Supersymmetry, quantum gauge anomalies and generalized Chern-Simons terms in chiral gauge theory

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München 2008

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Dissertation
an der Fakulät für Physik
der Ludwig-Maximilians-Universität
München

vorgelegt von
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aus München

München, den 1.Dezember 2008

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Tag der mündlichen Prüfung:	13. Mai 2009

Abstract

The purpose of this thesis is to investigate the interplay of anomaly cancellation and generalized Chern-Simons terms in four-dimensional chiral gauge theory. The inclusion of generalized Chern-Simons terms and additional axionic couplings allows to relax the constraints which are otherwise imposed by anomaly-freedom. There has been a lot of recent interest in the phenomenology of these additional couplings. Possible models that make use of this are provided by intersecting brane models in orientifold compactifications of the type II string theories. If the mass of the anomalous $U(1)$-gauge boson is low enough, these models predict small signals that might be detectable in near-future collider experiments.

We start with a detailed discussion of generalized Chern-Simons terms and establish the connection of generalized Chern-Simons terms with the cancellation of anomalies via the Green-Schwarz mechanism. With this at hand, we investigate the situation in general $\mathcal{N} = 1$ supersymmetric field theories with generalized Chern-Simons terms. Two simple consistency conditions are shown to encode strong constraints on the allowed anomalies for different types of gauge groups. The results even apply to $\mathcal{N} = 1$ matter-coupled supergravity generalizing previously known actions.

In $\mathcal{N} = 1$ supersymmetry or in theories without supersymmtry, the rigid symmtries of the vector and scalar sector are not directly related. The rigid symmetry group is a subset of the product of the symplectic duality transformations that act on the vector fields and the isometry group of the scalar manifold of the chiral multiplets. If nontrivial electic/magnetic duality transformations are involved, the fields before and after such a symmetry operation are not related by a local field transformation. In order to use the standard procedure for gauging a rigid symmetry, one therefore first has to switch to a symplectic duality frame in which the relevant symmetries act by local field transformations only. This obviously breaks the original duality covariance. Recently an alternative method has been proposed that allows one to formally maintain the full duality covariance at each step of the gauging procedure. This method requires the extension of the usual gauge degrees of freedom and the particle content, which leads to a new formulation of four-dimensional gauge theories.

In one major part of this thesis we are going to display to what extent one has to modify the existing formalism in order to allow for the cancellation of quantum gauge anomalies via the Green-Schwarz mechanism. The results might be relevant for certain $\mathcal{N} = 1$ flux compactifications with anomalous fermionic spectrum.

At the end of this thesis we comment on a puzzle in the literature on supersymmetric field theories with massive tensor fields. These occur naturally in the low-energy effective action of certain IIB orientifold compactifications with fluxes, where they give rise to scalar potentials that are not of the standard supersymmetry form. The potential contains a term that does not arise from eliminating an auxiliary field. We will clarify the origin of this term and display the relation to a standard D-term potential. In an appendix it is explicitly shown how these low energy effective actions might be connected to the formulation of four-didmensional gauge theories discussed at earlier stages of this thesis.

Inhaltsangabe

In dieser Dissertation untersuchen wir die Rolle verallgemeinerte Chern-Simons Terme in vierdimensionalen chiralen Eichtheorien, genauer, wie Anomlien weggehoben werden können. Unter Einbeziehung von verallgemeinerten Chern-Simons Termen und zusätzlichen axionischen Kopplungen ist man in der Lage die Bedingungen, die Abwesenheit von Anomalien garantieren, zu entschärfen. Phänomenologische Modelle, die gerade diese Art von Kopplungen beinhalten, sind seit einiger Zeit Mittelpunkt reger Untersuchungen. Mögliche Realisierungen für entsprechende Modelle sind zum Beispiel durch sich schneidende Branen-Modelle in Orientifoldkompaktifizierungen von Typ II Stringtheorien gegeben. Die Vorhersagen der phänomenologischen Untersuchungen dieser Modelle könnten sogar in naher Zukunft in Kollisionsexperimenten nachgeprüft werden, falls nur die Masse des anomalen $U(1)$-Eichbosons klein genug ist.

Nach einer kurzen Einführung in Quantenanomalien diskutieren wir im Detail die verallgemeinerten Chern-Simons Terme und erläutern unter welchen Umständen sie mit Hilfe eines Mechanismus nach Green und Schwarz zum Wegfall von Anomalien führen können. Diese ersten Ergebnisse erlauben eine umfassende Untersuchung der entsprechenden Situation in allgemeinen $\mathcal{N} = 1$ supersymmetrischen Feldtheorien mit verallgemeinerten Chern-Simons Termen. Wie gezeigt wird, können die starken Anforderungen, die sich aus der Abwesenheit von Anomalien unterschiedlicher Eichgruppen ergeben, durch zwei einfache Bedingungen zum Ausdruck gebracht werden. Dies gilt ebenfalls in $\mathcal{N} = 1$ Supergravitationstheorien mit Kopplungen an massive Felder, bekannte Wirkungen verallgemeinernd.

Globale Symmetrien jener Sektoren, die Vektorfelder und Skalarfelder enthalten, stehen in $\mathcal{N} = 1$ Supersymmetrie oder in nicht supersymmetrischen Theorien in keiner direkten Verbindung. Die globale Symmetriegruppe ist eine Untergruppe des Produkts der symplektischen Dualitätstransformationen, die auf die Vektorfelder wirken und der Isometriegruppe der skalaren Mannigfaltigkeit der chiralen Multipletts dar. Nichtriviale Transformationen der elektisch/magnetischen Dualität wirken derart auf Felder, dass diese nicht mehr in einer lokalen Beziehung mit den transformierten Feldern stehen. Wenn man nun eine

globale Symmetrie standardgemäß eichen will, dann muß man erst in einen symplektischen Dualitätsrahmen wechseln, in dem die Felder über lokale Transformationen untereinander in Beziehung stehen. Dies bricht offensichtlich die ursprüngliche Dualitätskovarianz. Vor nicht all zu langer Zeit wurde eine alternative Methode vorgeschlagen, die es erlaubt, bei jedem Schritt des Eichprozesses die volle formale Dualitätskovarianz zu bewahren. Diese Methode verlangt eine Erweiterung der gewöhnlichen Eichfreiheitsgrade und die Einführung neuer Felder. Auf diese Art wird eine neue Formulierung der Eichtheorien in vier Dimensionen erreicht. In einem der Hauptteile der Dissertation werden wir sehen, wie genau nun dieser Formalismus modifiziert werden muss, damit auch Quantenanomalien mit Hilfe des Mechanismus nach Green und Schwarz entfernt werden können. Diese Resultate sind relevant für gewisse $\mathcal{N} = 1$ Flusskompaktifizierungen mit anomalem Fermionspektrum.

Am Ende der Dissertation wenden wir uns einem Punkt zu, der in der Literatur zu supersymmetrischen Feldtheorien mit massiven Tensorfeldern angemerkt wurde. Diese Theorien erscheinen für gewöhnlich in den effektiven Niederenergie-Wirkungen gewisser IIB Orientifoldflusskompatifizierungen und erzeugen Potentiale für Skalarfelder von aussergewöhnlicher Form. Diese Potentiale enthalten einen Term, der nicht aus der Elimination eines Hilfsfeldes resultiert. Wir werden diesen Punkt klären und auch die Beziehung dieser Potentiale zu gewöhnlichen D-Term Potentialen aufzeigen. Im Anhang zu dieser Arbeit ist dargestellt, wie genau diese effektiven Niederenergie-Wirkungen mit einigen der zuvor erwähnten vierdimensionalen Eichtheorien in Zusammenhang stehen.

Contents

1 Introduction

In quantum physics an anomaly is the failure of a symmetry of the *classical* theory to be a symmetry of the full quantum theory. In chiral gauge theories an anomaly of the gauge symmetry may occur because the chirality of the gauge interactions may cause loop contributions (e.g. to n-point functions) that violate the symmetries of the classical action. For quantum gauge theories this is fatal, as such a gauge anomaly leads to a loss of renormalizeability. To avoid this, one usually has to impose a number of nontrivial constraints on the possible charges of the chiral fermions in such a way that the anomaly is absent. Without introducing any new particle or interaction, this amounts to demanding that the anomalous Feynman diagrams cancel. The vanishing of all anomalous one-loop diagrams already provides a sufficient condition for anomaly-freedom to all loop orders [1].

It is possible to relax these constraints if gauge variations of the classical action are able to cancel some of the anomalous one-loop contributions. In this case the classical action itself cannot be gauge invariant, of course. In the simplest example, the action contains an axionic coupling of a scalar $a(x)$ to the field strength of some vector field of the form $aF \wedge F$, where $a(x)$ transforms with a shift under some Abelian gauge symmetry with gauge parameter $\Lambda(x)$, i.e. $\delta a(x) \propto \Lambda(x)$. An Abelian anomaly may be exactly cancelled by the gauge variation of this axionic coupling, which is proportional to $\Lambda F \wedge F$. This is a simple four-dimensional example of the Green-Schwarz mechanism [2].

The scalar $a(x)$ is usually called "axion" and its kinetic term has to be of Stückelberg-type in order to be gauge invariant, i.e. proportional to $(\partial_\mu a - A_\mu)^2$. The Stückelberg coupling implements the shift symmetry via an Abelian gauge boson that gains a mass due to its coupling to the axion. If the mass of such a gauge boson is low enough and if it has suitable interactions with the Standard Model particles, it may lead to observable signals in near-future collider experiments. There has recently been quite some interest in the phenomenological studies of such anomalous Z'-type bosons [3–16]. A natural framework for such models is provided by intersecting brane models in type II orientifolds[1] because the four-dimensional Green-Schwarz mechanism is rather generic in these kind of models [23].

Interestingly, the Green-Schwarz mechanism alone is often not enough to cancel all con-

[1]More details on intersecting brane models can be found in, e.g., [17–22] and references therein.

tributions from gauge anomalies in these orientifold compactifications [12, 13][2]. Especially the cancellation of mixed Abelian anomalies between anomalous and non-anomalous Abelian factors is in general not achieved by the Green-Schwarz mechanism alone. Instead, one needs the help of topological terms, so-called generalized Chern-Simons terms, which are not gauge invariant. In general, it is the combination of the Green-Schwarz mechanism and the generalized Chern-Simons terms which possibly cancels the complete gauge anomaly. In [12][3] the question was raised, how to generate the generalized Chern-Simons terms from certain string compactifications. It was shown that the generalized Chern-Simons terms are a generic feature of the orientifold models we referred to above and may lead to new observable signals of Z'-bosons. Another possibility was mentioned in [26] where certain flux and generalized Scherk-Schwarz compactifications [27, 28] were used to explain possible origins. There is also the possibility to obtain $\mathcal{N}=2$ supergravity theories with generalized Chern-Simons terms from ordinary dimensional reduction of certain five dimensional $\mathcal{N}=2$ supergravity theories with tensor multiplets[4] [29].

It should be emphasized that the generalized Chern-Simons terms need not necessarily appear in combination with the Green-Schwarz mechanism and anomalies. Originally, these terms were first discovered in extended gauged supergravity theories [32] which are manifestly free of anomalies due to the usual incompatibility of chiral gauge interactions with extended supersymmetry in four dimensions. This motivated the discussions in [26–29, 33–39] which demonstrated how generalized Chern-Simons terms cancel axionic shifts in different classical setups. In all these cases the absence of gauge anomalies imposes strong restrictions on the form of possible gauged axionic shift symmetries.

In light of the above mentioned possible phenomenological applications and given their generic occurence in various string theory compactifications, it is surprising that the general interplay between the Green-Schwarz mechanism, generalized Chern-Simons terms and $\mathcal{N}=1$ supersymmetry was not very well understood until rather recently. It is the purpose of this thesis to give a systematic account of these issues as they were developed in [88] during the past years.

[2]For related phenomenological work, see also [14–16, 24, 25]

[3]The basic ideas are presented by means of a simple toy model in [13].

[4]These five dimensional $\mathcal{N}=2$ supergravity theories are discussed in [30, 31].

The outline of this thesis is as follows. In section 2, we review the most important facts about quantum anomalies in chiral gauge theories. We will illustrate how the triangle diagram causes a violation of the conservation law of axial currents. Then we will review how the anomaly can also be understood by the Jacobian of the path integral measure under axial transformations. With this at hand we will present the Wess-Zumino consistency condition and, at the end of section 2, we will shortly comment on some general aspects of anomaly cancellation.

In section 3, we construct generalized Chern-Simons terms along the lines of [34]. We will further show that there are no nontrivial generalized Chern-Simons terms for semisimple gauge groups. This motivates a short discussion of the example of a gauge group with the structure Abelian$\times$semisimple. The section ends with a generalization of the method developed in [34] so as to be able to incorporate anomalies into the formalism.

Section 4 summarizes the most important formulae concerning the gauge sector of global and local $\mathcal{N} = 1$ supersymmetry which will be of major concern in the subsequent section 5.

After the introductory sections 2 to 4, we will apply, in section 5, the results of section 3 to gauged isometries on the target manifold of scalar fields in global and local $\mathcal{N} = 1$ supersymmetry and generalize previous work. Therefore, we begin by gauging an Abelian isometry in global $\mathcal{N} = 1$ supersymmetry and show when it is necessary to add generalized Chern-Simons terms to the gauge sector presented in section 4 such that the resulting action is invariant under the gauged isometries. After having generalized the results to gauged nonabelian isometries, we will display under which conditions gauge anomalies are possibly cancelled. Furthermore, we investigate the conservation of supersymmetry in presence of gauged isometries. After this is accomplished, we will extend the results to $\mathcal{N} = 1$ supergravity. We will illustrate the cancellation procedure for a gauge group of the form Abelian$\times$semisimple.

In section 6, we will show that four-dimensional gauge theories with Green-Schwarz anomaly cancellation and possible generalized Chern-Simons terms admit a formulation that is manifestly covariant with respect to electric/magnetic duality transformations. This generalizes previous work on the symplectically covariant formulation of *anomaly-free* gauge theories and may have interesting applications, e.g., for flux compactification with intersecting branes.

In section 7 we discuss the action for a massive tensor multiplet coupled to chiral and vector multiplets in the $\mathcal{N} = 1$ superfield formalism. We compute the D-term potential and show that it is equivalent to a potential in standard form explaining an earlier result by [90]. The action can be regarded as the supersymmetrization of a special Abelian gauge of the theory presented in section 6. The precise connection is illustrated in appendix E.

The conclusion is found in section 8, and notations and conventions, as well as technical details to several calculations, are summarized in the appendices.

2 Quantum anomalies

A quantum theory is called anomalous if there is an exact symmetry of the classical action which is not preserved as a symmetry after quantization. When for gauge theories the quantum action is not gauge invariant, then the quantum theory is not renormalizable. The reason is that so-called Ward-identities, which are absolutely necessary for the renormalization procedure to be well-defined, do not hold.

Anomalies are not only a possible feature of gauge symmetries, but may also arise for global symmetries of the classical action. Contrary to quantum gauge theories, in the case of the global symmetry this is not necessarily a problem but may instead lead to interesting measurable physical effects as, for example, the decay of the pion into gamma rays shows. Historically, the observed decay rates in experiments did not match the theoretical predictions. Only once the contribution of the global anomaly was considered, very good agreement between experiment and theory could be obtained. The anomaly does not spoil renormalization here because no Ward-identity is violated. This example also shows that an anomaly is not simply a mathematical problem caused by the formalism but has a clear physical interpretation. In fact, an anomaly is a consequence of the non-invariance of the quantum measure in the path integral formulation as demonstrated by Fujikawa [41]. Nevertheless, already triangle diagrams show whether a given theory is anomalous or free of anomalies, which will be reviewed in the next section. In section 2.2, we illustrate how the anomaly appears in the path integral formalism. The consistent anomaly is explained in section 2.3 and the Wess-Zumino consistency condition is presented. Finally, in section 2.4, we comment briefly on the cancellation of anomalies.

2.1 Triangle anomaly

Gauge symmetry and renormalization are closely related topics. In gauge theory, the renormalization procedure makes use of identities that relate different Green's functions. These identities were proven by Ward [42] and Takahashi [43] and are hence called "Ward-Takahashi identities". The validity of the Ward-Takahashi identities is not automatic when chiral fermions are in the theory. More explicitly, one has to check whether there are diagrams that introduce anomalous terms, preventing the Ward-Takahashi identities from reproducing themselves recursively at higher orders in perturbation theory. In a theory with chiral fermions[5] the three-point functions

$$T_{\rho\mu\nu}(q,k_1,k_2) \equiv \langle 0|T[J^5_\rho(q)J_\mu(k_1)J_\nu(k_2)]|0\rangle\ , \tag{2.1}$$

$$T_{\mu\nu}(q,k_1,k_2) \equiv \langle 0|T[P(q)J_\mu(k_1)J_\nu(k_2)]|0\rangle \tag{2.2}$$

cause such anomalous terms that violate the Ward-Takahashi identities. Here $P(q)$ represents the pseudoscalar current which is explicitly given by $P = \bar{\psi}\gamma_5\psi$. The Feynman graphs that illustrate (2.1) and (2.2) are, to lowest order, triangle graphs with two external photons and one axial vector in the first case and a pseudoscalar (if present) for the second case.

Applying the standard Feynman rules to the Feynman diagrams displayed in figure 1 allows

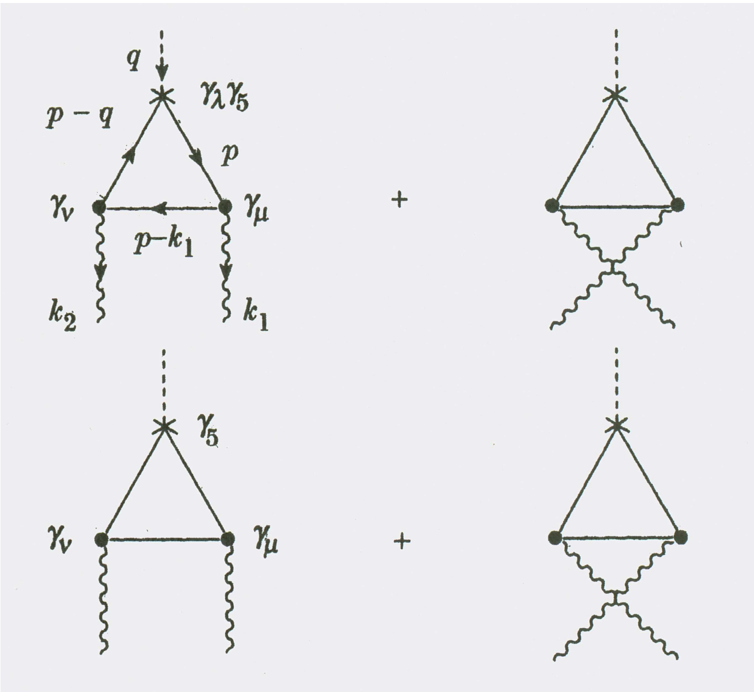

Figure 1: These diagrams cause contributions that violate explicitly the Ward-Takahashi identities. The graphic is taken from [1].

[5]Consider a Lagrangian where the fermion is denoted by ψ and couples to a vector field A_μ and to an axial vector field A^5_μ. The Lagrangian is given by $\mathcal{L}(A^5_\mu, A_\mu) = \bar{\psi}(\partial_\mu\gamma^\mu + A_\mu\gamma^\mu + A^5_\mu\gamma_\mu\gamma_5)\psi$, for example. Note that the given Lagrangian describes also the coupling of a vector field to the electromagnetic current represented by $J^\mu = \bar{\psi}\gamma^\mu\psi$ and of an axial vector field coupling to the axial vector current $J^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$.

one to write down the explicit expressions for (2.1) and (2.2), which are given by

$$
\begin{aligned}
T_{\mu\nu\rho}(q,k_1,k_2) &= -\mathrm{i}\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{tr}\frac{\mathrm{i}}{p^\mu\gamma_\mu - m}\gamma_\rho\gamma_5\frac{\mathrm{i}}{(p-q)^\mu\gamma_\mu - m}\gamma_\nu\frac{\mathrm{i}}{(p-k_1)^\mu\gamma_\mu - m}\gamma_\mu + \right.\\
&\quad\left. +\mathrm{tr}\frac{\mathrm{i}}{p^\mu\gamma_\mu - m}\gamma_\rho\gamma_5\frac{\mathrm{i}}{(p-q)^\mu\gamma_\mu - m}\gamma_\mu\frac{\mathrm{i}}{(p-k_2)^\mu\gamma_\mu - m}\gamma_\nu\right] \qquad (2.3)\\
T_{\mu\nu}(q,k_1,k_2) &= -\mathrm{i}\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{tr}\frac{\mathrm{i}}{p^\mu\gamma_\mu - m}\gamma_5\frac{\mathrm{i}}{(p-q)^\mu\gamma_\mu - m}\gamma_\nu\frac{\mathrm{i}}{(p-k_1)^\mu\gamma_\mu - m}\gamma_\mu + \right.\\
&\quad\left. +\mathrm{tr}\frac{\mathrm{i}}{p^\mu\gamma_\mu - m}\gamma_5\frac{\mathrm{i}}{(p-q)^\mu\gamma_\mu - m}\gamma_\mu\frac{\mathrm{i}}{(p-k_2)^\mu\gamma_\mu - m}\gamma_\nu\right] \qquad (2.4)
\end{aligned}
$$

where $q := k_1 + k_2$. In order to find the Ward-Takahashi identity for the axial vector, one has to compute $q^\rho T_{\rho\mu\nu}$. A useful identity is

$$
\begin{aligned}
\frac{1}{p^\mu\gamma_\mu - m}q^\rho\gamma_\rho\gamma_5\frac{1}{p^\mu\gamma_\mu - q^\mu\gamma_\mu - m} &= \frac{1}{p^\mu\gamma_\mu - m}\gamma_5 + \gamma_5\frac{1}{p^\mu\gamma_\mu - q^\mu\gamma_\mu - m}\\
&\quad +2m\frac{1}{p^\mu\gamma_\mu - m}\gamma_5\frac{1}{p^\mu\gamma_\mu - q^\mu\gamma_\mu - m}\,, \qquad (2.5)
\end{aligned}
$$

which can be easily proven by multiplying (2.5) from the left side by $(p^\mu\gamma_\mu - m)$ and from the right side by $(p^\mu\gamma_\mu - q^\mu\gamma_\mu - m)$. With the help of the identity (2.5) one can replace the first two fractions in (2.3) by the right hand side of (2.5), and it is not difficult to see that we have

$$
q^\rho T_{\mu\nu\rho} = R^1_{\mu\nu} + R^2_{\mu\nu} + 2m\,T_{\mu\nu}\,, \qquad (2.6)
$$

where $R^1_{\mu\nu}$ and $R^2_{\mu\nu}$ denote integrals that are caused by the first two terms on the right hand side of (2.5). The axial Ward-Takahashi identity is

$$
q^\rho T_{\mu\nu\rho} = 2m\,T_{\mu\nu}\,, \qquad (2.7)
$$

and we see that (2.6) violates (2.7) by the remaining terms $R^1_{\mu\nu}$ and $R^2_{\mu\nu}$. These remaining terms do not vanish because, when written out with the help of Feynman rules, they result in linearly divergent integrals that lead to ambiguities in the momentum route of the triangle graph.

The amplitude $T_{\mu\nu}$ (2.2) is convergent because the apparent linear and logarithmic divergencies disappear in the actual computation. The calculation is not repeated here but can be found in the classical lectures on anomalies by Jackiw ([44,45]) and in any textbook on quantum field theory, e.g. [46,47]. An additional useful reference is the book of Bertlmann [1].

The resulting anomalous Ward-Takahashi identity is equivalent to the modified conservation law for the axial current

$$\partial^\mu J^5_\mu \;=\; 2mP(x) + \mathcal{A}\,, \tag{2.8}$$

where the anomaly, $\mathcal{A}$, is given by

$$\mathcal{A} \;=\; \frac{e^2}{(4\pi)^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,. \tag{2.9}$$

This is the famous Adler-Bell-Jackiw anomaly [48,49], where $F_{\mu\nu}$ is the Abelian field strength defined by $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$.[6]

The anomaly (2.9) is independent of the fermion mass and therefore violates the current conservation of the massless theory.

The Ward-Takahashi identity of the vector currents is fulfilled which is a consequence of a chosen momentum route.

Observe that attaching new photon lines to one loop diagrams, which is equivalent to turning the triangle diagram into a quadrangle or in general n-angle diagram, generates an integral that is at least logarithmically divergent: $T_{\mu\nu\rho\sigma\dots}$ for fermionic loops with more than four external photons attached to it. This can be understood heuristically by noting that the superficial degrees of divergence of the higher order diagrams are less than one and the momentum-routing ambiguity does not exist for those diagrams. This summarizes the theorem by Adler and Bardeen [50], that states that radiative corrections in higher orders do not alter (2.8) and, thus, the anomaly is already totally determined by the triangle diagram.

2.2 Path integral and anomaly

Adler and Bardeen proposed in their theorem that the full structure of the chiral anomaly is given by the triangle anomaly [50] and does not receive contributions from further radiative corrections. This suggests that the anomaly should even exist beyond perturbation theory. Fujikawa was the first to recognize that in the path integral formalism the anomaly corresponds to the Jacobian of a γ_5-transformation of the quantum measure [41]. One can see

[6] Here and in the following, [] and () denote, respectively, antisymmetrization and symmetrization with "strength one", i.e., $[ab] = \frac{1}{2}(ab - ba)$ etc.

this as follows: Let there be massless fermionic fields in the theory transforming nontrivially under chiral gauge transformations as

$$\begin{aligned} \psi &\rightarrow e^{\mathrm{i}\theta\gamma_5}\psi\,, \\ \bar{\psi} &\rightarrow \bar{\psi}e^{\mathrm{i}\theta\gamma_5}\,. \end{aligned} \tag{2.10}$$

The important steps in Fujikawa's method are first to define the path integral measure more accurately by decomposing the spinors ψ and $\bar{\psi}$ into eigenfunctions of the Dirac operator and second to determine the Jacobian of the path integral measure under chirality transformations. The Jacobian of infinitesimal transformations will be exactly the anomaly.

The eigenvectors $|n\rangle$ of the operator $\mathcal{D}$ are given by:

$$\mathcal{D}^\mu\gamma_\mu|n\rangle = \lambda_n|n\rangle\,, \tag{2.11}$$

and the spinors decompose according to

$$\psi(x) = \sum_n a_n\langle x|n\rangle\,, \tag{2.12}$$

$$\bar{\psi}(x) = \sum_n \langle n|x\rangle\bar{b}_n\,, \tag{2.13}$$

where the decomposition coefficients a_n and $\bar{b}_n$ are independent Grassmann objects. These coefficients at hand, we are able to re-express the path integral measures $D\psi D\bar{\psi}$ according to

$$D\psi D\bar{\psi} = \prod_n Da_n D\bar{b}_n\,, \tag{2.14}$$

because the set of eigenvectors $|n\rangle$ is complete and orthonormal, i.e. $\langle n|m\rangle = \delta_{nm}$. In order to determine the behaviour of the objects a_n and $\bar{b}_n$ under chiral transformations, we consider the rotated spinor

$$\psi'(x) = e^{\mathrm{i}\theta\gamma_5}\psi(x)\,. \tag{2.15}$$

After decomposing both sides of (2.15) into the eigenvectors $|n\rangle$, and using the orthonormality of the eigenvectors, one finds that

$$\begin{aligned} a'_n &= \sum_m C_{nm}a_m\,, \\ C_{nm} &:= \int dx\,\langle n|x\rangle e^{\mathrm{i}\theta(x)\gamma_5}\langle x|m\rangle\,. \end{aligned} \tag{2.16}$$

The Grassmann measure transforms with the inverse determinant and, therefore, the path integral measure transforms with $\left[\det(C_{nm})^{-1}\right]^2$, which has to be determined. Making use of $\det C = e^{\mathrm{tr}\log(C)}$ and considering infinitesimally small transformations, one can decompose the logarithm around the unity matrix. Then, the Jacobian J of infinitesimal chirality transformations is given by

$$J = e^{-2\mathrm{i}\int dx\,\theta\,\tilde{\mathrm{tr}}(\gamma_5)}\,. \tag{2.17}$$

Observe, that the functional trace $\tilde{\mathrm{tr}}(\gamma_5)$ is defined through the eigenvectors $\tilde{\mathrm{tr}}(\gamma_5) := \sum_n \langle n|x\rangle\gamma_5\langle x|n\rangle$.[7] This trace is actually divergent, and we have to regulate the sum. As the regulator we use the convergent factor $\exp\left[-(\frac{\lambda_n}{M})^2\right]$ and take the limit $M\to\infty$. Then, we can manipulate the regulated exponent of (2.17) and after introducing unity operators of the form $\int d^4k\,|k\rangle\langle k|$ and by using completeness of the set $\{|n\rangle\}$, we find

$$\lim_{M\to\infty}\tilde{\mathrm{tr}}(\gamma_5 e^{-(\frac{\lambda_n}{M})^2}) = \lim_{M\to\infty}\int\frac{d^4k}{(2\pi)^4}\,e^{\mathrm{i}k\cdot x}(\gamma_5 e^{-(\frac{\mathcal{D}^\mu\gamma_\mu}{M})^2})e^{-\mathrm{i}k\cdot x}\,.$$

We decompose the operator $\left[\mathcal{D}^\mu\gamma_\mu\right]^2$ into an odd piece proportional to $[\gamma^\mu,\gamma^\nu]$ and an even piece proportional to $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ so that we have $\left[\mathcal{D}^\mu\gamma_\mu\right]^2 = \mathcal{D}^\mu\mathcal{D}_\mu + \frac{1}{4}[\gamma_\mu,\gamma_\nu]F^{\mu\nu}$. After rescaling the momentum and decomposing the exponential, there is only one term that survives in the limit $M\to\infty$ (the term quadratic in the field strength), and we obtain:

$$\tilde{\mathrm{tr}}(\gamma_5) = -\frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,. \tag{2.18}$$

Inserting this back into (2.17) we indeed find the anomaly (2.9), or in other words, the path integral measure transforms with the Jacobian

$$J = e^{\frac{\mathrm{i}}{16\pi^2}\int dx\,\theta(x)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}\,. \tag{2.19}$$

However, as we did not expand the path integral, this result is valid beyond any perturbative expansion. In the path integral picture the anomaly is explained by the non-invariance of the path integral measure under chirality transformations. The formal reason for the non-invariance can be traced back to the functional trace $\tilde{\mathrm{tr}}\gamma_5$, which is singular.

[7] Note, that $\tilde{\mathrm{tr}}(\gamma_5)$ is not equal to the γ-matrix trace $\mathrm{tr}(\gamma_5) = 0$.

2.3 Consistent anomaly

So far we have only considered Abelian symmetries. If we want to generalize the above concepts to the nonabelian case, then the expression (2.18) will of course no longer represent the full anomaly. The naive extension of (2.9), in which the field strength is replaced by its covariant counterpart, is not correct because the contribution of quadrangle diagrams and pentagon diagrams, though finite, violates the nonabelian structure. The access through diagrams becomes now more complicated and so let us choose the more convenient way by means of the path integral. As a first step, we define Green's functions with the help of the generating functional, which is given by

$$Z[A_\mu] \quad = \quad \int D\bar{\psi} D\psi \, e^{-\int d^4x \, (\bar{\psi}\gamma^\mu \partial_\mu \psi + A_\mu \bar{\psi}\gamma^\mu \psi)} \,, \tag{2.20}$$

where the gauge fields are treated as external fields and sources for the fermions are ignored. For the proof of renormalizeability it is suitable to use connected Green functions, but the generating functional $Z[A]$ contains both connected and disconnected diagrams. The connected Green functions are generated by $W[A]$ defined by

$$Z[A_\mu] \quad = \quad e^{-W[A_\mu]} \,. \tag{2.21}$$

For the anomaly we only need to consider the fermionic part of the theory, so (2.21), given by (2.20), is really all we need from the full quantum action. Let the gauge group be generated by T_A satisfying the algebra $[T_A, T_B] = f_{AB}{}^C T_C$, where $f_{AB}{}^C$ are the structure constants. Infinitesimal gauge transformations that act on the action (2.21) are defined by the operators

$$\begin{aligned} X_A(x) \quad &= \quad D_{\mu\,A}^{C} \frac{\delta}{\delta A_\mu^C(x)} \\ &:= \quad \left[\partial_\mu \delta_A^C + f_{AB}{}^C A_\mu^B(x)\right] \frac{\delta}{\delta A_\mu^C(x)} \,. \end{aligned} \tag{2.22}$$

It can be shown that these operators fulfil the algebra given by

$$[X_A(x), X_B(y)] \quad = \quad f_{AB}{}^C X_C(x) \delta(x-y) \,, \tag{2.23}$$

and that the gauge variation of $W[A]$ is given by

$$\begin{aligned} \delta W[\Lambda] \quad &:= \quad \int d^4x \, \Lambda^A(x) X_A(x) W[A_\mu] \\ &= \quad \int d^4x \, \Lambda^A(x) \langle D_{\mu\,A}^{C} j_C^\mu(x) \rangle_{\text{con.}} \end{aligned} \tag{2.24}$$

where $\langle j^\mu_C\rangle_{\rm con.} = \frac{1}{Z[A]}\int D\bar\psi D\psi\,(\bar\psi\gamma^\mu T_C\psi)e^{-\int d^4x\mathcal{L}(\bar\psi,\psi,A)}$ is the expectation value of the connected current. We can easily see that for an invariant quantum action, $\delta W[\Lambda] = 0$, the current is covariantly conserved, $\langle D^C_{\mu\,A} j^\mu_C\rangle_{\rm con.} = 0$. However, if the theory is anomalous, then the generating functional of connected Green functions satisfies $\delta W[\Lambda] = \Lambda^A\mathcal{A}_A$, and in order to be consistent with the gauge algebra (2.23), the anomaly has to obey the condition

$$X_A(x)\mathcal{A}_B - X_B(y)\mathcal{A}_A \;\; = \;\; f_{AB}{}^C\mathcal{A}_C\,\delta(x-y)\,, \tag{2.25}$$

which is the so-called "Wess-Zumino consistency condition" [51]. We can also see that the naive nonabelian extension of (2.9), where the Abelian field strengths are replaced by their nonabelian counterparts, is not correct because it violates (2.25).

An explicit solution of (2.25) is given by

$$\mathcal{A}_C \;\; = \;\; \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}{\rm tr}[T_C\partial_\mu(A_\nu\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma)]\,, \tag{2.26}$$

which represents exactly Bardeen's result [52] found from fermion loop computations. This solution is not unique because one can add local polynomials of the external gauge fields $f[A]$ to (2.26) and obtain another solution. These local polynomials can be induced, e.g., when the renormalization procedure is changed. The 2-point Green functions of two vector currents, for example, have a renormalization ambiguity because their Lorentz invariant extensions to test functions are not unique [53]. For the quantum action, this means that $\tilde W[A] = W[A] + f[A]$ and the generating functional receives a phase factor $Z[A]e^{if[A]}$. A phase factor, however, does not affect the transition probability and is not observable. Consequently, we can also call a theory anomalous, if there does not exist a local polynomial of the external gauge fields, such that (2.26) is effectively cancelled. Possible local polynomials are given by Chern-Simons terms or generalized Chern-Simons terms (depending on the dimension). In the following section we will discuss these topological terms, especially the generalized Chern-Simons terms because these are of special interest in four dimensions.

2.4 Cancellation of anomalies

Although there are attempts to live with anomalous theories, see for example [54] and [55], in renormalizable theories, anomalies must not occur. This implies severe restrictions on the physical content of a theory. In vector-like models all fermions couple symmetrically in both chiral sectors and any potential gauge anomaly in the left-handed sector is cancelled by the anomaly of the right-handed fermions. In chiral gauge theories, by contrast, anomaly cancellation is not automatic and the cancellation requires a careful balance of the fermionic gauge quantum numbers, as, e.g., in the standard model.

Another possibility to cancel anomalies is to introduce a counterterm into the action, with particles that transform appropriately under gauge transformations such that the anomaly is compensated. As mentioned in the introduction, a simple Abelian example is given by the interaction

$$\varepsilon^{\mu\nu\rho\sigma}\mathrm{i}a(x)F_{\mu\nu}F_{\rho\sigma}\,, \tag{2.27}$$

where the scalar, $a(x)$ varies under the gauge symmetry according to

$$\delta a(x) \quad = \quad \mathrm{i}\Lambda(x)\,. \tag{2.28}$$

Then the variation of the interaction (2.27) is able to cancel the Abelian anomaly (2.9). When the gauge theory is nonabelian then the full consistent anomaly cannot be cancelled by this mechanism. The Green-Schwarz anomaly cancellation mechanism in 10-dimensional supergravity and super Yang-Mills theory is a sophisticated generalization of this simple example, see for example [2] and [56].

3 Lie algebra cohomology and generalized Chern-Simons terms

In generic effective field theories one has scalar field dependent functions appearing in front of the gauge kinetic terms, i.e. in front of $\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}$ and $\mathcal{F}\wedge\mathcal{F}$. Here in general, the nonabelian field strength two form is defined as

$$\mathcal{F}^C \;:=\; dA^C - \frac{1}{2} f_{AB}{}^C A^A \wedge A^B \,. \tag{3.1}$$

Supersymmetric theories, for example, often generalize the gauge sector to incorporate a nontrivial gauge kinetic function f_{AB} that depends on a set of scalar fields, as is further explained in section 4.1. Compatibility with supersymmetry constrains this function and so, for instance, in $\mathcal{N} = 1$ supersymmetry it is required to be a holomorphic function of the complex scalars of the chiral multiplets.

The Lagrangian will contain a nontrivial $\mathcal{F}\wedge\mathcal{F}$ term when the imaginary part of the gauge kinetic function is nontrivial. In the literature this term is sometimes referred to as a "Peccei-Quinn term" and reads

$$\mathcal{L}_{\mathrm{PQ}} \;=\; \mathrm{i}\,\mathrm{Im}\,\mathrm{f}_{AB}\,\mathcal{F}^A \wedge \mathcal{F}^B \,. \tag{3.2}$$

The interaction given in equation (2.27) actually represents a special case of (3.2) where we just have a $U(1)$ gauge symmetry (and hence only one index, which may be dropped), and the gauge kinetic function is given by the axionic scalar $a(x)$, i.e. $\mathrm{f} = 4\,a(x)$.

In the remainder, the exterior product $\wedge$ is understood and will no longer be written out explicitly.

Under the gauge transformation of the connection one-forms $A^C = A^C_\mu dx^\mu$, which read

$$\delta A^C \;=\; D\Lambda^C := d\Lambda^C + f_{AB}{}^C \Lambda^A A^B \,, \tag{3.3}$$

the field strength two forms (3.1) transform covariantly, i.e. if

$$\delta\mathcal{F}^C \;=\; f_{AB}{}^C \Lambda^A \mathcal{F}^B \,. \tag{3.4}$$

Clearly, the Lagrangian (3.2) is invariant under (3.3) if the gauge kinetic function transforms in the symmetric product of two adjoint representations, i.e. if

$$\delta\mathrm{f}_{AB} \;=\; 2\Lambda^C f_{C(A}{}^D \mathrm{f}_{B)D} \,. \tag{3.5}$$

More generally, however, there is still the possibility to relax (3.5) according to

$$\delta \mathrm{f}_{AB} \;=\; 2\Lambda^C f_{C(A}{}^D \mathrm{f}_{B)D} + \mathrm{i} C_{AB,D}\Lambda^D \,, \tag{3.6}$$

so as to allow for constant shifts in the gauge kinetic function. Here $C_{AB,D}$ is a real constant tensor satisfying the constraints

$$C_{(AB,D)} \;=\; 0\,, \tag{3.7}$$

$$\frac{1}{2}C_{AB,D}f_{EF}{}^D - C_{DB,[E}f_{F]A}{}^D - C_{DA,[E}f_{F]B}{}^D \;=\; 0\,. \tag{3.8}$$

This more general transformation (3.6) can be induced if the scalar fields transform nontrivially under the gauge group and appear in a certain way in f_{AB}, but we will address this later in more detail.

Obviously, once we allow for these shifts, the Lagrangian (3.2) is no longer invariant under (3.3) and (3.6). Its variation is instead given by

$$\delta\mathcal{L}_{\mathrm{PQ}} \;=\; \mathrm{i}C_{AB,D}\Lambda^D\mathcal{F}^A\mathcal{F}^B\,. \tag{3.9}$$

If we only consider the classical action, the variation (3.9) can only be cancelled by new terms added to $\mathcal{L}_{\mathrm{PQ}}$, the so called generalized Chern-Simons terms [32, 34]. In this section we will show how a classically gauge invariant action generalizing (3.2) can be constructed by using the techniques of [34]. In the following subsection we introduce Lie algebra valued forms $C(A,\mathcal{F})$ and analyze them by means of cohomological techniques. This method allows one to understand the origin of the constraints (3.7) and (3.8). The constraint (3.7) demands the forms $C(A,\mathcal{F})$ to be homogeneous in the field strength and the gauge connection separately. Then, for forms $C(A,\mathcal{F})$ whose coefficients satisfy (3.7) we can identify the constraint (3.8) as the constraint demanding $C(A,\mathcal{F})$ to be closed with respect to the exterior derivative. After specifying the transformation properties of the gauge kinetic function, we are able to construct the gauge invariant extension of the Peccei-Quinn term, which is obtained by including generalized Chern-Simons terms.

In subsection 3.2 we find that there are no non-trivial generalized Chern-Simons terms for semisimple gauge groups and present the example of a gauge group that has the form Abelian$\times$semisimple in section 3.3. The results of these subsections are discussed in more detail in appendix A, where the results are proven by methods of Lie algebra cohomology.

Finally, in subsection 3.4, we generalize the formalism developed in [34] in order to allow for forms that do not need to satisfy the constraint (3.7). We will see that the Peccei-Quinn term and the generalized Chern-Simons term are no longer gauge invariant once we give up the constraint (3.7). The only possibility to cancel the gauge non-invariance in such a case is to consider anomalies.

Before we construct the generalized Chern-Simons terms, I would like to give a few comments on 'ordinary' Chern-Simons terms [57] that should illustrate the difference between ordinary and generalized Chern-Simons terms. The construction of 'ordinary' Chern-Simons forms is usually done by means of so called characteristic or invariant polynomials P_n. The characteristic polynomials $P_n(\mathcal{F})$ are symmetric functions of degree n in the field strength form $\mathcal{F}$ and invariant under the action of the gauge symmetry group. Therefore, the characteristic polynomials satisfy $P_n(\mathcal{F}^g) = P_n(\mathcal{F})$ where we denoted the gauge transformed field strength by $\mathcal{F}^g$. With the help of the Bianci identity

$$D\mathcal{F} := d\mathcal{F} + [A, \mathcal{F}] \quad = \quad 0\,, \tag{3.10}$$

it can be proven that the invariant polynomials are closed, i.e. $dP_n(\mathcal{F}) = 0$. A theorem by Chern and Weil states that the cohomology classes of $P_n(\mathcal{F})$ do not depend on the choice of the connection form A and characterize the de Rham cohomology group [58]. Then, the cohomology classes of invariant polynomials $P_n(\mathcal{F})$ of degree n are further characterized by the Chern-Simons terms $Q_{n-1}(A, \mathcal{F})$ which are forms of degree $(n-1)$, i.e.

$$P_n(\mathcal{F}) \quad = \quad dQ_{n-1}(A, \mathcal{F}) \tag{3.11}$$

Integrals of characteristic polynomials are topological invariants. Let us consider, for example, in four dimensions a characteristic polynomial of the form $P_4(\mathcal{F}) = \mathrm{tr}(\mathcal{F}\mathcal{F})$ which is invariant because of $P_4(\mathcal{F}^g) = \mathrm{tr}(g\mathcal{F}g^{-1}g\mathcal{F}g^{-1}) = \mathrm{tr}(\mathcal{F}\mathcal{F}) = P_4(\mathcal{F})$. Then this characteristic polynomial leads to the three-dimensional Chern-Simons form $Q_3(A, \mathcal{F}) = \mathrm{tr}[AdA + \frac{3}{2}A^3]$.[8] Observe, that Chern-Simons forms are in general odd dimensional while generalized Chern-Simons forms live in even dimensions as we will see.

[8]Since the determinant is invariant under the adjoint of the gauge symmetry, i.e. $\det(\,) = \det\left[g^{-1}(\,)g\right]$ if g represents an element of the gauge group, one can also obtain invariant polynomials with the help of the determinant. However, the corresponding Chern-Simons forms are not related to the one obtained from $P_4(\mathcal{F}) = \mathrm{tr}(\mathcal{F}\mathcal{F})$ as considered in the example.

3.1 Generalized Chern-Simons forms

Generalized Chern-Simons terms cannot be constructed from characteristic polynomials because there are no odd dimensional invariant polynomials in the field strength. To set the stage we consider a five-form $C(A,\mathcal{F})$ defined as

$$C(A,\mathcal{F}) \quad := \quad C_{AB,D}A^D\mathcal{F}^A\mathcal{F}^B \,, \tag{3.12}$$

and do not limit ourselves to four spacetime dimensions.

Note the peculiar structure of the indices of the constant tensor $C_{AB,D}$: the index corresponding to that carried by the gauge connection is separated from the indices that are carried by the field strengths by a comma. Therefore, the constant tensor is symmetric in its first two indices which is also consistent with (3.6). Furthermore, observe that the form $C(A,\mathcal{F})$ does not represent an invariant or characteristic polynomial as mentioned in the context of ordinary Chern-Simons terms because $C(A,\mathcal{F})$ depends explicitly on the gauge connection. There is no problem in generalizing (3.12) to forms of arbitrary degree in A and $\mathcal{F}$ by introducing constant tensors of the form $C_{A_1\ldots A_n,D_1\ldots D_m}$. Nevertheless, here we focus on the form (3.12), which leads to the gauge invariant generalization of (3.2) in four dimensions. Using (3.1) and (3.10) we can compute the exterior derivative of (3.12), which leads to

$$dC(A,\mathcal{F}) \quad = \quad C_{AB,D}\mathcal{F}^D\mathcal{F}^A\mathcal{F}^B + \left[\frac{1}{2}C_{AB,D}f_{EF}^D + f_{AE}^D C_{DB,F} + f_{BE}^D C_{AD,F}\right] A^E A^F \mathcal{F}^A\mathcal{F}^B \,.$$

Comparing this result with the constraints (3.7) and (3.8) shows that these correspond to demanding that $C(A,\mathcal{F})$ is homogenous[9] and closed, i.e. $dC(A,\mathcal{F})=0$. On the other hand, we can define an algebraic operator

$$(\mathcal{D}C)_{AB,EF} \quad := \quad \frac{1}{2}C_{AB,D}f_{EF}^D - C_{DB,[F}f_{E]A}^D - C_{AD,[F}f_{E]B}^D \,, \tag{3.13}$$

satisfying $\mathcal{D}^2=0$ because of $d^2=0$ (this can also be directly proven from (3.13) by using the Jacobi identity on the structure constants). Hence, we can say that as d^2 leads to the de Rham cohomology, $\mathcal{D}^2=0$ leads to Lie algebra cohomology of forms $C(A,\mathcal{F})$ satisfying the constraints (3.7) and (3.8). For a closed form C, i.e. if $C_{AB,D}$ fulfils the equations

[9] Observe that we call $C(A,\mathcal{F})$ a homogeneous form, following [34], if $dC(A,\mathcal{F})$ is homogeneous in A and $\mathcal{F}$ separately. The constraint (3.7) is satisfied by homogeneous forms. Homogeneity enables one to define algebraic operators acting on the coefficients $C_{AB,D}$ of homogeneous forms.

(3.7) and (3.8), the equivalence classes of all C' in the cohomology are, for some four-form $Z = Z_{AB}\mathcal{F}^A\mathcal{F}^B$, given by $C' = C + dZ$. So if the cohomology class is trivial, then we have $C = dZ$ and C is d-exact.[10] We will see later when this is the case.

At this point it is suitable to discuss the transformation properties of the scalars that appear in the gauge kinetic function f_{AB}. We assume that the scalar fields z^i transform under gauge transformation as

$$\delta z^i \quad = \quad \Lambda^A k^i_A(z)\,, \tag{3.14}$$

where the vector fields[11] $k_A = k^i_A\partial_i$ define a (possibly nonlinear) realization of the gauge group and satisfy

$$k^j_A\partial_j k^i_B - k^j_B\partial_j k^i_A \quad = \quad f_{AB}{}^C k^i_C\,. \tag{3.15}$$

As transformations of the scalars in general induce transformations of the gauge kinetic function, let us assume that (3.14) induces the transformation (3.6), i.e.,

$$\begin{aligned}\delta(\mathrm{Im}\,\mathrm{f}_{AB}) \quad &:= \quad k^j_D\partial_j(\mathrm{Im}\,\mathrm{f}_{AB})\Lambda^D \\ &= \quad 2f_{D(A}{}^E\left[\mathrm{Im}\,\mathrm{f}_{B)E}\right]\Lambda^D + C_{AB,D}\Lambda^D\,.\end{aligned} \tag{3.16}$$

Then, in order to make use of the form $C(A,\mathcal{F})$ as defined in (3.12), let us consider the following Lie algebra-valued form

$$k^j_D\partial_j(\mathrm{Im}\,\mathrm{f}_{AB})A^D\mathcal{F}^A\mathcal{F}^B\,. \tag{3.17}$$

With the help of the Bianchi identity (3.10) and the variation of the gauge kinetic function (3.16), this can be written as

$$k^j_D\partial_j(\mathrm{Im}\,\mathrm{f}_{AB})A^D\mathcal{F}^A\mathcal{F}^B \quad = \quad -\,\mathrm{Im}\,\mathrm{f}_{AB}d(\mathcal{F}^A\mathcal{F}^B) + C_{AB,D}A^D\mathcal{F}^A\mathcal{F}^B\,. \tag{3.18}$$

Due to the chain-rule, we furthermore have

$$d(\mathrm{Im}\,\mathrm{f}_{AB})(z)\mathcal{F}^A\mathcal{F}^B \quad = \quad \partial_j(\mathrm{Im}\,\mathrm{f}_{AB})dz^j\mathcal{F}^A\mathcal{F}^B\,, \tag{3.19}$$

[10] Note that from dZ we can define the action of an algebraic operator on Z_{AB} in total analogy with equation (3.13) for $C_{AB,D}$, such that $C_{AB,D} = (\mathcal{D}Z)_{AB,D}$. The algebraic operator $(\mathcal{D}Z)_{AB,D}$ is defined as in equation (A.5), which for the case at hand reads $(\mathcal{D}Z)_{AB,D} = 2f_{D(A}{}^E Z_{B)E}$.

[11] These vector fields are not spacetime vector fields, but are vectors on the scalar manifold.

from which we subtract (3.18) to finally obtain

$$\partial_j(\operatorname{Im} f_{AB})(dz^j - k^j_D A^D)\mathcal{F}^A\mathcal{F}^B = d\left[(\operatorname{Im} f_{AB})\mathcal{F}^A\mathcal{F}^B\right] - C_{AB,D}A^D\mathcal{F}^A\mathcal{F}^B\,. \tag{3.20}$$

Let us have a closer look at this result and find out about its implications.

Firstly, the left hand side of (3.20) is gauge invariant because $dz^j - k^j_D A^D$ is the gauge covariant derivative for the scalar fields z^i, and from (3.16) we see that $\partial_j(\operatorname{Im} f_{AB})$ transforms covariantly as $C_{AB,D}$ is a constant. Consequently, the left hand side of (3.20) represents an invariant Lagrangian in 5 dimensions.

Secondly, let us consider the right hand side of (3.20). We can see that any shift of $C_{AB,D}$ by an exact (in the Lie algebra cohomology) piece $(\mathcal{D}Z)_{AB,D} = 2f_{D(A}{}^E Z_{B)E}$ leads to a shift of the five form $C(A,\mathcal{F})$ by an exact form dZ, as was explained in footnote 10. According to (3.20), this exact form dZ can then be absorbed by a shift $\operatorname{Im} f_{AB} \to \operatorname{Im} f_{AB} + Z_{AB}$, as is also suggested by (3.16). Therefore, we can say that any exact contributions of C can be absorbed by a redefinition of the gauge kinetic function by a constant imaginary shift.

Now, that we have an invariant action in five dimensions, we want to pave the way to obtain invariance in four dimensions. If we demand that $C_{AB,D}$ satisfies the constraints (3.7) and (3.8), we know that $C(A,\mathcal{F})$ is closed. It then follows from Poincaré's lemma that locally there exists a form ω, such that $C = d\omega$. In order to find an explicit expression for ω, we single out one coordinate t and require $A^D(t) = tA^D$ with A^D depending only on the remaining coordinates. After introducing $d^t = d + \partial_t dt$ and defining

$$H^A(t) := tdA - \frac{1}{2}t^2 f_{BC}{}^A A^B A^C\,, \tag{3.21}$$

we can verify the following formulae

$$\mathcal{F}^C(t) = H^C(t) + dt\,A^C\,, \tag{3.22}$$

$$\mathcal{F}^A(t)\mathcal{F}^B(t) = H^A(t)H^B(t) + 2dtA^{(B}H^{A)}(t)\,. \tag{3.23}$$

As by assumption $C(A,\mathcal{F})$ is a closed form, the particular t-dependent form $C(A(t),\mathcal{F}(t))$, constructed from the definitions made above, is closed, too (the reason is that the constants $C_{AB,D}$ satisfy the constraints (3.7) and (3.8)). Then it is not difficult to prove that

$$\begin{aligned} 0 &= d^tC(A(t),\mathcal{F}(t)) = dt\partial_t C(A(t),\mathcal{F}(t)) + dC(A(t),\mathcal{F}(t)) \\ &= dt\partial_t C(A(t),\mathcal{F}(t)) + dC(A(t),H(t)) + 2t\,dt\,d(C_{AB,C}A^C A^B H^A)\,. \end{aligned} \tag{3.24}$$

The second term in the last line vanishes, which one sees very easily once the term is written in its component form

$$dC(A(t), H(t)) \;=\; d[C_{AB,D} A^D(t) H^A(t) H^B(t)]\,. \tag{3.25}$$

If we now absorb the factor t by rescaling, $At \to A$, then it follows from the definition (3.21), that $H(t) \to \mathcal{F}$, and (3.25) becomes $dC(A, \mathcal{F})$ which vanishes because $C(A, \mathcal{F})$ is closed. Finally, integrating (3.24) over t leaves us with

$$C(A, \mathcal{F}) \;=\; d\left[-2C_{AB,D} \int_0^1 dt\, t A^D A^B H^A(t)\right]\,. \tag{3.26}$$

Inserting (3.21), the integral can be computed, and we find

$$\omega \;=\; -\frac{2}{3} C_{BC,D} A^D A^B (dA^C - \frac{3}{8} f_{EF}{}^C A^E A^F)\,. \tag{3.27}$$

From the arguments below (3.20) we know that $d[\mathrm{Im}\, \mathrm{f}_{AB} \mathcal{F}^A \mathcal{F}^B] - C(A, \mathcal{F})$ is a gauge invariant expression in five dimensions and, consequently, $\left(\mathrm{Im}\, \mathrm{f}_{AB} \mathcal{F}^A \mathcal{F}^B - \omega\right)$ represents a gauge invariant Lagrangian in four dimensions. Concretely, the gauge invariant extension of the Peccei-Quinn Lagrangian reads

$$\mathcal{L}_{\mathrm{PQ}} + \mathcal{L}_{\mathrm{GCS}} \;=\; \mathrm{i}\, \mathrm{Im}\, \mathrm{f}_{AB} \mathcal{F}^A \mathcal{F}^B + \frac{2\mathrm{i}}{3} C_{BC,D} A^D A^B (dA^C - \frac{3}{8} f_{EF}{}^C A^E A^F)\,, \tag{3.28}$$

where the second term is the so called generalized Chern-Simons term.

These considerations are quite general and allow the extension of the transformation law for the gauge kinetic function by a constant imaginary shift $\mathrm{i}C_{AB,D}$ when at the same time the Peccei-Quinn term is accompanied by the generalized Chern-Simons term. The procedure is not limited to four dimensions and can be easily generalized to arbitrary even dimensions. The generalized Peccei-Quinn term then becomes the $2n$ form $\mathrm{f}_{A_1 A_2 \ldots A_n} \mathcal{F}^{A_1} \mathcal{F}^{A_2} \ldots \mathcal{F}^{A_n}$ and starting from the $(2n+1)$ form $C(A, \mathcal{F}) = C_{A_1 \ldots A_n, D} A^D \mathcal{F}^{A_1} \ldots \mathcal{F}^{A_n}$ the same procedure as outlined above determines the corresponding generalized Chern-Simons form to be

$$\omega \;=\; -\int_0^1 dt\, n\, t\, C_{A_1 A_2 \ldots A_n, D}\, A^D A^{A_1} H^{A_2}(t) \ldots H^{A_n}(t)\,. \tag{3.29}$$

The Abelian case is simply obtained by setting all structure constants to zero, and the generalized Chern-Simons term for an Abelian gauge theory is given by

$$\mathcal{L}_{\mathrm{GCS}} = \frac{2\mathrm{i}}{3} C_{BC,D} A^D A^B dA^C\,. \tag{3.30}$$

3.2 Generalized Chern-Simons terms and semisimple groups

As we presented in the previous subsection, when $C_{AB,D}$ is $\mathcal{D}$-exact it can be absorbed by redefining the gauge kinetic function and, as a consequence, the new Peccei-Quinn term becomes gauge invariant. Now, we will show this is the case for semisimple algebras, which means that the main application of generalized Chern-Simons terms is for non-semisimple gauge algebras.

We start with the result that if

$$C_{AB,C} = 2f_{C(A}{}^{D}Z_{B)D}\,, \tag{3.31}$$

for a constant real symmetric matrix Z_{AB}, the Chern-Simons term can be reabsorbed into the Peccei-Quinn term using

$$f'_{AB} = f_{AB} + \mathrm{i}Z_{AB}\,. \tag{3.32}$$

In fact, one easily checks that with the substitution (3.31) in the transformation law of the gauge kinetic function (3.6), the C-terms are absorbed by the redefinition (3.32). Equation (3.31) can be written as

$$C_{AB,C} = T_{C,AB}{}^{DE}Z_{DE}\,, \qquad T_{C,AB}{}^{DE} \equiv 2f_{C(A}{}^{(D}\delta_{B)}^{E)}\,. \tag{3.33}$$

In the case that the algebra is *semisimple*, one can always construct a Z_{AB} such that this equation is valid for any $C_{AB,C}$:

$$Z_{AB} = C_2(T)^{-1}_{AB}{}^{CD}T_{E,CD}{}^{GH}g^{EF}C_{GH,F}\,, \tag{3.34}$$

where g^{AB} and $C_2(T)^{-1}$ are the inverses of the Cartan-Killing metric

$$g_{AB} \;=\; f_{AC}{}^{D}f_{BD}{}^{C}\,, \tag{3.35}$$

and, respectively, the Casimir operator defined by

$$C_2(T)_{CD}{}^{EF} \;:=\; g^{AB}T_{A,CD}{}^{GH}T_{B,GH}{}^{EF}\,. \tag{3.36}$$

These inverses exist for semisimple groups. To show that (3.34) leads to (3.33) one needs the constraint (3.8), which can be brought to the following form

$$g^{HD}T_H \cdot \left(\tfrac{1}{2}C_C f_{DE}{}^{C} + T_{[D} \cdot C_{E]}\right) = 0\,. \tag{3.37}$$

We have dropped doublet symmetric indices here, using the notation $\cdot$ for contractions of such double indices. Furthermore, this implies

$$g^{AB} T_E \cdot T_B \cdot C_A = C_2(T) \cdot C_E \,, \tag{3.38}$$

with which the mentioned conclusions can easily be obtained.

This result can be also obtained from a cohomological analysis and we refer the interested reader to appendix A.1.

3.3 Application: Abelian×semisimple

The simplest nontrivial application are gauge groups of the form Abelian×semisimple for which one obtains an interesting result. Abelian generalized Chern-Simons terms are not trivial, but as we could show, the purely semisimple terms are. However, the direct product of an Abelian gauge group with a semisimple gauge group is not trivial again, especially it has a nontrivial mixed sector, which is going to be investigated in greater detail in the following. To reflect the product structure, we split the adjoint indices $A, B, ...$ into indices $a, b, c, ...$ for the Abelian part and adjoint indices $x, y, z, w, ...$ for the semisimple part. Due to the group structure, only the structure constants of the type ${f_{xy}}^z$ are nonzero. As before, we define a homogeneous five-form $C(A, \mathcal{F})$, which is given by

$$C(A,\mathcal{F}) \;=\; 2C_{(xb),a} A^a \mathcal{F}^x F^b + C_{xy,a} A^a \mathcal{F}^x \mathcal{F}^y + 2C_{(ax),y} A^y F^a \mathcal{F}^x \,, \tag{3.39}$$

with constants $C_{xb,a}$, $C_{bx,a}$, $C_{xy,a}$, $C_{ax,y}$ and $C_{ya,x}$. The closure relations can be directly obtained from (3.8) by simply inserting Abelian and semisimple indices[12] and we are led to

$$f^v_{xu} C_{vb,a} \;=\; 0 \tag{3.40}$$

$$f^v_{xy} C_{bv,a} \;=\; 0 \tag{3.41}$$

$$f_{u(y}{}^v C_{x)v,a} \;=\; 0 \tag{3.42}$$

$${f_{uy}}^v C_{ax,v} + {f_{xy}}^v C_{av,u} - {f_{xu}}^v C_{av,y} \;=\; 0 \tag{3.43}$$

$${f_{uy}}^v C_{xa,v} + {f_{xy}}^v C_{va,u} - {f_{xu}}^v C_{va,y} \;=\; 0 \,. \tag{3.44}$$

These relations already lead to various interesting results. By definition, a semisimple Lie algebra has no Abelian ideals. This implies, in particular, that there cannot be any non-trivial

[12] In appendix A.2 we apply the developed formalism and demonstrate that it leads to the same result.

null eigenvector of the structure constants, so that (3.40) and (3.41) imply

$$C_{xb,a} \equiv 0\,, \tag{3.45}$$

$$C_{bx,a} \equiv 0\,. \tag{3.46}$$

Equation (3.42) means that $C_{xy,a}$ is for each a, a symmetric invariant tensor in the adjoint representation of the semisimple part of the gauge group. $C_{xy,a}$ therefore has to be proportional to the Cartan-Killing metric g_{xy} of the semisimple Lie algebra. Thus, we have $C_{xy,a} = B_a g_{xy}$ where the B_a's are arbitrary but constant. The only nontrivial part of (3.39) is

$$C(A,\mathcal{F}) = 2C_{(xy),a}A^a\mathcal{F}^x\mathcal{F}^y + (C_{ya,x} + C_{ay,x})A^xF^a\mathcal{F}^y\,. \tag{3.47}$$

What we have done is to simply apply the formalism developed earlier in this section to the mixed part of a gauge group with the structure Abelian×semisimple. The purely Abelian part is not trivial and leads to the Chern-Simons term (3.30). After the cohomological analysis we found that the only nontrivial generalized Chern-Simons terms in the mixed sector of Abelian×semisimple are determined by the five form (3.47) and, consequently, the generalized Chern-Simons terms of the mixed sector read

$$\begin{aligned}\mathcal{L}_{\rm GCS} = {} & \frac{4\mathrm{i}}{3}C_{(xy),a}A^aA^x(dA^y - \frac{3}{8}f_{rs}{}^yA^rA^s) + \frac{2\mathrm{i}}{3}C_{ya,x}A^xA^ydA^a + \\ & + \frac{2\mathrm{i}}{3}C_{ay,x}A^xA^a(dA^y - \frac{3}{8}f_{rs}{}^yA^rA^s)\,.\end{aligned} \tag{3.48}$$

because all the other components of the constant tensor C vanish due to cohomological reasons. Observe, that if we do not allow for off-diagonal elements of the gauge kinetic function, i.e. $\mathrm{f}_{ax} = \mathrm{f}_{xa} = 0$, then the generalized Chern-Simons term in the mixed sector is given by

$$\mathcal{L}_{\rm GCS} = \frac{4\mathrm{i}}{3}C_{(xy),a}A^aA^x(dA^y - \frac{3}{8}f_{rs}{}^yA^rA^s)\,. \tag{3.49}$$

The purely semisimple part of C can be absorbed into the gauge kinetic function by redefinition. This matches the situation encountered in [12] without anomalies.

3.4 Nonhomogeneous forms and anomalies

In terms of Lie algebra cohomology, the constraints on $C(A,\mathcal{F})$, the equations (3.7) and (3.8), have a clear meaning. The first equation constrains $C(A,\mathcal{F})$ to be a homogeneous form which is closed under the algebraic operator $\mathcal{D}$ defined in (3.13) if it satisfies the constraint (3.8). However, is the formalism still valid for nonhomogeneous forms or, in other words, can the constraint (3.7) be relaxed?

In order to understand this, let $C(A,\mathcal{F})$ be nonhomogeneous, i.e. $C_{(AB,D)} \neq 0$. Consequently, $C(A,\mathcal{F})$ cannot be closed either, but is instead

$$dC(A,\mathcal{F}) = C_{(AB,D)}\mathcal{F}^D\mathcal{F}^A\mathcal{F}^B\,. \tag{3.50}$$

Clearly, the computation that led to the generalized Chern-Simons term (3.27) cannot be valid anymore. More precisely, instead of (3.24) one now has

$$\begin{aligned} C_{(AB,D)}\mathcal{F}^D(t)\mathcal{F}^A(t)\mathcal{F}^B(t) &= dt\partial_t C(A(t),\mathcal{F}(t)) + dC(A(t),H(t)) + \\ &\quad +2t\,dt\,d\big(C_{AB,D}A^DA^BH^A\big)\,. \end{aligned} \tag{3.51}$$

By using (3.22) one can prove easily that the left hand side decomposes according to

$$\begin{aligned} C_{(AB,D)}\mathcal{F}^D(t)\mathcal{F}^A(t)\mathcal{F}^B(t) &= C_{(AB,D)}H^D(t)H^A(t)H^B(t) + \\ &\quad +3dt\,C_{(AB,D)}A^D(t)H^A(t)H^B(t)\,. \end{aligned} \tag{3.52}$$

Of course the second term on the right hand side of (3.51) no longer vanishes either but causes the contribution $C_{(AB,D)}H^D(t)H^A(t)H^B(t)$ that cancels the corresponding term in equation (3.52). Therefore, (3.26) receives an extra contribution and is replaced by

$$\begin{aligned} C(A,\mathcal{F}) &= 3C_{(D,AB)}\int_0^1 dt\,A^D(t)H^A(t)H^B(t) - \\ &\quad -2C_{AB,D}d\left[\int_0^1 dtA^D(t)A^B(t)H^A(t)\right]\,. \end{aligned} \tag{3.53}$$

We see, that the nonvanishing totally symmetric part of $C_{AB,D}$ introduces the five-dimensional form

$$Q_5(A,\mathcal{F}) = 3C_{(D,AB)}\int_0^1 dt\,A^D(t)H^A(t)H^B(t) \tag{3.54}$$

This form is nothing else but the five-dimensional Chern-Simons term corresponding to the invariant polynomial $P_6(\mathcal{F}) = C_{(D,AB)}\mathcal{F}^D\mathcal{F}^A\mathcal{F}^B$. As the nonhomogeneous form $C(A,\mathcal{F})$ is no longer closed, there does not exist a form ω, such that $C = d\omega$ or, equivalently, the Chern-Simons form Q_5 is not representable by a coboundary, i.e. there is no ω' such that $Q_5 = d\omega'$.

Consequently, the five-dimensional form $d(\mathrm{Im}\,\mathrm{f}_{AB}\mathcal{F}^A\mathcal{F}^B) - C(A,\mathcal{F})$ cannot be represented by the coboundary (3.28) of homogeneous forms. Furthermore, it is no longer gauge invariant because Q_5 is not gauge invariant. However, this is only a problem in theories that are free of quantum anomalies. The solution is given by the descent equations [59–62]. By means of this set of equations, Stora and Zumino could relate the Chern-Simons forms Q_{2n-1} to the consistent anomaly $\mathcal{A}_{2n-2}(\Lambda, A)$ in $2n-2$ dimensions. The descent equation relevant for our case is

$$\delta_\Lambda Q_5(A,\mathcal{F}) \quad = \quad d\mathcal{A}(\Lambda, A)\,, \tag{3.55}$$

representing the gauge variation of the Chern-Simons form as the coboundary of the four-dimensional consistent anomaly. Applying a gauge variation to $d(\mathrm{Im}\,\mathrm{f}_{AB}\mathcal{F}^A\mathcal{F}^B) - C(A,\mathcal{F})$, we have

$$d\left[\delta_\Lambda(\mathrm{f}_{AB}\mathcal{F}^A\mathcal{F}^B)\right] - d\left[2C_{AB,D}\delta_\Lambda\left(\int_0^1 dt A^D(t)A^B(t)H^A(t)\right)\right] + d\left[\mathcal{A}(\Lambda,A)\right]\,, \tag{3.56}$$

which is equal to zero because of (3.20) as the steps leading to (3.20) are quite general and do not depend on $C(A,\mathcal{F})$ being homogeneous or not. The tensor $C_{AB,D}$ in (3.6), however, is no longer restricted to its mixed symmetric part alone but now also contains a totally symmetric part. Therefore, it can be decomposed into its totally symmetric part $C^{(\mathrm{s})}_{AB,D}$ and a part of mixed symmetry $C^{(\mathrm{m})}_{AB,D}$, i.e.

$$C_{AB,D} \quad = \quad C^{(\mathrm{s})}_{AB,D} + C^{(\mathrm{m})}_{AB,D}\,. \tag{3.57}$$

The generalized Chern-Simons term is still only proportional to the mixed symmetric part. The totally symmetric part is to be exactly cancelled by the anomaly as (3.56) shows. Note that (3.54) can only be consistent with (3.55) if the totally symmetric part of $C_{AB,D}$, $C^{(\mathrm{s})}_{AB,D} = C_{(AB,D)}$ is related to the quantum anomaly (we will discuss this in greater detail in section 5.2).

We see that the constraint (3.7) can be relaxed to allow for nonhomogeneous forms $C(A,\mathcal{F})$. As a consequence, the four-dimensional action (3.28) is no longer gauge invariant because the generalized Chern-Simons term is still only proportional to the mixed symmetric part of the tensor $C_{AB,D}$. The left over variation proportional to $C_{(AB,D)}$ may be cancelled by the anomaly if a suitable fermion spectrum exists. Hence, nonhomogeneous forms $C(A,\mathcal{F})$

are the appropriate forms necessary in applications to anomalous theories in order to absorb the anomaly. The cohomological reason is that the nonhomogeneous forms introduce the five-dimensional Chern-Simons form Q_5 into the cohomological discussion, which in turn is related to the anomaly in four dimensions by the Stora-Zumino descent equation (3.55). Consequently, the gauge variation of (3.28) does no longer vanish, but is given by the negative of the gauge anomaly, i.e.

$$\delta_\Lambda\big(\mathcal{L}_{\rm PQ} + \mathcal{L}_{\rm GCS}\big) \quad = \quad -\mathcal{A}(\Lambda, A)\,. \tag{3.58}$$

This result goes beyond the work of [34] and allows for nonhomogeneous forms.

At the end of this section, let us discuss again the example of a gauge group with the structure Abelian×semisimple. We set all off-diagonal elements of the gauge kinetic function to zero, i.e. $\mathrm{f}_{ax} = 0$. The constraints (3.40) to (3.44) do not change for nonhomogeneous forms (although they do not imply closure anymore), but are now valid for the full coefficient $C_{AB,D} = C^{(\rm s)}_{AB,D} + C^{(\rm m)}_{AB,D}$. Nevertheless, the implications drawn from (3.40) to (3.44) are still valid and, consequently, the only nontrivial part of a five-dimensional nonhomogeneous form $C(A, \mathcal{F})$ is determined by $C_{xy,a}$, i.e. $C_{xy,a} \neq 0$. Decomposing $C_{xy,a}$, we obtain

$$C^{(\rm s)}_{xy,a} = C^{(\rm s)}_{ax,y} \quad = \quad \frac{1}{3} C_{xy,a}\,, \tag{3.59}$$

$$C^{(\rm m)}_{xy,a} \quad = \quad \frac{2}{3} C_{xy,a}\,, \tag{3.60}$$

$$C^{(\rm m)}_{ax,y} \quad = \quad -\frac{1}{3} C_{xy,a}\,. \tag{3.61}$$

Thus, we see that the generalized Chern-Simons term in the mixed sector is still given by (3.49). However, there are new contributions due to the totally symmetric tensors $C^{(\rm s)}_{xy,a}$ and $C^{(\rm s)}_{ax,y}$ which cause nontrivial gauge variations of $\mathcal{L}_{\rm PQ} + \mathcal{L}_{\rm GCS}$. Cancellation of these remaining contributions can only be achieved with the help of mixed gauge anomalies, but we will discuss this example in more detail in section 5.4, where we will explicitly clarify the relation of the symmetric coefficients $C^{(\rm s)}$ to the quantum anomaly and show how mixed quantum anomalies cancel the remaining gauge variations.

4 $\mathcal{N} = 1$ Supersymmetry

In the early 1960s, Gell-Mann and Ne'eman, proposed a way to arrange the known hadrons into a unified framework and, in this way, brought some order into a whole zoo of particles that had been found until then [63]. The success of their model is based on a global $SU(3)$ symmetry which puts particles of the same spin into $SU(3)$-multiplets. This model caused a lot of enthusiasm, and efforts were made to unite particles of different spin as well. In the non-relativistic regime this could be achieved by an $SU(6)$ model, which made predictions that were quite well approximated by experimental data [64–66]. Unfortunately, further attempts to construct the relativistic versions of such models, in which the internal symmetry group is nontrivially entangled with the Poincaré group to form a so-called Master group, failed. All these efforts to create a Master group did not succeed because the Master groups always had nonphysical properties such as an infinite number of particles in each irreducible representation or continuous mass spectra. After Coleman and Mandula proved a no-go theorem, that stated that every nontrivial union of the Poincaré group with an internal symmetry group within the framework of ordinary Lie algebras would yield an essentially trivial S-matrix [67], all these efforts seemed to be leading nowhere.

In 1971, a new symmetry was found from the Neveu-Schwarz-Ramond superstring [68–72] that Wess and Zumino extended to quantum field theories in four dimensions [73].[13] As a novel feature, some of the generators of the symmetry algebra satisfy anticommuting relations instead of commutation relations. This, however, evaded the Coleman-Mandula theorem because the assumptions made in its proof considered only symmetry generators with commutation relations. This new symmetry, called supersymmetry, does not only represent a mathematical oddity, but provided the grounds for nontrivially entangling the Poincaré group with internal symmetry groups. To date, there is no direct experimental hint for supersymmetry being realized in nature but it has many properties that justify further investigation. It is for example the only known symmetry, that can protect fundamental scalars, such as the Higgs field, from obtaining huge radiative corrections up to very high energy scales (this

[13]Unknown to Wess and Zumino at that time, this symmetry had already appeared in a pair of papers published in the Soviet Union. In 1971, Gol'fand and Likhtman had extended the algebra of the Poincaré group to a superalgebra and had even constructed supersymmetric field theories in four dimensions [74]. The paper displayed few details and was generally ignored until much later.

is the so-called "hierarchy problem") where more fundamental theories like grand unifying theories or superstring theory could supersede the standard model.
Another feature of supersymmetry is the improved renormalization evolution of the three gauge coupling constants of the standard model. These coupling constants do not exactly meet at a common energy scale if we use the renormalization group equations obtained from the standard model. With the addition of supersymmetry, gauge coupling unification can be achieved in consistency with phenomenological constraints.
There is extensive observational evidence for an additional component of the matter density in the universe that goes under the name dark matter. Dark matter particles must be electrically neutral, otherwise they would scatter light and, thus, be directly observable. The lightest of the additional hypothetical particles found in supersymmetric models (called "lightest supersymmetric particle") is a possible candidate for dark matter.

In section 4.1 we introduce global $\mathcal{N}=1$ supersymmetry and discuss briefly supersymmetry in the gauge sector. We will see that a nontrivial gauge kinetic function induces several new interactions in the gauge sector. For future reference we quote the supersymmetric gauge sector and the necessary supersymmetry transformations.

In section 4.2 we briefly motivate $\mathcal{N}=1$ supergravity and we present the gauge sector of $\mathcal{N}=1$ supergravity together with the supergravity transformations.

4.1 Global Supersymmetry

Supersymmetry is a symmetry relating bosons and fermions and, therefore, we can make an ansatz for infinitesimal supersymmetry transformations with parameter ε to behave roughly as

$$\delta f = \varepsilon b \,, \tag{4.1}$$

$$\delta b = \bar{\varepsilon} f \,. \tag{4.2}$$

These transformation laws are only schematic and bosons are represented by b, while f stands for fermions. Although, equations (4.1) and (4.2) are of a rather symbolic nature, we can already draw several important conclusions from them. The first is, that the transformation parameter ε is anticommuting, instead of commuting as in usual symmetry transformations, because the left hand side of (4.1), and therefore also the right hand side, has to be fermionic, i.e. anticommuting. The parameter ε carries spin $\frac{1}{2}$ in supersymmetry [75].

In natural units ($\hbar = c = 1$) the action becomes dimensionless and the dimension of mass and length are inverse to one another. The derivative operator has then positive mass dimension (inverse length), i.e. $[\partial_\mu] = 1$. From the Dirac action for the fermion and the Klein-Gordon action for the scalar we therefore obtain the canonical mass dimension for fermionic and bosonic fields in four spacetime dimensions: $[f] = \frac{3}{2}$ and $[b] = 1$. The transformation law for bosons (4.2) would then lead us to $[\varepsilon] = -\frac{1}{2}$, which would be inconsistent with (4.1). The simplest way to obtain an algebra linear in the elementary fields without introducing new dimensionful parameters is to assume

$$\delta f = \gamma^\mu \varepsilon \partial_\mu b\,, \tag{4.3}$$

which together with (4.2) is consistent with $[\varepsilon] = -\frac{1}{2}$. Thus, already for dimensional reasons, transformation laws for a symmetry relating fermions and bosons must have the form (4.1) and (4.3), and the derivative in (4.3) can be understood as the mismatch in derivatives between the Dirac and the Klein-Gordon equation. The last implication of this concerns the commutator of two transformations, which we can expect to have the form

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]b \propto (\bar{\varepsilon}_2 \gamma^\mu \varepsilon_1)\partial_\mu b \tag{4.4}$$

for bosons and equivalently for fermions. The commutator of two supersymmetry transformations causes a translation in spacetime and this result is found in any globally supersymmetric model.

Now let us construct a globally supersymmetric model with gauge fields, as this plays an important role in section 5. The Abelian case is convenient to begin with, and it leads to results that are straightforwardly generalized to the nonabelian case.

Supersymmetry relates fermions and bosons, and, consequently, the gauge fields come together with fermionic partners, so-called gaugini[14]. A first ansatz for a supersymmetric gauge kinetic action is

$$\mathcal{L}_{gk} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda \tag{4.5}$$

[14] The gaugini are particles of spin $\frac{1}{2}$.

where we incorporated the gaugino, λ, by means of a kinetic term. Notations and conventions are summarized in appendix B. The first term represents the usual Maxwell Lagrangian. Let us define the transformation laws of the fields in accordance with (4.1) and (4.3) by

$$\delta\lambda = \sigma^{\mu\nu}\varepsilon\partial_\mu A_\nu = \frac{1}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} \tag{4.6}$$

$$\delta\bar{\lambda} = -\frac{1}{2}\bar{\varepsilon}\sigma^{\mu\nu}F_{\mu\nu} \tag{4.7}$$

$$\delta A_\mu = -\frac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda\,. \tag{4.8}$$

Here, $\sigma^{\mu\nu} := \frac{1}{4}[\gamma^\mu,\gamma^\nu]$ are the generators of SO(1,3) in the spinor representation. The transformation behaviour of the field strength can be read off from (4.24) to be

$$\delta F_{\mu\nu} = \bar{\varepsilon}\gamma_{[\mu}\partial_{\nu]}\lambda\,. \tag{4.9}$$

Using this, the variation of the Maxwell term in (4.5) is then easily written down

$$\delta(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{2}F^{\mu\nu}\bar{\varepsilon}\gamma_\mu\partial_\nu\lambda\,. \tag{4.10}$$

The variation of the second term of (4.5) is a little bit more involved, and relations such as (B.4) and (B.5) are convenient for the relevant computations. The variation of the second term of (4.5) is found to be

$$\delta(-\frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda) = \frac{1}{2}F^{\mu\nu}\bar{\varepsilon}\gamma_\mu\partial_\nu\lambda - \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\bar{\varepsilon}\gamma_5\gamma_\sigma\partial_\rho\lambda\,. \tag{4.11}$$

Altogether, the variation of (4.5) gives

$$\delta\mathcal{L}_{\mathrm{gk}} = -\frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\bar{\varepsilon}\gamma_5\gamma_\sigma\partial_\rho\lambda\,. \tag{4.12}$$

Observe, that (4.12) actually vanishes, because after a partial integration the variation becomes proportional to $\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho}$ which is identically zero due to the Bianchi-identity. Thus, we have proven that (4.5) is invariant under the transformations (4.6) and (4.8). We are not finished yet because counting the degrees of freedom, we find for the fermion 4 degrees of freedom, while the vector field only provides 3 degrees of freedom off-shell. On-shell, however, the number of degrees of freedom for the gaugino is 2, just as for the vector field. So on-shell the degrees of freedom are equal for fermions and bosons. To balance the degrees of freedom, we introduce another real scalar field D[15] that has algebraic equations of motion and, thus,

[15] The auxiliary field D is also needed for the supersymmetry algebra to close off-shell.

can be eliminated on-shell. The additional term in the Lagrangian containing the auxiliary field is $\frac{1}{2}D^2$. This auxiliary field has to transform into the gaugino, and the transformation law for the fermion has to be extended by a term containing D. Note, that D is a real field.

The Lagrangian[16]

$$\mathcal{L}_{\mathrm{gk}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \tag{4.13}$$

is indeed invariant under the variations

$$\delta\lambda = \frac{1}{2}\sigma^{\mu\nu}\varepsilon F_{\mu\nu} + \frac{\mathrm{i}}{2}\gamma_5\varepsilon D \tag{4.14}$$

$$\delta D = \frac{\mathrm{i}}{2}\bar{\varepsilon}\gamma_5\gamma^\mu\partial_\mu\lambda \tag{4.15}$$

and (4.8) because the extra variation of the Dirac action proportional to D precisely cancels against the variation of the auxiliary Lagrangian.

The action (4.5) can be generalized by means of a gauge kinetic function $\mathrm{f}(z)$. The gauge kinetic function depends on a set of scalar fields and if then again supersymmetry is demanded, the superpartners of these scalars must be taken into account, too. So let there be scalar fields z^i and their corresponding superpartners χ^i. In complete analogy, one finds that the Lagrangian

$$\mathcal{L}_{\mathrm{matter}} = \sum_i \left[\partial_\mu z^i \partial^\mu z^i + 2\bar{\chi}^i_L\gamma^\mu\partial_\mu\chi^i - F^i F^i\right] \tag{4.16}$$

which consists of complex scalar fields z^i and their corresponding fermionic superpartners χ^i. The matter Lagrangian is invariant under the following supersymmetry transformations

$$\delta z^i = \bar{\varepsilon}_L\chi^i_L\,, \tag{4.17}$$

$$\delta\chi^i_L = \frac{1}{2}\gamma^\mu\varepsilon_R\partial_\mu z^i + \frac{1}{2}F^i\varepsilon_L\,. \tag{4.18}$$

$$\delta F^i = \bar{\varepsilon}_R\gamma_\mu\partial^\mu\chi^i_L\,. \tag{4.19}$$

We used the chiral projections $\chi^i_L = \frac{1}{2}(1+\gamma_5)\chi^i$ and $\varepsilon_R = \frac{1}{2}(1-\gamma_5)\varepsilon$. The supermultiplet containing this scalar and this fermion is accompanied by a complex auxiliary field, F^i, that

[16]The Lagrangian (4.13) can be obtained by superspace methods, too. Superspace is introduced in appendix C.

balances the off-shell degrees of freedom. It is important to note that, according to (4.17) and the chain rule, the gauge kinetic function will transform under supersymmetry, i.e.,

$$\delta \mathrm{f}(z) \quad = \quad \partial_i \mathrm{f}(z) \bar{\varepsilon} \chi^i \,. \tag{4.20}$$

Observe that the gauge kinetic function is implicitly spacetime dependent through its dependence on scalar fields. At several steps that led to (4.11) we used a partial integration, which in presence of a nontrivial gauge kinetic function will produce new terms in (4.12) proportional to $\partial_\mu \mathrm{f}(z) = \partial_i \mathrm{f}(z) \partial_\mu z^i$ where $\partial_i = \partial / \partial z^i$. Observe that especially the term (4.12) will not vanish anymore, but will contribute with $\frac{\mathrm{i}}{8} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \operatorname{Re} \mathrm{f}(z) \bar{\varepsilon} \gamma_5 \gamma_\nu \lambda F_{\rho\sigma}$ to the supersymmetry variation. In addition to these contributions, one has to take $\delta[\operatorname{Re} \mathrm{f}(z)] F_{\mu\nu} F^{\mu\nu}$ into account, which has to be cancelled, too. Adding counterterms that cancel these variations and taking the variations of the counterterms into account, one is led inductively to an invariant Lagrangian after a finite number of steps.[17] The computation is standard and will not be repeated here but instead let us give the final result as given in, e.g., [76, 77]. The supersymmetric Lagrangian containing n_V vectormultiplets $(\mathcal{F}^A, \lambda^A, D^A)$, $A = 1 \ldots n_V$, and a nontrivial gauge kinetic function f_{AB} is given by

$$\begin{aligned} \mathcal{L}_{\mathrm{gk}} \quad = \quad & -\frac{1}{4} \operatorname{Re} \mathrm{f}(z)_{AB} \mathcal{F}^A_{\mu\nu} \mathcal{F}^{\mu\nu\, B} - \frac{1}{2} \operatorname{Re} \mathrm{f}(z)_{AB} \bar{\lambda}^A \gamma^\mu D_\mu \lambda^B + \frac{1}{2} \operatorname{Re} \mathrm{f}(z)_{AB} D^A D^B + \\ & + \frac{1}{8} \operatorname{Im} \mathrm{f}(z)_{AB} \varepsilon^{\mu\nu\rho\sigma} \mathcal{F}^A_{\mu\nu} \mathcal{F}^B_{\rho\sigma} + \frac{\mathrm{i}}{4} (D_\mu \operatorname{Im} \mathrm{f}(z)_{AB}) \bar{\lambda}^A \gamma_5 \gamma^\mu \lambda^B + \\ & + [\frac{\mathrm{i}}{2} \partial_i \mathrm{f}(z)_{AB} \bar{\chi}^i_L \lambda^A_L D^B - \frac{1}{2} \partial_i \mathrm{f}(z)_{AB} \mathcal{F}^A_{\mu\nu} \, \bar{\chi}^i_L \sigma^{\mu\nu} \lambda^B_L \\ & - \frac{1}{4} F^i \partial_i \mathrm{f}(z)_{AB} \bar{\lambda}^A_L \lambda^B_L + \frac{1}{4} \bar{\chi}^i_L \chi^j_L \partial_i \partial_j \mathrm{f}(z)_{AB} \bar{\lambda}^A_L \lambda^B_L + \mathrm{h.c.}] \end{aligned} \tag{4.21}$$

where we defined the covariant derivatives

$$D_\mu \operatorname{Im} \mathrm{f}_{AB} \quad = \quad \partial_\mu \operatorname{Im} \mathrm{f}_{AB} - 2 A^C_\mu f_{C(A}{}^D \mathrm{f}_{B)D} \,, \tag{4.22}$$

$$D_\mu \lambda^A \quad = \quad \partial_\mu \lambda^A - A^B_\mu \lambda^C f_{BC}{}^A \,. \tag{4.23}$$

The Lagrangian (4.21) is invariant under the supersymmetry transformations of the gauge

[17] Note that the superfield formalism as introduced in the appendix C leads also to the result that will be presented in equation (4.21).

sector

$$\begin{aligned} \delta A_\mu^C &= -\frac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda^C\,. && (4.24)\\ \delta\lambda^C &= \frac{1}{2}\sigma^{\mu\nu}\varepsilon\mathcal{F}_{\mu\nu}^C + \frac{\mathrm{i}}{2}\gamma_5\varepsilon D^C && (4.25)\\ \delta D^C &= \frac{\mathrm{i}}{2}\bar{\varepsilon}\gamma_5\gamma^\mu\partial_\mu\lambda && (4.26) \end{aligned}$$

and (4.17), (4.18), (4.19) of the matter sector. Observe, that chiral projections appear in the invariant Lagrangian, where $\lambda_L = \frac{1}{2}(1+\gamma_5)\lambda$ and $\lambda_R = \frac{1}{2}(1-\gamma_5)\lambda$.

Observe that a nontrivial gauge kinetic function introduces a CP-violating coupling $\frac{1}{8}\,\mathrm{Im}\,\mathrm{f}(z)_{AB}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}^A\mathcal{F}_{\rho\sigma}^B$ which is exactly of the form as the Peccei-Quinn term discussed in the previous section.

Note that the nonabelian field strength $\mathcal{F}$ appears in equation (4.21). In order that the Lagrangian be invariant under gauge and supersymmetry, the gauge kinetic function must transform in the symmetric product of two adjoint representations. It is one of the main topics of this thesis to generalize the transformation property of the gauge kinetic function and to discuss the compatibility with $\mathcal{N}=1$ supersymmetry.

The discussion of section 3 showed, that a generalization of the gauge transformation of f needs new terms in the bosonic part of the effective action. Before we come to a discussion of the their consequences, let us first also briefly introduce the salient features of local supersymmetry.

4.2 The gauge sector of $\mathcal{N}=1$ supergravity

We considered global $\mathcal{N}=1$ supersymmetry in the previous subsection. The transformation parameter ε was a constant spinor. In local supersymmetry, however, the transformation parameter ε is no longer a constant spinor but becomes spacetime dependent, i.e. $\varepsilon = \varepsilon(x)$. Then it follows immediately from (4.4) that also the translations become spacetime dependent through $\varepsilon(x)$ and differ from point to point as general coordinate transformations (the commutator of two supersymmetry transformations causes translations over distances $d^\mu \propto \bar{\varepsilon}_2\gamma^\mu\varepsilon_1$). Thus, a theory that is symmetric under local supersymmetry needs gravity and for that reason is called supergravity. The fermionic superpartner of the metric is called gravitino, ψ_μ, and carries spin $\frac{3}{2}$. It is a vectorial spinor (or a spinorial vector). In supergravity the transformation law relating the metric to its superpartner is given by[18]

$$\delta g^{\mu\nu} = -\bar{\psi}^{(\mu}\gamma^{\nu)}\varepsilon\,. \tag{4.27}$$

As gravity is present, the action of supergravity must contain the Einstein-Hilbert action which represents the kinetic term of the metric while the corresponding term for the gravitino is given by the so called Rarita-Schwinger action written down in 1941 by Rarita and Schwinger [78]. The Rarita-Schwinger action is quadratic in the gravitino and contains one spacetime derivative.

In general, the action of supergravity is a complicated Lagrangian that is divided into different sectors [76, 77] such as, for instance, a sector containing the Einstein-Hilbert action and the Rarita-Schwinger action together with four-fermion terms that are necessary to render this pure supergravity sector invariant under local supersymmetry. The sector of main interest to us is the gauge sector with nontrivial gauge kinetic function because it is investigated further in section 5.3. This sector contains the kinetic terms of the gauge supermultiplet. The gauge sector of $\mathcal{N}=1$ supergravity is given in [77], for example, and here we repeat it for future reference:

$$\begin{aligned}\mathcal{L}_{gauge} = {} & \mathrm{Re}\,\mathrm{f}_{AB}(z)[-\frac{e}{4}F^A_{\mu\nu}F^{\mu\nu\,A} - \frac{e}{2}\bar{\lambda}^A\gamma^\mu\hat{\mathcal{D}}_\mu\lambda^B + \frac{1}{2}D^AD^B + \frac{e}{4}\bar{\psi}_\mu\sigma^{\nu\rho}\gamma^\mu\lambda^B\,(F^A_{\nu\rho} + F^{\mathrm{cov}\,A}_{\nu\rho})] - \\ & -\frac{e}{4}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Im}\,\mathrm{f}_{AB}(z)F^A_{\mu\nu}F^B_{\rho\sigma} + \mathrm{i}\frac{e}{4}(\mathcal{D}_\mu\,\mathrm{Im}\,\mathrm{f}_{AB}(z))\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B + \\ & +\{\frac{e}{2}\partial_i\mathrm{f}_{AB}(z)[\bar{\chi}^i(-\sigma^{\mu\nu}\hat{F}^{\mathrm{cov}\,A}_{\mu\nu} + \mathrm{i}D^A)\lambda^B_L - \frac{1}{2}(F^i + \bar{\psi}^\mu_R\gamma_\mu\chi^i_L)\bar{\lambda}^A_L\lambda^B_L] + \\ & +\frac{e}{4}\partial_i\partial_j\mathrm{f}_{AB}(z)\bar{\chi}^i\chi^j\bar{\lambda}^A_L\lambda^B_L + \mathrm{h.c.}\}\,.\end{aligned} \tag{4.28}$$

[18] In this subsection, as well as in section 5.3, we deal with curved space and adopt a different notation in these subsections. Greek indices $\mu, \nu, \rho, \ldots$ will represent curved spacetime indices, while Latin indices $a, b, c, \ldots$ denote flat Lorentz-indices. Note that in this notation $\gamma^\mu = \gamma^a e^\mu_a$ is spacetime dependent via the vierbein e^μ_a (see footnote 19 for further information on the vierbein), contrary to γ^a which is a constant Dirac matrix. Furthermore, note that the Planck mass is set to one.

In this expression,

$$\mathcal{D}_\mu = \partial_\mu - A_\mu^C \delta_C + \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab} \tag{4.29}$$

$$\hat{\mathcal{D}}_\mu = \partial_\mu - A_\mu^C \delta_C + \frac{1}{2}\omega_\mu{}^{ab}(\psi)\sigma_{ab} \tag{4.30}$$

$$\omega_\mu{}^{ab}(\psi) = \omega_\mu{}^{ab} + \frac{1}{2}\bar{\psi}_\mu \gamma^{[a}\psi^{b]} + \frac{1}{4}\bar{\psi}^a \gamma_\mu \psi^b \tag{4.31}$$

$$\hat{F}_{\mu\nu}^{\mathrm{cov}\,A} = F_{\mu\nu}^{\mathrm{cov}\,A} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\mathrm{cov}\,\rho\sigma\,A} \tag{4.32}$$

$$F_{\mu\nu}^{\mathrm{cov}} = 2\partial_{[\mu}A_{\nu]} + \bar{\psi}_{[\mu}\gamma_{\nu]}\lambda\,, \tag{4.33}$$

and we have defined the determinant of the vierbein[19] $e := \det(e_\mu^a) = \sqrt{|\det(g_{\mu\nu})|}$. Note that the spin connection $\omega_\mu{}^{ab}(\psi)$ contains ψ-torsion. δ_C denotes infinitesimal transformations of the Yang-Mills symmetry. The object denoted by F^{cov} is called "supercovariant field strength" because it transforms under supersymmetry in an expression that does not contain any term proportional to $\partial_\mu\varepsilon$. The transformation laws for local supersymmetry on the independent fields (transformation laws for the auxiliary fields are omitted) are given by

$$\delta e_\mu^a = \frac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu \tag{4.34}$$

$$\delta\psi_\mu = (\partial_\mu + \frac{1}{2}\omega_\mu{}^{ab}(\psi)\sigma_{ab})\varepsilon \tag{4.35}$$

$$\delta z^i = \bar{\varepsilon}_L\chi^i \tag{4.36}$$

$$\delta\chi^i = \frac{1}{2}\gamma^\mu(\mathcal{D}_\mu z^i - \bar{\psi}_\mu\chi)\varepsilon_R + \frac{1}{2}F_i\varepsilon_L \tag{4.37}$$

$$\delta A_\mu^C = -\frac{1}{2}\bar{\varepsilon}\gamma_\mu\lambda^C \tag{4.38}$$

$$\delta\lambda^A = \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu}^{\mathrm{cov}\,A}\varepsilon + \frac{1}{2}\mathrm{i}\gamma_5\varepsilon D^A \tag{4.39}$$

The gauge sector (4.28) contains besides four-fermion interactions an interaction of the form $\psi\lambda F_{\mu\nu}$, which is not renormalizable. In contrast to global supersymmetry, where renormalizable models exist, in supergravity nonrenormalizable couplings are always present, but these couplings are suppressed by powers of the Planck mass.

[19] The vierbein defines local orthonormal frames in which $g_{\mu\nu} = \eta_{ab}e_\mu^a e_\nu^b$. From (4.27) one finds $\delta e_\mu^a = \frac{1}{2}\bar{\varepsilon}\gamma^a\psi_\mu$. It is not difficult to prove that $\delta e = \frac{e}{2}\bar{\varepsilon}\gamma^\mu\psi_\mu$ because for a matrix M, the variation of the determinant is given by $\delta\det(M) = \det(M)\mathrm{tr}(M^{-1}\delta M)$. If one takes the vierbein as a matrix, then the variation of its determinant δe is easily found.

5 Generalized Chern-Simons terms and chiral anomalies in $\mathcal{N}=1$ Supersymmetry

In the previous section we presented the supersymmetric gauge sector consisting of the Yang-Mills action together with kinetic terms for the superpartners and displayed the supersymmetry transformation laws. We saw that with a nontrivial gauge kinetic function that depends on a set of scalars, several new couplings of the gauge fields and gaugini to these scalar fields and their superpartners arise. Among those new terms in the Lagrangian is a CP-violating term of the form $(\mathrm{Im}\, \mathrm{f}_{AB})\mathcal{F}^A \wedge \mathcal{F}^B$ which is often referred to as "Peccei-Quinn term". Obviously, the Lagrangian (4.21) is not only invariant under global supersymmetry but also under nonabelian gauge transformations, if only the gauge kinetic function transforms appropriately [76]. This corresponds to the transformation law (3.5) given in section 3. In that section, however, we also presented a possible extension by means of a constant shift (3.6), under which the Peccei-Quinn term is no longer invariant,[20] and the generalized Chern-Simons terms had to be added to the Peccei-Quinn term in order to restore gauge-invariance. A superfield expression corresponding to generalized Chern-Simons terms was introduced in [26], but the authors restricted themselves to the special case of a linear gauge kinetic function and only considered Abelian gauge fields and global supersymmetry.[21] As we will see, the superfield formalism is only applicable for shift tensors $C_{AB,C}$ that are mixed symmetric in its indices while the discussion of section 3.4 proved that it is the symmetric part of $C_{AB,C}$ that can possibly cancel anomalies. A first complete discussion of generalized Chern-Simons terms and chiral anomalies in $\mathcal{N}=1$ supersymmetry and supergravity was done in [88] (the supersymmetrization of section 3) and will be discussed in this section. This is a new result and it is one of the major topics of this thesis.

In subsection 5.1 we will consider anomaly-free theories and we will allow for gauged isometries on the scalar manifold in global supersymmetry. It will be shown, that the presence of the gauged isometries violates the supersymmetry transformation laws as displayed in section 4. The supersymmetry transformation laws will be covariantized with respect to

[20] The possible extension (3.6) was already mentioned for $\mathcal{N}=2$ supergravity in [32] and later in [77] for $\mathcal{N}=1$ supersymmetry, but the extra terms necessary for its consistency were not considered.

[21] A superfield expression for the nonabelian generalized Chern-Simons term in Wess-Zumino gauge is given in the end of [26].

the gauged isometries, which is done according to [84]. As the Peccei-Quinn term has to be accompanied by the generalized Chern-Simons term, we will add an $\mathcal{N}=1$ superfield expression of the generalized Chern-Simons term to the gauge sector of $\mathcal{N}=1$ supersymmetry. The proof that this new action is indeed invariant under supersymmetry (where some of the supersymmetry transformations are covariantized with respect to the gauged isometries) is done in computing the variations that appear due to the modifications in the transformation laws and in showing that these contributions cancel.

In subsection 5.2 we will allow for a symmetric part of the tensor $C_{AB,C}$ and show how this can possibly cancel anomalies. To do so, we introduce a derivative that is covariant with respect to the gauged isometries and give up the $\mathcal{N}=1$ superfield expression of the generalized Chern-Simons term.

In subsection 5.3 we show how the results found for $\mathcal{N}=1$ supersymmetry can be extended to $\mathcal{N}=1$ supergravity.

In subsection 5.4 we will illustrate the results found in this section by means of the example of a gauge group of the form Abelian×semisimple. This completes earlier discussions of section 3

Finally, the results of this section are summarized in subsection 5.5.

5.1 Gauged isometries and generalized Chern-Simons terms in global supersymmetry

For simplicity, let us consider a $U(1)^n$ gauge theory, where the gauge fields are labelled by indices $A, B, \ldots = 1, \ldots, n$. Furthermore, let us assume that the scalar fields z^i transform nontrivially under the gauge symmetry as

$$\delta_\Lambda z^i = k_C^i(z)\Lambda^C\,, \tag{5.1}$$

where $k_C^i(z)$ are the Killing-vectors of the isometry on the target space of the scalar fields. A direct consequence is that the gauge kinetic function will in general no longer transform trivially. Instead, by applying the chain rule, one obtains

$$\delta_\Lambda \mathrm{f}_{AB}(z) = \partial_i \mathrm{f}_{AB} k_C^i \Lambda^C\,. \tag{5.2}$$

Also the fields χ^i transform under the isometry group because they are the superpartners of the scalars z^i. Let δ_ε denote the supersymmetry transformations and let δ_Λ stand for transformations with gauge parameter Λ. Then, on the one hand, we have from (3.14)

$$\delta_\Lambda \delta_\varepsilon z^i = \bar{\varepsilon}_L \delta_\Lambda \chi^i_L \,, \tag{5.3}$$

while, on the other hand, it is

$$\delta_\varepsilon \delta_\Lambda z^i = \partial_j k^i_C \Lambda^C \delta_\varepsilon z^j = \bar{\varepsilon}_L (\partial_j k^i_C \Lambda^C \chi^j)_L \,. \tag{5.4}$$

As supersymmetry transformations and gauge transformations commute [75], we find from comparing the above expressions that χ^i transforms under the isometry as

$$\delta_\Lambda \chi^i_L = \partial_j k^i_C \Lambda^C \chi^j_L \,. \tag{5.5}$$

If the transformations (5.1) and (5.5) are present, then (4.21) is no longer invariant and new terms have to be introduced in order to restore invariance under supersymmetry. We could already see in section 3 that once the gauge kinetic function transforms with a constant shift, new terms must be added in order to restore gauge invariance. These terms, the generalized Chern-Simons terms, were at that point explicitly calculated but not in the context of supersymmetry. The experience with supersymmetry suggests that the bosonic generalized Chern-Simons term will be accompanied with a term involving couplings to gaugini. However, from the discussion in section 3 we know that the generalized Chern-Simons term alone cannot be gauge invariant (otherwise it could not be used to cancel gauge variations) and, therefore, a manifest supersymmetric extension of the generalized Chern-Simons term by itself cannot exist. This is a crucial point, so let us discuss it in more detail: If an action is invariant under supersymmetry, it should also be gauge invariant. So, for example, the supersymmetry transformation $\delta_\varepsilon \chi^i$ as given in (4.18) does not commute with the gauge transformation (5.5) anymore[22]. Starting from (4.18), we find for the commutator $[\delta_\varepsilon, \delta_\Lambda]\chi^i_L = \frac{1}{2}\gamma^\mu \varepsilon_R k^i_C \partial_\mu \Lambda^C$ and in order that the commutator vanishes, the partial derivative in (4.18) has to be replaced by a covariant derivative. The action of the generators of supersymmetry on χ^i in the presence of (5.1) is no longer given by (4.18) because the commutation relations of the supersymmetry algebra are no longer satisfied. In the presence of (5.1) the action of the generator of supersymmetry on χ^i is obtained from $[\delta_\varepsilon, \delta_\Lambda]\chi^i = 0$ instead. The same is found for the variation

[22] $\delta_\varepsilon z^i$ is not altered, i.e., it is the same as in equation (4.17) because $[\delta_\varepsilon, \delta_\Lambda] z^i = 0$.

of the auxiliary field $\delta_\varepsilon F^i$ and in total the transformation laws consistent with (5.1) and (5.5) are

$$
\begin{aligned}
\delta_\varepsilon z^i &= \bar{\varepsilon}_L \chi_L^i && (5.6)\\
\delta_\varepsilon \chi_L^i &= \frac{1}{2}\gamma^\mu \varepsilon_R D_\mu z^i + \frac{1}{2} F^i \varepsilon_L && (5.7)\\
\delta_\varepsilon \bar{\chi}_L^i &= -\frac{1}{2}\bar{\varepsilon}_R \gamma^\mu D_\mu z^i + \frac{1}{2} F^i \bar{\varepsilon}_L && (5.8)\\
\delta_\varepsilon F^i &= \bar{\varepsilon}_R \gamma^\mu D_\mu \chi_L^i + \bar{\varepsilon}_R \lambda_R^A k_A^i && (5.9)
\end{aligned}
$$

where the covariant derivatives are defined as follows

$$
\begin{aligned}
D_\mu z^i &= \partial_\mu z^i - A_\mu^C k_C^i\,, && (5.10)\\
D_\mu \chi^i &= \partial_\mu \chi^i - A_\mu^C \partial_j k_C^i \chi^j\,. && (5.11)
\end{aligned}
$$

The new supersymmetry transformations (5.6), (5.7) and (5.9) take explicitly the gauge transformations into account, as demonstrated by the gauge covariant derivatives and the last term in (5.9). It originates in the requirement that supersymmetry does not only respect the gauge invariance of the auxiliary field, but both symmetries still commute with each other[23]. The Abelian generalized Chern-Simons terms of global $\mathcal{N}=1$ supersymmetry were given in [26][24] and for future reference we quote the result

$$
\mathcal{L}_{GCS}^{\mathcal{N}=1} = \frac{1}{6} C_{AB,C}\varepsilon^{\mu\nu\rho\sigma} A_\mu^C A_\nu^B F_{\rho\sigma}^A - \frac{\mathrm{i}}{4} C_{AB,C} A_\mu^C \bar{\lambda}^A \gamma_5 \gamma^\mu \lambda^B \qquad (5.12)
$$

where $C_{AB,C}$ is a real constant tensor that has to satisfy the constraint

$$
C_{(AB,C)} = 0 \qquad (5.13)
$$

[23]The whole problematic is present in superspace formalism, too. There this subtlety arises because after the Wess-Zumino gauge is fixed, the original supersymmetry of superspace (transformations induced by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ acting on the superfields) is broken and has to be replaced by a combination of the superspace supersymmetry and the gauge symmetry. The Wess-Zumino gauge is violated by supersymmetry transformations induced by $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ and only after applying a gauge transformation one is brought back into Wess-Zumino gauge again. This can also be understood from the supersymmetry algebra. After the Wess-Zumino gauge is fixed, the anticommutation relation $\{\mathcal{Q}_\alpha, \mathcal{Q}_{\dot\alpha}^\dagger\} = \sigma^\mu_{\alpha\dot\alpha}(\partial_\mu - A_\mu^A \delta_A)$ [84] shows mixing between supersymmetry and gauge symmetries (δ_A denotes the gauge transformation). This implies that if an action is invariant under supersymmetry, it should also be gauge invariant.

[24]The authors restricted themselves to linear gauge kinetic functions.

in agreement with equation (3.7) found in section 3 in the context of Lie algebra cohomology. The first term of (5.12) is the Abelian version of the generalized Chern-Simons term encountered in section 3 as equation (3.30), while the other term represents the coupling of the vector field to the pseudovector current of the gaugini. The possible coupling $A_\mu^C \bar{\lambda}^A \gamma^\mu \lambda^B$ vanishes identically because of $C_{[AB],C}=0$. Note, that the tensor $C_{AB,C}$ is mixed symmetric in the sense that its total symmetric part vanishes but it is symmetric in its first two indices. Observe further that equation (5.12) is neither gauge invariant nor supersymmetric.[25]

It remains to show that (4.21) together with (5.12) is invariant under (5.6), (5.7) and (5.9). This is easily done by observing that if we replace the covariant derivatives in the supersymmetry transformation laws (5.6), (5.7) and (5.9) by partial derivatives and remove the last term in $\delta_\varepsilon F^i$ then of course we obtain back the supersymmetry transformations under which $\mathcal{L}_{\rm gk}$, given by (4.21), is invariant. Therefore, we have to check whether the extra terms that appear in the variation of $\mathcal{L}_{\rm gk}$ cancel against $\delta_\varepsilon \mathcal{L}_{GCS}^{\mathcal{N}=1}$ when the scalars transform nontrivially under gauged isometries (5.1).

There are the following three terms in (4.21) that cause new contributions to the variation $\delta_\varepsilon \mathcal{L}_{\rm gk}$:

- The term $-\frac{1}{4}F^i\partial_i \mathrm{f}_{AB}\bar{\lambda}_L^A\lambda_L^B$ and its hermitian conjugate $-\frac{1}{4}F^{*\,i}\partial_i \mathrm{f}^*_{AB}\bar{\lambda}_R^A\lambda_R^B$ causes an extra variation of the form

$$-\frac{1}{4}\partial_i \mathrm{f}_{AB} k_C^i \bar{\varepsilon}_R \lambda_R^C \bar{\lambda}_L^A \lambda_L^B - \frac{1}{4}\partial_i \mathrm{f}^*_{AB} k_C^{*\,i} \bar{\varepsilon}_L \lambda_L^C \bar{\lambda}_R^A \lambda_R^B \tag{5.14}$$

  due to the last term in (5.9).

- Another new variation comes from $\delta_\varepsilon \chi$ in $\frac{\mathrm{i}}{2}\partial_i \mathrm{f}_{AB}\bar{\chi}_L^i \lambda_L^A D^B$ and its hermitian conjugate $-\frac{\mathrm{i}}{2}\partial_i \mathrm{f}^*_{AB}\bar{\chi}_R^i \lambda_R^A D^B$ which is

$$+\frac{\mathrm{i}}{4}\partial_i \mathrm{f}_{AB} k_C^i A_\mu^C D^B \bar{\varepsilon}_R \gamma^\mu \lambda_L^A - \frac{\mathrm{i}}{4}\partial_i \mathrm{f}^*_{AB} k_C^{*\,i} A_\mu^C D^B \bar{\varepsilon}_L \gamma^\mu \lambda_R^A\,. \tag{5.15}$$

- The term $-\frac{1}{2}\partial_i \mathrm{f}_{AB} F_{\mu\nu}^A \bar{\chi}_L^i \sigma^{\mu\nu}\lambda_L^B$ and its hermitian conjugate $-\frac{1}{2}\partial_i \mathrm{f}^*_{AB} F_{\mu\nu}^A \bar{\chi}_R^i \sigma^{\mu\nu}\lambda_R^B$ contribute to $\delta_\varepsilon \mathcal{L}_{\rm gk}$ with

$$\begin{aligned} &-\frac{1}{4}\partial_i \mathrm{f}_{AB} k_C^i A^{\mu\,C} \bar{\varepsilon}_R \gamma^\nu \lambda_L^B F_{\mu\nu}^A + \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma} k_C^i \partial_i \mathrm{f}_{AB} F_{\mu\nu}^A A_\rho^C \bar{\varepsilon}_R \gamma_5 \gamma_\sigma \lambda_L^B - \\ &-\frac{1}{4}\partial_i \mathrm{f}^*_{AB} k_C^{*\,i} A^{\mu\,C} F_{\mu\nu}^A \bar{\varepsilon}_L \gamma^\nu \lambda_R^B + \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma} k_C^{*\,i} \partial_i \mathrm{f}^*_{AB} F_{\mu\nu}^A A_\rho^C \bar{\varepsilon}_L \gamma_5 \gamma_\sigma \lambda_R^B\,. \end{aligned} \tag{5.16}$$

[25] In [26] the authors give a superspace expression for (5.12) (in Wess-Zumino gauge) but we will see that it is not manifestly supersymmetric.

Now let us compute the variations of the Chern-Simons terms. For the bosonic term we have

$$C_{AB,C}\delta_\varepsilon(\varepsilon^{\mu\nu\rho\sigma}A_\mu^C A_\nu^B F_{\rho\sigma}^A) \;=\; +\frac{3}{2}\cdot C_{AB,C}\varepsilon^{\mu\nu\rho\sigma}\bar\varepsilon\gamma_\mu\lambda^B A_\nu^C F_{\rho\sigma}^A \tag{5.17}$$

while the variation of the vector potential in the fermionic term gives

$$\begin{aligned} -\frac{\mathrm{i}}{4}C_{AB,C}(\delta_\varepsilon A_\mu^C)\bar\lambda^A\gamma_5\gamma^\mu\lambda^B &= -\frac{\mathrm{i}}{2}C_{AB,C}(\bar\varepsilon\lambda^B)(\bar\lambda^A\gamma_5\lambda^C) \\ &= -\frac{\mathrm{i}}{4}C_{AB,C}\bar\varepsilon\lambda^B\bar\lambda^C\gamma_5\lambda^A + \frac{\mathrm{i}}{4}C_{AB,C}\bar\varepsilon\gamma_5\lambda^B\bar\lambda^C\lambda^A \end{aligned} \tag{5.18}$$

which is proven by help of the rearrangement formulae given in appendix B. The remaining contribution of the fermionic part of the generalized Chern-Simons term is caused by the extra variation in the transformation law for the gaugini. We have to compute $-\frac{\mathrm{i}}{4}C_{AB,C}A_\mu^C\delta_\varepsilon(\bar\lambda^A\gamma_5\gamma^\mu\lambda^B)$ which is found to be

$$-\frac{\mathrm{i}}{4}C_{AB,C}A^{\nu\,C}F_{\mu\nu}^A\bar\varepsilon\gamma^\mu\gamma_5\lambda^B + \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}F_{\mu\nu}^A A_\rho^C\bar\varepsilon\gamma_\sigma\lambda^B + \frac{1}{4}C_{AB,C}A_\mu^C\bar\varepsilon\gamma^\mu\lambda^B D^A\,. \tag{5.19}$$

If the gauge sector together with the generalized Chern-Simons terms (5.12) is invariant under supersymmetry, then the variations determined above have to cancel among themselves. Obviously, the variations of the generalized Chern-Simons terms do not cancel among themselves (the contribution (5.18) can only be cancelled by another three gaugini interaction which is given by (5.14)). Thus, the generalized Chern-Simons term of global $\mathcal{N}=1$ supersymmetry given in equation (5.12) is not by itself invariant under supersymmetry and the superspace expression from which it originates [26] is not manifestly supersymmetric.

In order that variations of the gauge sector cancel against variations from the generalized Chern-Simons terms, the constants $C_{AB,C}$ and $\partial_i \mathrm{f}_{AB}k_C^i$ have to be related. A closer look at (5.14) and (5.18) shows, that if

$$\partial_i \mathrm{f}_{AB}k_C^i \;=\; \mathrm{i}C_{AB,C} \tag{5.20}$$

$$\partial_i \mathrm{f}_{AB}^* k_C^{*\,i} \;=\; -\mathrm{i}C_{AB,C} \tag{5.21}$$

then both variations add up to zero. The reason is that (5.14) can be brought to the form

$$-\frac{\mathrm{i}}{4}C_{AB,C}(\bar\varepsilon_R\lambda_R^B\bar\lambda_L^C\lambda_L^A + \bar\varepsilon_L\lambda_L^B\bar\lambda_R^C\lambda_R^A) \;=\; -\frac{\mathrm{i}}{8}C_{AB,C}\bar\varepsilon\lambda^B\bar\lambda^C\gamma_5\lambda^A + \frac{\mathrm{i}}{8}C_{AB,C}\bar\varepsilon\lambda^A\bar\lambda^C\gamma_5\lambda^A$$

which taken together with (5.18) leads to the equation for the shift tensor

$$2C_{CA,B} + C_{AB,C} \;=\; 3C_{(AB,C)} = 0 \tag{5.22}$$

where the constraint (5.13) is used in the last equality. Thus, the contribution of (5.14) and (5.18) is proportional to the symmetric part of the shift tensor $C_{(AB,C)}$ and vanishes. Furthermore, it is not difficult to see that the last term of (5.19) cancels against (5.15).[26] The first term of the first and the second line of (5.16) together with the first term of (5.19) add up to zero.[27] The remaining contributions from (5.16) and (5.19) add up to give[28]

$$\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}F^{A}_{\mu\nu}A^{C}_{\rho}\bar{\varepsilon}\gamma_{\sigma}\lambda^{B}\,, \tag{5.23}$$

which cancels exactly the variation of the purely bosonic generalized Chern-Simons term (5.17) given by $-\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}F^{A}_{\mu\nu}A^{C}_{\rho}\bar{\varepsilon}\gamma_{\sigma}\lambda^{B}$.

This, however, does not yet complete the proof that (4.21) together with the generalized Chern-Simons terms (5.12) is indeed invariant under supersymmetry. The supersymmetry variation of the four-fermion interaction $\frac{1}{4}\bar{\chi}^{i}_{L}\chi^{j}_{L}\partial_i\partial_j\mathrm{f}_{AB}\bar{\lambda}^{A}_{L}\lambda^{B}_{L}$ receives the contribution from the covariant derivative in (5.7), too, and causes the variation

$$\frac{1}{4}\bar{\varepsilon}_{R}\gamma^{\mu}\chi^{i}_{L}A^{C}_{\mu}k^{i}_{C}\partial_i\partial_j\mathrm{f}_{AB}\bar{\lambda}^{A}_{L}\lambda^{B}_{L}\,. \tag{5.24}$$

The same happens to the term $-\frac{1}{4}F^{i}\partial_i\mathrm{f}_{AB}\bar{\lambda}^{A}_{L}\lambda^{B}_{L}$ which causes the variation

$$\frac{1}{4}\bar{\varepsilon}_{R}\gamma^{\mu}\chi^{j}_{L}A^{C}_{\mu}\partial_j k^{i}_{C}\partial_i\mathrm{f}_{AB}\bar{\lambda}^{A}_{L}\lambda^{B}_{L} \tag{5.25}$$

due to the covariant derivative in (5.9). Note, that because $C_{AB,C}$ is constant,taking a derivative of (5.20) with respect to ∂_j yields

$$\partial_j k^{i}_{C}\partial_i\mathrm{f}_{AB} \;=\; -k^{i}_{C}\partial_j\partial_i\mathrm{f}_{AB}\,, \tag{5.26}$$

and the two variations drop out without the need of extra terms. So, indeed, the gauge sector (4.21) together with the generalized Chern-Simons terms (5.12) is invariant under the supersymmetry transformations (5.6) to (5.9) in the presence of gauged isometries.

Now we are going to show that the fermionic term (5.12) can be used to define a new derivative that is covariant with respect to gauged isometries. The isometries actually induce shifts because from (5.2) and (5.20) we can see that the gauge kinetic function is shifted by an imaginary constant:

$$\delta_{\Lambda}\mathrm{f}_{AB} \;=\; \mathrm{i}C_{AB,C}\Lambda^{C} \tag{5.27}$$

[26]A useful relation is $\bar{\varepsilon}_{R}\gamma^{\mu}\lambda_{L}+\bar{\varepsilon}_{L}\gamma^{\mu}\lambda_{R}=\bar{\varepsilon}\gamma^{\mu}\lambda$.

[27]This can be seen from $-\bar{\varepsilon}_{R}\gamma^{\nu}\lambda_{L}+\bar{\varepsilon}_{L}\gamma^{\nu}\lambda_{R}=-\bar{\varepsilon}\gamma^{\nu}\gamma_5\lambda$.

[28]This makes use of $-\bar{\varepsilon}_{R}\gamma_5\gamma_{\sigma}\lambda_{L}+\bar{\varepsilon}_{L}\gamma_5\gamma_{\sigma}\lambda_{R}=\bar{\varepsilon}\gamma_{\sigma}\lambda$.

The only terms of (4.21) that are affected by this shift are found in the second line. The first term is the Peccei-Quinn term, that was treated in 3, and the second term is proportional to the axial gaugino current. All the other terms are either proportional to $\mathrm{Re}\, \mathrm{f}_{AB}$ or its derivative and, thus, are not affected.

The term proportional to the axial gaugino current transforms under the gauged isometry as

$$\frac{1}{4}\delta_\Lambda(\partial_\mu \operatorname{Im} \mathrm{f}_{AB})\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B \quad = \quad \frac{\mathrm{i}}{4}\delta_\Lambda A^C_\mu C_{AB,C}\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B \tag{5.28}$$

which is cancelled by the variation of the corresponding fermionic generalized Chern-Simons term. If we now introduce the new derivative

$$D_\mu \mathrm{f}_{AB} \quad := \quad \partial_\mu \mathrm{f}_{AB} - \mathrm{i}C_{AB,C}A^C_\mu \tag{5.29}$$

which transforms covariantly under the shift symmetry (5.27), then we have done nothing else but absorbed the fermionic generalized Chern-Simons term of (5.12) into (5.29). From this point of view, it does not surprise that [26] found (5.12), though it was obtained for linear gauge kinetic functions and through superspace techniques.

Now let us turn to the nonabelian isometries. The Lagrangian corresponding to (4.21) but invariant under local nonabelian gauge symmetries is obtained from (4.21) by substituting partial derivatives by covariant derivatives and the Abelian field strengths by their covariant counterparts $\mathcal{F}^A_{\mu\nu} = 2\partial_{[\mu}A^A_{\nu]} + f_{BC}{}^A A^B_\mu A^C_\nu$. The same is valid for the supersymmetry transformations. The fermionic part of the generalized Chern-Simons term (5.12) is made invariant under nonabelian gauge transformations by introducing a covariant derivative. The pure bosonic generalized Chern-Simons term was determined in section 3 and is given by

$$\mathcal{L}_{\mathrm{GCS}} \quad = \quad \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}A^B_\mu A^C_\nu F^A_{\rho\sigma} + \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}f_{DE}{}^A A^D_\mu A^E_\nu A^C_\rho A^B_\sigma \tag{5.30}$$

where $F^A_{\mu\nu}$ represents the Abelian part of the nonabelian field strength $\mathcal{F}^A_{\rho\sigma} = F^A_{\rho\sigma} + f_{DE}{}^A A^D_\rho A^E_\sigma$. The constant tensors $C_{AB,C}$ have to fulfil two constraints, as given in (3.7) and (3.8):

$$C_{(AB,C)} \quad = \quad 0\,, \tag{5.31}$$

$$C_{CB,A}f_{DE}{}^A + 2C_{AC,[E}f_{D]B}{}^A + 2C_{AB,[E}f_{D]C}{}^A \quad = \quad 0\,. \tag{5.32}$$

The supersymmetry variation of the first term of (5.30) was computed in (5.17) so it only remains to vary the second term under supersymmetry. With the help of the constraints (5.31) and (5.32) one can show that

$$\begin{aligned} C_{AB,C} f_{DE}{}^A \varepsilon^{\mu\nu\rho\sigma} \delta\big(A_\mu^D A_\nu^E A_\rho^C A_\sigma^B\big) &= \varepsilon^{\mu\nu\rho\sigma} \big(2C_{AE,D} f_{BC}{}^A - C_{AE,B} f_{DC}{}^A - \\ &\quad - C_{AB,D} f_{EC}{}^A\big)\,(\delta A_\mu^B) A_\nu^C A_\rho^D A_\sigma^E \\ &= 2C_{AB,C} f_{DE}{}^A\, \varepsilon^{\mu\nu\rho\sigma} \bar\varepsilon \gamma_\mu \lambda^B A_\nu^C A_\rho^D A_\sigma^E \,. \end{aligned} \tag{5.33}$$

The variation (5.33) completes the Abelian field strength in (5.17) to form the nonabelian field strength $\mathcal{F}^A_{\rho\sigma}$. Therefore we find for the variation of the nonabelian generalized Chern-Simons term

$$\delta\mathcal{L}_{\rm GCS} = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} C_{AB,D} \bar\varepsilon\gamma_\mu \lambda^B A_\nu^D \mathcal{F}^A_{\rho\sigma} \tag{5.34}$$

and we see that the results of the Abelian discussion can be straight-forwardly extended to the nonabelian case.

5.2 Gauged isometries and anomalies in global $\mathcal{N}=1$ supersymmetry

In the previous subsection we found that once isometries on the target space of the scalar fields are gauged, the original supersymmetry transformations no longer commute with gauge symmetries. The new supersymmetry transformations are obtained from the old ones by replacing partial derivatives by gauge covariant derivatives. Furthermore, one has to introduce a new term into the transformation of the auxiliary field F^i that couples gaugini to Killing vectors. After these extensions in the transformation laws we saw that the Lagrangian (4.21) is no longer invariant under supersymmetry. In order to restore supersymmetry we had to covariantize the derivative[29] in the term $(D'_\mu \,{\rm Im}\, {\rm f}_{AB})\bar\lambda^A\gamma_5\gamma^\mu\lambda^B$ with respect to the gauged isometries and we added generalized Chern-Simons terms (5.30) to the action. We showed that this new action is indeed invariant under supersymmetry again. This is only a special case because the infinitesimal shift can in general have a nontrivial totally symmetric part, i.e.,

$$\partial_i {\rm f}_{(AB} k^i_{C)} \neq 0 \,. \tag{5.35}$$

[29]The derivative D'_μ is defined by $D'_\mu \,{\rm Im}\,{\rm f}_{AB} := \partial_\mu \,{\rm Im}\,{\rm f}_{AB} - 2A_\mu^C f_{C(A}{}^D\, {\rm Im}\,{\rm f}_{B)D}$ in accordance with (4.22).

Then, $C_{AB,C}$ as defined in (5.20) and the constant tensor used for the generalized Chern-Simons terms (5.30) are no longer identical. As noted before, the constant tensor of the generalized Chern-Simons terms, from now on denoted by $C^{\rm CS}_{AB,C}$, is mixed symmetric. That means that it is symmetric in its first two indices and its totally symmetric part vanishes. From the decomposition of a tensor of degree three it follows that for vanishing totally symmetric part it must be antisymmetric in its last two indices. This discussion shows that there is a difference between $C^{\rm CS}_{AB,C}$ from the generalized Chern-Simons terms and the shift $C_{AB,C}$ if that contains in addition to a part of mixed symmetry a part that is totally symmetric in all indices (this is consistent with the discussion in section 3.4). Hence, the generalized Chern-Simons terms (5.30) can only possibly cancel contributions from $\delta_\varepsilon\mathcal{L}_{\rm gk}$, if the mixed symmetric part of the shifts $C_{AB,C}$ is equal to $C^{\rm CS}_{AB,C}$, i.e. if

$$C_{AB,C} \;=\; C_{(AB,C)} + C^{\rm CS}_{AB,C}\,. \tag{5.36}$$

It is important to observe that the term

$$-\frac{\rm i}{4} C_{AB,C} A^C_\mu \bar\lambda^A \gamma_5 \gamma^\mu \lambda^B \tag{5.37}$$

is needed to render the derivative of the imaginary part of the gauge kinetic function, i.e. of $\left(D'_\mu \,{\rm Im}\,{\rm f}_{AB} \bar\lambda^A \gamma^\mu \gamma_5 \lambda\right)$, covariant with respect to the gauged isometry. This goes beyond the treatment of [26], where the mixed symmetric part of the term (5.37) was found to be a member of the Chern-Simons superfield in superspace. The part proportional to $C_{(AB,C)}$ cannot be obtained in a known way from a superfield expression for the generalized Chern-Simons term as given in [26], due to the symmetry properties of $C^{\rm CS}_{AB,C}$, i.e. the constraint (5.13). As equation (5.12) is not supersymmetric in the Wess-Zumino gauge, it is better not to follow the lines of [26] and to still consider (5.30) as the generalized Chern-Simons term for supersymmetric theories. The fermionic term (5.37) is then used to gauge $D'_\mu \,{\rm Im}\,{\rm f}_{AB}$ with respect to the shift symmetry. Another important point to note is that now (5.22) does not vanish anymore and leaves an uncancelled contribution to the supersymmetry variation given by

$$-\frac{3{\rm i}}{4} C_{(AB,C)} [\bar\varepsilon_R \lambda^B_R \bar\lambda^C_L \lambda^A_L - \bar\varepsilon_L \lambda^B_L \bar\lambda^C_R \lambda^A_R]\,. \tag{5.38}$$

In the same way, the generalized Chern-Simons terms cannot cancel the corresponding variations in (5.16) and (5.19) but it leaves the contribution to the supersymmetry variation

$$-\frac{1}{4}C_{(AB,C)}\varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}\gamma_\mu\lambda^B A_\nu^C \mathcal{F}_{\rho\sigma}^A + \frac{1}{8}C_{(AB,C)}f_{DE}{}^A\varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}\gamma_\mu\lambda^B A_\nu^C A_\rho^D A_\sigma^E\,. \tag{5.39}$$

Hence, for general shifts, where $C_{(AB,C)} \neq 0$, the action $\mathcal{L}_{\rm gk} + \mathcal{L}_{\rm GCS}$ is no longer supersymmetric. This tells us that we cannot even expect the action to be gauge invariant. In fact, the gauge variation leads to a non-invariance that is given by

$$\frac{\rm i}{8}\varepsilon^{\mu\nu\rho\sigma}\left[C_{(AB,C)}F_{\mu\nu}^A F_{\rho\sigma}^B + (C_{(AB,D)}f_{CE}{}^B + \frac{3}{2}C_{(AB,C)}f_{DE}{}^B)A_\mu^D A_\nu^E F_{\rho\sigma}^A\right]\Lambda^C\,. \tag{5.40}$$

This expression is similar to the consistent form of the anomaly. The total anomaly, however, is given by the supersymmetry anomaly and the gauge anomaly. A full cohomological analysis of anomalies in supergravity was made by Brandt in [79] and [80]. His result is that the total anomaly consisting of the gauge anomaly $\mathcal{A}_C\Lambda^C$ and the supersymmetry anomaly $\bar{\varepsilon}\mathcal{A}_\varepsilon$ is given by

$$\begin{aligned}
\mathcal{A}_C &= -\frac{\rm i}{8}\varepsilon^{\mu\nu\rho\sigma}[d_{ABC}F_{\mu\nu}^A F_{\rho\sigma}^B + (d_{ABD}f_{CE}{}^B + \frac{3}{2}d_{ABC}f_{DE}{}^B)A_\mu^D A_\nu^E F_{\rho\sigma}^A] && (5.41)\\
\bar{\varepsilon}\mathcal{A}_\varepsilon &= \frac{3\rm i}{4}d_{ABC}[\bar{\varepsilon}_R\lambda_R^B\bar{\lambda}_L^C\lambda_L^A - \bar{\varepsilon}_L\lambda_L^B\bar{\lambda}_R^C\lambda_R^A] + \frac{1}{4}d_{ABC}\varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}\gamma_\mu\lambda^B A_\nu^C\mathcal{F}_{\rho\sigma}^A - \\
&\quad -\frac{1}{8}d_{ABC}f_{DE}{}^A\varepsilon^{\mu\nu\rho\sigma}\bar{\varepsilon}\gamma_\mu\lambda^B A_\nu^C A_\rho^D A_\sigma^E && (5.42)
\end{aligned}$$

where d_{ABC} denote total symmetric tensors that characterize the anomaly and are determined by the Wess-Zumino consistency condition (2.25). The gauge anomaly given by (5.41) leads to the consistent anomaly (2.26), if one chooses the symmetric tensor to be of the form $d_{ABC} = \frac{\rm i}{24\pi^2}{\rm tr}\left[T_A\{T_B,T_C\}\right]$. The anomaly originates from chiral fermions in the matter sector.

In comparing the expressions (5.38) and (5.39) with the supersymmetry anomaly (5.42) and the gauge variation (5.40) with the consistent gauge anomaly (5.41), we see that the anomalies cancel the left over contributions in the supersymmetry and gauge variation precisely if $C_{(AB,C)} = d_{ABC}$.

Hence, generalized Chern-Simons terms and gauged isometries that introduce shifts in the gauge kinetic function cancel chiral anomalies if the shifts satisfy

$$C_{AB,C} = d_{ABC} + C_{AB,C}^{\rm GCS}\,. \tag{5.43}$$

This confirms the discussion in section (3.4). There the totally symmetric part of C caused the Chern-Simons five-form that again is related to the anomaly by the descent equation (3.55), implying that the anomaly can be cancelled if $d_{ABC} = C_{(AB,C)}$.

5.3 Generalized Chern-Simons terms in Supergravity

In going from global supersymmetry to supergravity, there appear terms in the gauge sector of supergravity that were not there in global supersymmetry. As it was demonstrated in the previous section, the Lagrangian (4.28) is invariant under the local supersymmetry transformations (4.34) to (4.39). In total analogy to the rigid case, when isometries on the target space are gauged, the derivatives in the transformation laws for the chiral fermions χ^i and the auxiliary fields F^i have to be covariantized with respect to the gauged isometries and the last term of (5.9) is present, too. This in turn causes again new contributions in the variation of (4.28) under local supersymmetry. Also the term proportional to $D_\mu \,\mathrm{Im}\,\mathrm{f}_{AB}$ has to be extended to transform covariantly under gauged isometries by introducing the new term

$$e\frac{\mathrm{i}}{4}C_{AB,C}A_\rho^C\bar{\lambda}^A\gamma^\rho\gamma_5\lambda^B\,. \tag{5.44}$$

This term causes new variations[30] under supersymmetry due to $\delta_\varepsilon e$, $\delta_\varepsilon\gamma^\rho$ and the term

$$\delta_\varepsilon^{extra}\lambda^A \;=\; \frac{1}{2}\sigma^{\mu\nu}\varepsilon\left(\bar{\psi}_\mu\gamma_\nu\lambda\right) \tag{5.45}$$

that arises because of $\mathcal{F}_{\mu\nu}\to\mathcal{F}_{\mu\nu}^{\mathrm{cov}}$ in supergravity (4.39). The contributions due to $\delta_\varepsilon e$, $\delta_\varepsilon\gamma^\rho$ and (5.45) are found to be equal to

$$\begin{aligned}&e\frac{\mathrm{i}}{8}C_{AB,C}A_\mu^C[\bar{\varepsilon}\gamma^\nu\psi_\nu\bar{\lambda}^A\gamma^\mu\gamma_5\lambda^B-\bar{\psi}_\nu\gamma^\mu\lambda^A\bar{\varepsilon}\gamma^\nu\gamma_5\lambda^B-\bar{\psi}^\mu\gamma^\nu\lambda^A\bar{\varepsilon}\gamma_5\gamma_\nu\lambda^B+\bar{\lambda}^A\gamma^\nu\gamma_5\lambda^B\bar{\varepsilon}\gamma_\nu\psi^\mu]+\\&+e\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}A_\rho^C\bar{\psi}_\mu\gamma_\nu\lambda^A\bar{\varepsilon}\gamma_\sigma\lambda^B\,.\end{aligned} \tag{5.46}$$

The contribution from covariantizing the derivative with respect to gauged isometries in the transformation law of χ^i will cause extra variations in the variation of terms that couple to χ^i. There are two relevant terms coupling to χ^i:

- The first term is $-\frac{1}{2}e\partial_i\mathrm{f}_{AB}\bar{\chi}_L^i\sigma^{\mu\nu}\lambda_L\hat{\mathcal{F}}_{\mu\nu}^A$+h.c. which gives rise to the term

$$\frac{\mathrm{i}}{8}eC_{AB,C}A_\mu^C\bar{\varepsilon}\gamma^\nu\gamma_5\lambda^A\left[\bar{\psi}_\nu\gamma^\mu\lambda^B-\bar{\psi}^\mu\gamma_\nu\lambda^B\right]-\frac{1}{8}e\varepsilon^{\mu\nu\rho\sigma}C_{AB,C}A_\rho^C\bar{\varepsilon}\gamma_\sigma\lambda^B\bar{\psi}_\mu\gamma_\nu\lambda^A\,. \tag{5.47}$$

[30] We recall that the matrices γ^a represent the flat space Dirac matrices and are constant, as opposed to γ^μ which are dressed with a vierbein and, consequently, it is $\delta_\varepsilon\gamma^\rho = \delta_\varepsilon e_a^\rho\gamma^a$.

These terms already cancel the second, third and fifth term of (5.46).

- Another contribution is caused by the term which is given by $-\frac{1}{4}e\partial_i \mathrm{f}_{AB}\bar{\psi}^{\mu}_R\gamma_\mu\chi^i_L\bar{\lambda}^A_L\lambda^B_L$+h.c. and couples the gravitino to χ^i. It leaves the uncancelled variation

$$\frac{\mathrm{i}}{16}eC_{AB,C}A^C_\nu\left[\bar{\psi}_\mu\gamma^\mu\gamma^\nu\varepsilon\bar{\lambda}^A\gamma_5\lambda^B-\bar{\psi}_\mu\gamma^\mu\gamma^\nu\gamma_5\varepsilon\bar{\lambda}^A\lambda^B\right] \tag{5.48}$$

- The last contribution that has to be considered originates from the variation of the auxiliary field F^i in $-\frac{1}{4}\,e\big(\partial_i\mathrm{f}_{AB}\big)F^i\bar{\lambda}^A_L\lambda^B_L$+h.c., i.e. through the covariant derivative of χ in δF^i:

$$\begin{aligned}\delta_\varepsilon F^i &= \bar{\varepsilon}_R\gamma^\mu D_\mu\chi^i+\ldots=-\frac{1}{2}\bar{\varepsilon}_R\gamma^\mu\gamma^\nu\hat{D}_\nu z^i\psi_{\mu\,R}+\ldots\\ &= \frac{1}{2}k^i_C A^C_\nu\bar{\varepsilon}_R\gamma^\mu\gamma^\nu\psi_{\mu\,R}+\ldots\,.\end{aligned} \tag{5.49}$$

Therefore, the extra variation is given by

$$+\frac{\mathrm{i}}{16}eC_{AB,C}A^C_\nu\left[\bar{\varepsilon}\gamma^\mu\gamma^\nu\gamma_5\psi_\mu\bar{\lambda}^A\lambda^B-\bar{\varepsilon}\gamma^\mu\gamma^\nu\psi_\mu\bar{\lambda}^A\gamma_5\lambda^B\right]\,. \tag{5.50}$$

With help of the rearrangement formulae for spinor bilinears, one finds that (5.48) and (5.50) cancel the first and the fourth term of (5.46). Also in $\mathcal{N}=1$ supergravity all the extra contributions to the supersymmetry variation that were not present in the supersymmetry variation of the supergravity action (4.28) vanish without the need of extra terms (e.g. generalizations of the generalized Chern-Simons terms due to supergravity). The variation of the generalized Chern-Simons terms themselves is not influenced by the transition from rigid supersymmetry to supergravity because it depends only on the vector fields A^C_μ, whose supersymmetry transformations have no gravitino corrections in $\mathcal{N}=1$ supergravity.

When checking the gauge invariance of terms proportional to the gravitino, one finds that neither terms involving the real part of the gauge kinetic function, $\mathrm{Re}\,\mathrm{f}_{AB}$, nor its derivatives violate the gauge invariance of (4.28). The only contributions that violate gauge invariance come from the purely imaginary parts of the gauge kinetic function $\mathrm{Im}\,\mathrm{f}_{AB}$. On the other hand, no extra terms proportional to $\mathrm{Im}\,\mathrm{f}_{AB}$ appear when one goes from rigid supersymmetry to supergravity. Hence, the gauge variation of (4.28) does not contain any gravitino which is consistent with the result that neither the supersymmetry variation of (4.28) nor the generalized Chern-Simons term (5.30) contain gravitini.

Consequently, the method of gauging isometries of the target space as developed in the previous subsection for rigid supersymmetry can be applied straightforwardly to $\mathcal{N} = 1$ supergravity, and anomalies are cancelled in accordance with rigid supersymmetry.

5.4 Reducing to Abelian×semisimple

Semisimple groups do not lead to non-trivial generalized Chern-Simons terms as shown in section 3.2. Furthermore, in section 3.3 we discussed the example of the direct product of an Abelian gauge group with a semisimple gauge group. Now we want to further restrict ourselves to the product of a *one-dimensional* Abelian factor and a semisimple group, denoted by G: $U(1) \times G$. This will allow us to clarify the relation between the results developed here and in previous work, in particular [81, 82]. In these papers, the authors study the structure of quantum consistency conditions of $\mathcal{N} = 1$ supergravity. More precisely, they clarify the anomaly cancellation conditions (required by the quantum consistency) for a $U(1) \times G$ gauge group. We introduce the notations $F_{\mu\nu}$ and $\mathcal{G}^x_{\mu\nu}$ for the Abelian and semisimple field strengths, respectively.

In this case, one can look at "mixed" anomalies, which are the ones proportional to $\mathrm{Tr}(QT_xT_y)$, where Q is the $U(1)$ charge operator and T_x are the generators of the semisimple algebra. Following [82, Sect.2.2], one can add counterterms represented by $\mathcal{L}_{\rm ct}$ such that the mixed anomalies proportional to Λ_x cancel and one remains with those that are of the form $\Lambda_0 \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,(Q\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma})$, where Λ_0 is the Abelian gauge parameter. Schematically, this corresponds to

$$\begin{array}{r|ccc} \text{Anomalies:} & \Lambda_x \mathcal{A}^x_{\text{mixed con}} & + & \Lambda_0 \mathcal{A}^0_{\text{mixed con}} \\ \hline \delta(\Lambda)\mathcal{L}_{\rm ct}: & -\Lambda_x \mathcal{A}^x_{\text{mixed con}} & - & \Lambda_0 \mathcal{A}^0_{\text{mixed con}} \\ & & + & \Lambda_0 \mathcal{A}^0_{\text{mixed cov}} \\ \hline\hline \text{sum:} & 0 & + & \Lambda_0 \mathcal{A}^0_{\text{mixed cov}} \end{array} \tag{5.51}$$

where the subscripts "con" and "cov" denote the consistent and covariant anomalies, respectively. The counterterms $\mathcal{L}_{\rm ct}$ have the following form:

$$\mathcal{L}_{\rm ct} = \tfrac{1}{3} Z \varepsilon^{\mu\nu\rho\sigma} C_\mu\, \mathrm{Tr}\left[Q \left(A_\nu \partial_\rho A_\sigma + \tfrac{3}{4} A_\nu A_\rho A_\sigma \right) \right], \qquad Z = \frac{1}{4\pi^2}, \tag{5.52}$$

where C_μ and A_μ are the gauge fields for the Abelian and semisimple gauge groups respectively. The expressions for the anomalies are:

$$\begin{aligned}
\mathcal{A}^x_{\text{mixed con}} &= -\tfrac{1}{3} Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big[T^x Q\partial_\mu\left(C_\nu\partial_\rho W_\sigma + \tfrac{1}{4}C_\nu W_\rho W_\sigma\right)\Big]\,,\\
\mathcal{A}^0_{\text{mixed con}} &= -\tfrac{1}{6} Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big[Q\partial_\mu\left(W_\nu\partial_\rho W_\sigma + \tfrac{1}{2}W_\nu W_\rho W_\sigma\right)\Big]\,,\\
\mathcal{A}^0_{\text{mixed cov}} &= -\tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big[Q\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}\Big]\,.
\end{aligned} \tag{5.53}$$

The remaining anomaly $\mathcal{A}^0_{\text{mixed cov}}$ is typically cancelled by the Green-Schwarz mechanism.

This will be now compared with the results of the current section and section 3 reduced to the case $U(1)\times G$. The index A is split into 0 for the $U(1)$ and x for the semisimple group generators. We expect the generalized Chern-Simons terms (5.30) to be equivalent to the counterterms in [82] and the role of the Green-Schwarz mechanism is played by a $U(1)$ variation of the kinetic terms f_{xy}, hence by a C-tensor with non-trivial components $C_{xy,0}$.

The discussion that led to (3.45) and (3.46) can be transferred to the present case and it follows that

$$C_{0x,0} = C_{00,x} = 0\,. \tag{5.54}$$

The $C_{xy,0}$'s are proportional to the Cartan-Killing metric in each simple factor as explained in section 3.3 and we write here

$$C_{xy,0} = Z\,\mathrm{Tr}(QT_xT_y)\,, \tag{5.55}$$

where Z could be arbitrary, but our results will match the results of [82] for the value of Z in (5.52). Note that this is in total agreement with section 3.3

If we do not allow for off-diagonal elements of the gauge kinetic function f_{AB}, we have

$$f_{0x} = 0 \quad\Rightarrow\quad C_{0x,y} = 0\,. \tag{5.56}$$

The components $C_{00,0}$ and $C_{xy,z}$ may be nonzero, but here we shall be only concerned with the mixed components, i.e. we have only (5.55) different from zero.

If we reduce the gauge variation $\delta_\Lambda\big(\mathrm{Im}\,\mathrm{f}_{AB}F^A\wedge F^B\big)$ using (5.54) and (5.55), we obtain

$$\Big[\delta(\Lambda)\hat{S}_f\Big]_{\text{mixed}} = \int \mathrm{d}^4x\Big[\tfrac{1}{8}Z\Lambda_0\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(Q\mathcal{G}_{\mu\nu}\mathcal{G}_{\rho\sigma}\right)\Big]\,. \tag{5.57}$$

It is suitable to split (5.55) into a totally symmetric and a part of mixed symmetry, which leads to

$$C^{(\mathrm{s})}_{xy,0} = C^{(\mathrm{s})}_{0x,y} = \tfrac{1}{3}C_{xy,0} = \tfrac{1}{3}Z\,\mathrm{Tr}(QT_xT_y)\,,$$
$$C^{(\mathrm{m})}_{xy,0} = \tfrac{2}{3}C_{xy,0} = \tfrac{2}{3}Z\,\mathrm{Tr}(QT_xT_y)\,, \qquad C^{(\mathrm{m})}_{0x,y} = -\tfrac{1}{3}C_{xy,0} = -\tfrac{1}{3}Z\,\mathrm{Tr}(QT_xT_y)\,. \quad (5.58)$$

Note that this is consistent with the discussion in section 3.4, i.e. with the equations (3.59) to (3.61). In the previous sections, it was shown that for a final gauge and supersymmetry invariant theory the mixed symmetric part has to be identified with the constant tensor in front of the generalized Chern-Simons term, i.e. $C^{\mathrm{CS}} = C^{(\mathrm{m})}$. Therefore, the mixed part of the generalized Chern-Simons term, (5.30), becomes in this case:

$$[S_{\mathrm{CS}}]_{\mathrm{mixed}} \;=\; \int \mathrm{d}^4x \left[\tfrac{1}{3}ZC_\mu\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[Q\left(A_\nu\partial_\rho A_\sigma + \tfrac{3}{4}A_\nu A_\rho A_\sigma\right)\right]\right], \qquad (5.59)$$

which matches (5.52) and is consistent with equation (3.49).

Finally, from reducing the consistent anomaly (5.41) we find, using $d_{ABC} = C^{(\mathrm{s})}_{ABC}$, that the mixed anomalies are given by

$$\mathcal{A}^0 \;=\; -\tfrac{1}{6}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[Q\partial_\mu\left(A_\nu\partial_\rho A_\sigma + \tfrac{1}{2}A_\nu A_\rho A_\sigma\right)\right]\,,$$
$$\mathcal{A}^x \;=\; -\tfrac{1}{3}Z\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left[T^xQ\partial_\mu\left(C_\nu\partial_\rho A_\sigma + \tfrac{1}{4}C_\nu A_\rho A_\sigma\right)\right]\,, \qquad (5.60)$$

which match exactly (5.53).

Let us summarize the results of our comparison with [82]:

(i) The mixed part of the GCS action (5.59) is indeed equal to the counterterms (5.52), introduced in [82].

(ii) The consistent anomalies (5.60) match those in the first two lines of (5.53). As we mentioned above, the counterterm has modified the resulting anomaly to the covariant form in the last line of (5.53).

(iii) We see that the variation of the kinetic term for the vector fields (5.57) may cancel this mixed covariant anomaly (this is the Green-Schwarz mechanism).

Taking all together, we can summarize the cancellation procedure schematically as follows:

$$\begin{array}{r|ccc}
\text{Anomalies:} & \Lambda_x \mathcal{A}^x_{\text{mixed con}} & + & \Lambda_0 \mathcal{A}^0_{\text{mixed con}} \\
\hline
\delta(\Lambda)\mathcal{L}_{(\text{CS})} : & -\Lambda_x \mathcal{A}^x_{\text{mixed con}} & - & \Lambda_0 \mathcal{A}^0_{\text{mixed con}} \\
 & & + & \Lambda_0 \mathcal{A}^0_{\text{mixed cov}} \\
\hline
\delta(\Lambda)\hat{S}_f : & & - & \Lambda_0 \mathcal{A}^0_{\text{mixed cov}} \\
\hline\hline
\text{sum:} & 0 & + & 0
\end{array} \qquad (5.61)$$

5.5 Summary

In the beginning of this section we showed that gauged isometries on the target space of scalar fields modified the supersymmetry transformations of the gauge supermultiplet found in section 4.1. We had to extend the partial derivative in the supersymmetry transformation $\delta\chi^i_L$ to a covariant derivative (5.7) and to introduce the term $\bar{\varepsilon}_R \lambda^A_R k^i_A$ into the supersymmetry transformation of F^i according to [84]. We know from the discussion in section 3 that the gauge transformation $\delta_\Lambda z^i$ (5.1) in general causes a gauge variation of the Peccei-Quinn-type term $\operatorname{Im} \mathrm{f}_{AB} \mathcal{F}^A \mathcal{F}^B$, which may be cancelled in certain cases by a generalized Chern-Simons term. This motivated to add a term to the gauge sector of global $\mathcal{N} = 1$ supersymmetry, that is equal to the extension of the generalized Chern-Simons term to $\mathcal{N} = 1$ supersymmetry presented in [26]. The new term consists of the usual bosonic Chern-Simons term (5.30) together with the fermionic term

$$\frac{\mathrm{i}}{4} C_{AB,C} A^C_\mu \bar{\lambda}^A \gamma_5 \gamma^\mu \lambda^B \,, \qquad (5.62)$$

where $C_{AB,C}$ is mixed symmetric in its indices. We showed that if the gauged isometries induce an imaginary shift in the gauge kinetic function (5.27), then the variations of the gauge sector, the generalized Chern-Simons terms and the fermionic term (5.62) under supersymmetry cancel provided the constraint $C_{(AB,C)} = 0$ holds. If ont the other hand, $C_{(AB,C)} \neq 0$, it is suitable to use the fermionic term in order to define the gauge covariant derivative

$$D_\mu \operatorname{Im} \mathrm{f}_{AB} \;=\; \partial_\mu \operatorname{Im} \mathrm{f}_{AB} - 2A^C_\mu f_{C(A}{}^D \operatorname{Im} \mathrm{f}_{B)D} - \mathrm{i} A^C_\mu C_{AB,C} \,, \qquad (5.63)$$

and not to add it to the generalized Chern-Simons term (5.30). Note that now there is the full tensor $C_{AB,C}$ in equation (5.63), i.e. $C_{AB,C} = C^{(s)}_{AB,C} + C^{(m)}_{AB,C}$.

Now that we have relaxed the constraint $C_{(AB,C)} = 0$ and allowed for a nontrivial totally symmetric part $C_{(AB,C)}$, this causes new contributions to the gauge and supersymmetry variations that no longer vanish. The important observation is that the gauge and supersymmetry non-invariance induced by $C_{(AB,C)} \neq 0$ can only be cancelled if there are gauge and supersymmetry anomalies and we demand

$$C_{(AB,C)} \quad = \quad d_{ABC}\,, \tag{5.64}$$

where the symmetric tensor d_{ABC} characterizes the anomaly.

After performing the analysis in global $\mathcal{N}=1$ supersymmetry, we could extend our results to $\mathcal{N}=1$ supergravity. It turns out that the generalized Chern-Simons term (5.30) does not need any gravitino correction and can thus be added as such to matter-coupled supergravity actions.

Thus, the results of this section provide an extension to the general framework of coupled chiral and vector multiplets in global and local $\mathcal{N}=1$ supersymmetry to include the general form of gauged axionic shifts, generalized Chern-Simons terms and anomalies.[31]

[31] We should emphasize that we only considered anomalies of gauge symmetries that are gauged by elementary vector fields. The interplay with Kähler anomalies in supergravity theories can be an involved subject [81, 82], which was not studied. Also we did not consider gravitational anomalies.

6 Symplectically covariant formalism and anomalies in chiral gauge theories

In this section we introduce a formulation of chiral gauge theories which is manifestly covariant with respect to electric/magnetic duality. For anomaly-free gauge theories as they occur in extended supergravity, this formulation was first presented in [83]. Maintaining covariance at each step is achieved by introducing the so-called embedding tensor. A set of constraints on the embedding tensor and extra gauge invariances make sure that the degrees of freedom remain unchanged. We will see that in addition to the usual gauge variations of gauge theory extra gauge variations appear which cause violations of the Bianchi identity and the Jacobi identity. Consequently, the field strength tensor corresponding to the vector fields will no longer transform covariantly. Therefore, the authors of [83] introduce tensor fields which transform under the gauge variation such that the combination of the field strength tensor together with the tensor fields transforms covariantly again. For this to work, one has to add two topological terms in order to obtain a gauge invariant action that is invariant with respect to the gauge transformations. The gauge invariance relies heavily on the set of constraints of the embedding tensor given in [83]. We will show that it is possible to relax one of these constraints in order to a allow for a nontrivial totally symmetric tensor. We will display how this totally symmetric tensor leads to a gauge non-invariance of the Lagrangian given in [83]. We will further show how one can cancel this gauge non-invariance by gauge anomalies, if the totally symmetric tensor describes anomalies in a symplectically covariant way and give the necessary condition. In this sense we can say that the results of this section generalize the Green-Schwarz mechanism [2] to become a "symplectically covariant Green-Schwarz mechanism". In making a special choice for the embedding tensor one recovers the results of the previous chapter for the purely bosonic sector. In subsection 6.4 we give an explicit example that goes beyond the discussion of [83] and show how the relaxation of one constraint allows a possible cancellation by anomalies. This section represents another major topic of this thesis and is based on the work [40].

The outline of this section is as follows. In subsection 6.1 we will give the symplectically covariant framework of [83] in a more general treatment. Then in subsection 6.2 we show how the formalism of [83] has to be modified in order to accommodate quantum anomalies.

In subsection 6.3 we choose purely electric gaugings and obtain back earlier results.We flesh out our results with a simple nontrivial example in subsection 6.4. The main results of this section are summarized in subsection 6.5.

In this section the notation is changed to the one of [40] so as to make the generalization of [83] more transparent.

6.1 Electric/magnetic duality without anomalies

In this subsection we will introduce electric/magnetic duality and display the main results of [83].

6.1.1 Electric/magnetic duality and the conventional gauging

In the absence of charged fields, a gauge invariant four-dimensional Lagrangian of n Abelian vector fields $A_\mu{}^\Lambda(\Lambda = 1, \dots, n)$ only depends on their curls $F_{\mu\nu}{}^\Lambda \equiv 2\partial_{[\mu} A_{\nu]}{}^\Lambda$. Defining the dual magnetic field strengths

$$G_{\mu\nu\,\Lambda} \;\; := \;\; \varepsilon_{\mu\nu\rho\sigma} \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}\,, \tag{6.1}$$

the Bianchi identities and field equations can be brought to the following form

$$\partial_{[\mu} F_{\nu\rho]}{}^\Lambda \;\; = \;\; 0\,, \tag{6.2}$$

$$\partial_{[\mu} G_{\nu\rho]\,\Lambda} \;\; = \;\; 0\,. \tag{6.3}$$

This formulation allows to combine the electric Abelian field strengths, $F_{\mu\nu}{}^\Lambda$, and their magnetic duals, $G_{\mu\nu\,\Lambda}$, into a $2n$-plet, $F_{\mu\nu}{}^M$, such that $F^M = (F^\Lambda, G_\Lambda)$. Therefore, (6.2) and (6.3) can be written in the following compact way:

$$\partial_{[\mu} F_{\nu\rho]}{}^M \;\; = \;\; 0\,. \tag{6.4}$$

It is rather obvious that equation (6.4) is invariant under general linear transformations

$$F^M \to F'^M = \mathcal{S}^M{}_N F^N\,, \;\text{ where } \mathcal{S}^M{}_N \; = \; \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix}\,, \tag{6.5}$$

but a relation of the type (6.1) is only possible for symplectic matrices $\mathcal{S}^M{}_N \in Sp(2n, \mathbb{R})$. Thus, the admissible rotations $\mathcal{S}^M{}_N$ form the group $Sp(2n, \mathbb{R})$:

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega, \tag{6.6}$$

with the symplectic metric, Ω_{MN}, given by

$$\Omega_{MN} = \begin{pmatrix} 0 & \Omega_\Lambda{}^\Sigma \\ \Omega^\Lambda{}_\Sigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_\Lambda^\Sigma \\ -\delta_\Sigma^\Lambda & 0 \end{pmatrix} . \tag{6.7}$$

We define Ω^{MN} via $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Note that the components of Ω^{MN} should not be written as $\Omega^\Lambda{}_\Sigma$ etc., as these differ from (6.7) by the factor of (-1).

Starting point is a kinetic Lagrangian of the form

$$\mathcal{L}_{\rm gk} \;=\; +\frac{1}{4}\,{\rm Im}\,\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}{}^\Lambda F^{\mu\nu\Sigma} - \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,{\rm Re}\,\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}{}^\Lambda F_{\rho\sigma}{}^\Sigma , \tag{6.8}$$

where $\mathcal{N}_{\Lambda\Sigma}$ denotes the gauge kinetic function[32]. Applying an electric/magnetic duality transformation to (6.8) leads to a new Lagrangian, $\mathcal{L}'_{\rm gk}(F')$, which is of a similar form, but with a new gauge kinetic function

$$\mathcal{N}_{\Lambda\Sigma} \;\to\; \mathcal{N}'_{\Lambda\Sigma} = (V\mathcal{N}+W)_{\Lambda\Omega}\big[(U+Z\mathcal{N})^{-1}\big]^\Omega{}_\Sigma . \tag{6.9}$$

The subset of $Sp(2n,\mathbb{R})$ symmetries (of field equations and Bianchi identities) for which the Lagrangian remains unchanged, in the sense that $\mathcal{L}'(F'(F)) = \mathcal{L}(F)$, are invariances of the action. In a different duality frame, the Lagrangian might have a different set of invariances.

From the spacetime point of view, these are all rigid ("global") symmetries and sometimes these global symmetries can be gauged. For the conventional gaugings [26] one has to restrict to the transformations that leave the Lagrangian invariant, which implies that $Z^{\Lambda\Sigma}$ in the matrices $\mathcal{S}^M{}_N$ of (6.5) has to vanish. In the context of symplectically covariant gaugings [83], however, this restriction can be relaxed. We will come back to these more general gaugings in section 6.1.2.

When the symmetry is gauged, covariant derivatives and field strengths are introduced as usual. In the standard way of gauging, this can be implemented solely with the electric vector fields $A_\mu{}^\Omega$ and the corresponding electric gauge parameters Λ^Ω. The gaugeable symplectic transformation, $\mathcal{S}$, must thus be of the infinitesimal form

$$\mathcal{S}^M{}_N = \delta^M{}_N - \Lambda^\Omega \mathcal{S}_\Omega{}^M{}_N . \tag{6.10}$$

[32] The gauge kinetic function f_{AB}, as used so far, corresponds in this section to $-{\rm i}\mathcal{N}^*_{\Lambda\Sigma}$.

According to our definition (6.5), these infinitesimal symplectic transformations act on the field strengths by multiplication with the matrices $\mathcal{S}_\Lambda{}^M{}_N$ from the left. Following the conventions of [83], however, we will use matrices $X_{\Omega M}{}^N$ to describe the infinitesimal symplectic action via multiplication from the right:

$$\delta F_{\mu\nu}{}^M \;=\; F'_{\mu\nu}{}^M - F_{\mu\nu}{}^M = -\Lambda^\Omega F_{\mu\nu}{}^N X_{\Omega N}{}^M\,, \qquad \text{i.e.} \qquad X_{\Omega N}{}^M = S_\Omega{}^M{}_N\,. \tag{6.11}$$

Then, for standard electric gaugings we have the transformation

$$\delta \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\mu\nu\,\Lambda} \end{pmatrix} \;=\; -\Lambda^\Omega \begin{pmatrix} X_{\Omega\Xi}{}^\Lambda & 0 \\ X_{\Omega\Lambda\Xi} & X_\Omega{}^\Xi{}_\Lambda \end{pmatrix} \begin{pmatrix} F^\Xi_{\mu\nu} \\ G_{\mu\nu\,\Xi} \end{pmatrix}, \tag{6.12}$$

where $X_{\Omega\Sigma}{}^\Lambda = -X_\Omega{}^\Lambda{}_\Sigma = f_{\Omega\Sigma}{}^\Lambda$ must be the structure constants of the gauge algebra[33], and $X_{\Sigma\Xi\Gamma} = X_{\Sigma(\Xi\Gamma)}$ would give rise to the axionic shifts[34] mentioned in sections 3 and 5.

Then the gauging proceeds in the usual way by introducing covariant derivatives $(\partial_\mu - A_\mu{}^\Lambda\delta_\Lambda)$, where the δ_Λ are the gauge generators in a suitable representation of the matter fields (see (5.1), for example). One also introduces covariant field strengths and possibly GCS terms as described below. As we assume the absence of quantum anomalies in this subsection, we have to require $X_{(\Lambda\Sigma\Gamma)} = 0$ in accordance with the results found in sections 3 and 5.

6.1.2 The symplectically covariant gauging

We will now turn to the more general gauging of symmetries. The group that will be gauged is a subgroup of the rigid symmetry group. What we mean by the rigid symmetry group is a bit more subtle in $\mathcal{N}=1$ supersymmetry (or theories without supersymmetry) than in extended supersymmetry. This is due to the fact that in extended supersymmetry the vectors are supersymmetrically related to scalar fields, and therefore their rigid symmetries are connected to the symmetries of scalar manifolds.

In $\mathcal{N}=1$ supersymmetry or in theories without supersymmetry, the rigid symmetries of the vector and scalar sector are not directly related. Then the rigid symmetry group, G_{rigid}, is a subset of the product of the symplectic duality transformations that act on the vector fields and the isometry group of the scalar manifold of the chiral multiplets: $G_{\text{rigid}} \subseteq$

[33]In previous sections denoted by $f_{AB}{}^C$.

[34]The shifts $C_{AB,C}$ are translated by $X_{\Lambda\Sigma\Omega} = C_{\Sigma\Omega,\Lambda}$ for the choice made in (6.10).

$Sp(2n,\mathbb{R})\times \mathrm{Iso}(\mathcal{M}_{\mathrm{scalar}})$. In $\mathcal{N}=1$ supergravity, this means that the action of the symmetries is given by elements $(g_1,\, g_2)$ of $Sp(2n,\mathbb{R})\times \mathrm{Iso}(\mathcal{M}_{\mathrm{scalar}})$ that are compatible with (6.9) in the sense that the symplectic action (6.9) of g_1 on the matrix $\mathcal{N}$ is induced by the isometry g_2 on the scalar manifold. These are rigid ("global") symmetries provided they also leave the rest of the theory (deriving from scalar potentials, etc.) invariant [85]. In this sense, the relevant isometries are those that respect the Kähler structure (i.e. the isometries have to be generated by holomorphic Killing vectors) and that also leave the superpotential invariant (in supergravity, the superpotential should transform according to the Kähler transformations).[35]

The generators of G_{rigid} will be denoted by δ_α, $\alpha=1,\dots,\dim(G_{\mathrm{rigid}})$. These generators act on the different fields of the theory either via Killing vectors $\delta_\alpha = K_\alpha = K_\alpha^i\frac{\partial}{\partial\phi^i}$ defining infinitesimal isometries on the scalar manifold, or with certain matrix representations[36], e.g. $\delta_\alpha\phi^i = -\phi^j(t_\alpha)_j{}^i$.

On the field strengths $F_{\mu\nu}{}^M = (F_{\mu\nu}{}^\Lambda,\, G_{\mu\nu\,\Lambda})$, these rigid symmetries must act by multiplication with infinitesimal symplectic matrices[37] $(t_\alpha)_M{}^P$, i.e., we have

$$(t_\alpha)_{[M}{}^P\Omega_{N]P} \;=\; 0\,. \tag{6.13}$$

In order to gauge a subgroup, $G_{\mathrm{local}}\subset G_{\mathrm{rigid}}$, the $2n$-dimensional vector space spanned by the vector fields[38] $A_\mu{}^M$ has to be projected onto the Lie algebra of G_{local}, which is formally done with the so-called embedding tensor $\Theta_M{}^\alpha = (\Theta_\Lambda{}^\alpha, \Theta^{\Lambda\,\alpha})$. Equivalently, $\Theta_M{}^\alpha$ completely determines the gauge group G_{local} via the decomposition of the gauge group generators, which we will denote by $\tilde{X}_M$, into the generators of the rigid invariance group G_{rigid}:

$$\tilde{X}_M \;:=\; \Theta_M{}^\alpha\delta_\alpha. \tag{6.14}$$

[35] Note that this may include cases where either the symplectic transformation g_1 or the isometry g_2 is trivial. Another special case is when the isometry g_2 is non-trivial, but $\mathcal{N}$ does not transform under it, as happens, e.g, when $\mathcal{N}=\mathrm{i}\mathbb{1}$ is constant. G_{rigid} is in general a genuine subgroup of $Sp(2n,\mathbb{R})\times\mathrm{Iso}(\mathcal{M}_{\mathrm{scalar}})$, even in the latter case of constant $\mathcal{N}$.

[36] The structure constants defined by $[\delta_\alpha,\delta_\beta]=f_{\alpha\beta}{}^\gamma\delta_\gamma$ lead for the matrices to $[t_\alpha,t_\beta]=-f_{\alpha\beta}{}^\gamma t_\gamma$.

[37] These matrices might be trivial, e.g., for Abelian symmetry groups that only act on the scalars (and/or the fermions) and that do not give rise to axionic shifts of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$.

[38] The equations of motion (6.3) imply the existence of magnetic gauge potentials, $A_{\mu\,\Lambda}$, via $G_{\mu\nu\,\Lambda} = 2\partial_{[\mu}A_{\nu]\Lambda}$. The magnetic gauge potentials obtained in this way are in turn related to the electric vector potentials, $A_\mu{}^\Lambda$, by nonlocal field redefinitions. The electric and magnetic vector fields can be combined into a $2n$-plet, A_μ^M, such that $A^M=(A^\Lambda,A_\Lambda)$.

The gauge generators $\tilde{X}_M$ enter the gauge covariant derivatives of matter fields,

$$\mathcal{D}_\mu = \partial_\mu - A_\mu{}^M \tilde{X}_M = \partial_\mu - A_\mu{}^\Lambda \Theta_\Lambda{}^\alpha \delta_\alpha - A_{\mu\Lambda} \Theta^{\Lambda\alpha} \delta_\alpha \,, \tag{6.15}$$

where the generators δ_α are meant to either act as representation matrices on the fermions or as Killing vectors on the scalar fields, as mentioned above. On the field strengths of the vector potentials, the generators δ_α act by multiplication with the matrices $(t_\alpha)_N{}^P$, so that (6.14) is represented by matrices $(X_M)_N{}^P$ whose elements we denote as $X_{MN}{}^P$ and whose antisymmetric part in the lower indices appears in the field strengths

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M + X_{[NP]}{}^M A_\mu{}^N A_\nu{}^P \,, \qquad X_{NP}{}^M = \Theta_N{}^\alpha (t_\alpha)_P{}^M \,. \tag{6.16}$$

The symplectic property (6.13) implies

$$X_{M[N}{}^Q \Omega_{P]Q} = 0 \,, \qquad X_{MQ}{}^{[N} \Omega^{P]Q} = 0 \,. \tag{6.17}$$

In the remainder of this paper, the symmetrized contraction $X_{(MN}{}^Q \Omega_{P)Q}$ will play an important rôle. We therefore give this tensor a special name and denote it by D_{MNP}:

$$D_{MNP} \;\; = \;\; X_{(MN}{}^Q \Omega_{P)Q} \,. \tag{6.18}$$

Note that this is really just a definition and no new constraint. Using the definition (6.18), one can check that

$$\begin{aligned} & 2X_{(MN)}{}^Q \Omega_{RQ} + X_{RM}{}^Q \Omega_{NQ} = 3D_{MNR} \,, \\ \text{i.e.} \qquad & X_{(MN)}{}^P = \tfrac{1}{2} \Omega^{PR} X_{RM}{}^Q \Omega_{NQ} + \tfrac{3}{2} D_{MNR} \Omega^{RP} \,. \end{aligned} \tag{6.19}$$

6.1.2.1 Constraints on the embedding tensor The embedding tensor $\Theta_M{}^\alpha$ has to satisfy a number of consistency conditions. Closure of the gauge algebra and locality require, respectively, the quadratic constraints

$$\text{closure:} \qquad f_{\alpha\beta}{}^\gamma \Theta_M{}^\alpha \Theta_N{}^\beta = (t_\alpha)_N{}^P \Theta_M{}^\alpha \Theta_P{}^\gamma \,, \tag{6.20}$$

$$\text{locality:} \qquad \Omega^{MN} \Theta_M{}^\alpha \Theta_N{}^\beta = 0 \quad \Leftrightarrow \quad \Theta^{\Lambda[\alpha} \Theta_\Lambda{}^{\beta]} = 0 \,, \tag{6.21}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants of the rigid invariance group G_{rigid}, see footnote 36. The constraint (6.20) also expresses the invariance of the embedding tensor under G_{rigid}.

Another constraint, besides (6.20) and (6.21), was inferred in [83] from supersymmetry constraints in $\mathcal{N}=8$ supergravity

$$D_{MNR} \equiv X_{(MN}{}^{Q}\Omega_{R)Q} = 0\,. \tag{6.22}$$

This constraint eliminates some of the representations of the rigid symmetry group and is therefore sometimes called the "representation constraint". One can actually show that the locality constraint is not independent of (6.20) and (6.22), apart from specific cases where $(t_\alpha)_M{}^N$ has a trivial action on the vector fields.

However, we will neither use the locality constraint (6.21) nor the representation constraint (6.22). We will, instead, need another constraint in section 6.1.2.4, whose meaning we will discuss in section 6.2. Before coming to that new constraint, we thus only use the closure constraint (6.20). This constraint reflects the invariance of the embedding tensor under $G_{\rm local}$ and it implies for the matrices X_M the relation

$$[X_M, X_N] = -X_{MN}{}^P X_P\,. \tag{6.23}$$

This clearly shows that the gauge group generators commute into each other with 'structure constants' given by $X_{[MN]}{}^P$. In general, $X_{MN}{}^P$ also contains a non-trivial symmetric part, $X_{(MN)}{}^P$. The antisymmetry of the left hand side of (6.23) only requires that the contraction $X_{(MN)}{}^P\Theta_P{}^\alpha$ vanishes, as is also directly visible from (6.20). Therefore one has

$$X_{(MN)}{}^P\Theta_P{}^\alpha = 0 \qquad \rightarrow \qquad X_{(MN)}{}^P X_{PQ}{}^R = 0\,. \tag{6.24}$$

Writing out (6.23) explicitly gives

$$X_{MQ}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{MP}{}^R + X_{MN}{}^P X_{PQ}{}^R = 0\,. \tag{6.25}$$

Antisymmetrizing in $[MNQ]$, we can split the second factor of each term into the antisymmetric and symmetric part, $X_{MN}{}^P = X_{[MN]}{}^P + X_{(MN)}{}^P$, and this gives a violation of the Jacobi identity for $X_{[MN]}{}^P$ as

$$\begin{aligned} & X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R \\ & = -\tfrac{1}{3}\left(X_{[MN]}{}^P X_{(QP)}{}^R + X_{[QM]}{}^P X_{(NP)}{}^R + X_{[NQ]}{}^P X_{(MP)}{}^R\right)\,. \end{aligned} \tag{6.26}$$

Other relevant consequences of (6.25) can be obtained by (anti)symmetrizing in MQ. This gives, using also (6.24), the two equations

$$X_{(MQ)}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{(MP)}{}^R - X_{NM}{}^P X_{(QP)}{}^R = 0\,,$$
$$X_{[MQ]}{}^P X_{NP}{}^R - X_{NQ}{}^P X_{[MP]}{}^R + X_{NM}{}^P X_{[QP]}{}^R = 0\,. \tag{6.27}$$

6.1.2.2 Gauge transformations An important consequence of the nonvanishing symmetric part $X_{(MN)}{}^P$ is the violation of the Jacobi identity (6.26). This is the prize one has to pay for the symplectically covariant treatment in which both electric and magnetic vector potentials appear at the same time. In order to compensate for this violation and in order to make sure that the number of propagating degrees of freedom is the same as before, one imposes an additional gauge invariance in addition to the usual non-Abelian transformation $\partial_\mu \Lambda^M + X_{[PQ]}{}^M A_\mu{}^P \Lambda^Q$ and extends the gauge transformation of the vector potentials to

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M - X_{(NP)}{}^M \Xi_\mu{}^{NP}\,, \qquad \mathcal{D}_\mu \Lambda^M = \partial_\mu \Lambda^M + X_{PQ}{}^M A_\mu{}^P \Lambda^Q\,, \tag{6.28}$$

where we introduced the covariant derivative $\mathcal{D}_\mu \Lambda^M$, and new vector-like gauge parameters $\Xi_\mu{}^{NP}$, symmetric in the upper indices. The extra terms $X_{(PQ)}{}^M A_\mu{}^P \Lambda^Q$ and the Ξ-transformations contained in (6.28) allow one to gauge away the vector fields that correspond to the directions in which the Jacobi identity is violated, i.e., directions in the kernel of the embedding tensor (see (6.24)).

It is important to notice that the modified gauge transformations (6.28) still close on the gauge fields and thus form a Lie algebra. Indeed, if we split (6.28) into two parts,

$$\delta A_\mu{}^M = \delta(\Lambda) A_\mu{}^M + \delta(\Xi) A_\mu{}^M\,, \tag{6.29}$$

the commutation relations are

$$[\delta(\Lambda_1), \delta(\Lambda_2)]\, A_\mu{}^M = \delta(\Lambda_3) A_\mu{}^M + \delta(\Xi_3) A_\mu{}^M,$$
$$[\delta(\Lambda), \delta(\Xi)]\, A_\mu{}^M = [\delta(\Xi_1), \delta(\Xi_2)]\, A_\mu{}^M = 0\,, \tag{6.30}$$

with

$$\Lambda_3^M = X_{[NP]}{}^M \Lambda_1^N \Lambda_2^P\,,$$
$$\Xi_{3\mu}{}^{PN} = \Lambda_1^{(P} \mathcal{D}_\mu \Lambda_2^{N)} - \Lambda_2^{(P} \mathcal{D}_\mu \Lambda_1^{N)}\,. \tag{6.31}$$

To prove that the terms that are quadratic in the matrices X_M in the left-hand side of (6.30) follow this rule, one uses (6.27). Due to (6.24) and (6.28), however, the usual properties of the field strength

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + X_{[PQ]}{}^M A_\mu{}^P A_\nu{}^Q \tag{6.32}$$

are changed. In particular, it will no longer fulfill the Bianchi identity, which now must be replaced by

$$\mathcal{D}_{[\mu}\mathcal{F}_{\nu\rho]}{}^M \quad = \quad X_{(NP)}{}^M A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P - \frac{1}{3}X_{(PN)}{}^M X_{[QR]}{}^P \; A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R\,. \tag{6.33}$$

Furthermore, $\mathcal{F}_{\mu\nu}{}^M$ does not transform covariantly under a gauge transformation (6.28). Instead, we have

$$\begin{aligned}\delta\mathcal{F}_{\mu\nu}{}^M &= 2\mathcal{D}_{[\mu}\delta A_{\nu]}{}^M - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q \\ &= X_{NQ}{}^M \; \mathcal{F}_{\mu\nu}{}^N \Lambda^Q - 2X_{(NP)}{}^M \mathcal{D}_{[\mu}\Xi_{\nu]}{}^{NP} - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q\,,\end{aligned} \tag{6.34}$$

where the covariant derivative is (both expressions are useful and related by (6.27))

$$\begin{aligned}X_{(NP)}{}^M \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu \left(X_{(NP)}{}^M \Xi_\nu{}^{NP}\right) + A_\mu{}^R X_{RQ}{}^M X_{(NP)}{}^Q \Xi_\nu{}^{NP}\,,\\ \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu \Xi_\nu{}^{NP} + X_{QR}{}^P A_\mu{}^Q \Xi_\nu{}^{NR} + X_{QR}{}^N A_\mu{}^Q \Xi_\nu{}^{PR}\,.\end{aligned} \tag{6.35}$$

Therefore, if we want to deform the gauge kinetic Lagrangian $\mathcal{L}_{\rm gk}$ and accommodate electric and magnetic gauge fields, $\mathcal{F}_{\mu\nu}{}^M$ cannot be used to construct gauge-covariant kinetic terms.

For this reason, the authors of [83] introduced tensor fields $B_{\mu\nu\,\alpha}$, later in [86] to be described by $B_{\mu\nu}{}^{MN}$, symmetric in (MN), and with them modified field strengths

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + X_{(NP)}{}^M B_{\mu\nu}{}^{NP}\,. \tag{6.36}$$

We will consider gauge transformations of the antisymmetric tensors of the form

$$\delta B_{\mu\nu}{}^{NP} = 2\mathcal{D}_{[\mu}\Xi_{\nu]}{}^{NP} + 2A_{[\mu}{}^{(N}\delta A_{\nu]}{}^{P)} + \Delta B_{\mu\nu}{}^{NP}\,, \tag{6.37}$$

where $\Delta B_{\mu\nu}{}^{NP}$ depends on the gauge parameter Λ^Q, but we do not fix it further at this point. Together with (6.34), this then implies[39]

$$\delta\mathcal{H}_{\mu\nu}{}^M = X_{NQ}{}^M \Lambda^Q \mathcal{H}_{\mu\nu}{}^N + X_{(NP)}{}^M \Delta B_{\mu\nu}{}^{NP}\,. \tag{6.38}$$

[39]Note that $\mathcal{F}_{\mu\nu}{}^N$ in the second line of (6.34) can be replaced by $\mathcal{H}_{\mu\nu}{}^N$ due to (6.24).

6.1.2.3 The kinetic Lagrangian As the field strength does not transform covariantly anymore, the Lagrangian (6.8) cannot be invariant. Invariance can be restored in extending (6.8) as we will show now. The first step towards a gauge invariant action is to replace ${\cal F}_{\mu\nu}{}^{\Lambda}$ in ${\cal L}_{\rm g.k.}$, (6.8), by ${\cal H}_{\mu\nu}{}^{\Lambda}$ because if $\Delta B_{\mu\nu}{}^{NP}=0$, then ${\cal H}^M_{\mu\nu}$ transforms covariantly under (6.28). So in this case the new kinetic Lagrangian

$$
{\cal L}_{\rm g.k.} = \tfrac14 e{\cal I}_{\Lambda\Sigma}{\cal H}_{\mu\nu}{}^{\Lambda}{\cal H}^{\mu\nu\Sigma} - \tfrac18{\cal R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}{\cal H}_{\mu\nu}{}^{\Lambda}{\cal H}_{\rho\sigma}{}^{\Sigma}\,, \tag{6.39}
$$

is indeed invariant. Here again ${\cal I}_{\Lambda\Sigma}$ and ${\cal R}_{\Lambda\Sigma}$ denote, respectively, ${\rm Im}\,{\cal N}_{\Lambda\Sigma}$ and ${\rm Re}\,{\cal N}_{\Lambda\Sigma}$. The dual field strength to ${\cal H}^{\Lambda}_{\mu\nu}$ is given by

$$
{\cal G}_{\mu\nu\,\Lambda} \equiv \varepsilon_{\mu\nu\rho\sigma}\frac{\partial{\cal L}}{\partial{\cal H}_{\rho\sigma}{}^{\Lambda}} = {\cal R}_{\Lambda\Gamma}{\cal H}_{\mu\nu}{}^{\Gamma} + \frac12 e\varepsilon_{\mu\nu\rho\sigma}\,{\cal I}_{\Lambda\Gamma}\,{\cal H}^{\rho\sigma\,\Gamma}\,, \tag{6.40}
$$

and, consequently, the Lagrangian and its transformations can be written as

$$
\begin{aligned}
{\cal L}_{\rm g.k.} &= -\tfrac18\varepsilon^{\mu\nu\rho\sigma}{\cal H}^{\Lambda}_{\mu\nu}{\cal G}_{\rho\sigma\,\Lambda}\,,\\
\delta{\cal L}_{\rm g.k.} &= -\tfrac14\varepsilon^{\mu\nu\rho\sigma}{\cal G}_{\mu\nu\,\Lambda}\delta{\cal H}^{\Lambda}_{\rho\sigma}\\
&\quad +\tfrac18\varepsilon^{\mu\nu\rho\sigma}\Lambda^Q\left({\cal H}^{\Lambda}_{\mu\nu}X_{Q\Lambda\Sigma}{\cal H}^{\Sigma}_{\rho\sigma} - 2{\cal H}^{\Lambda}_{\mu\nu}X_{Q\Lambda}{}^{\Sigma}{\cal G}_{\rho\sigma\,\Sigma} - {\cal G}_{\mu\nu\,\Lambda}X_Q{}^{\Lambda\Sigma}{\cal G}_{\rho\sigma\,\Sigma}\right)\,,
\end{aligned} \tag{6.41}
$$

In the third line, we used the infinitesimal form of (6.9):

$$
\delta(\Lambda){\cal N}_{\Lambda\Sigma} \;=\; \Lambda^M\Big[-X_{M\Lambda\Sigma} + 2\,X_{M(\Lambda}{}^{\Gamma}{\cal N}_{\Sigma)\Gamma} + {\cal N}_{\Lambda\Gamma}\,X_M{}^{\Gamma\Xi}\,{\cal N}_{\Xi\Sigma}\Big]\,. \tag{6.42}
$$

The second line of (6.41) can be rewritten as a covariant expression when

$$
{\cal G}_{\mu\nu}{}^{M} = \left({\cal G}_{\mu\nu}{}^{\Lambda}\,,\,{\cal G}_{\mu\nu\Lambda}\right) \quad \text{with} \quad {\cal G}_{\mu\nu}{}^{\Lambda} \equiv {\cal H}_{\mu\nu}{}^{\Lambda}\,, \tag{6.43}
$$

is introduced. In using (6.38), we obtain for the variation of the gauge kinetic Lagrangian

$$
\begin{aligned}
\delta{\cal L}_{\rm g.k.} = \varepsilon^{\mu\nu\rho\sigma}\quad &\left[-\tfrac14{\cal G}_{\mu\nu\,\Lambda}\left(\Lambda^Q X_{PQ}{}^{\Lambda}{\cal H}_{\rho\sigma}{}^{P} + X_{(NP)}{}^{\Lambda}\Delta B_{\rho\sigma}{}^{NP}\right)\right.\\
&\left. +\tfrac18{\cal G}_{\mu\nu}{}^{M}{\cal G}_{\rho\sigma}{}^{N}\Lambda^Q X_{QM}{}^{R}\Omega_{NR}\right]\,.
\end{aligned} \tag{6.44}
$$

Even if $\Delta B_{\mu\nu}{}^{NP}=0$, the newly proposed form for ${\cal L}_{\rm g.k.}$ in (6.39) is still not gauge invariant. This should not come as a surprise because (6.42) contains a constant shift (i.e., the term proportional to $X_{M\Lambda\Sigma}$), which requires the addition of extra terms to the Lagrangian (in section 5 and 3 we had to add the generalized Chern-Simons terms to absorb constant shifts in the gauge kinetic function). Also the last term on the right hand side of (6.42) gives extra contributions that are quadratic in the kinetic function. In the next steps we will see that besides GCS terms, also terms linear and quadratic in the tensor field are required to restore gauge invariance. We start with the discussion of the latter terms.

6.1.2.4 Topological terms for the B-field and a new constraint The second step towards gauge invariance is made by adding topological terms linear and quadratic in the tensor field ${B_{\mu\nu}}^{NP}$ to the gauge kinetic term (6.39), namely

$$\mathcal{L}_{\text{top},B} = \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\, {X_{(NP)}}^{\Lambda}\, {B_{\mu\nu}}^{NP} \left(\mathcal{F}_{\rho\sigma\,\Lambda} + \tfrac{1}{2}\, X_{(RS)\Lambda}\, {B_{\rho\sigma}}^{RS}\right). \tag{6.45}$$

Note that this term vanishes for purely electric gaugings because there one has ${X_{(NP)}}^{\Lambda} = 0$ (as can be seen from the discussion around (6.12)). Consequently, the tensor fields decouple from the theory in electric gaugings.

We recall that, up to now, only the closure constraint (6.20) has been used. Now, however, one new but not independent constraint is imposed:

$$ {X_{(NP)}}^{M}\Omega_{MQ}{X_{(RS)}}^{Q} = 0\,. \tag{6.46}$$

It will be shown later that this constraint is actually implied by the locality constraint (6.21) and the original representation constraint of [83], i.e. (6.22). As it turns out, even the relaxation of the constraint (6.22) to allow for nontrivial $D_{MNR} \neq 0$ will still imply (6.46). The constraint (6.46) simply means that

$$ {X_{(NP)}}^{\Lambda}X_{(RS)\Lambda} = X_{(NP)\Lambda}{X_{(RS)}}^{\Lambda}\,. \tag{6.47}$$

A consequence of this constraint that is quite useful for computations follows from the first of (6.19) and (6.24):

$$ {X_{(PQ)}}^{R}D_{MNR} = 0\,. \tag{6.48}$$

The variation of $\mathcal{L}_{\text{top},B}$ is

$$\begin{aligned}
\delta\mathcal{L}_{\text{top},B} &= \tfrac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}{X_{(NP)}}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta {B_{\rho\sigma}}^{NP} + {B_{\rho\sigma}}^{NP}\delta\mathcal{F}_{\mu\nu\Lambda}\right] \\
&= \tfrac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}{X_{(NP)}}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta {B_{\rho\sigma}}^{NP} + 2{B_{\rho\sigma}}^{NP}\left(\mathcal{D}_{\mu}\delta A_{\nu\Lambda} - X_{(RS)\Lambda}A_{\mu}^{R}\delta A_{\nu}^{S}\right)\right]\,.
\end{aligned} \tag{6.49}$$

6.1.2.5 Generalized Chern-Simons terms If there is a constant shift by $X_{M\Lambda\Sigma}$ in (6.42) we know from the arguments in section 5 that generalized Chern-Simons terms are necessary. In [83], the authors introduced a generalized Chern-Simons term of the form (these are the last two lines in what they called $\mathcal{L}_{top}$ in their equation (4.3))

$$\mathcal{L}_{\rm GCS} = \varepsilon^{\mu\nu\rho\sigma} A_\mu{}^M A_\nu{}^N \left(\frac{1}{3} X_{MN\,\Lambda}\,\partial_\rho A_\sigma{}^\Lambda + \frac{1}{6} X_{MN}{}^\Lambda \partial_\rho A_{\sigma\Lambda} + \frac{1}{8} X_{MN\,\Lambda} X_{PQ}{}^\Lambda A_\rho{}^P A_\sigma{}^Q\right) . \tag{6.50}$$

Using (6.25) antisymmetrized in $[MNQ]$ and the definition of D_{MNP} in (6.18), one can write its variation as

$$\begin{aligned}\delta\mathcal{L}_{\rm GCS} \;&= \varepsilon^{\mu\nu\rho\sigma} \;\; \left[\tfrac{1}{2}\mathcal{F}_{\mu\nu}{}^\Lambda \mathcal{D}_\rho \delta A_{\sigma\Lambda} - \tfrac{1}{2}\mathcal{F}_{\mu\nu\Lambda} X_{(NP)}{}^\Lambda A_\rho{}^N \delta A_\sigma{}^P \right.\\ &\quad \left. - D_{MNP} A_\mu{}^M \delta A_\nu{}^N \left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8} X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\right] .\end{aligned} \tag{6.51}$$

modulo total derivatives. Finally, combining the variation of the generalized Chern-Simons term with (6.49) results in

$$\begin{aligned}\delta\left(\mathcal{L}_{{\rm top},B} + \mathcal{L}_{\rm GCS}\right) \;&=\; \varepsilon^{\mu\nu\rho\sigma} \left[\tfrac{1}{2}\mathcal{H}_{\mu\nu}{}^\Lambda \mathcal{D}_\rho \delta A_{\sigma\Lambda} + \tfrac{1}{4}\mathcal{H}_{\mu\nu\Lambda} X_{(NP)}{}^\Lambda \left(\delta B_{\rho\sigma}{}^{NP} - 2 A_\rho{}^N \delta A_\sigma{}^P\right)\right.\\ &\quad \left. - D_{MNP} A_\mu{}^M \delta A_\nu{}^N \left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8} X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\right] .\end{aligned} \tag{6.52}$$

6.1.2.6 Variation of the total action The results of the previous paragraphs allow us to discuss the symmetry variation of the total Lagrangian

$$\mathcal{L}_{VT} = \mathcal{L}_{\rm g.k.} + \mathcal{L}_{{\rm top},B} + \mathcal{L}_{\rm GCS} , \tag{6.53}$$

built from (6.39), (6.45) and (6.50). In agreement with [83] we will find that (6.53) is indeed invariant under (6.28). In order to see this, we first check the invariance of (6.53) with respect to the Ξ-transformations. One can see directly from (6.44) that the gauge-kinetic terms are invariant as no Ξ-term appears in their variation. The second line of (6.52) also clearly vanishes because any Ξ-transformation is proportional to the symmetric part $X_{(MN)}{}^P$ and is projected to zero by D_{RSP} due to (6.48). This leaves us with the first line of (6.52). If we use (6.37) and (6.28), this can be written in a symplectically covariant form:

$$\delta_\Xi \mathcal{L}_{VT} = -\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^M X_{(NP)}{}^Q \Omega_{MQ} \mathcal{D}_\rho \Xi_\sigma{}^{NP} . \tag{6.54}$$

The B-terms in $\mathcal{H}$, see (6.36), are proportional to $X_{(RS)}{}^M$ and thus give a vanishing contribution due to our new constraint (6.46). For the $\mathcal{F}$ terms we can perform an integration by parts[40] and then according to (6.33) there are again only terms proportional to $X_{(RS)}{}^M$ leading to the same conclusion. Therefore, the Ξ-variation of the total action vanishes.

Thus, we only have to consider the Λ^M gauge transformations. In accordance with (6.34), the $\mathcal{D}_\rho \delta A_{\sigma\Lambda}$-term in (6.52) can be replaced by $\frac{1}{2}\Lambda^Q X_{NQ\Lambda}\mathcal{H}_{\rho\sigma}{}^N$ (see again footnote 39). One can then obtain a symplectically covariant expression when this is combined with the first term of (6.44) (the first term on the right hand side of (6.55) below). Adding also the remaining terms of (6.52) and (6.44), one obtains, using (6.37),

$$\begin{aligned}\delta\mathcal{L}_{VT} \;=\; \varepsilon^{\mu\nu\rho\sigma} \;\; & \big[\tfrac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}\mathcal{H}_{\rho\sigma}{}^N + \tfrac{1}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q X_{QM}{}^R\Omega_{NR} \\ & + \tfrac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu\,\Lambda}X_{(NP)}{}^\Lambda\Delta B_{\rho\sigma}{}^{NP} \\ & - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big] \, . \end{aligned} \tag{6.55}$$

We observe that if the $\mathcal{H}$ in the first line was a $\mathcal{G}$, eqs. (6.17) and (6.19) would allow one to write the first line as an expression proportional to D_{MNP}. This leads to the first line in (6.56) below. The second observation is that the identity $(\mathcal{H}-\mathcal{G})^\Lambda = 0$ allows one to rewrite the second line of (6.55) in a symplectically covariant way, so that, altogether, we have

$$\begin{aligned}\delta\mathcal{L}_{VT} \;=\; \varepsilon^{\mu\nu\rho\sigma} \;\; & \big[\tfrac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \tfrac{3}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q D_{QMN} \\ & - \tfrac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^M\Omega_{MR}X_{(NP)}{}^R\Delta B_{\rho\sigma}{}^{NP} \\ & - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big] \, . \end{aligned} \tag{6.56}$$

By choosing

$$\Delta B_{\rho\sigma}{}^{NP} = -\Lambda^N\mathcal{G}_{\rho\sigma}{}^P - \Lambda^P\mathcal{G}_{\rho\sigma}{}^N \, , \tag{6.57}$$

the result (6.56) becomes

$$\begin{aligned}\delta\mathcal{L}_{VT} \;=\; \varepsilon^{\mu\nu\rho\sigma} \;\; & \big[\tfrac{3}{8}\Lambda^Q D_{MNQ}\left(2\mathcal{G}_{\mu\nu}{}^M(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\right) \\ & - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big] \, , \end{aligned} \tag{6.58}$$

[40] Integration by parts with the covariant derivatives is allowed because (6.25) can be read as the invariance of the tensor X and (6.17) as the invariance of Ω.

which is then proportional to D_{MNP}, and hence zero when the original representation constraint (6.22) is imposed.

Our goal is to generalize this for theories with quantum anomalies. These anomalies depend only on the gauge vectors. However, the field strengths $\mathcal{G}$, (6.40) also depends on the matrix $\mathcal{N}$ which itself generically depends on scalar fields. Therefore, we want to consider modified transformations of the antisymmetric tensors such that $\mathcal{G}$ does not appear in the final result.

To achieve this, we would like to replace (6.57) by a transformation such that

$$X_{(NP)}{}^{R}\Delta B_{\rho\sigma}{}^{NP} = -2X_{(NP)}{}^{R}\Lambda^{N}\mathcal{G}_{\rho\sigma}{}^{P} + \tfrac{3}{2}\Omega^{RM}D_{MNQ}\Lambda^{Q}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^{N}\,. \tag{6.59}$$

Indeed, inserting this in (6.56) would lead to

$$\begin{aligned}\delta\mathcal{L}_{VT} &= \varepsilon^{\mu\nu\rho\sigma}\quad\big[\tfrac{3}{8}\Lambda^{Q}D_{MNQ}\mathcal{F}_{\mu\nu}{}^{M}\mathcal{F}_{\rho\sigma}{}^{N}\\ &\qquad -D_{MNP}A_{\mu}{}^{M}\mathcal{D}_{\nu}\Lambda^{N}\left(\partial_{\rho}A_{\sigma}{}^{P}+\tfrac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\big]\,,\end{aligned} \tag{6.60}$$

where we have used (6.48) to delete contributions coming from the $B_{\mu\nu}{}^{NP}$ term in $\mathcal{H}_{\mu\nu}{}^{M}$ (cf. (6.36)).

The first term on the right hand side of (6.59) would follow from (6.57), but the second term cannot in general be obtained from assigning transformations to $B_{\rho\sigma}{}^{NP}$ (compare with (6.19)). Indeed, self-consistency of (6.59) requires that the second term on the right hand side be proportional to $X_{(NP)}{}^{R}$, which imposes a further constraint on D_{MNP}. We will see in section 6.2.3 how we can nevertheless justify the transformation law (6.59) by introducing other antisymmetric tensors. For the moment, we just accept (6.59) and explore its consequences.

Expanding (6.60) using (6.16) and (6.28) and using a partial integration, (6.60) can be rewritten as

$$\delta\mathcal{L}_{VT} = -\mathcal{A}[\Lambda]\,, \tag{6.61}$$

where

$$\begin{aligned}\mathcal{A}[\Lambda] \;=\; & -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda^{P}D_{MNP}\partial_{\mu}A_{\nu}{}^{M}\partial_{\rho}A_{\sigma}{}^{N}\\ & -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda^{P}\big(D_{MNR}X_{[PS]}{}^{N}+\frac{3}{2}D_{MNP}X_{[RS]}{}^{N}\big)\partial_{\mu}A_{\nu}{}^{M}A_{\rho}{}^{R}A_{\sigma}{}^{S}\,.\end{aligned} \tag{6.62}$$

This expression formally looks like a symplectically covariant generalization of the electric consistent anomaly (5.41) which we encountered in section 5. Notice, however, that at this point this is really only a formal analogy, as the tensor D_{MNP} has, a priori, no connection with quantum anomalies. We will study the meaning of this analogy in more detail in the next section. To prove (6.61), one uses (6.48) and the preservation of D_{MNP} under gauge transformations, which follows from preservation of X, see (6.25), and of Ω, see (6.17), and reads

$$X_{M(N}{}^{P} D_{QR)P} = 0 \,. \tag{6.63}$$

For the terms quartic in the gauge fields, one needs the following consequence of (6.63):

$$\begin{aligned}(X_{RS}{}^{M} X_{PQ}{}^{N} D_{LMN})_{[RSPL]} &= -(X_{RS}{}^{M} X_{PM}{}^{N} D_{LQN} + X_{RS}{}^{M} X_{PL}{}^{N} D_{QMN})_{[RSPL]} \\ &= -(X_{RS}{}^{M} X_{PL}{}^{N} D_{QMN})_{[RSPL]} \,, \end{aligned} \tag{6.64}$$

where the final line uses (6.26) and again (6.48).

Let us summarize the result of our calculation up to the present point. We have used the action (6.53) and considered its transformations under (6.28) and (6.37), where $\Delta B_{\mu\nu}{}^{NP}$ was undetermined. We used the closure constraint (6.20) and one new constraint (6.46). It was shown that the choice (6.57) leads to invariance if D_{MNP} vanishes, which is the representation constraint (6.22) used in the anomaly-free case studied in [83]. However, when we use the more general transformation (6.59) in the case $D_{MNP} \neq 0$ instead, we obtain the non-vanishing classical variation (6.61). The corresponding expression (6.62) formally looks very similar to a symplectically covariant generalization of the electric consistent quantum anomaly.

In order to fully justify and understand this result, we are then left with the following three open issues, which we will discuss in the following section:

(i) The expression (6.62) for the non-vanishing classical variation of the action has to be related to quantum anomalies so that gauge invariance can be restored at the level of the quantum effective action, in analogy to the electric case described in sections 3 and 5. This will be done in section 6.2.1.

(ii) The meaning of the new constraint (6.46) that was used to obtain (6.61) has to be clarified. This is subject of section 6.2.2.

(iii) We have to show how the transformation (6.59), which also underlies the result (6.61), can be realized. This will be done in section 6.2.3.

6.2 Gauge invariance of the effective action with anomalies

6.2.1 Symplectically covariant anomalies

In section 6.1, we discussed the algebraic constraints that were imposed on the embedding tensor in ref. [83] and that allowed the construction of a gauge invariant Lagrangian with electric and magnetic gauge potentials as well as tensor fields. Two of these constraints, (6.20) and (6.21), had a very clear physical motivation and ensured the closure of the gauge algebra and the mutual locality of all interacting fields. The physical origin of the third constraint, the representation constraint, (6.22), on the other hand, remained a bit obscure. In order to understand its meaning, we specialize it to its purely electric components,

$$X_{(\Lambda\Sigma\Omega)} = 0\,. \tag{6.65}$$

Given that the components $X_{\Lambda\Sigma\Omega}$ generate axionic shift symmetries (remember the first term on the right hand side of (6.42)), we can identify them with the corresponding symbols C_{ABC} in section 5, and recognize (6.65) as the condition for the absence of quantum anomalies for the electric gauge bosons (see (5.43)). It is therefore suggestive to interpret (6.22) as the condition for the absence of quantum anomalies for all gauge fields (i.e. for the electric and the magnetic gauge fields), and one expects that in the presence of quantum anomalies, this constraint can be relaxed. We will show that the relaxation consists in assuming that the symmetric tensor D_{MNP} defined by (6.18) is of the form[41]

$$D_{MNP} = d_{MNP}\,, \tag{6.66}$$

for a symmetric tensor d_{MNP} which describes the quantum gauge anomalies due to an anomalous spectrum of chiral fermions. In fact, one expects quantum anomalies from the loops of these fermions, ψ, which interact with the gauge fields via minimal couplings

$$\bar{\psi}\gamma^{\mu}(\partial_\mu - A_\mu{}^{\Lambda}\Theta_\Lambda{}^{\alpha}\delta_\alpha - A_{\mu\Lambda}\Theta^{\Lambda\alpha}\delta_\alpha)\psi\,. \tag{6.67}$$

Therefore, the anomalies contain – for each external gauge field (or gauge parameter) – an embedding tensor, i.e. d_{MNP} has the following particular form:

$$d_{MNP} = \Theta_M{}^{\alpha}\Theta_N{}^{\beta}\Theta_P{}^{\gamma}d_{\alpha\beta\gamma}\,, \tag{6.68}$$

[41] The possibility to impose a relation such as (6.66) is by no means guaranteed for all types of gauge groups (see e.g. [87] for a short discussion in the purely electric case studied in [88]).

with $d_{\alpha\beta\gamma}$ being a constant symmetric tensor. In the familiar context of a theory with a flat scalar manifold, constant fermionic transformation matrices, t_α, and the corresponding minimal couplings, the tensor d_{MNP} is simply proportional to

$$d_{MNP} \propto \Theta_M{}^\alpha \Theta_N{}^\beta \Theta_P{}^\gamma \mathrm{Tr}(\{t_\alpha, t_\beta\} t_\gamma\}, \tag{6.69}$$

where the trace is over the representation matrices of the fermions.[42]

We showed that the generalization of the consistent anomaly (5.41) in a symplectically covariant way leads to an expression of the form (6.62) with the D_{MNP}-tensor replaced by d_{MNP}. Indeed, the constraint (6.66) implies the cancellation of this quantum gauge anomaly by the classical gauge variation (6.61). Note that it is necessary for this cancellation that the anomaly tensor d_{MNP} is really constant (i.e., independent of the scalar fields). We expect this constancy to be generally true for the same topological reasons that imply the constancy of $d_{\Lambda\Gamma\Omega}$ in the conventional electric gaugings. In this way we have already addressed the first issue of the end of the previous section. We are now going to show how the constraint (6.66) suffices also to address the other two issues, (ii) and (iii).

6.2.2 The new constraint

We now comment on the constraint (6.46):

$$X_{(NP)}{}^M \Omega_{MQ} X_{(RS)}{}^Q = 0\,. \tag{6.70}$$

We will show that this equation holds if the locality constraint is satisfied, and (6.66) is imposed on D_{MNP} with d_{MNP} of the particular form given in (6.68). To clarify this, we introduce as in [83] the 'zero mode tensor'[43]

$$Z^{M\alpha} = \frac{1}{2}\Omega^{MN}\Theta_N{}^\alpha\,, \qquad \text{i.e.} \quad \begin{cases} Z^{\Lambda\alpha} = \frac{1}{2}\Theta^{\Lambda\alpha}\,, \\ Z_\Lambda{}^\alpha = -\frac{1}{2}\Theta_\Lambda{}^\alpha\,. \end{cases} \tag{6.71}$$

One then obtains, using (6.19), the definition of X in (6.16) and (6.68) that

$$X_{(NP)}{}^M = Z^{M\alpha}\Delta_{\alpha NP}\,, \tag{6.72}$$

[42] One might wonder how the magnetic vector fields $A_{\mu\Lambda}$ can give rise to anomalous triangle diagrams, as they have no propagator due to the lack of a kinetic term. However, it is the *amputated* diagram with internal fermion lines that one has to consider.

[43] Note that the components of Ω^{MN} have signs opposite to those of Ω_{MN} as given in (6.7).

for some tensor $\Delta_{\alpha NP} = \Delta_{\alpha PN}$. Due to the fact that we allow the symmetric tensor D_{MNP} in (6.18) to be non-zero and impose the constraint (6.66), this tensor $\Delta_{\alpha NP}$ is not the analogous quantity called $d_{\alpha MN}$ in [83][44], but can be written as

$$\Delta_{\alpha NP} = (t_\alpha)_N{}^Q \Omega_{PQ} - 3 d_{\alpha\beta\gamma} \Theta_N{}^\beta \Theta_P{}^\gamma \,. \tag{6.73}$$

However, the explicit form of this expression will not be relevant. We will only need that $X_{(NP)}{}^M$ is proportional to $Z^{M\alpha}$.

Now we will finally use the locality constraint (6.21), which implies

$$Z^{\Lambda[\alpha} Z_\Lambda{}^{\beta]} = 0 \,, \qquad \text{i.e.} \qquad Z^{M\alpha} Z^{N\beta} \Omega_{MN} = 0 \,. \tag{6.74}$$

and, thus, leads to the desired result (6.70).

The tensor $Z^{M\alpha}$ can be called zero-mode tensor as e.g. the violation of the usual Jacobi identity (second line of (6.26)) is proportional to it. We now show that it also defines zero modes of D_{MNR}. Indeed, another consequence of the locality constraint is

$$X_{MN}{}^P \Omega^{MQ} \Theta_Q^\alpha = 0 \qquad \rightarrow \qquad X_{MN}{}^P Z^{M\alpha} = 0 \,, \qquad X_{QM}{}^P \Omega^{QS} X_{SN}{}^R = 0 \,. \tag{6.75}$$

With (6.19) and (6.24) this implies

$$D_{MNR} Z^{R\alpha} = 0 \,. \tag{6.76}$$

Note that we did not need (6.66) to achieve this last result, but that the equation is consistent with it.

6.2.3 New antisymmetric tensors

Finally, in this section we will justify the transformation (6.59), without requiring further constraints on the D-tensor. That transformation gives an expression for $X_{(NP)}{}^R \Delta B_{\rho\sigma}{}^{NP}$ that is not obviously a contraction with the tensor $X_{(NP)}{}^R$ (due to the second term on the right hand side of (6.59)). We can therefore in general not assign a transformation of $B_{\rho\sigma}{}^{NP}$ such that its contraction with $X_{(NP)}{}^R$ gives (6.59). To overcome this problem, we will have

[44] We use $\Delta_{\alpha MN}$ in this work to denote the analogue (or better: generalization) of what was called $d_{\alpha MN}$ in [83], because $d_{\alpha MN}$ is reserved in the present paper to denote the quantity $\Theta_M{}^\beta \Theta_N{}^\gamma d_{\alpha\beta\gamma}$ (cf eq. (6.84)) related to the quantum anomalies.

to change the set of independent antisymmetric tensors. The ${B_{\mu\nu}}^{MN}$ cannot be considered as independent fields in order to realize (6.59). We will, as it was done along the lines of [83], introduce a new set of independent antisymmetric tensors, given by $B_{\mu\nu\,\alpha}$ for any α denoting a rigid symmetry.

The fields ${B_{\mu\nu}}^{NP}$ and their associated gauge parameters Ξ^{NP} appeared in the relevant formulae in the form ${X_{(NP)}}^M {B_{\mu\nu}}^{NP}$ or ${X_{(NP)}}^M \Xi^{NP}$, see e.g. in (6.28), (6.34), (6.36) and (6.45). Now, as we have the form (6.72), this can be written as

$$ {X_{(NP)}}^M {B_{\mu\nu}}^{NP} = Z^{M\alpha} \Delta_{\alpha MN} {B_{\mu\nu}}^{MN} \,. \tag{6.77} $$

Therefore, we will replace

$$ \Delta_{\alpha MN} {B_{\mu\nu}}^{MN} \quad \to \quad B_{\mu\nu\,\alpha} \,. \tag{6.78} $$

and consider the $B_{\mu\nu\,\alpha}$ as the independent antisymmetric tensors. Thus, there is one tensor for every generator of the rigid symmetry group and the replacement implies that

$$ {X_{(NP)}}^M {B_{\mu\nu}}^{NP} \quad \to \quad Z^{M\alpha} B_{\mu\nu\,\alpha} \,. \tag{6.79} $$

We also introduce a corresponding set of independent gauge parameters $\Xi_{\mu\,\alpha}$ through the substitution:

$$ \Delta_{\alpha MN} {\Xi_\mu}^{MN} \quad \to \quad \Xi_{\mu\,\alpha} \,. \tag{6.80} $$

This allows us to reformulate all the equations in the previous subsections in terms of $B_{\mu\nu\,\alpha}$ and $\Xi_{\mu\,\alpha}$. It is now, for instance,:

$$ \delta {A_\mu}^M = \mathcal{D}_\mu \Lambda^M - Z^{M\alpha} \Xi_{\mu\,\alpha} \,, \tag{6.81} $$

$$ {\mathcal{H}_{\mu\nu}}^M = {\mathcal{F}_{\mu\nu}}^M + Z^{M\alpha} B_{\mu\nu\,\alpha} \,, \tag{6.82} $$

$$ \mathcal{L}_{\text{top},B} = \tfrac{1}{4} \varepsilon^{\mu\nu\rho\sigma} \, Z^{\Lambda\alpha} B_{\mu\nu\,\alpha} \left(\mathcal{F}_{\rho\sigma\,\Lambda} + \tfrac{1}{2} \, {Z_\Lambda}^\alpha B_{\mu\nu\,\alpha} \right) . \tag{6.83} $$

We will show that considering $B_{\mu\nu\,\alpha}$ as the independent variables, we are ready to solve the remaining third issue mentioned at the end of section 6.1. To this end, we first note that all the calculations in section 6.1 remain valid when (6.79) and (6.81)-(6.83) are used to express everything in terms of the new variables $B_{\mu\nu\,\alpha}$ and $\Xi_{\mu\,\alpha}$. The equations (6.46) and (6.48) we used in section 6.1 are now simply replaced by (6.74) and (6.76), respectively.

Following (6.68), we are able to set

$$d_{MNP} = \Theta_M{}^{\alpha} d_{\alpha NP}\,, \qquad d_{\alpha NP} = d_{\alpha\beta\gamma}\Theta_N{}^{\beta}\Theta_P{}^{\gamma}\,, \tag{6.84}$$

and, consequently, can define (bearing in mind (6.72))

$$\begin{aligned} \delta B_{\mu\nu\,\alpha} &= 2\,\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} + 2\Delta_{\alpha\,NP}A_{[\mu}{}^{N}\delta A_{\nu]}{}^{P} + \Delta B_{\mu\nu\,\alpha}\,, \\ \Delta B_{\mu\nu\,\alpha} &= -2\Delta_{\alpha NP}\Lambda^{N}\mathcal{G}_{\mu\nu}{}^{P} + 3d_{\alpha NP}\Lambda^{N}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^{P}\,, \end{aligned} \tag{6.85}$$

to reproduce (6.59). Here the left-hand side of (6.59) is replaced according to (6.79) and the covariant derivative is defined as

$$\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} = \partial_{[\mu}\Xi_{\nu]\,\alpha} + f_{\alpha\beta}{}^{\gamma}\Theta_P{}^{\beta}A_{[\mu}{}^{P}\Xi_{\nu]\,\gamma}\,. \tag{6.86}$$

Of course, (6.85) is only fixed modulo terms that vanish upon contraction with the embedding tensor.

So let us summarize what we have found out. In this section we have seen, so far, that it is possible to relax the representation constraint (6.22) used in ref. [83] to the more general condition (6.66) if one allows for quantum anomalies. The physical interpretation of the original representation constraint (6.22) of [83] is thus the absence of quantum anomalies.

Due to these constraints we obtained the equation (6.72), which allowed us to introduce the $B_{\mu\nu\,\alpha}$ as independent variables. All the calculations of section 6.1.2 are then valid with the substitutions given in (6.79) and (6.80). We did not impose (6.72) in section 6.1.2, and therefore we could at that stage only work with $B_{\mu\nu}{}^{NP}$. However, now we conclude that we need the $B_{\mu\nu\,\alpha}$ as independent fields and will further only consider these antisymmetric tensors.

The results of this section can alternatively be viewed as a covariantization of the results of section 5 and [12, 88] with respect to electric/magnetic duality transformations.[45] To further check the consistency of our results, we will in the next section reduce our treatment to a purely electric gauging and show that the results of section 5 can be reproduced.

[45]We have not discussed the complete embedding into $\mathcal{N}=1$ supersymmetry here, which would include all fermionic terms as well as the supersymmetry transformations of all the fields. This is beyond the scope of this thesis.

6.3 Purely electric gaugings

Let us first explicitly write down D_{MNP} in its electric and magnetic components:

$$\begin{aligned} D_{\Lambda\Sigma\Gamma} &= X_{(\Lambda\Sigma\Gamma)}\,, \\ 3D^{\Lambda}{}_{\Sigma\Gamma} &= X^{\Lambda}{}_{\Sigma\Gamma} - 2X_{(\Sigma\Gamma)}{}^{\Lambda}\,, \\ 3D^{\Lambda\Sigma}{}_{\Gamma} &= -X_{\Gamma}{}^{\Lambda\Sigma} + 2X^{(\Lambda\Sigma)}{}_{\Gamma}\,, \\ D^{\Lambda\Sigma\Gamma} &= -X^{(\Lambda\Sigma\Gamma)}\,. \end{aligned} \tag{6.87}$$

In the case of a purely electric gauging, the only non-vanishing components of the embedding tensor are electric:

$$\Theta_M{}^{\alpha} = (\Theta_{\Lambda}{}^{\alpha},\ 0)\ . \tag{6.88}$$

Therefore also $X^{\Lambda}{}_N{}^P = 0$ and (6.68) implies that the only non-zero components of $D_{MNP} = d_{MNP}$ are $D_{\Lambda\Sigma\Omega}$. Therefore, (6.87) reduce to

$$D_{\Lambda\Sigma\Omega} = X_{(\Lambda\Sigma\Omega)}\,, \qquad X_{(\Sigma\Omega)}{}^{\Lambda} = 0\,, \qquad X_{\Omega}{}^{\Lambda\Sigma} = 0\,. \tag{6.89}$$

The non-vanishing entries of the gauge generators are $X_{\Lambda\Sigma\Gamma}$ and $X_{\Sigma\Omega}{}^{\Lambda} = -X_{\Sigma}{}^{\Lambda}{}_{\Omega} = X_{[\Sigma\Omega]}{}^{\Lambda}$, the latter satisfying the Jacobi identities since the right hand side of (6.26) for $MNQR$ all electric indices vanishes. The $X_{[\Sigma\Omega]}{}^{\Lambda}$ can be identified with the structure constants of the gauge group that were called $f_{AB}{}^{C}$ in section 5. The $X_{\Lambda\Sigma\Omega}$ correspond to the shifts in (5.20). The first relation in (6.89) then corresponds to $C_{(AB,C)} = d_{ABC}$.

The locality constraint is trivially satisfied and the closure relation reduces to (5.32) as expected.

At the level of the action $\mathcal{L}_{\rm VT}$, all tensor fields drop out since, when we express everything in terms of the new tensors $B_{\mu\nu\,\alpha}$, these tensors always appear contracted with a factor $\Theta^{\Lambda\,\alpha} = 0$. In particular, the topological terms $\mathcal{L}_{{\rm top},B}$ vanish and the modified field strengths for the electric vector fields $\mathcal{H}_{\mu\nu}{}^{\Lambda}$ reduce to ordinary field strengths:

$$\mathcal{H}_{\mu\nu}{}^{\Lambda} \;=\; 2\partial_{[\mu}A_{\nu]}{}^{\Lambda} + X_{[\Omega\Sigma]}{}^{\Lambda}A_{\mu}{}^{\Omega}A_{\nu}{}^{\Sigma}\,. \tag{6.90}$$

Also the GCS terms (6.50) reduce to the analogue form of (5.30) in purely electric gaugings. Finally, the gauge variation of $\mathcal{L}_{\rm VT}$ reduces to minus the ordinary consistent gauge anomaly.

This concludes our reinvestigation of the electric gauging with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies as it follows from our more general symplectically covariant treatment. We showed that the more general theory reduces consistently to the known case of a purely electric gauging.

6.4 A simple example of magnetic gauging

The above results can be shown by means of a simple example that already provides a nontrivial symmetric tensor D_{MNP}. Let us now briefly illustrate the above results by means of a simple example. We consider a theory with a rigid symmetry group embedded in the electric/magnetic duality group $Sp(2,\mathbb{R})$. The embedding into the symplectic transformations is given by

$$t_{1M}{}^N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad t_{2M}{}^N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad t_{3M}{}^N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{6.91}$$

i.e. $t_2{}^{11} = 1$. Let us consider the following subset of duality transformations:

$$\mathcal{S}^M{}_N = \delta^M{}_N - \Lambda^P X_{PN}{}^M, \quad \text{with generators} \quad X_{PM}{}^N = \begin{pmatrix} 0 & 0 \\ X_P{}^{11} & 0 \end{pmatrix}, \tag{6.92}$$

where Λ^P is the rigid transformation parameter. The tensor X is related to the embedding of the symmetries in the symplectic algebra using the embedding tensor,

$$X_{PM}{}^N = \sum_{\alpha=1}^{3} \Theta_P{}^\alpha t_{\alpha M}{}^N . \tag{6.93}$$

We have thus chosen the embedding tensor

$$\Theta_P{}^1 = 0 , \qquad \Theta_P{}^2 = X_P{}^{11} , \qquad \Theta_P{}^3 = 0 . \tag{6.94}$$

The task is to promote $\mathcal{S}^M{}_N$ to a gauge transformation, i.e., to take $\Lambda^N = \Lambda^N(x)$ spacetime dependent and to identify the $X_{PM}{}^N$ with the gauge generators. This obviously corresponds to a magnetic gauging, because (6.89) is violated. However, the locality constraint (6.21) is automatically satisfied, as only the index value $\alpha = 2$ appears, and closure of the gauge algebra spanned by the $X_{PM}{}^N$ requires that (6.20) is imposed, where only the right-hand side is non-trivial. It is necessary that $\Theta_1{}^2 = 0$, and the only gauge generators that are consistent with this constraint are

$$X_{PM}{}^N = (X_{1M}{}^N,\, X^1{}_M{}^N)\,, \quad \text{with} \quad X_{1M}{}^N = 0\,, \quad X^1{}_M{}^N = \begin{pmatrix} 0 & 0 \\ X^{111} & 0 \end{pmatrix}. \tag{6.95}$$

Note that this choice still violates the original linear representation constraint (6.22) because (6.87) leads to $D^{111} = -X^{111} \neq 0$. However, this is not an obstacle in performing the gauging with generators $X_{PM}{}^N$ given in (6.95). In order to do so we introduce a symplectic vector field $A_\mu{}^M$ which contains an electric and a magnetic part, $A_\mu{}^1$ and $A_{\mu 1}$. Only the magnetic vector field couples to matter via covariant derivatives since the embedding tensor projects out the electric part. In what follows, we also assume the presence of anomalous couplings between the magnetic vector field and chiral fermions which justifies the nonzero $X^{111} \neq 0$ because it will give rise to anomaly cancellation terms in the classical gauge variation of the action. More precisely, we will have to require that

$$\Theta^{12} = X^{111}\,, \qquad -X^{111} = d^{111} = (X^{111})^3 \tilde{d}_{222}\,, \tag{6.96}$$

where we introduced $\tilde{d}_{222}$ as the component of $d_{\alpha\beta\gamma}$.

There is the kinetic term for the electric vector fields:

$$\mathcal{L}_{\text{g.k.}} = \tfrac{1}{4}\, e\, \mathcal{I}\, \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}^{\mu\nu\,1} - \tfrac{1}{8}\, \mathcal{R}\, \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^1 \mathcal{H}_{\rho\sigma}{}^1, \tag{6.97}$$

where we introduced the modified field strength (6.82)

$$\mathcal{H}_{\mu\nu}{}^1 = 2\partial_{[\mu} A_{\nu]}{}^1 + \frac{1}{2} X^{111} B_{\mu\nu 2}\,, \tag{6.98}$$

and whose variation has to be computed. Observe that it depends on a tensor field $B_{\mu\nu 2}$ because in (6.94) it was chosen a magnetic gauging. However, it transforms covariantly under

$$\begin{aligned}
\delta A_\mu{}^1 &= \partial_\mu \Lambda^1 + X^{111} A_{\mu\,1} \Lambda_1 - \frac{1}{2} X^{111} \Xi_{\mu 2}\,, \\
\delta B_{\mu\nu 2} &= 2\partial_{[\mu} \Xi_{\nu]2} + 4 A_{[\mu\,1} \partial_{\nu]} \Lambda_1 - 6 \Lambda_1 \partial_{[\mu} A_{\nu]\,1} - \Lambda_1 \mathcal{G}_{\mu\nu\,1}\,, \\
\delta A_{\mu 1} &= \partial_\mu \Lambda_1\,.
\end{aligned} \tag{6.99}$$

which follows from (6.85) since the only nonzero component of Δ_{2MN} is $\Delta_2{}^{11} = 2$ and for d_{2MN} we have only $d_2{}^{11} = -1$. One can check that

$$\begin{aligned} \delta\mathcal{H}_{\mu\nu}{}^1 &= -\tfrac{1}{2}X^{111}\Lambda_1(\mathcal{H}+\mathcal{G})_{\mu\nu\,1}\,, \quad \text{with} \\ & \mathcal{H}_{\mu\nu\,1} = \mathcal{F}_{\mu\nu\,1} = 2\partial_{[\mu}A_{\nu]1}\,, \qquad \mathcal{G}_{\mu\nu\,1} \equiv \mathcal{R}\mathcal{H}_{\mu\nu}{}^1 + \frac{1}{2}e\mathcal{I}\varepsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\rho\sigma\,1}\,. \end{aligned} \quad (6.100)$$

Under gauge variations, the real and imaginary part of the kinetic function transform as follows (cf. (6.42)):

$$\delta\mathcal{I} = 2\Lambda_1 X^{111}\mathcal{R}\mathcal{I}\,, \qquad \delta\mathcal{R} = \Lambda_1 X^{111}\left(\mathcal{R}^2 - \mathcal{I}^2\right)\,. \quad (6.101)$$

From this one obtains the gauge variation of the kinetic term, given by

$$\delta\mathcal{L}_{\text{g.k.}} = \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 X^{111}\mathcal{G}_{\mu\nu\,1}\partial_\rho A_{\sigma 1}\,. \quad (6.102)$$

which corresponds to (6.44) in our present gauge (6.94).

In a second step, we add the topological term (6.83)

$$\mathcal{L}_{\text{top},B} = \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}X^{111}B_{\mu\nu 2}\partial_{[\rho}A_{\sigma]\,1}\,. \quad (6.103)$$

The gauge variation of this term is equal to (up to a total derivative)

$$\delta\mathcal{L}_{\text{top},B} = -\tfrac{1}{4}\Lambda_1 X^{111}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A_{\nu\,1}\right)\left(2\partial_\rho A_{\sigma\,1} + \mathcal{G}_{\rho\sigma\,1}\right)\,. \quad (6.104)$$

Note that the generalized Chern-Simons term (6.50) vanishes in this case. In combining (6.102) and (6.104), one derives

$$\delta\left(\mathcal{L}_{\text{g.k.}} + \mathcal{L}_{\text{top},B}\right) = -\tfrac{1}{2}\Lambda_1 X^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right)\varepsilon^{\mu\nu\rho\sigma}\,. \quad (6.105)$$

This cancels the magnetic gauge anomaly whose form can be derived from (6.62),

$$\mathcal{A}[\Lambda] \;=\; -\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda_1 d^{111}\left(\partial_\mu A_{\nu\,1}\right)\left(\partial_\rho A_{\sigma\,1}\right) \quad (6.106)$$

if we remember that $X^{111} = -D^{111} = -d^{111}$. Note that the electric gauge fields do not appear reflecting the fact that the electric gauge fields do not couple to chiral fermions.

A simple fermionic spectrum that could yield such an anomaly (6.106) is given by, e.g., three chiral fermions with canonical kinetic terms and quantum numbers $Q = (-1),(-1),(+2)$ under the $U(1)$ gauged by $A_{\mu\,1}$. Indeed, with this spectrum, we would have $\text{Tr}(Q) = 0$, i.e., vanishing gravitational anomaly, but a cubic Abelian gauge anomaly $d^{111} \propto \text{Tr}(Q^3) = +6$.

6.5 Summary

In section 6.1.2 we argued that the rigid symmetry group $G_{\rm rigid}$ is a subset of the product of the symplectic duality transformations that act on the vector fields and the isometry group of the scalar manifold of the chiral multiplets in $\mathcal{N}=1$ supersymmetry or in theories without supersymmetry. The reason is that the rigid symmetries of the vector and scalar sector are not directly related in these theories. On the fields strengths $F_{\mu\nu}{}^M = (F_{\mu\nu}{}^\Lambda, G_{\mu\nu\,\Lambda})$ these rigid symmetries act by multiplication with infinitesimal symplectic matrices $(t_\alpha)_M{}^P$ for which we have $(t_\alpha)_{[M}{}^P\Omega_{N]P}=0$ where Ω_{NP} is the symplectic metric given by (6.7). The gauging of a subgroup, $G_{\rm local}\subset G_{\rm rigid}$, is achieved by projecting the $2n$-dimensional vector space spanned by the vector fields $A_\mu{}^M$ onto the Lie algebra of $G_{\rm local}$ which is done by the embedding tensor $\Theta_M{}^\alpha$. The generators of $G_{\rm local}$ decompose according to $(X_M)_N{}^P = \Theta_M{}^\alpha(t_\alpha)_N{}^P$ whose components are denoted by $X_{MN}{}^P$. The embedding tensor has to satisfy a number of consistency conditions:

$$f_{\alpha\beta}{}^\gamma\Theta_M{}^\alpha\Theta_N{}^\beta = (t_\alpha)_N{}^P\Theta_M{}^\alpha\Theta_P{}^\gamma\,,$$

$$\Omega^{MN}\Theta_M{}^\alpha\Theta_N{}^\beta = 0 \qquad\Leftrightarrow\qquad \Theta^{\Lambda[\alpha}\Theta_\Lambda{}^{\beta]} = 0\,. \tag{6.107}$$

Closure of the gauge algebra requires the first line of (6.107), while the constraint displayed in the second line of (6.107) is required by locality. The closure constraint reflects the invariance of the embedding tensor under $G_{\rm local}$ and it implies for the matrices X_M the relation

$$[X_M, X_N] \;=\; -X_{MN}{}^P X_P\,. \tag{6.108}$$

It is crucial to observe that the 'structure constants' given by $X_{MN}{}^P$ contain also an in general nontrivial symmetric part $X_{(MN)}{}^P$. The antisymmetry of the left hand side of (6.108) only requires that the contraction $X_{(MN)}{}^P\Theta_P{}^\alpha$ vanishes. This gives a violation of the Jacobi identity (6.26) which can be compensated in extending the gauge transformation of the vector potentials to

$$\delta A_\mu{}^M = \mathcal{D}_\mu\Lambda^M - X_{(NP)}{}^M\Xi_\mu{}^{NP}\,,\qquad \mathcal{D}_\mu\Lambda^M = \partial_\mu\Lambda^M + X_{PQ}{}^M A_\mu{}^P\Lambda^Q\,, \tag{6.109}$$

where we introduced the covariant derivative $\mathcal{D}_\mu\Lambda^M$, and new vector-like gauge parameters $\Xi_\mu{}^{NP}$, symmetric in the upper indices. Consequently, the field strength $\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M +$

$X_{[PQ]}{}^M A_\mu{}^P A_\nu{}^Q$ does no longer transform covariantly (6.34) and violates the Bianchi-identity (6.33). As another consequence we find that a gauge kinetic Lagrangian of the form

$$\mathcal{L}_{\rm gk} = +\frac{1}{4}\,{\rm Im}\,\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}{}^{\Lambda}F^{\mu\nu\Sigma} - \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\,{\rm Re}\,\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}{}^{\Lambda}F_{\rho\sigma}{}^{\Sigma}\,, \tag{6.110}$$

cannot be gauge invariant under transformations (6.109) either. In [83] it was shown that the Lagrangian

$$\begin{aligned}\mathcal{L}_{VT} = {} & \tfrac{1}{4}e\mathcal{I}_{\Lambda\Sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}^{\mu\nu\Sigma} - \tfrac{1}{8}\mathcal{R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{\Sigma} + \\ & + \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,X_{(NP)}{}^{\Lambda}\,B_{\mu\nu}{}^{NP}\left(\mathcal{F}_{\rho\sigma\,\Lambda} + \tfrac{1}{2}\,X_{(RS)\Lambda}\,B_{\rho\sigma}{}^{RS}\right) + \\ & + \varepsilon^{\mu\nu\rho\sigma}A_\mu{}^M A_\nu{}^N\left(\frac{1}{3}\,X_{MN\,\Lambda}\,\partial_\rho A_\sigma{}^{\Lambda} + \frac{1}{6}X_{MN}{}^{\Lambda}\partial_\rho A_{\sigma\Lambda} + \right. \\ & \left. + \frac{1}{8}X_{MN\,\Lambda}X_{PQ}{}^{\Lambda}A_\rho{}^P A_\sigma{}^Q\right)\,,\end{aligned} \tag{6.111}$$

with $\mathcal{H}$ as in (6.36), is indeed invariant under the gauge transformations (6.109) if the embedding tensor satisfies the additional constraint

$$D_{MNR} := X_{(MN}{}^{Q}\Omega_{R)Q} = 0 \tag{6.112}$$

We could show in this thesis that the gauge variation of (6.111) for nontrivial $D_{MNR} \neq 0$ does no longer vanish but is instead given by

$$\begin{aligned}\delta\mathcal{L}_{VT} = {} & -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P D_{MNP}\partial_\mu A_\nu{}^M\partial_\rho A_\sigma{}^N \\ & -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P\big(D_{MNR}X_{[PS]}{}^N + \frac{3}{2}D_{MNP}X_{[RS]}{}^N\big)\partial_\mu A_\nu{}^M A_\rho{}^R A_\sigma{}^S\,.\end{aligned} \tag{6.113}$$

which formally looks like the consistent anomaly [40]. Cancellation of (6.113) is only possible in presence of anomalies and if one relaxes the constraint (6.112) according to $D_{MNR} = d_{MNR}$ where the symmetric tensor d_{MNR} describes gauge anomalies. In fact, one can expect gauge anomalies due to an anomalous spectrum of chiral fermions ψ which interact with gauge fields via minimal couplings $\bar{\psi}\gamma^\mu(\partial_\mu - A_\mu{}^{\Lambda}\Theta_\Lambda{}^{\alpha}\delta_\alpha - A_{\mu\,\Lambda}\Theta^{\Lambda\,\alpha}\delta_\alpha)\psi$. In the discussion of section 2 we learned that the coupling of gauge fields to chiral fermions causes anomalous contributions to the conservation law of the axial currents which are to lowest order given by triangle diagrams. The electric vector fields and the magnetic vector fields generate such anomalous contributions due to their coupling to chiral fermions in total analogy to the discussion of section 2.1. The constraint $D_{MNR} = d_{MNR}$ implies the cancellation of this quantum anomaly by the classical variation of (6.111). In this sense we showed how the Green-Schwarz mechanism is applied in a symplectically covariant way (symplectically covariant Green-Schwarz mechanism).

In section 6.4 we explicitly displayed how an Abelian magnetic gauge violates (6.112). Furthermore, we gave the example of an anomalous spectrum of chiral fermions that possibly cancels the classical gauge variation in this example.

The results of this section are new and generalize the work [83], as presented in [40].

7 Abelian gauging and D-term potential in $\mathcal{N}=1$ supersymmetry

In the previous section, we discussed the symplectically covariant formulation of the gauge sector with a nontrivial gauge kinetic function that transforms with a shift under gauged isometry transformations of the target space. The symplectically covariant treatment required the introduction of magnetic vector fields that are dual to the original electric vector fields and do not possess a kinetic term. The additional degrees of freedom represented by the magnetic vector fields are compensated by additional gauge transformations. Invariance of the Lagrangian under the gauge transformations is restored by new couplings. Among these new couplings is a topological term (6.45) that couples an antisymmetric tensor field to the magnetic vector fields. In appendix E we show that the action (6.53) reduces to (E.11) for Abelian gaugings if the gauge sector is coupled to a nonlinear sigma model with gauged shift symmetries and the magnetic vector fields are eliminated by their equations of motion. We observe that in the Lagrangian (E.11) there are electric vectors and tensor fields left, i.e. (E.11) contains a kinetic term and a mass term for the tensor field, where the mass is given by the embedding tensor. There is also a topological coupling of the tensor field to the electric vectors among other couplings which are of minor interest. The topological coupling of the tensor field to the electric vectors is of similar type as (6.45). Interestingly, in [90] the authors discuss theories with massive tensor multiplets in global $\mathcal{N}=1$ supersymmetry where the bosonic sector contains exactly those couplings encountered in (E.11). Partly motivated by the results of orientifold compactifications [91, 92], the authors of [90] proposed an $\mathcal{N}=1$ superfield action for a massive tensor multiplet coupled to several vector chiral multiplets. Furthermore, the authors computed the component form of that action and deduced the following potential

$$
\begin{aligned}
V \quad \propto \quad & \left[\left(e_\Sigma + 2m^\Lambda \mathrm{Im} f_{\Lambda\Sigma}\right)\left[(\mathrm{Re}f)^{-1}\right]^{\Sigma\Omega}\left(e_\Omega + 2\mathrm{Im} f_{\Omega\Gamma} m^\Gamma\right) + \right. \\
& \left. +4\,\mathrm{Re} f_{\Lambda\Sigma} m^\Lambda m^\Sigma\right] \ , \qquad (7.1)
\end{aligned}
$$

where $f_{\Lambda\Sigma}$ denotes the gauge kinetic function. The potential (7.1) is not only determined by auxiliary fields but contains a direct mass term for the scalar in the tensor multiplet (the last term in (7.1)) which does not arise from eliminating an auxiliary field. In this sense, the discussion that follows is connected to the Abelian gauge of the symplectically covariant formalism presented in the previous section.

In this section we will show that the potential (7.1) is actually equivalent to a D-term potential in its standard form as given in [93], for example. In order to do so, we will add a total derivative to the action used in [90]. The advantage over the procedure used in [90] is that we are now able to absorb the topological couplings of the tensor fields to the electric vectors into the gauge kinetic term by a suitable redefintion of the gauge kinetic function. The gauge kinetic Lagrangian has the advantage that it is easier dualized than the Lagrangian used in [90]. It is suitable to rotate the fields and couplings to a special frame and then to determine the D-term potential. In this frame the potential is given in its standard form and a component expression is obtained after eliminating the auxiliary fields with help of their equations of motion. The potential (7.1) is found once we rotate the special frame back to its original form and, thus, the potential (7.1) is equivalent to a D-term potential in standard form.

The authors of [90] start from the following Lagrangian

$$2\big[K(L)\big]_D + \big[f_{\Lambda\Sigma}(N)(W^\Lambda - 2\mathrm{i}m^\Lambda\Phi)^T\epsilon(W^\Sigma - 2\mathrm{i}m^\Sigma\Phi) + 2e_\Lambda\Phi^T\epsilon(W^\Lambda - \mathrm{i}m^\Lambda\Phi) + \text{h.c.}\big]_F \quad (7.2)$$

where the tensor field is contained in the spinor superfield[46] Φ and the electric field strength is part of the superfield W^Λ where the index Λ counts the number of $U(1)$ vectorfields, $\Lambda = 1, \ldots, k$ (for notational issues consult appendix C). The field strength of the tensor field is part of the linear multiplet L and the kinetic term, which in the simplest case would be of the form L^2, is generalized by the real function $K(L)$. The gauge kinetic coupling function $f_{\Lambda\Sigma}(N)$ represents now a function of chiral superfields N and, thus, is itself a superfield. Before dualizing the theory of the massive tensor multiplet we add to (7.2) a total derivative term $2\mathrm{i}\,\mathrm{Im}\,\big(W^T\epsilon W\big)|_{\theta\bar\theta} = -\frac{\mathrm{i}}{2}F\wedge F$ which, therefore, does not affect the equations of motion. Thus, in simply adding $-2\frac{e_\Lambda e_\Sigma}{2e_\Omega m^\Omega}\cdot\mathrm{Im}\,\big[(W^\Lambda)^T\epsilon W^\Sigma\big]$, we obtain the following Lagrangian

$$2\big[K(L)\big]_D + \big[\tilde f_{\Lambda\Sigma}(N)(W^\Lambda - 2\mathrm{i}m^\Lambda\Phi)^T\epsilon(W^\Sigma - 2\mathrm{i}m^\Sigma\Phi) + \text{h.c.}\big]_F \quad (7.3)$$

where the gauge kinetic function is redefined according to

$$\tilde f_{\Lambda\Sigma}(N) \;:=\; f_{\Lambda\Sigma}(N) + \frac{\mathrm{i}}{2}\cdot\frac{e_\Lambda e_\Sigma}{e_\Omega m^\Omega} \quad (7.4)$$

[46] Consult appendix C for more details on the spinor superfield and the linear multiplet.

The Lagrangian (7.3) is more convenient to dualize because the last term in (7.2) is now absorbed into the redefinition of the couplings. We have to construct a first order Lagrangian before we can dualize (7.3). In the first order Lagrangian one does not consider L to be a linear multiplet, instead one imposes a constraint on L by means of a real Lagrangian multiplier Ω [89]. Then, by eliminating the Lagrange multiplier, L is constrained to be a linear superfield. The first order Lagrangian reads

$$2\big[K(L)\big]_D + \big[\tilde{f}_{\Lambda\Sigma}(N)(W^A - 2\mathrm{i}m^\Lambda\Phi)^T\epsilon(W^\Sigma - 2\mathrm{i}m^\Sigma\Phi) + \text{h.c.}\big]_F - \\ -2\big[e_\Lambda m^\Lambda \Omega L - \frac{\mathrm{i}}{4}e_\Lambda \Omega D^T\epsilon(W^\Lambda - 2\mathrm{i}m^\Lambda\Phi)\big]_D \tag{7.5}$$

where L represents an arbitrary real superfield. Elimination of Φ_α from the action is done by varying L and Φ_α in (7.5), leading to

$$\partial_L K = e_\Lambda m^\Lambda \Omega\,, \tag{7.6}$$

$$0 = \tilde{f}_{\Lambda\Sigma}m^\Sigma(W^\Lambda_\alpha - 2\mathrm{i}m^\Lambda\Phi_\alpha)^T\epsilon + \frac{\mathrm{i}}{8}e_\Lambda m^\Lambda(\bar{D}^T\epsilon D)D_\alpha\Omega\,. \tag{7.7}$$

For further computation it is convenient to rotate the k vector fields by an operator S such that the "vector" $(Sm)^\Lambda$ has only one component denoted by m, i.e. $(Sm)^\Lambda = (m, 0, ..., 0)^T$. Furthermore we make the definitions

- $e'_\Lambda := (eS^{-1})_\Lambda$
- $W'^\Lambda := (SW)^\Lambda$
- $g_{\Lambda\Sigma} := [(S^{-1})^T\tilde{f}S^{-1}]_{\Lambda\Sigma}$.

The purpose to introduce the new basis is to considerably simplify the dualization. After rescaling Ω according to $e'_1\Omega \;\to\; \Omega$ we see that (7.5) and (7.6) imply a Legendre transformation in the sense that

$$U(\Omega) := [-K(L) + m\Omega L] \tag{7.8}$$

defines a real function of $U(\Omega)$. The Legendre transformed Lagrangian is then

$$-2\big[U(\Omega)\big]_D + \big[g_{11}(N)(W'^1 - 2\mathrm{i}m\Phi)^T\epsilon(W'^1 - 2\mathrm{i}m\Phi) + \\ +2g_{a1}(N)(W'^a)^T\epsilon(W'^1 - 2\mathrm{i}m\Phi) + \text{h.c.}\big]_F + \big[-\frac{\mathrm{i}}{4}\Omega D^T\epsilon(W'^1 - \\ -2\mathrm{i}m\Phi_\alpha)\big]_D + \big[g_{ab}(W'^a)^T\epsilon W'^b + \text{h.c.}\big]_F, \tag{7.9}$$

$a, b = 2, \ldots, k$

With the help of the other equation of motion for the spinor superfield (7.7) one obtains as a result the Lagangian of the dual theory given by

$$\Big[\begin{pmatrix} \bar{D}^2 D\Omega & W'^a \end{pmatrix} \begin{pmatrix} (64 g_{11})^{-1} & -\frac{\mathrm{i}}{8}\frac{g_{a1}}{g_{11}} \\ -\frac{\mathrm{i}}{8}\frac{g_{a1}}{g_{11}} & g_{ab} - \frac{g_{a1}g_{b1}}{g_{11}} \end{pmatrix} \begin{pmatrix} \bar{D}^2 D\Omega \\ W'^a \end{pmatrix} + \text{h.c.}\Big]_F - \\ -2\big[U(\Omega)\big]_D \tag{7.10}$$

In what follows, the matrix describing the couplings of Ω and W'^a will be denoted by $\hat{f}_{\Lambda\Sigma}$. In supersymmetric theories the D-term contribution to the scalar potential is

$$V = \frac{1}{2}\mathrm{Re}\hat{f}_{\Lambda\Sigma} D^\Lambda D^\Sigma \tag{7.11}$$

and the D-term is given by

$$D^\Lambda \sim (\mathrm{Re}\hat{f}_{\Lambda\Sigma})^{-1}\big(\frac{\mathrm{i}}{2}k^j_\Sigma \frac{\partial K}{\partial \phi^j} + \text{c.c.}\big) \tag{7.12}$$

Here K is the Kählerpotential and k^j_Σ denotes the Killing vector of the gauged isometry. The Killing vector is constant for the shift symmetry $\phi \to \phi + \mathrm{i}$. Furthermore, the Kählerpotential can only depend on the real part of the scalar field ϕ because otherwise the shift symmetry could not be an isometry of the scalar manifold. Hence, $K(\phi,\bar{\phi}) = K(\mathrm{Re}\,\phi)$ and it follows directly for the scalar potential that

$$V \sim (\mathrm{Re}\,\hat{f}_{11})^{-1}(K')^2 \tag{7.13}$$

At this point it is suitable to rescale again Ω in order to get rid of the factor $\frac{1}{8}$. Furthermore, let us display $(\mathrm{Re}\hat{f})^{-1}$

$$(\mathrm{Re}\hat{f})^{-1} = \begin{pmatrix} \mathrm{Re}g_{11} + \mathrm{Im}g_{1\Lambda}[(\mathrm{Re}g)^{-1}]^{\Lambda\Sigma}\mathrm{Im}g_{\Sigma 1} & -\mathrm{Im}g_{1\Lambda}[(\mathrm{Re}g)^{-1}]^{\Lambda a} \\ -\mathrm{Im}g_{1\Lambda}[(\mathrm{Re}g)^{-1}]^{\Lambda b} & [(\mathrm{Re}g)^{-1}]^{ab} \end{pmatrix} \tag{7.14}$$

In order to fix scaling factors in front of terms involving Ω, the kinetic term $U(\Omega)$ has to be expanded into its component fields. The expansion is carried out in more detail in appendix F, and the result is

$$\begin{aligned} U(\Omega)|_D &= \frac{1}{2}U'(C)\cdot[D(x) + \frac{1}{2}\Box C(x)] + \frac{1}{4}U''(C)\cdot[\frac{1}{2}(M^2+N^2) - \bar{\omega}\lambda - \frac{1}{2}A_\mu A_\nu \eta^{\mu\nu} + \frac{1}{4}\bar{\omega}\gamma^\mu\partial_\mu\omega] + \\ &+ \frac{\mathrm{i}}{16}U'''(C)\cdot[-M\bar{\omega}\gamma_5\omega + N\bar{\omega}\omega + \bar{\omega}\gamma^\mu\gamma_5\omega^m A_\mu] + \frac{1}{64}U^{(4)}(C)\cdot\bar{\omega}\omega\bar{\omega}\omega \end{aligned} \tag{7.15}$$

Here, $U^{(n)}$ denotes the nth derivative after $C(x)$. At this point, it is convenient to rescale the real superfield Ω once again but this time we absorb the factor m appearing in the Legendre transformation

$$\Omega \quad \rightarrow \quad \frac{1}{m}\Omega \tag{7.16}$$

This rescaling does not only bring the Legendre transformation into a normalized form, but will also simplify the reverse transformation to the old basis system.
Next, the auxiliary field D has to be eliminated. In order to achieve this, we introduce a new field strength superfield, corresponding to Ω, according to:

$$W_\alpha^{(\Omega)} \quad := \quad -\frac{1}{4}(\bar{D}\gamma_5 D)D_\alpha\Omega. \tag{7.17}$$

If we want to still keep the normalization of the F-term in formula (7.10) then another rescaling is necessary. Now, in considering all the scalings, done so far, then altogether the superfield Ω must be rescaled after (7.10) as follows

$$\Omega \quad \rightarrow \quad -\frac{2}{m}\Omega\,. \tag{7.18}$$

Having a look at the decomposition into component fields, we can read off the D-terms of the Lagrangian where the tensor fields are eliminated by their equations of motion. The D dependent terms are found in (7.15) and according to (7.11) in the potential. Collecting all these terms leaves us with

$$\begin{aligned}&\frac{1}{2}U'D^\Omega + \frac{1}{2}\cdot\frac{4}{m^2}\cdot(\mathrm{Re}\hat{f})_{11}(D^\Omega)^2 + \frac{2}{m}\cdot(\mathrm{Re}\hat{f})_{1a}D^\Omega D^a + \\ &+\frac{1}{2}\cdot(\mathrm{Re}\hat{f})_{ab}D^a D^b\end{aligned} \tag{7.19}$$

The equations of motion for the auxiliary field are obtained from the variation after D^Ω and D^a, respectively, and are given by

$$\frac{1}{2}U' + \frac{4}{m^2}\cdot(\mathrm{Re}\hat{f})_{11}D^\Omega + \frac{2}{m}\cdot(\mathrm{Re}\hat{f})_{1a}D^a \quad = \quad 0 \tag{7.20}$$

$$(\mathrm{Re}\hat{f})_{ab}D^b + \frac{2}{m}\cdot(\mathrm{Re}\hat{f})_{1a}D^\Omega \quad = \quad 0 \tag{7.21}$$

These equations are equivalent to the following equations at the component level of the coupling matrix $\hat{f}$ (see appendix F for more details)

$$\begin{aligned} 0 &= \frac{1}{2}U' + \frac{4}{m^2}\cdot\frac{\mathrm{Re}g_{11}}{|g_{11}|^2}\cdot D^{\Omega} - \frac{2}{m}\cdot\frac{\mathrm{Im}g_{1a}\mathrm{Re}g_{11} - \mathrm{Re}g_{1a}\mathrm{Im}g_{11}}{|g_{11}|^2}\cdot D^a \qquad (7.22)\\ 0 &= [\mathrm{Re}g_{ab} + \frac{1}{|g_{11}|^2}\cdot(\mathrm{Re}g_{11}\mathrm{Im}g_{1a}\mathrm{Im}g_{1b} - \mathrm{Re}g_{11}\mathrm{Re}g_{1a}\mathrm{Re}g_{1b} - \\ &\quad -\mathrm{Im}g_{1a}\mathrm{Re}g_{1b}\mathrm{Im}g_{11} - \mathrm{Re}g_{1a}\mathrm{Im}g_{1b}\mathrm{Im}g_{11})]D^b - \frac{2}{m}\cdot\frac{1}{|g_{11}|^2}\cdot \\ &\quad \cdot(\mathrm{Im}g_{1a}\mathrm{Re}g_{11} - \mathrm{Re}g_{1a}\mathrm{Im}g_{11})\cdot D^{\Omega} \qquad (7.23)\end{aligned}$$

After some calculation (given in appendix F), one finds that the auxiliary fields are given by

$$D^b = -\frac{m}{4}U'\cdot[(\mathrm{Re}g)^{-1}]^{bA}\mathrm{Im}g_{A1} \qquad (7.24)$$

$$D^{\Omega} = -\frac{m^2}{8}U'\cdot(\mathrm{Im}g_{1A}[(\mathrm{Re}g)^{-1}]^{AB}\mathrm{Im}g_{B1} + \mathrm{Re}g_{11}) \qquad (7.25)$$

These expressions can be compared with the expression for the inverse of the coupling matrix $\left(\mathrm{Re}\,\hat{f}\right)$ (7.14) and the components of the inverse matrix can be identified in the following way

$$[(\mathrm{Re}\hat{f})^{-1}]^{11} = -\frac{8}{m^2U'}\cdot D^{\Omega} \qquad (7.26)$$

$$[(\mathrm{Re}\hat{f})^{-1}]^{1a} = \frac{4}{mU'}\cdot D^a \qquad (7.27)$$

Reintroducing these identifications into the D-terms (7.19) of the component decomposition and carrying out the calculation leads to the following result for the potential

$$V = \frac{4(U')^2m^2}{32}\cdot[\mathrm{Re}g_{11} + \mathrm{Im}g_{1\Lambda}[(\mathrm{Re}g)^{-1}]^{\Lambda\Sigma}\mathrm{Im}g_{\Sigma 1}]\,. \qquad (7.28)$$

At this point we make the rotation $S^{-1}(m,0,...,0)^T = m^A$ that brings us back to the original basis for the electric and magnetic coupling, and we obtain

$$V = \frac{4(U')^2}{32}\cdot[\mathrm{Re}\tilde{f}_{\Lambda\Sigma}\; m^{\Lambda}m^{\Sigma} + m^{\Lambda}\;\mathrm{Im}\tilde{f}_{\Lambda\Sigma}[(\mathrm{Re}\tilde{f})^{-1}]^{\Sigma\Omega}\mathrm{Im}\tilde{f}_{\Omega\Gamma}\; m^{\Gamma}]\,. \qquad (7.29)$$

After expanding $\tilde{f}$ according to (7.4) we find that this is nothing else but

$$\begin{aligned} V &= \frac{1}{32}(U')^2\left[\left(e_{\Sigma} + 2m^{\Lambda}\mathrm{Im}f_{\Lambda\Sigma}\right)[(\mathrm{Re}f)^{-1}]^{\Sigma\Omega}\left(e_{\Omega} + 2\mathrm{Im}f_{\Omega\Gamma}m^{\Gamma}\right) + \right.\\ &\quad \left. +4\,\mathrm{Re}f_{\Lambda\Sigma}m^{\Lambda}m^{\Sigma}\right] \qquad (7.30)\end{aligned}$$

which is in total agreement with the expression that was obtained for the potential in [90]. The discussion shows that actually the potential (7.30) is symplectically equivalent to a D-term potential in its standard form (7.11). In [90] the authors could not connect this potential with a potential of standard form. However, we demonstrated that the explicit mass term for the scalar in the tensor multiplet is absorbed by the redefinition of the gauge kinetic function (7.4) due to the possibility to add total derivatives to the action. This confirms that neither generalized Chern-Simons terms nor the topological B-term cause any new nontrivial contribution to the standard D-term potential in $\mathcal{N} = 1$ supersymmetry.

8 Conclusion

In this thesis we studied quantum anomalies and generalized Chern-Simons terms in chiral gauge theory. We discussed this topic in global and local $\mathcal{N} = 1$ supersymmetry and in general gauge theories that are covariant with respect to electric/magnetic duality. This generalized previous works [26, 34, 83], in which only *classically* gauge invariant theories with anomaly-free fermionic spectra were considered.

We began our discussion with generalized Chern-Simons terms along the lines of [34]. The authors of that paper showed how generalized Chern-Simons terms can cancel certain constant shifts of the gauge kinetic function in the context of Lie algebra cohomology. The generalized Chern-Simons terms originate from Lie algebra valued forms $C(A, \mathcal{F})$ that are defined by constant tensors $C_{AB,C}$ which have to satisfy the constraint $C_{(AB,C)} = 0$ inter alia. It is possible to show that the constraints correspond to the requirement of $C(A, \mathcal{F})$ being closed with respect to the exterior derivative. The complicated formalism leads then directly to the result that in semisimple gauge theories one can always absorb the generalized Chern-Simons terms by a redefinition of the gauge kinetic function. We generalized the forms $C(A, \mathcal{F})$ by relaxing the closure condition such that we allowed for forms with nontrivial symmetric part $C_{(AB,C)}$. Consequently, these more general forms were no longer closed, which apparently violated the procedure to construct generalized Chern-Simons terms. However, with the help of the Stora-Zumino descent equations we were able to show that only the generalized Chern-Simons terms together with suitable gauge anomalies could cancel the constant shifts of the gauge kinetic function. This generalizes the results of [34] and concludes section 3.

In section 5, we studied the consistency conditions that ensure the gauge and supersymmetry invariance of global and local matter coupled $\mathcal{N} = 1$ supersymmetry theories with Peccei-Quinn terms, generalized Chern-Simons terms and quantum anomalies. Each of these three ingredients defines a constant three index tensor:

(i) The gauge non-invariance of the Peccei-Quinn terms is proportional to a constant imaginary shift of the gauge kinetic function parameterized by a tensor $C_{AB,C}$. This tensor in general splits into a completely symmetric part and a part of mixed symmetry, $C^{(\mathrm{s})}_{AB,C} + C^{(\mathrm{m})}_{AB,C}$.

(ii) Generalized Chern-Simons terms are defined by a tensor, $C^{(\mathrm{CS})}_{AB,C}$, of mixed symmetry.

(iii) Quantum gauge anomalies of chiral fermions are proportional to a completely symmetric tensor d_{ABC}.

We found that the full quantum effective action is only gauge invariant and supersymmetric if

$$C_{AB,C} = C_{AB,C}^{(\mathrm{CS})} + d_{ABC} \, . \tag{8.1}$$

The inclusion of the quantum anomalies encoded in a non-trivial tensor d_{ABC} is the key feature that distinguishes $\mathcal{N} = 1$ theories from theories with extended supersymmetry as the latter theories cannot have chiral gauge interactions and hence no quantum anomalies.
First we performed our analysis in global $\mathcal{N} = 1$ supersymmetry and later also in $\mathcal{N} = 1$ supergravity. The interesting result is that the Chern-Simons term does not need any gravitino corrections when added as such to the matter-coupled supergravity actions. This completes the comprehension of $\mathcal{N} = 1$ supersymmetry, generalizing earlier work of [26] on *Abelian* generalized Chern-Simons terms in *global* $\mathcal{N} = 1$ supersymmetry *without* quantum anomalies.
In [12], orientifold compactifications with anomalous fermion spectra were studied, in which the chiral anomalies are cancelled by a mixture of the Green-Schwarz mechanism and generalized Chern-Simons terms. The analysis in [12] was mainly concerned with the gauge invariance of the bosonic part of the action and revealed the generic presence of a completely symmetric and a mixed part in $C_{AB,C}$ and the generic necessity of generalized Chern-Simons terms. Our results show how such theories can be embedded into the framework of $\mathcal{N} = 1$ supergravity and supplements the phenomenological discussions of [12] by the fermionic couplings in a supersymmetric setting. The fermionic couplings were used in the presentation of [96] where the discussion of [12] was lifted to an extension of the MSSM based on our results.

In section 6 we have shown how general gauge theories with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies [88] can be formulated in a way that is covariant with respect to electric/magnetic duality transformations. This generalizes previous work of [83], in which only *classically* gauge invariant theories with anomaly-free fermionic spectra were considered. Whereas the work [83] was modelling extended (and hence automatically anomaly-free) gauged supergravity theories, our results here can be applied to general $\mathcal{N} = 1$ gauged supergravity theories with possibly anomalous fermionic spectra. Such anomalous fermionic spectra are a natural feature of many string compactifications, notably of intersecting brane models in type II orientifold compactifications [16–22], where also GCS terms frequently occur [12]. Especially in combination with background fluxes, such compactifications may naturally lead to four-dimensional actions with tensor fields and gaugings in unusual duality frames. Our formulation accommodates all these non-standard formulations, just as ref. [83] does in the anomaly-free case.

At a technical level, our results were obtained by relaxing the so-called representation constraint to allow for a symmetric three-tensor d_{MNP} that parameterizes the quantum anomaly. In contrast to the other constraints for the embedding tensor, this modified representation constraint is not homogeneous in the embedding tensor, which is a novel feature in this formalism. Also our treatment gave an interpretation for the physical meaning of the "representation" constraint: In its original form used in [83], it simply states the absence of quantum anomalies. It is interesting, but in retrospect not surprising, that the extended supergravity theories from which the original constraint has been derived in [83], need this constraint for their internal classical consistency.

In section 7 we reinvestigated the result of [90] who proposed an $\mathcal{N} = 1$ superfield action for one massive tensor multiplet coupled to vector and chiral multiplets. The potential corresponding to this theory displayed a direct mass term for the scalar in the tensor multiplet which apparently violated the form of the D-term potential. We demonstrated that this 'unusual' form of the potential is actually equivalent to a standard form. The reason is that the direct mass term for the scalar in the tensor multiplet can be absorbed by a suitable redefinition of the gauge kinetic function by means of a total derivative.

The theory of massive tensor multiplets represents the supersymmetrization of a special Abelian gauging of the manifestly symplectically covariant framework proposed in [83] and presented in appendix E.

We are led to the conclusion that neither the generalized Chern-Simons terms nor the topological couplings to the tensor fields cause contributions that violate the standard form of the D-term potential.

In this thesis we have neither touched the topic of gravitational anomalies nor of Kähler anomalies [81, 82, 97–105] in $\mathcal{N} = 1$ supergravity.

The results of this thesis can be taken as the starting point for phenomenological models such as [96]. We could show that in the framework $\mathcal{N} = 1$ supersymmetry, as discussed in [81, 82, 96], one has to take additional fermionic couplings

$$C_{(AB,D)} A_\mu^D \bar{\lambda}^A \gamma_5 \gamma^\mu \lambda^B \tag{8.2}$$

into account. These new fermionic couplings had not been considered before (probably because to date it is not clear how they could originate from a superfield expression) and it would be interesting to study explicit $\mathcal{N} = 1$ string compactifications within the framework used in this thesis.

A Technical details on Lie algebra cohomology

A.1 The Laplace equation of Lie algebra cohomology

In section 3.2 we found that generalized Chern-Simons terms are trivial for semisimple algebras. In this appendix we want to demonstrate how cohomological arguments lead to the same result.

The Cartan-Killing metric is defined as $g_{AB} := -f_{AL}{}^{K} f_{BK}{}^{L}$ and assumed to exist as well as to be invertible. This will allow us to construct another operator $\mathcal{I}$ besides the algebraic operator $\mathcal{D}$. We still introduce some notational issues, that will simplify some of the upcoming computations and allow easily for generalizations. It is again suitable to introduce the operator

$$(\mathrm{T}_A)_{CD}^{EF} \quad := \quad f_{AC}{}^{E}\, \delta_D^F + f_{AD}{}^{F}\, \delta_C^E\,. \tag{A.1}$$

Note, that with this generator at hand, we can bring (3.5) into the form $\delta \mathrm{f}_{AB} = \Lambda^C (\mathrm{T}_C)_{AB}^{DF} \mathrm{f}_{DF}$. In order to reduce clutter, we further introduce a single Greek multi-index $\alpha := AB$, representing the two indices A and B that it is now $(\mathrm{T}_A)_{CD}^{EF} \to (\mathrm{T}_A)_{\alpha}^{\beta}$. This is equivalent to the $\cdot$ notation for contractions of such double indices introduced in section 3.2. Observe that with this multi-index at hand we can write $C_{\alpha,D}$ instead of $C_{AB,D}$ and, furthermore, $\mathcal{F}^A\mathcal{F}^B \to \mathcal{F}^{\alpha}$. When we compute the Bianchi identity for $\mathcal{F}^{\alpha}$, starting from $d(\mathcal{F}^A\mathcal{F}^B)$, we find

$$d\mathcal{F}^{\alpha} \quad = \quad -(T_D)_{\beta}^{\alpha} A^D \mathcal{F}^{\beta}\,. \tag{A.2}$$

With help of the Cartan-Killing metric we can introduce a new operator, called $\mathcal{I}$, which is defined as

$$(\mathcal{I}C)_{\alpha,D_1\ldots D_n} \quad := \quad (n+1) C_{\beta,ED_1\ldots D_n} g^{EF} (\mathrm{T}_F)_{\alpha}^{\beta}\,. \tag{A.3}$$

Note that this operator lowers the amount of indices after the comma by one, as opposed to $\mathcal{D}$, which increases the amount of indices after the comma by one. There is still another possibility to generate this operator, by introducing a new operation τ which acts trivially on the gauge connection, $\tau A^D = 0$, while on the multilinear field strength form it acts as $\tau \mathcal{F}^{\alpha} = [(T_A)\mathcal{F}]^{\alpha} g^{AB} \frac{\partial}{\partial A^B}$. This form will be convenient to use in applications such as the example of an Abelian×semisimple gauge group. Therefore, the operator that is not changing the amount of indices is given as $\mathcal{DI}+\mathcal{ID}$, that is formally similar to the Laplace operator in Cartan calculus. Let us try to evaluate this Laplace operator acting on $C_{\alpha,D}$. Following the definitions (3.13) and (A.3), we find the following relevant relations

$$(\mathcal{D}C)_{\alpha,AB} = \frac{1}{2}\, f_{AB}{}^{D} C_{\alpha,D} + C_{\beta,[B}(\mathrm{T}_{A]})^{\beta}_{\alpha}\,, \tag{A.4}$$

$$(\mathcal{D}C)_{\alpha,A} = (\mathrm{T}_A)^{\beta}_{\alpha}\, C_{\beta}\,, \tag{A.5}$$

$$(\mathcal{I}C)_{\alpha} = (\mathrm{T}_A)^{\beta}_{\alpha}\, C_{\beta,B}\, g^{AB}\,, \tag{A.6}$$

$$(\mathcal{I}C)_{\alpha,A} = 2\,(\mathrm{T}_C)^{\beta}_{\alpha}\, C_{\beta,BA}\, g^{BC}\,. \tag{A.7}$$

The equation (A.5) is proven by acting on $C_\alpha \mathcal{F}^\alpha$ with the exterior derivative and making use of (A.2). From (A.5) and (A.6) we can read off

$$[\mathcal{D}(\mathcal{I}C)]_{\alpha,A} = (\mathrm{T}_A)^{\beta}_{\alpha}\, C_{\gamma,D}\, (\mathrm{T}_B)^{\gamma}_{\beta}\, g^{BD}\,, \tag{A.8}$$

while (A.4) and (A.7) give

$$[\mathcal{I}(\mathcal{D}C)]_{\alpha,A} = C_{\beta,E}\, g^{BD}\, f_{DA}{}^{E}\, (\mathrm{T}_B)^{\beta}_{\alpha} + 2\,(\mathrm{T}_B)^{\beta}_{\alpha}\, g^{BD} C_{\gamma,[A}(\mathrm{T}_{D]})^{\gamma}_{\beta}\,. \tag{A.9}$$

The first term in (A.9) can be manipulated as follows

$$\begin{aligned} g^{BD}\, f_{AD}{}^{E}\, (\mathrm{T}_B)^{\beta}_{\alpha} &= g_{AG}\, f^{EG}{}_{D}\, (\mathrm{T}^D)^{\beta}_{\alpha} \\ &= g_{AG}\, [\mathrm{T}^E, \mathrm{T}^G]^{\beta}_{\alpha} \\ &= -g^{EB}\, [\mathrm{T}_A, \mathrm{T}_B]^{\beta}_{\alpha}\,, \end{aligned}$$

where in the first line we used the Cartan-Killing metric to pull indices up and down. In the second line we made use of $[\mathrm{T}^A, \mathrm{T}^B] = f^{AB}{}_C \mathrm{T}^C$ which is true when the metric is invertible. Inserting this result back into (A.9), we obtain

$$[\mathcal{I}(\mathcal{D}C)]_{\alpha,A} = C_{\gamma,A} g^{BD} (\mathrm{T}_B)^{\beta}_{\alpha} (\mathrm{T}_D)^{\gamma}_{\beta} - C_{\gamma,D}\, (T_B)^{\gamma}_{\beta}\, (T_A)^{\beta}_{\alpha}\, g^{BD}\,. \tag{A.10}$$

Adding (A.10) and (A.8) together, we find the Laplace equation

$$[(\mathcal{DI}+\mathcal{ID})C]_{\alpha,D} = C_{\beta,D}\, \mathcal{C}_2(T)^{\beta}_{\alpha}\,, \tag{A.11}$$

where we had defined the Casimir operator

$$\mathcal{C}_2(T)^{\gamma}_{\alpha} := g^{AB}\, (T_A)^{\gamma}_{\beta}\, (T_B)^{\beta}_{\alpha} \tag{A.12}$$

of the gauge group. We see, that the action of the Laplacian on the forms C is proportional to the Casimir operator. From (A.11) we can read of a very important result. For gauge groups that possess a nonsingular Cartan-Killing metric, i.e. semisimple gauge groups, every closed form C is exact. The Laplace equation does not only tell us that the cohomology class is trivial but provides us with an explicit expression for C, namely,

$$C_{\alpha,D} \quad = \quad [\mathcal{D}(\mathcal{I}C)]_{\beta,D}\, C_2^{-1}(T)^{\beta}_{\alpha} \tag{A.13}$$

which is equivalent to (3.34). There are no generalized Chern-Simons terms for semisimple groups necessary because they can be always absorbed into a redefinition of the gauge kinetic function itself.

A.2 Application: Abelian×semisimple

The results of section 3.3 can be obtained with help of Lie algebra cohomology as well. It might not be too instructive to do so, but it is a consistency check for the developed formalism and shows, how the formalism is applied.

The product structure is again reflected by splitting the adjoint indices $A, B, ...$ into indices $a, b, c, ...$ for the Abelian part and adjoint indices $x, y, z, w, ...$ for the semisimple part. Due to the group structure, only the structure constants of the type $f_{xy}{}^{z}$ are nonzero. The five-form $C(A, \mathcal{F})$ corresponding to the mixed group structure is defined by

$$C(A,\mathcal{F}) \quad = \quad 2C_{(xb),a}A^a\mathcal{F}^xF^b + C_{xy,a}A^a\mathcal{F}^x\mathcal{F}^y + 2C_{(ax),y}A^yF^a\mathcal{F}^x\,, \tag{A.14}$$

with constants $C_{xb,a}$, $C_{bx,a}$, $C_{xy,a}$, $C_{ax,y}$ and $C_{ya,x}$. In order to be able to define the operator $\mathcal{D}$, we have to evaluate the exterior derivative acting on C. The computation which makes use of the structure equations and the Bianchi identities of Abelian×semisimple leads to

$$\begin{aligned} dC(A,\mathcal{F}) \quad = \quad & (C_{xy,a} + C_{ax,y} + C_{ya,x})\mathcal{F}^x\mathcal{F}^yF^a + \\ & +(C_{vb,a}f_{xy}{}^{v} + C_{bv,a}f_{xy}{}^{v})A^aA^x\mathcal{F}^yF^b + \\ & +(f_{u(y}{}^{v}C_{x)v,a} + f_{u(y}{}^{v}C_{x)v,a})A^uA^a\mathcal{F}^x\mathcal{F}^y + \\ & +(f_{uy}{}^{v}C_{v,(xa)} + 2C_{(va),[y}f_{u]x}{}^{v})A^uA^y\mathcal{F}^xF^a \end{aligned} \tag{A.15}$$

Observe that for $C(A, \mathcal{F})$ to be homogenous in the field strength forms, the first line has to vanish, which requires the components of $C(A,\mathcal{F})$ to satisfy

$$C_{(ya,x)} = 0\,. \tag{A.16}$$

Now we can write down the action of an algebraic operator on the coefficients of $C(A,\mathcal{F})$

$$\begin{aligned} dC(A,\mathcal{F}) &= 2(\mathcal{D}C)_{(yb),ax}A^aA^x\mathcal{F}^yF^b + (\mathcal{D}C)_{xy,ua}A^uA^a\mathcal{F}^x\mathcal{F}^y + \\ &\quad +2(\mathcal{D}C)_{(xa),uy}A^uA^y\mathcal{F}^xF^a\,, \end{aligned} \tag{A.17}$$

where we define

$$(\mathcal{D}C)_{ax,(yb)} := C_{(vb),a}f_{xy}{}^v \tag{A.18}$$

$$(\mathcal{D}C)_{xy,ua} := 2f^v_{u(y}C_{x)v,a} \tag{A.19}$$

$$(\mathcal{D}C)_{(xa),uy} := \frac{1}{2}C_{(xa),v}f_{uy}{}^v + C_{(va),[y}f_{u]x}{}^v\,. \tag{A.20}$$

Hence, the algebraic condition for $C(A,\mathcal{F})$ being closed are obtained in setting above relations to zero and we obtain

$$f^v_{xu}C_{vb,a} = 0 \tag{A.21}$$

$$f^v_{xy}C_{bv,a} = 0 \tag{A.22}$$

$$f_{u(y}{}^vC_{x)v,a} = 0 \tag{A.23}$$

$$f_{uy}{}^vC_{ax,v} + f_{xy}{}^vC_{av,u} - f_{xu}{}^vC_{av,y} = 0 \tag{A.24}$$

$$f_{uy}{}^vC_{xa,v} + f_{xy}{}^vC_{va,u} - f_{xu}{}^vC_{va,y} = 0\,, \tag{A.25}$$

which are exactly equal to the relations (3.40) to (3.44).

As before, we have to define the operator $\mathcal{I}$ which allows us to compute the Laplace equation for this case. It is convenient to do this by means of τ which is defined by the following relations

$$\tau[F^c\mathcal{F}^x] := (f_{vy}{}^x\delta^c_b)F^b\mathcal{F}^yg^{zv}\frac{\partial}{\partial A^z}\,, \tag{A.26}$$

$$\tau[\mathcal{F}^x\mathcal{F}^y] := (f_{uv}{}^x\delta^y_z + f_{uz}{}^y\delta^x_v)\mathcal{F}^v\mathcal{F}^zg^{uw}\frac{\partial}{\partial A^w}\,, \tag{A.27}$$

while $\tau A = 0$ for Abelian or semisimple gauge connections. It is not a difficult but a little lengthy computation to verify that the action of the operators $\mathcal{I}\mathcal{D}$ and $\mathcal{D}\mathcal{I}$ on the different components of $C(A,\mathcal{F})$ and the only nonvanishing contributions are

$$\mathcal{I}(\mathcal{D}C)_{xy,a} = 2C_{sv,a}f_{uy}{}^{v}f_{rx}{}^{s}g^{ru} + 2C_{xv,a}\mathcal{C}_2(f)_y^v \tag{A.28}$$

$$\mathcal{I}(\mathcal{D}C)_{(ax),y} = C_{(av),y}\mathcal{C}_2(f)_x^v - C_{(as),v}f_{uk}{}^{s}f_{yx}{}^{k}g^{ru} \tag{A.29}$$

$$\mathcal{D}(\mathcal{I}C)_{(ax),y} = C_{(av),u}f_{rs}{}^{v}f_{yx}{}^{s}g^{ru}\,. \tag{A.30}$$

These relations lead directly to the Laplace equation for the example of an Abelian×semisimple gauge group

$$(\tau d + d\tau)C(A,\mathcal{F}) = 2C_{sv,a}f^v_{uy}f^s_{rx}g^{ru} + 2C_{xv,a}\mathcal{C}_2(f)^v_y + C_{(av),y}\mathcal{C}_2(f)^v_x \tag{A.31}$$

Observe that the first term represents an inhomogenous term. This means that a possible generalized Chern-Simons term proportional to $C_{xy,a}$ cannot be trivial, i.e., it cannot be absorbed into a redefinition of the gauge kinetic function, even if C is a closed form. In section 3.3 we could show from the closure constraint that it is actually of the form $C_{xy,a} = B_a g_{xy}$ with the B_a's being arbitrary but constant.

B Notation and conventions

Notations and conventions are chosen in agreement with [106]. We use the Minkowski metric $\eta = \mathrm{diag}(-1,+1,+1,+1)$ and the epsilon tensor $\varepsilon^{0123} = +1$. The Dirac matrices satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ and $\gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. The generators of the spinor representation of SO(1,3) are defined as $\sigma^{\mu\nu} := \frac{1}{4}[\gamma^\mu,\gamma^\nu]$. Obviously, the gamma matrices fulfil

$$\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = -\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\gamma_5 \tag{B.1}$$

because both sides are completely antisymmetric. The factor $-\mathrm{i}$ appears due to our definition of γ_5. Contracting both sides with γ_σ from the right, one can derive

$$\gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = -\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma\,. \tag{B.2}$$

Another useful relation is

$$2\sigma^{\mu\nu}\gamma^\rho = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} + \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu \tag{B.3}$$

which is almost trivial if one considers the three nontrivial cases $\mu \neq \nu \neq \rho$, $\nu = \rho \neq \mu$ and $\nu \neq \rho = \mu$ separately. Then, it is not difficult to derive the following two relations

$$\sigma^{\mu\nu}\gamma^{\rho} = \frac{1}{2}(g^{\nu\rho}\gamma^{\mu} - g^{\mu\rho}\gamma^{\nu} - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma)\,, \tag{B.4}$$

$$\gamma^{\rho}\sigma^{\mu\nu} = \frac{1}{2}(g^{\rho\mu}\gamma^{\nu} - g^{\rho\nu}\gamma^{\mu} - \mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma)\,, \tag{B.5}$$

that are quite useful for actual computations.

The Dirac matrices are given in a special representation by

$$\gamma^0 = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = -\mathrm{i}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.6}$$

with the usual Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.7}$$

The Dirac conjugated spinor is defined by $\bar{u} = u^{\dagger}\beta$ where

$$\beta := \mathrm{i}\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.8}$$

and $u^{\dagger} = u^{*\mathrm{T}}$. We use Majorana spinors $u = \begin{pmatrix} e\xi \\ \xi^* \end{pmatrix}$ where e is the antisymmetric matrix

$$e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.9}$$

Majorana spinors are required to fulfil

$$u^* = -\beta\gamma_5\epsilon u \tag{B.10}$$

and the matrix ϵ is defined according to $\epsilon = \mathrm{diag}(e, e)$. The charge conjugation matrix is defined by

$$\mathcal{C} = \gamma_2\beta = -\epsilon\gamma_5 \tag{B.11}$$

and, then, for a Majorana spinor one has $\bar{u} = u^{\mathrm{T}}\mathcal{C}$. The gamma matrices obey

$$\begin{aligned} \mathcal{C}\gamma_\mu\mathcal{C}^{-1} &= -\gamma_\mu^T , &\text{(B.12)}\\ \mathcal{C}\gamma_5\mathcal{C}^{-1} &= \gamma_5^T , &\text{(B.13)}\\ \mathcal{C}\sigma_{\mu\nu}\mathcal{C}^{-1} &= -\sigma_{\mu\nu}^T , &\text{(B.14)}\\ \mathcal{C}\gamma_\mu\gamma_5\mathcal{C}^{-1} &= (\gamma_5\gamma_\mu)^T . &\text{(B.15)} \end{aligned}$$

This allows one to prove easily $\bar{\lambda}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\lambda$ and similar relations because $\mathcal{C}$ is antisymmetric $\mathcal{C}^T = -\mathcal{C}$ and $\mathcal{C}^2 = -\mathbf{1}$. In total we have for anticommuting Majorana spinors

$$\begin{aligned} \bar{\lambda}\chi &= \bar{\chi}\lambda &\text{(B.16)}\\ \bar{\lambda}\gamma_\mu\chi &= -\bar{\chi}\gamma_\mu\lambda &\text{(B.17)}\\ \bar{\lambda}\sigma_{\mu\nu}\chi &= -\bar{\chi}\sigma_{\mu\nu}\lambda &\text{(B.18)}\\ \bar{\lambda}\gamma_5\chi &= \bar{\chi}\gamma_5\lambda &\text{(B.19)}\\ \bar{\lambda}\gamma_5\gamma_\mu\chi &= \bar{\chi}\gamma_5\gamma_\mu\lambda &\text{(B.20)}\\ & &\text{(B.21)} \end{aligned}$$

A very useful tool in order to manipulate bilinear of spinors are rearrangement formulas. For spinors θ, they are obtained from $\theta\bar{\theta}$ and the fact that the set of 16 covariant matrices $\{\mathbf{1},\, \gamma^\mu,\, \sigma^{\mu\nu},\, \gamma_5,\, \gamma_5\gamma^\mu\}$ is complete and $\mathbf{1}$ represents the unity. This means that any 4×4 matrix can be decomposed into a superposition of these, especially $\theta\bar{\theta}$. Taking Lorentz invariance into account, an expansion is given by

$$\begin{aligned} \theta\bar{\theta} &= a \cdot (\bar{\theta}\theta) + b \cdot \gamma^\mu(\bar{\theta}\gamma_\mu\theta) + c \cdot \sigma^{\mu\nu}(\bar{\theta}\sigma_{\mu\nu}\theta) + d \cdot \gamma_5(\bar{\theta}\gamma_5\theta) + \\ &\quad + e \cdot \gamma_5\gamma^\mu(\bar{\theta}\gamma_5\gamma_\mu\theta) \end{aligned} \tag{B.22}$$

where a, b, c, d, e are constants that have to be determined. It is immediately obvious that $b = c = 0$ because of (B.17) and (B.18). The remaining constants are found in multiplying from the right with $\{\mathbf{1},\, \gamma_5,\, \gamma_5\gamma_\mu\}$ and taking the trace, one obtains

$$\theta\bar{\theta} = \frac{1}{4}(\bar{\theta}\theta) + \frac{1}{4}\gamma_5\gamma^\mu(\bar{\theta}\gamma_5\gamma_\mu\theta) - \frac{1}{4}\gamma_5(\bar{\theta}\gamma_5\theta) . \tag{B.23}$$

A lot of useful relations can be obtained from (B.23) in multiplying on the right with $\mathcal{C}$ or decomposing θ into its left- and right-handed parts. In this way one finds

$$(\bar{\varepsilon}\theta)(\bar{\theta}\theta) = -(\bar{\varepsilon}\gamma_5\theta)(\bar{\theta}\gamma_5\theta) \tag{B.24}$$

$$(\bar{\varepsilon}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta) = -\eta_{\mu\nu}(\bar{\varepsilon}\gamma_5\theta)(\bar{\theta}\gamma_5\theta) \tag{B.25}$$

$$(\bar{\varepsilon}\gamma^\mu\theta)(\bar{\theta}\gamma_5\gamma_\mu\theta) = -4(\bar{\varepsilon}\theta)(\bar{\theta}\gamma_5\theta) \tag{B.26}$$

$$(\bar{\varepsilon}\gamma_5\theta)(\bar{\theta}\theta) = -(\bar{\varepsilon}\theta)(\bar{\theta}\gamma_5\theta) \tag{B.27}$$

$$(\bar{\theta}\theta)^2 = -(\bar{\theta}\gamma_5\theta)^2 \tag{B.28}$$

$$(\bar{\theta}\gamma_5\gamma_\mu\theta)(\bar{\theta}\gamma_5\gamma_\nu\theta) = -\eta_{\mu\nu}(\bar{\theta}\gamma_5\theta)^2 \tag{B.29}$$

$$\theta_R\bar{\theta}_L\theta_L = \theta_R\bar{\theta}\gamma_5\theta \tag{B.30}$$

$$2\bar{\theta}_L\gamma^\mu\theta_R = -\bar{\theta}\gamma_5\gamma^\mu\theta \tag{B.31}$$

Note that because spinors are anticommuting objects, products of more than two spinors vanish, i.e. $\theta_L\bar{\theta}_L\theta_L = 0$ and $\theta_R\bar{\theta}_R\theta_R = 0$.
Any product of five and more components of θ vanishes, so that the list of nontrivial products of spinor components is given by

$$\theta_\alpha\theta_\beta = \frac{1}{4}(\epsilon\gamma_5)_{\alpha\beta}(\bar{\theta}\theta) + \frac{1}{4}(\gamma_\mu\epsilon)_{\alpha\beta}(\bar{\theta}\gamma_5\gamma^\mu\theta) + \frac{1}{4}\epsilon_{\alpha\beta}(\bar{\theta}\gamma_5\theta) \tag{B.32}$$

$$\begin{aligned}\theta_\alpha\theta_\beta\theta_\gamma = & -\frac{1}{4}(\bar{\theta}\gamma_5\theta)[\epsilon_{\alpha\beta}\theta_\gamma - (\epsilon\gamma_5)_{\alpha\beta}(\gamma_5\theta)_\gamma - \epsilon_{\alpha\gamma}\theta_\beta + \\ & +(\epsilon\gamma_5)_{\alpha\gamma}(\gamma_5\theta)_\beta + \epsilon_{\beta\gamma}\theta_\alpha - (\epsilon\gamma_5)_{\beta\gamma}(\gamma_5\theta)_\alpha]\end{aligned} \tag{B.33}$$

$$\begin{aligned}\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta = & \frac{1}{16}(\bar{\theta}\theta)^2[\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} - (\epsilon\gamma_5)_{\alpha\beta}(\epsilon\gamma_5)_{\gamma\delta} - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \\ & +(\epsilon\gamma_5)_{\alpha\gamma}(\epsilon\gamma_5)_{\beta\delta} + \epsilon_{\beta\gamma}\epsilon_{\alpha\delta} - (\epsilon\gamma_5)_{\beta\gamma}(\epsilon\gamma_5)_{\alpha\delta}]\end{aligned} \tag{B.34}$$

From the relation (B.34) it follows that

$$(\bar{\theta}\theta)(\bar{\theta}\gamma_5\theta) = 0 \tag{B.35}$$

which is a useful relation once computations in superspace are performed.

The Lagrangian is real but some fields are described by complex valued objects. Terms involving these complex fields come always together with their hermitian conjugates so that effectively the real part of these terms appear. For bilinears of Majorana spinors one finds

$$(\bar{\lambda}_1 M\lambda_2) = (\bar{\lambda}_1\beta\epsilon\gamma_5 M^*\beta\epsilon\gamma_5\lambda_2) \tag{B.36}$$

where we made use of (B.10) and the fact that complex conjugation interchanges the spinors. In the representation for the gamma matrices given in (B.6) it is $\beta\epsilon\gamma_5\gamma_\mu^*\beta\epsilon\gamma_5 = \gamma_\mu$ and, thus,

$$\beta C M^* C^1 \beta \;=\; +M \text{ for } M = 1, \gamma_\mu, \sigma_{\mu\nu} \tag{B.37}$$

$$\beta C M^* C^1 \beta \;=\; -M \text{ for } M = \gamma_5, \gamma_\mu\gamma_\mu \tag{B.38}$$

from which one finds the properties of bilinears under complex conjugation

$$(\bar{\lambda}_1 M \lambda_2)^* \;=\; +(\bar{\lambda}_1 M \lambda_2) \text{ for } M = 1, \gamma_\mu, \sigma_{\mu\nu} \tag{B.39}$$

$$(\bar{\lambda}_1 M \lambda_2)^* \;=\; -(\bar{\lambda}_1 M \lambda_2) \text{ for } M = \gamma_5, \gamma_5\gamma_\mu \tag{B.40}$$

C Superspace

A convenient tool to treat computations in global $\mathcal{N} = 1$ supersymmetry is given by superspace. For supergravity it is not so helpful anymore, because before one can enjoy the convenience of superspace, one has to introduce differential geometry in curved superspace and one has to impose constraints which is a lot of work. However, for global supersymmetry there are not so many new concepts necessary before one is able to use the advantages of superspace. Hereby, the spacetime coordinates x^μ are extended by fermionic coordinates that are represented by a four-component Majorana spinor θ_α. Due to the symmetry properties of bilinears of spinors (B.17) and (B.18) it is immediately clear that $\bar{\theta}\gamma_\mu\theta = \bar{\theta}\sigma_{\mu\nu}\theta = 0$. Furthermore, as θ has only four components, any power series in θ terminates after quartic order. The formulas (B.32), (B.33) and (B.34) suggest, that any product of two spinors is proportional to a linear combination of $(\bar{\theta}\theta)$, $(\bar{\theta}\gamma_\mu\gamma_5\theta)$ and $(\theta\gamma_5\theta)$, while a product of three spinors only is proportional to $(\bar{\theta}\gamma_5\theta)$ and a product of four θs is proportional to $(\bar{\theta}\theta)^2$. With this at hand, we can express the most general function of x^μ and θ, called a superfield, as

$$\begin{aligned} S(x,\theta) \;=\; & C(x) - \mathrm{i}\bar{\theta}\gamma_5\psi(x) - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)M(x) - \frac{1}{2}(\bar{\theta}\theta)N(x) + \\ & + \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)A^\mu(x) - \mathrm{i}(\bar{\theta}\gamma_5\theta)\bar{\theta}\lambda(x) - \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\theta)\bar{\theta}\gamma^\mu\partial_\mu\psi(x) + \\ & + \frac{1}{4}(\bar{\theta}\theta)^2(D(x) + \frac{1}{2}\partial_\mu\partial^\mu C(x)) \,. \end{aligned} \tag{C.1}$$

It is convenient to separate $\partial_\mu\partial^\mu C(x)$ and $\gamma^\mu\partial_\mu\psi(x)$ from D and λ, respectively, as will become clear in a moment. The component fields λ and ψ are fermionic while A_μ is a vector field. The remaining fields are scalar or pseudoscalar fields, depending also on whether $S(x,\theta)$ is a scalar field.

Supersymmetry transformations are generated by the infinitesimal operators

$$\mathcal{Q} = \gamma_5\epsilon\frac{\partial}{\partial\theta} + \gamma^\mu\theta\frac{\partial}{\partial x^\mu}\,, \tag{C.2}$$

$$\bar{\mathcal{Q}} = \frac{\partial}{\partial\theta} - \gamma_5\epsilon\gamma^\mu\theta\frac{\partial}{\partial x^\mu}\,. \tag{C.3}$$

The transformation laws of the fields contained in $S(x,\theta)$ are found from

$$\delta S = (\bar{\varepsilon}\mathcal{Q})S \tag{C.4}$$

and in using the rearrangement formulas from the previous section to put each term into its corresponding form proportional to a standard bilinear in θ. The advantage of superfields is that if S_1 and S_2 are superfields, then $S = S_1S_2$ is again a superfield. Also superfields automatically provide representations of the supersymmetry algebra on fields. Note that the superpartners are also characterized by the expansion in θ.

Besides (C.2) and (C.3), one can define another differential operators in superspace by

$$\mathcal{D} = \gamma_5\epsilon\frac{\partial}{\partial\theta} - \gamma^\mu\theta\frac{\partial}{\partial x^\mu} \tag{C.5}$$

$$\bar{\mathcal{D}} = \frac{\partial}{\partial\theta} + \gamma_5\epsilon\gamma^\mu\theta\frac{\partial}{\partial x^\mu} \tag{C.6}$$

where the only difference to (C.2) and (C.3) is a change in sign. This change, however, is responsible that the anticommuter

$$\{\mathcal{D},\mathcal{Q}\} = 0 \tag{C.7}$$

between the generator of supersymmetry and the differential operator $\mathcal{D}$ vanishes. In turn, $(\bar{\varepsilon}\mathcal{Q})$ commutes with $\mathcal{D}$ and besides the arbitrary polynomial function of $S(x,\theta)$ being a superfield, their superderivatives $\mathcal{D}S$, $\mathcal{D}\mathcal{D}S$, etc. are superfields as well. In other words, the superderivatives are used in order to impose constraints on the general superfield $S(x,\theta)$. Requiring that the superfield is real, i.e. $S(x,\theta) = S^*(x,\theta)$, one obtains the so-called real superfield. Usually it is denoted by $V(x,\theta)$ and formally given by the same expression (C.1), but with only real component fields. There is a certain arbitrariness in the expansion of the most general superfield and we can chose as well

$$\begin{aligned}\Omega(x,\theta) &= B(x) - \frac{1}{2}\bar{\theta}(1+\gamma_5)\omega - \frac{1}{2}(\bar{\theta}(1+\gamma_5)\theta)P(x) + \\ &+\frac{1}{2}(\bar{\theta}\gamma_5\gamma_\mu\theta)\partial^\mu W(x) - \frac{1}{2}(\bar{\theta}\gamma_5\theta)\bar{\theta}(1+\gamma_5)\gamma^\mu\partial_\mu\omega(x) + \\ &+\frac{1}{8}(\bar{\theta}\theta)^2\partial_\mu\partial^\mu B(x))\end{aligned} \tag{C.8}$$

where $B(x)$, $P(x)$ and $W(x)$ are arbitrary complex functions of spacetime and the spinor is an arbitrary Majorana spinor. With help of the complex conjugation formulas for spinor bilinears, (B.39) and (B.40), one can determine the complex conjugated superfield to (C.8). It is not difficult to see that $V(x,\theta) + \operatorname{Im}\Omega(x,\theta)$ allow to gauge away the nonphysical degrees of freedom in the real superfield according to

$$C(x) \rightarrow C(x) - \operatorname{Im} B(x) \tag{C.9}$$

$$\psi(x) \rightarrow \psi(x) + \omega(x) \tag{C.10}$$

$$M(x) \rightarrow M(x) - \operatorname{Re} P(x) \tag{C.11}$$

$$N(x) \rightarrow N + \operatorname{Im} P(x) \tag{C.12}$$

$$A_\mu \rightarrow A_\mu + \partial_\mu W(x) \tag{C.13}$$

while the transformation of the vector field acts like a gauge transformation. Thus, we see that the arbitrariness in the general superfield allows one to put the real superfield into the form

$$V(x,\theta) = \frac{\mathrm{i}}{2}(\bar{\theta}\gamma_5\gamma^\mu\theta)A_\mu - \mathrm{i}(\bar{\theta}\gamma_5\theta)\bar{\theta}\lambda + \frac{1}{4}(\bar{\theta}\theta)^2 D(x) \tag{C.14}$$

which is the so called "Wess-Zumino gauge". So far we have treated Abelian gauge theories. The nonabelian generalization is not as easy, but in Wess-Zumino gauge, the corresponding real superfield is obtained from (C.14) by replacing the Abelian objects with the corresponding nonabelian counterparts.

The supersymmetry transformations, obtained from applying the operator $\bar{\varepsilon}\mathcal{Q}$ to the real superfield, are the same transformations as found in the text by the Noether method, i.e. (4.24), (4.25) and (4.26). We observe that the transformation of the auxiliary field, (4.26), is proportional to a derivative and on the other side, the auxiliary field is given by the highest component in the decomposition after θ, the so called D-term.

The chiral multiplet is given by the chiral superfield B which is determined by

$$0 \;=\; \bar{\mathcal{D}}B\,. \tag{C.15}$$

In its decomposition we find the scalar field z, the fermionic superpartner χ and the auxiliary field F which is as in the gauge sector, the highest component proportional to $(\bar{\theta}\theta)^2$.In chiral multiplets, the term proportional to $(\bar{\theta}\theta)$ is called the F-term. The transformations under supersymmetry are the same as given in (4.17) to (6.34) in the context of the Noether method. We will not go into too much detail, but focus on the superfield, that contains the field strength. The curl multiplet, as it is occasionally called, is defined through the relation

$$W_\alpha \;:=\; -\frac{1}{4}(\mathcal{D}^T\epsilon\mathcal{D})\mathcal{D}_\alpha V(x,\theta)\,. \tag{C.16}$$

A rather simple form of this superfield is found, when one uses coordinates $x_+^\mu := x^\mu + 1/2(\bar{\theta}\gamma_5\gamma^\mu\theta)$. Then the left chiral superfield, containing the field srength, reads

$$W_L(x,\theta) \;=\; \mathrm{i}\lambda_L(x_+) + 2\mathrm{i}\sigma^{\mu\nu}\theta_L F_{\mu\nu}(x_+) + \mathrm{i}(\theta_L^T\epsilon\theta_L)\gamma^\mu\partial_\mu\lambda_R(x_+) + \theta_L D(x_+)\,. \tag{C.17}$$

With help of the formulae given in appendix B, one finds for the projection to the F-term

$$\frac{1}{2}\,\mathrm{Re}[W_{L\alpha}\epsilon_{\alpha\beta}W_{L\beta}]_F \;=\; -\frac{1}{2}\bar{\lambda}\gamma^\mu\partial_\mu\lambda - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2 \tag{C.18}$$

which is the same as (4.13). Note, that the imaginary part $\mathrm{Im}\,\bar{W}_L\gamma_5 W_L$ contains the total derivative $\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$.

Another superfield which is used in section 7 contains the antisymmetric tensor field. As the curl superfield, it is a chiral superfield and in accordance with (C.15) it is defined via

$$\mathcal{D}_R^T\Phi \;=\; 0\,. \tag{C.19}$$

The tensor field provides off shell 6 degrees of freedom of which 2 are rendered unphysical because $B_{\mu\nu}$ there is the freedom to add $2\partial_{[\mu}\Lambda_{\nu]}$ for suitable Λ_μ. One four component Majorana spinor is not enough to balance the degrees of freedom off-shell but two Majorana spinors are. The two bosonic degrees of freedom that are still missing are provided by a complex scalar. From this one can write down the θ-expansion of the spinor superfield as

$$\Phi \;=\; \chi - \big(\frac{1}{2}C + \sigma^{\mu\nu}B_{\mu\nu}\big)\theta + \bar{\theta}\theta(\eta + \gamma^\mu\gamma_5\partial_\mu\chi)\,. \tag{C.20}$$

The field strength of $B_{\mu\nu}$ is contained in the linear superfield L which is obtained from the spinor superfield by

$$L \;:=\; \frac{1}{2}\mathcal{D}^T_\alpha \epsilon_{\alpha\beta}\Phi_\beta \tag{C.21}$$

The explicit expansion of L is not of immediate importance and that is why it is not quoted here. An expansion can be found in [90], for example.

D Some direct calculations of section 6

In this appendix we prove some formulas used in section 6 by direct calculations. We will try to keep this appendix as self-contained as possible that the reader is not forced to thumb too much back and forth.

D.1 The Bianchi identity

In this appendix we want to prove the Bianchi identity (6.33).

In order to do so we have to compute the action of the covariant derivative on the field strength (6.32):

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^M \;=\; \partial_{[\mu}\mathcal{F}_{\nu\rho]}{}^M + X_{NP}{}^M A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P \tag{D.1}$$

The two terms of (D.1) are calculated separately:

- The computation of the first term in (D.1) gives

$$\begin{aligned}\partial_{[\mu}\mathcal{F}_{\nu\rho]}{}^M &= 2\partial_{[\mu}\partial_\nu A_{\rho]}{}^M + X_{[NP]}{}^M \partial_{[\mu}(A_\nu{}^N A_{\rho]}{}^P) \\ &= -2X_{[NP]}{}^M A_{[\mu}{}^N \partial_\nu A_{\rho]}{}^P\,.\end{aligned} \tag{D.2}$$

- The second term is found to be given by

$$X_{NP}{}^M A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P \;=\; X_{NP}{}^M A_{[\mu}{}^N \partial_\nu A_{\rho]}{}^P + X_{[QR]}{}^P X_{NP}{}^M A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R \tag{D.3}$$

  where we used the result of the auxiliary calculation (D.4).

Putting (D.2) and (D.3) together we obtain the Bianchi identity:

$$D_{[\mu}\mathcal{F}_{\nu\rho]}{}^M \;=\; 2X_{(NP)}{}^M A_{[\mu}{}^N \partial_\nu A_{\rho]}{}^P + X_{[QR]}{}^P X_{NP}{}^M A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R$$

Auxiliary calculation:

$$\begin{aligned}
X_{[QR]}{}^P X_{NP}{}^M A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R &= X_{[QR]}{}^P X_{NP}{}^M \cdot \frac{1}{6} \cdot (A_\mu{}^N A_\nu{}^Q A_\rho{}^R + A_\nu{}^N A_\rho{}^Q A_{\mu]}{}^R + \\
&\quad + A_\rho{}^N A_\mu{}^Q A_{\nu]}{}^R - A_\nu{}^N A_\mu{}^Q A_\rho{}^R - A_\rho{}^N A_\nu{}^Q A_{\mu]}{}^R - A_\mu{}^N A_\rho{}^Q A_\nu{}^R) \\
&= \frac{1}{3} A_\mu{}^N A_\nu{}^Q A_\rho{}^R \, (X_{[QR]}{}^P X_{NP}{}^M + X_{[NQ]}{}^P X_{RP}{}^M + X_{[RN]}{}^P X_{QP}{}^M) \\
&= \frac{1}{3} A_\mu{}^N A_\nu{}^Q A_\rho{}^R \big(X_{[QR]}{}^P X_{(NP)}{}^M + X_{[NQ]}{}^P X_{(RP)}{}^M + \\
&\quad + X_{[RN]}{}^P X_{(QP)}{}^M - X_{(P[N)}{}^M X_{QR]}{}^P \big) \\
&= \frac{2}{3} X_{(PN)}{}^M X_{[QR]}{}^P A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R \qquad \text{(D.4)}
\end{aligned}$$

We made use of the Jacobi identity (6.26) in order to obtain (D.4).

D.2 Gauge variation of $\mathcal{F}_{\mu\nu}{}^M$

In this appendix we will compute the gauge variation of the field strength with respect to (6.28). The relevant formulae are:

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M + X_{[PQ]}{}^M \; A_\mu{}^P A_\nu{}^Q \,, \qquad \text{(D.5)}$$

$$\delta A_\mu{}^M = \mathcal{D}_\mu \Lambda^M - Z^{M\alpha} \Xi_{\mu\alpha} \,, \qquad \text{(D.6)}$$

$$\mathcal{D}_\mu \Lambda^M = \partial_\mu \Lambda^M + X_{PQ}{}^M \; A_\mu{}^P \Lambda^Q \,, \qquad \text{(D.7)}$$

$$[X_M, X_N] = -X_{[MN]}{}^P X_P \,. \qquad \text{(D.8)}$$

Furthermore, we make use of the auxiliary calculation:

- Let us determine the gauge variation of the first term in (D.5). In order to do so we compute

$$\begin{aligned}
\delta(\partial_\mu A_\nu{}^M) &= \partial_\mu(\delta A_\nu{}^M) \\
&= \partial_\mu \mathcal{D}_\nu \Lambda^M - Z^{M\alpha} \partial_\mu \Xi_{\nu\alpha} \\
&= \partial_\mu \partial_\nu \Lambda^M + X_{PQ}{}^M \; \partial_\mu A_\nu{}^P \Lambda^Q + X_{PQ}{}^M \; A_\nu{}^P \partial_\mu \Lambda^Q - Z^{M\alpha} \partial_\mu \Xi_{\nu\alpha}
\end{aligned} \qquad \text{(D.9)}$$

and in using (D.5), we obtain from (D.9):

$$\begin{aligned}
\delta(2\partial_{[\mu}A_{\nu]}{}^M) &= X_{PQ}{}^M\ [\partial_\mu A_\nu{}^P - \partial_\nu A_\mu{}^P]\Lambda^Q + X_{PQ}{}^M\ [A_\nu{}^P\partial_\mu\Lambda^Q - A_\mu{}^P\partial_\nu\Lambda^Q] - \\
&\quad -Z^{M\alpha}[\partial_\mu\Xi_{\nu\alpha} - \partial_\nu\Xi_{\mu\alpha}] \\
&= X_{PQ}{}^M\ \mathcal{F}_{\mu\nu}{}^P\Lambda^Q -{}^2\ X_{PQ}{}^M X_{[RS]}{}^P A_\mu{}^R A_\nu{}^S\Lambda^Q - 2X_{PQ}{}^M\ A_{[\mu}{}^P\partial_{\nu]}\Lambda^Q - \\
&\quad -2Z^{M\alpha}\partial_{[\mu}\Xi_{\nu]\alpha}\,. \qquad \text{(D.10)}
\end{aligned}$$

- Next we calculate the variation of the second term of (D.5):

$$\begin{aligned}
\delta(X_{[PQ]}{}^M\ A_\mu{}^P A_\nu{}^Q) &= X_{[PQ]}{}^M\ (\delta A_\mu{}^P A_\nu{}^Q + A_\mu{}^P\delta A_\nu{}^Q) \\
&= X_{[PQ]}{}^M\ (\mathcal{D}_\mu\Lambda^P A_\nu{}^Q - Z^{P\alpha}\Xi_{\mu\alpha}A_\nu{}^Q + A_\mu{}^P\mathcal{D}_\nu\Lambda^Q - A_\mu{}^P\Xi_{\nu\alpha}Z^{Q\alpha}) \\
&= X_{[PQ]}{}^M\ (A_\nu{}^Q\partial_\mu\Lambda^P + X_{RS}{}^P A_\mu{}^R A_\nu{}^Q\Lambda^S + A_\mu{}^P\partial_\nu\Lambda^Q + \\
&\quad +X_{RS}{}^Q A_\mu{}^P A_\nu{}^R\Lambda^S - Z^{P\alpha}\Xi_{\mu\alpha}A_\nu{}^Q - A_\mu{}^P\Xi_{\nu\alpha}Z^{Q\alpha}) \\
&= X_{[PQ]}{}^M\ (A_\nu{}^Q\partial_\mu\Lambda^P + A_\mu{}^P\partial_\nu\Lambda^Q) + \\
&\quad +X_{[PQ]}{}^M\ (X_{RS}{}^P A_\mu{}^R A_\nu{}^Q\Lambda^S + X_{RS}{}^Q A_\mu{}^P A_\nu{}^R\Lambda^S) - \\
&\quad -X_{[PQ]}{}^M\ (Z^{P\alpha}\Xi_{\mu\alpha}A_\nu{}^Q + A_\mu{}^P\Xi_{\nu\alpha}Z^{Q\alpha}) \\
&= X_{[PQ]}{}^M\ (A_\mu{}^P\partial_\nu\Lambda^Q - A_\nu{}^P\partial_\mu\Lambda^Q) + \\
&\quad +X_{[QR]}{}^M X_{PS}{}^Q A_\mu{}^P A_\nu{}^R\Lambda^S + X_{[PQ]}{}^M X_{RS}{}^Q A_\mu{}^P A_\nu{}^R\Lambda^S - \\
&\quad -X_{[PQ]}{}^M\ (-Z^{Q\alpha}\Xi_{\mu\alpha}A_\nu{}^P + A_\mu{}^P\Xi_{\nu\alpha}Z^{Q\alpha}) \\
&= 2X_{[PQ]}{}^M\ A_{[\mu}{}^P\partial_{\nu]}\Lambda^Q + \\
&\quad +(-X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M)A_\mu{}^P A_\nu{}^R\Lambda^S - \\
&\quad -X_{[PQ]}{}^M Z^{Q\alpha}\ A_{[\mu}{}^P\Xi_{\nu]\alpha} \qquad \text{(D.11)}
\end{aligned}$$

The variation of the field strength (D.5) under (D.6) is given by (D.10) and (D.11), which is added up and simplified according to:

$$\begin{aligned}
\delta\mathcal{F}_{\mu\nu}{}^M &= X_{PQ}{}^M\ \mathcal{F}_{\mu\nu}{}^P\Lambda^Q - X_{PQ}{}^M X_{[RS]}{}^P A_\mu{}^R A_\nu{}^S\Lambda^Q - 2X_{PQ}{}^M\ A_{[\mu}{}^P\partial_{\nu]}\Lambda^Q - \\
&\quad -2Z^{M\alpha}\partial_{[\mu}\Xi_{\nu]\alpha} + 2X_{[PQ]}{}^M\ A_{[\mu}{}^P\partial_{\nu]}\Lambda^Q + \\
&\quad +(-X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M)A_\mu{}^P A_\nu{}^R\Lambda^S - 2X_{[PQ]}{}^M Z^{Q\alpha}\ A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= X_{PQ}{}^M\ \mathcal{F}_{\mu\nu}{}^P\Lambda^Q - 2X_{(PQ)}{}^M\ A_{[\mu}{}^P\partial_{\nu]}\Lambda^Q + \\
&\quad +(-X_{[PR]}{}^Q X_{QS}{}^M - X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M)A_\mu{}^P A_\nu{}^R\Lambda^S - \\
&\quad -2Z^{M\alpha}\partial_{[\mu}\Xi_{\nu]\alpha} - 2X_{[PQ]}{}^M Z^{Q\alpha}\ A_{[\mu}{}^P\Xi_{\nu]\alpha}
\end{aligned} \tag{D.12}$$

Now, let us have a closer look at the last line of (D.12) but before let us remember the closure constraint (6.20) which is displayed again:

$$f_{\gamma\beta}{}^\alpha\ \Theta_P{}^\gamma\Theta_S{}^\beta + \Theta_Q{}^\alpha\Theta_P{}^\gamma(t_\gamma)_S{}^Q \quad = \quad 0\,. \tag{D.13}$$

With help of (D.13), we can manipulate the last term of the last line of (D.12) as follows:

$$\begin{aligned}
-2X_{[PQ]}{}^M Z^{Q\alpha}A_{[\mu}{}^P\Xi_{\nu]\alpha} &= -X_{PQ}{}^M Z^{Q\alpha}A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -[\frac{1}{2}X_{PQ}{}^M\Omega^{QR}\Theta_R{}^\alpha]A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -[\frac{1}{2}\Theta_Q{}^\alpha(-X_{PR}{}^L\Omega^{MS}\Omega_{LS}\Omega^{QR})]A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -[\frac{1}{2}\Theta_Q{}^\alpha(-X_{PS}{}^L\Omega^{MS}\Omega_{LR}\Omega^{QR})]A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -[-\frac{1}{2}\Theta_Q{}^\alpha X_{PS}{}^Q\Omega^{MS}]A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -[\frac{1}{2}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma\Theta_S{}^\beta\Omega^{MS}]A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -Z^{M\beta}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -2Z^{M\beta}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma A_{[\mu}{}^P\Xi_{\nu]\alpha} + Z^{Q\alpha}X_{PQ}{}^M A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -2Z^{M\beta}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma A_{[\mu}{}^P\Xi_{\nu]\alpha} + Z^{Q\alpha}X_{PQ}{}^M A_{[\mu}{}^P\Xi_{\nu]\alpha} \\
&= -2Z^{M\beta}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma A_{[\mu}{}^P\Xi_{\nu]\alpha} + 2X_{(PQ)}{}^M Z^{Q\alpha}A_{[\mu}{}^P\Xi_{\nu]\alpha}
\end{aligned} \tag{D.14}$$

where we made use of $X_{P[R}{}^L\Omega_{S]L} = 0$ and of locality in the form $X_{QP}{}^M Z^{Q\alpha} = 0$. Note that we have just proven the relation

$$2X_{[PQ]}{}^M Z^{Q\alpha} \quad = \quad Z^{M\beta}f_{\gamma\beta}{}^\alpha\Theta_P{}^\gamma \tag{D.15}$$

If we define the covariant derivative of a tensor field according to

$$\mathcal{D}_\mu \Xi_{\nu\alpha} \;:=\; \partial_\mu X_\nu + f_{\gamma\beta}{}^\alpha \Theta_P{}^\gamma A_\mu{}^P \Xi_{\nu\alpha}\,, \tag{D.16}$$

then we obtain for (D.12) the following expression:

$$\begin{aligned}\delta \mathcal{F}_{\mu\nu}{}^M \;=\;& X_{PQ}{}^M\, \mathcal{F}_{\mu\nu}{}^P \Lambda^Q - 2Z^{M\alpha}\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} + \\ & +(-X_{[PR]}{}^Q X_{QS}{}^M - X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M) A_\mu{}^P A_\nu{}^R \Lambda^S - \\ & -2X_{(PQ)}{}^M A_{[\mu}{}^P \left(\partial_{\nu]}\Lambda^Q - Z^{Q\alpha}\Xi_{\nu]\alpha}\right)\,.\end{aligned} \tag{D.17}$$

Now, let us have a closer look at the terms in (D.17) proportional to $A_\mu{}^P A_\nu{}^R \Lambda^S$:

$$\begin{aligned} & -X_{[PR]}{}^Q X_{QS}{}^M - X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M = \\ =\; & -X_{[PR]}{}^Q X_{[QS]}{}^M - X_{[PS]}{}^Q X_{[RQ]}{}^M + X_{[RS]}{}^Q X_{[PQ]}{}^M - \\ & -X_{[PR]}{}^Q X_{(QS)}{}^M - X_{(PS)}{}^Q X_{[RQ]}{}^M + X_{(RS)}{}^Q X_{[PQ]}{}^M = \\ =\; & X_{[PR]}{}^Q X_{[SQ]}{}^M + X_{[SP]}{}^Q X_{[RQ]}{}^M + X_{[RS]}{}^Q X_{[PQ]}{}^M - \\ & -X_{[PR]}{}^Q X_{(QS)}{}^M - X_{(PS)}{}^Q X_{[RQ]}{}^M + X_{(RS)}{}^Q X_{[PQ]}{}^M \end{aligned} \tag{D.18}$$

The first line of (D.18) satisfies the modified Jacobi identity (6.26) (repeated down in equation (D.19)) due to (D.6)

$$X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R \;=\; -X_{(P[Q)}{}^R X_{MN]}{}^P \;. \tag{D.19}$$

Then we can manipulate the first line of (D.18) as follows:

$$\begin{aligned} X_{[PR]}{}^Q X_{[SQ]}{}^M + X_{[SP]}{}^Q X_{[RQ]}{}^M + X_{[RS]}{}^Q X_{[PQ]}{}^M \;&=\; -X_{(Q[S)}{}^M X_{PR]}{}^Q \\ = -\frac{1}{3}\cdot [X_{(QS)}{}^M X_{[PR]}{}^Q + X_{(QR)}{}^M X_{[SP]}{}^Q + X_{(QP)}{}^M X_{[RS]}{}^Q] & \\ = -\frac{1}{3}\cdot [X_{[PR]}{}^Q X_{(QS)}{}^M + X_{[SP]}{}^Q X_{(QR)}{}^M + X_{[RS]}{}^Q X_{(QP)}{}^M] & \end{aligned}$$

Let us continue with the second line in (D.18)

$$
\begin{aligned}
& -X_{[PR]}{}^Q X_{(QS)}{}^M - X_{(PS)}{}^Q X_{[RQ]}{}^M + X_{(RS)}{}^Q X_{[PQ]}{}^M = \\
= \;& \frac{1}{4}\cdot[-X_{PR}{}^Q X_{QS}{}^M - X_{PR}{}^Q X_{SQ}{}^M + X_{RP}{}^Q X_{QS}{}^M + X_{RP}{}^Q X_{SQ}{}^M - X_{PS}{}^Q X_{RQ}{}^M - X_{SP}{}^Q X_{RQ}{}^M \\
& + X_{PS}{}^Q X_{QR}{}^M + X_{SP}{}^Q X_{QR}{}^M + X_{RS}{}^Q X_{PQ}{}^M + X_{SR}{}^Q X_{PQ}{}^M - X_{RS}{}^Q X_{QP}{}^M - X_{SR}{}^Q X_{QP}{}^M] \\
= \;& \frac{1}{4}\cdot[X_{RP}{}^Q X_{SQ}{}^M - X_{SP}{}^Q X_{RQ}{}^M + X_{SR}{}^Q X_{PQ}{}^M - X_{PR}{}^Q X_{SQ}{}^M + X_{RS}{}^Q X_{PQ}{}^M - X_{PS}{}^Q X_{RQ}{}^M \\
& + X_{PS}{}^Q X_{QR}{}^M + X_{SP}{}^Q X_{QR}{}^M - X_{RS}{}^Q X_{QP}{}^M - X_{SR}{}^Q X_{QP}{}^M + X_{RP}{}^Q X_{QS}{}^M - X_{PR}{}^Q X_{QS}{}^M] \\
= \;& \frac{1}{4}\cdot[-X_{[RS]}{}^Q X_{QP}{}^M - X_{[SP]}{}^Q X_{QR}{}^M - X_{[RP]}{}^Q X_{QS}{}^M + \\
& + X_{PS}{}^Q X_{QR}{}^M + X_{SP}{}^Q X_{QR}{}^M - X_{RS}{}^Q X_{QP}{}^M - X_{SR}{}^Q X_{QP}{}^M + X_{RP}{}^Q X_{QS}{}^M - X_{PR}{}^Q X_{QS}{}^M] \\
= \;& \frac{1}{4}\cdot[-X_{[RS]}{}^Q X_{QP}{}^M - X_{[SP]}{}^Q X_{QR}{}^M + X_{[RP]}{}^Q X_{QS}{}^M] \\
= \;& \frac{1}{4}\cdot[X_{[RS]}{}^Q X_{[PQ]}{}^M + X_{[SP]}{}^Q X_{[RQ]}{}^M + X_{[PR]}{}^Q X_{[SQ]}{}^M - X_{[RS]}{}^Q X_{(PQ)}{}^M - X_{[SP]}{}^Q X_{(RQ)}{}^M - \\
& - X_{[PR]}{}^Q X_{(SQ)}{}^M] \\
= \;& \frac{1}{4}\cdot[-X_{[PR}{}^Q X_{(S]Q)}{}^M - 3\cdot X_{[PR}{}^Q X_{(S]Q)}{}^M] \\
= \;& -X_{[PR}{}^Q X_{(S]Q)}{}^M
\end{aligned}
\tag{D.20}
$$

where we made use of (D.8), (D.19) and locality in form of $X_{QP}{}^R Z^{Q\alpha}$. If we add the first line in (D.18) to (D.20), we obtain

$$
(-X_{[PR]}{}^Q X_{QS}{}^M - X_{PS}{}^Q X_{[RQ]}{}^M + X_{RS}{}^Q X_{[PQ]}{}^M) A_\mu{}^P A_\nu{}^R \Lambda^S \;=\; -2X_{[RS}{}^Q X_{(P]Q)}{}^M A_\mu{}^P A_\nu{}^R \Lambda^S ,
$$

which completes the partial derivative in the last line of (D.17) to form a covariant derivative. From (D.6) we finally see, how the field strength $\mathcal{F}_{\mu\nu}{}^M$ transforms under the gauge transformation (6.99), i.e., we have

$$
\delta\mathcal{F}_{\mu\nu}{}^M \;=\; X_{PQ}{}^M \mathcal{F}_{\mu\nu}{}^P \Lambda^Q - 2Z^{M\alpha}\mathcal{D}_{[\mu}\Xi_{\nu]\alpha} - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q . \tag{D.21}
$$

D.3 Gauge variation of the generalized Chern-Simons term

We want to show that the gauge variation of the generalized Chern-Simons term (6.50) is given by (6.51), i.e.,

$$
\begin{aligned}
\delta\mathcal{L}_{\rm GCS} \;=\;& \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\left[\mathcal{F}_{\mu\nu}{}^\Lambda D_\rho \delta A_{\sigma\Lambda} - \mathcal{F}_{\mu\nu\Lambda} X_{(PQ)}{}^\Lambda A_\rho^P \delta A_\sigma^Q\right] - \\
& - \epsilon^{\mu\nu\rho\sigma} D_{MNP} A_\mu^M \delta A_\nu^N \left(\partial_\rho A_\sigma^P + \frac{3}{8} X_{RS}{}^P A_\rho^R A_\sigma^S\right) .
\end{aligned}
\tag{D.22}
$$

For future reference we want to denote the first line of (D.22) by Δ and expand Δ in terms of the vector fields. Consequently, we have

$$\begin{aligned}\Delta \;:=\;& \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\left[\mathcal{F}_{\mu\nu}{}^{\Lambda}\,D_\rho\delta A_{\sigma\Lambda} - \mathcal{F}_{\mu\nu\Lambda}\,X_{(PQ)}{}^{\Lambda}\,A_\rho^P\,\delta A_\sigma^Q\right] = \\ =\;& \epsilon^{\mu\nu\rho\sigma}\,\Bigg[\partial_\mu\,A_\nu^\Lambda\,\partial_\rho\delta A_{\sigma\Lambda} + \frac{1}{2}\,X_{PQ}{}^{\Lambda}\,A_\mu^P\,A_\nu^Q\partial_\rho\delta A_{\sigma\Lambda} + \partial_\mu\,A_\nu^\Lambda\,A_\rho^P\,\delta A_\sigma^Q\,X_{[PQ]\Lambda} \\ & -X_{(PQ)}{}^{M}\,\Omega_{MN}\,\partial_\mu A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q + \frac{1}{2}\,X_{PQ}{}^{\Lambda}\,X_{RS\Lambda}\,A_\mu^P\,A_\nu^Q\,A_\rho^R\,\delta A_\sigma^S \\ & -\frac{1}{2}\,X_{PQ\Lambda}\,X_{(RS)}{}^{\Lambda}\,A_\mu^P\,A_\nu^Q\,A_\rho^R\,\delta A_\sigma^S\Bigg]\,. \qquad \text{(D.23)}\end{aligned}$$

Let us prove the equation (D.22) first for the A^3 (e.g. $A\,A\partial\delta A$) and then for the A^4 terms (i.e. $A\,A\,A\,\delta A$).

The A^3-terms.

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{GCS}|\,A^3} &= (a)+(b)+(c)+(d)\,, \\ (a) &= \frac{2}{3}\,\epsilon^{\mu\nu\rho\sigma}\,X_{[MN]\Lambda}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^\Lambda\,, \\ (b) &= \frac{1}{3}\,\epsilon^{\mu\nu\rho\sigma}\,X_{MN\Lambda}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_\sigma^\Lambda\,, \\ (c) &= \frac{1}{3}\,\epsilon^{\mu\nu\rho\sigma}\,X_{[MN]}{}^{\Lambda}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_{\sigma\Lambda}\,, \\ (d) &= \frac{1}{6}\,\epsilon^{\mu\nu\rho\sigma}\,X_{MN}{}^{\Lambda}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_{\sigma\Lambda}\,. \qquad \text{(D.24)}\end{aligned}$$

We can group some of those terms in symplectic invariant expressions:

$$\begin{aligned}(b)+(d) &= \epsilon^{\mu\nu\rho\sigma}\,\left[-\frac{1}{3}\,X_{MN}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_\sigma^Q + \frac{1}{2}\,X_{MN}{}^{\Lambda}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_{\sigma\Lambda}\right]\,, \\ (a)+(c) &= \epsilon^{\mu\nu\rho\sigma}\,\left[\frac{1}{3}\,X_{[MN]}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q + X_{[MN]\Lambda}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^\Lambda\right]\,. \qquad \text{(D.25)}\end{aligned}$$

We can now compute the sum

$$\begin{aligned}(a)+(b)+(c)+(d) =\;& \epsilon^{\mu\nu\rho\sigma}\,\Bigg[\frac{1}{2}\,X_{MN}{}^{\Lambda}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_{\sigma\Lambda} + X_{[MN]\Lambda}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^\Lambda \\ & +\frac{1}{3}\,X_{[MN]}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q - \frac{1}{3}\,X_{MN}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_\sigma^Q\Bigg]\,. \qquad \text{(D.26)}\end{aligned}$$

Modulo total derivatives we can rewrite the last two terms of (D.26) in the following way:

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{3}X_{[MN]}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q-\frac{1}{3}X_{MN}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_\sigma^Q\right]=\\
&=\frac{1}{3}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{[MN]}{}^{P}\,\Omega_{PQ}-2\,X_{[MQ]}{}^{P}\,\Omega_{PN}\right)\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q=\\
&=\frac{1}{6}\,\epsilon^{\mu\nu\rho\sigma}\,\left[X_{MN}{}^{P}\,\Omega_{PQ}-X_{NM}{}^{P}\,\Omega_{PQ}-2\,X_{MQ}{}^{P}\,\Omega_{PN}\right.\\
&\left.+2\,(-3\,D_{MNQ}-X_{MN}{}^{P}\,\Omega_{PQ}-X_{NM}{}^{P}\,\Omega_{PQ})\right]\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q=\\
&=-\epsilon^{\mu\nu\rho\sigma}\,(X_{(MN)}{}^{P}\,\Omega_{PQ}+D_{MNQ})\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q\,.
\end{aligned}\tag{D.27}$$

Where we have used the definition of D_{MNQ} (6.18). From this we conclude that

$$\begin{aligned}
\delta\mathcal{L}_{\mathrm{GCS}|\,A^3} &= (a)+(b)+(c)+(d)=\epsilon^{\mu\nu\rho\sigma}\left[\frac{1}{2}X_{MN}{}^{\Lambda}\,A_\mu^M\,A_\nu^N\,\partial_\rho\delta A_{\sigma\Lambda}+X_{[MN]\Lambda}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^\Lambda\right.\\
&\left.-X_{(MN)}{}^{P}\,\Omega_{PQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q\right]-\epsilon^{\mu\nu\rho\sigma}\,D_{MNQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q=\\
&= \Delta_{|A^3}-\epsilon^{\mu\nu\rho\sigma}\,D_{MNQ}\,A_\mu^M\,\delta A_\nu^N\,\partial_\rho A_\sigma^Q\,.
\end{aligned}\tag{D.28}$$

The A^4 terms in $\delta\mathcal{L}_{GCS}$ are readily computed by noting that

$$\mathcal{L}_{\mathrm{GCS}|\,A^4} = \frac{1}{8}\,\epsilon^{\mu\nu\rho\sigma}\,X_{MN\Lambda}\,X_{PQ}{}^{\Lambda}\,A_\mu^M\,A_\nu^N\,A_\rho^P\,A_\sigma^Q\,.\tag{D.29}$$

We can then write

$$\delta\mathcal{L}_{\mathrm{GCS}|\,A^4} = \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{MN\Lambda}\,X_{[PQ]}{}^{\Lambda}+X_{MN}{}^{\Lambda}\,X_{[PQ]\Lambda}\right)\,A_\mu^M\,A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q\,.\tag{D.30}$$

Now let us compare the above expression with the A^4 terms in (D.23):

$$\begin{aligned}
\Delta_{|A^4} &= \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{MN}{}^{\Lambda}\,X_{PQ\Lambda}-X_{MN\Lambda}\,X_{(PQ)}{}^{\Lambda}\right)\,A_\mu^M\,A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q=\\
&= \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{MN}{}^{R}\,X_{PQ}{}^{S}\,\Omega_{RS}+X_{MN}{}^{\Lambda}\,X_{PQ\Lambda}-X_{MN\Lambda}\,X_{QP}{}^{\Lambda}\right)\,A_\mu^M\,A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q=\\
&= \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{MN}{}^{R}\,X_{PQ}{}^{S}\,\Omega_{RS}+X_{MN}{}^{R}\,X_{(PQ)}{}^{S}\,\Omega_{RS}+X_{MN}{}^{\Lambda}\,X_{[PQ]\Lambda}\right.\\
&\left.+\,X_{MN\Lambda}\,X_{[PQ]}{}^{\Lambda}\right)\,A_\mu^M\,A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q=\\
&= \delta\mathcal{L}_{\mathrm{GCS}|\,A^4}+\\
&+\frac{1}{4}\,\Omega_{RS}\,\epsilon^{\mu\nu\rho\sigma}\,\left(X_{MN}{}^{R}\,X_{PQ}{}^{S}+X_{MN}{}^{R}\,X_{(PQ)}{}^{S}\right)\,A_\mu^M\,A_\nu^N\,A_\rho^P\,\delta A_\sigma^Q\,.
\end{aligned}\tag{D.31}$$

Now let us consider the last term of (D.31) and show that it is proportional to the D_{MNP}. We use the following properties

$$\begin{aligned}
& \left(X_{MN}{}^R X_{PQ}{}^S \Omega_{RS} + X_{MN}{}^R X_{(PQ)}{}^S \Omega_{RS}\right)_{[MNP]} = \\
= & \frac{1}{2}\left(3 X_{MN}{}^R X_{PQ}{}^S \Omega_{RS} + X_{MN}{}^R X_{QP}{}^S \Omega_{RS}\right)_{[MNP]} = \\
= & \frac{1}{2}\left(3 X_{MN}{}^R X_{PQ}{}^S \Omega_{RS} + 3 X_{MN}{}^R d_{QPR} - X_{MN}{}^R X_{RP}{}^S \Omega_{QS} - X_{MN}{}^R X_{PQ}{}^S \Omega_{RS}\right)_{[MNP]} = \\
= & \frac{3}{2} X_{[MN}{}^R D_{P]QR} + \frac{1}{2}\left(2 X_{MN}{}^R X_{PR}{}^S \Omega_{QS} - X_{MN}{}^R X_{RP}{}^S \Omega_{QS}\right)_{[MNP]} = \\
= & \frac{3}{2} X_{[MN}{}^R d_{P]QR}\,,
\end{aligned} \tag{D.32}$$

where we have used the constraint (6.20) in the form:

$$\left(2 X_{MN}{}^R X_{PR}{}^S \Omega_{QS} - X_{MN}{}^R X_{RP}{}^S \Omega_{QS}\right)_{[MNP]} \;=\; 0\,. \tag{D.33}$$

From the above result and equation (D.31) we conclude that

$$\Delta_{|A^4} \;=\; \delta\mathcal{L}_{\mathrm{GCS}|\,A^4} + \frac{3}{8}\,\epsilon^{\mu\nu\rho\sigma}\, X_{MN}{}^R d_{PQR}\, A_\mu^M\, A_\nu^N\, A_\rho^P\, \delta A_\sigma^Q\,. \tag{D.34}$$

Equations (D.28) and (D.34) imply together (D.22) which concludes the proof.

E Abelian Gauging

In this appendix we are going to show how the Abelian gauging of the symplectically covariant formalism of section 6 leads to models with massive tensor fields as discussed in [29, 89, 90, 107–110]. The discussion is along the lines of [83] but presented in more detail because then the connection to [90] becomes more clear.

The Abelian gauging is obtained from the ungauged rigid symmetry group G_{rigid} by decomposing it into $G_V \times G_M$ where G_V and G_M act exclusively on vector and matter fields, respectively. Therefore, the generators decompose into mutually commuting sets $\{t_\alpha\} = \{t_A\} \oplus \{t_a\}$ where only t_A acts nontrivially on vector fields and the generators t_a act merely in the matter sector. Also the embedding tensor $\Theta_M{}^\alpha$ decomposes defining the generators of the gauge group, $X_M = \Theta_M{}^A t_A + \Theta_M{}^a t_a$. The closure constraint (6.20) splits up into two separate equations

$$f_{AB}{}^C \Theta_M{}^A \Theta_N{}^B - (t_A)_N{}^P \Theta_M{}^A \Theta_P{}^C \;=\; 0 \tag{E.1}$$

$$f_{ab}{}^c \Theta_M{}^a \Theta_N{}^b - (t_a)_N{}^P \Theta_M{}^a \Theta_P{}^c \;=\; 0 \tag{E.2}$$

and the second equation leads to an additional and independent constraint

$$\Theta^{\Lambda[a}\Theta_{\Lambda}{}^{b]} \;=\; 0\,. \tag{E.3}$$

It is $\Theta_M{}^A = 0$ for Abelian gaugings without axionic shifts. Consequently, the generalized Chern-Simons terms vanish and the Lagrangian takes the simple form

$$\begin{aligned}\mathcal{L}_{VT} \;=\;& \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\,\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{\Sigma} - \\ & -\frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda a}\;B_{\mu\nu\,a}F_{\rho\sigma\,\Lambda} + \frac{1}{32}\varepsilon^{\mu\nu\rho\sigma}\Theta^{\Lambda a}\Theta_{\Lambda}{}^{b}\;B_{\mu\nu\,a}B_{\rho\sigma\,b}\end{aligned} \tag{E.4}$$

where

$$\mathcal{H}_{\mu\nu}{}^{\Lambda} \;:=\; F_{\mu\nu}{}^{\Lambda} + \frac{1}{2}\Theta^{\Lambda a}B_{\mu\nu\,a} \tag{E.5}$$

The magnetic vector fields can be eliminated from (E.4) by their equations of motion

$$\frac{\delta\mathcal{L}}{\delta A_{\Lambda}} = d\frac{\delta\mathcal{L}}{\delta dA_{\Lambda}} \tag{E.6}$$

which are algebraic and not dynamical because the magnetic vector fields do not possess any kinetic terms. Before we can do so, let us specify the coupling to the matter sector. In the case that we consider we will assume that the matter couplings are given by a nonlinear sigma model with gauged isometries of its target space. The coordinates of the target space are represented by $\{\phi^x, q^i\}$ and its metric G_{nm} only depends on the subset $\{\phi^x\}$. In other words, the sigma model is invariant under constant shifts of $\{q^i\}$. Then the gauging of the isometries leads to the covariant derivatives

$$D^{\mu}q^i \;:=\; \partial^{\mu}q^i - \Theta^{\Lambda i}A^{\mu}_{\Lambda} - \Theta_{\Lambda}{}^{\;i}\;A^{\mu\Lambda} \tag{E.7}$$

and, hence, the nonlinear sigma model is given by

$$\begin{aligned}\mathcal{L}_{\text{matter}} \;=\;& \frac{1}{2}G_{xy}(\phi)\partial^{\mu}\phi^x\partial_{\mu}\phi^y + \frac{1}{2}G_{xi}(\phi)\partial^{\mu}\phi^x D_{\mu}q^i + \frac{1}{2}G_{ix}(\phi)D^{\mu}q^i\partial_{\mu}\phi^x + \\ & + \frac{1}{2}G_{ij}(\phi)D^{\mu}q^iD_{\mu}q^j\end{aligned} \tag{E.8}$$

Now we are able to use (E.6) and determine the equations of motion for the magnetic vector fields:

$$G_{xi}(\phi)\Theta^{\Lambda i}\partial^{\mu}\phi^x + G_{ij}(\phi)\Theta^{\Lambda j}D^{\mu}q^i \;=\; \frac{1}{2}\Theta^{\Lambda i}\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}B_{\rho\sigma\;i}\,. \tag{E.9}$$

Solving this equation for the magnetic vector field

$$\Theta^{\Sigma i} A^{\mu}_{\Sigma} = (\partial^{\mu} q^i - \Theta_{\Sigma}{}^{i} A^{\mu\Sigma}) - \frac{1}{2}(G^{-1})^{ji}(\phi) \cdot \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma\, j} \tag{E.10}$$

it is possible to eliminate the magnetic vector field from the Lagrangian and we obtain

$$\begin{aligned}\mathcal{L} = & \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\, \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}^{\mu\nu\,\Sigma} + \frac{1}{8}\mathcal{R}_{\Lambda\Sigma} \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\mu\nu}{}^{\Lambda} \mathcal{H}_{\rho\sigma}{}^{\Sigma} + \frac{1}{8}\Theta_{\Lambda}{}^{i}\, \varepsilon^{\mu\nu\rho\sigma}\, B_{\mu\nu\, i}\, F_{\rho\sigma}{}^{\Lambda} + \\ & + \frac{1}{8}(G^{-1})^{ji}(\phi) \cdot \varepsilon^{\mu\nu\rho\sigma} \partial_{\nu} B_{\rho\sigma\, i} \cdot \varepsilon_{\mu\lambda\alpha\beta}\, \partial^{\lambda} B^{\alpha\beta}{}_{j} + \\ & + \frac{1}{32}\Theta^{\Lambda i}\, \Theta_{\Lambda}{}^{j}\, \varepsilon^{\mu\nu\rho\sigma}\, B_{\mu\nu\, i}\, B_{\rho\sigma\, j} + \ldots \end{aligned} \tag{E.11}$$

where ellipsis denote the rest of the coupling terms that are not of concern for us. We see that for the Abelian gauging after eliminating the magnetic vector fields by its equations of motion one ends up with a Lagrangian that, in addition to electric fields, consists of a kinetic term for the tensor fields and the topological coupling of $B_{\mu\nu\, a}$ to the electric field strength. The Lagrangian (E.11) reproduces the results of [94] and [95]. In [90] the authors discussed the supersymmetrization of (E.11) which will be reinvestigated in section 7.

F Details on the calculations of section 7

In agreement with (C.1), let the expansion of a real superfield be given as

$$\begin{aligned}\Omega(x,\theta,\bar\theta) = & C(x) - \mathrm{i}\bar\theta\gamma_5\omega(x) - \frac{i}{2}\bar\theta\gamma_5\theta M(x) - \frac{1}{2}\bar\theta\theta N(x) \\ & + \frac{\mathrm{i}}{2}\bar\theta\gamma^{\mu}\gamma_5\theta A_{\mu}(x) - \mathrm{i}(\bar\theta\gamma_5\theta)\bar\theta\lambda(x) - \frac{\mathrm{i}}{2}(\bar\theta\gamma_5\theta)\bar\theta\gamma^{\mu}\partial_{\mu}\omega(x) + \quad (\text{F.1}) \\ & + \frac{1}{4}\bar\theta\theta\bar\theta\theta(D(x) + \frac{1}{2}\Box C(x)). \quad (\text{F.2})\end{aligned}$$

In order to evaluate the D-term of $U(\Omega)$ we Taylor expand $U(\Omega)$ around $\Omega|_{\theta=\bar\theta=0}$ and project out the $\theta\theta\bar\theta\bar\theta$ component. In order to do so, it is convenient to introduce $X := \Omega - \Omega|$ and we can see that $X^5 = 0$. Hence,

$$U(\Omega) = U(\Omega| + X) = U(\Omega)| + \frac{\partial U}{\partial C}X + \frac{1}{2!}\frac{\partial^2 U}{\partial C^2}X^2 + \ldots + \frac{1}{4!}\frac{\partial^4 U}{\partial C^4}X^4. \tag{F.3}$$

As we are interested in the D-term of this expression, we need the following results

$$\begin{aligned}
X|_{\bar\theta\theta\bar\theta\theta} &= \frac{1}{4}(D+\frac{1}{2}\Box C) && \text{(F.4)}\\
X^2|_{\bar\theta\theta\bar\theta\theta} &= \frac{1}{8}(M^2+N^2)-\frac{1}{4}\bar\omega\lambda-\frac{1}{8}A_\mu A_\nu\eta^{\mu\nu}+\frac{1}{16}\bar\omega\gamma^\mu\partial_\mu\omega && \text{(F.5)}\\
X^3|_{\bar\theta\theta\bar\theta\theta} &= -\frac{3i}{16}M\bar\omega\gamma_5\omega+\frac{3\mathrm{i}}{16}N\bar\omega\omega+\frac{3\mathrm{i}}{16}\bar\omega\gamma^\mu\gamma_5\omega^m A_\mu && \text{(F.6)}\\
X^4|_{\bar\theta\theta\bar\theta\theta} &= \frac{3}{16}\bar\omega\omega\bar\omega\omega && \text{(F.7)}
\end{aligned}$$

where we made use of the rearrangement formulae presented in appendix B. Inserting this back into the Taylor expansion leaves us with

$$\begin{aligned}
U(\Omega)|_{\theta\theta\bar\theta\bar\theta} = {}& \frac{1}{4}U'(C)\left[D(x)+\frac{1}{2}\Box C(x)\right]+\frac{1}{8}U''(C)\left[\frac{1}{2}(M^2+N^2)-\bar\omega\lambda-\frac{1}{2}A_\mu A_\nu\eta^{\mu\nu}+\right.\\
&\left.+\frac{1}{4}\bar\omega\gamma^\mu\partial_\mu\omega\right]+\frac{\mathrm{i}}{32}U'''(C)\left[-M\bar\omega\gamma_5\omega+N\bar\omega\omega+\bar\omega\gamma^\mu\gamma_5\omega^m A_\mu\right]+\\
&+\frac{1}{128}U^{(4)}(C)\bar\omega\omega\bar\omega\omega && \text{(F.8)}
\end{aligned}$$

Other important relations for evaluating the gauge coupling matrix are

$$\begin{aligned}
(\operatorname{Re}\hat f)_{11} &= \frac{\operatorname{Re}g_{11}}{|g_{11}|^2} && \text{(F.9)}\\
(\operatorname{Re}\hat f)_{1a} &= \frac{\operatorname{Re}(ig_{1a}g_{11}^*)}{|g_{11}|^2}=-\frac{\operatorname{Im}g_{1a}\operatorname{Re}g_{11}-\operatorname{Re}g_{1a}\operatorname{Im}g_{11}}{|g_{11}|^2} && \text{(F.10)}\\
(\operatorname{Re}\hat f)_{ab} &= \operatorname{Re}(g_{ab}-\frac{g_{1a}g_{1b}g_{11}^*}{|g_{11}|^2})=\\
&= \operatorname{Re}g_{ab}+\frac{1}{|g_{11}|^2}\cdot(\operatorname{Re}g_{11}\operatorname{Im}g_{1a}\operatorname{Im}g_{1b}-\operatorname{Re}g_{11}\operatorname{Re}g_{1a}\operatorname{Re}g_{1b}-\\
&\quad -\operatorname{Im}g_{1a}\operatorname{Re}g_{1b}\operatorname{Im}g_{11}-\operatorname{Re}g_{1a}\operatorname{Im}g_{1b}\operatorname{Im}g_{11}) && \text{(F.11)}
\end{aligned}$$

With the help of the above given relations, we get from (7.20), (7.21) to (7.22), (7.23). Now if we solve (7.22) in terms of D^Ω

$$D^\Omega=D^b\frac{m}{2}\,(\operatorname{Im}g_{1b}-\frac{\operatorname{Re}g_{1b}\operatorname{Im}g_{11}}{\operatorname{Re}g_{11}})-\frac{m^2}{8}\frac{|g_{11}|^2}{\operatorname{Re}g_{11}}U' \qquad \text{(F.12)}$$

and insert it into (7.23), we find that

$$D^b[\operatorname{Re}g_{ab}-\frac{\operatorname{Re}g_{1a}\operatorname{Re}g_{b1}}{\operatorname{Re}g_{11}}]=-\frac{m}{4}U'(\operatorname{Im}g_{1a}-\frac{\operatorname{Re}g_{1a}\operatorname{Im}g_{11}}{\operatorname{Re}g_{11}}) \qquad \text{(F.13)}$$

The equation can be further modified in acting with $[(\mathrm{Re}\, g)^{-1}]^{\Sigma a}$ from the left. The left hand side therefore gives

$$
\begin{aligned}
&[(\mathrm{Re}\, g)^{-1}]^{\Sigma a}(\mathrm{Re}\, g_{ac} - \frac{\mathrm{Re}\, g_{1a}\,\mathrm{Re}\, g_{c1}}{\mathrm{Re}\, g_{11}})D^c = [(\mathrm{Re}\, g)^{-1}]^{\Sigma a}\,\mathrm{Re}\, g_{ac}D^c - [(\mathrm{Re}\, g)^{-1}]^{\Sigma a}\frac{\mathrm{Re}\, g_{1a}\,\mathrm{Re}\, g_{c1}}{\mathrm{Re}\, g_{11}}D^c \\
&= (\delta^\Sigma_c - [(\mathrm{Re}\, g)^{-1}]^{\Sigma 1}\,\mathrm{Re}\, g_{1c})D^c - (\delta^\Sigma_1 - [(\mathrm{Re}\, g)^{-1}]^{\Sigma 1}\,\mathrm{Re}\, g_{11})\frac{\mathrm{Re}\, g_{1c}D^c}{\mathrm{Re}\, g_{11}} = (\delta^\Sigma_c - \delta^\Sigma_1\frac{\mathrm{Re}\, g_{1c}}{\mathrm{Re}\, g_{11}})D^c
\end{aligned} \tag{F.14}
$$

while on the other hand we have for the right hand side

$$
\begin{aligned}
\left[(\mathrm{Re}\, g)^{-1}\right]^{\Sigma a}\left(\mathrm{Im}\, g_{1a} - \frac{\mathrm{Re}\, g_{1a}\,\mathrm{Im}\, g_{11}}{\mathrm{Re}\, g_{11}}\right) &= -\left(\delta^\Sigma_1 - [(\mathrm{Re}\, g)^{-1}]^{\Sigma 1}\,\mathrm{Re}\, g_{11}\right)\frac{\mathrm{Im}\, g_{11}}{\mathrm{Re}\, g_{11}} + \\
&\quad + \left[(\mathrm{Re}\, g)^{-1}\right]^{\Sigma a}\,\mathrm{Im}\, g_{a1} \\
&= [(\mathrm{Re}\, g)^{-1}]^{\Sigma\Lambda}\,\mathrm{Im}\, g_{\Lambda 1} - \delta^\Sigma_1
\end{aligned} \tag{F.15}
$$

and, thus, we have

$$
(\delta^\Sigma_c - \delta^\Sigma_1\frac{\mathrm{Re}\, g_{1c}}{\mathrm{Re}\, g_{11}})D^c = -\frac{m}{4}U' \cdot \left(\left[(\mathrm{Re}\, g)^{-1}\right]^{\Sigma\Lambda}\mathrm{Im}\, g_{\Lambda 1} - \delta^\Sigma_1\right) \tag{F.16}
$$

Now we see that if $\Sigma = b$, then we have

$$
D^b = -\frac{m}{4}U' \cdot [(\mathrm{Re}\, g)^{-1}]^{b\Lambda}\,\mathrm{Im}\, g_{\Lambda 1} \tag{F.17}
$$

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