general. Note that for these relations our Conjecture 6.1.1 is true. We denote the images of $[x_i^r x_j],[x_i x_j^r]\in\Bbbk\langle X\rangle$ $(r\geq 1)$ of the algebra map in Section 5.1 by $[a_i^r a_j],[a_i a_j^r]$:

Lemma 6.1.3. *Let A be a lifting of $\mathfrak{B}(V)$ with braiding matrix (q_{ij}) and Cartan matrix (a_{ij}). Further let $1\leq i<j\leq\theta$ and $N_i:=\mathrm{ord}q_{ii}$. We may assume $q_{ii}\neq 1$ for all $1\leq i\leq\theta$.*

(1) *We have*

$$a_i^{N_i}=\mu_i(1-g_i^{N_i}).$$

Moreover, if $q_{ji}^{N_i}\neq 1$, then $a_i^{N_i}=0$. Especially, if $q_{ii}=-1$ and $q_{ji}\neq\pm1$, then $a_i^2=0$.

(2) (a) *If $q_{ij}q_{ji}=q_{ii}^{a_{ij}}$, $N_i>1-a_{ij}$, then*

$$\left[a_i^{1-a_{ij}}a_j\right]=\lambda_{i^{1-a_{ij}}j}(1-g_i^{1-a_{ij}}g_j).$$

Moreover, if $\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix} \neq \begin{pmatrix} q_{ii} & q_{ii}^{-(1-a_{ij})} \\ q_{ii} & q_{ii}^{-(1-a_{ij})} \end{pmatrix}$, *then* $[a_i^{1-a_{ij}} a_j] = 0$*; in particular the latter claim holds if* $q_{jj} = -1$ *and* $q_{ii}^{1-a_{ij}} \neq -1$ *(e.g.,* N_i *is odd).*

(b) *If* $N_i = 1 - a_{ij}$*, then*

$$\left[a_i^{N_i} a_j\right] = \mu_i (1 - q_{ij}^{N_i}) a_j.$$

(3) (a) *If* $q_{ij} q_{ji} = q_{jj}^{a_{ji}}$, $N_j > 1 - a_{ji}$*, then*

$$\left[a_i a_j^{1-a_{ji}}\right] = \lambda_{ij^{1-a_{ji}}} (1 - g_i g_j^{1-a_{ji}}).$$

Moreover, if $\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix} \neq \begin{pmatrix} q_{jj}^{-(1-a_{ji})} & q_{jj} \\ q_{jj}^{-(1-a_{ji})} & q_{jj} \end{pmatrix}$, *then* $[a_i a_j^{1-a_{ji}}] = 0$*; in particular the latter claim holds if* $q_{ii} = -1$ *and* $q_{jj}^{1-a_{ji}} \neq -1$ *(e.g.,* N_j *is odd).*

(b) *If* $N_j = 1 - a_{ji}$*, then*

$$\left[a_i a_j^{N_j}\right] = \mu_j (q_{ji}^{N_j} - 1) a_i g_j^{N_j}.$$

Proof. (1) This is a consequence of Lemma 5.5.4(1)-(3) and Eq. (6.1).

(2a) and (3a) follow from Lemma 5.5.4(4)-(6) and Eq. (6.1).

(2b) and (3b) follow from the restricted q-Leibniz formula of Proposition 3.2.3 and (1) above: For example

$$\left[a_i a_j^{N_j}\right] = \left[a_i, a_j^{N_j}\right]_{q_{ij}^{N_j}} = \left[a_i, \mu_j (1 - g_j^{N_j})\right]_{q_{ij}^{N_j}} = \mu_j \Big((1 - q_{ij}^{N_j}) a_i - (1 - q_{ij}^{N_j} q_{ji}^{N_j}) a_i g_j^{N_j} \Big).$$

Now either $\mu_j = 0$ or $q_{ij}^{N_j} = 1$ by (1), from where the claim follows. □

From now on let $\theta = 2$, i.e., $\mathfrak{B}(V)$ is of rank 2.

6.2 Lifting of $\mathfrak{B}(V)$ with Cartan matrix $A_1 \times A_1$

Let $\mathfrak{B}(V)$ be a finite-dimensional Nichols algebras with Cartan matrix $(a_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ of type $A_1 \times A_1$, i.e., the braiding matrix (q_{ij}) fulfills

$$q_{12} q_{21} = 1,$$

since we may suppose that $\mathrm{ord} q_{ii} \geq 2$ [27, Sect. 2], especially $q_{ii} \neq 1$. The Dynkin diagram is $\overset{q}{\circ} \qquad \overset{r}{\circ}$ with $q := q_{11}$ and $r := q_{22}$. Then the Nichols algebra is given by

$$\mathfrak{B}(V) = T(V) / \big([x_1 x_2],\, x_1^{N_1},\, x_2^{N_2} \big)$$

with basis $\{x_2^{r_2}x_1^{r_1} \mid 0 \leq r_i < N_i\}$ where $N_i = \mathrm{ord} q_{ii} \geq 2$ [30]. It is well-known [6] that any lifting A is of the form

$$A \cong (T(V)\#\Bbbk[\Gamma])/\big(\begin{array}{l} [x_1x_2] - \lambda_{12}(1-g_{12}), \\ x_1^{N_1} - \mu_1(1-g_1^{N_1}), \\ x_2^{N_2} - \mu_2(1-g_2^{N_2}) \quad \big) \end{array}$$

with basis $\{x_2^{r_2}x_1^{r_1}g \mid 0 \leq r_i < N_i, g \in \Gamma\}$ and $\dim_\Bbbk A = N_1N_2 \cdot |\Gamma|$; we prove the statement for the basis in Section 9.1.

6.3 Lifting of $\mathfrak{B}(V)$ with Cartan matrix A_2

Let $\mathfrak{B}(V)$ be a Nichols algebras with Cartan matrix $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ of type A_2, i.e., the braiding matrix (q_{ij}) fulfills

$$q_{12}q_{21} = q_{11}^{-1} \text{ or } q_{11} = -1, \text{ and } q_{12}q_{21} = q_{22}^{-1} \text{ or } q_{22} = -1.$$

The Nichols algebras are given explicitly in [29]. As mentioned above, it is crucial to know the redundant relations for the computation of the liftings. Therefore we give the ideals without redundant relations which are detected by the PBW Criterion 7.3.1:

Proposition 6.3.1 (Nichols algebras with Cartan matrix A_2). *The finite-dimensional Nichols algebras $\mathfrak{B}(V)$ with Cartan matrix of type A_2 are exactly the following:*

(1) $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$ *(Cartan type A_2). Let $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$.*

(a) *If $q_{11} = -1$, then*

$$\mathfrak{B}(V) = T(V)/\big(x_1^2,\ [x_1x_2]^2,\ x_2^2\big)$$

with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \leq r_2, r_{12}, r_1 < 2\}$ and $\dim_\Bbbk \mathfrak{B}(V) = 2^3 = 8$.

(b) *If $N := \mathrm{ord} q_{11} \geq 3$, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2],\ [x_1x_2x_2],\ x_1^N,\ [x_1x_2]^N,\ x_2^N\big)$$

with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \leq r_2, r_{12}, r_1 < N\}$ and $\dim_\Bbbk \mathfrak{B}(V) = N^3$.

(2) $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}$. *If $q_{12}q_{21} = q_{11}^{-1}$, $N := \mathrm{ord} q_{11} \geq 3$, $q_{22} = -1$, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2],\ x_1^N,\ x_2^2\big)$$

with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \leq r_1 < N,\ 0 \leq r_2, r_{12} < 2\}$ and $\dim_\Bbbk \mathfrak{B}(V) = 4N$.

(3) $\overset{-1}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$. *If $q_{11} = -1$, $q_{12}q_{21} = q_{22}^{-1}$, $N := \mathrm{ord} q_{22} \geq 3$, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_2x_2],\ x_1^2,\ x_2^N\big)$$

with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \leq r_2 < N,\ 0 \leq r_1, r_{12} < 2\}$ and $\dim_\Bbbk \mathfrak{B}(V) = 4N$.

(4) $\overset{-1}{\circ}\overset{q}{\text{———}}\overset{-1}{\circ}$. *If* $q_{11} = q_{22} = -1$, $N := \mathrm{ord} q_{12}q_{21} \geq 3$, *then*

$$\mathfrak{B}(V) = T(V)/\big(x_1^2,\ [x_1x_2]^N,\ x_2^2\big)$$

with basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid 0 \leq r_2, r_1 < 2,\ 0 \leq r_{12} < N\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 4N$.

We prove this later in Section 9.2.

Remark 6.3.2. The Nichols algebras of Proposition 6.3.1 all have the PBW basis $[L] = \{x_1, [x_1x_2], x_2\}$, and (1) resp. (2)-(4) form the standard Weyl equivalence class of row 2 resp. 3 in Table 2.1, where the latter is not of Cartan type. They build up the tree type T_2 of [29].

Theorem 6.3.3 (Liftings of $\mathfrak{B}(V)$ with Cartan matrix A_2)**.** *For any lifting A of $\mathfrak{B}(V)$ as in Proposition 6.3.1, we have*

$$A \cong (T(V)\#\Bbbk[\Gamma])/I,$$

where I is specified as follows:

(1) $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$ *(Cartan type A_2). Let* $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$.

(a) *If* $q_{11} = -1$, *then I is generated by*

$$\begin{aligned} x_1^2 &- \mu_1(1 - g_1^2),\\ [x_1x_2]^2 - 4\mu_1 q_{21}x_2^2 &- \mu_{12}(1 - g_{12}^2),\\ x_2^2 &- \mu_2(1 - g_2^2). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_2, r_{12}, r_1 < 2,\ g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^3 \cdot |\Gamma| = 8 \cdot |\Gamma|$.

(b) *If* $\mathrm{ord} q_{11} = 3$, *then I is generated by, see* [14],

$$\begin{aligned} &[x_1x_1x_2] - \lambda_{112}(1 - g_{112}),\\ &[x_1x_2x_2] - \lambda_{122}(1 - g_{122}),\\ &x_1^3 - \mu_1(1 - g_1^3),\\ &[x_1x_2]^3 + (1 - q_{11})q_{11}\lambda_{112}[x_1x_2x_2]\\ &\qquad - \mu_1(1 - q_{11})^3x_2^3 - \mu_{12}(1 - g_{12}^3),\\ &x_2^3 - \mu_2(1 - g_2^3). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_2, r_{12}, r_1 < 3,\ g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 3^3 \cdot |\Gamma| = 27 \cdot |\Gamma|$.

(c) *If* $N := \mathrm{ord} q_{11} \geq 4$, *then I is generated by, see* [8],

$$\begin{aligned} &[x_1x_1x_2],\\ &[x_1x_2x_2],\\ &x_1^N - \mu_1(1 - g_1^N),\\ &[x_1x_2]^N - \mu_1(q_{11} - 1)^N q_{21}^{\frac{N(N-1)}{2}} x_2^N - \mu_{12}(1 - g_{12}^N),\\ &x_2^N - \mu_2(1 - g_2^N). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_2, r_{12}, r_1 < N,\ g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = N^3 \cdot |\Gamma|$.

(2) $\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ}$. *Let* $q_{12}q_{21} = q_{11}^{-1}$, $q_{22} = -1$.
(a) *If* $4 \neq N := \mathrm{ord} q_{11} \geq 3$, *then* I *is generated by*

$$\begin{aligned} &[x_1x_1x_2], \\ &x_1^N - \mu_1(1 - g_1^N), \\ &x_2^2 - \mu_2(1 - g_2^2). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_1 < N, 0 \leq r_2, r_{12} < 2, g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^2N \cdot |\Gamma| = 4N \cdot |\Gamma|$.
(b) *If* $\mathrm{ord} q_{11} = 4$, *then* I *is generated by*

$$\begin{aligned} &[x_1x_1x_2] - \lambda_{112}(1 - g_{112}), \\ &x_1^4 - \mu_1(1 - g_1^4), \\ &x_2^2 - \mu_2(1 - g_2^2). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_1 < 4, 0 \leq r_2, r_{12} < 2, g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^24 \cdot |\Gamma| = 16 \cdot |\Gamma|$.

(3) $\overset{-1}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$. *Let* $q_{11} = -1$, $q_{12}q_{21} = q_{22}^{-1}$.
(a) *If* $4 \neq N := \mathrm{ord} q_{22} \geq 3$, *then* I *is generated by*

$$\begin{aligned} &[x_1x_2x_2], \\ &x_1^2 - \mu_1(1 - g_1^2), \\ &x_2^N - \mu_2(1 - g_2^N). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_2 < N, 0 \leq r_1, r_{12} < 2, g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^2N \cdot |\Gamma| = 4N \cdot |\Gamma|$.
(b) *If* $\mathrm{ord} q_{22} = 4$, *then* I *is generated by*

$$\begin{aligned} &[x_1x_2x_2] - \lambda_{122}(1 - g_{122}), \\ &x_1^2 - \mu_1(1 - g_1^2), \\ &x_2^4 - \mu_2(1 - g_2^4). \end{aligned}$$

A basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \leq r_2 < 4, 0 \leq r_1, r_{12} < 2, g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^24 \cdot |\Gamma| = 16 \cdot |\Gamma|$.

(4) $\overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$. *Let* $q_{11} = q_{22} = -1$ *and* $N := \mathrm{ord} q_{12}q_{21} \geq 3$.
(a) *If* $q_{12} \neq \pm 1$, *then* I *is generated by*

$$\begin{aligned} &x_1^2 - \mu_1(1 - g_1^2), \\ &[x_1x_2]^N - \mu_{12}(1 - g_{12}^N), \\ &x_2^2. \end{aligned}$$

(b) *If* $q_{12} = \pm 1$*, then* I *is generated by*

$$\begin{gathered} x_1^2, \\ [x_1x_2]^N - \mu_{12}(1 - g_{12}^N), \\ x_2^2 - \mu_2(1 - g_2^2). \end{gathered}$$

In both cases a basis is $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1}g \mid 0 \le r_2, r_1 < 2,\, 0 \le r_{12} < N,\, g \in \Gamma\}$ *and* $\dim_{\Bbbk} A = 2^2N \cdot |\Gamma| = 4N \cdot |\Gamma|$.

Proof. At first we show that in each case $(T(V)\#\Bbbk[\Gamma])/I$ is a pointed Hopf algebra with coradical $\Bbbk[\Gamma]$ and claimed basis and dimension such that $\operatorname{gr}((T(V)\#\Bbbk[\Gamma])/I) \cong \mathfrak{B}(V)\#\Bbbk[\Gamma]$. Then we show that a lifting A is necessarily of this form.

• $(T(V)\#\Bbbk[\Gamma])/I$ is a Hopf algebra: We show that in every case I is generated by skew-primitive elements, thus I is a Hopf ideal. The elements $x_i^{N_i} - \mu_i(1 - g_i^{N_i})$ and $[x_1x_1x_2] - \lambda_{112}(1 - g_{112})$ are skew-primitive if $q_{12}q_{21} = q_{11}^{-1}$ by Lemma 5.5.1. So we have a Hopf ideal in (2) and (3).

For the elements $[x_1x_2]^{N_{12}} - d_{12}$ we argue as follows: In (1a) we directly calculate that $[x_1x_2]^2 - 4\mu_1q_{21}x_2^2 \in P_{g_{12}^2}^{\chi_{12}^2}$. (1b),(1c) is treated in [14, 8].

For (4a): By induction on N (the induction basis $N = 2$ is Lemma 6.1.3(2b))

$$[x_1(x_1x_2)^{N-1}] = \mu_1\Big(\prod_{i=0}^{N-2}(1 - q_{12}^{i+2}q_{21}^{i})\Big)x_2[x_1x_2]^{N-2}.$$

Further $q_{12,12} = q_{12}q_{21}$ is of order N and $q_{21}^2 = 1$ (or $\mu_1 = 0$), we have $q_{12}^Nq_{21}^{N-2} = (q_{12}q_{21})^r = 1$ and thus $[x_1(x_1x_2)^{N-1}] = 0$. Hence $[x_1x_2]^N$ is skew-primitive by Lemma 5.5.2(1).

(4b) works in a similar way because of $q_{12}^2 = 1$: Again by induction (the induction basis $N = 2$ is Lemma 6.1.3(3b))

$$[(x_1x_2)^{N-1}x_2] = \mu_2\Big(\prod_{i=0}^{N-2}(1 - q_{12}^{i+2}q_{21}^{i+2})\Big)[x_1x_2]^{N-2}x_1g_2^2,$$

which is 0 since $(q_{12}q_{21})^N = q_{12,12}^N = 1$. Now $[x_1x_2]^N$ is skew-primitive by Lemma 5.5.2(2).

• We prove the statement on the basis and dimension of $(T(V)\#\Bbbk[\Gamma])/I$ later in Section 9.2 with the help of the PBW Criterion 7.3.1.

•The algebra $\Bbbk[\Gamma]$ embeds in $(T(V)\#\Bbbk[\Gamma])/I$ and the coradical of the latter is $((T(V)\#\Bbbk[\Gamma])/I)_0 = \Bbbk[\Gamma]$ [43, Lem. 5.5.1], so $(T(V)\#\Bbbk[\Gamma])/I$ is pointed.

• We consider the Hopf algebra map

$$T(V)\#\Bbbk[\Gamma] \to \operatorname{gr}((T(V)\#\Bbbk[\Gamma])/I)$$

which maps x_i onto the residue class of x_i in the homogeneous component of degree 1, namely $((T(V)\#\Bbbk[\Gamma])/I)_1/\Bbbk[\Gamma]$. It is surjective, since $(T(V)\#\Bbbk[\Gamma])/I$ is generated as an algebra by x_1, x_2 and Γ. Further it factorizes to

$$\mathfrak{B}(V)\#\Bbbk[\Gamma] \xrightarrow{\sim} \operatorname{gr}((T(V)\#\Bbbk[\Gamma])/I).$$

This is a direct argument looking at the coradical filtration as in [6, Cor. 5.3]: all equations of I are of the form $[uv]-c_{uv}$, $[u]^{N_u}-d_u$ with $c_{uv},d_u\in\Bbbk[\Gamma]=((T(V)\#\Bbbk[\Gamma])/I)_0$, hence $[uv]=0$, $[u]^{N_u}=0$ in $\mathrm{gr}((T(V)\#\Bbbk[\Gamma])/I)$. The latter surjective Hopf algebra map must be an isomorphism because the dimensions coincide.

• The other way round, let A be a lifting of $\mathfrak{B}(V)$ with $a_i\in P_{g_i}^{\chi_i}$ as in the beginning of this chapter. We consider the Hopf algebra map

$$T(V)\#\Bbbk[\Gamma]\to A$$

which takes x_i to a_i and g to g. It is surjective since A is generated by a_1,a_2 and Γ [6, Lem. 2.2]. We have to check whether this map factorizes to

$$(T(V)\#\Bbbk[\Gamma])/I\xrightarrow{\sim}A.$$

Then we are done since the dimension implies that this is an isomorphism.

But this means we have to check that the relations of I hold in A: By Lemma 6.1.3 the relations concerning the elements $a_i^{N_i}$, $[a_1a_1a_2]$ and $[a_1a_2a_2]$ are of the right form. We are left to check those for $[a_1a_2]^{N_{12}}$, which appear in (1) and (4):

In (1a) we have $[a_1a_2]^2-4\mu_1q_{21}a_2^2\in P_{g_{12}^2}^{\chi_{12}^2}$ like before. Now since $q_{1,12}^2=q_{12}^2\neq-1=q_{11}$ or $q_{12,2}^2=q_{12}^2\neq q_{12}$, and $q_{1,12}^2=q_{12}^2\neq q_{12}$ or $q_{12,2}^2=q_{12}^2\neq-1=q_{22}$ (otherwise we get the contradiction $q_{12}=1$ and $q_{12}^2=-1$), we have $[a_1a_2]^2=4\mu_1q_{21}a_2^2+\mu_{12}(1-g_{12}^2)$ by Lemma 6.0.6(2). (1b),(1c) work in the same way; see [14, 8]. For (4): As shown before $[a_1a_2]^N\in P_{g_{12}^N}^{\chi_{12}^N}$. Again we deduce from Lemma 6.0.6(2) that $[a_1a_2]^N=\mu_{12}(1-g_{12}^N)$. □

Remark 6.3.4. The Conjecture 6.1.1 is true in the situation of Theorem 6.3.3: the r_{uv} of the non-redundant relations $[uv]-c_{uv}$ are 0 ($r_{112}=r_{122}=0$ if the Serre relations are not redundant) and $s_{12}\in\Bbbk[\Gamma]$ in (1), otherwise $s_u=0$ if $[u]^{N_u}-d_u$ is not redundant.

6.4 Lifting of $\mathfrak{B}(V)$ with Cartan matrix B_2

In this section we lift some of the Nichols algebras of standard type with associated Cartan matrix B_2 (in the next Section also of non-standard type B_2). At first we recall the Nichols algebras (see [29]), but again we give the ideals without redundant relations:

Proposition 6.4.1 (Nichols algebras with Cartan matrix B_2)**.** *The following finite-dimensional Nichols algebras $\mathfrak{B}(V)$ of standard type with braiding matrix (q_{ij}) and Cartan matrix of type B_2 are represented as follows:*

(1) $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$ *(Cartan type B_2). Let $q_{12}q_{21}=q_{11}^{-2}=q_{22}^{-1}$ and $N:=\mathrm{ord}q_{11}$.*

(a) *If $N=3$, then*

$$\mathfrak{B}(V)=T(V)/\big([x_1x_2x_2],\ x_1^3,\ [x_1x_1x_2]^3,\ [x_1x_2]^3,\ x_2^3\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}\mid 0\le r_1,r_{12},r_{112},r_2<3\big\}$ and $\dim_\Bbbk\mathfrak{B}(V)=3^4=81$.

(b) *If $N=4$, then*

$$\mathfrak{B}(V)=T(V)/\big([x_1x_1x_1x_2],\ x_1^4,\ [x_1x_1x_2]^2,\ [x_1x_2]^4,\ x_2^2\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12} < 4,\ 0 \le r_2, r_{112} < 2\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 2^2 \cdot 4^2 = 64$.

(c) *If* $N \ge 5$ *is odd, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_1x_2],\ [x_1x_2x_2],\ x_1^N,\ [x_1x_1x_2]^N,\ [x_1x_2]^N,\ x_2^N\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12}, r_{112}, r_2 < N\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = N^4$.

(d) *If* $N \ge 6$ *is even, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_1x_2],\ [x_1x_2x_2],\ x_1^N,\ [x_1x_1x_2]^{\frac{N}{2}},\ [x_1x_2]^N,\ x_2^{\frac{N}{2}}\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12} < N,\ 0 \le r_2, r_{112} < \frac{N}{2}\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = \frac{N^4}{4}$.

(2) $\overset{q}{\circ}\overset{q^{-2}}{-\!\!-\!\!-\!\!-}\overset{-1}{\circ}$, $\overset{-q^{-1}}{\circ}\overset{q^{2}}{-\!\!-\!\!-\!\!-}\overset{-1}{\circ}$. *Let* $q_{12}q_{21} = q_{11}^{-2}$, $q_{22} = -1$ *and* $N := \mathrm{ord} q_{11}$.

(a) *If* $N = 3$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2],\ x_1^3,\ [x_1x_2]^6,\ x_2^2\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1 < 3,\ 0 \le r_{12} < 6,\ 0 \le r_2, r_{112} < 2\ \right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 72$.

(b) *If* $N \ge 5$ *(*$N = 4$ *is* (1b)*), then for* $N' := \mathrm{ord}(-q_{11}^{-1})$

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_1x_2],\ x_1^N,\ [x_1x_2]^{N'},\ x_2^2\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1 < N,\ 0 \le r_{12} < N',\ 0 \le r_2, r_{112} < 2\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 4NN'$.

(3) $\overset{\zeta}{\circ}\overset{q^{-1}}{-\!\!-\!\!-\!\!-}\overset{q}{\circ}$, $\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{-\!\!-\!\!-\!\!-}\overset{\zeta q^{-1}}{\circ}$. *Let* $\mathrm{ord} q_{11} = 3$, $q_{12}q_{21} = q_{22}^{-1}$ *and* $N := \mathrm{ord} q_{22}$.

(a) *If* $N = 2$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2],\ x_1^3,\ [x_1x_1x_2]^6,\ x_2^2\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12} < 3,\ 0 \le r_2 < 2,\ 0 \le r_{112} < 6\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 108$.

(b) *If* $N \ge 4$ *(*$N = 3$ *is* (1) *or Proposition 6.3.1*(1)*), then for* $N' := \mathrm{ord} q_{11}q_{22}^{-1}$

$$\mathfrak{B}(V) = T(V)/\big([x_1x_2x_2],\ x_1^3,\ [x_1x_1x_2]^{N'},\ x_2^N\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12} < 3,\ 0 \le r_2 < N,\ 0 \le r_{112} < N'\right\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 9NN'$.

(4) $\overset{\zeta}{\circ}\overset{-\zeta}{-\!\!-\!\!-\!\!-}\overset{-1}{\circ}$, $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{-\!\!-\!\!-\!\!-}\overset{-1}{\circ}$. *Let* $\mathrm{ord} q_{11} = 3$, $q_{12}q_{21} = -q_{11}$, $q_{22} = -1$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2],\ x_1^3,\ x_2^2\big)$$

with basis $\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{12} < 3,\ 0 \le r_2, r_{112} < 2\right\}$ *and* $\dim_{\Bbbk} A = 36$.

We prove this later in Section 9.3 with help of the PBW Criterion 7.3.1.

Remark 6.4.2. The Nichols algebras of Proposition 6.4.1 all have the PBW basis $[L] = \{x_2, [x_1x_2], [x_1x_1x_2], x_1\}$, and (1)-(4) form the standard Weyl equivalence classes of row 4-7 in Table 2.1, where the rows 5-7 are not of Cartan type. They build up the tree type T_3 of [29].

Theorem 6.4.3 (Liftings of $\mathfrak{B}(V)$ with Cartan matrix B_2)**.** *For any lifting A of $\mathfrak{B}(V)$ as in Proposition 6.4.1, we have*

$$A \cong (T(V)\#\Bbbk[\Gamma])/I,$$

where I is specified as follows:

(1) $\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^{2}}{\circ}$ *(Cartan type B_2). Let $q_{12}q_{21} = q_{11}^{-2} = q_{22}^{-1}$.*

(a) *If* $\mathrm{ord} q_{11} = 4$ *and $q_{12} \neq \pm 1$, then I is generated by*

$$\begin{aligned} &[x_1x_1x_1x_2],\\ &x_1^4 - \mu_1(1-g_1^4),\\ &[x_1x_1x_2]^2,\\ &[x_1x_2]^4 - \mu_{12}(1-g_{12}^4),\\ &x_2^2. \end{aligned}$$

(b) *If* $\mathrm{ord} q_{11} = 4$ *and $q_{12} = \pm 1$, then I is generated by*

$$\begin{aligned} &[x_1x_1x_1x_2],\\ &x_1^4 - \mu_1(1-g_1^4),\\ &[x_1x_1x_2]^2 - 8q_{11}\mu_1x_2^2 - \mu_{112}(1-g_{112}^2),\\ &[x_1x_2]^4 - 16\mu_1x_2^4 + 4\mu_{112}q_{11}x_2^2 - \mu_{12}(1-g_{12}^4),\\ &x_2^2 - \mu_2(1-g_2^2). \end{aligned}$$

In both (a) *and* (b) *a basis is*

$$\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_{12} < 4,\ 0 \le r_2, r_{112} < 2,\ g \in \Gamma\right\}$$

and $\dim_{\Bbbk} A = 2^2 4^2 \cdot |\Gamma| = 128 \cdot |\Gamma|$.

(2) $\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ}$, $\overset{-q^{-1}}{\circ}\overset{q^{2}}{\text{——}}\overset{-1}{\circ}$. *Let $q_{12}q_{21} = q_{11}^{-2}$, $q_{22} = -1$.*

(a) *If* $\mathrm{ord} q_{11} = 3$ *and $q_{12} \neq \pm 1$, then I is generated by*

$$\begin{aligned} &[x_1x_1x_2x_1x_2],\\ &x_1^3 - \mu_1(1-g_1^3),\\ &[x_1x_2]^6 - \mu_{12}(1-g_{12}^6),\\ &x_2^2. \end{aligned}$$

(b) *If* $\mathrm{ord} q_{11} = 3$ *and* $q_{12} = -1$*, then* I *is generated by*

$$\begin{aligned} &[x_1x_1x_2x_1x_2], \\ &x_1^3, \\ &[x_1x_2]^6 - \mu_{12}(1 - g_{12}^6), \\ &x_2^2 - \mu_2(1 - g_2^2). \end{aligned}$$

(c) *If* $\mathrm{ord} q_{11} = 3$ *and* $q_{12} = 1$*, then* I *is generated by*

$$\begin{aligned} &[x_1x_1x_2x_1x_2] + 3\mu_1(1 - q_{11})x_2^2 - \lambda_{11212}(1 - g_{11212}), \\ &x_1^3 - \mu_1(1 - g_1^3), \\ &[x_1x_2]^6 - s_{12} - \mu_{12}(1 - g_{12}^6), \\ &x_2^2 - \mu_2(1 - g_2^2), \end{aligned}$$

where

$$\begin{aligned} s_{12} := -3\mu_2\Big\{ &(\lambda_{11212}(1 - q_{11}) + 9\mu_1\mu_2q_{11})[x_1x_2]^2x_1g_2^2 \\ &- q_{11}(\lambda_{11212}(1 - q_{11}) + 9\mu_1\mu_2q_{11})[x_1x_2][x_1x_1x_2]g_2^2 \\ &+ (\lambda_{11212}^2q_{11}^2 + 3\mu_1\mu_2\lambda_{11212}(1 - q_{11}^2) - 9\mu_1^2\mu_2^2)g_1^6g_2^6 \\ &+ 3\mu_1\mu_2(\lambda_{11212}(1 - q_{11}^2) - 3\mu_1\mu_2)g_1^3g_2^6 \\ &+ \lambda_{11212}(3\mu_1\mu_2(q_{11} - 1) + \lambda_{11212})g_1^3g_2^4 \\ &- 9\mu_1^2\mu_2^2g_2^6 \\ &+ 3\mu_1\mu_2(\lambda_{11212}(q_{11} - 1) - 9\mu_1\mu_2q_{11})g_2^4 \\ &+ q_{11}(\lambda_{11212}^2 - 6\mu_1\mu_2\lambda_{11212}(1 - q_{11}) - 27\mu_1^2\mu_2^2q_{11})g_2^2\Big\}. \end{aligned}$$

In (a),(b),(c) *a basis is*

$$\Big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1 < 3,\ 0 \le r_{12} < 6,\ 0 \le r_2, r_{112} < 2,\ g \in \Gamma\Big\}$$

and $\dim_{\Bbbk} A = 72 \cdot |\Gamma|$.
(d) *Let* $N := \mathrm{ord} q_{11} > 4$ *(*$N = 4$ *is* (1)*), and* $q_{12} \neq \pm 1$*. Denote*

$$N' := \mathrm{ord}(-q_{11}^{-1}) = \begin{cases} 2N, & \text{if } N \text{ odd}, \\ N/2, & \text{if } N \text{ even and } N/2 \text{ odd}, \\ N, & \text{if } N, N/2 \text{ even}. \end{cases}$$

Then I *is generated by*

$$\begin{aligned} &[x_1x_1x_1x_2], \\ &x_1^N - \mu_1(1 - g_1^N), \\ &[x_1x_2]^{N'} - \mu_{12}(1 - g_{12}^{N'}), \\ &x_2^2. \end{aligned}$$

A basis is

$$\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \leq r_1 < N,\ 0 \leq r_{12} < N',\ 0 \leq r_2, r_{112} < 2,\ g \in \Gamma\right\}$$

and $\dim_{\Bbbk} A = 4NN' \cdot |\Gamma|$.

(3) $\overset{\zeta}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$, $\overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{——}}\overset{\zeta q^{-1}}{\circ}$. *Let* $\operatorname{ord}q_{11} = 3$, $q_{12}q_{21} = q_{22}^{-1}$.
(a) *If* $q_{22} = -1$ *and* $q_{12} \neq \pm 1$, *then* I *is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2],\\
&x_1^3 - \mu_1(1 - g_1^3),\\
&[x_1x_1x_2]^6 - \mu_{112}(1 - g_{112}^6)\\
&x_2^2.
\end{aligned}$$

(b) *If* $q_{22} = -1$ *and* $q_{12} = 1$, *then* I *is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2],\\
&x_1^3,\\
&[x_1x_1x_2]^6 - \mu_{112}(1 - g_{112}^6)\\
&x_2^2 - \mu_2(1 - g_2^2).
\end{aligned}$$

(c) *If* $q_{22} = -1$ *and* $q_{12} = -1$, *then* I *is generated by*

$$\begin{gathered}
[x_1x_1x_2x_1x_2] + 4\mu_2 x_1^3 g_2^2 - \lambda_{11212}(1 - g_1^3 g_2^2),\\
x_1^3 - \mu_1(1 - g_1^3),\\
[x_1x_1x_2]^6 - s_{112} - \mu_{112}(1 - g_{112}^6)\\
x_2^2 - \mu_2(1 - g_2^2),
\end{gathered}$$

where

$$
\begin{aligned}
s_{112} := -2\mu_1\Big\{ & 2(-\lambda_{11212}+4\mu_1\mu_2)q_{11}(1-q_{11})x_2[x_1x_1x_2]^3 g_1^3 g_2^2 \\
&+ 2(\lambda_{11212}-4\mu_1\mu_2)q_{11}(1-q_{11})[x_1x_2]^2[x_1x_1x_2]^2 g_1^3 g_2^2 \\
&+ 2(\lambda_{11212}^2-8\mu_1\mu_2\lambda_{11212}+16\mu_1^2\mu_2^2)q_{11}(1-q_{11})[x_1x_2][x_1x_1x_2] g_1^6 g_2^4 \\
&+ 8\mu_1\mu_2(\lambda_{11212}-4\mu_1\mu_2)q_{11}(1-q_{11})[x_1x_2][x_1x_1x_2] g_1^3 g_2^4 \\
&+ 2\lambda_{11212}(-\lambda_{11212}+4\mu_1\mu_2)q_{11}(1-q_{11})[x_1x_2][x_1x_1x_2] g_1^3 g_2^2 \\
&+ 2(-\lambda_{11212}^3+6\mu_1\mu_2\lambda_{11212}^2-16\mu_1^2\mu_2^2\lambda_{11212}+16\mu_1^3\mu_2^3) g_1^{12} g_2^6 \\
&+ (-\lambda_{11212}^3+12\mu_1\mu_2\lambda_{11212}^2-48\mu_1^2\mu_2^2\lambda_{11212}+64\mu_1^3\mu_2^3)q_{11}(1-q_{11}) g_1^9 g_2^6 \\
&+ 10\mu_1\mu_2(-\lambda_{11212}^2+8\mu_1\mu_2\lambda_{11212}-16\mu_1^2\mu_2^2) g_1^6 g_2^6 \\
&+ 2(\lambda_{11212}^3-7\mu_1\mu_2\lambda_{11212}^2+8\mu_1^2\mu_2^2\lambda_{11212}+16\mu_1^3\mu_2^3) g_1^6 g_2^4 \\
&+ 16\mu_1^2\mu_2^2(\lambda_{11212}-4\mu_1\mu_2)q_{11}(1-q_{11}) g_1^3 g_2^6 \\
&+ 8\mu_1\mu_2\lambda_{11212}(-\lambda_{11212}+4\mu_1\mu_2)q_{11}(1-q_{11}) g_1^3 g_2^4 \\
&+ 32\mu_1^3\mu_2^3 g_2^6 \\
&+ \lambda_{11212}^2(\lambda_{11212}-4\mu_1\mu_2)q_{11}(1-q_{11}) g_1^3 g_2^2 \\
&+ 32\mu_1^2\mu_2^2(-\lambda_{11212}+\mu_1\mu_2) g_2^4 \\
&+ 4\mu_1\mu_2(3\lambda_{11212}^2-8\mu_1\mu_2\lambda_{11212}+8\mu_1^2\mu_2^2) g_2^2\Big\}
\end{aligned}
$$

In (a),(b),(c) *a basis is*

$$\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_{12} < 3,\ 0 \le r_2 < 2,\ 0 \le r_{112} < 6,\ g \in \Gamma\right\}$$

and $\dim_{\Bbbk} A = 108 \cdot |\Gamma|$.

(4) $\overset{\zeta}{\circ}\!\!\frac{\quad -\zeta \quad}{}\!\!\overset{-1}{\circ}$, $\overset{\zeta^{-1}}{\circ}\!\!\frac{\quad -\zeta^{-1} \quad}{}\!\!\overset{-1}{\circ}$. *Let* $\operatorname{ord} q_{11} = 3$, $q_{12}q_{21} = -q_{11}$ *of order* 6, $q_{22} = -1$.

(a) *If* $q_{12} \neq \pm 1$, *then* I *is generated by*

$$
\begin{gathered}
[x_1x_1x_2x_1x_2], \\
x_1^3 - \mu_1(1-g_1^3), \\
x_2^2.
\end{gathered}
$$

(b) *If* $q_{12} = 1$, *then* I *is generated by*

$$
\begin{gathered}
[x_1x_1x_2x_1x_2], \\
x_1^3, \\
x_2^2 - \mu_2(1-g_2^2).
\end{gathered}
$$

(c) *If* $q_{12} = -1$, *then* I *is generated by*

$$
\begin{gathered}
[x_1x_1x_2x_1x_2] - \mu_2(1+q_{11})x_1^3 g_2^2 - \lambda_{11212}(1-g_{11212}), \\
x_1^3 - \mu_1(1-g_1^3), \\
x_2^2 - \mu_2(1-g_2^2).
\end{gathered}
$$

A basis in (a),(b),(c) *is*

$$\left\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_{12} < 3,\ 0 \le r_2, r_{112} < 2,\ g \in \Gamma\right\}$$

and $\dim_{\mathbb{k}} A = 36 \cdot |\Gamma|$.

Proof. We proceed as in the proof of Theorem 6.3.3.

• $(T(V)\#\mathbb{k}[\Gamma])/I$ is a Hopf algebra, since I is generated by skew-primitive elements: Again the elements $x_i^{N_i} - \mu_i(1 - g_i^{N_i})$ and $[x_1x_1x_1x_2] - \lambda_{1112}(1 - g_{1112})$ are skew-primitive if $q_{12}q_{21} = q_{11}^{-2}$ by Lemma 5.5.1.

(1a) By Lemma 5.5.2(1) $[x_1x_2]^4 \in P_{g_{12}^4}^{\chi_{12}^4}$ and hence also $[x_1x_2]^4 - \mu_{12}(1 - g_{12}^4) \in P_{g_{12}^4}^{\chi_{12}^4}$. A direct computation yields $[x_1x_1x_2]^2 \in P_{g_{112}^2}^{\chi_{112}^2}$.

(1b) Again direct computation shows that $[x_1x_1x_2]^2 - 8q_{11}\mu_1x_2^2 - \mu_{112}(1 - g_{112}^2)$ and $[x_1x_2]^4 - 16\mu_1x_2^4 + 4\mu_{112}q_{11}x_2^2 - \mu_{12}(1 - g_{12}^4)$ are skew primitive; we used the computer algebra system FELIX, see Appendix A.

(2a) We have $[x_1x_1x_2x_1x_2] \in P_{g_{11212}}^{\chi_{11212}}$ by Lemma 5.5.3(2). Further $[x_1x_2]^6 \in P_{g_{12}^6}^{\chi_{12}^6}$ by Lemma 5.5.2(1).

(2b) Again $[x_1x_1x_2x_1x_2] \in P_{g_{11212}}^{\chi_{11212}}$ by Lemma 5.5.3(2) and a direct computation yields $[x_1x_2]^6 - \mu_{12}(1 - g_{12}^6) \in P_{g_{12}^6}^{\chi_{12}^6}$.

(2c) Using FELIX we get that all elements are skew-primitive; see Appendix A.

(2d) This is again Lemma 5.5.2(1).

(3a) and (3b): $[x_1x_1x_2x_1x_2] \in P_{g_{11212}}^{\chi_{11212}}$ by Lemma 5.5.3(1). Straightforward calculation shows that $[x_1x_1x_2]^6 - \mu_{112}(1 - g_{112}^6) \in P_{g_{112}^6}^{\chi_{112}^6}$; here again we used FELIX.

(3c) is computed using FELIX.

(4a) and (4b): $[x_1x_1x_2x_1x_2] \in P_{g_{11212}}^{\chi_{11212}}$ by Lemma 5.5.3(1).

(4c) Looking at the coproduct computed in Lemma 5.5.3(1) we deduce that the element $[x_1x_1x_2x_1x_2] - \mu_2(1 + q_{11})x_1^3g_2^2$ and hence $[x_1x_1x_2x_1x_2] - \mu_2(1 + q_{11})x_1^3g_2^2 - \lambda_{11212}(1 - g_{11212})$ is skew-primitive.

• We prove the statement on the basis and dimension of $(T(V)\#\mathbb{k}[\Gamma])/I$ later in Section 9.3 with the help of the PBW Criterion 7.3.1.

• $(T(V)\#\mathbb{k}[\Gamma])/I$ is pointed by the same argument as in the proof of Theorem 6.3.3.

• The surjective Hopf algebra map as given in the proof of Theorem 6.3.3

$$T(V)\#\mathbb{k}[\Gamma] \to \mathrm{gr}((T(V)\#\mathbb{k}[\Gamma])/I)$$

factorizes to an isomorphism $\mathfrak{B}(V)\#\mathbb{k}[\Gamma] \xrightarrow{\sim} \mathrm{gr}((T(V)\#\mathbb{k}[\Gamma])/I)$: Again we look at the coradical filtration. All equations of I are of the form $[uv] - c_{uv}$, $[u]^{N_u} - d_u$ with c_{uv} resp. d_u of lower degree in $\mathrm{gr}((T(V)\#\mathbb{k}[\Gamma])/I)$, hence $[uv] = 0$, $[u]^{N_u} = 0$ in $\mathrm{gr}((T(V)\#\mathbb{k}[\Gamma])/I)$.

• Like before, for a lifting A we have to check whether the surjective Hopf algebra map

$$T(V)\#\mathbb{k}[\Gamma] \to A$$

which takes x_i to a_i and g to g factorizes to

$$(T(V)\#\mathbb{k}[\Gamma])/I \xrightarrow{\sim} A.$$

By Lemma 6.1.3 the relations concerning the elements $a_i^{N_i}$ and $[a_1a_1a_1a_2]$ are of the right form. We deduce from Lemma 6.0.6 that the relations also hold in A: this is just combinatorics on the braiding matrices which we want to demonstrate for the following.

(1a) We have $\chi_{112}^2 \neq \varepsilon$ by Lemma 6.0.6(2a), since $q_{1,112}^2 = q_{12}^2 \neq 1$. Further $P_{g_{112}^2}^{\chi_{112}^2} = 0$ by Lemma 6.0.6(2b): Suppose $q_{21}^4 q_{22}^2 = q_{21}$, $q_{12}^4 q_{22}^2 = q_{12}$, then $q_{21}^3 = q_{12}^3 = 1$, which contradicts $q_{12}q_{21} = q_{11}^{-2} = -1$; also if $q_{11}^4 q_{12}^2 = q_{12}$, $q_{11}^4 q_{21}^2 = q_{21}$, then $q_{12} = q_{21} = 1$, again a contradiction to $q_{12}q_{21} = q_{11}^{-2} = -1$. Hence $[a_1a_1a_2]^2 = 0$. The other cases work in exactly the same manner. $\square$

Remark 6.4.4. The Conjecture 6.1.1 is true in the above cases. Further note that in (2c) $s_{12} \notin \Bbbk[\Gamma]$ and in (3c) $s_{112} \notin \Bbbk[\Gamma]$.

Further we want to note the cases not treated in the theorem above:

1. The case (1) when $5 \neq N := \mathrm{ord} q_{11} \geq 3$ is odd is treated in [14], and the case $N = 5$ in [19].
2. There is no general method for (1) in the case $N := \mathrm{ord} q_{11} \geq 6$ is even. Here $\mathrm{ord} q_{22} = \mathrm{ord} q_{11}^2 = \frac{N}{2}$.
3. There is no general method for (2d) in the case $q_{12} = \pm 1$.
4. There is no general method for (3) in the case $N := \mathrm{ord} q_{22} \geq 4$. The case $N = 3$ is (1) of the theorem above or (1) of Theorem 6.3.3.

6.5 Lifting of $\mathfrak{B}(V)$ of non-standard type

In this section we want to lift some of the Nichols algebras of the Weyl equivalence classes of rows 8 and 9 of Table 2.1 which are not of standard type, namely for $\mathrm{ord}\zeta = 12$ we lift

$$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ},\quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ},\quad \overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ},\quad \overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ},\quad \overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$$

of row 8, and

$$\overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{———}}\overset{-1}{\circ},\quad \overset{-\zeta^{-1}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-1}{\circ}$$

of row 9. Again, at first we give a nice presentation of the ideal cancelling the redundant relations of the ideals given in [29]:

Proposition 6.5.1 (Nichols algebras of rows 8 and 9)**.** *The following finite-dimensional Nichols algebras $\mathfrak{B}(V)$ with braiding matrix (q_{ij}) of rows 8 and 9 of Table 2.1 are represented as follows: Let $\zeta \in \Bbbk^\times$, $\mathrm{ord}\zeta = 12$.*

(1) $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$. *Let $q_{11} = -\zeta^{-2}$, $q_{12}q_{21} = -\zeta^3$, $q_{22} = -\zeta^2$, then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_2] - \frac{1}{2}q_{11}q_{12}(q_{12}q_{21} - q_{11})(1 - q_{12}q_{21})[x_1x_2]^2, x_1^3,\ x_2^3\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2x_2]^{r_{122}}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \leq r_1, r_2 < 3,\ 0 \leq r_{112}, r_{122} < 2,\ 0 \leq r_{12} < 4\big\}$ *and* $\dim_\Bbbk \mathfrak{B}(V) = 144$.

(2) $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$, $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^2$, $q_{12}q_{21} = -\zeta$, $q_{22} = -1$, *or* $q_{11} = -\zeta^{-2}$, $q_{12}q_{21} = \zeta^{-1}$, $q_{22} = -1$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2x_1x_2],\ x_1^3,\ x_2^2\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2x_1x_2]^{r_{11212}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{112} < 3,\ 0 \le r_2, r_{11212} < 2,\ 0 \le r_{12} < 4\big\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 144$.

(3) $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$, $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^3$, $q_{12}q_{21} = \zeta$, $q_{22} = -1$, *or* $q_{11} = -\zeta^3$, $q_{12}q_{21} = -\zeta^{-1}$, $q_{22} = -1$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2],\ x_1^4,\ x_2^2\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}[x_1x_1x_1x_2]^{r_{1112}}x_1^{r_1} \mid 0 \le r_1 < 4,\ 0 \le r_{12}, r_{112} < 3,\ 0 \le r_2, r_{1112} < 2\big\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 144$.

(4) $\overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{———}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^2$, $q_{12}q_{21} = \zeta^3$, $q_{22} = -1$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_2x_1x_2x_1x_2],\ x_1^3,\ [x_1x_2]^{12},\ x_2^2\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2x_1x_2]^{r_{11212}}[x_1x_1x_2]^{r_{112}}x_1^{r_1} \mid 0 \le r_1, r_{112} < 3,\ 0 \le r_2, r_{11212} < 2,\ 0 \le r_{12} < 12\big\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 432$.

(5) $\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^{-1}$, $q_{12}q_{21} = -\zeta^3$, $q_{22} = -1$, *then*

$$\mathfrak{B}(V) = T(V)/\big([x_1x_1x_1x_1x_2],\ [x_1x_1x_2x_1x_2],\ x_1^{12},\ x_2^2\big)$$

with basis $\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}[x_1x_1x_1x_2]^{r_{1112}}x_1^{r_1} \mid 0 \le r_1 < 12,\ 0 \le r_{12}, r_{112} < 3,\ 0 \le r_2, r_{1112} < 2\big\}$ *and* $\dim_{\Bbbk} \mathfrak{B}(V) = 432$.

We prove this in Sections 9.4, 9.5, 9.6 with the PBW Criterion 7.3.1.

Remark 6.5.2. The Nichols algebras of Proposition 6.5.1 have different PBW bases, also if they are in the same Weyl equivalence class. They build up the tree types T_4, T_5 and T_7 of [29].

Theorem 6.5.3 (Liftings of $\mathfrak{B}(V)$ of rows 8 and 9)**.** *For any lifting A of $\mathfrak{B}(V)$ as in Proposition 6.5.1, we have*

$$A \cong (T(V)\#\Bbbk[\Gamma])/I,$$

where I is specified as follows: Let $\zeta \in \Bbbk^\times$, $\mathrm{ord}\zeta = 12$.

(1) $\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$. *Let* $q_{11} = -\zeta^{-2}$, $q_{12}q_{21} = -\zeta^3$, $q_{22} = -\zeta^2$.

(a) *If* $q_{12}^3 \neq 1$, *then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_2] - \frac{1}{2}q_{11}q_{12}(q_{12}q_{21} - q_{11})(1 - q_{12}q_{21})[x_1x_2]^2,\\
&x_1^3 - \mu_1(1 - g_1^3),\\
&x_2^3.
\end{aligned}$$

(b) *If $q_{12}^3 = 1$, then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_2] - \frac{1}{2}q_{11}q_{12}(q_{12}q_{21} - q_{11})(1 - q_{12}q_{21})[x_1x_2]^2,\\
&x_1^3,\\
&x_2^3 - \mu_2(1 - g_2^3).
\end{aligned}$$

In (a),(b) *a basis is*

$$\big\{x_2^{r_2}[x_1x_2x_2]^{r_{122}}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_2 < 3,\\ 0 \le r_{112}, r_{122} < 2,\ 0 \le r_{12} < 4,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 144 \cdot |\Gamma|$.

(2) $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$, $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$. *Let $q_{11} = -\zeta^2$, $q_{12}q_{21} = -\zeta$, $q_{22} = -1$, or $q_{11} = -\zeta^{-2}$, $q_{12}q_{21} = \zeta^{-1}$, $q_{22} = -1$.*

(a) *If $q_{12} \neq \pm 1$, then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2x_1x_2],\\
&x_1^3 - \mu_1(1 - g_1^3),\\
&x_2^2.
\end{aligned}$$

(b) *If $q_{12} = \pm 1$, then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2x_1x_2] + \mu_2q_{12}(q_{11}q_{12}q_{21} + q_{12}q_{21} - 1)[x_1x_1x_2]x_1^2g_2^2,\\
&x_1^3,\\
&x_2^2 - \mu_2(1 - g_2^2).
\end{aligned}$$

In (a),(b) *a basis is*

$$\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2x_1x_2]^{r_{11212}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_{112} < 3,\\ 0 \le r_2, r_{11212} < 2,\ 0 \le r_{12} < 4,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 144 \cdot |\Gamma|$.

(3) $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$, $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$. *Let $q_{11} = -\zeta^3$, $q_{12}q_{21} = \zeta$, $q_{22} = -1$, or $q_{11} = -\zeta^3$, $q_{12}q_{21} = -\zeta^{-1}$, $q_{22} = -1$.*

(a) *If $q_{12} \neq \pm 1$, then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2],\\
&x_1^4 - \mu_1(1 - g_1^3),\\
&x_2^2.
\end{aligned}$$

(b) *If $q_{12} = \pm 1$, then I is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2] - \mu_2q_{12}(q_{11} + 2q_{12}^2q_{21}^2 - q_{12}q_{21})x_1^3g_2^2,\\
&x_1^4,\\
&x_2^2 - \mu_2(1 - g_2^2).
\end{aligned}$$

In (a),(b) *a basis is*

$$\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}[x_1x_1x_1x_2]^{r_{1112}}x_1^{r_1}g \mid 0 \le r_1 < 4,$$
$$0 \le r_{12}, r_{112} < 3,\ 0 \le r_2, r_{1112} < 2,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 144 \cdot |\Gamma|$.

(4) $\overset{-\zeta^2}{\circ}\overset{\zeta^3}{\frac{\qquad}{}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^2$, $q_{12}q_{21} = \zeta^3$, $q_{22} = -1$.
(a) *If* $q_{12} \neq \pm 1$, *then* I *is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2x_1x_2],\\
&x_1^3 - \mu_1(1 - g_1^3),\\
&[x_1x_2]^{12} - \mu_{12}(1 - g_{12}^{12}),\\
&x_2^2.
\end{aligned}$$

A basis is

$$\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2x_1x_2]^{r_{11212}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1, r_{112} < 3,$$
$$0 \le r_2, r_{11212} < 2,\ 0 \le r_{12} < 12,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 432 \cdot |\Gamma|$.
(b) (incomplete) $q_{12} = \pm 1$, *then* I *is generated by*

$$\begin{aligned}
&[x_1x_1x_2x_1x_2x_1x_2] + q_{12}2\mu_2(q_{12}q_{21} + 1)[x_1x_1x_2]x_1^2g_2^2,\\
&x_1^3,\\
&[x_1x_2]^{12} - d_{12},\\
&x_2^2 - \mu_2(1 - g_1^2).
\end{aligned}$$

(5) $\overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\frac{\qquad}{}}\overset{-1}{\circ}$. *Let* $q_{11} = -\zeta^{-1}$, $q_{12}q_{21} = -\zeta^3$, $q_{22} = -1$.
(a) *If* $q_{12} \neq \pm 1$, *then* I *is generated by*

$$\begin{aligned}
&[x_1x_1x_1x_1x_2],\\
&[x_1x_1x_2x_1x_2],\\
&x_1^{12} - \mu_1(1 - g_1^{12}),\\
&x_2^2.
\end{aligned}$$

(b) *If* $q_{12} = \pm 1$, *then* I *is generated by:*

$$\begin{aligned}
&[x_1x_1x_1x_1x_2],\\
&[x_1x_1x_2x_1x_2] + 2\mu_2q_{12}x_1^3g_2^2,\\
&x_1^{12} - \mu_1(1 - g_1^{12}),\\
&x_2^2 - \mu_2(1 - g_2^2).
\end{aligned}$$

In (a),(b) *a basis is*

$$\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}[x_1x_1x_1x_2]^{r_{1112}}x_1^{r_1}g \mid 0 \le r_1 < 12,$$
$$0 \le r_{12}, r_{112} < 3,\ 0 \le r_2, r_{1112} < 2,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 432 \cdot |\Gamma|$.

Proof. We argue exactly as in the proofs of Theorem 6.3.3 and 6.4.3.

• $(T(V)\#\Bbbk[\Gamma])/I$ is a Hopf algebra, since I is generated by skew-primitive elements: The elements $x_i^{N_i} - \mu_i(1-g_i^{N_i})$ and $[x_1x_1x_1x_1x_2] - \lambda_{11112}(1-g_{11112})$ are skew-primitive if $q_{12}q_{21} = q_{11}^{-2}$ by Lemma 5.5.1. For the elements $[x_1x_1x_2x_1x_2] - c_{11212}$ and $[x_1x_1x_2x_1x_2x_1x_2] - c_{1121212}$ we use Lemma 5.5.3 and for $[x_1x_2]^{N_{12}} - d_{12}$ Lemma 5.5.2(1). Further in (1) $[x_1x_1x_2x_2] - \frac{1}{2}q_{11}q_{12}(q_{12}q_{21} - q_{11})(1 - q_{12}q_{21})[x_1x_2]^2$ is skew-primitive by a straightforward calculation.

• The statement on the basis and dimension of $(T(V)\#\Bbbk[\Gamma])/I$ is proved in Sections 9.4, 9.5, 9.6 with the help of the PBW Criterion 7.3.1.

• $(T(V)\#\Bbbk[\Gamma])/I$ is pointed and $\mathrm{gr}((T(V)\#\Bbbk[\Gamma])/I) \cong \mathfrak{B}(V)\#\Bbbk[\Gamma]$ by the same arguments as in the proofs of Theorems 6.3.3 and 6.4.3.

• Also in the same way, the surjective Hopf algebra map $T(V)\#\Bbbk[\Gamma] \to A$ factorizes to an isomorphism

$$(T(V)\#\Bbbk[\Gamma])/I \xrightarrow{\sim} A$$

by Lemma 6.1.3 and 6.0.6, doing the combinatorics on the braiding matrices. □

Remark 6.5.4. The Conjecture 6.1.1 is true in the above cases. Further note that in (1) $r_{1122} \notin \Bbbk[\Gamma]$ (as well as $r_{1122} \neq 0$ in $\mathfrak{B}(V)$), in (2b) $r_{1121212} \notin \Bbbk[\Gamma]$, in (3b) $r_{11212} \notin \Bbbk[\Gamma]$, in (4b) $r_{1121212} \notin \Bbbk[\Gamma]$ and in (5b) $r_{11212} \notin \Bbbk[\Gamma]$.

Chapter 7

A PBW basis criterion

In this chapter we want to state a PBW basis criterion which is applicable for any character Hopf algebra. It can be adapted to other more general situations with an arbitrary bialgebra H instead of $\Bbbk[\Gamma]$, but then the conditions may become more technical.

At first we need to define several algebraic objects for the formulation of the PBW Criterion 7.3.1. The main idea is, not to work in the free algebra $\Bbbk\langle X\rangle$ but in the free algebra $\Bbbk\langle X_L\rangle$ where $\langle X_L\rangle$ is the free monoid of Section 4.5. In this way a super letter $[u]$ corresponds to a letter/variable x_u, making way for applying the diamond lemma to the (super) letters.

7.1 The free algebra $\Bbbk\langle X_L\rangle$ and $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

Let $L \subset \mathcal{L}$ be Shirshov closed. In Section 4.5 we associated to a super letter $[u] \in [L]$ a new variable $x_u \in X_L$, where X_L contains X. Hence the free algebra $\Bbbk\langle X_L\rangle$ also contains $\Bbbk\langle X\rangle$.

We define the action of Γ on $\Bbbk\langle X_L\rangle$ and q-commutators by

$$\begin{aligned} g \cdot x_u &:= \chi_u(g)x_u && \text{for all } g \in \Gamma, u \in L,\\ [x_u, x_v] &:= x_u x_v - q_{u,v} x_v x_u && \text{for all } u, v \in L. \end{aligned}$$

In this way $\Bbbk\langle X_L\rangle$ becomes a $\Bbbk[\Gamma]$-module algebra and we calculate

$$gx_u = \chi_u(g)x_u g$$

in the smash product $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$.

7.2 The subspace $I_{\prec U} \subset \Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

Via ρ of Eq. (4.2) we now define elements $c^{\rho}_{(u|v)}, d^{\rho}_u \in \Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ which correspond to $c_{(u|v)}, d_u \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$: For all $u, v \in L$ with Eq. (5.1) resp. $u \in L$ with $N_u < \infty$ we write $c_{uv} = \sum \alpha U + \sum \beta V g \prec_L [uv]$ resp. $d_u = \sum \alpha' U' + \sum \beta' V' g' \prec_L [u]^{N_u}$, with $\alpha, \alpha', \beta, \beta' \in \Bbbk$

and $U, U', V, V' \in [L]^{(\mathbb{N})}$ (such decompositions may not be unique; we just fix one). Then we define in $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

$$c^\rho_{uv} := \sum \alpha\rho(U) + \sum \beta\rho(V)g \quad \text{resp.} \quad d^\rho_u := \sum \alpha'\rho(U') + \sum \beta'\rho(V')g'.$$

If $\mathrm{Sh}(uv) = (u|v)$ we set

$$c^\rho_{(u|v)} := \begin{cases} x_{uv}, & \text{if } uv \in L, \\ c^\rho_{uv}, & \text{if } uv \notin L. \end{cases}$$

Else if $\mathrm{Sh}(uv) \neq (u|v)$ let $\mathrm{Sh}(u) = (u_1|u_2)$. Then we define inductively on the length of $\ell(u)$

$$c^\rho_{(u|v)} := \partial^\rho_{u_1}(c^\rho_{(u_2|v)}) + q_{u_2,v}c^\rho_{(u_1|v)}x_{u_2} - q_{u_1,u_2}x_{u_2}c^\rho_{(u_1|v)}, \tag{7.1}$$

where $\partial^\rho_{u_1}$ is defined $\Bbbk$-linearly by

$$\begin{aligned} &\partial^\rho_{u_1}(x_{l_1}\dots x_{l_n}) := c^\rho_{(u_1|l_1)}x_{l_2}\dots x_{l_n} + \sum_{i=2}^n q_{u_1,l_1\dots l_{i-1}}x_{l_1}\dots x_{l_{i-1}}\big[x_{u_1}, x_{l_i}\big]x_{l_{i+1}}\dots x_{l_n}, \\ &\partial^\rho_{u_1}(\rho(V)g) := \big[x_{u_1}, \rho(V)\big]_{q_{u_1,u_2v}\chi_{u_1}(g)}g, \end{aligned}$$

if the $c_{(u_2|v)}$ is a linear combination of $[l_1]\dots[l_n]$, Vg as in the proof of Lemma 5.3.1. Note that all the combinatorial properties of Lemma 5.3.1 are transferred to the just defined elements.

For any $U \in \langle X_L\rangle$ let $I_{\prec U}$ denote the subspace of $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ spanned by the elements

$$\begin{aligned} &Vg\big([x_u, x_v] - c^\rho_{(u|v)}\big)Wh && \text{for all } u, v \in L, u < v, \\ &V'g'\big(x_u^{N_u} - d^\rho_u\big)W'h' && \text{for all } u \in L, N_u < \infty \end{aligned}$$

with $V, V', W, W' \in \langle X_L\rangle$, $g, g', h, h' \in \Gamma$ such that

$$Vx_ux_vW \prec U \quad \text{and} \quad V'x_u^{N_u}W' \prec U.$$

Finally we want to define the following elements of $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ for $u, v, w \in L$, $u < v < w$, resp. $u \in L$, $N_u < \infty$, $u \leq v$, resp. $v < u$:

$$\begin{aligned} J(u<v<w) &:= [c^\rho_{(u|v)}, x_w]_{q_{uv,w}} - [x_u, c^\rho_{(v|w)}]_{q_{u,vw}} \\ &\qquad + q_{u,v}x_v[x_u, x_w] - q_{v,w}[x_u, x_w]x_v, \\ L(u, u<v) &:= \big[\underbrace{x_u, \dots [x_u}_{N_u-1}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q_{u,u}^{N_u-1}q_{u,v}} - [d^\rho_u, x_v]_{q_{u,v}^{N_u}}, \\ L(u, u\leq v) &:= \begin{cases} L(u, u<v), & \text{if } u < v, \\ L(u) := -[d^\rho_u, x_u]_1, & \text{if } u = v, \end{cases} \\ L(u, v<u) &:= \big[\dots[c^\rho_{(v|u)}, \underbrace{x_u]_{q_{v,u}q_{u,u}} \dots, x_u}_{N_u-1}\big]_{q_{v,u}q_{u,u}^{N_u-1}} - [x_v, d^\rho_u]_{q_{v,u}^{N_u}}. \end{aligned}$$

Remark 7.2.1. Note that

$$J(u<v<w) \in \big([x_u,x_v]-c^\rho_{(u|v)},[x_v,x_w]-c^\rho_{(v|w)}\big)$$

by the q-Jacobi identity of Proposition 3.2.3, and

$$L(u,u\le v) \in \big([x_u,x_v]-c^\rho_{(u|v)},\ x_u^{N_u}-d_u^\rho\big),\quad L(u,v<u) \in \big([x_v,x_u]-c^\rho_{(v|u)},\ x_u^{N_u}-d_u^\rho\big)$$

by the restricted q-Leibniz formula of Proposition 3.2.3.

7.3 The PBW criterion

Theorem 7.3.1. *Let $L\subset\mathcal{L}$ be Shirshov closed and I be an ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ as in Section 5.3. Then the following assertions are equivalent:*

(1) *The residue classes of $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$ with $t\in\mathbb{N}$, $u_i\in L$, $u_1>\dots>u_t$, $0<r_i<N_{u_i}$, $g\in\Gamma$, form a $\Bbbk$-basis of the quotient algebra $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I$.*

(2) *The algebra $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ respects the following conditions:*
(a) q-Jacobi condition: *$\forall\ u,v,w\in L$, $u<v<w$:*

$$J(u<v<w)\in I_{\prec x_u x_v x_w}.$$

(b) restricted q-Leibniz conditions: *$\forall\ u,v\in L$ with $N_u<\infty$, $u\le v$ resp. $v<u$:*

(i) *$L(u,u\le v)\in I_{\prec x_u^{N_u}x_v}$, resp.*

(ii) *$L(u,v<u)\in I_{\prec x_v x_u^{N_u}}$,*

(2') *The algebra $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ respects the following conditions:*
(a) *Condition* (2a) *only for $uv\notin L$ or $\mathrm{Sh}(uv)\neq(u|v)$.*
(b) (i) *Condition* (2bi) *only for $u=v$ and $u<v$ where $v\neq uv'$ for all $v'\in L$.*

(ii) *Condition* (2bii) *only for $v<u$ where $v\neq v'u$ for all $v'\in L$.*

We need to formulate several statements over the next sections. Afterwards the proof of Theorem 7.3.1 will be carried out in Section 7.7.

7.4 $(\Bbbk\langle X\rangle\#H)/I$ as a quotient of a free algebra

In order to make the diamond lemma applicable for $(\Bbbk\langle X\rangle\#H)/I$, also not just for the regular letters X but for some super letters $[L]$, we will define a quotient of a certain free algebra, which is the special case in Section 7.5 of the following general construction:

In this section let X,S be arbitrary sets such that $X\subset S$, and H be a bialgebra with $\Bbbk$-basis G. Then

$$\Bbbk\langle X\rangle\subset\Bbbk\langle S\rangle\quad\text{and}\quad H=\operatorname{span}_\Bbbk G\subset\Bbbk\langle G\rangle,$$

if we view the set G as variables. Further we set $\langle S, G\rangle := \langle S \cup G\rangle$ where we may assume that the union is disjoint. By omitting $\otimes$

$$\Bbbk\langle X\rangle \otimes H = \operatorname{span}_{\Bbbk}\{ug \mid u \in \langle X\rangle, g \in G\} \subset \Bbbk\langle S, G\rangle$$

Now let $\Bbbk\langle X\rangle$ be a H-module algebra. Next we define the ideals corresponding to the extension of the variable set X to S, and to the smash product structure and the multiplication of H, and study their properties afterwards.

Definition 7.4.1. (1) Let A be an algebra, $B \subset A$ a subset. Then let $(B)_A$ denote the ideal generated by the set B.

(2) Let $f_s \in \Bbbk\langle X\rangle$ for all $s \in S$. Further let $1_H \in G$ and $f_{gh} := gh \in H = \operatorname{span}_{\Bbbk} G$ for all $g, h \in G$. We then define the ideals

$$\begin{aligned} I_S &:= (s - f_s \mid s \in S)_{\Bbbk\langle S,G\rangle}, \\ I_G &:= \big(gs - (g_{(1)} \cdot f_s)g_{(2)},\ gh - f_{gh},\ 1_H - 1 \mid g, h \in G, s \in S\big)_{\Bbbk\langle S,G\rangle}, \end{aligned}$$

where 1 is the empty word in $\Bbbk\langle S, G\rangle$.

Remark 7.4.2. We may assume that $1_H \in G$, if $H \neq 0$: Suppose $1_H \notin G$ and write 1_H as a linear combination of G. Suppose all coefficients are 0, then $1_H = 0_H$ hence $H = 0$; a contradiction. So there is a g with non-zero coefficient and we can exchange this g with 1_H.

Example 7.4.3. Let $H = \Bbbk[\Gamma]$ be the group algebra with the usual bialgebra structure $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Here $G = \Gamma$, $f_{gh} \in \Gamma$ is just the product in the group, and

$$I_\Gamma = \big(gs - (g \cdot f_s)g,\ gh - f_{gh},\ 1_\Gamma - 1 \mid g, h \in \Gamma,\ s \in S\big).$$

Lemma 7.4.4. *For any $g \in \Gamma$ we have*

$$g(\Bbbk\langle S, G\rangle) \subset \operatorname{span}_{\Bbbk}\{ug \mid u \in \langle X\rangle, g \in G\} + I_G.$$

Proof. Let $a_1 \dots a_n \in \langle S, G\rangle$. We proceed by induction on n. If $n = 1$ then either $a_1 \in S$ or $a_1 \in G$. Then either $ga_1 \in (g_{(1)} \cdot f_{a_1})g_{(2)} + I_G \subset \operatorname{span}_{\Bbbk}\{ug \mid u \in \langle X\rangle, g \in G\} + I_G$ or $ga_1 \in f_{ga_1} + I_G \subset \operatorname{span}_{\Bbbk}\{ug \mid u \in \langle X\rangle, g \in G\} + I_G$. If $n > 1$, then let us consider $ga_1a_2 \dots a_n$. Again either $a_1 \in S$ or $a_1 \in G$ and we argue for ga_1 as in the induction basis; then by using the induction hypothesis we achieve the desired form. $\square$

Proposition 7.4.5. *Assume the above situation. Then*

$$\Bbbk\langle X\rangle \# H \cong \Bbbk\langle S, G\rangle/(I_S + I_G),$$

and for any ideal I of $\Bbbk\langle X\rangle \# H$ also $I_S + I_G + I$ is an ideal of $\Bbbk\langle S, G\rangle$ such that

$$(\Bbbk\langle X\rangle \# H)/I \cong \Bbbk\langle S, G\rangle/(I_S + I_G + I).$$

Further we have the following special cases:

$$H \cong \Bbbk: \qquad \Bbbk\langle X\rangle \cong \Bbbk\langle S\rangle/I_S, \qquad \Bbbk\langle X\rangle/I \cong \Bbbk\langle S\rangle/(I_S + I). \tag{7.2}$$

$$S = X: \qquad \Bbbk\langle X\rangle \# H \cong \Bbbk\langle X, G\rangle/I_G, \qquad (\Bbbk\langle X\rangle \# H)/I \cong \Bbbk\langle X, G\rangle/(I_G + I). \tag{7.3}$$

Proof. (1) The algebra map

$$\Bbbk\langle S,G\rangle \to \Bbbk\langle X\rangle\#H, \quad s\mapsto f_s\#1_H, \quad g\mapsto 1_{\Bbbk\langle X\rangle}\#g$$

is surjective and contains I_S+I_G in its kernel; this is a direct calculation using the definitions. Hence we have a surjective algebra map on the quotient

$$\Bbbk\langle S,G\rangle/(I_S+I_G) \longrightarrow \Bbbk\langle X\rangle\#H. \tag{7.4}$$

In order to see that this map is bijective, we verify that a basis is mapped to a basis.

(a) The residue classes of the elements of $\{ug\,|\,u\in\langle X\rangle, g\in G\}$ $\Bbbk$-generate $\Bbbk\langle S,G\rangle/(I_S+I_G)$: Let $A\in\langle S,G\rangle$. Then either $A\in\langle S\rangle$ or it contains an element of G. In the first case $A\in\Bbbk\langle X\rangle+I_S$ by definition of I_S, and then $A\in\Bbbk\langle X\rangle 1_H+I_S+I_G$ since $1_H-1\in I_\Gamma$. In the other case let $A=A_1gA_2$ with $A_1\in\langle S\rangle$, $g\in G$, $A_2\in\langle S,G\rangle$. We argue for A_1 like before, and $gA_2\in\operatorname{span}_\Bbbk\{ug\,|\,u\in\langle X\rangle, g\in G\}+I_G$ by Lemma 7.4.4.

(b) The residue classes of $\{ug\,|\,u\in\langle X\rangle, g\in G\}$ are mapped by Eq. (7.4) to the $\Bbbk$-basis $\langle X\rangle\#G$ of the right-hand side. Hence the residue classes are linearly independent, thus form a basis of $\Bbbk\langle S,G\rangle/(I_S+I_G)$.

(2) $I_S+I_\Gamma+I$ is an ideal: Let $A\in\langle S,G\rangle$ and $a\in I\subset\operatorname{span}_\Bbbk\{ug\,|\,u\in\langle X\rangle, g\in G\}$. Then by (1a) above $A\in\operatorname{span}_\Bbbk\{ug\,|\,u\in\langle X\rangle, g\in G\}+I_S+I_G$, and since I is an ideal of $\Bbbk\langle X\rangle\#H$, we have $Aa, aA\in I_S+I_G+I$ by the isomorphism Eq. (7.4).

Using the isomorphism theorem and part (1) we get

$$\Bbbk\langle S,G\rangle/(I_S+I_G+I) \;\cong\; \big(\Bbbk\langle S,G\rangle/(I_S+I_G)\big)\big/\big((I_S+I_G+I)/(I_S+I_G)\big) \;\cong\; (k\langle X\rangle\#H)/I,$$

where the last $\cong$ holds since $(I_S+I_G+I)/(I_S+I_G)$ is mapped to I by the isomorphism Eq. (7.4).

(3) The special cases follow from the facts that $I_S=0$ if $S=X$, and if $H\cong\Bbbk$ then $G=\{1_H\}$. Hence $I_G=(1_H-1)$ and $\Bbbk\langle X\rangle\cong\Bbbk\langle X\rangle\#\Bbbk\cong\Bbbk\langle S,\{1_H\}\rangle/(I_S+(1_H-1))\cong\Bbbk\langle S\rangle/I_S$. □

Proposition 7.4.5 has various applications for constructing isomorphisms, including classical Examples and Ore extensions:

Example 7.4.6 (Quantum plane)**.** For $0\neq q\in\Bbbk$ let $\mathbf{Q}(q):=\Bbbk\langle x,g\,|\,gx=qxg\rangle$. For $X=\{x,g\}$, $I=(gx-qxg)$, $S=\{x,g_0,g_1=g,g_2,g_3,\ldots\}$ and $I_S=(g_i-g^i\,|\,i\geq 0)$ by Eq. (7.2)

$$\mathbf{Q}(q) \;\cong\; \Bbbk\langle x,g_i;\ i\geq 0 \mid gx=qxg,\ g_i=g^i;\ i\geq 0\rangle.$$

Now let $X=S=\{x\}$, $G=\{g_0,g_1=g,g_2,g_3,\ldots\}$, the monoid $\Gamma=\langle g_i;\ i\geq 0\mid g_i=g^i\rangle\cong\langle g\rangle$ and $H=\Bbbk[\Gamma]\cong\Bbbk[g]$ as in Example 7.4.3. If we define the H-action on $\Bbbk[x]$ by $g\cdot x:=qx$, then $I_G=\big(g_ix-q^ixg_i,\ g_ig_j-g_{i+j},\ g_0-1\,|\,i,j\geq 0\big)=\big(gx-qxg,\ g_i=g^i\,|\,i\geq 0\big)$; the last $=$ is an easy inductive argument. By Eq. (7.3) and the latter isomorphism

$$\mathbf{Q}(q)\cong\Bbbk[x]\#\Bbbk[g].$$

Example 7.4.7 (Weyl algebra)**.** Let $\mathbf{W}:=\Bbbk\langle y,x\,|\,xy=1+yx\rangle$. In a similar way as in Example 7.4.6 we construct

$$\mathbf{W}\cong\Bbbk[y]\#\Bbbk[x],$$

if we set $\Delta(x):=x\otimes 1+1\otimes x$, $\varepsilon(x):=0$ and the action $x\cdot y:=1$.

Example 7.4.8 (Taft algebra)**.** Let $0 \neq q \in \Bbbk$ with $\operatorname{ord} q = N > 1$ and $\mathbf{T}_N(q) := \Bbbk\langle x, g \mid gx = qxg,\ g^N = 1,\ x^N = 0\rangle$. We take $X = \{x, g\}$, $S = \{x, g_0, g_1 = g, g_2, \ldots, g_{N-1}\}$ and $I = (gx - qxg,\ g^N - 1,\ x^N - 0)$. Then by Eq. (7.2)

$$\begin{aligned}\mathbf{T}_N(q) \ \cong\ \Bbbk\langle x, g_i;\ 0 \le i < N \mid gx = qxg,\ g^N = 1,\ x^N = 0,\\ g_i = g^i;\ 0 \le i < N\rangle\end{aligned}$$

Next let $X = S = \{x\}$, $G = \{g_0, g_1 = g, g_2, \ldots, g_{N-1}\}$, the group $\Gamma = \langle g_i;\ i \geq 0 \mid g_i = g^i,\ g^N - 1\rangle \cong \langle g \mid g^N = 1\rangle$ and $H = \Bbbk[\Gamma] \cong \Bbbk[g \mid g^N = 1]$ as in Example 7.4.3. Further let the H-action on $\Bbbk[x]$ be as in Example 7.4.6, and $I = (x^N)$. Then as in Example 7.4.6 by Eq. (7.3)

$$\mathbf{T}_N(q) \cong \big(\Bbbk[x]\#\Bbbk[g|g^N = 1]\big)/(x^N).$$

7.5 The case $S = X_L$ and $H = \Bbbk[\Gamma]$

We now return to the situation of Section 5 and rewrite Proposition 7.4.5:

Corollary 7.5.1. *Let $L \subset \mathcal{L}$ be Shirshov closed and*

$$\begin{aligned} I_L &:= \big(x_u - [x_v, x_w] \mid u \in L,\ \operatorname{Sh}(u) = (v|w)\big)_{\Bbbk\langle X_L, \Gamma\rangle}\\ I'_\Gamma &:= \big(gx_u - \chi_u(g)x_u g,\ gh - f_{gh},\ 1_\Gamma - 1 \mid g, h \in \Gamma,\ u \in L\big)_{\Bbbk\langle X_L, \Gamma\rangle}.\end{aligned}$$

Then for any ideal I of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ also $I_L + I'_\Gamma + I$ is an ideal of $\Bbbk\langle X_L, \Gamma\rangle$ such that

$$(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I \ \cong\ \Bbbk\langle X_L, \Gamma\rangle/(I_L + I'_\Gamma + I).$$

Further we have the analog special cases of Proposition 7.4.5.

Proof. We apply Proposition 7.4.5 to the case $S = X_L$, $H = \Bbbk[\Gamma]$, $f_{x_u} = [u]$ for all $u \in L$. Then $I_{X_L} = \big(x_u - [u] \mid u \in L\big)_{\Bbbk\langle X_L, \Gamma\rangle}$ and I_Γ is as in Example 7.4.3. We are left to prove $I_L + I'_\Gamma + I = I_{X_L} + I_\Gamma + I$, which follows from the Lemma below. □

Lemma 7.5.2. *We have*

(1) $[u] \in x_u + I_L$ *for all* $u \in L$*; hence* $I_{X_L} = I_L$.

(2) $I_\Gamma \subset I'_\Gamma + I_L$

Proof. (2) follows from (1), which we prove by induction on $\ell(u)$: For $\ell(u) = 1$ there is nothing to show. Let $\ell(u) > 1$ and $\operatorname{Sh}(u) = (v|w)$. Then by the induction assumption we have

$$\begin{aligned}[u] = [v][w] - q_{v,w}[w][v] &\in (x_v + I_L)(x_w + I_L) - q_{vw}(x_w + I_L)(x_v + I_L)\\ &\subset [x_v, x_w] + I_L = x_u - \underbrace{\big(x_u - [x_v, x_w]\big)}_{\in I_L} + I_L = x_u + I_L.\end{aligned}$$

□

Example 7.5.3. Let $X = \{x_1, x_2\} \subset L = \{x_1, x_1x_2, x_2\}$. Then $I_L = \big(x_{12} - [x_1, x_2]\big)$ and by Corollary 7.5.1

$$\Bbbk\langle x_1, x_2\rangle \;\cong\; \Bbbk\big\langle x_1, x_{12}, x_2 \;\big|\; x_{12} = [x_1, x_2]\big\rangle,$$

$$\begin{aligned}\Bbbk\langle x_1, x_2\rangle\#\Bbbk[\Gamma] \;\cong\; \Bbbk\langle x_1, x_{12}, x_2, \Gamma \mid x_{12} = [x_1, x_2],\\ gx_u = \chi_u(g)x_ug,\ gh = f_{gh},\ 1_\Gamma - 1; \forall u \in L, g, h \in \Gamma\rangle.\end{aligned}$$

For more Examples see Chapters 8 and 9.

7.6 Bergman's diamond lemma

Following Bergman [16], let Y be a set, $\Bbbk\langle Y\rangle$ the free $\Bbbk$-algebra and Σ an index set. We fix a subset $\mathcal{R} = \{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\} \subset \langle Y\rangle \times \Bbbk\langle Y\rangle$, and define the ideal

$$I_{\mathcal{R}} := (W_\sigma - f_\sigma \mid \sigma \in \Sigma)_{\Bbbk\langle Y\rangle}.$$

An *overlap of* $\mathcal{R}$ is a triple (A, B, C) such that there are $\sigma, \tau \in \Sigma$ and $A, B, C \in \langle Y\rangle\backslash\{1\}$ with $W_\sigma = AB$ and $W_\tau = BC$. In the same way an *inclusion of* $\mathcal{R}$ is a triple (A, B, C) such that there are $\sigma \neq \tau \in \Sigma$ and $A, B, C \in \langle Y\rangle$ with $W_\sigma = B$ and $W_\tau = ABC$.

Let $\preceq_\diamond$ be a *with* $\mathcal{R}$ *compatible well-founded monoid partial ordering* of the free monoid $\langle Y\rangle$, i.e.:

- $(\langle Y\rangle, \preceq_\diamond)$ is a partial ordered set.
- $B \prec_\diamond B' \Rightarrow ABC \prec_\diamond AB'C$ for all $A, B, B', C \in \langle Y\rangle$.
- Each non-empty subset of $\langle Y\rangle$ has a minimal element w.r.t. $\preceq_\diamond$.
- f_σ is a linear combination of monomials $\prec_\diamond W_\sigma$ for all $\sigma \in \Sigma$; in this case we write $f_\sigma \prec_\diamond W_\sigma$.

For any $A \in \langle Y\rangle$ let $I_{\prec_\diamond A}$ denote the subspace of $\Bbbk\langle Y\rangle$ spanned by all elements $B(W_\sigma - f_\sigma)C$ with $B, C \in \langle Y\rangle$ such that $BW_\sigma C \prec_\diamond A$. The next theorem is a short version of the diamond lemma:

Theorem 7.6.1. [16, Thm 1.2] *Let* $\mathcal{R} = \{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\} \subset \langle Y\rangle \times \Bbbk\langle Y\rangle$ *and* $\preceq_\diamond$ *be a with* $\mathcal{R}$ *compatible well-founded monoid partial ordering on* $\langle Y\rangle$. *Then the following conditions are equivalent:*

(1) (a) $f_\sigma C - Af_\tau \in I_{\prec_\diamond ABC}$ *for all overlaps* (A, B, C).
    (b) $Af_\sigma C - f_\tau \in I_{\prec_\diamond ABC}$ *for all inclusions* (A, B, C).

(2) *The residue classes of the elements of* $\langle Y\rangle$ *which do not contain any* W_σ *with* $\sigma \in \Sigma$ *as a subword form a* $\Bbbk$*-basis of* $\Bbbk\langle Y\rangle/I_{\mathcal{R}}$.

We now define the ordering for our situation, where $L \subset \mathcal{L}$ is Shirshov closed and $Y = X_L \cup \Gamma$: Let $\pi_L : \langle X_L, \Gamma\rangle \to \langle X_L\rangle$ be the monoid map with $x_u \mapsto x_u$ and $g \mapsto 1$ for all $u \in L$, $g \in \Gamma$ (π_L deletes all g in a word of $\langle X_L, \Gamma\rangle$).

Moreover, for a $A \in \langle X_L, \Gamma\rangle$ let $n_\Gamma(A)$ denote the number of letters $g \in \Gamma$ in the word A and $t(A)$ the $n_\Gamma(A)$-tuple of non-negative integers

$$\begin{aligned}&(\text{number of letters after the last } g \in \Gamma \text{ in } A, \dots,\\ &\qquad\qquad \dots, \text{number of letters after the first } g \in \Gamma \text{ in } A) \in \mathbb{N}^{n_\Gamma(A)}.\end{aligned}$$

Definition 7.6.2. For $A, B \in \langle X_L, \Gamma\rangle$ we define $A \prec_\diamond B$ by

- $\pi_L(A) \prec \pi_L(B)$, or
- $\pi_L(A) = \pi_L(B)$ and $n_\Gamma(A) < n_\Gamma(B)$, or
- $\pi_L(A) = \pi_L(B)$, $n_\Gamma(A) = n_\Gamma(B)$ and $t(A) < t(B)$ under the lexicographical order of $\mathbb{N}^{n_\Gamma(A)}$, i.e., $t(A) \neq t(B)$, and the first non-zero term of $t(B) - t(A)$ is positive.

$\preceq_\diamond$ is a well-founded monoid partial ordering of $\langle X_L, \Gamma\rangle$, which is straightforward to verify, and will be compatible with the later regarded $\mathcal{R}$.

Note that we have the following correspondence between $\prec$ of Section 4.4 and $\prec_\diamond$, which follows from the definitions: For any $U, V \in [L]^{(\mathbb{N})}$, $g, h \in \Gamma$ we have $\rho(U)g, \rho(V)h \in \langle X_L\rangle\Gamma$ and

$$U \prec V \iff \rho(U)g \prec_\diamond \rho(V)h. \tag{7.5}$$

7.7 Proof of Theorem 7.3.1

Again suppose the assumptions of Theorem 7.3.1. By Corollary 7.5.1

$$(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I \cong \Bbbk\langle X_L, \Gamma\rangle/(I_L + I'_\Gamma + I),$$

thus $(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I$ has the basis

$$[u_1]^{r_1}[u_2]^{r_2} \dots [u_t]^{r_t} g$$

if and only if $\Bbbk\langle X_L, \Gamma\rangle/(I_L + I'_\Gamma + I)$ has the basis

$$x_{u_1}^{r_1} x_{u_2}^{r_2} \dots x_{u_t}^{r_t} g$$

($t \in \mathbb{N}$, $u_i \in L$, $u_1 > \ldots > u_t$, $0 < r_i < N_u$, $g \in \Gamma$). The latter we can reformulate equivalently in terms of the Diamond Lemma 7.6.1:

- We define $\mathcal{R}$ as the set of the elements

$$(1_\Gamma,\ 1), \tag{7.6}$$
$$(gh,\ f_{gh}), \text{ for all } g, h \in \Gamma, \tag{7.7}$$
$$\big(gx_u,\ \chi_u(g)x_u g\big), \text{ for all } g \in \Gamma, u \in L, \tag{7.8}$$
$$\big(x_u x_v,\ c^\rho_{(u|v)} + q_{u,v} x_v x_u\big), \text{ for all } u, v \in L \text{ with } u < v, \tag{7.9}$$
$$\big(x_u^{N_u},\ d^\rho_u\big), \text{ for all } u \in L \text{ with } N_u < \infty, \tag{7.10}$$

where we again see $c^\rho_{(u|v)}, d^\rho_u \in \Bbbk\langle X_L\rangle \otimes \Bbbk[\Gamma] \subset \operatorname{span}_\Bbbk\{Ug \mid U \in \langle X_L\rangle, g \in \Gamma\} \subset \Bbbk\langle X_L, \Gamma\rangle$. Then the residue classes of $c^\rho_{(u|v)}, d^\rho_u$ modulo $I_L + I'_\Gamma$ correspond to $c_{(u|v)}$ and d_u by the isomorphism of Corollary 7.5.1, and we have $I_{\mathcal{R}} = I_L + I'_\Gamma + I$.

• Note that $\prec_\diamond$ is compatible with $\mathcal{R}$: In Eq. (7.6) resp. (7.7) we have $1 \prec_\diamond 1_\Gamma$ resp. $f_{gh} \prec_\diamond gh$ since $n_\Gamma(1) = 0 < 1 = n_\Gamma(1_\Gamma)$ resp. $n_\Gamma(f_{gh}) = 1 < 2 = n_\Gamma(gh)$ ($f_{gh} \in \Gamma$). Eq. (7.8): $t(x_u g) = (0) < (1) = t(gx_u)$, hence $x_u g \prec_\diamond gx_u$. Moreover, by Lemma 5.3.1 we have $c^\rho_{(u|v)} + q_{u,v}x_v x_u \prec_\diamond x_u x_v$, and $d^\rho_u \prec_\diamond x_u^{N_u}$ by assumption.

• By the Diamond Lemma 7.6.1 we have to consider all possible overlaps and inclusions of $\mathcal{R}$. The only inclusions happen with Eq. (7.6), namely $(1, 1_\Gamma, h)$, $(g, 1_\Gamma, 1)$, $(1, 1_\Gamma, x_u)$. But they all fulfill the condition (1b) of the Diamond Lemma 7.6.1: for example $h - f_{1_\Gamma h} = h - h = 0 \in I_{\prec_\diamond 1_\Gamma h}$, and $x_u - \chi_u(1_\Gamma)x_u 1_\Gamma = x_u(1_\Gamma - 1) \in I_{\prec_\diamond 1_\Gamma x_u}$.

So we are left to check the conditon (1a) for all overlaps: (g, h, k) with $g, h, k \in \Gamma$ fulfills it by the associativity of Γ; for (g, h, x_u) we have

$$f_{gh}x_u - \chi_u(h)gx_u h = \chi_u(gh)x_u f_{gh} - \chi_u(h)\chi_u(g)x_u gh = 0,$$

calculating modulo $I_{\prec_\diamond ghx_u}$ and using $\chi_u(f_{gh}) = \chi_u(gh)$ since $f_{gh} \in \Gamma$. The next overlap is (g, x_u, x_v) where $u < v$: Calculating modulo $I_{\prec_\diamond gx_u x_v}$ we get

$$\begin{aligned}\chi_u(g)x_u g x_v - g\big(c^\rho_{(u|v)} + q_{u,v}x_v x_u\big)&\\ = \chi_u(g)\chi_v(g)x_u x_v g - \chi_{uv}(g)\big(c^\rho_{(u|v)} + q_{u,v}x_v x_u\big)g&\\ = \chi_{uv}(g)\big(x_u x_v - \big(c^\rho_{(u|v)} + q_{u,v}x_v x_u\big)\big)g = 0,&\end{aligned}$$

since $c_{(u|v)} \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi_{uv}}$ and $x_u x_v g \prec_\diamond gx_u x_v$. For the overlap $(g, x_u, x_u^{N_u - 1})$ we obtain modulo $I_{\prec_\diamond g x_u^{N_u}}$

$$\chi_u(g)x_u g x_u^{N_u-1} - g d^\rho_u = \chi_u(g)^{N_u}\big(x_u^{N_u} - d^\rho_u\big)g = 0,$$

because $d_u \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi_u^{N_u}}$ and $x_u^{N_u}\vartheta_g \prec_\diamond \vartheta_g x_u^{N_u}$. The remaining overlaps are those with Eqs. (7.9) and (7.10); for these we formulate the following Lemmata:

Lemma 7.7.1. *The overlap (x_u, x_v, x_w), $u < v < w$, fulfills condition 7.6.1(1a), i.e., $a := \big(c^\rho_{(u|v)} + q_{u,v}x_v x_u\big)x_w - x_u\big(c^\rho_{(v|w)} + q_{v,w}x_w x_v\big) \in I_{\prec_\diamond x_u x_v x_w}$, if and only if $J(u < v < w) \in I_{\prec_\diamond x_u x_v x_w}$.*

Proof. We calculate in $\Bbbk\langle X_L, \Gamma\rangle$

$$\begin{aligned}J(u<v<w) &= c^\rho_{(u|v)}x_w - q_{uv,w}x_w c^\rho_{(u|v)} - \big(x_u c^\rho_{(v|w)} - q_{u,vw}c^\rho_{(v|w)}x_u\big)\\ &\quad + q_{u,v}x_v\big(x_u x_w - q_{u,w}x_w x_u\big) - q_{v,w}\big(x_u x_w - q_{u,w}x_w x_u\big)x_v,\\ a &= c^\rho_{(u|v)}x_w + q_{u,v}x_v x_u x_w - x_u c^\rho_{(v|w)} - q_{v,w}x_u x_w x_v,\end{aligned}$$

and show that the difference is zero modulo $I_{\prec_\diamond x_u x_v x_w}$:

$$\begin{aligned}J(u<v<w) - a &= q_{uv,w}x_w\big(x_u x_v - c^\rho_{(u|v)}\big) + q_{u,vw}\big(c^\rho_{(v|w)} - x_v x_w\big)x_u\\ &= q_{uv,w}x_w\big(q_{u,v}x_v x_u\big) - q_{u,vw}\big(q_{v,w}x_w x_v\big)x_u = 0.\end{aligned}$$

since $x_w x_u x_v, x_v x_w x_u \prec_\diamond x_u x_v x_w$. □

Lemma 7.7.2. *The overlaps* $\big(x_u^{N_u-1}, x_u, x_v\big)$ *resp.* $\big(x_u, x_v, x_v^{N_v-1}\big)$ *fulfill condition 7.6.1(1a), i.e.,* $d_u^\rho x_v - x_u^{N_u-1}\big(c^\rho_{(u|v)} + q_{u,v}x_vx_u\big) \in I_{\prec_\diamond x_u^{N_u}x_v}$ *resp.* $\big(c^\rho_{(u|v)} + q_{uv}x_vx_u\big)x_v^{N_v-1} - x_ud_v^\rho \in I_{\prec_\diamond x_ux_v^{N_v}}$ *if and only if* $L(u,u<v) \in I_{\prec_\diamond x_u^{N_u}x_v}$ *resp.* $L(u,u>v) \in I_{\prec_\diamond x_vx_u^{N_u}}$.

Proof. We prove it for $\big(x_u^{N_u-1}, x_u, x_v\big)$; the other overlap is proved analogously. We set $r := N_u - 1$, then $\operatorname{ord} q_{u,u} = r+1$. Using the q-Leibniz formula of Proposition 3.2.3 we get

$$
\begin{aligned}
&x_u^r\big(c^\rho_{(u|v)} + q_{u,v}x_vx_u\big) - d_u^\rho x_v = \\
&= \big[x_u^r, c^\rho_{(u|v)}\big]_{q^r_{u,u}q_{u,v}} + q^r_{u,u}q_{u,v}c^\rho_{(u|v)}x_u^r \\
&\qquad\qquad + q_{u,v}\big[x_u^r, x_v\big]_{q^r_{u,v}}x_u + q^{r+1}_{u,v}x_vx_u^{r+1} - d_u^\rho x_v \\
&= \sum_{i=0}^{r} q^i_{u,u}q^i_{u,v}\tbinom{r}{i}_{q_{u,u}}\big[\underbrace{x_u,\dots[x_u}_{r-i}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q^{r-i}_{u,u}q_{u,v}}x_u^i \\
&\quad + \sum_{i=0}^{r-1} q^{i+1}_{u,v}\tbinom{r}{i}_{q_{u,u}}\big[\underbrace{x_u,\dots[x_u}_{r-i}, x_v]_{q_{u,v}}\dots\big]_{q^{r-i-1}_{u,u}q_{u,v}}x_u^{i+1} + q^{r+1}_{u,v}x_vx_u^{r+1} - d_u^\rho x_v.
\end{aligned}
$$

Because of $x_u^{r-i}x_vx_u^{i+1} \prec_\diamond x_u^{r+1}x_v$ for all $0 \le i \le r$, this is modulo $I_{\prec_\diamond x_u^{r+1}x_v}$ equal to

$$
\begin{aligned}
&\sum_{i=0}^{r} q^i_{u,u}q^i_{u,v}\tbinom{r}{i}_{q_{u,u}}\big[\underbrace{x_u,\dots[x_u}_{r-i}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q^{r-i}_{u,u}q_{u,v}}x_u^i \\
&\quad + \sum_{i=0}^{r-1} q^{i+1}_{u,v}\tbinom{r}{i}_{q_{u,u}}\big[\underbrace{x_u,\dots[x_u}_{r-i-1}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q^{r-i-1}_{u,u}q_{u,v}}x_u^{i+1} - \big[d_u^\rho, x_v\big]_{q^{r+1}_{u,v}}.
\end{aligned}
$$

Now shifting the index of the second sum, we obtain

$$
\begin{aligned}
&\big[\underbrace{x_u,\dots[x_u}_{r}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q^r_{u,u}q_{u,v}} - \big[d_u^\rho, x_v\big]_{q^{r+1}_{u,v}} \\
&\quad + \sum_{i=1}^{r} q^i_{u,v}\Big(q^i_{u,u}\tbinom{r}{i}_{q_{u,u}} + \tbinom{r}{i-1}_{q_{u,u}}\Big)\big[\underbrace{x_u,\dots[x_u}_{r-i}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q^{r-i}_{u,u}q_{u,v}}x_u^i.
\end{aligned}
$$

Finally we obtain the claim, since $q^i_{u,u}\tbinom{r}{i}_{q_{u,u}} + \tbinom{r}{i-1}_{q_{u,u}} = \tbinom{r+1}{i}_{q_{u,u}} = 0$ for all $1 \le i \le r$, by Eq. (3.2) and $\operatorname{ord} q_{u,u} = r+1$. □

Lemma 7.7.3. *The overlaps* $\big(x_u^{N_u-i}, x_u^i, x_u^{N_u-i}\big)$ *fulfill condition 7.6.1(1a) for all* $1 \le i < N_u$*, if and only if the overlap* $\big(x_u^{N_u-1}, x_u, x_u^{N_u-1}\big)$ *fulfills condition 7.6.1(1a), if and only if* $L(u) \in I_{\prec_\diamond x_u^{N_u+1}}$.

Proof. This is evident. □

• Finally the assertions of the last three Lemmata are equivalent to (2) of the Theorem 7.3.1, which follows from the following Lemma:

Lemma 7.7.4. *Let $a \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ and $W \in [L]^{(\mathbb{N})}$ such that $a \prec_L W$. Further let $a^\rho \in \Bbbk\langle X_L\rangle\#\Bbbk[\Gamma] \subset \Bbbk\langle X_L, \Gamma\rangle$ as defined in Section 7.2. Then $a^\rho \in I_{\prec_\diamond \rho(W)}$ in $\Bbbk\langle X_L, \Gamma\rangle$ if and only if $a^\rho \in I_{\prec \rho(W)}$ in $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$.*

Proof. By definition a^ρ is a linear combination of $U \in \langle X_L\rangle$ with $\ell(U) = \ell(W)$, $U > \rho(W)$, and Vg, $V \in \langle X_L\rangle, g \in \Gamma$ with $\ell(V) < \ell(W)$. Note that the only elements Γ in a^ρ occur in monomials Vg with $\ell(V) < \ell(W)$. Thus the only relations Eqs. (7.6),(7.7),(7.8) of I'_Γ which apply to a^ρ are already contained in $I_{\prec_\diamond W}$ since $Vg \prec_\diamond W$, hence: $a^\rho \in I_{\prec_\diamond \rho(W)}$ in $\Bbbk\langle X_L, \Gamma\rangle$ $\Leftrightarrow a^\rho \in I_{\prec_\diamond \rho(W)} + I'_\Gamma$ in $\Bbbk\langle X_L, \Gamma\rangle \Leftrightarrow a^\rho \in I_{\prec \rho(W)}$ in $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$, the latter equivalence by the isomorphism $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma] \cong \Bbbk\langle X_L, \Gamma\rangle/I'_\Gamma$ of Eq. (7.3) applied for $X = X_L$. $\square$

• We are left to prove the equivalence of (2) to its weaker version (2') of Theorem 7.3.1: For (2'a) we show that if $uv \in L$ and $\mathrm{Sh}(uv) = (u|v)$, then conditon (2a) is already fulfilled: By definition $c^\rho_{(u|v)} = x_{uv}$ and

$$\left[c^\rho_{(u|v)}, x_w\right]_{q_{uv,w}} = \left[x_{uv}, x_w\right] = c^\rho_{(uv|w)}$$

modulo $I_{\prec x_u x_v x_w}$. Now certainly $\mathrm{Sh}(uvw) \neq (uv|w)$, thus

$$c^\rho_{(uv|w)} = \partial^\rho_u(c^\rho_{(v|w)}) + q_{v,w} c^\rho_{(u|w)} x_v - q_{u,v} x_v c^\rho_{(u|w)}$$

by Eq. (7.1). Hence in this case the q-Jacobi condition is fulfilled by the q-derivation formula of Proposition 3.2.3.

For (2'b) of Theorem 7.3.1 it is enough to show the following: Let condition (2bi) hold for $u = v$, i.e., $[x_u, d^\rho_u]_1 \in I_{\prec x_u^{N_u+1}}$. Then, if condition (2bi) holds for some $u < v$ with $N_u < \infty$, then (2bi) also holds for $u < uv$ (whenever $uv \in L$). Analogously, if (2bii) holds for $v < u$ with $N_u < \infty$, then also (2bii) holds for $vu < u$ (whenever $vu \in L$).

Note that if $u < v$, then $uv < v$: Either v does not begin with u, then $uv < v$ by Lemma 4.1.1; or let $v = uw$ for some $w \in \langle X\rangle$. Then $u < v = uw < w$ since $v \in \mathcal{L}$. Hence $uv = uuw < uw = v$.

We will prove the first part (2'bi), (2'bii) is the same argument. But before we formulate the following

Lemma 7.7.5. *Let $a \in \Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$, $A, W \in \langle X_L\rangle$ such that $a \preceq_L A \prec W$. Then $a \in I_{\prec W}$ if and only if $a \in I_{\preceq A}$.*

Proof. Clearly $I_{\preceq A} \subset I_{\prec W}$, since $A \prec W$. So denote by $\{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ the set of Eqs. (7.9) and (7.10) with $f_\sigma \prec_L W_\sigma$, and let $a \in I_{\prec W}$, i.e., a is a linear combination of $Ug(W_\sigma - f_\sigma)Vh$ with $U, V \in \langle X_L\rangle$ such that $UW_\sigma V \prec W$. Denote by E the $\prec$-biggest word of all $UW_\sigma V$ with non-zero coefficient. $E \succ A$ contradicts the assumption $a \preceq_L A \prec W$. Hence $E \preceq A$ and therefore $f \in I_{\preceq A}$. $\square$

Suppose (2bi) for $u < v$ with $N_u < \infty$ and $uv \in L$, i.e.,

$$\begin{aligned}
&\big[\underbrace{x_u, \ldots [x_u}_{N_u - 1}, x_{uv}]_{q_{u,u}q_{u,v}} \ldots\big]_{q_{u,u}^{N_u-1}q_{u,v}} - [d^\rho_u, x_v]_{q_{u,v}^{N_u}} \in I_{\prec x_u^{N_u} x_v} \\
\Leftrightarrow &\big[\underbrace{x_u, \ldots [x_u}_{N_u - 2}, c^\rho_{(u|uv)}]_{q_{u,u}^2 q_{u,v}} \ldots\big]_{q_{u,u}^{N_u-1}q_{u,v}} - [d^\rho_u, x_v]_{q_{u,v}^{N_u}} \in I_{\preceq x_u^{N_u-1} x_w U x_v},
\end{aligned}$$

for some $w \in L$ with $w > u$ and $U \in \langle X_L \rangle$ such that $\ell(U) + \ell(w) = \ell(u)$. Here we used the relation $[x_u, x_{uv}]_{q_{u,uv}} - c^{\rho}_{(u|uv)}$, and Lemma 7.7.5 since the above polynomial is $\preceq x_u^{N_u-1} x_w U x_v$ (by assumption $c_{(u|uv)} \preceq_L [uuv]$, $d_u \prec_L [u]^{N_u}$). Hence the condition (2bi) for $u < uv$ reads

$$\big[\underbrace{x_u, \dots [x_u}_{N_u-1}, c^{\rho}_{(u|uv)}]_{q^2_{u,u} q_{u,v}} \dots\big]_{q^{N_u}_{u,u} q_{u,v}} - [d^{\rho}_u, x_{uv}]_{q^{N_u}_{u,u} q^{N_u}_{u,v}} \in I_{\prec x_u^{N_u} x_{uv}}$$
$$\Leftrightarrow \big[x_u, [d^{\rho}_u, x_v]_{q^{N_u}_{u,v}}\big]_{q^{N_u}_{u,u} q_{u,v}} - [d^{\rho}_u, x_{uv}]_{q^{N_u}_{u,u} q^{N_u}_{u,v}} \in I_{\prec x_u^{N_u} x_{uv}},$$

since $x_u I_{\preceq x_u^{N_u-1} x_w U x_v}, I_{\preceq x_u^{N_u-1} x_w U x_v} x_u \subset I_{\prec x_u^{N_u} x_{uv}}$ ($w > u$ and w cannot begin with u since $\ell(w) \leq \ell(u)$, hence $w > uv$ by Lemma 4.1.1). By the q-Jacobi identity

$$\begin{aligned}\big[x_u, [d^{\rho}_u, x_v]_{q^{N_u}_{u,v}}\big]_{q^{N_u}_{u,u} q_{u,v}} &= \big[[x_u, d^{\rho}_u]_{q^{N_u}_{u,u}}, x_v\big]_{q^{N_u+1}_{u,v}} + q^{N_u}_{u,u} d^{\rho}_u [x_u, x_v] - q^{N_u}_{u,v} [x_u, x_v] d^{\rho}_u \\ &= \big[[x_u, d^{\rho}_u]_1, x_v\big]_{q^{N_u+1}_{u,v}} + [d^{\rho}_u, x_{uv}]_{q^{N_u}_{u,v}} = [d^{\rho}_u, x_{uv}]_{q^{N_u}_{u,v}}.\end{aligned}$$

For the last two "=" we used $q^{N_u}_{u,u} = 1$, the relation $[x_u, x_v] - x_{uv}$ and $[x_u, d^{\rho}_u]_1 \in I_{\prec x_u^{N_u+1}}$ (We can use this condition: Note that $[x_u, d^{\rho}_u]_1 \preceq x_u^{N_u} x_{w'} U'$ for some $w' \in L$, $w' > u$, $U' \in \langle X_L \rangle$, $\ell(U') + \ell(w') = \ell(u)$, hence $[x_u, d^{\rho}_u]_1 \in I_{\preceq x_u^{N_u} x_{w'} U'}$ by Lemma 7.7.5. Therefore $x_v I_{\preceq x_u^{N_u} x_{w'} U'}, I_{\preceq x_u^{N_u} x_{w'} U'} x_v \subset I_{\prec x_u^{N_u} x_{uv}}$, like before).

Chapter 8

PBW basis in rank one

Let V be a 1-dimensional vector space with basis x_1 and $\mathrm{ord} q_{11} = N \leq \infty$. Since $T(V) \cong \Bbbk[x_1]$ we have $\mathcal{L} = \{x_1\}$. We give the condition when

$$(T(V)\#\Bbbk[\Gamma])/\big(x_1^N - d_1\big)$$

has the PBW basis $\{x_1\}$. By the PBW Criterion 7.3.1 the only condition is

$$[d_1^\rho, x_1]_1 \in I_{\prec x_1^{N+1}}$$

in $\Bbbk[x_1]\#\Bbbk[\Gamma]$. This clearly is fulfilled if $d_1 = 0$ and we get directly the following examples:

Example 8.0.6 (Nichols algebra A_1)**.** The set $\{x_1^r \,|\, 0 \leq r < N\}$ is a basis of

$$\mathfrak{B}(V) = T(V)/\big(x_1^N\big),$$

the Nichols algebra of Cartan type A_1.

Example 8.0.7 (Taft algebra)**.** The set $\{x_1^r g \mid 0 \leq r < N, g \in \mathbb{Z}_N\}$ is a basis of

$$\mathbf{T}_N(q_{11}) \cong (T(V)\#\Bbbk[\mathbb{Z}_N])/(x_1^N),$$

see Example 7.4.8.

Example 8.0.8 (Liftings A_1)**.** The set $\{x_1^r g \,|\, 0 \leq r < N, g \in \Gamma\}$ is a basis of

$$(T(V)\#\Bbbk[\Gamma])/\big(x_1^N - \mu_1(1 - g_1^N)\big),$$

which are the liftings of $\mathfrak{B}(V)$ of Cartan type A_1.

Proof. It is $d_1 \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_1^N}$ by Definition 6.0.7 of μ_1. Further

$$\big[\mu_1(1 - g_1^N), x_1\big]_1 = \mu_1\big[1, x_1\big]_1 - \mu_1\big[g_1^N, x_1\big]_1 = -\mu_1(q_{11}^N - 1)x_1 g_1^N = 0,$$

since $\mathrm{ord} q_{11} = N$. $\square$

Chapter 9

PBW basis in rank two and redundant relations

Let V be a 2-dimensional vector space with basis x_1, x_2, hence $T(V) \cong \Bbbk\langle x_1, x_2\rangle$. In this chapter we apply the PBW Criterion 7.3.1 to verify for certain $L \subset \mathcal{L}$ that the algebra

$$(T(V)\#\Bbbk[\Gamma])/I,$$

with I as in Section 5.3, has the PBW basis $[L]$. In particular, we examine the Nichols algebras and their liftings of Chapter 6. Moreover, we will see how to find the redundant relations, and in addition, we will treat some classical examples.

9.1 PBW basis for $L = \{x_1 < x_2\}$

This is the easiest case and covers the Cartan Type $A_1 \times A_1$, see Section 6.2, as well as many other examples. We are interested when $[L]$ builds up a PBW Basis of

$$(T(V)\#\Bbbk[\Gamma])/\big([x_1x_2] - c_{12},\ x_1^{N_1} - d_1,\ x_2^{N_2} - d_2\big),$$

with $N_1 = \mathrm{ord} q_{11}, N_2 = \mathrm{ord} q_{22} \in \{2, 3, \dots, \infty\}$. If $N_1 = N_2 = \infty$, then by the PBW Criterion 7.3.1 there is no condition in $\Bbbk\langle x_1, x_2\rangle\#\Bbbk[\Gamma]$ such that we can choose c_{12} arbitrarily with $c_{12} \prec_L [x_1x_2]$ and $\deg_{\widehat{\Gamma}}(c_{12}) = \chi_1\chi_2$:

Example 9.1.1 (Quantum Plane)**.** See also Example 7.4.6.

$$\mathbf{Q}(q_{12}) \cong T(V)/([x_1x_2])$$

has the basis $\{x_2^{r_2}x_1^{r_1} \mid r_2, r_1 \geq 0\}$ since $c_{12} = 0$; of course this can be seen in Example 7.4.6 directly via the decomposition into a smash product.

Example 9.1.2 (Nichols algebra $A_1 \times A_1$)**.** In the case $q_{12}q_{21} = 1$, the latter example is the infinite dimensional Nichols algebra of Cartan Type $A_1 \times A_1$ with basis $\{x_2^{r_2}x_1^{r_1} \mid r_2, r_1 \geq 0\}$.

Example 9.1.3 (Weyl algebra)**.** If $q_{12} = 1$, then

$$\mathbf{W} \cong T(V)/([x_1x_2] - 1),$$

see Example 7.4.7. This relation is $\widehat{\Gamma}$-homogeneous if $\chi_1\chi_2 = \varepsilon$, e.g., take $\Gamma = \{1\}$. Then $\mathbf{W}$ has the basis $\{x_2^{r_2}x_1^{r_1} \mid r_2, r_1 \geq 0\}$; again this can be seen directly in Example 7.4.7.

If $\operatorname{ord} q_{11} = N_1 < \infty$ or $\operatorname{ord} q_{22} = N_2 < \infty$, then by the PBW Criterion 7.3.1 we have to check

$$\big[d_1^\rho, x_1\big]_1 \in I_{\prec x_1^{N_1+1}}, \quad \text{or} \quad \big[d_2^\rho, x_2\big]_1 \in I_{\prec x_2^{N_2+1}}, \text{ and} \tag{9.1}$$

$$\big[\underbrace{x_1, \dots \big[x_1}_{N_1-1}, c_{12}^\rho\big]_{q_{11}q_{12}} \dots\big]_{q_{11}^{N_1-1}q_{12}} - \big[d_1^\rho, x_2\big]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1}x_2}, \text{ or} \tag{9.2}$$

$$\big[\dots\big[c_{12}^\rho, \underbrace{x_2\big]_{q_{12}q_{22}} \dots, x_2}_{N_2-1}\big]_{q_{12}q_{22}^{N_2-1}} - \big[x_1, d_2^\rho\big]_{q_{12}^{N_2}} \in I_{\prec x_1 x_2^{N_2}}. \tag{9.3}$$

This is the case when $d_1 = d_2 = c_{12} = 0$:

Example 9.1.4 (Nichols algebra $A_1 \times A_1$)**.** The set $\{x_2^{r_2}x_1^{r_1} \mid 0 \le r_i < N_i\}$ is a basis of

$$T(V)/\big([x_1x_2],\, x_1^{N_1},\, x_2^{N_2}\big).$$

This includes the finite-dimensional Nichols algebra of Cartan type $A_1 \times A_1$, where $q_{12}q_{21} = 1$.

Example 9.1.5 (Liftings $A_1 \times A_1$)**.** Let $q_{12}q_{21} = 1$. Then $\{x_2^{r_2}x_1^{r_1}g \mid 0 \le r_i < N_i, g \in \Gamma\}$ is a basis of

$$(T(V)\#\Bbbk[\Gamma])/\big([x_1x_2] - \lambda_{12}(1-g_{12}),\ x_1^{N_1} - \mu_1(1-g_1^{N_1}),\ x_2^{N_2} - \mu_2(1-g_2^{N_2})\big),$$

which are the liftings of the Nichols algebras of Cartan type $A_1 \times A_1$.

Proof. By definition of $\lambda_{12}, \mu_1, \mu_2$ the elements have the required $\widehat{\Gamma}$-degree. As in Example 8.0.8 we show conditions Eq. (9.1).

Eq. (9.2): We have $\chi_1\chi_2 = \varepsilon$ if $\lambda_{12} \neq 0$, hence $q_{11}q_{12} = 1$ and then $q_{11} = q_{11}q_{12}q_{21} = q_{21}$, since $q_{12}q_{21} = 1$. Using these equations we calculate

$$\big[\underbrace{x_1, \dots \big[x_1}_{N_1-1}, \lambda_{12}(1-g_1g_2)\big]_{q_{11}q_{12}} \dots\big]_{q_{11}^{N_1-1}q_{12}} = -\lambda_{12}(1-q_{11}^2)\dots(1-q_{11}^{N_1})x_1^{N_1-1}g_1g_2 = 0.$$

Further $\chi_i^{N_i} = \varepsilon$ if $\mu_1 \neq 0$, thus $q_{21}^{N_1} = 1$; by taking $q_{12}q_{21} = 1$ to the N_1-th power, we deduce $q_{12}^{N_1} = 1$. Then

$$\big[\mu_1(1-g_1^{N_1}), x_2\big]_{q_{12}^{N_1}} = \mu_1(1-q_{12}^{N_1})x_2 = 0.$$

The remaining condition Eq. (9.3) works in a similar way. $\square$

Finally we want to mention that there are also many other non-graded quotient algebras for which our PBW Basis Criterion 7.3.1 works. Direct computation gives

Example 9.1.6. For $q_{12} = 1$ and $q_{22} = -1$ the set $\{x_2^{r_2}x_1^{r_1} \mid 0 \le r_2 < 2,\ 0 \le r_1 < N_1\}$ is a basis of

$$T(V)/\big([x_1x_2],\ x_1^{N_1} - x_2,\ x_2^2 - 1\big).$$

9.2 PBW basis for $L = \{x_1 < x_1x_2 < x_2\}$

We now examine the case when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$, where I is generated by the following elements

$$\begin{aligned} &[x_1x_1x_2] - c_{112}, & x_1^{N_1} - d_1,\\ &[x_1x_2x_2] - c_{122}, & [x_1x_2]^{N_{12}} - d_{12},\\ & & x_2^{N_2} - d_2, \end{aligned}$$

with $\mathrm{ord} q_{11} = N_1$, $\mathrm{ord} q_{12,12} = N_{12}$, $\mathrm{ord} q_{22} = N_2 \in \{2, 3, \dots, \infty\}$. We have in $\Bbbk\langle x_1, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ the elements

$$c^\rho_{(1|12)} = c^\rho_{112}, \quad c^\rho_{(1|2)} = x_{12}, \quad c^\rho_{(12|2)} = c^\rho_{122}.$$

At first we want to study the conditions in general. By Theorem 7.3.1(2') we have to check the following: The only Jacobi condition is for $1 < 12 < 2$, namely

$$\big[c^\rho_{112}, x_2\big]_{q_{112,2}} - \big[x_1, c^\rho_{122}\big]_{q_{1,122}} + (q_{1,12} - q_{12,2})x_{12}^2 \in I_{\prec x_1x_{12}x_2}. \tag{9.4}$$

There are the following restricted q-Leibniz conditions: If $N_1 < \infty$, then we have to check Eqs. (9.1) and (9.2) for $1 < 2$; note that we can omit the restricted Leibniz condition for $1 < 12$ in (2') of Theorem 7.3.1. In the same way if $N_2 < \infty$, then there are the conditions Eqs. (9.1) and (9.3) for $1 < 2$; we can omit the condition for $12 < 2$. Further Eq. (9.2) resp. (9.3) is equivalent to

$$[\underbrace{x_1, \dots [x_1}_{N_1-2}, c^\rho_{112}]_{q_{11}^2q_{12}} \dots]_{q_{11}^{N_1-1}q_{12}} - [d^\rho_1, x_2]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1}x_2}, \tag{9.5}$$

$$[\dots [c^\rho_{122}, \underbrace{x_2]_{q_{12}q_{22}^2} \dots, x_2}_{N_2-2}]_{q_{12}q_{22}^{N_2-1}} - [x_1, d^\rho_2]_{q_{12}^{N_2}} \in I_{\prec x_1x_2^{N_2}}. \tag{9.6}$$

In the case $N_1 = 2$ resp. $N_2 = 2$ then condition Eq. (9.5) resp. (9.6) is

$$c^\rho_{112} - [d^\rho_1, x_2]_{q_{12}^2} \in I_{\prec x_1^2x_2} \quad \text{resp.} \quad c^\rho_{122} - [x_1, d^\rho_2]_{q_{12}^2} \in I_{\prec x_1x_2^2}.$$

Here we see with Corollary 7.5.1 that by the restricted q-Leibniz formula $[x_1x_1x_2] - c_{112} \in (x_1^2 - d_1)$ resp. $[x_1x_2x_2] - c_{122} \in (x_2^2 - d_2)$, hence these two relations are redundant. Suppose $[d_1, x_2]_{q_{12}^2} \prec_L [x_1x_1x_2]$ resp. $[x_1, d_2]_{q_{12}^2} \prec_L [x_1x_2x_2]$. Thus if we define

$$c^\rho_{112} := [d^\rho_1, x_2]_{q_{12}^2} \quad \text{resp.} \quad c^\rho_{122} := [x_1, d^\rho_2]_{q_{12}^2}, \tag{9.7}$$

then condition Eq. (9.5) resp. (9.6) is fulfilled.

Finally, if $N_{12} < \infty$, then there are the conditions

$$\begin{aligned} \big[d^\rho_{12}, x_{12}\big]_1 &\in I_{\prec x_{12}^{N_{12}+1}},\\ [\dots [c^\rho_{112}, \underbrace{x_{12}]_{q_{1,12}q_{12,12}} \dots, x_{12}}_{N_{12}-1}]_{q_{1,12}q_{12,12}^{N_{12}-1}} - [x_1, d^\rho_{12}]_{q_{1,12}^{N_{12}}} &\in I_{\prec x_1x_{12}^{N_{12}}},\\ [\underbrace{x_{12}, \dots [x_{12}}_{N_{12}-1}, c^\rho_{122}]_{q_{12,12}q_{12,2}} \dots]_{q_{12,12}^{N_{12}-1}q_{12,2}} - [d^\rho_{12}, x_2]_{q_{12,2}^{N_{12}}} &\in I_{\prec x_{12}^{N_{12}}x_2}. \end{aligned} \tag{9.8}$$

Now we want to take a closer look at Eq. (9.4). Essentially, there are two cases: If $q_{11} = q_{22}$ we set $q := q_{112,2} = q_{1,122}$ and then Eq. (9.4) reads

$$\left[c^{\rho}_{112}, x_2\right]_q - \left[x_1, c^{\rho}_{122}\right]_q \in I_{\prec x_1 x_{12} x_2}. \tag{9.9}$$

Certainly this happens when $c_{112} = c_{122} = 0$, and in the case $N_1 = N_{12} = N_2 = \infty$ we have:

Example 9.2.1 (Nichols algebra A_2)**.** If $q_{11} = q_{22}$ then

$$T(V)/\big([x_1x_1x_2],\, [x_1x_2x_2]\big)$$

has basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid r_2, r_{12}, r_1 \geq 0\}$. This includes also the infinite dimensional Nichols algebra of Cartan type A_2, where $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$.

Else if $q_{11} \neq q_{22}$. Suppose $N_{12} = \mathrm{ord} q_{12,12} = 2$, then we define

$$d_{12} := -(q_{1,12} - q_{12,2})^{-1}\big(\left[c_{112}, x_2\right]_{q_{1,2}q_{12,2}} - \left[x_1, c_{122}\right]_{q_{1,122}}\big).$$

It is $[x_1x_2]^2 - d_{12} \in \big([x_1x_1x_2] - c_{112},\, [x_1x_2x_2] - c_{122}\big)$ by the q-Jacobi identity, see Eq. (9.4) and Corollary 7.5.1, i.e., this relation is redundant. Further $d_{12} \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi_{12}^2}$. Let us assume that $d_{12} \prec_L [x_1x_2]^2$, e.g., c_{122}, c_{112} are linear combinations of monomials of length < 3. Then for

$$d^{\rho}_{12} := -(q_{1,12} - q_{12,2})^{-1}\big(\left[c^{\rho}_{112}, x_2\right]_{q_{1,2}q_{12,2}} - \left[x_1, c^{\rho}_{122}\right]_{q_{1,122}}\big) \tag{9.10}$$

condition Eq. (9.4) is fulfilled. If $c_{122} = c_{112} = 0$ then also $d_{12} = 0$ and we have:

Example 9.2.2. If $q_{11} \neq q_{22}$ and $q_{12,12} = -1$, then

$$T(V)/\big([x_1x_1x_2],\, [x_1x_2x_2]\big)$$

has basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} \mid r_2, r_1 \geq 0,\ 0 \leq r_{12} < 2\}$.

Now we want to proof that the ideal given for the Nichols algebras in Proposition 6.3.1 and their liftings in Theorem 6.3.3 admit a PBW basis $[L]$. We could prove Proposition 6.3.1 directly very easily since all $c_{uv} = d_u = 0$, but instead we prove the more general statement for the liftings.

Proposition 9.2.3. *The liftings $(T(V)\#\Bbbk[\Gamma])/I$ of Theorem 6.3.3 have the PBW basis $\{x_2, [x_1x_2], x_1\}$ as claimed in this theorem.*

Proof. Note that all defined ideals are $\widehat{\Gamma}$-homogeneous by the definition of the coefficients. The conditions Eq. (9.1) are exactly as in Example 8.0.8.

The numeration refers to the one in Theorem 6.3.3:

(1a) We have $N_1 = N_2 = N_{12} = 2$. Since $d^{\rho}_1 = \mu_1(1 - g_1^2)$ we have by the argument preceding Eq. (9.7), that necessarily

$$c_{112} = [\mu_1(1 - g_1^2), x_2]_{q_{12}^2} \quad \text{and} \quad c_{122} = [x_1, \mu_2(1 - g_2^2)]_{q_{12}^2}$$

and the conditions Eqs. (9.5) and (9.6) are fulfilled. Note that $c_{112} = \mu_1(1 - q_{12}^2)x_2 = 0$: either $\mu_1 = 0$ or else $\mu_1 \neq 0$, but then $\chi_1^2 = \varepsilon$ and $q_{21}^2 = 1$. By squaring the assumption $q_{12}q_{21} = -1$, we obtain $q_{12}^2 = 1$. In the same way $c_{122} = 0$.

Then the conditions Eq. (9.8) are

$$\begin{aligned} \big[4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_{12}\big]_1 &\in I_{\prec x_{12}^3} \\ [0, x_{12}]_{q_{1,12}q_{12,12}} - [x_1, 4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2)]_{q_{1,12}^2} &\in I_{\prec x_1 x_{12}^2}, \\ [x_{12}, 0]_{q_{12,12}q_{12,2}} - [4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_2]_{q_{12,2}^2} &\in I_{\prec x_{12}^2 x_2}. \end{aligned}$$

Again, if $\mu_1 \neq 0$, then $q_{12}^2 = q_{21}^2 = 1$, hence $q_{1,12}^2 = 1$ and $q_{2,12}^2 = 1$. If $\mu_{12} \neq 0$, then $\chi_{12}^2 = \varepsilon$ and $q_{1,12}^2 = 1$; in this case also $q_{12}^2 = q_{21}^2 = 1$. Thus modulo $I_{\prec x_{12}^3}$ we have

$$\begin{aligned} \big[4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_{12}\big]_1 &= 4\mu_1 q_{21}\big[x_2^2, x_{12}\big]_1 - \mu_{12}(q_{12,12}^2 - 1)x_{12}g_{12}^2 \\ &= 4\mu_1\mu_2 q_{21}\big[1 - g_2^2, x_{12}\big]_1 = -4\mu_1\mu_2 q_{21}(q_{2,12}^2 - 1)x_{12}g_2^2 = 0. \end{aligned}$$

Further modulo $I_{\prec x_1 x_{12}^2}$ we get

$$\begin{aligned} [x_1, 4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2)]_1 &= 4\mu_1 q_{21}[x_1, x_2^2]_1 + \mu_{12}[x_1, 1 - g_{12}^2]_1 \\ &= 4\mu_1 q_{21} c_{122}^{\rho} - \mu_{12}(1 - q_{12,1}^2)x_1 g_{12}^2 = 0, \end{aligned}$$

which means that the second condition is fulfilled. The third one of Eq. (9.8) works analogously.

The last condition is Eq. (9.4), or equivalently condition Eq. (9.9) since $q_{11} = q_{22}$:

$$\big[0, x_2\big]_q - \big[x_1, 0\big]_q = 0 \in I_{\prec x_1 x_{12} x_2}.$$

(1b) Either $\lambda_{112} = \lambda_{122} = 0$, or $\chi_{112} = \varepsilon$ and/or $\chi_{122} = \varepsilon$, from where we conclude $q := q_{11} = q_{12} = q_{21} = q_{22}$. We start with Eq. (9.4): Since $q^3 = 1$ we have

$$\big[\lambda_{112}(1 - g_{112}), x_2\big]_1 - \big[x_1, \lambda_{122}(1 - g_{122})\big]_1 = 0.$$

We continue with Eq. (9.5): Either $\mu_1 = 0$ or $\chi_1^3 = \varepsilon$, hence $q_{21}^3 = 1$ and then also $q_{12}^3 = (q_{12}q_{21})^3 = q_{11}^{-3} = 1$. Then

$$\big[x_1, \lambda_{112}(1 - g_{112})\big]_1 - [\mu_1(1 - g_1^3), x_2]_1 = 0.$$

Next, Eq. (9.6): In the same way, $\mu_2 \neq 0$ or $q_{21}^3 = q_{12}^3 = 1$. Then

$$\big[\lambda_{122}(1 - g_{122}), x_2\big]_1 - [x_1, \mu_2(1 - g_2^3)]_1 = 0.$$

For Eq. (9.8) we have $q_{1,12}^3 = 1$ if $\mu_{12} \neq 0$. Thus $q_{12}^3 = 1$, moreover $q_{21}^3 = (q_{12}q_{21})^3 = q_{11}^{-3} = 1$. Hence modulo $I_{\prec x_1 x_{12}^3}$ we have

$$\begin{aligned} &\big[[\lambda_{112}(1 - g_{112}), x_{12}]_{q_{1,12}q_{12,12}}, x_{12}\big]_{q_{1,12}q_{12,12}^2} \\ &\qquad - \big[x_1, -(1 - q_{11})q_{11}\lambda_{112}\lambda_{122}(1 - g_{122}) + \mu_1(1 - q_{11})^3 x_2^3 + \mu_{12}(1 - g_{12}^3)\big]_{q_{1,12}^3} = 0, \end{aligned}$$

since each summand is zero. Further a straightforward calculation shows

$$\begin{aligned}&\big[x_{12},[x_{12},\lambda_{122}(1-g_{122})]_{q_{12,12}q_{12,2}}\big]_{q_{12,12}^2q_{12,2}}\\&\qquad-\big[-(1-q_{11})q_{11}\lambda_{112}\lambda_{122}(1-g_{122})+\mu_1(1-q_{11})^3x_2^3+\mu_{12}(1-g_{12}^3),x_2\big]_{q_{12,2}^2}=0.\end{aligned}$$

Finally, an easy calculation shows that

$$\big[-(1-q_{11})q_{11}\lambda_{112}\lambda_{122}(1-g_{122})+\mu_1(1-q_{11})^3x_2^3+\mu_{12}(1-g_{12}^3),x_{12}\big]_1=0$$

modulo $I_{\prec x_{12}^4}$, again by definition of the coefficients.

(1c) is a generalization of (1a) (and (1b) if $\lambda_{112}=\lambda_{122}=0$) and works completely in the same way (only the Serre-relations $[x_1x_1x_2]=[x_1x_2x_2]=0$ are not redundant, as they are (1a)). We leave this to the reader.

(2a) We leave this to the reader and prove the little more complicated (2b): Since we have $N_2=2$, as in (1a) we deduce from Eq. (9.7), that

$$c_{122}=[x_1,\mu_2(1-g_2^2)]_{q_{12}^2}=\mu_2(q_{21}^2-1)x_1g_2^2$$

and the condition Eq. (9.6) is fulfilled.

If $\lambda_{112}\neq 0$ then $q_{11}=q_{21}$ of order 4, $q_{12}=q_{22}=-1$; if $\mu_1\neq 0$ then $q_{12}^4=1$. Then Eq. (9.5) is fulfilled:

$$\big[x_1,[x_1,\lambda_{112}(1-g_{112})]_1\big]_{q_{11}}-[\mu_1(1-g_1^4),x_2]_1=0,$$

since both summands are zero.

It is $q_{11}\neq q_{22}$, $\mathrm{ord}q_{12,12}=2$ and c_{112}^ρ resp. c_{122}^ρ are linear combinations of monomials of length 0 resp. 1. By the discussion before Eq. (9.10), we see that $[x_1x_2]^2-d_{12}$ is redundant and for

$$\begin{aligned}d_{12}^\rho&:=-(q_{1,12}-q_{12,2})^{-1}\big(\big[\lambda_{112}(1-g_{112}),x_2\big]_{-1}-\big[x_1,\mu_2(q_{21}^2-1)x_1g_2^2\big]_{q_{11}}\big)\\&=-q_{12}^{-1}(q_{11}+1)^{-1}\big(\lambda_{112}2x_2-\mu_2\underbrace{(q_{21}^2-1)(1-q_{11}q_{21}^2)}_{=:q}x_1^2g_2^2\big)\end{aligned}$$

the condition Eq. (9.4) is fulfilled. We are left to show the conditions Eq. (9.8) $\big[d_{12}^\rho,x_{12}\big]_1\in I_{\prec x_{12}^3}$,

$$\big[c_{112}^\rho,x_{12}\big]_{q_{112,12}}-\big[x_1,d_{12}^\rho\big]_{q_{1,12}^2}\in I_{\prec x_1x_{12}^2}\quad\text{and}\quad\big[x_{12},c_{122}^\rho\big]_{q_{12,122}}-\big[d_{12}^\rho,x_2\big]_{q_{12,2}^2}\in I_{\prec x_{12}^2x_2}.$$

We calculate the first one: Modulo $I_{\prec x_{12}^3}$ we get

$$\big[d_{12}^\rho,x_{12}\big]_1=-q_{12}^{-1}(q_{11}+1)^{-1}\big(-\lambda_{112}2\underbrace{\big[x_{12},x_2\big]_1}_{=c_{122}^\rho}-\mu_2q\underbrace{\big[x_1^2g_2^2,x_{12}\big]_1}_{=q_{21}^2[x_1^2,x_{12}]_{q_{1,12}^2}g_2^2}\big).$$

Now by the q-derivation property $[x_1^2,x_{12}]_{q_{1,12}^2}=x_1c_{112}^\rho+q_{1,12}c_{112}^\rho x_1=\lambda_{112}(1-q_{11})x_1$. Because of the coefficient λ_{112} the two summands in the parentheses have the coefficient $\pm 4\lambda_{112}\mu_2$, hence cancel.

(3) works exactly as (2).

(4a) Since we have $N_1 = N_2 = 2$, as in (1a) we deduce from Eq. (9.7), that

$$c_{112} = [\mu_1(1-g_1^2), x_2]_{q_{12}^2} = \mu_1(1-q_{12}^2)x_2 \quad \text{and} \quad c_{122} = [x_1, 0]_{q_{12}^2} = 0$$

and the conditions Eqs. (9.5) and (9.6) are fulfilled.

For the second condition of Eq. (9.8) one can easily show by induction

$$\begin{aligned}
&[\dots[c^\rho_{112}, \underbrace{x_{12}]_{q_{1,12}q_{12,12}} \dots, x_{12}}_{N-1}]_{q_{1,12}q_{12,12}^{N-1}} \\
&= \mu_1(1-q_{12}^2)\big[\dots[x_2, \underbrace{x_{12}]_{q_{11}q_{12}^2q_{21}} \dots, x_{12}}_{N-1}\big]_{q_{11}q_{12}^Nq_{21}^{N-1}} \\
&= \mu_1 \prod_{i=0}^{N-1}(1-q_{12}^{i+2}q_{21}^i)x_2x_{12}^{N-1} = 0.
\end{aligned}$$

The last equation holds since for $i = N-2$ we have $1 - q_{12}^Nq_{21}^{N-2} = 0$: if $\mu_1 \neq 0$ then $q_{21}^2 = 1$ and $(q_{12}q_{21})^N = q_{12,12}^N = 1$. Further also $[x_1, d^\rho_{12}]_{q_{1,12}^N} = [x_1, \mu_{12}(1-g_{12}^N)]_1 = -\mu_{12}(1-q_{12,1}^N)x_1g_{12}^N = 0$, since either $\mu_{12} = 0$ or $q_{12}^N = q_{21}^N = (-1)^N$ such that $q_{12,1}^N = (-1)^N(-1)^N = 1$. This proves the second condition of Eq. (9.8). The third of Eq. (9.8) is easy since $c_{122} = 0$, and the first of Eq. (9.8) is a direct computation.

Finally, Eq. (9.4) is Eq. (9.9), since $q_{11} = q_{22}$:

$$\big[\mu_1(1-q_{12}^2)x_2, x_2\big]_{q_{112,2}} - \big[x_1, 0\big]_{q_{1,122}} = 0$$

because of the relation $x_2^2 = 0$.

(4b) works analogously to (4a). Note that here $c_{112} = 0$ and $c_{122} = [x_1, \mu_2(1-g_2^2)]_1 = \mu_2(q_{21}^2 - 1)x_1g_2^2$. $\square$

9.3 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_2 < x_2\}$

This PBW basis $[L]$ occurs in the Nichols algebras of Proposition 6.4.1 and their liftings of Theorem 6.4.3. Generally, we list the conditions when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned}
&[x_1x_1x_1x_2] - c_{1112}, && x_1^{N_1} - d_1, \\
&[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_1x_2]^{N_{112}} - d_{112}, \\
&[x_1x_2x_2] - c_{122}, && [x_1x_2]^{N_{12}} - d_{12}, \\
& && x_2^{N_2} - d_2.
\end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$\begin{aligned}
&c^\rho_{(1|2)} = x_{12}, && c^\rho_{(12|2)} = c^\rho_{122}, && c^\rho_{(112|12)} = c^\rho_{11212}, \\
&c^\rho_{(1|12)} = x_{112}, && c^\rho_{(1|112)} = c^\rho_{1112},
\end{aligned}$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ by Eq. (7.1)

$$\begin{aligned} c^{\rho}_{(112|2)} &= \partial^{\rho}_1(c^{\rho}_{(12|2)}) + q_{12,2}c^{\rho}_{(1|2)}x_{12} - q_{1,12}x_{12}c^{\rho}_{(1|2)}, \\ &= \partial^{\rho}_1(c^{\rho}_{122}) + (q_{12,2} - q_{1,12})x_{12}^2. \end{aligned}$$

We have for $1 < 112 < 2$, $1 < 112 < 12$ and $112 < 12 < 2$ the following q-Jacobi conditions (note that we can leave out $1 < 12 < 2$):

$$\begin{aligned} &\big[c^{\rho}_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, c^{\rho}_{(112|2)}\big]_{q_{1,1122}} \\ &\qquad + q_{1,112}x_{112}[x_1,x_2] - q_{112,2}[x_1,x_2]x_{112} \in I_{\prec x_1x_{112}x_2} \\ \Leftrightarrow &\big[c^{\rho}_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial^{\rho}_1(c^{\rho}_{122})\big]_{q_{1,1122}} \\ &\quad - (q_{12,2} - q_{1,12})c^{\rho}_{11212} - (q_{12,2} - q_{1,12})q_{1,12}(q_{12,12}+1)x_{12}x_{112} \\ &\qquad + q_{1,112}c^{\rho}_{11212} + q_{112,2}(q_{1,112}q_{112,1} - 1)x_{12}x_{112} \in I_{\prec x_1x_{112}x_2} \\ \Leftrightarrow &\big[c^{\rho}_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial^{\rho}_1(c^{\rho}_{122})\big]_{q_{1,1122}} + \underbrace{q_{12}(q_{11}^2 - q_{22} + q_{11})}_{=:q} c^{\rho}_{11212} \\ &\quad + \underbrace{q_{12}^2(q_{22}(q_{11}^4q_{12}q_{21} - 1) - q_{11}(q_{22} - q_{11})(q_{12,12}+1))}_{=:q'} x_{12}x_{112} \in I_{\prec x_1x_{112}x_2} \end{aligned} \tag{9.11}$$

If $q \neq 0$, we see that $[x_1x_1x_2x_1x_2] - c_{11212} \in \big([x_1x_1x_1x_2] - c_{1112}, [x_1x_2x_2] - c_{122}\big)$ is redundant with

$$c_{11212} = -q^{-1}\big(\big[c_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial_1(c_{122})\big]_{q_{1,1122}} + q'[x_1x_2][x_1x_1x_2]\big)$$

by Corollary 7.5.1 and the q-Jacobi identity of Proposition 3.2.3. We have $\deg_{\widehat{\Gamma}}(c_{11212}) = \chi_{11212}$; suppose that $c_{11212} \prec_L [x_1x_1x_2x_1x_2]$ (e.g. c_{1112} resp. c_{122} are linear combinations of monomials of length < 4 resp. < 3) then condition Eq. (9.11) is fulfilled for

$$c^{\rho}_{11212} := -q^{-1}\big(\big[c^{\rho}_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial^{\rho}_1(c^{\rho}_{122})\big]_{q_{1,1122}} + q'x_{12}x_{112}\big).$$

There are three cases, where the coefficients q, q' are of a better form for our setting: Since

$$q = q_{12}\big((3)_{q_{11}} - (2)_{q_{22}}\big), \quad q' = q_{12}\big(q(1 + q_{11}^2q_{12}q_{21}q_{22}) - q_{11}q_{12}(2)_{q_{22}}\big),$$

we have

$$\begin{aligned} &q = q_{12}q_{11} \neq 0, && q' = -q_{12}q_{11}^2q(1 - q_{11}^2q_{12}q_{21}), && \text{if } q_{11}^2 = q_{22}, \\ &q = q_{12}(3)_{q_{11}}, && q' = q_{12}q(1 - q_{11}^2q_{12}q_{21}), && \text{if } q_{22} = -1, \\ &q = -q_{12}(2)_{q_{22}}, && q' = -q_{12}q(1 + q_{11} + q_{11}^2q_{12}q_{21}q_{22}), && \text{if } \mathrm{ord}q_{11} = 3. \end{aligned}$$

The second q-Jacobi condition for $1 < 112 < 12$ reads

$$\begin{aligned} &\big[c^{\rho}_{1112}, x_{12}\big]_{q_{1112,12}} - \big[x_1, c^{\rho}_{11212}\big]_{q_{1,11212}} \\ &\qquad + q_{1,112}x_{112}[x_1,x_{12}] - q_{112,12}[x_1,x_{12}]x_{112} \in I_{\prec x_1x_{112}x_{12}} \\ \Leftrightarrow &\big[c^{\rho}_{1112}, x_{12}\big]_{q_{1112,12}} - \big[x_1, c^{\rho}_{11212}\big]_{q_{1,11212}} + \underbrace{q_{11}^2q_{12}(1 - q_{12}q_{21}q_{22})}_{=:q''} x_{112}^2 \in I_{\prec x_1x_{112}x_{12}} \end{aligned} \tag{9.12}$$

If $q'' \neq 0$ then we see that $[x_1x_1x_2]^2 - d_{112} \in \big([x_1x_1x_1x_2] - c_{11212}, [x_1x_1x_2x_1x_2] - c_{11212}\big)$ is redundant with $d_{112} = -q''^{-1}\big(\big[c_{1112}, [x_1x_2]\big]_{q_{1112,12}} - \big[x_1, c_{11212}\big]_{q_{1,11212}}\big)$ by Corollary 7.5.1 and the q-Jacobi identity of Proposition 3.2.3. It is $\deg_{\widehat{\Gamma}}(d_{112}) = \chi_{112}^2$; suppose that $d_{112} \prec_L [x_1x_1x_2]^2$ then condition Eq. (9.13) is fulfilled for

$$d_{112}^\rho := -q''^{-1}\big(\big[c_{1112}^\rho, x_{12}\big]_{q_{1112,12}} - \big[x_1, c_{11212}^\rho\big]_{q_{1,11212}}\big)$$

If further $\mathrm{ord} q_{112,112} = 2$ then we have to consider the restricted q-Leibniz conditions for d_{112}^ρ (see below).

The last q-Jacobi condition for $112 < 12 < 2$ is

$$\begin{aligned}
&\big[c_{11212}^\rho, x_2\big]_{q_{11212,2}} - \big[x_{112}, c_{122}^\rho\big]_{q_{112,122}} \\
&\qquad + q_{112,12}x_{12}[x_{112}, x_2] - q_{12,2}[x_{112}, x_2]x_{12} \in I_{\prec x_{112}x_{12}x_2} \\
\Leftrightarrow &\big[c_{11212}^\rho, x_2\big]_{q_{11212,2}} - \big[x_{112}, c_{122}^\rho\big]_{q_{112,122}} \\
&\quad + q_{112,12}x_{12}\partial_1^\rho(c_{122}^\rho) - q_{12,2}\partial_1^\rho(c_{122}^\rho)x_{12} \\
&\qquad + \underbrace{q_{12}^2q_{22}(q_{22} - q_{11})(q_{11}^2q_{12}q_{21} - 1)}_{=:q'''}x_{12}^3 \in I_{\prec x_{112}x_{12}x_2}
\end{aligned} \tag{9.13}$$

If $q''' \neq 0$ then we see that $[x_1x_2]^3 - d_{12} \in \big([x_1x_1x_2x_1x_2] - c_{11212}, [x_1x_2x_2] - c_{122}\big)$ is redundant with $d_{12} := -q'''^{-1}\big(\big[c_{11212}, x_2\big]_{q_{11212,2}} - \big[[x_1x_1x_2], c_{122}\big]_{q_{112,122}} + q_{112,12}[x_1x_2]\partial_1(c_{122}) - q_{12,2}\partial_1(c_{122})[x_1x_2]\big)$ by Corollary 7.5.1 and the q-Jacobi identity of Proposition 3.2.3. It is $\deg_{\widehat{\Gamma}}(d_{12}) = \chi_{12}^3$; suppose that $d_{12} \prec_L [x_1x_1]^3$ (e.g., c_{11212} resp. c_{122} are linear combinations of monomials of length < 5 resp. < 3) then condition Eq. (9.13) is fulfilled for

$$\begin{aligned}
d_{12}^\rho : = -q''^{-1}\big(&\big[c_{11212}^\rho, x_2\big]_{q_{11212,2}} - \big[x_{112}, c_{122}^\rho\big]_{q_{112,122}} \\
&+ q_{112,12}x_{12}\partial_1^\rho(c_{122}^\rho) - q_{12,2}\partial_1^\rho(c_{122}^\rho)x_{12}\big)
\end{aligned}$$

If further $\mathrm{ord} q_{12,12} = 3$ then we have to consider the q-Leibniz conditions for d_{12}^ρ (see below).

There are the following restricted q-Leibniz conditions: If $N_1 < \infty$, then $\big[d_1^\rho, x_1\big]_1 \in I_{\prec x_1^{N_1+1}}$ and for $1 < 2$ (we can omit $1 < 12, 1 < 112$)

$$[\underbrace{x_1, \dots [x_1}_{N_1-3}, c_{1112}^\rho]_{q_{11}^3q_{12}} \dots]_{q_{11}^{N_1-1}q_{12}} - [d_1^\rho, x_2]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1}x_2}. \tag{9.14}$$

If $N_2 < \infty$, then $\big[d_2^\rho, x_2\big]_1 \in I_{\prec x_2^{N_2+1}}$ and for $1 < 2$ (we can omit $12 < 2, 112 < 2$)

$$[\dots [c_{122}^\rho \underbrace{x_2]_{q_{12}q_{22}^2} \dots, x_2}_{N_2-2}]_{q_{12}q_{22}^{N_2-1}} - [x_1, d_2^\rho]_{q_{12}^{N_2}} \in I_{\prec x_1x_2^{N_2}}. \tag{9.15}$$

If $N_{12} < \infty$, then $\big[d_{12}^\rho, x_{12}\big]_1 \in I_{\prec x_{12}^{N_{12}+1}}$ and for $1 < 12$, $12 < 2$ (we can omit $112 < 12$)

$$\begin{aligned}
&[\dots [c_{112}^\rho, \underbrace{x_{12}]_{q_{1,12}q_{12,12}} \dots, x_{12}}_{N_{12}-1}]_{q_{1,12}q_{12,12}^{N_{12}-1}} - [x_1, d_{12}^\rho]_{q_{1,12}^{N_{12}}} \in I_{\prec x_1x_{12}^{N_{12}}}, \\
&[\underbrace{x_{12}, \dots [x_{12}}_{N_{12}-1}, c_{122}^\rho]_{q_{12,12}q_{12,2}} \dots]_{q_{12,12}^{N_{12}-1}q_{12,2}} - [d_{12}^\rho, x_2]_{q_{12,2}^{N_{12}}} \in I_{\prec x_{12}^{N_{12}}x_2}.
\end{aligned} \tag{9.16}$$

If $N_{112} < \infty$, then $\left[d^\rho_{112}, x_{112}\right]_1 \in I_{\prec x_{112}^{N_{112}+1}}$ and for $1 < 112$, $112 < 12$, $112 < 2$

$$
\begin{aligned}
&\big[\dots[c^\rho_{1112}, \underbrace{x_{112}]_{q_{1,112}q_{112,112}}\dots, x_{112}}_{N_{112}-1}\big]_{q_{1,112}q_{112,112}^{N_{112}-1}} - [x_1, d^\rho_{112}]_{q_{1,112}^{N_{112}}} \in I_{\prec x_1 x_{112}^{N_{112}}}\\
&\big[\underbrace{x_{112},\dots[x_{112}}_{N_{112}-1}, c^\rho_{11212}]_{q_{112,112}q_{112,12}}\dots\big]_{q_{112,112}^{N_{112}-1}q_{112,12}} - [d^\rho_{112}, x_{12}]_{q_{112,12}^{N_{112}}} \in I_{\prec x_{112}^{N_{112}}x_{12}}\\
&\big[\underbrace{x_{112},\dots[x_{112}}_{N_{112}-1}, c^\rho_{(112|2)}]_{q_{112,112}q_{112,2}}\dots\big]_{q_{112,112}^{N_{112}-1}q_{112,2}} - [d^\rho_{112}, x_{2}]_{q_{112,2}^{N_{112}}} \in I_{\prec x_{112}^{N_{112}}x_{2}}
\end{aligned}
\tag{9.17}
$$

Now we see that the ideals of the Nichols algebras of Proposition 6.4.1 are of the given form. It is again easy to check that they have the PBW basis $\{x_1, [x_1x_1x_2], [x_1x_2], x_2\}$, since all $c^\rho_{uv} = 0$ and $d^\rho_u = 0$.

The proof that the liftings of Theorem 6.4.3 have the PBW basis $\{x_1, [x_1x_1x_2], [x_1x_2], x_2\}$ consists in plugging the c^ρ_{uv} and d^ρ_u in the conditions above, like it was done before in Proposition 9.2.3. We leave this to the reader.

9.4 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_2 < x_1x_2x_2 < x_2\}$

This PBW basis $[L]$ appears in the Nichols algebras of Proposition 6.5.1(1) and their liftings of Theorem 6.5.3(1). More generally, we ask for the conditions when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$
\begin{aligned}
&[x_1x_1x_1x_2] - c_{1112}, && x_1^{N_1} - d_1,\\
&[x_1x_1x_2x_2] - c_{1122}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\
&[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_2]^{N_{12}} - d_{12},\\
&[x_1x_2x_1x_2x_2] - c_{12122}, && [x_1x_2x_2]^{N_{122}} - d_{122},\\
&[x_1x_2x_2x_2] - c_{1222}, && x_2^{N_2} - d_2.
\end{aligned}
$$

In $\Bbbk\langle x_1, x_{112}, x_{12}, x_{122}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$
\begin{aligned}
c^\rho_{(1|2)} &= x_{12}, & c^\rho_{(1|112)} &= c^\rho_{1112}, & c^\rho_{(112|12)} &= c^\rho_{11212},\\
c^\rho_{(1|12)} &= x_{112}, & c^\rho_{(1|122)} &= c^\rho_{1122}, & c^\rho_{(12|122)} &= c^\rho_{12122},\\
c^\rho_{(12|2)} &= x_{122}, & c^\rho_{(122|2)} &= c^\rho_{1222},
\end{aligned}
$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ and $\mathrm{Sh}(112122) \neq (112|122)$ by Eq. (7.1)

$$
\begin{aligned}
c^\rho_{(112|2)} &= \partial^\rho_1(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)}\\
&= c^\rho_{1122} + (q_{12,2} - q_{1,12})x_{12}^2,\\
c^\rho_{(112|122)} &= \partial^\rho_1(c^\rho_{12122}) + q_{12,122}c^\rho_{1122}x_{12} - q_{1,12}x_{12}c^\rho_{1122}.
\end{aligned}
$$

We have to check the q-Jacobi conditions for $1 < 112 < 2$ (like Eq. (9.11)), $1 < 112 < 12$ (like Eq. (9.12)), $1 < 112 < 122$, $1 < 122 < 2$, $112 < 12 < 2$ (like Eq. (9.13)), $112 < 12 < 122$, $112 < 122 < 2$, $12 < 122 < 2$ (note that we can omit $1 < 12 < 2$, $1 < 12 < 122$). The restricted q-Leibniz conditions are treated like before (note that we can leave out those for $1 < 112$, $1 < 12$, $1 < 122$ if $N_1 < \infty$, $112 < 12$, $12 < 122$ if $N_{12} < \infty$, $112 < 2$, $12 < 2$, $122 < 2$ if $N_2 < \infty$).

Both types of conditions detect many redundant relations like before. The proof that the given ideals of the Nichols algebras of Proposition 6.5.1 and their liftings of Theorem 6.5.3 admit the PBW basis $\{x_1, [x_1x_1x_2], [x_1x_2], [x_1x_2x_2], x_2\}$ is again a straightforward but rather expansive calculation.

9.5 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_1x_2x_1x_2 < x_1x_2 < x_2\}$

This PBW basis $[L]$ shows up in the Nichols algebras of Proposition 6.5.1(2) and (4) and their liftings of Theorem 6.5.3(2) and (4). More generally, we examine when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned}
[x_1x_1x_1x_2] - c_{1112}, &\qquad x_1^{N_1} - d_1,\\
[x_1x_1x_1x_2x_1x_2] - c_{111212}, &\qquad [x_1x_1x_2]^{N_{112}} - d_{112},\\
[x_1x_1x_2x_1x_1x_2x_1x_2] - c_{11211212}, &\qquad [x_1x_1x_2x_1x_2]^{N_{11212}} - d_{11212},\\
[x_1x_1x_2x_1x_2x_1x_2] - c_{1121212}, &\qquad [x_1x_2]^{N_{12}} - d_{12},\\
[x_1x_2x_2] - c_{122}, &\qquad x_2^{N_2} - d_2.
\end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{11212}, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$\begin{aligned}
c^\rho_{(1|2)} &= x_{12}, & c^\rho_{(1|112)} &= c^\rho_{1112}, & c^\rho_{(11212|12)} &= c^\rho_{1121212},\\
c^\rho_{(1|12)} &= x_{112}, & c^\rho_{(112|12)} &= x_{11212}, & c^\rho_{(112|11212)} &= c^\rho_{11211212},\\
c^\rho_{(12|2)} &= c^\rho_{122}, & c^\rho_{(1|11212)} &= c^\rho_{111212},
\end{aligned}$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ and $\mathrm{Sh}(112122) \neq (11212|2)$ by Eq. (7.1)

$$\begin{aligned}
c^\rho_{(112|2)} &= \partial^\rho_1(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)}\\
&= c^\rho_{1122} + (q_{12,2} - q_{1,12})x_{12}^2,\\
c^\rho_{(11212|2)} &= \partial^\rho_{112}(c^\rho_{122}) + q_{12,2}c^\rho_{(112|2)}x_{12} - q_{112,12}x_{12}c^\rho_{(112|2)}\\
&= \partial^\rho_{112}(c^\rho_{122}) + q_{12,2}c^\rho_{1122}x_{12} - q_{112,12}x_{12}c^\rho_{1122}\\
&\quad + (q_{12,2} - q_{112,12})(q_{12,2} - q_{1,12})x_{12}^3.
\end{aligned}$$

Again we have to consider all q-Jacobi conditions and restricted q-Leibniz conditions, from where we detect again many redundant relations. Like before, we omit the proof for the examples in Proposition 6.5.1(2) and (4) resp. Theorem 6.5.3(2) and (4), where we just have to put the given c^ρ_{uv} and d^ρ_u in the conditions.

9.6 PBW basis for $L = \{x_1 < x_1x_1x_1x_2 < x_1x_1x_2 < x_1x_2 < x_2\}$

The Nichols algebras of Proposition 6.5.1(3) and (5) and their liftings of Theorem 6.5.3(3) and (5) have this PBW basis $[L]$. We study the situation, when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$
\begin{aligned}
&[x_1x_1x_1x_1x_2] - c_{11112}, && x_1^{N_1} - d_1,\\
&[x_1x_1x_1x_2x_1x_1x_2] - c_{1112112}, && [x_1x_1x_1x_2]^{N_{1112}} - d_{1112},\\
&[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\
&[x_1x_2x_2] - c_{122}, && [x_1x_2]^{N_{12}} - d_{12},\\
& && x_2^{N_2} - d_2.
\end{aligned}
$$

In $\Bbbk\langle x_1, x_{112}, x_{11212}, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$
\begin{aligned}
&c^\rho_{(1|2)} = x_{12}, && c^\rho_{(1|112)} = x_{1112}, && c^\rho_{(1112|112)} = c^\rho_{1121212},\\
&c^\rho_{(1|12)} = x_{112}, && c^\rho_{(112|12)} = c^\rho_{11212}, &&\\
&c^\rho_{(12|2)} = c^\rho_{122}, && c^\rho_{(1|1112)} = c^\rho_{11112}, &&
\end{aligned}
$$

and for $\mathrm{Sh}(1122) \neq (112|2)$, $\mathrm{Sh}(11122) \neq (1112|2)$ and $\mathrm{Sh}(111212) \neq (1112|12)$ by Eq. (7.1)

$$
\begin{aligned}
c^\rho_{(112|2)} &= \partial^\rho_1(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)}\\
&= \partial^\rho_1(c^\rho_{122}) + (q_{12,2} - q_{1,12})x_{12}^2,\\
c^\rho_{(1112|2)} &= \partial^\rho_1(c^\rho_{(112|2)}) + q_{112,2}c^\rho_{(1|2)}x_{112} - q_{1,112}x_{112}c^\rho_{(1|2)},\\
&= \partial^\rho_1(\partial^\rho_1(c^\rho_{122})) + (q_{12,2} - q_{1,12})(x_{112}x_{12} + q_{1,12}x_{12}[x_1, x_{12}])\\
&\quad + q_{112,2}x_{12}x_{112} - q_{1,112}x_{112}x_{12},\\
&= \partial^\rho_1(\partial^\rho_1(c^\rho_{122})) + q_{12}(q_{22} - q_{11} - q_{11}^2)x_{112}x_{12}\\
&\quad + q_{12}^2(q_{11}(q_{22} - q_{11}) + q_{22})x_{12}x_{112},\\
c^\rho_{(1112|12)} &= \partial^\rho_1(c^\rho_{11212}) + (q_{112,2} - q_{1,112})x_{112}^2.
\end{aligned}
$$

Note that for the fifth equation we used the relation $[x_1, x_{12}] - x_{112}$.

The assertion concerning the PBW basis and the redundant relations of Proposition 6.5.1(3) and (5) and Theorem 6.5.3(3) and (5) are again straightforward to verify.

Appendix A

Program for FELIX

A.1 Example

As an example we give the source code, which we used for the computation of the lifting of Theorem 6.4.3 (2c), namely we show the following in the spirit of Section 6.1:

Let $\mathrm{ord} q_{11} = 3$, $q_{12} = 1$, $q_{12}q_{21} = q_{11}^{-2}$ and $q_{22} = -1$, then we demonstrate how we compute $s_{12} \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ such that

$$[x_1x_2]^6 - s_{12} \in P_{g_{12}^6}^{\chi_{12}^6}$$

modulo the ideal $I_{[x_1x_2]^6}$ generated by

$$\begin{aligned}
[x_1x_1x_2x_1x_2] + 3\mu_1(1-q_{11})x_2^2 - \lambda_{11212}(1-g_{11212}),\\
x_1^3 - \mu_1(1-g_1^3),\\
x_2^2 - \mu_2(1-g_2^2).
\end{aligned}$$

At first we calculate $\Delta([x_1x_2]^6)$ modulo $I_{[x_1x_2]^6}$ and from the output we take the term occuring with $\_\otimes 1$, namely

$$\begin{aligned}
s_{12} := -3\mu_2\Big\{&(\lambda_{11212}(1-q_{11}) + 9\mu_1\mu_2 q_{11})[x_1x_2]^2 x_1 g_2^2\\
&- q_{11}(\lambda_{11212}(1-q_{11}) + 9\mu_1\mu_2 q_{11})[x_1x_2][x_1x_1x_2]g_2^2\\
&+ (\lambda_{11212}^2 q_{11}^2 + 3\mu_1\mu_2\lambda_{11212}(1-q_{11}^2) - 9\mu_1^2\mu_2^2)g_1^6 g_2^6\\
&+ 3\mu_1\mu_2(\lambda_{11212}(1-q_{11}^2) - 3\mu_1\mu_2)g_1^3 g_2^6\\
&+ \lambda_{11212}(3\mu_1\mu_2(q_{11}-1) + \lambda_{11212})g_1^3 g_2^4\\
&- 9\mu_1^2\mu_2^2 g_2^6\\
&+ 3\mu_1\mu_2(\lambda_{11212}(q_{11}-1) - 9\mu_1\mu_2 q_{11})g_2^4\\
&+ q_{11}(\lambda_{11212}^2 - 6\mu_1\mu_2\lambda_{11212}(1-q_{11}) - 27\mu_1^2\mu_2^2 q_{11})g_2^2\Big\}.
\end{aligned}$$

In the next step we add the relation $[x_1x_2]^6 - s_{12} - \mu_{12}(1-g_{12}^6)$ to $I_{[x_1x_2]^6}$ and obtain I. We know by the PBW Criterion 7.3.1 that this set of relations making up I is enough to get the basis

$$\big\{x_2^{r_2}[x_1x_2]^{r_{12}}[x_1x_1x_2]^{r_{112}}x_1^{r_1}g \mid 0 \le r_1 < 3,\ 0 \le r_{12} < 6,\ 0 \le r_2, r_{112} < 2,\ g \in \Gamma\big\}$$

and $\dim_{\Bbbk} A = 72 \cdot |\Gamma|$. For the implementation we use the isomorphism of Corollary 7.5.1:

$$(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I \;\cong\; \Bbbk\langle x_2, x_{12}, x_{112}, x_1, \Gamma\rangle/(I_L + I'_\Gamma + I).$$

A.2 Short introduction to FELIX

The program presented is written for the computer algebra system *FELIX* [13], which can be downloaded at http://felix.hgb-leipzig.de/. For multiplying tensors (e.g. for calculating coproducts) one needs to treat $\otimes$ as a variable. We want to thank István Heckenberger for providing his extension module tensor.cmp which realizes this. We want to give some comments on the program:

- Using a terminal, we execute FELIX with the command felix. A *file* is compiled with the command felix <*file*. Also a module is compiled in the same way, e.g., felix <tensor.cmp.

- The program starts with the inclusion of the *modules*: link("*module*").

- Then the definition of the non-commutative polynomial ring over the rational numbers with *parameters* and *variables* together with a *matrix*, which determines the ordering of the variables, is given by select rat(*parameters*)<*variables*;*matrix*>. We order the variables by $\preceq$ from Section 4.4.

  Notabene: The right ordering-matrix is crucial for the termination of the Gröbner basis, since the algorithm is non-deterministic in the non-commutative case.

- We treat a root of unity q as a variable and assign to it the degree zero in the ordering-matrix. Further we give the minimal polynomial in the ideal defined by ideal(*relations*).

- The variable ixi is treated by the tensor module as $\otimes$. We use the function ttimes of tensor.cmp for the product of two tensors.
  Further, we will denote coproducts like $\Delta([x_1x_2])$ by variables like delx12, and powers like $\Delta([x_1x_2]^6)$ by del6x12.

- The function standard(*ideal*) computes the Gröbner basis of the *ideal* from which we can read off the basis easily. Then the function remainder(*polynomial* , *ideal*) computes the *polynomial* modulo the *ideal* w.r.t. the ordering matrix.

- To multiply two variables/parameters one needs to put the product sign *. To end an command with an output we type $, to suppress the output we type _ .

- Finally, the command print("*string*") prints the *string* in the output, comments are given by % *comment* %, and bye ends the program.

A.3 Program

At first we execute the following program which calculates $\Delta([x_1x_2]^6)$.

```
link("tensor.mdl")$

select rat(mu1,mu2,mu12,lam11212)<x2,x12,x112,x1,g1,g2,ixi,q;
  { {0,0,0,0,0,0,1,0},
    {1,2,3,1,0,0,0,0},
    {1,0,0,0,0,0,0,0},
    {0,1,0,0,0,0,0,0},
    {0,0,1,0,0,0,0,0},
    {0,0,0,0,1,1,0,0},
    {0,0,0,0,0,1,0,0}, }
>$

q11:=q$
q12:=1$
q21:=q$
q22:=-1$

si:=ideal(
  q^2+q+1, q*ixi-ixi*q,

  q*x1-x1*q, q*x2-x2*q,
  q*g1-g1*q, q*g2-g2*q,

  g1*g2-g2*g1,

  g1*x1-q11*x1*g1,   g1*x2-q12*x2*g1,
  g2*x1-q21*x1*g2,   g2*x2-q22*x2*g2,

  x1*x2-q12*x2*x1-x12,
  x1*x12-q11*q12*x12*x1-x112,
  x112*x12-q11^2*q12^2*q21*q22*x12*x112
    +3*(1-q)*mu1*x2^2 -lam11212*(1-g1^3*g2^2),

  x2^2-mu2*(1-g2^2),
  x1^3-mu1*(1-g1^3)
)$

si:=standard(si)$

print("Computation of the coproducts:")_
```

```
delx1:=x1*ixi+g1*ixi*x1$
delx2:=x2*ixi+g2*ixi*x2$
delx12:=remainder(
    ttimes(delx1,delx2)-ttimes(q12*ixi,ttimes(delx2,delx1))
  ,si)$
delx112:=remainder(
    ttimes(delx1,delx12)-ttimes(q11*q12*ixi,ttimes(delx12,delx1))
  ,si)$
del2x1:=remainder(ttimes(delx1,delx1),si)$
del3x1:=remainder(ttimes(del2x1,delx1),si)$
del2x2:=remainder(ttimes(delx2,delx2),si)$
del2x12:=remainder(ttimes(delx12,delx12),si)$
del4x12:=remainder(ttimes(del2x12,del2x12),si)$

% (1) %
print("Delta(x12^6) =")_
del6x12:=remainder(ttimes(del4x12,del2x12),si)$

bye$
```

The code after (1) genereates the output

```
          ⋮
 > Delta(x12^6) =
 @ :=  ...
```

From the ... we take the term occuring with `_ *ixi*1`, which will be s_{12} as said before. We then define in (2) below the variable `w` which corresponds to $[x_1 x_2]^6 - s_{12} - \mu_{12}(1 - g_{12}^6)$:

```
link("tensor.mdl")$

select rat(mu1,mu2,mu12,lam11212)<x2,x12,x112,x1,g1,g2,ixi,q;
  { {0,0,0,0,0,0,1,0},
    {1,2,3,1,0,0,0,0},
    {1,0,0,0,0,0,0,0},
    {0,1,0,0,0,0,0,0},
    {0,0,1,0,0,0,0,0},
    {0,0,0,0,1,1,0,0},
    {0,0,0,0,0,1,0,0}, }
>$

q11:=q$
```

```
q12:=1$
q21:=q$
q22:=-1$

si:=ideal(
  q^2+q+1, q*ixi-ixi*q,

  q*x1-x1*q, q*x2-x2*q,
  q*g1-g1*q, q*g2-g2*q,

  g1*g2-g2*g1,

  g1*x1-q11*x1*g1,   g1*x2-q12*x2*g1,
  g2*x1-q21*x1*g2,   g2*x2-q22*x2*g2,

  x1*x2-q12*x2*x1-x12,
  x1*x12-q11*q12*x12*x1-x112,
  x112*x12-q11^2*q12^2*q21*q22*x12*x112
    +3*(1-q)*mu1*x2^2 -lam11212*(1-g1^3*g2^2),

  x2^2-mu2*(1-g2^2),
  x1^3-mu1*(1-g1^3)
)$

si:=standard(si)$

print("Computation of the coproducts:")_
delx1:=x1*ixi+g1*ixi*x1$
delx2:=x2*ixi+g2*ixi*x2$
delx12:=remainder(
    ttimes(delx1,delx2)-ttimes(q12*ixi,ttimes(delx2,delx1))
  ,si)$
delx112:=remainder(
    ttimes(delx1,delx12)-ttimes(q11*q12*ixi,ttimes(delx12,delx1))
  ,si)$
del2x1:=remainder(ttimes(delx1,delx1),si)$
del3x1:=remainder(ttimes(del2x1,delx1),si)$
del2x2:=remainder(ttimes(delx2,delx2),si)$
del2x12:=remainder(ttimes(delx12,delx12),si)$
del4x12:=remainder(ttimes(del2x12,del2x12),si)$

% (1) %
print("Delta(x12^6) =")_
del6x12:=remainder(ttimes(del4x12,del2x12),si)$
```

```
% (2) %
print("Definition of w:=x12^6-s12-mu12*(1-g1^6*g2^6)")_
w:=x12^6
  +((-3*mu2*lam11212+27*mu1*mu2^2)*q+3*mu2*lam11212)
    *x12^2*x1*g2^2
  +((-6*mu2*lam11212+27*mu1*mu2^2)*q+(-3*mu2*lam11212+27*mu1*mu2^2))
    *x12*x112*g2^2
  +((-3*mu2*lam11212^2+9*mu1*mu2^2*lam11212)*q
    +(-3*mu2*lam11212^2+18*mu1*mu2^2*lam11212-27*mu1^2*mu2^3))
    *g1^6*g2^6
  +(9*mu1*mu2^2*lam11212*q+(18*mu1*mu2^2*lam11212-27*mu1^2*mu2^3))
    *g1^3*g2^6
  +(9*mu1*mu2^2*lam11212*q+(3*mu2*lam11212^2-9*mu1*mu2^2*lam11212))
    *g1^3*g2^4
  -27*mu1^2*mu2^3
    *g2^6
  +((9*mu1*mu2^2*lam11212-81*mu1^2*mu2^3)*q-9*mu1*mu2^2*lam11212)
    *g2^4
  +((3*mu2*lam11212^2-36*mu1*mu2^2*lam11212+81*mu1^2*mu2^3)*q
    +(-18*mu1*mu2^2*lam11212+81*mu1^2*mu2^3))
    *g2^2
  -mu12*(1-g1^6*g2^6)$

delw:=remainder(
  del6x12
  +((-3*mu2*lam11212+27*mu1*mu2^2)*q+3*mu2*lam11212)
    *ttimes(ttimes(del2x12,delx1),g2^2*ixi*g2^2)
  +((-6*mu2*lam11212+27*mu1*mu2^2)*q+(-3*mu2*lam11212+27*mu1*mu2^2))
    *ttimes(delx12,ttimes(delx112,g2^2*ixi*g2^2))
  +((-3*mu2*lam11212^2+9*mu1*mu2^2*lam11212)*q
    +(-3*mu2*lam11212^2+18*mu1*mu2^2*lam11212-27*mu1^2*mu2^3))
    *g1^6*g2^6*ixi*g1^6*g2^6
  +(9*mu1*mu2^2*lam11212*q+(18*mu1*mu2^2*lam11212-27*mu1^2*mu2^3))
    *g1^3*g2^6*ixi*g1^3*g2^6
  +(9*mu1*mu2^2*lam11212*q+(3*mu2*lam11212^2-9*mu1*mu2^2*lam11212))
    *g1^3*g2^4*ixi*g1^3*g2^4
  -27*mu1^2*mu2^3
    *g2^6*ixi*g2^6
  +((9*mu1*mu2^2*lam11212-81*mu1^2*mu2^3)*q-9*mu1*mu2^2*lam11212)
    *g2^4*ixi*g2^4
  +((3*mu2*lam11212^2-36*mu1*mu2^2*lam11212+81*mu1^2*mu2^3)*q
    +(-18*mu1*mu2^2*lam11212+81*mu1^2*mu2^3))
    *g2^2*ixi*g2^2
  -mu12*(ixi-g1^6*g2^6*ixi*g1^6*g2^6)
,si) $
```

```
print("Is w=x12^6-s12-mu12*(1-g1^6*g2^6) skew-primitive? (0 = Yes)")_
remainder(delw-w*ixi-g1^6*g2^6*ixi*w,si)$

bye$
```

As a final result we obtain the output

```
            ⋮
> Is w=x12^6-s12-mu12*(1-g1^6*g2^6) skew-primitive?  (0 = Yes)
@ := 0
```

which confirms that the chosen s_{12} is correct.

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